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LOCALLY QUASI-NILPOTENT ELEMENTARY OPERATORS

NADIA BOUDI AND MARTIN MATHIEU

ABSTRACT. Let A be a unital dense algebra of linear mappings on a complex vector space X. Let $\phi = \sum_{i=1}^n M_{a_i,b_i}$ be a locally quasi-nilpotent elementary operator of length n on A. We show that, if $\{a_1, \ldots, a_n\}$ is locally linearly independent, then the local dimension of $V(\phi) = \mathrm{span}\{b_i a_j : 1 \leq i,j \leq n\}$ is at most $\frac{n(n-1)}{2}$. If $\mathsf{ldim}\, V(\phi) = \frac{n(n-1)}{2}$, then there exists a representation of ϕ as $\phi = \sum_{i=1}^n M_{u_i,v_i}$ with $v_i u_j = 0$ for $i \geq j$. Moreover, we give a complete characterization of locally quasi-nilpotent elementary operators of length 3.

1. INTRODUCTION

Let A be a unital complex algebra. In order to gain information on the range of a linear mapping $\phi \colon A \to A$ one can impose conditions on the size of the spectrum of the elements $\phi(x)$, $x \in A$. Various classes of linear mappings such as (inner) derivations or generalised (inner) derivations with small spectrum in the range, especially when the spectrum of every element in the range consists only of zero, have been studied by Aupetit, Brešar, Le Page, Pták, Šemrl, the present authors and many others; see, e.g., [3, 9, 5, 6, 18] and the references contained therein. Quite often such results are connected with commutativity criteria; see, e.g., [3, §2 in Chapter V].

An attractive and fairly large class of linear mappings are the elementary operators, that is, those which can be written as $\phi(x) = \sum_{i=1}^n a_i x b_i$, $x \in A$. In this paper we intend, on the one hand, to extend various results that were available only for special elementary operators, for instance, for generalised inner derivations $x \mapsto ax - xb$, to general elementary operators; on the other hand, we want to uncover the algebraic structure behind arguments which were at times used in an analytic setting.

Clearly, every elementary operator leaves each ideal of A invariant and thus induces an elementary operator on each primitive quotient of A. As a result, we focus our attention here solely on the case of dense algebras of linear mappings on a complex vector space. The general case then is mainly a question of putting the information from primitive quotients together to a global picture which can, e.g., be done with the aid of the extended centroid of a semisimple algebra A.

In the setting of a dense algebra A of linear mappings on a vector space X, it turns out that the local dimension of $\operatorname{span}\{a_1, \ldots, a_n\}$ plays an important role in such descriptions. In the case where $\{a_1, \ldots, a_n\}$ is locally linearly independent, we will determine the maximal local dimension of $\operatorname{span}\{b_i a_j : 1 \leq i, j \leq n\}$, and we shall give a complete characterization if the local dimension is maximal. Our arguments depend heavily on Gerstenhaber's well-known theorem concerning linear spaces of nilpotent matrices [14]. Lacking a general description of maximal nilpotent spaces, and since the classification of minimal locally linearly dependent spaces is a highly non-trivial matter [8, 17], we will, in the last part of the paper (Section 4), have to limit ourselves to elementary operators of length at most three.

The main idea in our approach is as follows. Starting with a locally quasi-nilpotent elementary operator ϕ (that is, the spectrum of $\phi(x)$ is the singleton 0 for every $x \in A$), we aim to find finite-dimensional $\phi(x)$-invariant subspaces $L \subseteq X$, for certain x, so that $\phi(x)_{|L}$ will be nilpotent. If the dimension of L is large relative to the local dimension of $\operatorname{span}\{a_1, \ldots, a_n\}$, we will get full information on how to represent ϕ with suitable coefficients. The details of this approach will be explained at the start of Sections 3 and 4.

In a sequel to this paper, [7], we shall apply our results to the study of spectrally bounded elementary operators on Banach algebras continuing the line of research in [11] and [6], for example.

2. PRELIMINARIES

Let X, Y be two complex vector spaces. The space of all linear mappings from X to Y will be designated by $L(X, Y)$, and $L(X)$ stands for the algebra $L(X, X)$. Throughout this paper, A will denote a dense algebra (in the sense of Jacobson) of linear mappings on X.

Though not quite standard, we will use the term *elementary operator* for every linear mapping $\phi \colon A \to L(X)$ that can be written in the form

$$\phi(x) = \sum_{i=1}^{n} a_i x b_i \quad (x \in A), \tag{2.1}$$

for some $a_i, b_i \in L(X)$ and some $n \in \mathbb{N}$. Special cases are $L_a \colon x \mapsto ax$, $R_b \colon x \mapsto xb$ and $M_{a,b} = L_a R_b$. Clearly the representation of ϕ in a sum as in (2.1) is not unique; but it is well known that, in our setting, whenever we change the coefficients a_i, b_i to new ones u_i, v_i (and possibly the n to some m) we obtain $u_i \in \operatorname{span}\{a_1, \ldots, a_n\}$ and $v_i \in \operatorname{span}\{b_1, \ldots, b_n\}$. The smallest $n \in \mathbb{N}$ such that the elementary operator ϕ can be written as $\phi = \sum_{i=1}^{n} M_{a_i, b_i}$ is called *the length of* ϕ and will be abbreviated as $\ell(\phi)$. If $\phi = \sum_{i=1}^{n} M_{a_i, b_i}$ and $\ell(\phi) = n$ then, evidently, the sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are

linearly independent. Moreover, if $\phi = \sum_{i=1}^{n} M_{u_i, v_i}$, there exists an invertible matrix $P \in M_n(\mathbb{C})$ such that

$$(v_i u_j)_{1 \leq i,j \leq n} = P^{-1} (b_i a_j)_{1 \leq i,j \leq n} \, P.$$

We will denote by $\mathscr{El}(A, L(X))$ and $\mathscr{El}_n(A, L(X))$, respectively the set of elementary operators from A to $L(X)$ of arbitrary length and of fixed length n. If $\phi = \sum_{i=1}^{n} M_{a_i, b_i} \in \mathscr{El}(A, L(X))$, we define a new elementary operator ϕ^* by $\phi^* = \sum_{i=1}^{n} M_{b_i, a_i}$.

We shall use the following notation for $\phi = \sum_{i=1}^{n} M_{a_i, b_i}$:

$$L(\phi) = \mathrm{span}\{a_1, \ldots, a_n\},$$
$$R(\phi) = \mathrm{span}\{b_1, \ldots, b_n\}, \text{and}$$
$$V(\phi) = \mathrm{span}\{b_i a_j : 1 \leq i,j \leq n\}.$$

A linear mapping $y \in L(X)$ is said to be *quasi-nilpotent* if $\lambda - y$ is bijective for each $\lambda \in \mathbb{C} \setminus \{0\}$ and y itself is not bijective. We shall call $\phi \in \mathscr{El}(A, L(X))$ *locally quasi-nilpotent* if $\phi(x)$ is quasi-nilpotent for every $x \in A$.

