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Reflecting on Absolute Infinity

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Abstract

This article is concerned with reflection principles in the context of Cantor's conception of the set theoretic universe. We argue that within such a conception reflection principles can be formulated that confer intrinsic plausibility to strong axioms of infinity.[1]

Introduction

The perspective adopted throughout most of this article may be taken for the purposes of discussion to be that of set theoretic platonism, which holds that the mathematical universe consists of *collections*. Often that universe is conceived as built over a collection of Urelemente, (as Zermelo did) such as the natural numbers, or the reals, or, ... that are not themselves sets. For brevity's sake however we assume here that there are no Urelemente. This entails that we take even mathematical objects that are often assumed to be somehow irreducible, such as real numbers and natural numbers, to be represented by collections in the universe of sets. This assumption does not affect our arguments, but it does reflect the fact that we are not concerned with the problems of indeterminacy of reference to which Benacerraf has drawn attention.[2] Zermelo held that the mathematical universe forms a potentially infinite sequence of sets of a special kind, which he called 'normal domains'. Quantification over sets is necessarily restricted: we cannot quantify over all sets.

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[2] Paul Benacerraf, *What numbers could not be.* Philosophical Review **74**(1965), p. 47–73.

Zermelo's viewpoint allows the motivation of set theoretical principles that go beyond the standard axioms of *ZFC*.

Cantor, by contrast, held that the set theoretic universe exists as a *completed* absolute infinity. The Burali-Forti paradox and Russell's paradox were initially interpreted as showing that Cantor's 'naive' set theory, as it is still sometimes called, is inconsistent. It was thought (erroneously) that Cantor had failed to recognise that the mathematical universe cannot itself constitute a set. Cantor himself protested that he never took the set theoretic universe as a whole to be a set. Nowadays, Cantor is rarely accused of having defended an outright inconsistent theory of sets. Nevertheless, according to the received view, Cantor's ideas about the set theoretic universe as a whole are outdated, and ultimately philosophically untenable.[3]

Certainly there are, as we shall see, tensions in Cantor's view of the nature of the existence of the set theoretic universe. However we claim that a Cantorian viewpoint, when appropriately understood, can be nevertheless more powerful and fruitful than Zermelo's view of the set theoretic universe. This is manifested in the motivation of reflection principles in set theory. It is known that on Zermelo's conception of the set theoretic universe,[4] only weak reflection principles can be motivated, which give rise to small large cardinal principles such as strongly inaccessible cardinals, and perhaps arguably with some more work, to so-called Mahlo cardinals but to little more. It is also known that a Cantorian conception of the set theoretic universe, as formalised by von Neumann, allows for the further motivation of somewhat stronger reflection principles.[5]

I The Zermelean and Cantorian Universes

Zermelo was the first to hold that, Urelemente aside, the mathematical universe consists

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[3]For one expression of this view, cf. Ignacio Jané,. *The role of the Absolutely Infinite in Cantor's conception of set.* Erkenntnis **42**(1995), p. 375–402.

[4]Ernst Zermelo, *On boundary numbers and domains of sets*, 1930 Translated by M. Hallett. In: W. Ewald, *From Kant to Hilbert: a source book in mathematics. Volume 2*, Oxford University Press, 1996, p. 1208–1233.

[5]Paul Bernays, *Zur Frage der Unendlichkeitsschemata in der axiomatische Mengenlehre.* In: *Essays on the Foundations of Mathematics*, Magnus Press, 1961, p. 3–49, and William Tait, *Constructing cardinals from below.* In: Tait, W. *The provenance of pure reason: essays in the philosophy of mathematics and its history.* Oxford University Press, 2005, p. 133–154.

only of sets. Through the work of Zermelo, Skolem, Fraenkel, and von Neumann, it became established in the 1920s that sets are governed by the laws of ZFC. This has become the most prevalent form of set theoretic platonism: there are only sets, and they obey the principles of ZFC.

The question is thus raised as to how the sets are related to the mathematical universe. Zermelo's viewpoint can be canvassed as follows.[6] When we are engaged in set theory, our quantifiers always range over a domain of discourse D, which Zermelo calls a 'normal domain'. The entities over which our set theoretic quantifiers range are sets: they are governed by the principles of standard set theory (ZFC). Our domain of discourse D itself is also a collection. Since there are no collections other than sets, our domain of discourse must also be governed by the principles of ZFC. However, on pain of contradiction, D can then not be included as an element in our domain of discourse. Nonetheless, we can expand our domain of discourse so that it includes D as an element. The expanded domain of discourse D' can even be taken to be such that it also satisfies the principles of ZFC. However the expanded domain D' will again be a set. So the previous considerations apply to D' also. In sum, even though the domain of discourse can always be expanded, it never comprises all sets. The outcome of this viewpoint for Zermelo, is that the mathematical universe is a potential infinite sequence of (actually infinite) domains of discourse that satisfy the principles of ZFC.[7] There thus was for Zermelo a hierarchy of normal domains (that we should now call 'models of set theory') indexed by Cantor's ordinal numbers. He wrote of 'The two opposite tendencies of the thinking spirit, the idea of creative *advancement* and that of collective completion [*Abschluss*] [...] are symbolically represented and reconciled [here]':[8] as we advance through the Cantorian ordinals we see more sets are created which

[6]p. 1231–1233 in Ernst Zermelo, *op.cit.* For a detailed description of Zermelo's technical results and philosophical view as articulated here, see Section 6 of Akihiro Kanamori, *Zermelo and set theory*, Bulletin of Symbolic Logic **10**(2004), p. 487–553. For an account of the role of Fraenkel and von Neumann in this development, in particular with respect to the axiom of Replacement, see Section 5 of Akihiro Kanamori, *The higher infinite. Large cardinals in set theory from their beginnings.* Springer, 1994.

[7]A similar view is expressed at p. 219 of Parsons, *Sets and Classes.* Noûs **8**(1974). Reprinted in Parsons, C. *Mathematics in Philosophy. Selected Essays.* Cornell University Press, 1983, p. 209–220.

[8]*Op.cit.*, p. 1233.

are collected into this unending tower of completed domains.

There are basic structural insights about the set theoretic universe that escaped Cantor. For instance, Zermelo in his later years after adopting the Axiom of Foundation viewed the set theoretic universe as structured into a layered hierarchy of initial segments V_α (with α ranging over the ordinals) that are sets.[9] Zermelo also saw that the above picture yields initial segments of the universe of the form V_α in which all the axioms of *ZFC* are true. Cantor, perforce in this informal, pre-axiomatic phase of the subject's development did not see that far. Nonetheless, we shall argue that Cantor's conception of the set theoretic universe as a completed infinity is more powerful and more plausible than that of Zermelo.

In discussing Cantor's ideas one has to be careful, as Tait warns us, of not falling into a trap of anachronistic thinking: Cantor did not have the concept of von Neumann ordinal, nor of a V_α, before him. The 'transfinite numbers' constituting Ω were of an independent nature as order types. Tait gives a careful analysis in what way Cantor would have interpreted a Burali-Forti-like paradox in his (Cantor's) own terms, given that he did not have the ordinals.[10] Nevertheless Tait concludes Cantor would doubtless have seen the contradictoriness of claiming Ω as a set.

