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**NORMALIZABILITY, INTEGRABILITY AND
MONODROMY MAPS OF SINGULARITIES
IN THREE-DIMENSIONAL VECTOR FIELDS**

by

NIAZY HADY HUSSEIN

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School of Computing, Electronics and Mathematics

Faculty of Science and Environment

Plymouth University

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Abstract

In this thesis we consider three-dimensional dynamical systems in the neighbourhood of a singular point with rank-one and rank-two resonant eigenvalues.

We first introduce and generalize here a new technique extending previous work which was described by Aziz and Christopher (2012), where a second first integral of a 3D system can be found if the system has a Darboux-analytic first integral and an inverse Jacobi multiplier. We use this new technique to find two independent first integrals one of which contains logarithmic terms, allowing for non-zero resonant terms in the formal normal form of vector field.

We also consider sufficient conditions for the existence of one analytic first integral for three dimensional vector fields around a singularity. Starting from the generalized Lotka-Volterra system with rank-one resonant eigenvalues, using the normal form method, we find an inverse Jacobi multiplier of the system under suitable conditions. Moreover, these conditions are sufficient conditions for the existence of one analytic first integral of the system. We apply this to demonstrate the sufficiency of the conditions in Aziz and Christopher (2014).

In the case of two-dimensional systems, Christopher et al. (2003) addressed the question of orbital normalizability, integrability, normalizability and linearizability of a complex differential system in the neighbourhood at a critical point. We here address the question of normalizability, orbital normalizability, and integrability of three-dimensional systems in the neighbourhood at the origin for rank-one

resonance system.

We consider the case when the eigenvalues of three-dimensional systems have rank-one resonance satisfying the condition $\lambda + \mu + \nu = 0$ as a typical example, and we use a further change of coordinates to bring the formal normal form for three-dimensional systems into a reduced normal form which contains a finite number of resonant monomials. By using this technique, we can find two independent first integrals formally. The first one of these first integrals is of Darboux-analytic type, and other first integral contains logarithmic terms corresponding to non-zero resonant monomials of the original system.

We introduce the monodromy map in three-dimensional vector fields by using these two independent first integrals to study a relationship between normalizability and integrability of systems. In the case of rank-one resonant eigenvalues, we get a monodromy map which is in normal form, and then in the same way as the case of vector fields, we use a further change of coordinates to reduce this map into a reduced map which contains only a finite number of resonant monomials.

This thesis also examines briefly the case of rank-two resonant eigenvalues of three-dimensional systems. The normal form in this case contains an infinite number of resonant monomials, we were not able to find a reduced normal form with a finite number of resonant monomials. This situation is therefore much more complex than the rank-one case. Thus, we simplify the investigation by truncating the 3D system to a 3D homogeneous cubic system as a first step to understanding the general case. Even though we can find two independent first integrals, the second one involves the hypergeometric function, leading to some interesting topics for further investigation.

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Dedication

This dissertation is lovingly dedicated to these people:

- To my father and mother,

Hady and Najiba

- To my wife,

Hasiba

- To my children,

Muhammad, Madina, Aala, Zeena, Sarah and Saddwan

- To the memory of my friend,

Rizgar H. Younis

- My brothers and sisters

- My friends, who have encouraged and supported me.

Author's Declaration

At no time during the registration for the degree of Doctor of Philosophy has the author been registered for any other University award without prior agreement of the Graduate Committee.

Work submitted for this research degree at the Plymouth University has not formed part of any other degree either at Plymouth University or at another establishment.

Relevant scientific seminars and conferences were frequently attended at which work was usually presented.

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List of Abbreviations

2D	Two Dimensional
3D	Three Dimensional
2D-LV	Two Dimensional Lotka-Volterra system
3D-LV	Three Dimensional Lotka-Volterra system
3DDS	Three Dimensional Dynamical System
GLV	Generalized LotkaVolterra system
IJM	Inverse Jacobi Multiplier
LV	Lotka-Volterra system
PID	Principal Ideal Domain
RM	Reduced Map
RNFS	Reduced Normal Form Systems
SHFC	Singular Holomorphic Foliation by Curve
\mathcal{R}_λ	Resonant condition
r_λ	Rank of eigenvalues
\mathbb{N}	The set of all natural numbers without zero
\mathcal{N}	The set of all natural numbers with zero
\mathbb{Z}	The set of all integer numbers
$\mathbb{Z}_{\geq 0}$	The set of all positive integer numbers with zero
\mathbb{Q}	The set of all rational numbers

\mathbb{R}	The set of all real numbers
\mathbb{C}	The set of all complex numbers
\mathcal{F}	Foliation of a manifold
\mathcal{L}	Leaf of a foliation

Chapter 1

Introduction

Normal forms have been applied to problems in many areas of mathematics in order to simplify the underlying calculation. We shall consider here normal forms of analytic vector fields and maps.

In this thesis, we shall develop the theory of normal forms for three-dimensional dynamical systems (3DDS) in a neighbourhood of a singular point.

We first briefly review the theory of normal forms for vector fields and diffeomorphism.

In these cases, a normal form is a simpler form near a singular point obtained by using a suitable change of coordinates. The aim of this is to display the local dynamical properties better. In many cases these transformations are formal and not analytic, but still provide valuable information about the dynamics of the system.

There have been many of studies on the application of normal forms in two-dimensional (2D) vector fields. However, the application of normal forms in three-dimensional (3D) vector fields has been the subject of much less research, and in particular little is known about their possible reduced normal forms. Our aim here to investigate such reduced normal forms with application to the integrability of

3D vector fields.

Among many historical references we note the following. Birkhoff (1966), in the early stages of the theory, was interested in Hamiltonian systems, and the normalizing transformations considered were canonical transformation. Bruno (1989) obtained in detail the convergence and divergence criteria for the normalizing transformations. Ashkenazi and Chow (1988) have presented the basic theory of normal forms based on the classical ideas of Poincaré and Birkhof; they also discuss the relationship between vector fields and diffeomorphisms by using normal form methods. Basic references on normal forms and their applications may be found in Chow and Hale (1982), Guckenheimer and Holmes (1983) and Arnold (1988).

In this thesis, we shall examine the issue of orbital normalizability and integrability in the neighbourhood of the origin for the 3D system,

$$\dot{x} = \lambda x + \sum_{n=2}^{\infty} P_n(x, y, z), \quad \dot{y} = \mu y + \sum_{n=2}^{\infty} Q_n(x, y, z), \quad \dot{z} = \nu z + \sum_{n=2}^{\infty} R_n(x, y, z), \quad (1.1)$$

where $\lambda, \mu, \nu \neq 0$ and P_n, Q_n, R_n are homogeneous polynomials in $\mathbb{C}[x, y, z]$ of degree n .

We are interested if it is possible to perform a change of coordinates in the neighbourhood of the origin to bring the system (1.1) to the system,

$$\begin{aligned} \dot{x} &= x(\lambda + \sum_{n=1}^{\infty} \tilde{P}_n(x, y, z)), \\ \dot{y} &= y(\mu + \sum_{n=1}^{\infty} \tilde{Q}_n(x, y, z)), \\ \dot{z} &= z(\nu + \sum_{n=1}^{\infty} \tilde{R}_n(x, y, z)). \end{aligned} \quad (1.2)$$

This system is known as generalized Lotka-Volterra (GLV) equations, and are

also sometimes referred to as Kolmogorov equations. The system (1.2) is more general than either predator-prey or the competitive Lotka-Volterra (LV) systems, for more detail see Rand et al. (1994) and Hofbauer and Sigmund (1998).

From the system (1.2), when $z = 0$, we get the 2D generalized Lotka-Volterra system,

$$\dot{x} = x(\lambda + \sum_{n=1}^{\infty} \tilde{P}_n(x, y)), \quad \dot{y} = y(\mu + \sum_{n=1}^{\infty} \tilde{Q}_n(x, y)). \quad (1.3)$$

These systems have been considered by many authors especials in the case of polynomial systems. Giné and Romanovski (2010) found the necessary and sufficient conditions for a singular point to be integrable for planar quintic LV equations with $(1 : -1)$ -resonance. Żołądek (1997) discussed some conditions for the existence of a local meromorphic first integral, $H = x^{\lambda_1} y^{-\lambda_2} + \dots$, of the system (1.3). Christopher et al. (2003) addressed the question of orbital normalizability, integrability, normalizability and linearizability of a complex differential system in the neighbourhood at a critical point for the system (1.3) with eigenvalues in the ratio $1 : -\lambda$ with $\lambda \in \mathbb{R}^+$. There are other works relative to this topic, and one can find many information in Christopher and Rousseau (2004) and Christopher et al. (2004). Moreover, other investigations about the 2D-LV systems can be found in Romanovski and Shafer (2008) and Wang and Liu (2008).

The qualitative properties of the system (1.2) when $\tilde{P}_n = \tilde{Q}_n = \tilde{R}_n = 0$ for $n > 1$ has been widely investigated, and we only mention a few references. Bobiński and Żołądek (2005) considered the case when a centre does not lie on any coordinate plane. The use of the Darboux method to show the integrability of LV systems were investigated by several researcher, such as Christodoulides and Damianou (2009), Llibre and Valls (2011), Aziz and Christopher (2012) and Hu et al. (2013). Another question is to realize how many limit cycles can be obtained when perturbing systems in the centre variety in the class of the 3D-LV systems.

Salih (2015) used a new technique to prove that two and four limit cycles can be bifurcated for the centre on a planar and a conic invariant surface, respectively, of the LV systems.

In Chapter 2, we give some background material for this thesis. Then, in Chapter 3 and 4, we consider the case when the eigenvalues for the system (1.2) have rank-one resonance satisfying the condition $\lambda + \mu + \nu = 0$. In this case, a normal form of the system (1.2) can be generated by exactly one resonant monomial of the form $u = xyz$. This result is a powerful way to find two independent first integrals under appropriate conditions on the resonant coefficients.

In Chapter 3, we are interested to find one analytic first integral for the system (1.2). The integrability of non-Hamiltonian systems are, in general, very complicated to detect. The Darbouxian theory of integrability can be used to find first integrals of vector fields. This kind of integrability gives a link between the integrability of vector fields and the number of invariant algebraic surfaces that they have. If the system (1.2) is locally analytically completely integrable in \mathbb{C}^3 , then the system has two functionally independent analytic first integrals, see Zhang (2008). This proves that the system can be brought to an orbitally linearizable system by an invertible change of coordinates. Also, Zhang (2014) showed that if an n -dimensional polynomial differential system has $n - 1$ functionally independent Darboux Jacobian multiplier, then it has $n - 1$ functionally independent Liouvillian first integrals. Llibre et al. (2015b) proved that in n -dimensional integrable systems, any Jacobian multiplier is functional independent of these $n - 1$ independent first integrals such that the divergence of the system is not zero.