For a finite-dimensional subspace V of $L(X,Y)$, let $\mathsf{ldim}\, V = \max\{\dim V\zeta : \zeta \in X\}$ denote the *local dimension of* V, cf. [16]. Recall that V (or, equivalently, a basis of V) is said to be *locally linearly dependent* if $\mathsf{ldim}\, V < \dim V$. For background on these notions, the reader is referred to [8, 17]. In the case that $\mathsf{ldim}\, V = \dim V$, any vector satisfying $\dim V\zeta = \dim V$ is called a *separating vector of* V.

Spaces of matrices with bounded rank have been extensively studied by many authors, see, e.g., [2], and their classification is still a challenging problem. The connection with locally linearly dependent spaces is given by the following lemma the proof of which is straightforward and hence omitted.

Lemma 2.1. *Let X, Y be complex vector spaces, and let V be a finite-dimensional subspace of $L(X,Y)$. Suppose that $\mathsf{ldim}\, V = r$ and set $\widehat{X} = \{\widehat{\zeta} : \zeta \in X\}$, where $\widehat{\zeta} : V \to Y$ is defined by $\widehat{\zeta}(T) = T\zeta$ for all $T \in V$. Then $\widehat{X}$ is a subspace of $L(V, Y)$ of rank at most r.*

Let $F(X)$ denote the ideal of $L(X)$ consisting of those linear mappings with finite-dimensional range and let X^* be the space of all linear functionals on X. For a vector $\xi \in X$ and a linear functional $f \in X^*$, let $\xi \otimes f \in F(X)$ denote the mapping $(\xi \otimes f)(\eta) = f(\eta)\xi$, $\eta \in X$.

The first of the next two auxiliary results is standard while the second follows from [8, Lemma 2.1]; see also [15].

Lemma 2.2. *Let $\phi = \sum_{i=1}^{m} M_{a_i, b_i} \in \mathscr{El}(A, L(X))$. Suppose that $\{u_1, \ldots, u_n\}$ is a basis of $L(\phi)$. Then there exists a basis $\{v_1, \ldots, v_n\}$ of $R(\phi)$ such that $\phi = \sum_{i=1}^{n} M_{u_i, v_i}$.*

Lemma 2.3. *Let X be a vector space and let $V_1, \ldots, V_k$ be finite-dimensional subspaces of $L(X)$. Then there exists $\zeta \in X$ such that $\dim V_i\zeta = \operatorname{ldim} V_i$ for all $1 \leq i \leq k$.*

Proposition 2.4. *Let $\phi = \sum_{i=1}^{n} \phi_i$, where $\phi_i \colon A \to L(X)$ are linear mappings satisfying $\phi_j(x)\phi_i(y) = 0$ for all $j \geq i$ and $x, y \in A$. Then $\prod_{t=1}^{n+1} \phi(x_t) = 0$ for all $x_1, \ldots, x_{n+1} \in A$.*

Proof. We proceed by induction on n. The case $n = 1$ is trivial. Assume that $n > 1$ and the desired conclusion holds for $n - 1$. Since $\phi_n(x)\phi_i(y) = 0$ for all i and all $x, y \in A$, we have for each n,

$$\phi(x)\phi(y) = \sum_{i=1}^{n-1} \phi_i(x) \sum_{i=1}^{n-1} \phi_i(y) + \sum_{i=1}^{n-1} \phi_i(x)\phi_n(y)$$

and analogously for all r and $x_1, \ldots, x_r \in A$,

$$\prod_{t=1}^{r} \phi(x_t) = \prod_{t=1}^{r} \sum_{i=1}^{n-1} \phi_i(x_t) + \prod_{t=1}^{r-1} \sum_{i=1}^{n-1} \phi_i(x_t)\phi_n(x_r).$$

The induction hypothesis implies that $\prod_{t=1}^{n} \sum_{i=1}^{n-1} \phi_i(x_t) = 0$. Thus

$$\prod_{t=1}^{n} \phi(x_t) = \prod_{t=1}^{n-1} \sum_{i=1}^{n-1} \phi_i(x_t)\phi_n(x_n).$$

Consequently, $\prod_{t=1}^{n+1} \phi(x_t) = 0$. □

The following consequence of the above proposition explains a local nilpotency property of elementary operators and is in part the motivation for our main result in Section 4 of this paper.

Corollary 2.5. *Let $\phi = \sum_{i=1}^{n} M_{a_i,b_i} \in \mathcal{E}\ell(A, L(X))$. Suppose that $b_i a_j = 0$ for all $i \geq j$. Then $(\phi(x))^{n+1} = 0$ for all $x \in A$.*

3. Locally nilpotent elementary operators

The purpose of this section is to find necessary conditions for local nilpotency of an elementary operator.

Let $\phi = \sum_{i=1}^{n} M_{a_i,b_i}$ be an elementary operator on A of length n, where $a_i, b_i \in L(X)$. Let $\zeta \in X$ and $x \in A$ be such that $xV(\phi)\zeta \subseteq \mathbb{C}\zeta$. Then

$$\phi(x)L(\phi)\zeta \subseteq L(\phi)\zeta. \tag{3.1}$$

Let $\pi \colon \mathbb{C}\zeta \to \mathbb{C}$, $\pi(\zeta) = 1$ denote the canonical map. Suppose moreover $\{a_1\zeta, \ldots, a_n\zeta\}$ is linearly independent; in such a situation, we shall abbreviate this set to $\boldsymbol{a}\zeta$, if convenient. Let $M(\phi(x), \boldsymbol{a}\zeta)$ be the corresponding matrix representation of $\phi(x)$ with respect to $\boldsymbol{a}\zeta$. Then

$$M(\phi(x), \boldsymbol{a}\zeta) = (\pi(xb_i a_j \zeta))_{1 \leq i,j \leq n}. \tag{3.2}$$

Proposition 3.1. *Let* $\phi = \sum_{i=1}^{n} M_{a_i,b_i} \in \mathscr{El}(A, L(X))$ *be an elementary operator. Suppose that for every* $\zeta \in X$ *and* $x \in A$ *such that* $xV(\phi)\zeta \subseteq \mathbb{C}\zeta$, $\phi(x)_{|L(\phi)\zeta}$ *is nilpotent. Then* $\sum_{i=1}^{n} b_i a_i = 0$.

Proof. Suppose that there exist $\zeta \in X$ and $j \in \{1, \ldots, n\}$ such that $b_j a_j \zeta \neq 0$. Set $s = \dim L(\phi)\zeta$. Rearranging the order of the a_i's, if necessary, we may assume that $\{a_1\zeta, \ldots, a_s\zeta\}$ is linearly independent. For $t > s$ we can thus write $a_t\zeta = \sum_{i=1}^{s} \lambda_{ti} a_i \zeta$ with unique $\lambda_{ti} \in \mathbb{C}$, $s < t \leq n$. Replacing the coefficients a_t, $t > s$ by

$$a'_t = a_t - \sum_{i=1}^{s} \lambda_{ti} a_i, \quad s < t \leq n$$

it is clear that $a_t\zeta = 0$ for $t > s$. As it is easily verified that

$$\sum_{i=1}^{n} M_{a_i,b_i} = \sum_{i=1}^{s} M_{a_i, b_i + \sum_{t=s+1}^{n} \lambda_{ti} b_t} + \sum_{t=s+1}^{n} M_{a'_t, b_t}, \tag{3.3}$$

we can write ϕ in this new representation assuming these two additional properties of the coefficients a_i. (This argument will be used repeatedly in the sequel.)