Zermelo's picture does raise some difficult questions. For one thing, it is not clear in which dimension the mathematical universe is supposed to vary. His notion of *creative advancement* suggests some form of progression or growth but this is not a metaphor that can be stretched too far. It seems that Zermelo was thinking of iterating the power set in some autonomous fashion along the ordinals of each normal domain internally, rather than along some external class of ordinals. For another, there is the question how Zermelo can get his view across to us. Zermelo calls his theory about normal domains 'meta-set theory'.[11] However he does not elaborate on the question what the domain of discourse of meta-set theory is. (*What* is it that we cannot quantify over?) This does not imply that Zer-

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[9]Mirimanoff anticipated this hierarchy in Dmitry Mirimanov *Les antinomies de Russell et Burali-Forti et le problème fondamental de la théorie des ensembles*, L'Enseignement Mathematique, 1917, p. 37–52; the work of Von Neumann was also crucial: see again p. 518 of Kanamori *op.cit.*.

[10]William Tait, *Cantor's Grundlagen and the Paradoxes of Set Theory* In: Tait, W. *Ibid., p. 265.*

[11]*Op. cit.*, p. 1233.

melo's thesis about the essential restrictedness of quantification is false; however, it does seem difficult to see how this thesis can be communicated. He nevertheless hypothesised, as he says, in his 'meta-set theory', that there is a one-to-one correspondence between Cantor's ordinals and his normal domains.[12] Indeed, for both these reasons it seems somewhat hard to motivate second order or higher order principles on Zermelo's 'potentialist' viewpoint.[13] Such principles involve quantification over a completed collection of all classes over V - which is just what Zermelo does not have. One may perhaps formulate principles based on a universal class quantification only, but principles involving further use of such quantifiers beg a domain of quantification. We note that this can be taken as a reason for preferring Cantor's view over Zermelo's. Tait suggests the alternative that we have to consider the second order quantifiers as applying over each normal domain in turn: thus ranging over over $V_{\kappa+1}$ for each 'normal domain' V_κ.[14]. We shall return to his point when we discuss reflection below.

Cantor's theory of the nature of the set theoretic universe as a whole is not easy to summarise. His views seem to have undergone a transformation around 1895. We first discuss his earlier views, and then turn to his later views.

Cantor's basic convictions preclude Zermelo's potential infinity of (completed) normal domains ever to be the final word about the nature of the set theoretic universe. The set theoretic universe could not, in Cantor's view, form a potential infinity of actual infinities because of what Hallett calls Cantor's *domain principle*:

> There is no doubt that we cannot do without variable quantities in the sense
> of the potential infinite; and from this the necessity of the actual infinite can
> also be proven, as follows: In order for there to be a variable quantity in some
> mathematical inquiry, the 'domain' of its variability must strictly speaking be
> known beforehand through a definition. However, this domain cannot itself be

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[12]*Op. cit.* Last paragraph p.1233 *"To the unbounded series of Cantor ordinals there corresponds a similarly unbounded [double-]series of essentially different set-theoretic models . . .*

[13]For a discussion of various aspects of this problem, see also the essays in Agustín Rayo and Gabriel Uzquiano (eds) *Absolute Generality.* Oxford University Press, 2006.

[14]William Tait, *op.cit.* p. 142.

something variable, since otherwise each fixed support for the inquiry would collapse. Thus, this 'domain' is a definite, actually infinite set of values. Thus, each potential infinite, if it is rigorously applicable mathematically, presupposes an actual infinite. [15]

In this quotation, Cantor speaks of the necessity of 'knowing' the domain of variation through a 'definition'. Surely Cantor is merely sloppy here, and we are should discount the epistemological overtones. Another slip can be detected in Cantor's use of the word 'set' in this quotation. Cantor means the argument to be applicable not just to sets but also to absolute infinities. In particular, this means that the infinity of transfinite sets that potentially exists (as for Zermelo) presupposes an actual, completed absolutely infinite domain as its range of variation.

Admittedly Cantor was in his writings not very explicit about what he did take the set theoretic universe as a whole to be. One problem is that it is not in every instance clear whether he has a theological or a mathematical conception of absolute infinity in mind. Indeed, he argues that it is the task not of mathematics but of 'speculative theology' to investigate what can be humanly known about the absolutely infinite.[16] Nonetheless, in many instances it is clear that Cantor has a mathematical conception of the absolutely infinite:

The transfinite, with its wealth of arrangements and forms, points with necessity to an absolute, to the 'true infinite', whose magnitude is not subject to any increase or reduction, and for this reason it must be quantitatively conceived as an absolute maximum.[17]

We shall concentrate on Cantor's conception of the 'quantitatively absolute maximum', which is the set theoretic universe as a whole. From the passages discussed above, we

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[15]*Mitteilungen zur Lehre vom Transfiniten VII* (1887): p. 410–411, in Georg Cantor *Abhandlungen mathematischen und philosophischen Inhalts. Herausgegeben von Ernst Zermelo,* Verlag Julius Springer, 1932 (our translation). For an extended discussion of the domain principle, see p. 7–8 and Ch. 1 of Michael Hallett, *Cantorian set theory and limitation of size.* Clarendon Press, Oxford,1984.

[16]p. 378 in Georg Cantor, *ibid.*

[17]*Mitteilungen zur Lehre vom Transfiniten V* (1887), in Georg Cantor, *ibid*, p. 405, (our translation).

conclude that he attributes to it the following properties. It is a fully determinate, fully actual ('completed'), inaugmentable totality. It is composed of objects (sets) that are of a mental nature ('ideas'). Moreover unlike the sets in the mathematical universe, the universe as a whole cannot be uniquely characterised.

Burali-Forti's 'paradox' was published in 1897, and from this time one finds a subtle change of terminology in Cantor's writings. Whereas before, Cantor used the expression 'the Absolutely Infinite' for characterising the set theoretic universe, he now categorises the set theoretic universe and other proper classes (such as the class of all ordinals) as *inconsistent multiplicities*. In a letter to Dedekind[18] he wrote that it had proven to be necessary to distinguish two kinds of multiplicity. A multiplicity can be of such a nature, that the assumption of the 'togetherness' ('Zusammenseins') of a multiplicity's elements leads to a contradiction, so that it is impossible to conceive the multiplicity as a unity, as a 'finished thing'. He called such multiplicities absolutely infinite or inconsistent multiplicities. In latter day jargon we should call such 'proper classes'.[19] However the passages do not show that Cantor no longer believed that the set theoretic universe did not form an inaugmentable totality that forms the domain of our mathematical discourse past, present, and future. Hauser[20] has argued that by 'existing together' or Zusammensein, Cantor means 'existing together as elements of a "finished" set'. Thus, what he is saying is merely that the totality of all transfinite numbers (or all alephs) does not constitute a set and therefore cannot be an element of some other set. However, Hauser continues, Cantor is not denying that the transfinite numbers coexist in some other form, namely as *apeiron*, which is math-