In our work, by using the normal form method, we find an inverse Jacobi multiplier for the system (1.2) under appropriate conditions, and then we are able to find the sufficient conditions for existence of one formal analytic first integral. In this way, we can prove a sufficient condition for the existence of one first integral

for some cases of the 3D-LV system. These cases have been left as conjectural in Aziz and Christopher (2014), who gave a number of necessary conditions for one first integral, but could not prove their sufficiency.

In Chapter 4, we consider reduced normal form systems (RNFS), by which we mean that, by a further change of coordinates, a formal normal form of the system (1.2) can be brought to a simpler form containing a finite number of resonant monomials. In the case of 2D vector fields, Ilyashenko and Yakovenko (2008) discuss in detail of the formal normal forms for 2D vector fields. Christopher et al. (2003) considered a reduced normal form system for 2D vector fields in the cases when the vector fields were normalizable and orbital normalizable. We are interested in similarly looking for a further change of coordinates for which the system (1.2) can be brought to a RNFS which only contains a finite number of resonant monomials. We show that this RNFS has two independent first integrals. The second one of these first integrals is found by a new technique. This technique extends previous work which was described by Aziz and Christopher (2012). This new technique is detailed in Theorem 6.

Another approach considered here is to use the monodromy map of the system (1.2) to determine the type of the singular point. Applications of monodromy are pervasive in mathematics, even playing an important role in arithmetic algebraic geometry. Many topics associated with monodromy in differential equations will be found in Żołądek (2006). We apply a monodromy map in the neighbourhood of one of the separatrices using the two independent first integrals near a non-trivial loop surrounding the singular point to obtain a 2D monodromy map. The idea of using monodromy to examine differential equations were first explained by Mattei and Moussu (1980), see also Rousseau (2004) and Ilyashenko and Yakovenko (2008). In the last few years, lots of researches have been concentrated on the monodromy map for 2D vector fields. More details about monodromy can be

found in the works of Arnold (1988), Zakeri (2001), Christopher et al. (2003), and Christopher and Rousseau (2004).

Aziz (2013) considered the use of the monodromy techniques in 3D-LV systems. In our work, we use a different method to understand the monodromy map of a 3D vector field. This is detailed in Section 4.3.

In the same way as the case of vector fields, we also apply a further transformation to reduce the 2D monodromy map into a simpler map. The idea of a reduced normal form for 2D map was investigated by Chen and Della Dora (1999), Wang and Liu (2008) and Abate and Raissy (2013).

In Chapter 5, we consider the case when the eigenvalues of the system (1.2) have rank-two resonance. In this case, a normal form of the system can be generated by two independent resonant monomials. However, the normal form still contains an infinite number of resonant monomials even when reduced. This situation is much more complex than the rank-one case. Therefore, we simplify the investigation by truncating the 3D system to a 3D homogeneous cubic system as a first step to understand the general case.

Investigation of cubic centres for 2D systems is still a subject of current research. Sibirskii (1965) showed that the cyclicity of a linear centre or focus perturbed by homogeneous polynomials of the third degree is at most five. Chavarriga and Giné (1998) gave a simple characterisation for all integrable cases for 2D of cubic systems with degenerate infinity in polar coordinates. Schlomiuk (1993) investigated to make a distinction between a focus and a weak focus of a general cubic system by using Poincaré-Lyapunov constant. See also, Chavarriga et al. (1999), Romanovskii and Shcheglova (2000), Liu and Chen (2002), Llibre and Vulpe (2006), Hu et al. (2008), Llibre et al. (2015a) and the references therein.

Here, we find two independent first integrals of the 3D homogeneous cubic system by using the Darboux method with an inverse Jacobi multiplier. The second

one of these first integrals involves the hypergeometric function and interesting some things which have been unable solve in this thesis.

Chapter 2

Background

In this chapter, we give some definitions, theorems and a brief explanation of some of the methods we use, such as the Darboux method and normal forms as a background to the thesis.

2.1 The basic definitions

We consider the three dimensional dynamical system (3DDS):

$$\begin{aligned}\dot{x} &= \lambda_1 x + \sum_{n=2}^{\infty} P_n(x, y, z), \\ \dot{y} &= \lambda_2 y + \sum_{n=2}^{\infty} Q_n(x, y, z), \\ \dot{z} &= \lambda_3 z + \sum_{n=2}^{\infty} R_n(x, y, z),\end{aligned}\tag{2.1}$$

where P_n, Q_n, R_n are homogeneous polynomials in $\mathbb{C}[x, y, z]$ of degree n . We let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$.

Definition 1. System (2.1) is said to have resonant eigenvalues at the origin if

$\lambda_i, i = 1, 2, 3$, satisfy the following arithmetic condition

$$(\lambda, n) - \lambda_i = \lambda_1 n_1 + \lambda_2 n_2 + \lambda_3 n_3 - \lambda_i = 0, \quad \text{for some } i \in \{1, 2, 3\}, \quad (2.2)$$

for some $n = (n_1, n_2, n_3) \in \mathcal{N}^3 = (\mathbb{N} \cup \{0\})^3$.

The monomial $X^n = x^{n_1} y^{n_2} z^{n_3}$ in the system (2.1) with the order $|n| \geq 2$ is said to be a resonant monomial if n satisfies equation (2.2), where $|n| = n_1 + n_2 + n_3$. The coefficient of the monomial X^n in the system (2.1) is called a resonant coefficient and the corresponding term is called a resonant term.

Note that if condition (2.2) does not hold, then the eigenvalues are called nonresonant and then by a formal change of coordinates the system (2.1) can be brought formally into its linear system, (more details can be found in Ilyashenko and Yakovenko (2008)).

By normalizability of the system (2.1) in a neighbourhood of the origin, we mean that there is a change of coordinates, transforming the system (2.1) into a system in *normal form* (see section (2.4)) which is one in the following form

$$\begin{aligned} \dot{X} &= X(\lambda_1 + \sum_{n=1}^{\infty} \tilde{P}_n(X, Y, Z)), \\ \dot{Y} &= Y(\lambda_2 + \sum_{n=1}^{\infty} \tilde{Q}_n(X, Y, Z)), \\ \dot{Z} &= Z(\lambda_3 + \sum_{n=1}^{\infty} \tilde{R}_n(X, Y, Z)), \end{aligned} \quad (2.3)$$

where $\tilde{P}, \tilde{Q}, \tilde{R} \in \mathbb{C}[X, Y, Z]$ only containing the resonant terms.

Theorem 1 (Poincaré-Dulac theorem). A formal vector field is formally equivalent to a vector field whose a linear part is in Jordan normal form and which has only resonant monomials in the nonlinear part.

According to the above theorem, there is an invertible formal power series tangent to identity $(X, Y, Z) = \varphi(x, y, z) = (x + o(x, y, z), y + o(x, y, z), z + o(x, y, z))$, transforming the system (2.1) to the system (2.3). In this state, we denote $\varphi(x, y, z)$ transforms the system (2.1) to a normal form.

By χ we denote the corresponding vector field of system (2.1)

$$\chi = P(x, y, z) \frac{\partial}{\partial x} + Q(x, y, z) \frac{\partial}{\partial y} + R(x, y, z) \frac{\partial}{\partial z},$$

where $P = \lambda_1 x + \sum_{n=2}^{\infty} P_n(x, y, z)$, $Q = \lambda_2 y + \sum_{n=2}^{\infty} Q_n(x, y, z)$ and $R = \lambda_3 z + \sum_{n=2}^{\infty} R_n(x, y, z)$.

Definition 2. A singular point (x_0, y_0, z_0) of the system (2.1) is a point that satisfies the equation $P(x_0, y_0, z_0) = Q(x_0, y_0, z_0) = R(x_0, y_0, z_0) = 0$, otherwise is called an ordinary point.

Definition 3. A continuously differentiable function $\varphi(x, y, z)$ in a neighbourhood of a singular point is said to be a first integral of the system (2.1) if $\varphi(x, y, z)$ is a constant on the trajectories of the system (2.1). That is

$$\chi(\varphi) = \frac{\partial \varphi}{\partial x} P + \frac{\partial \varphi}{\partial y} Q + \frac{\partial \varphi}{\partial z} R = 0.$$

Definition 4. By a Darboux-analytic first integral, we mean that the first integral is of the form

$$\varphi = x^\alpha y^\beta z^\gamma (k + o(x, y, z)),$$

where α, β, γ are constant and $k \neq 0$.

If we identify the vector field χ with the following the 2-form:

$$\Omega = P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy, \tag{2.4}$$

where \wedge is the exterior product or wedge product of vectors. The function $\varphi(x, y, z)$ is a first integral, if it satisfies

$$d\varphi \wedge \Omega = 0, \quad \text{where } d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz.$$

We are interested mainly in Darboux-analytic first integrals. Also, we say a term in the first integral φ with order $|n| = |n_1 + n_2 + n_3| \geq 1$ is the resonant monomial if the order of this monomial satisfies the following equation

$$(\lambda, n) = \lambda_1 n_1 + \lambda_2 n_2 + \lambda_3 n_3 = 0, \quad \lambda = (\lambda_1, \lambda_2, \lambda_3), \quad n = (n_1, n_2, n_3) \in \mathcal{N}^3$$

where $\mathcal{N} = \mathbb{N} \cup \{0\}$.

The system (2.1) is partially integrable if it has only one first integral, or it is completely integrable if it has two independent first integrals (Cairo and Llibre, 2000). The concept of integrability for the system (2.1) is based on the existence of first integrals. It is crucial to find conditions on the parameter values (usually the coefficients of the monomials) for which the system (2.1) is partially or completely integrable.

Definition 5. By the integrability of the system (2.1) at the origin, we mean that there is an analytic change of coordinates around zero, transforming the system (2.1) into the system:

$$\dot{X} = \lambda_1 X \xi(X, Y, Z), \quad \dot{Y} = \lambda_2 Y \xi(X, Y, Z), \quad \dot{Z} = \lambda_3 Z \xi(X, Y, Z), \quad (2.5)$$

where $\xi(X, Y, Z) = 1 + o(X, Y, Z)$. If the change of coordinates can be chosen so that $\xi(X, Y, Z) = 1$, then we say the system (2.1) is linearizable.