Choose $i_1, \ldots, i_r$ such that $\{b_{i_1} a_{i_1} \zeta, \ldots, b_{i_r} a_{i_r} \zeta\}$ is linearly independent, r being maximal. Fix $j \in \{1, \ldots, r\}$. Choose $x \in A$ such that

$$xV(\phi)\zeta \subseteq \mathbb{C}\zeta, \quad x(b_{i_j} a_{i_j} \zeta) = \zeta \quad \text{and} \quad x(b_{i_t} a_{i_t} \zeta) = 0 \quad (1 \leq t \leq r, \ t \neq j). \tag{3.4}$$

Let $J = \{1, \ldots, s\} \setminus \{i_1, \ldots, i_r\}$. Write $b_i a_i \zeta = \sum_{t=1}^{r} \alpha_t^i b_{i_t} a_{i_t} \zeta$ for all $i \in J$. Then $x b_i a_i \zeta = \alpha_j^i \zeta$ for all $i \in J$. Moreover,

$$\phi(x)L(\phi)\zeta \subseteq L(\phi)\zeta \quad \text{and} \quad \mathsf{Tr}(\phi(x)_{|L(\phi)\zeta}) = 1 + \sum_{i \in J} \alpha_j^i,$$

where Tr denotes the trace. By our hypothesis, we must have $1 + \sum_{i \in J} \alpha_j^i = 0$. From

$$\sum_{i=1}^{n} b_i a_i \zeta = \sum_{i=1}^{s} b_i a_i \zeta = \sum_{t=1}^{r} b_{i_t} a_{i_t} \zeta + \sum_{i \in J} b_i a_i \zeta$$
$$= \sum_{t=1}^{r} b_{i_t} a_{i_t} \zeta + \sum_{i \in J} \sum_{t=1}^{r} \alpha_t^i b_{i_t} a_{i_t} \zeta$$

we obtain

$$\sum_{i=1}^{n} b_i a_i \zeta = \sum_{t=1}^{r} \left(1 + \sum_{i \in J} \alpha_t^i\right) b_{i_t} a_{i_t} \zeta. \tag{3.5}$$

As a result, $\sum_{i=1}^{n} b_i a_i \zeta = 0$.

Since this entails that $\sum_{i=1}^{n} b_i a_i \zeta = 0$ for every $\zeta \in X$, $\sum_{i=1}^{n} b_i a_i = 0$ as desired. $\qquad \square$

Proposition 3.2. *Let $\phi \in \mathcal{E\ell}(A, L(X))$ be an elementary operator such that $\mathsf{dim}\, V(\phi) = 1$. Set $\mathsf{ldim}\, L(\phi) = r$ and suppose that for every $\zeta \in X$ and $x \in A$ such that $xV(\phi)\zeta \subseteq \mathbb{C}\zeta$, $\phi(x)_{|L(\phi)\zeta}$ is nilpotent. Then ϕ admits a representation of the form $\phi = \sum_{i=1}^{n} M_{u_i, v_i}$ with*

$$(v_i u_j)_{1 \leq i,j \leq n} = \begin{pmatrix} T & 0 \\ * & 0 \end{pmatrix}, \tag{3.6}$$

where T is a strictly upper triangular matrix of order r. Moreover, $\phi^* \phi = 0$ and $(\phi(x))^{r+2} = 0$ for all $x \in A$.

Proof. Write $\phi = \sum_{i=1}^{n} M_{a_i, b_i}$. It follows from Lemma 2.3 that there exists $\zeta \in X$ such that $V(\phi)\zeta \neq 0$ and $\mathsf{dim}\, L(\phi)\zeta = r$. As in the proof of Proposition 3.1, we can suppose that $\{a_1\zeta, \ldots, a_r\zeta\}$ is linearly independent and that $a_k\zeta = 0$ for every $k > r$. It then follows from our assumption on $V(\phi)$ that $R(\phi)a_k = 0$ for all $k > r$. Let $s, t \in \{1, \ldots, n\}$ be such that $b_s a_t \neq 0$. Then $V(\phi) = \mathbb{C}\, b_s a_t$. Pick $x \in A$ with the property that $xb_s a_t\zeta = \zeta$. Then $\phi(x)\big(\mathrm{span}\{a_1\zeta, \ldots, a_r\zeta\}\big) \subseteq \mathrm{span}\{a_1\zeta, \ldots, a_r\zeta\}$ and we have

$$M(\phi(x), \boldsymbol{a}\zeta) = (\pi(xb_i a_j\zeta))_{1 \leq i,j \leq r}. \tag{3.7}$$

Choose a basis $\{u_1\zeta, \ldots, u_r\zeta\}$ of $L(\phi)\zeta$ such that $M(\phi(x), \boldsymbol{u}\zeta)$ is upper triangular. Write $u_i\zeta = \sum_{t=1}^{r} \alpha_t^i a_t\zeta$. Replacing u_i by $u_i = \sum_{t=1}^{r} \alpha_t^i a_t$, if necessary, we have $\mathrm{span}\{u_1, \ldots, u_r\} = \mathrm{span}\{a_1, \ldots, a_r\}$. For $k > r$, set $u_k = a_k$ and write $\phi = \sum_{i=1}^{n} M_{u_i, v_i}$ for some suitable $v_1, \ldots, v_n \in L(X)$. It is clear that the matrix $(v_i u_j)$ has the desired form. Next observe that ϕ is the sum of two elementary operators ϕ_1, ϕ_2, such that $\ell(\phi_1) = r$, $\ell(\phi_1) + \ell(\phi_2) = \ell(\phi)$ and $R(\phi)L(\phi_2) = 0$. Therefore

$$(\phi(x))^k = (\phi_1(x))^k + \phi_2(x)(\phi_1(x))^{k-1} \quad (k \in \mathbb{N},\ x \in A). \tag{3.8}$$

It is now easy to get the final assertion. $\qquad\square$

The main result of this section gives a description of locally nilpotent elementary operators where the coefficient spaces have maximal local dimension.

Theorem 3.3. *Let $\phi \in \mathcal{E\ell}_n(A, L(X))$ be an elementary operator of length n such that $\mathsf{ldim}\, L(\phi) = n$. Suppose that for every $\zeta \in X$ and $x \in A$ such that $xV(\phi)\zeta \subseteq \mathbb{C}\zeta$, $\phi(x)_{|L(\phi)\zeta}$ is nilpotent. Then $\mathsf{ldim}\, V(\phi) \leq \frac{n(n-1)}{2}$. Moreover, if $\mathsf{ldim}\, V(\phi) = \frac{n(n-1)}{2}$, then ϕ admits a representation of the form $\phi = \sum_{i=1}^{n} M_{u_i, v_i}$, where $v_i u_j = 0$ for every $i \geq j$. In particular, $\mathsf{dim}\, V(\phi) = \frac{n(n-1)}{2}$.*

Proof. Write $\phi = \sum_{i=1}^{n} M_{a_i, b_i}$, set $\mathsf{ldim}\, V(\phi) = r$ and choose $\zeta \in X$ such that $\mathsf{dim}\, V(\phi)\zeta = r$ and $\mathsf{dim}\, L(\phi)\zeta = n$ (Lemma 2.3). Let $\{b_{i_t} a_{j_t}\zeta\}_{1 \leq t \leq r}$ be a basis of $V(\phi)\zeta$. Pick $x_1, \ldots, x_r \in A$ with $x_k b_{i_t} a_{j_t}\zeta = \delta_{kt}\zeta$, for each $1 \leq t, k \leq r$. Let N be the vector subspace of $M_n(\mathbb{C})$ generated by $M(\phi(x_t), \boldsymbol{a}\zeta)$. Then N is nilpotent and $\mathsf{dim}\, N = r$.