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[18] Letter to Dedekind (1899), p. 443 of Cantor *ibid.*

[19] Jané has argued that passages such as this letter (and another in 1899 to Hilbert) indicate that Cantor no longer believed that the set theoretic universe formed a completed infinity (sections 6 and 7 of Ignacio Jané, *The role of the Absolutely Infinite in Cantor's conception of set.* Erkenntnis **42**(1995), p. 375–402) and speculates that Cantor tacitly moved to a conception of the set theoretic universe as an irreducibly potential entity, whereby he arrived at a pre-figuration of Zermelo's conception of the mathematical universe. In his more recent work, (*Idealist and realist elements in Cantor's approach to set theory.* Philosophia Mathematica **18**(2010), p. 193–226), he no longer claims that Cantor actually gave up the thesis of the existence of the mathematical universe as a completed infinity. However Jané rightly stresses that there remains a tension between Cantor's earlier commitments and Cantor's later terminology of inconsistent multiplicities.

[20] Section 3 of Kai Hauser, *Cantor's Absolute in metaphysics and mathematics,* International Philosophical Quarterly **53**(2013), p. 161–188.

ematically indeterminate, meaning that one cannot assign a cardinal or ordinal number to the totality of all numbers.

The content of the notion 'apeiron' that one finds in Plato is notoriously unclear. So this does not really help much in the clarification of the nature of the set theoretic universe. In other words, there is an unresolved interpretative debt at this point on the side of the defender of the Cantorian viewpoint. Such a defender is facing the choice of either upholding Cantor's earlier view of the set theoretical universe and trying to make good philosophical sense of it, or else taking Cantor's characterisation of the mathematical universe as an inconsistent multiplicity as the final word, and trying to make sense of that. However both cannot be done at the same time.

What we propose to do is, in the first instance, to ignore Cantor's description of the set theoretic universe as an inconsistent multiplicity. In the following sections, we shall adopt Cantor's characterisation of the set theoretic universe as a completed whole, and discuss how it can be used to motivate 'top down' reflection principles. Then we shall formulate a stronger reflection principle. We shall see that to make sense of this stronger reflection principle, elements both of Cantor's earlier views and elements of Cantor's later views on the nature of the set theoretic universe can be used.

II Reflection

Cantor came to speak about mathematical absolute infinities in terms redolent of ineffability. More modern interpretations of this quality of unknowability are known as *reflection principles*. The starting point of set theoretic reflection is the early Cantorian view that the mathematical Absolutely Infinite is unknowable.[21]

As it stands, this 'unknowability' is indeed a negative statement. However it can be given a positive interpretation as follows. Let us provisionally identify the mathematical Absolutely Infinite with the set theoretic universe V. The set theoretic universe is unknowable in the sense that we cannot single it out or pin it down by means of any of our

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[21]'The Absolute can only be acknowledged, but never known, not even approximately known' (Endnote to section 4: p. 205 in Georg Cantor, *op.cit.*, our translation).

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assertions: no true assertion about V can be made that excludes other unintended interpretations that make the assertion true. In particular—and this is stronger than the previous sentence—no assertion that we make about V can ensure that we are talking about the mathematical universe rather than an object *in* this universe. So if we do make a true assertion ϕ about V, then there exist sets s such that ϕ is also true when it is interpreted over s.

In the late 1890s the Burali-Forti theorem made it abundantly clear that V is not the only actual whole that is absolutely infinite. So in light of this we must say that the mathematical absolutely infinite comprises, in addition to the mathematical universe as a whole, all other proper classes.[22] However in fact, the above argument should hold true for any proper class. They can then be said to be unknowable in the sense that no assertion in the language of sets can be true of only some proper classes. So if we do make a true assertion concerning a proper class, then there exists sets about which this assertion is already true. If we truly describe mathematical absolute infinities, then there are set proxies for the absolute infinities such that our description can also truly be taken to range over the proxies. Cantor did not explicitly articulate this line of argument (although we shall).[23]

On the face of it, Zermelo's viewpoint uses a form of set theoretic reflection: every admissible domain of discourse in set theory is a 'normal domain', and this can by a reflective movement be seen to be a set. We cannot quantify over, or in any way make use of, proper classes, for, in his view, no such things exist, we only have normal domains of the form V_κ that are models of second order ZFC. The set theoretic universe as a whole is not something we can talk about, according to Zermelo, for it never exists as a completed realm. So, literally speaking, Zermelo cannot, according to his own view, truly say that "*the* set theoretic universe is so rich that it contains many normal domains".

The best Zermelo can do, as mentioned above, is simply to postulate that above ev-

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[22]Cantor's 1899 argument that the ordinals form an inconsistent totality is critically discussed in Jané ibid. p. 395–396.

[23]There is a view that Cantor used an application of a proto-reflection argument to deduce the existence of the natural numbers a set. See p. 117–118 of Hallett *op.cit.*

ery ordinal, there is an ordinal which is the 'boundary number' of a normal domain. In modern terms, this is expressed as an axiom that postulates unboundedly many strongly inaccessible cardinals:

Axiom 1 $\forall \alpha \exists \beta : \beta > \alpha \wedge$ " β is a strongly inaccessible cardinal ".

This *seems* to say exactly what is required. It says that a fundamental property of the set theoretic universe, namely making *second-order* Zermelo-Fraenkel set theory (ZFC^2) true, is reflected in arbitrarily large set-sized domains. However closer inspection reveals that this cannot exactly be the case: there must be ordinal numbers that fall outside the quantifiers in this axiom. By Zermelo's own lights, the quantifiers in Axiom 1 must range over a domain of discourse that forms a set in a wider domain of discourse. There will be ordinals in this wider domain of discourse that do not belong to the 'earlier' domain of discourse. Hence Axiom 1 can only be regarded as our slightly anachronistic formalization of Zermelo's 'meta-set theoretic' postulate, that the boundary numbers are in one-to-one correspondence with the Cantorian ordinals. It is only a reflection principle if we allow ourselves the absolute infinity of the whole of V as a ZFC^2 model, (which Zermelo did not) and *then* posit reflection on that.

Stronger reflection principles can be formulated if we take Cantor's view of the set theoretic universe as a completed infinity seriously. However, to study these reflection principles in a precise setting, logical laws governing them have to be formulated. The language that is assumed is the language of second-order (or two-sorted, if you will) set theory, where the membership symbol is expressing the only fundamental non-logical relation, and where we have two types of variables: the first-order variables range over sets ($x, y, \ldots$) and the second-order variables range over (proper and improper) classes ($X, Y, \ldots$). We shall from now on take the sets and classes to be governed by the principles of Von Neumann-Bernays-Gödel (NBG) class theory (and worry about the justification for this later). Indeed, von Neumann's class theory, the pre-cursor to Bernays' formulation of NBG, can be seen as a formalisation of Cantor's viewpoint (but not as a conceptual clarifi-

cation).[24]

If we take the point of view of Cantor's early theory of the mathematical universe, and that there are more absolutely infinite collections than V alone, then we can express the reflection idea as follows:

Axiom 2 $\forall X : \Phi(X) \rightarrow \exists \alpha : \Phi^{V_\alpha}(X \cap V_\alpha),$

where Φ^{V_α} is obtained by relativising all first- and second-order quantifiers to V_α and its power set, respectively, and where α does not occur free in Φ.