If the system (2.1) is integrable, then it has two first integrals

$$\varphi_1 = X^{-\lambda_2} Y^{\lambda_1}, \quad \varphi_2 = Y^{\lambda_3} Z^{-\lambda_2}, \quad (2.6)$$

which pulled back to original coordinates

$$\tilde{\varphi}_1 = x^{-\lambda_2} y^{\lambda_1} (1 + o(x, y, z)), \quad \tilde{\varphi}_2 = y^{\lambda_3} z^{-\lambda_2} (1 + o(x, y, z)), \quad (2.7)$$

given two independent first integrals of the system (2.1). Conversely, if we have two first integrals of the form (2.7), then by an invertible change of coordinates, we can bring the two first integral (2.7) into the form (2.6), and hence the transformed system is of the form (2.5) for some $\xi(X, Y, Z) = 1 + O(X, Y, Z)$.

Definition 6. A function $M(x, y, z) : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ is said to be an inverse Jacobi multiplier (IJM) of 2-form Ω if it is not locally null and satisfied the equation

$$d(M) \wedge \Omega = M d(\Omega), \quad (2.8)$$

where $d(M) = \frac{\partial M}{\partial x} dx + \frac{\partial M}{\partial y} dy + \frac{\partial M}{\partial z} dz$. According to the vector field χ , we can use the partial differential equation below to find an inverse Jacobi multiplier

$$\chi(M) = M \operatorname{div}(\chi),$$

where $\operatorname{div}(\chi)$ stands for the divergence operator of the system (2.1), $\chi(M) = \frac{\partial M}{\partial x} P + \frac{\partial M}{\partial y} Q + \frac{\partial M}{\partial z} R$, and similar for Ω .

In the 3DDS, if there two independent first integrals $\varphi(x, y, z)$ and $\psi(x, y, z)$ exist, then $\Omega = M(d\varphi \wedge d\psi)$, where Ω is the 2-form as in (2.4), and the function M is an inverse Jacobi multiplier.

The foundation of material on the IJM used here can be found in Berrone and Giacomini (2003).

2.2 Integrability and normalizability of systems

To understand the results are achieved in this thesis we need a basic background of the normal form method for vector fields. We relate this to the concept of integrable and normalizable systems for the system (2.1) with rank-one and rank-two resonant eigenvalues at the origin (see Definition 7).

Let us start with a set

$$\mathcal{R}_\lambda = \{n = (n_1, n_2, n_3) \in \mathbb{Z}_{\geq 0}^3 : (\lambda, n) = \lambda_i, i = 1, 2, 3, |n| \geq 2\}$$

where $\lambda_i, i = 1, 2, 3$, are the eigenvalues, $|n| = |n_1 + n_2 + n_3|$ and $(\lambda, n) = \sum_{i=1}^3 \lambda_i n_i$. We call *the rank* of vectors in the set \mathcal{R}_λ , by r_λ , clearly if $\lambda_1, \lambda_2, \lambda_3 \neq \{0\}$ then $r_\lambda \leq 2$

Zhang (2008) proved that if the origin of the system (2.1) has no eigenvalues equal to zero, (even at least one of the eigenvalues not equal to zero, (see Zhang (2013))), then the system (2.1) has two locally independent analytic first integrals if and only if $r_\lambda = 2$, and by an analytic change of coordinates, the system (2.1) is analytically equivalent to its normal form (2.3).

Definition 7. The eigenvalues $\lambda_i, i = 1, 2, 3$, are said to have rank-one resonance if there is exactly one independent linear dependency, and rank-two resonance if there is exactly two independent linear dependencies over \mathbb{Q} of these three eigenvalues.

Definition 8. Following Arnold (1988), the Poincaré domain in \mathbb{C}^3 is the collection of all tuples $(\lambda_1, \lambda_2, \lambda_3)$ such that the convex hull of the point set $\{\lambda_1, \lambda_2, \lambda_3\}$ does not contain the origin inside or on the boundary.

The Siegel domain is the complement of the Poincaré domain.

We are interested in the case where a singular point whose non-zero eigenvalues lie in the Siegel domain with rank-one resonance. Now, if we fix the value of these eigenvalues, then any resonant monomials in normal form of the system (2.1) should satisfy the condition

$$a\lambda_1 + b\lambda_2 + c\lambda_3 = 0, \quad (2.9)$$

where $(a, b, c) \in \mathbb{Z}^3 \setminus \{0\}$. Moreover, zero lies inside a triangle formed by the eigenvalues and can be written as a convex combination of these eigenvalues (take a look at the triangle in Figure (2.1, a))

Now, we want to show that $a, b, c \in \mathbb{Z}_{\geq 0}$. Suppose, this does not hold. Firstly, if all $a, b, c < 0$, then we can multiply by -1 , we get all elements are positive. So, all three are not positive the only remain options, are that, two elements of a, b, c are positive and other element is negative, or one of them is positive and other elements are negative. But if we choose the formal case, then by multiplying the coefficients by -1 , this brings us to the later case. Therefore, without loss of generality, we can assume that $b < 0$ and $a, c > 0$. By definition of the convex hull, we want to find the location of zero. We have

$$a\lambda_1 + b\lambda_2 + c\lambda_3 = a\lambda_1 + (-b)(-\lambda_2) + c\lambda_3 = 0, \text{ where } a, -b, c \geq 0,$$

by multiplying above equation by $\frac{1}{a-b+c}$, we get

$$\frac{a}{a-b+c}\lambda_1 + \frac{-b}{a-b+c}(-\lambda_2) + \frac{c}{a-b+c}\lambda_3 = a_1\lambda_1 + b_1\lambda_2 + c_1\lambda_3 = 0,$$

where $a_1, b_1, c_1 > 0$, and $a_1 + b_1 + c_1 = 1$, hence the zero is located inside of the

point set $\{\lambda_1, -\lambda_2, \lambda_3\}$. This implies that zero located in two different places, which is impossible, hence we can choose all a, b, c in $\mathbb{Z}_{\geq 0}$, see Figure (2.1, b), unless zero located on the boundary. If zero located on the boundary, we have two possibilities,

- i. If these three eigenvalues do not lie on the line, in which case zero lies between two eigenvalues, let zero between λ_1 and λ_3 , and hence from equation (2.9), we get $b = 0$ and the equation (2.9) becomes

$$a\lambda_1 + c\lambda_3 = 0, \quad \text{where } (a, c) \in \mathbb{Z}^2 \setminus \{0\}.$$

In the same way to the above argument. If $a, b < 0$, we can multiply by -1 , we get $a, b > 0$. So, if a and c are not positive at the same time the only remain option, is that, one of them is positive and other element is negative, then we can assume that $a > 0$ and $c < 0$, thus we have

$$a\lambda_1 + c\lambda_3 = a\lambda_1 + (-c)(-\lambda_3) = 0, \quad a, -c > 0,$$

multiplying by $\frac{1}{a-c}$, yields

$$\frac{a}{a-c}\lambda_1 + \frac{-c}{a-c}(-\lambda_3) = a_1\lambda_1 + c_1(-\lambda_3) = 0,$$

where $a_1, c_1 > 0$ and $a_1 + c_1 = 1$, hence the zero is located inside of the point set $\{\lambda_1, -\lambda_3\}$. This implies that zero located in two different places, which is impossible, hence we can choose all a, c in $\mathbb{Z}_{\geq 0}$.

- ii. If these three eigenvalues lie on the line, in which case zero also lies between two eigenvalues, in the same way as above, we can choose all a, b, c in $\mathbb{Z}_{\geq 0}$.

Also, we consider the case where the linear dependency involves all three eigen-

values. Therefore, all a, b and c are not equal to zero in the same time. If two of these elements are zero, we get one zero eigenvalue, but we consider non-zero eigenvalues of the system (2.1). The last case, if we can suppose that one of a, b, c is zero (let $c = 0$), then from equation (2.9) we get

$$a\lambda_1 + b\lambda_2 = 0,$$

where $(a, b) \in \mathbb{Z}^2 \setminus \{0\}$. In the same way to Case (i). If $a, b < 0$, we can multiply by -1 , we get both are positive. So, the only remain option, is that, one element of a, b is positive and other element is negative. By which in this case, we can assume that $a > 0$ and $b < 0$, then we have

$$a\lambda_1 + b\lambda_2 = a\lambda_1 - b(-\lambda_2) = 0,$$

multiplying by $\frac{1}{a-b}$, yields

$$\frac{a}{a-b}\lambda_1 + \frac{-b}{a-b}(-\lambda_2) = a_1\lambda_1 + b_1(-\lambda_2) = 0,$$

where $a_1, b_1 > 0$. This implies that zero located in two different places, which is also impossible, hence we can choose all a, b in $\mathbb{Z}_{\geq 0}$,

In order to find all the resonant monomials in the Poincaré-Dulac Theorem 1, we should choose the vector $A = (a, b, c) \in \mathbb{Z}_{\geq 0}^3$, where $X^A = x^a y^b z^c$ is the resonant monomial. Convex hull of the point set of eigenvalues In the next theorem, resonant monomials can be easily described in normal forms. In which we mean that the normal form can be generated by a finite number of resonant monomials which depend on the ranks of eigenvalues.

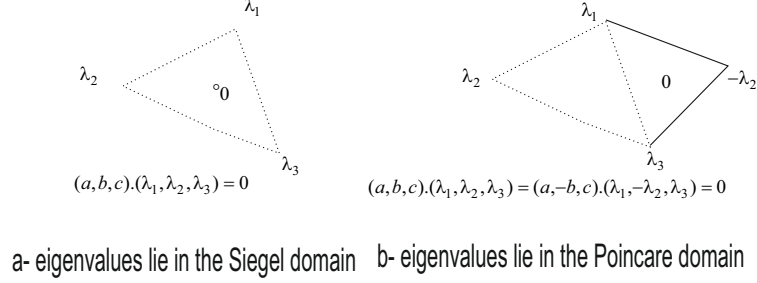


Figure 2.1: Convex hull of the point set of eigenvalues

Theorem 2. If the system (2.1) is normalizable at the origin with rank-one resonant eigenvalues $C = (\lambda_1, \lambda_2, \lambda_3)$, $\lambda_i \neq 0$, $i = 1, 2, 3$ and there is $A = (a, b, c) \in \mathbb{Z}_{\geq 0}^3$, such that $A \cdot C = 0$. Then the normal form (2.3) corresponding to the system (2.1) can be generated by only one resonant monomial, $u = X^A = X^a Y^b Z^c$, that is, the only terms in system (2.3) are u^k , $k \in \mathbb{Q}$.