Applying Gerstenhaber's theorem on maximal spaces of nilpotent matrices [14], we infer that $r \leq \frac{n(n-1)}{2}$ and if $r = \frac{n(n-1)}{2}$, there exists a basis B of $L(\phi)\zeta$ such that $M(\phi(x), B)$ is upper triangular for each $x \in \text{span}\{x_1, \ldots, x_r\}$. Next suppose that $r = \frac{n(n-1)}{2}$ and set $B = \{u_1\zeta, \ldots, u_n\zeta\}$. Then $L(\phi) = \text{span}\{u_1, \ldots u_n\}$. Applying Lemma 2.2, we get the existence of a basis $\{v_1, \ldots, v_n\}$ of $R(\phi)$ such that $\phi = \sum_{i=1}^{n} M_{u_i, v_i}$. Since $M(\phi(x), B) = (\pi(xv_iu_j\zeta))_{1 \leq i,j \leq n}$, we have $\pi(xv_iu_j\zeta) = 0$ for all $i \geq j$ and $x \in \text{span}\{x_1, \ldots, x_r\}$, thus $v_iu_j\zeta = 0$ for all $i \geq j$. Since $\dim V(\phi)\zeta = \frac{n(n-1)}{2}$, the set $\{v_iu_j\zeta : i < j\}$ is a basis of $V(\phi)\zeta$.

Next we shall show by induction on n that $\dim V = \frac{n(n-1)}{2}$. The case $n = 1$ is trivial. Suppose that the desired result is true for elementary operators of length $n-1$, $n > 1$. We first claim that $R(\phi)u_1 = 0$. To prove the claim suppose that there exists $\zeta' \in X$ such that $R(\phi)u_1\zeta' \neq 0$. Let i_1 be maximal (among $1, \ldots, n$) such that $v_{i_1}u_1\zeta' \neq 0$. It is clear that $i_1 > 1$. Let i_2 be maximal such that $\{v_{i_1}u_1\zeta', v_{i_2}u_1\zeta'\}$ is linearly independent. If $i_2 < t < i_1$, replace v_t by $v_t' = v_t - \alpha v_{i_1}$ and u_{i_1} by $u_{i_1}' = u_{i_1} + \alpha u_t$ such that $v_t'u_1\zeta' = 0$. We recursively find $i_1 > \ldots > i_r$ and define u_i', v_i' such that $u_1' = u_1$, $\phi = \sum M_{u_i', v_i'}$, $v_j'u_i'\zeta = 0$ for all $j \geq i$, the sets $\{v_i'u_j'\zeta : i < j\}$ and $\{v_{i_1}'u_1'\zeta', \ldots, v_{i_r}'u_1'\zeta'\}$ are linearly independent and if $t \notin \{i_1, \ldots, i_r\}$, $v_t'u_1'\zeta' = 0$. Choose a non-zero scalar λ such that $\{v_i'u_j'(\zeta + \lambda\zeta') : i < j\}$ is linearly independent. Replacing ζ' by $\zeta + \lambda\zeta'$ if needed, we may assume with no loss of generality that $\{v_i'u_j'\zeta' : i < j\}$ is linearly independent. Thus, by the above argument, there exists $c_1 \in L(\phi)$ such that $R(\phi)c_1\zeta' = 0$. In particular, $c_1 \notin \mathbb{C}u_1'$. Write $c_1 = \sum_{i=1}^{n} \beta_iu_i'$. Since $v_1'c_1\zeta' = 0$ and $\{v_1'u_2'\zeta', \ldots, v_1'u_n'\zeta'\}$ is linearly independent, we infer that $\beta_1v_1'u_1'\zeta' \neq 0$. Consequently, $i_r = 1$. We get a contradiction by choosing $x \in A$ such that $xv_{i_t}'u_1'\zeta' = 0$ for $t = 1, \ldots, r-1$ and $xv_1'u_1'\zeta' = \zeta'$ (indeed, we have $\phi(x)u_1'\zeta' = u_1'\zeta'$). The claim is proved. Now set $\psi = \sum_{i=2}^{n} M_{u_i, v_i}$. Then for every $x \in A$ and $\zeta' \in X$ such that $\psi(x)V(\psi)\zeta' \subseteq \mathbb{C}\zeta'$, the restriction of $\psi(x)$ to $L(\psi)\zeta'$ must be nilpotent; otherwise, there exists $y \in A$ such that $yV(\phi)\zeta' \subseteq \mathbb{C}\zeta'$ and $y_{|V(\psi)\zeta'} = x_{|V(\psi)\zeta'}$ such that $\phi(y)_{|V(\phi)\zeta'}$ is not nilpotent. Our induction hypothesis yields the desired result. $\square$

Proposition 3.4. *Let* $\phi = \sum_{i=1}^{n} M_{a_i, b_i}$ *be an elementary operator of length* n *such that* $\text{ldim}\, L(\phi) = n$ *and* $\text{ldim}\, V(\phi) = 1$. *Suppose that for every* $\zeta \in X$ *and* $x \in A$ *such that* $xV(\phi)\zeta \subseteq \mathbb{C}\zeta$, $\phi(x)_{|L(\phi)\zeta}$ *is nilpotent. Then* $\dim V(\phi) \leq \frac{n(n-1)}{2}$. *Moreover, if* $\dim V(\phi) = \frac{n(n-1)}{2}$, *then* ϕ *admits a representation of the form* $\phi = \sum_{i=1}^{n} M_{u_i, v_i}$ *where* $v_iu_j = 0$ *for every* $i \geq j$.

Proof. Choose i, j such that $b_ia_j \neq 0$. The case $\dim V(\phi) = 1$ follows from Proposition 3.2, so suppose that $\dim V(\phi) > 1$. Then there exists k, l such that $b_ka_l \notin \mathbb{C}b_ia_j$. We have $\{b_ia_j\zeta, b_ka_l\zeta\}$ linearly dependent for each $\zeta \in X$. It follows from [8, Theorem 2.3] that b_ia_j has rank one. Thus, there exists $\zeta_0 \in X$ and linear functionals f_{kl} such