The Zermelian reflection principle (Axiom 1), if it can be so regarded, only expresses that certain true class theoretical statements are reflected downwards (*e.g.* the axioms of ZFC^2). Axiom 2 states that *every* true (second order parametrised) class theoretic statement is reflected down to some set sized domain. Axiom 2 is stronger than Axiom 1: it implies large cardinal principles that postulate indescribable cardinals.[25]

Of course it is then natural to formulate reflection principles of orders higher than two in an analogous manner. However already the full third-order class reflection principle is inconsistent, at least for formulae that involve general parameters.[26] Tait (*op.cit.* 2005) returns to a "bottom up" style of Cantorian reflection. He points out that Cantor's Principle with which he introduced the class of transfinite numbers Ω:

> If the initial segment Σ of Ω is a set, then it has least strict upper bound $S(\Sigma) \in$
> Ω.

has a clearly imprecise, or impredicative, content: for we have to know what a set is beforehand. Tait suggests therefore a *Relativized Cantorian Principle*: one formulates an *existence* condition C, and the principle asserts that if an initial segment of the ordinals satisfies C

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[24]See von Neumann, *An axiomatization of set theory*, 1925, In: J. Van Heijenoort (ed) *From Frege to Gödel. A source book in mathematical logic (1879–1931)*, Harvard University Press, 1967.

[25]This axiom and its relatives were discussed in Paul Bernays, *op.cit.*

[26]See William Reinhardt, *Remarks on reflection principles, large cardinals, and elementary embeddings.* in *Proceedings of Symposia in Pure Mathematics. Vol. 10* American Mathematical Society, p. 189–205,1974. But parameter free sentences of higher orders are unproblematic. See Victoria Marshall, *Higher order reflection principles*, Journal of Symbolic Logic **54**(1969), p. 474–489 for an account of higher order reflection with restricted kinds of parameters.

then it has its supremum in Ω. Existence principles are then provided which yield third-order reflection restricted to a certain class of "positive" formulae, on models of the form V_κ, which are consistent and stronger than second-order class reflection. Tait thereby demonstrates that Zermelo's 'potentialism' does not require Zermelo to restrict himself to only second order language: potentialism can be detached from a purely second order viewpoint. Such principles then entail the existence of (strengthenings of) ineffable cardinals, for example, but do not prove the existence of the least ω-Erdős cardinal: $\kappa(\omega)$.[27] Fourth-order reflection is inconsistent even when restricted to "positive" formulae.[28] Indeed, Koellner gives an argument to the effect that no "internal reflection principle" can ever entail large cardinal principle axioms that are as strong as the principle that postulates the existence of $\kappa(\omega)$; the import here is that the latter is consistent with V being Gödel's constructible universe L.

In sum, the situation is this. From Zermelo's conception of the set theoretic universe as a potential infinity of sets, the region of small large cardinals in the neighbourhood of inaccessible cardinals can be motivated. Due to its recognition of proper classes alongside of sets, the Cantorian point of view can be said to have lead *via* the work of Bernays *et al.*, to the above stronger reflection principles of class reflection, strengthened yet further by the work of Tait, using "bottom up"' reflection and the relativized Cantorian principles sketched above. However none of those principles get us near our desideratum, as these do not take us beyond the small large cardinal principles consistent with $V = L$. In particular, the class reflection principles fall below the strength of postulating so-called measurable cardinals. Indeed, Koellner's (tentative) conclusion is that intrinsically motivated set or class theoretic reflection principles arising from the iterative concept of set are either relatively weak or inconsistent.

Gödel thought that *all* sound large cardinal principles should be reducible to reflection

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[27] See section 4 of Peter Koellner, *On reflection principles.* Annals of Pure and Applied Logic, **157**(2009), p. 206–219.

[28] Section 5, Koellner *op.cit*.

principles.[29] This sentiment goes against the conclusions that Koellner reached, and is often regarded as implausible, due to the weakness of all the familiar reflection principles. Nonetheless, we shall now argue that from a Cantorian point of view Gödel's thought can be given a freer rein. Gödel himself was an adherent of Cantor's actualist viewpoint regarding the set theoretic universe in contrast to Zermelo's potentialist viewpoint.[30] We have seen that the set theoretic universe as a whole and all proper classes of sets are recognised by Cantor to (actually) exist: let us call this informal structure $(V, \in, \mathcal{C})$, where $\mathcal{C}$ denotes the collection of all classes. Then the reflection idea tells us that we cannot single this structure out by means of any of our assertions. Positively put, any (with some qualification) assertions that hold in $(V, \in, \mathcal{C})$ must also hold in some set-size structure.

There are various possible ways of trying to making this more precise. We shall not try to give a catalogue of the *pro*'s and *contra*'s of various options. Rather, we shall concentrate on one way that seems to us especially powerful, natural, and fruitful. Consider the following principle:

Axiom 3 *There is an initial segment of the universe V_κ, together with D, some classes over V_κ, and a nontrivial elementary embedding*

$$j : (V_\kappa, \in, D) \longrightarrow (V, \in, \mathcal{C})$$

with critical point κ.[31]

It turns out that the strength of such an assumption turns on the relationship between D and the set of all classes over V_κ, which of course is $V_{\kappa+1}$. What the embedding function does is to send the elements of D, thus regarded as a sub-collection of the proper classes of V_κ, to elements of $\mathcal{C}$: $j(\kappa) = On, j(V_\kappa) = V, j(Card \cap \kappa) = Card, \ldots$ *etc.* whilst not moving any sets of V_κ. Such principles consistently follow from, and indeed, by varying D,

[29]Section 8.7.9 of Hao Wang, *A Logical Journey. From Gödel to philosophy,* MIT Press, 1996.
[30]*Cfr. section 8.3.4* of Wang, *op.cit.*
[31]I.e., $j(\kappa) > \kappa$ whereas for $z \in V_\kappa$, j is the identity transformation: $j(z) = z$.

imply, reflection principles involving higher order language indescribability as discussed by Bernays; in such cases $D \subset \mathcal{P}(V_\kappa)^L \subseteq V_{\kappa+1}$, where L denotes Gödel's universe of constructible sets (*N.B.* the proper inclusion).