Proof. We bring the system (2.1) to the normal form (2.3)

$$\begin{aligned}\dot{X} &= X(\lambda_1 + \sum_{n=1}^{\infty} \tilde{P}_n(X, Y, Z)), \\ \dot{Y} &= Y(\lambda_2 + \sum_{n=1}^{\infty} \tilde{Q}_n(X, Y, Z)), \\ \dot{Z} &= Z(\lambda_3 + \sum_{n=1}^{\infty} \tilde{R}_n(X, Y, Z)),\end{aligned}$$

where $\tilde{P}, \tilde{Q}, \tilde{R} \in \mathbb{C}[X, Y, Z]$ only containing the resonant terms.

Let $A' = (a', b', c')$ be any non-zero vector in \mathbb{Z}^3 such that $C \cdot A' = 0$. Since, by definition of the rank-one resonance, we have exactly one linear dependency, therefore A must be multiple by A' , it means that, $A = k A'$, for some $k \in \mathbb{Q}$. If we

start with $u^A = X^a Y^b Z^c$, we see by rewrite this term as the resonant monomials of the form

$$u^A = u^{kA'} = (u^{A'})^k,$$

hence, for each A' there is $k \in \mathbb{Q}$. □

Now let us describe a connection between normalizable system and a system is orbitally normalizable. When we want to investigate an orbital normalizability, we work with analytic orbital equivalence. We say that two systems are *orbitally analytically equivalent* if by an analytic change of coordinates one can be changed to a multiple of the other. In particular, the system (2.1) is orbitally normalizable with rank-one resonance if there is a change of coordinates in the neighbourhood of the origin, transforming the system (2.1) into the following form

$$\begin{aligned}\dot{X}_i &= X_1 (\lambda_1 + h_1(u)) k(X, Y, Z), \\ \dot{X}_2 &= X_2 (\lambda_2 + h_2(u)) k(X, Y, Z), \\ \dot{X}_3 &= X_3 (\lambda_3 + h_3(u)) k(X, Y, Z),\end{aligned}$$

where $h_i(u)$, $i = 1, 2, 3$, $k(X, Y, Z) = 1 + o(X, Y, Z)$ are analytic functions, and $u = X_1^a X_2^b X_3^c$ is the resonant monomial.

Proposition 1. If it is possible to find an analytic change of coordinates, we can bring the system (2.1) into the following form

$$\begin{aligned}\dot{x} &= x(\lambda_1 + \sum_{n \geq 1} P_n(x, y, z)), \\ \dot{y} &= y(\lambda_2 + \sum_{n \geq 1} Q_n(x, y, z)), \\ \dot{z} &= z(\lambda_3 + \sum_{n \geq 1} R_n(x, y, z)),\end{aligned} \tag{2.10}$$

where $P_n, Q_n, R_n \in \mathbb{C}[x, y, z]$. Then the above system is orbitally normalizable at

the origin with rank-one resonance if and only if the system obtained by dividing by $1 + \frac{1}{\lambda_1} \sum_{n \geq 1} P_n(x, y, z)$ is normalizable,

$$\dot{x} = \lambda_1 x, \quad \dot{y} = y \frac{\lambda_2 + \sum_{n \geq 1} Q_n(x, y, z)}{1 + \frac{1}{\lambda_1} \sum_{n \geq 1} P_n(x, y, z)}, \quad \dot{z} = z \frac{\lambda_3 + \sum_{n \geq 1} R_n(x, y, z)}{1 + \frac{1}{\lambda_1} \sum_{n \geq 1} P_n(x, y, z)}. \quad (2.11)$$

We prove this in Section 4.2.1. If (2.10) is orbitally normalizable with rank-one resonance, then it has two independent first integrals. One of these first integrals is of Darboux-analytic type and other first integral contains logarithm terms allowing for non-zero resonant terms in the normal form. (see Chapter 3).

In addition, if the system (2.11) is normalizable then there is a change of coordinates, transforming the system into a system in normal form which is one of the following form

$$\dot{X} = \lambda_1 X, \quad \dot{Y} = Y(\lambda_2 + \sum_{n \geq 1} \tilde{Q}_n(X, Y, Z)), \quad \dot{Z} = Z(\lambda_3 + \sum_{n \geq 1} \tilde{R}_n(X, Y, Z)). \quad (2.12)$$

where $\tilde{Q}_n, \tilde{R}_n \in \mathbb{C}[x, y, z]$ only containing the resonant monomials. We can start with the system (2.11), if it is orbitally normalizable with rank-one resonance, then by a change of coordinates $(X, Y, Z) = (x, y + o_1(x, y, z), z + o_2(x, y, z))$, where $o_i(0, 0, 0) = 1$, $i = 1, 2$, the system (2.11) can be transformed to the form (2.12), where $X^a Y^b Z^c$ is the resonant monomial in the system. (In fact we can choose $X = x$ in such a transformation for (2.11)).

2.3 Darboux method in 3D

In this section we discuss the Darboux integrability. In 1878, a seminal work on the integrability of polynomial differential equations in the plane was published by Darboux. He showed how the integrability of a polynomial system can be obtained from sufficient invariant algebraic curves. There are some application for the Dar-

boux method in two-dimensional systems (Christopher and Llibre (1999, 2000), Christopher et al. (2007)). This method also has been used for higher dimensional systems, for instance (Moulin-Ollagnier (1997), Cairo and Llibre (2000)).

We denote the corresponding vector field of (2.1) by χ ,

$$\chi = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y} + R \frac{\partial}{\partial z}.$$

where $P, Q, R \in \mathbb{C}[x, y, z]$. An invariant algebraic surface of the system (2.1) is a surface in \mathbb{C}^3 given by the equation $f(x, y, z) = 0$, such that there exist $K_f(x, y, z) \in \mathbb{C}[x, y, z]$ satisfying

$$\chi(f) = f(x, y, z)K_f(x, y, z), \quad \text{where } \chi(f) = \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R, \quad (2.13)$$

$K_f(x, y, z)$ is called the cofactor of the invariant algebraic curve $f(x, y, z) = 0$. Note that the degree of K_f less than and equal to the degree of the polynomial vector field. If the invariant surface does not pass through a singular point, then it has a cofactor which must vanish at this point.

For any point on the invariant algebraic surface $f(x, y, z) = 0$, the inner product of the two vectors $\nabla f(x, y, z)$ and χ is zero, where $\nabla f(x, y, z) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$. Then the vector field $\chi = (P, Q, R)$ is tangent to the plane $f(x, y, z) = 0$, hence $f = 0$ is shaped by trajectories of the vector field χ . This explain the name ‘invariant’ which satisfies equation (2.13) for some cofactors, as it is invariant under the flow which defined by χ .

The exponential factor has the same role as the invariant algebraic surfaces when two surfaces merge together. When this happened, we will need an exponential factor to calculate the integrability of systems. Let $E = \exp(f/g)$ where

$f, g \in \mathbb{C}[x, y, z]$, then we call E an *exponential factor* if

$$E(\chi) = K_E E, \quad \text{where } E(\chi) = \frac{\partial E}{\partial x}P + \frac{\partial E}{\partial y}Q + \frac{\partial E}{\partial z}R,$$

for some polynomial $K_E \in \mathbb{C}[x, y, z]$ of degree one less than the vector field and it is called the *cofactor* of E . All exponential factors have cofactors vanishing at the critical points not in the denominator of their exponent. An exponential factor in some references is also referring to as degenerate algebraic curves. For example (see Christopher (1994), Christopher and Llibre (1999, 2000))

The following propositions are generalized directly from Christopher (1994) or see Aziz and Christopher (2012).

Proposition 2. If $E = \exp(g/f)$ is an exponential factor for a vector field χ , then g satisfies the equation below with $f = 0$ is an invariant algebraic surface of the vector field,

$$g(\chi) = gK_f + fK_E,$$

where K_f , and K_E are the cofactors of f and E , respectively.

Proposition 3. Assume that f_1 and f_2 are invariant algebraic surfaces with their cofactors K_{f_1} and K_{f_2} , respectively. Then

- i. $f_1 f_2 = 0$, is an invariant algebraic surface with cofactor $K_{f_1} + K_{f_2}$.
- ii. $f^r, r \in \mathbb{C}$, is an invariant algebraic surface with respect to the cofactor rK_f .
- iii. If the two cofactors $K_{f_1} = K_{f_2}$, then the ratio of f_1 and f_2 , is a first integral.

Proposition 4. Assume that $F(x) = f_1^{c_1} \cdots f_r^{c_r} \in \mathbb{C}[x, y, z]$ such that each f_i is a irreducible factor over $\in \mathbb{C}[x, y, z]$. Then, $F = 0$ is an invariant algebraic curve with cofactor K_F of a vector field χ if and only if $f_i, i = 1, 2, \dots, r$ are an invariant algebraic curves with cofactors K_{f_i} . Moreover $K_{f_i} = c_1 K_{f_1} + \cdots + c_r K_{f_r}$.

Darboux's idea, to find first integrals of the vector field χ , is to look for first integrals $F(x, y, z)$ of χ on regions U of \mathbb{C} , which is of the form

$$F = \prod_{i=1}^r f_i^{c_i} E^{c_0}$$

where f_i are algebraic solutions of the equation (2.13), and E is exponential factor. Such integrals called a *Darboux first integral*, or sometimes is said to be a *Darboux function*, with cofactors satisfying

$$\sum_{i=1}^r c_i K_{f_i} + c_0 K_E = 0.$$

where c_i , and c_0 are analytic functions. The function F is a non-trivial first integral of the system if and only if the cofactors are linearly dependent

Given a Darboux function F , we can calculate

$$DF = F\left(\sum_{i=1}^r c_i \frac{Df_i}{f_i} + c_0 \frac{DE}{E}\right) = F\left(\sum_{i=1}^r c_i K_{f_i} + c_0 K_E\right),$$

if $div = \sum_{i=1}^r c_i K_{f_i} + c_0 K_E$, then by definition F is an IJM of the vector field χ , in this situation F is said to be a *Darboux inverse Jacobi multiplier*. The role of the integrating factor is taken by the Jacobi Multiplier when the degree of dimensions of a system more than two. In the context of the Darboux integrability, we usually consider the corresponding reciprocals: inverse integrating factors, and inverse Jacobi multipliers, (see Berrone and Giacomini (2003)).

2.4 Normal forms

We now shortly present a normal form method for a vector field and apply it in some cases of the system (2.1).