that $b_k a_l = \zeta_0 \otimes f_{kl}$ for all $1 \leq k, l \leq n$. Set $M(\zeta) = (f_{ij}(\zeta))_{1 \leq i,j \leq n}$ for every $\zeta \in X$. Clearly, $M(\zeta) + M(\zeta') = M(\zeta + \zeta')$. Observe that if $\mathsf{dim}\, L(\phi)\zeta = n$, and $x \in A$ satisfies $x\zeta_0 = \zeta$, then $x(V(\phi))\zeta \subseteq \mathbb{C}\zeta$ and $M(\phi(x), \boldsymbol{a}\zeta) = M(\zeta)$. Hence $M(\zeta)$ is nilpotent. Set $\mathsf{dim}\, V(\phi) = r$. Then $\mathsf{dim}\, \mathrm{span}\{f_{ij} : 1 \leq i, j \leq n\} = r$. Choose $\zeta_1, \ldots, \zeta_r$ with the property that the set $\{M(\zeta_1), \ldots, M(\zeta_r)\}$ is linearly independent and pick $\zeta \in X$ such that $\mathsf{dim}\, L(\phi)\zeta = n$. It follows from [8, Lemma 2.1] that for all but finitely many $r + 1$-tuples $(\lambda_0, \lambda_1, \ldots, \lambda_r) \in \mathbb{C}^n$, $\mathsf{dim}\, L(\phi)(\lambda_0\zeta + \sum_{t=1}^{r} \lambda_t \zeta_t) = n$ and $\{M(\zeta_1 + \lambda_0\zeta), \ldots, M(\zeta_r + \lambda_0\zeta)\}$ is linearly independent. In particular, for all but finitely many $(\lambda_0, \lambda_1, \ldots, \lambda_r) \in \mathbb{C}^n$, $M(\lambda_0\zeta + \sum_{t=1}^{r} \lambda_t \zeta_t)$ is nilpotent. Let N be the vector subspace of $M_n(\mathbb{C})$ generated by

$$\{M(\lambda_0\zeta + \sum_{t=1}^{r} \lambda_t \zeta_t) : (\lambda_0, \lambda_1, \ldots, \lambda_r) \in \mathbb{C}^n\}.$$

Then it is easy to see that N is nilpotent and $\mathsf{dim}\, N \geq r$. Applying once again Gerstenhaber's theorem on maximal spaces of nilpotent matrices [14], we infer that $r \leq \frac{n(n-1)}{2}$ and if $r = \frac{n(n-1)}{2}$, there exists an invertible matrix $P \in M_n(\mathbb{C})$ such that $P^{-1}NP$ equals the subspace of nilpotent upper triangular matrices. Suppose that $r = \frac{n(n-1)}{2}$ and let $B = \{u_1, \ldots, u_n\}$ be the basis corresponding to P of $L(\phi)$. Then $\phi = \sum_i M_{u_i, v_i}$ for a convenient basis $\{v_i\}$ of $R(\phi)$. Set $v_j u_i = \zeta_0 \otimes g_{ij}$. Then $g_{ij} = 0$ for $i \geq j$. This completes the proof. $\qquad\square$

The first non-trivial case allows a particularly nice description.

Corollary 3.5. *Let ϕ be an elementary operator of length 2. Suppose that for every $\zeta \in X$ and $x \in A$ such that $x(V(\phi))\zeta \subseteq \mathbb{C}\zeta$, $\phi(x)_{|L(\phi)\zeta}$ is nilpotent. Then $\phi = M_{a,b} + M_{c,d}$, where $ba = dc = bc = 0$.*

Proof. Write $\phi = \sum_{i=1}^{2} M_{a_i, b_i}$. Suppose first that $\mathsf{ldim}\, L(\phi) = 2$. Then the desired result follows from Theorem 3.3 and Proposition 3.4. Now suppose that $\{a_1, a_2\}$ is locally linearly dependent. Then, by [8, Theorem 2.3], there exists $\zeta_0 \in X$ and linear functionals f_1, f_2 such that $a_i = \zeta_0 \otimes f_i$. We have

$$\phi(x)\zeta = \sum_{i=1}^{2} f_i(xb_i\zeta)\zeta_0. \tag{3.9}$$

Suppose for a moment that $b_i\zeta_0 \neq 0$ for some i. Since $\{f_1, f_2\}$ is linearly independent, we can find $x \in A$ such that $f_i(xb_i\zeta_0) = 1$ and $f_j(xb_j\zeta_0) = 0$ for $j \neq i$, and we get a contradiction. Thus $b_i a_j = 0$ for $i, j = 1, 2$. $\qquad\square$

For each $y \in L(X)$, we define the *point spectrum* $\sigma_p(y)$ by

$$\sigma_p(y) = \{\lambda \in \mathbb{C} : \lambda - y \text{ is not injective}\}.$$

The following result reveals the relationship between the two conditions "quasi-nilpotent" and "of finite rank".

Proposition 3.6. *Let $\phi = \sum_{i=1}^{n} M_{a_i,b_i}$ be an elementary operator of length n, and let $N \in \mathbb{N}$. Suppose that $\#\sigma_p(\phi(x)) < N$ for all $x \in A$. Then either $b_i a_j$ has finite rank for all $1 \leq i, j \leq n$ or there exists a finite-codimensional subspace Y of X such that for every $\zeta \in Y$ and $x \in A$ satisfying $xV(\phi)\zeta \subseteq \mathbb{C}\zeta$, $\phi(x)_{|L(\phi)\zeta}$ is nilpotent.*

Proof. Suppose that there exists $i, j \in \{1, \ldots, n\}$ such that $b_i a_j \notin F(X)$ and choose $r > 0$ maximal with the property that $\{b_{i_1} a_{j_1}, \ldots, b_{i_r} a_{j_r}\}$ is linearly independent modulo $F(X)$. Then for each i, j, there exist scalars $\alpha_{ij}^1, \ldots, \alpha_{ij}^r$ and $y_{ij} \in F(X)$ such that $b_i a_j = \sum_{t=1}^{r} \alpha_{ij}^t b_{i_t} a_{j_t} + y_{ij}$. Set $Y = \bigcap \ker y_{ij}$. Then it is easy to see that Y has finite codimension. Suppose that there exists $\zeta \in Y$ and $x \in A$ satisfying $xV(\phi)\zeta \subseteq \mathbb{C}\zeta$ and $\phi(x)_{|L(\phi)\zeta}$ is not nilpotent. With no loss of generality, we may suppose that there exists $u \in L(\phi)$ such that $(\phi(x))u\zeta = u\zeta$. Set $xb_i u\zeta = \lambda_i \zeta$ and $a = \sum \lambda_i a_i$. Then $u\zeta = a\zeta$. Applying [4, Lemma 2.1], we can find $\zeta_1, \ldots, \zeta_N \in Y$ such that $\{b_{i_1} a_{j_1} \zeta_t, \ldots, b_{i_r} a_{j_r} \zeta_t\}_{1 \leq t \leq N}$ is linearly independent. Let $\mu_1, \ldots, \mu_N$ be distinct numbers. Choose $y \in A$ such that $yb_{i_k} a_{j_k} \zeta_t = \mu_t \pi(xb_{i_k} a_{j_k} \zeta)\zeta_t$. Since $\zeta, \zeta_1, \ldots, \zeta_N$ lie in Y, for all i, j, $yb_i a_j \zeta_t = \mu_t(\pi xb_i a_j \zeta)\zeta_t$. For each $c \in L(\phi)$, $yb_i c\zeta_t = \mu_t(\pi xb_i c\zeta)\zeta_t$. Now $\phi(y)a\zeta_t = \sum a_i yb_i a\zeta_t = \mu_t a\zeta_t$. In particular, $\{\mu_1, \ldots, \mu_N\} \subseteq \sigma_p(\phi(y))$, a contradiction. $\square$

4. LOCALLY QUASI-NILPOTENT ELEMENTARY OPERATORS OF LENGTH 3

In this section we shall pay special attention to the case of locally quasi-nilpotent elementary operators of length 3; shorter lengths have been treated elsewhere, see, e.g., [6].