Moreover new increased strength arises when we demand greater identity between $(V_\kappa, \in, D)$ and $(V, \in, \mathcal{C})$: if $D \supseteq \mathcal{P}(\kappa)^L$ then we obtain what set theorists often dub the "smallest large cardinal" (the indescribables *etc.* notwithstanding) but let us call it here the least *extra-constructible* large cardinal: it is that first principle that generates a fully truth preserving non-trivial map from L to L, and thus a large cardinal inconsistent with $V = L$. By making further demands on the closure of D within $V_{\kappa+1}$ and its corresponding j yet further principles of this nature can be derived and which encompass a spectrum of strong axioms of infinity connected with L-like inner models. For example, if K is some larger definable inner model, then requiring that $D \supseteq \mathcal{P}(\kappa)^K$ will ensure this model K also will embed to itself, and by known methods will engender a yet stronger large cardinal property. The formalism of the Axiom 3 is thus extremely flexible. However of particular interest for this paper is the natural completion of this line of thought where D is taken to be maximal:[32]

Axiom 4 *There is an initial segment of the universe V_κ and a nontrivial elementary embedding*

$$j : (V_\kappa, \in, V_{\kappa+1}) \longrightarrow_{\Sigma^0_\infty} (V, \in, \mathcal{C})$$

with critical point κ, where the subscript Σ^0_∞ indicates elementarity for first-order formulas (with class parameters).

As was mentioned earlier, it may be assumed that $(V, \in, \mathcal{C})$ makes at least the principles of *NBG* true. Let us call this principle the *Global Reflection Principle* (*GRP*).[33] So Axiom 3 says

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[32] Philip Welch, *Global reflection principles.* Paper prepared for *Exploring the Frontiers of Incompleteness.* in Isaac Newton Institute Pre-print Series, No. INI12051-SAS, June 2012; see also Philip Welch, *On Obtaining Woodin's Cardinals* to appear in Woodin's 60th Birthday Celebratory volume, American Mathematical Society Memoirs.

[33] A slightly stronger version of any *GRP* requires in Axiom 3 that for every ordinal α, there is a reflection $(V_\kappa, \in, V_{\kappa+1})$ that includes this ordinal α. Everything we say generalises straightforwardly to the stronger axiom.

that the set theoretic universe (with all its proper classes) is reflected in a particular way to a set-size initial segment of the universe.

The level of elementarity that is insisted upon can be varied. Mostly we shall only discuss the precise form of *GRP* as given above. If we need to distinguish the elementarity for formulae in some class Γ other than Σ^0_∞ posited in such principles, we may denote the resulting global reflection principle as GRP_Γ. We thus could also impose the stronger requirement of second order Σ^1_∞-elementarity. Alternatively, we could impose the apparently weaker requirement of Σ^0_1-elementarity. (However it can be shown that *GRP* with Σ^0_1-elementarity is equivalent to *GRP* with Σ^0_∞-elementarity.)

The principle *GRP* says that the universe with its parts is, to a certain degree, indistinguishable from at least one of its initial segments V_κ and *its* parts. It says that the whole set theoretic universe with all its proper classes is mirrored in a set-sized initial segment $(V_\kappa, \in, V_{\kappa+1})$, where the first-order quantifiers range over V_κ, and where the reflection of a proper class X is obtained by 'cutting it off' at level V_κ.

GRP expresses the idea of reflection in a substantially more powerful way than Axiom 2. Axiom 2 just says that each (second-order) statement is reflected from the set theoretic universe to some V_κ (where possibly different second-order statements are reflected in different V_κ's): therefore it does not entail that the universe as a totality particularly resembles any one *single* set-like initial segment. However *GRP* postulates that the whole universe $(V, \in, \mathcal{C})$ is indistinguishable from an initial 'cut' $(V_\kappa, \in, V_{\kappa+1})$ in a very specific way, namely in a way such that no proper class can be distinguished from a proper subset of itself (its intersection with $V_{\kappa+1}$). Thereby *GRP* is a more robustly ontological form of reflection than Axiom 2. In the class theoretic context there is a collection of sets which, together with its classes, forms a simulacrum of the universe with its classes.

However *is GRP* a reflection principle? In contrast with traditional reflection principles such as Axiom 2, the reflection effected by *GRP* is mediated by the posited embedding function *j* acting on *classes*. For this reason, Koellner suggests that therefore *GRP* is more

aptly called a *resemblance principle* (or perhaps a *projection principle*) than a reflection principle. However whichever choice of words one settles on, in our view, *GRP* captures the idea that our whole picture of the universe with its classes is nevertheless strongly reflected in an initial segment of that universe. We are not arguing that it is motivated by the iterative concept of set in isolation, but only with additional class-theoretic considerations.

Note that the relationship of 'similarity' between an initial segment of V, V_κ say, is not purely a first order elementary one about sets: that V_κ is *first order indistinguishable from V*, more usually put as being *elementarily equivalent to V*. This usual sense is relatively weak: it is not an extra-constructible principle (and indeed in one formulation is merely equiconsistent with *ZFC*). Nor is the 'similarity' that of the usual second order reflection, where some second order formula $\Phi(\vec{x}, \vec{X})$ about some sets $\vec{x}$ and classes $\vec{X}$, true in $(V, \in, \vec{X})$ reflects to some κ and is thus true in $(V_\kappa, \in, \vec{X} \cap V_\kappa)$, with $\vec{x} \in V_\kappa$. Nor would it be if we insisted this reflection was always to the same V_κ irrespective of choice of $\vec{X}$ or Φ (as long as $\vec{x} \in V_\kappa$), as is the case with *GRP* — such would still be consistent with $V = L$. We must have indistinguishability between $(V, \in, \mathcal{C})$ and $(V_\kappa, \in, V_{\kappa+1})$, that is with all the classes of V_κ included. *GRP* is thus stronger than all these, and expresses a first order indistinguishability between one single level of the ramified hierarchy and V for formulae containing named classes. The relationship between those classes $X \in \mathcal{C}$ needed, and the corresponding class X' in $V_{\kappa+1}$ is the simplest possible: $X \cap V_{\kappa+1} = X'$. But, and this is the point, any such $X' \in V_{\kappa+1}$ must arise in this way: it is a relationship between *all of* $(V_\kappa, \in, V_{\kappa+1})$ and $(V, \in, C)$.[34] That GRP is then extra-constructible is easily indicated. Define $\mathcal{U}$ by:

$$Y \in \mathcal{U} \Leftrightarrow Y \subseteq \kappa \wedge \kappa \in j(Y).$$

Then well known methods show that $\mathcal{U}$ is a *normal measure*, *i.e.* κ is, in V, a measurable cardinal.[35]

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[34]One perspective is to say that there is a collection $\mathcal{D}$ (of size $|V_{\kappa+1}|$) of classes in $\mathcal{C}$, so that $(V_\kappa, \in, \langle \dot{c} \rangle_{c \in V_{\kappa+1}})$ is elementarily equivalent to $(V, \in, \langle \dot{X} \rangle_{X \in \mathcal{D}})$. Of course all we are doing here is taking $\mathcal{D} = ran(j)$. Moreover $c = X \cap V_\kappa$ for the relevant c, X.

[35]For further details on the strength of *GRP* we refer the reader to Philip Welch, *op. citatis.*

Perhaps it is possible to 'split' the content of *GRP* into an embeddingless reflection principle on the one hand, and a strong class choice principle on the other hand (which yields the embedding function). We will not pursue this possibility in this paper, but merely note that Zermelo viewed choice axioms as *logical* principles.[36]

III Sets, parts, and pluralities

Now that the philosophical motivation behind, and the content of, *GRP* has been explained, we turn to the ontological assumptions of the framework in which it is formulated.