The starting point is a holomorphic system of differential equations with a

singular point taken to be the origin, expanded as a power series. In more detail, we assume that system (2.1) is in the following form

$$\dot{x} = Ax + \mathbf{X}(x) = Ax + P_2(x) + P_3(x) + \cdots, \quad (2.14)$$

where A is an $n \times n$ real or complex matrix, $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ and $\mathbf{X}(x) \in \mathbb{C}[[x_1, \dots, x_n]]$ is a vector power series over \mathbb{C} in n variables which defines in the following expression

$$\mathbf{X}(x) = (P_1(x), \dots, P_n(x)), \quad P_k(x) = \sum_{|n|=2}^{\infty} P_k^{(n)} x^n, \quad 1 \leq k \leq n, \quad P_k^{(n)} \in \mathbb{C}, \quad (2.15)$$

where $x^n = x_1^{n_1} \cdots x_n^{n_n}$, $|n| = n_1 + \cdots + n_n$, and $n = (n_1, \dots, n_n)$, $n_i \in \mathbb{Z}_{\geq 0}$. And each component $X_k(x) = P_k(x)$ of \mathbf{X} , $1 \leq k \leq n$, is a formal or convergent power series with complex coefficients.

If there exists a neighbourhood of the origin where all coordinate series are (absolutely) convergent then we say that the series $\mathbf{X}(x)$ converges, if there is no assertion of the convergence of a series then we say that $\mathbf{X}(x)$ is a formal series (computing a formal power-series is explained by Bochner and Martin (1948) in Chapter I).

We now describe a normal form, firstly we assume the linear term Ax to be already in the required normal form, usually it is in Jordan normal form or a real canonical form. We consider a change of coordinates to get a new variables y , having the form

$$x = \mathbf{H}(y) = y + h^k(y), \quad \mathbf{H}(0) = 0, \quad y \in \mathbf{N}(y) \subset \mathbb{C}^n, \quad (2.16)$$

where $\mathbf{N}(y)$ is a neighbourhood of the origin of \mathbb{C}^n , $\mathbf{H}(y)$ is a power series of n

variables, and $h^k(y) \in H_k^n$ of degree $k \geq 2$, where H_k^n is the space of homogeneous polynomials of degree k in n variables with coefficients in \mathbb{C} . We assume $\mathbf{H}(y)$ is tangent to identity. Our purpose is to transform the system (2.14), in a neighbourhood of the origin, into a simpler form. To achieve this, brings out the essential features of the flow near the singular point, in such a way that the Taylor expansion of the transformed non-linear vector field contains a minimal number of terms of every order. By substitution of (2.16) into (2.14), we get

$$\dot{y} = [\mathbf{H}_y(y)]^{-1} A \mathbf{H}(y) + [\mathbf{H}_y(y)]^{-1} P(\mathbf{H}(y)), \quad y \in \mathbf{N}(y), \quad (2.17)$$

having the same general form as the original system, where $\mathbf{H}_y(y)$ denotes the Jacobian matrix $\partial \mathbf{H}(y)/\partial y$. And then following Abate (2005)

$$\mathbf{H}_y(y) = I + h_y^k(y), \quad \text{where } h_y^k(y) = \partial h^k(y)/\partial y, [\mathbf{H}_y(y)]^{-1} = I - h_y^k(y) + O(|y|)^{2k-2}, \quad y \in \mathbf{N}(y), \quad (2.18)$$

and, therefore the system (2.17), becomes of the form

$$\dot{y} = Ay + Q(y) = Ay + Q_2(y) + \cdots + Q_k(y), \quad y \in \mathbf{N}(y), \quad (2.19)$$

where $Q(y) \in \mathbb{C}[[y_1, \dots, y_n]]$, it is the same expression with (2.15) for $y \in \mathbb{C}^n$, and each component $Q_k(y)$ is a formal or convergent power series. To get the best form, we should choose h^j carefully, so that the Q_j are “easier” in some sense than the P_j .

By acting a sequence of the change of coordinates (2.16), some terms can be removed of the original system. Therefore, the system (2.19) is *equivalent* to the system (2.14) in a neighbourhood of the singular point. Then the normal form will contain only the resonant terms, when these terms cannot be removed by the

change of coordinates (2.16).

To see, to what extent $Q(y)$ can be simplified. Expanding $P(x) = P_2(x) + \dots$, $P_k(x) \in H_k^n$, and substituting (2.16), (2.18), and (2.14) into (2.17), we get

$$\dot{y} = Ay + P_2(y) + \dots + P_{k-1} + [P_k(y) - \{h_y^k(y)Ay - Ah^k(y)\}] + O(|y|^{k+1}). \quad (2.20)$$

This suggests introducing the Lie derivative operator which represent the part in bracket

$$L_A^k : H_k^n \rightarrow H_k^n, \quad \text{for each } k \geq 2,$$

which is a map of the vector space H_n^k into itself with respect the matrix A defined by

$$(L_A^k h^k)(y) = h_y^k(y)Ay - Ah_y^k(y), \quad \text{for } k \geq 2, \quad (2.21)$$

where $h_y^k(y) = \partial h^k(y)/\partial y$.

Let \mathcal{R}^k denote the range of L_A^k . The relation between the P_j , Q_j and h^j is determined recursively by the homological equations

$$L_A h^j = K_j - Q_j, \quad j = 2, 3, \dots,$$

where $K_2 = P_2$ and K_j equals P_j plus a correction term computed from P_2, \dots, P_{j-1} and h^2, \dots, h^{j-1} .

Let \mathcal{L}^k be any choice of a complementary subspace to the image of $L_A \in H_n^k$, then it is possible to choose the Q_j so that each $Q_j \in \mathcal{L}^j$. (Let $Q_j = M_j K_j$, where $M_j : H^j \rightarrow \mathcal{L}^j$ is the projection map, and note that the homological equation can be solved, non-uniquely, for Q_j). The choice of \mathcal{L}^j is called a normal form style producing the system (2.19) from (2.14) associated to matrix $A_{n \times n}$ up to order $j \geq 2$ (with respect to the map M_j), or an A -normal form of (2.14) up to order j .

We note that the A -normal form equation is not unique for the fixed A . In fact, it depends on the choice of the complementary subspaces \mathcal{L}^j . Consequently, we get the following decomposition:

$$H_n^k = \mathcal{R}^k \oplus \mathcal{L}^k, \quad \text{for } k \geq 2. \quad (2.22)$$

Theorem 3. Let $X(x)$, $D X(0) = A \in \mathbb{C}^{n \times n}$ in the system (2.14) and the decomposition (2.22) be given with origin taken as a singular point. Then, by a change of coordinates (2.16), we can bring the system (2.14) into the following form

$$\dot{y} = Ay + Q_2(y) + \cdots + Q_r(y) + O(|y|^{r+1}), \quad y \in \mathbf{N}(y),$$

where $Q_k(y) \in \mathcal{L}^k$, for $k = 2, 3, \dots, r$.

Proof. Let begin with the system (2.20), when $k = 2$, we get

$$\dot{y} = Ay + (P_2(y) - L_A^2 h^2(y)) + O(|y|^3), \quad (2.23)$$

since $P_2 \in H_n^2$, then there is $f^2(y) \in \mathcal{R}^2$ and $Q_2(y) \in \mathcal{L}^2$ such that $P_2(y) = f^2(y) + Q_2(y)$. Therefore, a suitable $h^2(y)$ can be found by $L_A^2 h^2(y) = f^2(y)$, which gives

$$\dot{y} = Ay + Q_2(y) + O(|y|^{2+1}). \quad (2.24)$$

Therewith, we perform by mathematical induction. Assume that the theorem is true for degree $2 < k < s - 1 < r$, this means that the system (2.24) can be determined for degree k . That is, the system (2.14) is transformed to normal form for degree k . We may assume that by a change of coordinates the system

(2.14) becomes into the following form

$$\dot{x} = Ax + Q_2(x) + \cdots + Q_{s-1}(x) + P_s(x) + O(|x|^{s+1}), \quad x \in \mathbf{N}_{s-1},$$

where $Q_k \in \mathcal{L}^k$, for $k = 2, \dots, s-1$, $P_s \in H_s^n$, and \mathbf{N}_{s-1} is a neighbourhood of the origin.

Let now begin with a change of coordinates $x = y + h^s(y)$ which starts at degree $s \in \mathbb{N}$. By (2.20) and by choosing a suitable $h^s \in \mathbf{H}_n^s$. In a small neighbourhood $\mathbf{N}_s \in \mathbf{N}_{s-1}$ of the origin, we have

$$\begin{aligned} \dot{y} &= Ay + Q_2(y) + \cdots + Q_{s-1}(y) + [P_s(y) - L_A^s(h^s(y))] + O(|y|^{s+1}) \\ &= Ay + Q_2(y) + \cdots + Q_{s-1}(y) + Q_s(y) + O(|y|^{s+1}), \quad y \in \mathbf{N}_s, \end{aligned} \quad (2.25)$$

for some $Q_s \in \mathcal{L}^s$. □

Given the decomposition (2.22), then the following truncation of the system (2.25)

$$\dot{y} = Ay + Q_2(y) + \cdots + Q_r(y), \quad \text{where } Q_k(y) \in \mathcal{L}^k, k = 2, \dots, r, \quad y \in \mathbf{N}_r,$$

is called an A -normal form up to order r , or a normal form with respect to matrix A up to order $r \geq 2$. In fact, the A -normal form obtained by this way is not unique, it relies on the choice of the complement of the subspace \mathcal{R}^k .

Now let us explain the convergence of normalizing transformations. We consider the power series

$$X(x) = \sum_{i=1}^{\infty} a_i x^i. \quad (2.26)$$

Assume that the above series is convergent for some value of $x = x_0$. The convergence of $\sum_{i=1}^{\infty} a_i x_0^i$ yields $|a_i x_0^i| \rightarrow 0$ as $i \rightarrow \infty$, and therefore $|a_i| < M|x_0^i|^{-1}$ for

some sufficiently large M . If $|x| < |x_0|$ then $|a_i x^i| \leq M m^i$ where $m = |\frac{x}{x_0}| < 1$. Then, by comparison, the series (2.26) is absolutely convergent with $M \sum_{i=1}^{\infty} m^i$.

Therefore, any such power series has a radius of convergence R which is the largest number r such that $\sum_{i=1}^{\infty} a_i x^i$ converges if $|x| < r$. Or r may be zero. Otherwise, the series is divergent for any x with $|x| > r$.

Moreover, a complex function $f(x) = \sum_{i=0}^{\infty} a_i x^i$ can be represented by the series (2.26) within the disk of radius R . Then the function $f(x)$ is analytic for $|x| < R$, and its derivative is obtained by term-by-term differentiation of the series: $f'(x) = \sum_{i=0}^{\infty} i a_i x^{i-1}$, which has the same radius of convergence with the series for $f(x)$.

We denote by $\mathbb{C}[[x]] = \mathbb{C}[[x_1, \dots, x_n]]$ the set of formal power series in x_1, \dots, x_n with coefficients in \mathbb{C} , the coefficient of x^α in $X(x)$ will be denoted by $X^{(\alpha)}$.