Suppose that $\phi(x)$ is quasi-nilpotent for every $x \in A$. For every $x \in A$ and $\zeta \in X$ such that $xV(\phi)\zeta \subseteq \mathbb{C}\zeta$, we have $\phi(x)L(\phi)\zeta \subseteq L(\phi)\zeta$. As $\dim L(\phi)\zeta$ is finite, $\phi(x)_{|L(\phi)\zeta}$ is nilpotent and we can apply the results of the previous section. Assuming that $\ell(\phi) = n$, the best case occurs when $\mathsf{ldim}\, L(\phi) = n$, that is, when we have a separating vector for $L(\phi)$. If, in addition, $\mathsf{ldim}\, V(\phi)$ is maximal, then we obtain the best possible representation of ϕ; this was achieved in Theorem 3.3. In the case $n = 2$, $\mathsf{ldim}\, V(\phi)$ is at most 1 and therefore no complications can occur (Corollary 3.5). However, already in the case $n = 3$, the gap between $\mathsf{ldim}\, V(\phi)$ and $\mathsf{ldim}\, L(\phi)$ widens which results in further "exceptional" descriptions of ϕ as in the theorem below. Moreover, local nilpotency is no longer sufficient, as is illustrated by an example in Remark 4.5.

The proof of our main result is organised according to the three levels of difficulty: $\mathsf{ldim}\, L(\phi) = 3$, $\mathsf{ldim}\, L(\phi) = 2$ and finally $\mathsf{ldim}\, L(\phi) = 1$.

Theorem 4.1. *Let X be a complex vector space and let A be a dense algebra of linear operators on X. Then $\phi \in \mathscr{E}_3(A, L(X))$ is a locally quasi-nilpotent elementary operator*

if and only if there exists a representation of ϕ of the form $\phi = \sum_{i=1}^{3} M_{u_i,v_i}$, $u_i, v_i \in L(X)$ such that one of the following three cases occurs:

(i) $v_i u_j = 0$ *for all* $i \geq j$;

(ii) $(v_i u_j)_{1 \leq i,j \leq 3} = \begin{pmatrix} 0 & \zeta_1 \otimes f & 0 \\ \zeta_0 \otimes f & 0 & \zeta_1 \otimes f \\ 0 & -\zeta_0 \otimes f & 0 \end{pmatrix}$,

     where $\{\zeta_0, \zeta_1\} \subseteq X$ *is linearly independent and* $f \in X^*$;

(iii) $(v_i u_j)_{1 \leq i,j \leq 3} = \begin{pmatrix} 0 & \zeta_0 \otimes g & 0 \\ \zeta_0 \otimes f & 0 & \zeta_0 \otimes g \\ 0 & -\zeta_0 \otimes f & 0 \end{pmatrix}$,

     where $\zeta_0 \in X$ *and* $\{f, g\} \subseteq X^*$ *is linearly independent.*

Proof. Set $\phi = \sum_{i=1}^{3} M_{a_i,b_i}$, $a_i, b_i \in L(X)$. Suppose first that $L(\phi)$ has a separating vector. We distinguish three cases.

Case 1. $\mathsf{ldim}\, V(\phi) = 3$. This case follows from Theorem 3.3.

Case 2. $\mathsf{ldim}\, V(\phi) = 2$. Pick $\zeta \in X$ satisfying $\dim L(\phi)\zeta = 3$ and $\dim V(\phi)\zeta = 2$ (Lemma 2.3). Set

$$N = \{M(\phi(x), \boldsymbol{a}\zeta) : x \in A,\ xV(\phi)\zeta \subseteq \mathbb{C}\zeta\}. \tag{4.1}$$

Clearly, N is a nilpotent subspace of $M_3(\mathbb{C})$ of dimension 2. Let N' be a maximal subspace containing N. Then N' is conjugate to one of the following subspaces by [12, Proposition 3] (see also [13]):

$$\left\{ \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\} \quad \text{or} \quad \left\{ \begin{pmatrix} 0 & \beta & 0 \\ \alpha & 0 & \beta \\ 0 & -\alpha & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}.$$

Thus, we distinguish two subcases.

Subcase 2.1. There exists $\zeta \in X$ and a representation of ϕ of the form $\phi = \sum_{i=1}^{3} M_{u_i,v_i}$, $u_i \in L(\phi)$, $v_i \in R(\phi)$ such that $\dim L(\phi)\zeta = 3$, $\dim V(\phi)\zeta = 2$, and for every $x \in A$ such that $xV(\phi)\zeta \subseteq \mathbb{C}\zeta$, the matrix $(\pi(xv_i u_j \zeta))_{1 \leq i,j \leq 3}$ is upper triangular. Then

$$(v_i u_j \zeta)_{1 \leq i,j \leq 3} = \begin{pmatrix} 0 & v_1 u_2 \zeta & v_1 u_3 \zeta \\ 0 & 0 & v_2 u_3 \zeta \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly, we may assume that one of the sets $\{v_1 u_2 \zeta, v_1 u_3 \zeta\}$ or $\{v_1 u_3 \zeta, v_2 u_3 \zeta\}$ is linearly independent. Using an argument similar to the one used in the proof of Theorem 3.3, we show that $R(\phi)u_1 = 0$ or $v_3 L(\phi) = 0$ and arrive at the desired conclusion.

Subcase 2.2. For every $\zeta \in X$ such that $\dim L(\phi)\zeta = 3$ and $\dim V(\phi)\zeta = 2$, there exists a representation of ϕ of the form $\phi = \sum_{i=1}^{3} M_{u_i,v_i}$, $u_i \in L(\phi)$, $v_i \in R(\phi)$ depending on ζ

such that, for every $x \in A$ satisfying $xV(\phi)\zeta \subseteq \mathbb{C}\zeta$, the matrix $(\pi(xv_iu_j\zeta))_{1 \leq i,j \leq 3}$ has the form

$$\begin{pmatrix} 0 & \pi(xv_2u_3\zeta) & 0 \\ \pi(xv_2u_1\zeta) & 0 & \pi(xv_2u_3\zeta) \\ 0 & -\pi(xv_2u_1\zeta) & 0 \end{pmatrix}. \tag{4.2}$$

Then we have

$$(v_iu_j\zeta)_{1 \leq i,j \leq 3} = \begin{pmatrix} 0 & v_2u_3\zeta & 0 \\ v_2u_1\zeta & 0 & v_2u_3\zeta \\ 0 & -v_2u_1\zeta & 0 \end{pmatrix}. \tag{4.3}$$

Now fix a separating vector ζ of $L(\phi)$ such that $V(\phi)\zeta$ has maximal dimension. Set $v_2u_1\zeta = \eta_1$ and $v_2u_3\zeta = \eta_2$. Pick $\eta_3 \in X \setminus \operatorname{span}\{\eta_1, \eta_2\}$. Write $X = \operatorname{span}\{\eta_1, \eta_2, \eta_3\} \oplus Y$, for some subspace Y of X. Let $p_1 : X \to \operatorname{span}\{\eta_1, \eta_2, \eta_3\}$ and $p_2 : X \to Y$ be the natural projections. Fix $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{C} \setminus \{0\}$. Let $\zeta' \in X$. Choose $x_{\zeta'} \in X$ satisfying

$$x_{\zeta'}\eta_i = \gamma_i\zeta', \quad x_{\zeta'}p_2v_iu_j\zeta' = 0 \quad (1 \leq i,j \leq 3).$$