If we want to formally express *GRP*, then at first blush it seems that we need a language of third order: the function j that is postulated to exist pairs sets of V_κ with themselves and sets of $V_{\kappa+1}$ with proper classes (elements of $\mathcal{C}$). However the mapping j that is postulated by *GRP* can be regarded as a second-order object: as a proper class K consisting of ordered pairs such that its first element a is in the domain of j (namely: $V_{\kappa+1}$) and the second element $j(a)$ is an element of $V_\kappa \bigcup \mathcal{C}$.

We also need a satisfaction predicate to express the elementarity of the embedding. *GRP* deploys two notions of truth: truth in the structure $(V_\kappa, V_{\kappa+1}, \in)$, and truth in $(V, \mathcal{C}, \in)$. Truth in $(V_\kappa, V_{\kappa+1}, \in)$ can of course be defined in a two sorted language $\mathcal{L}^2_\in$, whereas truth in $(V, \mathcal{C}, \in)$ cannot. However for our proof-theoretic purposes, adding a Tarskian compositional satisfaction predicate T to $\mathcal{L}^2_\in$ and postulating that the compositional truth axioms hold for $\mathcal{L}^2_\in$ suffices to express what it means for a statement of $\mathcal{L}^2_\in$ to be true and to prove basic properties of truth. In sum, the fact that *GRP* postulates the elementarity of the embedding j does not necessitate us to go up to third order (and this remains so even if we were to take a strong version of *GRP* that is Σ^1_∞ preserving).

As mentioned earlier, Cantor's distinction between sets and Absolute Infinities is a prefiguration of the distinction between sets and proper classes, which was articulated explicitly by von Neumann. The difference with Cantor's theory is that von Neumann did take classes as well as sets to be governed by mathematical laws. It is just that classes are objects

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[36]Ernst Zermelo, *Proof that every set can be well-ordered*, 1904, translated in J. van Heijenoort (ed) *From Frege to Gödel. A source book in mathematical logic 1879–1931.* Harvard University Press, 1967, p. 139–141, p. 141.

sui generis: they obey different laws. Proper classes are objects that have elements, but they are not themselves elements. So, in particular, there is no analogue of the power set axiom for proper classes. However it remained an open question how talk of proper classes ought to be interpreted.

To this question we now turn. What we require is an interpretation of class talk that supports the global reflection idea and that is philosophically attractive. We will start by discussing views of the nature of classes that do not meet this requirement. Then we will articulate our preferred view, and explain why it should be preferred over its rivals.

If proper classes are taken to be *collections* of some sort ('super-sets', or 'collections that are too large to be sets'), then it is somewhat mysterious why they can have elements but not be elements. In Maddy's words:[37]

> The problem is that when proper classes are combinatorially determined just as sets are, it becomes very difficult to say why this layer of proper classes on top of V is not just another stage of sets we forgot to include. It looks like just another rank; saying it is not seems arbitrary. The only difference we can point to is that the proper classes are banned from set membership, but so is the κth rank banned from membership in sets of rank less than κ.

And then why is there no singleton, for instance, that contains the class of the ordinals as its sole element? An alternative would be to say that proper classes can be collected into new wholes, but that these could (for obvious reasons) not themselves be proper classes. They would be again a *sui generis* kind of objects: super-classes. However in this way we embark on a hierarchical road that few find worth traveling. On this picture, classes, super-classes, et cetera, look too much like sets. We seem to be replicating the cumulative hierarchy of sets whilst incurring the cost of introducing a host of different kinds of set-like objects. In sum, we reject views of classes as collections on philosophical grounds.

Parsons has defended a view according to which class talk can be interpreted as short-

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[37]Penelope Maddy *Proper classes.* The Journal of Symbolic Logic **48**(1983), p. 122.

hand for talk about *set theoretic predicates* (and a truth predicate for mathematical formulae).[38] The attractiveness of this view is that, in sharp contrast with the view that we have just discussed, this interpretation of class theory does not carry ontological baggage at all.

Unfortunately, this nominalist interpretation of classes is not open to us. On Parsons' interpretation of class theory, impredicative definitions of classes are inadmissible (for circular). In other words, it is crucial to his view that the class comprehension scheme is restricted to predicates that contain no bound class quantifiers, as is the case in NBG.[39] Now we have argued earlier that GRP does commit us to accepting all instances of the class comprehension scheme of NBG. But it is not hard to see that GRP commits us to at least *some* undefinable classes. In particular, the class embedding j that is posited by GRP can readily be shown not to be definable in NBG.[40]

Boolos and Uzquiano have argued that second-order quantification can be interpreted in such a way that it does not commit one to *classes* of elements as objects in their own right belonging to an overall underlying domain of discourse. This is done by reading a formula of the form $\exists X : \phi(X)$ as: "there are some entities such that ϕ holds of *them*". Such a *plural interpretation* can be applied to second-order ZFC, yielding a plural interpretation of class theory.[41] In other words, we can recognise *the truth* of systems of class theory without recognising anything beyond sets in our ontology: *class theory without proper classes*.[42]

The consensus in the literature seems to be that the plural interpretation of classes is

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[38]See Charles Parsons, *Sets and Classes.* Noûs **8**(1974). Reprinted in Parsons, C. *Mathematics in Philosophy. Selected Essays.* Cornell University Press, 1983, p. 209–220.

[39]See Charles Parsons also points out that with an interpretation of classes as first-order definable over sets it is possible to have theories slightly stronger than NBG, see his *Informal Axiomatization, Formalization, and the concept of Truth* also reprinted in *Mathematics in Philosophy*, p.88.

[40]Otherwise there will be an infinite descending sequence of critical points κ that meet the conditions of GRP, contradicting the Axiom of Foundation.

[41]Boolos viewed this in the context of interpreting set theory in a second-order *logical* setting: see George Boolos, *To be is to be a value of a variable (or some values of some variables).* Journal of Philosophy **81**(1984), p. 430–449, and *Nominalist platonism*, Philosophical Review **94**(1985), p. 327–344. Uzquiano does not tie this view to a particular view on logic: see Gabriel Uzquiano, *Plural quantification and classes.* Philosophia Mathematica **11**(2003), p. 67–81.