Definition 9. Given two power series $X(x) = \sum_i X^{(i)} x^i$, $Y(x) = \sum_i A^{(i)} x^i$ in $\mathbb{C}[[x]]$, we say that $Y(x)$ majorizes the series $X(x)$, and we write $X(x) \prec Y(x)$ if $|X^{(i)}| \leq A^{(i)}$, $i = 0, 1, 2, \dots$. More generally, by a vector power series, we mean an expression:

$$\mathbf{X}(x) = (X_1(x), \dots, X_n(x)), \quad X_k(x) = \sum_{|i|=2}^{\infty} X_k^{(i)} x^i, \quad (k = 1, \dots, n),$$

where $x^i = x_1^{i_1} \dots x_n^{i_n}$ and $X_k^{(i)} \in \mathbb{C}$. Then, a series $\mathbf{Y}(x)$ majorizes the series $\mathbf{X}(x)$ if $X_i(x) \prec Y_i(x)$ for all $i \in \{1, 2, \dots, n\}$, we then write $\mathbf{X}(x) \prec \mathbf{Y}(x)$.

We notice that from the above definition, the coefficients of the majorizing series are real and non-negative. In fact, if a convergent series $Y(x)$ majorizes a series $X(x)$, then $X(x)$ is convergent on some neighbourhood of zero.

Definition 10. If the system (2.14) $\dot{x} = Ax + P(x)$ and (2.19) $\dot{y} = Ay + Q(y)$ together with the change of coordinates $x = H(y)$ in the normal forms method are convergent, then we say that the systems (2.14) and (2.19) are analytically equivalent.

According to the previous definition, it is enough if we can prove that the change of coordinates $x = H(y)$ is analytic when we consider that the original system is analytic. To achieve this, we shall use the majorant series method. An extensive study of this problem via the majorant series method was used by Bibikov (1979).

Theorem 4 (Bibikov (1979)). Let $\lambda = (\lambda_1, \dots, \lambda_n)$, where λ_i are eigenvalues of the diagonal matrix A in the system (2.14), and suppose that there are $\epsilon, s > 0$ such that the following hold:

1. for all $n = (n_1, \dots, n_n) \in \mathbb{Z}_{\geq 0}^n$, if $(n, \lambda) - \lambda_i \neq 0$ for some $i \in \mathbb{N}$, then $|(n, \lambda) - \lambda_i| \geq \epsilon$, where ϵ does not depend on n and λ
2. for all $r = (r_1, \dots, r_i)$ in $\mathbb{Z}_{\geq 0}^n$ for which $2 \leq |r| \leq |n|$ and $r \leq n + e_i$, where $e_i = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$, such that $(n - r, \lambda) = 0$, the following is valid

$$\left| \sum_{i=1}^m r_i \mathbf{Q}_i^{(n-r+e_i)} \right| \leq s |(r, \lambda)| \sum_{i=1}^m |\mathbf{Q}_i^{(n-r+e_i)}|, \quad (2.27)$$

where s does not depend on n and λ , and $\mathbf{Q}_i^{(n)}$ is a resonant coefficient in the distinguished normal form.

Then, if the series $P(x) = P_2 + \dots + P_k(x) + \dots$ in the system (2.14) is analytic it follows that the series $x = H(y)$ is analytic in distinguished normalizing transformation (2.16).

2.5 Normal forms of maps near a fixed point

As we have already said, normal forms provide a significant tool for the investigating of dynamical systems (see for example Arnold (1988), Bryuno (1988)) as they can be used to investigate an essential simplification of local dynamics. The normal form theory also applies on the maps by using a change of variables near identity to transform the original map into a simpler form. We call this simpler form a *normal form* for map. Usually, the normal form is simpler than the original map and sometimes (but not always) allows an explicit study of the local dynamics. Classical normal forms are not always unique. In which case a further reduction is possible. Further reduction of normal form of map has been investigated by several methods in order to get a unique normal form, (see for examples, Wang et al. (2008) gave some sufficient conditions for uniqueness of normal forms of smooth maps, also they gave a recursive formula for the homogeneous terms of the transformed map. Gelfreich and Gelfreikh (2009) used non-linear grading functions to construct a resonant normal form for an area-preserving map near a generic resonant elliptic fixed point. Chen and Della Dora (1999) introduced an important refinement of normal forms for differentiable maps near a fixed point. Abate and Tovenà (2005) described a method for constructing formal normal forms of holomorphic maps with a hyper-surface of fixed points, and they obtained a list of formal normal forms for 2D holomorphic maps tangential to a curve of fixed points).

Now, let us consider a holomorphic map $F : U \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^n$, where $n \geq 1$ and U is an open neighbourhood of p . Here we always assume that the origin is a fixed point, by which we mean that $F(0) = 0$. Also, we always assume that $F \neq id$ (identity) in U . Then, a formal analytic map $F(x)$ at the origin can be written

as one of the following form

$$F(x) = Ax + f_2(x) + \cdots + f_r(x) + \cdots, \quad x \in \mathbb{C}^n, \quad (2.28)$$

where $A = \text{diag}(\lambda_1, \dots, \lambda_n) \in GL(n, \mathbb{C})$ and $f_k \in H_n^k$, where H_n^k is the linear space of n -dimensional vector valued homogeneous polynomials of degree $2 \leq k \leq n$, in n variables with coefficients in \mathbb{C} .

The problem here is to seek a change of coordinates

$$x = H(y) = y + h^2(y) + \cdots,$$

tangent to identity such that the transformed map

$$G(y) = H^{-1} \circ F(x) \circ H(y),$$

is in a simpler form (\circ is composing of two maps), where this simpler form only contains a monomials $x^k e_j$ satisfying the following condition

$$\lambda^k - \lambda_j = \lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j = 0, \quad \lambda = (\lambda_1, \dots, \lambda_n), \quad \text{for some } 1 \leq j \leq n, \quad (2.29)$$

for some $k = (k_1, \dots, k_n) \in \mathbb{Z}_{\geq 0}^n$, and $|k| = k_1 + \cdots + k_n \geq 2$. Equation (2.29) is called a *resonant condition* for the matrix A . Moreover, the monomial $x^k e_j$, $|k| \geq 2$ is said to be a *resonant monomial* if the condition (2.29) holds.

Definition 11. When $n = 1$, then the map (2.28) becomes of the following form

$$F(x) = \lambda x + a_2 x^2 + \cdots \in \mathbb{C}_0\{x\},$$

the number $\lambda = F'(0) \in \mathbb{C}$ is said to be the multiplier of $F(x)$. If $\lambda \neq 0, 1$, then

we say that the fixed point 0 of F is hyperbolic.

When $n = 1$, there is a resonance if and only if the multiplier is a root of unity, or zero. If $n > 1$ resonances may occur in the hyperbolic case too, see Abate et al. (2010). In addition, when $n > 1$, the fixed point 0 is said to be hyperbolic if all eigenvalues of F have modulus different from 1.

Resonances have an important role to obtain a simpler map in order to understand the local dynamical properties as we showed in the previous section. The resonances are the obstruction to formal linearization.

Definition 12. Let $F_1 : U_1 \rightarrow \mathbb{C}^2$ and $F_2 : U_2 \rightarrow \mathbb{C}^2$ be two maps at $p_1 \in \mathbb{C}^2$ and $p_2 \in \mathbb{C}^2$ respectively. We say that $F_1, F_2 \in \mathbb{C}[x, y]$ are formally conjugated if there is an invertible map $\varphi \in \mathbb{C}[x, y]$ such that

$$F_1 = \varphi^{-1} \circ F_2 \circ \varphi, \quad \text{in } \mathbb{C}[x, y].$$

In the following, we want to describe a normal form method for map.

At the beginning, by using if necessary an appropriate linear change of coordinates, we can transform the matrix A into a Jordan canonical form.

The basic idea of normal forms is to choose a transformation tangent to identity

$$x = T(y) = y + h(y) = y + h^2(y) + \cdots + h^k(y), \quad (2.30)$$

where $h^k(y) \in H_n^k$ for $2 \leq k \leq n$. The inverse transformation to (2.30) is

$$y = x - h^k(x) + O(|x|^{2k}), \quad \text{as } ||x|| \rightarrow 0, k \geq 2,$$

which is a smooth diffeomorphism in a neighbourhood of the origin. Then, the

transformed map of (2.28) takes the following

$$\begin{aligned} G(y) &= T^{-1} \circ F \circ T(y) \\ &= Ay + f_2(y) + \cdots + f_{r-1}(y) + [f_r(y) - \{h^r(Ay) - Ah^r(y)\}] + O(|y|)^{r+1} \end{aligned} \quad (2.31)$$

where we want it to be as simple as possible for $r \geq 2$, and $G(y)$ is defined in a neighbourhood $U' \subset U$ of the origin. According to the bracket in (2.31), we can suggest a linear map (the homological operator) $L_A^r : H_n^r \rightarrow H_n^r$ defined by

$$L_A^r h(x) = h(Ax) - Ah(x), h \in H_n^r, \quad \text{for } r > 1.$$

Let \mathcal{R}^r be the range of L_A^r in H_n^r . Similarly, as we have already shown that in the previous section \mathcal{R}^r has an complementary subspace, say \mathcal{C}^r via inner product in H_n^r , then we have the following decomposition

$$\mathcal{H}^r = \mathcal{R}^r \oplus \mathcal{C}^r, \quad 2 \leq r \leq n. \quad (2.32)$$

In our context, the following theorem is the groundwork of calculation of normal forms.

Theorem 5 (Ashkenazi and Chow (1988)). Given the decomposition (2.32), then there is a formal series transformation (2.30) tangent to identity which transforms the map (2.28) into

$$G(y) = T^{-1} \circ F \circ T(y) = Ay + g_2(y) + \cdots + g_r(y) + \cdots, \quad y \in U' \subseteq U, \quad (2.33)$$

where $g_r \in \mathcal{C}_n^r$ for $2 \leq r$, and U' is the neighbourhood of the origin.

To find an A-normal form up to order r , it is helpful to find a basis for $\mathcal{C}^k = \ker(L_{A^*}^k)$, $k \geq 2$. And, it can be easily shown that $L_{A^*}^k = (L_A^k)^*$, $\forall k \geq 2$ via the

inner product $\langle \cdot, \cdot \rangle_n$ of H_n^r , see Ashkenazi and Chow (1988).

Definition 13. (Abate et al. (2010)). The normal form $G(y)$ is called a Poincaré-Dulac normal form of the polynomial map $F(x)$.

Definition 14. Given the decomposition (2.32), then the truncated form of the map (2.33) from r^{th} -jet is called an A -normal form map of (2.28) up to order r .