Set

$$M(\zeta', \gamma_1, \gamma_2, \gamma_3) = \left(\pi x_{\zeta'}v_iu_j\zeta'\right)_{1 \leq i,j \leq 3}$$

and

$$E(\gamma_1, \gamma_2, \gamma_3) = \{M(\zeta', \gamma_1, \gamma_2, \gamma_3) : \zeta' \in X\}.$$

Observe that if ζ' is a separating vector of $L(\phi)$, we have

$$M(\zeta', \gamma_1, \gamma_2, \gamma_3) = M\left(\phi(x_{\zeta'}), \{u_1\zeta', u_2\zeta', u_3\zeta'\}\right)$$

which is nilpotent. Since for every $\zeta' \in X$, $\zeta' + \lambda\zeta$ is a separating vector of $L(\phi)$ for all but finitely many $\lambda \in \mathbb{C}$, we infer that $E(\gamma_1, \gamma_2, \gamma_3)$ is a nilpotent subspace of $M_3(\mathbb{C})$. Next apply once again [12, Proposition 3].

Suppose first that $E(\gamma_1, \gamma_2, \gamma_3)$ has dimension 2 for an open set O of values $(\gamma_1, \gamma_2, \gamma_3)$ of $\mathbb{C}^3$. Then, for every $(\gamma_1, \gamma_2, \gamma_3) \in O$, $\zeta_1, \zeta_2, \zeta_3 \in X$ and $w_1, w_2, w_3 \in \{v_iu_j : 1 \leq i,j \leq 3\}$, we must have $\det((x_{\zeta_j}w_i\zeta_j)_{1 \leq i,j \leq 3}) = 0$. Let $(i,j) \notin \{(1,2), (2,3), (2,1), (3,2)\}$. Suppose for a moment that $v_iu_j \neq 0$. Then $\{v_iu_j, v_2u_1, v_2u_3\}$ is locally linearly dependent. Since $v_iu_j\zeta = 0$, $v_iu_jX \subseteq \operatorname{span}\{\eta_1, \eta_2\}$. Hence we may suppose that $v_iu_j\zeta_2 = \eta_1$ and either $v_iu_j\zeta_3 = \eta_2$, or $v_iu_j\zeta_3 = 0$. Set $w_1 = v_2(\gamma_2u_1 - \gamma_1u_3)$, $w_2 = v_iu_j$, $w_3 = v_2u_1$, and $\zeta_1 = \zeta$. Computing the determinant, and using the fact that $x_\zeta v_2(\gamma_2u_1 - \gamma_1u_3)\zeta = 0$, we see that $v_2u_1\zeta_2 = 0$, which is impossible since we can replace ζ_2 by $\zeta_2 + \lambda\zeta$. Thus $v_iu_j = 0$. Let ζ' be a separating vector of $L(\phi)$. By considering kernels, we see that we must have $\gamma_2u_1 - \gamma_1u_3 \in \operatorname{span}\{u'_1, u'_3\}$ where u'_1, u'_3 now depend on ζ'. Therefore $\operatorname{span}\{u_1, u_3\} = \operatorname{span}\{u'_1, u'_3\}$. Now it is easy to see that we can put $u_i = u'_i$ for all i. This implies that we must have $v_2u_1 = -v_3u_2$ and $v_1u_2 = v_2u_3$.

Now suppose that the set of $(\gamma_1, \gamma_2, \gamma_3)$ for which $\mathsf{dim}\, E(\gamma_1, \gamma_2, \gamma_3) \neq 2$ is dense in $\mathbb{C}^3$. Then observe that $E(\gamma_1, \gamma_2, \gamma_3)$ must be triangularizable for an open subset $(\gamma_1, \gamma_2, \gamma_3)$ of $\mathbb{C}^3$. By considering a common eigenvector, we see that we must have $\gamma_2 u_1 - \gamma_1 u_3 \in$ $\mathrm{span}\{u_1^{\zeta'}, u_3^{\zeta'}\}$ for every separating vector ζ' of $L(\phi)$. As above, we deduce that we can put $u_i = u_i^{\zeta'}$ for all i.

Next suppose that there exists $\zeta' \in X$ with $v_2 u_1 \zeta' = \eta_3$. Choose $x \in A$ with $x\eta_3 = \zeta$, $x\eta_2 = \zeta'$ and $x\eta_1 = 0$. Then $\phi(x)(u_1 \zeta' + u_2 \zeta) = u_1 \zeta' + u_2 \zeta$, a contradiction. Hence $p_1 v_2 u_1 X \subseteq \mathrm{span}\{\eta_1, \eta_2\}$. Analogously for $p_1 v_2 u_3$. We have thereby shown that $v_i u_j X \subseteq$ $\mathrm{span}\{\eta_1, \eta_2\}$ for every i, j. Next choose $\zeta' \in X$ with $v_2 u_1 \zeta' = 0$ (this is possible since the dimension of X is greater than 2). Suppose for a moment that $v_2 u_3 \zeta' \neq 0$. Pick $x \in A$ such that $x\eta_1 = \zeta'$ and $x v_2 u_3 \zeta' = \zeta$. Then $\phi(x)(u_1 \zeta + u_2 \zeta') = u_1 \zeta + u_2 \zeta'$, a contradiction. Hence $v_2 u_1$ and $v_2 u_3$ have the same kernel. Since they are at most rank 2, $\mathrm{span}\{v_2 u_3, v_2 u_1\}$ is generated by rank one operators. Thus we can assume that $v_2 u_3, v_2 u_1$ are rank one. Since they must have the same kernel, we conclude that we must have the form (ii).

Case 3. $\mathsf{ldim}\, V(\phi) = 1$. It follows from Propositions 3.2 and 3.4 that we only need to treat the case where $\mathsf{dim}\, V(\phi) = 2$. So suppose that $\mathsf{dim}\, V(\phi) = 2$ and set $b_i a_j = \zeta_0 \otimes f_{ij}$ for all $1 \leq i, j \leq 3$. For every $\zeta \in X$, let $M(\zeta) = (f_{ij}(\zeta))_{1 \leq i, j \leq 3}$ and consider the vector space $N = \{M(\zeta) : \zeta \in X\}$. As above, we show that N is a nilpotent subspace of $M_3(\mathbb{C})$, and hence, either N is triangularizable, or

$$N = \left\{ \begin{pmatrix} 0 & \gamma' & 0 \\ \gamma & 0 & \gamma' \\ 0 & -\gamma & 0 \end{pmatrix} : \gamma, \gamma' \in \mathbb{C} \right\}.$$

Thus, there exists a representation of ϕ of the form $\phi = \sum_{i=1}^3 M_{u_i, v_i}$, $u_i, v_i \in L(X)$ such that either $v_i u_j = 0$ for all $i \geq j$ or

$$(v_i u_j)_{1 \leq i, j \leq 3} = \begin{pmatrix} 0 & \zeta_0 \otimes g & 0 \\ \zeta_0 \otimes f & 0 & \zeta_0 \otimes g \\ 0 & -\zeta_0 \otimes f & 0 \end{pmatrix}; \quad \zeta_0 \in X \text{ and } f, g \in X^*.$$