[42]Or so the argument goes. For arguments to the effect that appeal to plural reference to avoid commitment to proper classes ultimately is a cheat: see Michael Resnik, *Second-order logic still wild.* Journal of Philosophy **85**(1988), p. 75–87, Alan Hazen, *Against pluralism*, Australasian Journal of Philosophy **71**(1993), p. 132–144, Øystein Linnebo, *Plural Quantification Exposed.* Noûs **37**(2003), p. 71–92.

compatible with the existence of impredicative classes. Indeed, the plural interpretation is usually taken to support full Kelly-Morse class theory.[43] After all, there is no reason to think that any bunch of sets is predicatively definable. Thus the plural interpretation cannot be ruled out of being incompatible with *GRP*. The problem with the interpretation of classes as pluralities is that it is difficult to see how the intrinsic philosophical motivation of *GRP* in terms of the notion of *resemblance* between *entities* can be upheld if the class quantifiers are interpreted in a plural way. Put bluntly, how can there be a question of resemblance of $(V_\kappa, V_{\kappa+1}, \in)$ if there is no entity for it to resemble? Therefore in the present context the plural interpretation is also dismissed.[44]

Instead of the interpretations discussed above, we propose to adopt a *mereological* interpretation of proper classes. We hold that the mathematical universe is a mereological whole and classes, proper as well as improper, are *parts* of the mathematical universe. We identify those parts of V that are also parts of sets, *i.e.,* that are set sized, with sets. This way the threat of a hierarchy of super- and hyper-wholes is not looming. The fusion of the parts of a whole does not create a super-whole, but just the whole itself. So there is no mereological analogue of the creative force of the power set axiom: the mathematical 'power set of' concept does not apply to parts, but only to sets. The mereological interpretation of *classes* that we are proposing here is similar to David Lewis' interpretation of *sets*.[45] Lewis takes sets to be generated by the singleton function and unrestricted mereological fusion. So sets, in his view, have subsets as their mereological parts. Similarly, in our proposed interpretation, classes have sub-classes (and not their elements) as their proper parts. In contrast to proper classes, set *are* elements (of sets and of classes). Indeed we may think of, for example the abstraction term in an instance of the Comprehension Scheme as specifying a part of some set, and that instance as declaring that part a set. This device then allows us to declare any part of a set to be a set.

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[43]See Gabriel Uzquiano, *op. cit.*, section 1.

[44]One person's proof is another's *reductio*: if one believes that there are independent compelling reasons to think that the plural interpretation of classes is correct, then one might find intrinsic reasons to accept *GRP* lacking.

[45]David Lewis *Parts of Classes.* Basil Blackwell, 1986, and *Mathematics is megethology.* Philosophia Mathematica **3**, 1993, p. 3–23.

The difference between our proposed interpretation of the range of the second-order quantifiers and Lewis' theory of classes is that we do not take it upon ourselves to elucidate the nature of sets. Lewis regards the relation between an entity and its singleton as thoroughly "mysterious". Reluctantly, he takes it to be a structural relation (section 2.6, *op. cit.*). Derivatively, there is, in Lewis' approach, something mysterious about all sets. Given the singleton-relation that is part of set theory, the elementhood relation for classes can be explained in a straightforward way. For this reason Lewis thinks that the parthood relation sheds *some* light on the question what sets are and what the elementhood relation is. We are in a different position. As stated in the introduction, in this article we take set theory as given, and do not commit ourselves to any specific interpretation (reductive or non-reductive) of the membership relation: explaining the element-relation for sets is outside the scope of this article.[46]

The mereological interpretation does support the reflection idea that we are developing in this article. It allows us to give real content to the reflection thought that the whole (V) resembles the part (V_κ). As mentioned before, GRP commits us to some class impredicativity. But as with the plural interpretation, there is no reason to believe that only predicatively definable parts of V exist. Indeed, the mereological seems a perfectly acceptable interpretation of full Kelly-Morse class theory. Accepting $GRP_{\Sigma^0_\infty}$ does not commit us to *all* impredicative instances of comprehension. However, it is easy to see that the stronger principle $GRP_{\Sigma^1_\infty}$ does commit us to full impredicative comprehension.[47] Moreover, the mereological interpretation of classes satisfies the two key philosophical desiderata that according to Penelope Maddy an interpretation of class theory has to satisfy:[48] (1) Classes should be real, well-defined entities; (2) Classes should be significantly different from sets. The first desideratum is satisfied because classes are just as real and well-defined as sets. We treat parts as real entities, and the second desideratum is satisfied because the laws of

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[46]For a critique of Lewis' use of the parthood relation to help us understand the objects of set theory (as opposed to class theory) and the relations they stand in, see Alex Oliver *Parts of classes?* Analysis **54**(1994), p. 220–221.

[47]The axioms of Kelly-Morse hold at $(V_\kappa, \in V_{\kappa+1})$, and are then sent up by virtue of the Σ^1_∞ elementarity of j (Σ^0_ω elementary is insufficient).

[48]p. 123 of Penelope Maddy, *op. cit.*

parthood are significantly different from the laws governing sets.

IV Mathematical and mereological reflection

Even the weak versions of *GRP* yield strong large cardinal consequences that in some sense can be said to 'complete' the theory of countable sets.[49]

The strength of strong versions of *GRP* lie between the statement that postulates a 1-extendible cardinal and the statement that postulates the existence of the cardinal motivating the *GRP*: a subcompact cardinal.[50]

Large cardinal axioms can be formulated as postulating elementary embeddings from a model M of set theory into another model N of set theory ("$\exists j : M \longrightarrow N$"). The principle that postulates the existence of 1-extendible cardinals marks a watershed in the theory of large cardinals. For all weaker large cardinal axioms (with critical point κ), the embeddings that they postulate are *continuous* at κ^+, in the sense that $sup\, j``\kappa^+ = j(\kappa^+)$. However from the axiom of 1-extendible cardinals onward, the '=' in this equality must be replaced by '<'. This discontinuity property is exploited time and time again in the theory of large large cardinals.

As noted earlier, variants of *GRP* can be ordered according to the level of elementarity that they require. It can be shown that changing the level of elementarity required alters the strength of the resulting global reflection principle: $NBG + GRP_{\Sigma^0_\infty}$ can not prove *"there is a 1-extendible cardinal"* but raising the elementarity to Σ^1_1-preserving one can. So increasing

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[49]In fact even the weakest *GRP* yields an unbounded class of *measurable Woodin* cardinals in V. Such properties of V have been used repeatedly as the centrality of the notion of Woodin cardinal became apparent in the past two decades. It occurs many times as a hypothesis in the theorems of Woodin, and implies key determinacy properties in analysis via work of Martin, Steel (Donald Martin & John Steel, *A Proof of Projective Determinacy*, Journal of the American Mathematical Society, 1989, 2 , p. 71-125) and Woodin, (W. Hugh Woodin, *The Axiom of Determinacy, Forcing Axioms and the Non-stationary Ideal.* Logic and its Applications series, vol. 1, de Gruyter, 1999) and interestingly, in his proof of the fact of the absoluteness of the theory of the real continuum under all possible attempts to alter it *via* Cohen style set forcing. In W. Hugh Woodin (*The Continuum Hypothesis, Part I* , Notices of the American Mathematical Society, July 2001, p. 567-576) he outlines how the theory of projective determinacy 'completes' the theory of the hereditarily countable sets, as the Dedekind-Peano Axioms do for the natural numbers.