The methods and theorems related to the A -normal form map can be traced back to the study (Arnold (1988), Wang et al. (2008), Ashkenazi and Chow (1988)).

Also, for other (formal or analytic) normal forms of diffeomorphisms, one can refer to Martinet and Ramis (1982). In general, the classical normal forms are not unique with respect to formal conjugacy.

In Takens (1973), the relationship between normal forms for diffeomorphisms and vector fields is explained in \mathbb{R}^2 . We applied a method on 3D vector field with rank-one resonant eigenvalues to obtain a map in $(\mathbb{C}^2, 0)$. Hopefully, this map does not involve the quadratic terms and is not tangent to identity which is given the details this in chapter (4). Therefore, this map is different from the maps were investigated by Abate (2005), Abate and Raissy (2013). Basic references on normal forms and their applications will be found in Arnold (1988), Briuno (1979), Chow and Hale (1982).

Here we also apply a further transformation to reduce map into a simpler map. The idea of reduced normal forms for maps can be obtained by the renormalizing Poincaré-Dulac normal forms, where the renormalizing Poincaré-Dulac normal forms is well-defined in the context of vector fields. Through this redaction, we will see the relation between the concept of the orbital normalizability and integrability on one side and reduced normal forms and normalizable maps on the other side.

To study a further reduction on normal forms for map, one can see in the following. Chen and Della Dora (1999) gave a method to obtain further reduction of classical normal forms. Abate and Raissy (2013) described a general reduced method for germs of holomorphic (or formal) self-maps, also they provided a list of normal forms for quadratic bi-dimensional super-attracting germs. Wang and Liu (2008) used the recursive formula for further reduction of normal forms for maps, also the concepts of normal forms up to order N and infinite-order normal forms of smooth maps were developed.

2.6 Monodromy map

Before starting to define the monodromy map, it is convenient to describe how two-dimensional vector fields give rise to a foliation in \mathbb{CP}^2 . Firstly, we want to start with the definition of \mathbb{CP}^2 . We consider $\mathbb{C}^3 \setminus \{0\}$ with the action C^* defined by $\lambda \cdot (x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda x_3)$, the orbit of (x_1, x_2, x_3) is denoted by $[x_1, x_2, x_3]$. The complete projective plane \mathbb{CP}^2 is the quotient space of $\mathbb{C}^3 \setminus \{0\}$ modulo the action C^* , with the natural projective $\pi : \mathbb{C}^3 \setminus \{0\} \rightarrow \mathbb{CP}^2$, where $\pi(x_1, x_2, x_3) = [x_1 : x_2 : x_3]$.

On \mathbb{CP}^2 , we have three affine charts which give a structure of a two-dimensional compact complex manifold of the following way:

$$U_i = \{[x_1, x_2, x_3]\}, \quad i = 1, 2, 3,$$

and define homomorphisms $\phi_i : \mathbb{C}^2 \rightarrow U_i$ by

$$\begin{cases} \phi_1(x, y) = [1 : x : y], & (x, y) \in \mathbb{C}^2, \\ \phi_2(u, v) = [u : 1 : v], & (u, v) \in \mathbb{C}^2, \\ \phi_3(r, s) = [r : s : 1], & (r, s) \in \mathbb{C}^2. \end{cases} \quad (2.34)$$

Observe that the change of coordinates $\phi_{ij} = \phi_j^{-1} \circ \phi_i$ is given by

$$\begin{cases} \phi_{12} = \phi_2^{-1} \circ \phi_1(x, y) = (\frac{1}{x}, \frac{y}{x}), & x \neq 0, \\ \phi_{23} = \phi_3^{-1} \circ \phi_2(u, v) = (\frac{1}{u}, \frac{v}{u}), & u \neq 0, \\ \phi_{31} = \phi_1^{-1} \circ \phi_3(r, s) = (\frac{s}{r}, \frac{1}{r}), & r \neq 0. \end{cases}$$

these maps are holomorphic. A unique complex structure on \mathbb{CP}^2 is determined by the atlas $\{(U_i, \phi_i)\}$, $i = 1, 2, 3$ such that ϕ_i are biholomorphic, and each (U_i, ϕ_i) , $i = 1, 2, 3$, is said to be an affine chart of \mathbb{CP}^2 .

We can define lines $L_i = \mathbb{CP}^2 \setminus \phi_i(\mathbb{C}^2)$, $i = 1, 2, 3$. Each line L_i is called the line at infinity (projective lines) associated to affine chart (U_i, ϕ_i^{-1}) .

The sets $L_i = \mathbb{CP}^2 \setminus \phi_i(\mathbb{C}^2)$ have the complex structure of the Riemann sphere $\overline{\mathbb{C}}$. For instance, we can identify $L_1 = \{[0, x, y] : (x, y) \in \mathbb{C}^2\}$ with $\{[x, y] : (x, y) \in \mathbb{C}^2\} \simeq \mathbb{CP}^1$ under the restriction to \mathbb{C}^2 . Then, \mathbb{CP}^2 is a one-line compactification of \mathbb{C}^2 (see Figure (2.2) for the corresponding location of these three lines).

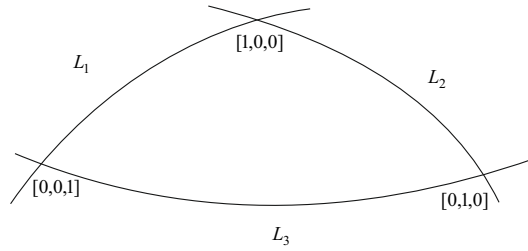


Figure 2.2: Geometry of projective space

In general, a projective line is the image under the map π of a plane in \mathbb{C}^3 which passes through the origin. For example, the plane $ax_1 + bx_2 + cx_3 = 0$ is sent under

π to the projective line having as affine equations $a + bx + cy = 0$, $au + b + cv = 0$ and $ar + bs + c = 0$. The projective lines are the simplest algebraic curves in \mathbb{CP}^2 .

More generally, suppose that $P(x_1, x_2, x_3)$ is a homogeneous polynomial of degree k which defines the algebraic curve $C = \{[x_1 : x_2 : x_3] \in \mathbb{CP}^2 : P(x_1, x_2, x_3) = 0\}$ under the action C^* , where $P(x, y) = \sum a_{i,j} x^i y^j$ is a homogeneous polynomial of degree k on the chart (x, y) . By using the change of coordinates, we write P in two other affine charts as

$$\begin{aligned} P \circ \phi_{12} &= P\left(\frac{1}{u}, \frac{v}{u}\right) = u^{-k} \sum a_{i,j} u^{k-i-j} v^j, \\ P \circ \phi_{23} &= P\left(\frac{s}{r}, \frac{1}{r}\right) = r^{-k} \sum a_{i,j} r^{k-i-j} s^j. \end{aligned}$$

The affine equations for C are $P(x, y) = P(1, x, y) = 0$, $P'(u, v) = P(u, 1, v) = 0$ and $P''(r, s) = P(r, s, 1) = 0$. Therefore, the algebraic curve C in \mathbb{CP}^2 can be defined as the compact set

$$\phi_1\{(x, y) : P(x, y) = 0\} \cup \phi_2\{(u, v) : P'(u, v) = 0\} \cup \phi_3\{(r, s) : P''(r, s) = 0\},$$

where $P(u, v) = \sum a_{i,j} u^{k-i-j} v^j$ and $P(r, s) = \sum a_{i,j} r^{k-i-j} s^j$.

In more detail, there is another way of viewing this curve is by introducing the homogeneous polynomial $\tilde{P}(x_1, x_2, x_3)$ in \mathbb{C}^3 of degree k which is of the following

$$\tilde{P}(x_1, x_2, x_3) = x_1^k P\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right) = \sum a_{i,j} x_1^i x_2^j x_3^{k-i-j}.$$

It is then easily verified that the algebraic curve C over P , under the map π , is the curve

$$C = \pi\{(x_1, x_2, x_3) \mid \tilde{P}(x_1, x_2, x_3) = 0\}. \quad (2.35)$$

Then a projective line is an algebraic curve C for a polynomial $P(x_1, x_2, x_3)$ of

degree one.

It turns out that the space \mathbb{CP}^2 is the natural ambient space to study the 1-form as

$$\Omega = P(x, y) dy - Q(x, y) dx, \quad (2.36)$$

where P, Q are polynomials.

Definition 15. A point $x_0 \in \mathbb{C}^2$ is a singular point (singularity) of Ω , if $P(x_0) = Q(x_0) = 0$. Otherwise the point is non-singular.

Indeed, a solution of the 1-form $\Omega = 0$, which passes through a non-singular point $a \in \mathbb{C}^2$ is a maximum curve L_a satisfying the condition that for any point $b \in L_a$, there is a local parametrization of L_a and a holomorphic map $h = (h_1, h_2) : U \rightarrow L_a$ such that $P(h(t))h'_1(t) + Q(h(t))h'_2(t) = 0$ for all $t \in U$. A solution L_a always exists which passes through a non-singular point $a \in \mathbb{C}^2$. Additionally, if $a, b \in \mathbb{C}^2$, either $L_a \cap L_b = \emptyset$ or $L_a \equiv L_b$.

In regard to the geometrical view, we can define the foliation by using a complex manifold; Let M be a complex n -manifold, a holomorphic foliation \mathcal{F} of co-dimension m on M is a partition of M into disjoint path-connected subsets by $\mathcal{F} = \{L_a\}$ such that for any $p \in M$ there is a chart (U, φ) around p and open poly-disks $A \subset \mathbb{C}^{n-m}$ and $B \subset \mathbb{C}^m$ with maps $\varphi : U \rightarrow A \times B$ which takes the connected components of $F_a \cap U$ to the level sets $A \times \{b\}$, $b \in B$. We call L_a the leaves of the foliation.

Remark 1. The definition which follows is equivalent to the definition above of the foliation.

A holomorphic foliation \mathcal{F} of co-dimension m of M is an analytic maximal atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ for M which satisfies the following

1. For each i , φ_i is a biholomorphism $U_i \rightarrow A_i \times B_i$, where $A_i \subset \mathbb{C}^{n-m}$ and $B_i \subset \mathbb{C}^m$

are open poly-disks.

2. If $(U, \varphi), (V, \psi)$ are in \mathcal{A} with $U \cap V \neq \emptyset$, then the maps $\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$ has the form

$$(\psi \circ \varphi^{-1})(z, w) = (h_1(z, w), h_2(w)), \quad (2.37)$$

where $(z, w) \in \mathbb{C}^{n-m} \times \mathbb{C}^m$, and h_1, h_2 are holomorphic mappings into \mathbb{C}^{n-m} and \mathbb{C}^m , respectively.