We now move to the case $\mathsf{ldim}\, L(\phi) = 2$. Let $\zeta \in X$ be such that the vector spaces $L(\phi)\zeta$ and $V(\phi)\zeta$ have maximal dimensions. Choose $u \in L(\phi)$ such that $u \neq 0$ and $u\zeta = 0$. Write $\phi = \sum_{i=1}^3 M_{u_i, v_i}$ where $u_1 = u$. Set $\psi = \sum_{i=2}^3 M_{u_i, v_i}$. Then arguing as in the proof of Theorem 3.3, we see that for every $x \in A$ satisfying $x V(\psi)\zeta \subseteq \mathbb{C}\zeta$, $\phi(x)_{|L(\psi)\zeta}$ must be nilpotent. It follows from Corollary 3.5 that there exists a representation of $\psi = \sum_{i=2}^3 M_{c_i, d_i}$ such that $d_j c_i \zeta = 0$ for $j \geq i$. We have thereby shown that there exists a representation of $\phi = \sum_{i=1}^3 M_{u_i, v_i}$ such that $v_j u_i \zeta = 0$ for $j \geq i$. Next we adapt the argument used in the proof of Theorem 3.3 to show that either $R(\phi)u_1 = 0$

or $v_3L(\phi) = 0$ (by our assumption on ζ and $V(\phi)$, $\mathsf{ldim}\, V(\phi) \leq 3$). Treating the three occurring cases $\mathsf{ldim}\, V(\phi) = 1$, 2 or 3 separately, it is easy to complete the argument.

Suppose finally that $\mathsf{ldim}\, L(\phi) = 1$. Then there exists $\zeta_0 \in X$ such that $a_i = \zeta_0 \otimes f_i$. Therefore, for each $x \in A$ and $\zeta \in X$, we have

$$\phi(x)\zeta = \sum_{i=1}^{3} f_i(xb_i\zeta)\zeta_0. \tag{4.4}$$

Suppose that $b_i\zeta_0 \neq 0$ for some i and choose $x \in A$ such that $f_i(xb_i\zeta_0) = 1$, $f_j(xb_i\zeta_0) = 0$ for $j \neq i$ (this is possible since the set $\{f_1, f_2, f_3\}$ is linearly independent). Then we get a contradiction. Thus, $b_ia_j = 0$ for each i, j.

To prove the converse, observe that if $\phi = \sum M_{u_i,v_i}$ and $v_iu_j = 0$ for all $i \geq j$, then it follows from Corollary 2.5 that $\phi(x)^4 = 0$ for all $x \in A$. A straightforward computation checking the other cases shows that $\phi(x)^5 = 0$ for all $x \in A$. This completes the proof. $\qquad \square$

Corollary 4.2. *Let $\phi \in \mathcal{E}\ell_3(A, L(X))$ be a locally quasi-nilpotent elementary operator. Then $\phi^*\phi = 0$ and $\phi(x)^5 = 0$ for all $x \in A$.*

Corollary 4.3. *Let $\phi \in \mathcal{E}\ell_3(A, L(X))$ be a locally quasi-nilpotent elementary operator. Then one of the following two cases occurs:*

  (i) *there exists a representation of ϕ of the form $\phi = \sum_{i=1}^{3} M_{u_i,v_i}$, where $v_iu_j = 0$ for all $i \geq j$;*

  (ii) *the space $\phi(x)L(\phi)X$ has at most dimension 3 for all $x \in A$, and hence $\mathsf{dim}\, \phi(x)^2 X \leq 3$ for all $x \in A$.*

Proof. Suppose first that there exists a representation of ϕ of the form $\phi = \sum_{i=1}^{3} M_{u_i,v_i}$ such that

$$(v_iu_j)_{1 \leq i,j \leq 3} = \begin{pmatrix} 0 & \zeta_1 \otimes f & 0 \\ \zeta_0 \otimes f & 0 & \zeta_1 \otimes f \\ 0 & -\zeta_0 \otimes f & 0 \end{pmatrix},$$

where $\zeta_0, \zeta_1 \in X$, $f \in X^*$ and $\{\zeta_0, \zeta_1\}$ is linearly dependent. Then

$$\phi(x)L(\phi)X \subseteq \mathrm{span}\{u_2x\zeta_0, u_2x\zeta_1, u_1x\zeta_1 - u_3x\zeta_0\} \quad (x \in A).$$

Next suppose that there exists a representation of ϕ of the form $\phi = \sum_{i=1}^{3} M_{u_i,v_i}$ such that

$$(v_iu_j)_{1 \leq i,j \leq 3} = \begin{pmatrix} 0 & \zeta_0 \otimes g & 0 \\ \zeta_0 \otimes f & 0 & \zeta_0 \otimes g \\ 0 & -\zeta_0 \otimes f & 0 \end{pmatrix},$$

where $\{f, g\} \subseteq X^*$ is linearly independent and $\zeta_0 \in X$. Then

$$\phi(x)L(\phi)X \subseteq \mathrm{span}\{u_2x\zeta_0, u_1x\zeta_0, u_3x\zeta_0\} \quad (x \in A).$$

Now the desired conclusion follows from Theorem 4.1. □

Remark 4.4. In the above proof, if $\mathsf{ldim}\, V(\phi) = 2$ and $\mathsf{ldim}\, L(\phi) = 3$, then the space $\widehat{X}$ (see Lemma 2.1) has rank at most 2. Hence one can use the classification of spaces of matrices of rank at most 2 [1]. On the other hand, if $\mathsf{ldim}\, L(\phi) = 2$, one can use the structure of locally linearly dependent spaces of dimension 3 which is clearly stated in [10, Theorem 3.1]. However, in both cases, the proof will be longer. Our approach here is direct and relatively elementary.

Remark 4.5. In Section 3, our standard assumption on the elementary operator $\phi \in \mathcal{E}\ell(A, L(X))$ was that, for all $x \in A$ and $\zeta \in X$ satisfying $xV(\phi)\zeta \subseteq \mathbb{C}\zeta$, $\phi(x)_{|L(\phi)\zeta}$ is nilpotent. Suppose that $\phi = \sum_{i=1}^{3} M_{u_i,v_i}$ is an elementary operator on A satisfying

$$(v_i u_j)_{1 \leq i,j \leq 3} = \begin{pmatrix} 0 & v_2 u_3 & 0 \\ v_2 u_1 & 0 & v_2 u_3 \\ 0 & -v_2 u_1 & 0 \end{pmatrix}.$$

Then ϕ satisfies the above condition; however, in general, ϕ is not locally quasi-nilpotent. This shows that, in contrast to the case of length two, see Corollary 3.5, for length three these conditions are no longer equivalent.

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Département de Mathématiques, Université Moulay Ismail, Faculté des Sciences, Meknès, Maroc

E-mail address: nadia_boudi@hotmail.com

Pure Mathematics Research Centre, Queen's University Belfast, University Road, Belfast BT7 1NN, Northern Ireland

E-mail address: m.m@qub.ac.uk