[50]Welch (*EFI paper, op.cit.*) emphasises that although the consistency of 1-extendibles implies that of the weakest of the *GRP*'s, the motivation was very different from that of Reinhardt, briefly because subcompact cardinals embody this kind of top-down reflection, while Reinhardt was interested in 'projecting' the universe upwards.

the elementarity-requirement from Σ^0_∞ to Σ^1_1 carries us over an important dividing line in the theory of large cardinals.

We have classified Σ^0_∞ statements as *mathematical* statements because they only quantify over sets. We have classified Σ^1_∞ statements as *mereological* statements because they quantify over proper classes, which we regard as extra- or supra-mathematical objects. In other words, we might call $GRP_{\Sigma^0_\infty}$ *mathematical global reflection*, whereas $GRP_{\Sigma^1_1}$ must already be regarded as *mereological global reflection*. The divide between mathematical and mereological reflection then coincides with the discontinuity threshold above.

In fact, even $GRP_{\Sigma^1_\infty}$ does not express the reflection idea in its strongest form. Recall that the guiding idea was that the set theoretic universe is *absolutely indistinguishable* from some set-like initial segment of V. $GRP_{\Sigma^1_\infty}$ requires that the embedding j is elementary with respect to the second-order language of set theory (that is, one without a satisfaction predicate). If we have a satisfaction predicate in our arsenal, we might require even stronger elementarity, viz. with respect to the second-order language *including the satisfaction predicate as a primitive*. Since this same satisfaction predicate is used to express the elementarity of j, it will have to be a non-Tarskian, type-free satisfaction predicate. However it is known from the theory of the semantic paradoxes that type-free truth and satisfaction quickly lead to contradictions. So care must be taken here.

But there are philosophical reasons for being cautious and not to go even as far as endorsing $GRP_{\Sigma^1_\infty}$. If *everything* we truly say about V and $\mathcal{C}$ is also true about some set V_κ and its subsets, then what makes it the case that when we are using 'V' and '$\mathcal{C}$' in this article, these terms refer to the set-theoretic universe and its classes, respectively, rather than to some set and its subsets? If we insist on articulating GRP as requiring full Σ^1_∞ elementarity, then if we only have the primitive notion of Satisfaction to single out V and $\mathcal{C}$, this is also true about some set V_κ and its subsets. However, if we articulate GRP as insisting only on a form of mathematical elementarity, that is, Σ^0_1 elementarity (which can be shown to be self-strengthening and to imply Σ^0_∞ elementarity), then this worry is not pressing. Then

we can say that mereological statements allow us to distinguish V and $\mathcal{C}$ from every set together with its subsets. In view of this, we refrain from endorsing mereological global reflection, but rather argue for the mathematical global reflection of *GRP*.

Conclusion

According to many, Cantor's early view of the mathematical universe as a whole is hopelessly entangled with his theological views.[51] In contrast, his later view of the set theoretic universe and proper classes more generally as 'inconsistent multiplicities' is taken to be less so, and can be seen as a first step in the direction of a modern view of the set theoretic reality. It can then either be seen as a prefiguration of a plural interpretation of proper classes (as per George Boolos) or as a potentialist conception of the mathematical universe in the spirit of Zermelo (as per Jané).

In this article we have argued that good secular sense can be made of Cantor's earlier view of the set theoretic universe. It is an ontological view on which proper classes as Absolute Infinities are recognised. Sets are all the mathematical objects there are. All the sets together form, as Cantor The Younger said, a completed whole: the mathematical universe V. However V itself is not a *mathematical* object. Proper classes are parts of the universe. Every part of V is a completed whole. Every set is an element of V. The parthood relation corresponds to the subclass relation, which is a transitive relation, and thus is not the same as the set membership relation, which is not. The language of sets and parts of V is the language of second-order set theory $\mathcal{L}^2_{\in}$. The first-order quantifiers range over all sets. The second-order quantifiers range over all the parts of V. So we are ontologically committed to the existence of sets, the universe of all sets (V), and a rich enough collection of parts of V: we make no further ontological commitments. The sets certainly satisfy *ZFC*. The parts of V satisfy at least predicative second-order comprehension. And the class replacement axiom also holds. So we are licensed to postulate *NBG* class theory in the language $\mathcal{L}^2_{\in}$.

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[51]Christian Tapp, *Absolute Infinity—a bridge between Mathematics and Theology? In N. Tennant (Ed.), Foundational Adventures: Essays in Honor of Harvey M. Friedman. London: College Publications, 2012.*

Not only is this interpretation of Cantor's earlier view perfectly coherent: it is also mathematically fruitful. It allows us to indirectly motivate strong principles of infinity, i.e., large cardinal principles which play an important role in contemporary set theory.

Gödel argued that mathematical axioms can be motivated in two ways: intrinsically, and extrinsically.[52] Extrinsic support for an axiom derives from its consequences. Thus extrinsic motivations are success arguments; they are instances of *inference to the best explanation*. Many believe that intrinsic justification for mathematical principles is more reliable than extrinsic justification. Indeed, many do not think that external motivation for a mathematical axiom can provide strong confirmation of its truth. So it is an important question to what extent large cardinal principles can be motivated intrinsically.

Mathematical reflection principles are intrinsically motivated. In particular *GRP* is not motivated by the iterative concept alone, nor by the nature of set membership relation. It is motivated by expanding our viewpoint and considering the whole structure of the universe V with its classes. The thought in this paper is that the mathematical universe with its parts is so ineffable that there is a rank κ such that $(V_\kappa, V_{\kappa+1}, \in)$ is elementarily equivalent with V with all its classes, and moreover can be connected, or embedded, in a sufficiently truth preserving way, into the whole universe $(V, \mathcal{C}, \in)$. Cantor had no vantage point to see anywhere near this far. In general, he mostly referred to the epistemic transcendence of the set theoretic universe as a whole instead of drawing consequences from that transcendence in the form of reflection principles.

The global reflection principle in its stronger forms is essentially a second-order reflection postulate. So to interpret it, we have to assign a clear meaning to the second-order quantifiers. On Zermelo's potentialist picture, this seems a tall order. Perhaps what Zermelo calls 'meta-set theory' allows quantification over absolute infinities, but this seems counter to the potentialist spirit of Zermelo's view, and his 'meta-set theory' has never been articulated in any detail anyway. The pluralist interpretation of second-order quan-

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[52] Kurt Gödel, *What is Cantor's continuum problem?* In: Gödel, K., *Collected Works, Volume II: Publications 1938–1974.* Oxford University Press, 1990, p. 254–270.1964

tification fares better. It may well give us a fairly clear interpretation of the second-order quantifiers. However on this interpretation, and therefore also on the interpretation of the second order quantifiers as ranging not only over sets but also over 'inconsistent multiplicities', the motivation for *GRP* becomes opaque: we have argued that it is hard to make sense of the motivation for *GRP* in terms of a notion of resemblance, if we adopt an attitude of classes as plurals. As far as we can see, it is only in terms of the interpretation of the second-order quantifiers as ranging over parts of the universe that the intrinsic motivation of *GRP* in our expanded sense can be articulated. For this reason we conclude that the early Cantorian view of the set theoretical universe is mathematically the most fruitful one.