A one-dimensional foliation \mathcal{F} in the two-dimensional manifold can be explained in Figure (2.3).

Given any chart (U_i, φ_i) of a foliation \mathcal{F} , then the plaques \mathcal{F} in U are the set

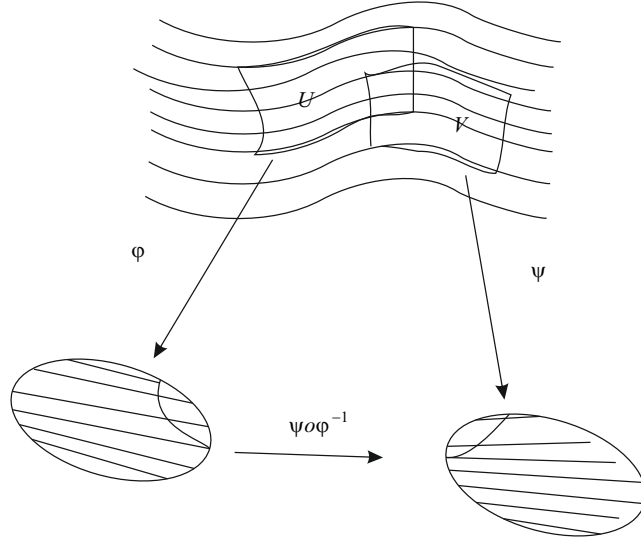


Figure 2.3: Charts defining a foliation

of the form $\varphi_i^{-1}(A_i \times \{c\}), c \in B_i$. Each $a \in M$ lies in at least one plaques. Two points a, b are called equivalent if there are a sequence $\alpha_1, \dots, \alpha_n$ of plaque such that $a \in \alpha_1, b \in \alpha_n$ and $\alpha_i \cap \alpha_{i+1} \neq \emptyset, i = 1, \dots, n - 1$.

Now, we want to define a singular holomorphic foliation by curves on a complex manifold M . Here, the foliation is allowed to have singularities. Firstly, we consider a chart (U, φ) , a subset $E \subset M$ is called an analytic subvariety if each $p \in M$ has a neighbourhood U on which there are holomorphic functions $f_i : U \rightarrow \mathbb{C}, 1 \leq i \leq n$ such that $E \cap U = \{x \in U : f_i(x) = 0, 1 \leq i \leq n\}$. Then, a singular holomorphic foliation by curves \mathcal{F} (SHFC) on M is a holomorphic foliation by curves on $M \setminus E$, where E is an analytic subvariety of M of co-dimension ≥ 2 . In fact, a foliation with singularity in \mathbb{CP}^2 can be induced by $\Omega = 0$, see Zakeri (2001).

To introduce the notion of a singular holomorphic foliation by curves on the \mathbb{CP}^2 , assume that the singular set has co-dimension ≥ 2 . When the underlying manifold is \mathbb{CP}^2 , then the very special geometry of the space permits us to apply some standard algebraic geometry to show that all such foliations are induced by a polynomial vector field on \mathbb{CP}^2 , see Zakeri (2001).

Now we want to extend the 1-form (2.36) on \mathbb{CP}^2 . We consider the 1-form (2.36) on \mathbb{C}^2 and its corresponding SHFC \mathcal{F}_Ω (by definition, the singular foliation induced by Ω , $\mathcal{F}_\Omega : \{\Omega = 0\}$). Using the coordinate map (2.34), we can transfer \mathcal{F}_Ω to U_1 by ϕ_{23} . To achieve this, first transfer \mathcal{F}_Ω to the affine chart (u, v) by ϕ_2 of the following; In the coordinated (u, v) we can write $\tilde{\Omega} = 0$ by using a change of coordinated $x = \frac{1}{u}, y = \frac{v}{u}$ in the following

$$\begin{aligned} \tilde{\Omega}(u, v) &= P\left(\frac{1}{u}, \frac{v}{u}\right) d\left(\frac{v}{u}\right) - Q\left(\frac{1}{u}, \frac{v}{u}\right) d\left(\frac{1}{u}\right) \\ &= u^{-1}P\left(\frac{1}{u}, \frac{v}{u}\right) dv - u^{-2}[vP\left(\frac{1}{u}, \frac{v}{u}\right) - Q\left(\frac{1}{u}, \frac{v}{u}\right)] du. \end{aligned} \tag{2.38}$$

Putting $R(x, y) = yP(x, y) - xQ(x, y)$. Then

$$\tilde{\Omega}(u, v) = u^{-1} \left(P\left(\frac{1}{u}, \frac{v}{u}\right) dv - R\left(\frac{1}{u}, \frac{v}{u}\right) du \right).$$

Let k be the smallest positive integer such that $\Omega' = u^{k+1}\tilde{\Omega}$ is the 1-form on $(u, v) \in \mathbb{C}^2$. Then, the foliations $\mathcal{F}_{\Omega'}$ and $\mathcal{F}_{\tilde{\Omega}}$ are identical on $\{(u, v) \in \mathbb{C}^2 : u \neq 0\}$. Now transport $\mathcal{F}_{\Omega'}$ to U_2 by ϕ_2 . It is easily checked that on $U_1 \cap U_2$ the foliations induced by $(\mathcal{F}_{\Omega}, \phi_1)$ and $(\mathcal{F}_{\Omega'}, \phi_2)$ are coincide.

In the same way, \mathcal{F}_{Ω} could be transported to the affine chart (r, s) by ϕ_{31} to get a foliation $\mathcal{F}_{\Omega''}$ induced by a 1-form Ω'' on $(r, s) \in \mathbb{C}^2$. Therefore, $\mathcal{F}_{\Omega''}$ is transported to U_3 by ϕ_3 .

By \mathcal{F} and repeatedly coincide $\mathcal{F}_{\Omega}, \mathcal{F}_{\Omega'}, \mathcal{F}_{\Omega''}$ with their transported companions on \mathbb{CP}^2 we denote the extended foliation on \mathbb{CP}^2 . Hence, the affine charts $(x, y), (u, v)$ and (r, s) are considered as subsets of \mathbb{CP}^2 by identifying them with U_1, U_2, U_3 respectively.

It follows from the above structure that in each affine chart, \mathcal{F}_{Ω} is given by the solution of the following vector fields:

$$\begin{aligned}\chi_1 &= P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}, \quad (x, y) \in U_1, \\ \chi_2 &= u^k \left(P\left(\frac{1}{u}, \frac{v}{u}\right) \frac{\partial}{\partial u} + R\left(\frac{1}{u}, \frac{v}{u}\right) \frac{\partial}{\partial v} \right), \quad (u, v) \in U_2, \\ \chi_3 &= -r^l \left(Q\left(\frac{s}{r}, \frac{1}{r}\right) \frac{\partial}{\partial r} - R\left(\frac{s}{r}, \frac{1}{r}\right) \frac{\partial}{\partial s} \right), \quad (r, s) \in U_3,\end{aligned}\tag{2.39}$$

where k and l are the least positive integers making the above into polynomial vector fields.

Now, we can also define a singularity of foliation; A finite number of leaves reduced to points are called the singularity of foliation, and the set of them is denoted by $\text{sing}(\mathcal{F})$. We have extended the foliation of Ω across the “line at infinity” L_1 . In which we mean that we obtain a \mathcal{F}_{Ω} in \mathbb{CP}^2 with $\text{sing}(\mathcal{F})$ from Ω according to the process described above. On the other hand, we can say that \mathcal{F}_{Ω} is an SHFC on \mathbb{CP}^2 with $\text{sing}(\mathcal{F}_{\Omega}) = C_P \cap C_Q \cap C_R$, where C are algebraic curve which are defined in (2.35).

Now, we want to find conditions on a 1-form Ω which ensure that the line at infinity $L_1 = \mathbb{CP}^2 \setminus U_1$ with singular points of foliation \mathcal{F}_Ω can be removed as a leaf. Firstly, we consider an SHFC \mathcal{F} which is induced by a 1-form Ω in the chart $(x, y) \in U_1$ of the following

$$\Omega = (P(x, y) + xf(x, y)) dy - (Q(x, y) + yf(x, y)) dx, \quad (2.40)$$

where $P(x, y) = \sum_{k=1}^n P_k(x, y)$, $Q(x, y) = \sum_{k=0}^n Q_k(x, y)$ with P_k, Q_k are homogeneous polynomials of degree k , and either $f(x, y)$ is a non-zero homogeneous polynomial of degree n , or $f(x, y) \equiv 0$, but $yP(x, y) - xQ(x, y) \neq 0$.

Proposition 5 (Zakeri (2001)). We consider the 1-form (2.40), then the line at infinity L_1 with singular points of \mathcal{F}_Ω can be removed as a leaf of the foliation \mathcal{F} if and only if $f(x, y) \equiv 0$.

Proof. Firstly, we consider the affine chart $(u, v) = (\frac{1}{x}, \frac{y}{x}) \in U_2$ in which L_1 is given by the line $\{v = 0\}$. Using this chart, we get

$$\tilde{P}_n(u, v) = -u^{n+1}P_n(\frac{1}{u}, \frac{v}{u}), \quad \tilde{Q}_n(u, v) = -u^n[vP_n(\frac{1}{u}, \frac{v}{u}) - Q_n(\frac{1}{u}, \frac{v}{u})].$$

We obtain a foliation which is described by 1-form $\Omega' = \tilde{P}(u, v) dv - \tilde{Q}(u, v) du$, where

$$\begin{aligned} \tilde{P}(u, v) &= \sum_{k=0}^n u^{n+1-k} P_k(u, v) - f(1, v), \\ \tilde{Q}(u, v) &= \sum_{k=0}^n u^{n-k} [vP_k(u, v) - Q_k(1, v)]. \end{aligned}$$

Indeed, $L_1 \setminus \text{sing}(\mathcal{F})$ is a leaf if and only if the line $\{u = 0\}$ is a solution of $\Omega' = 0$. This occurs if and only if $f(1, v) \equiv 0$. We suppose that $f(x, y)$ is a homogeneous polynomial, then the later condition is equivalent to $f \equiv 0$. \square

In general, the 1-form in (2.36) has an invariant line at infinity $L_1 \setminus \text{sing}(\mathcal{F})$,

and we denote $L_0 \setminus \text{sing}(\mathcal{F})$ by \mathcal{L}_∞ . However, the 1-form in (2.40) hasn't any invariant \mathcal{L}_∞ if $f(x, y) \equiv 0$. For more information, one can refer to Zakeri (2001), Camacho and Lins Neto (1985).

A tool to study so-called transverse dynamics of the foliation is required for describing a leaf. The concept of the monodromy of a leaf is the fundamental tool in describing the transverse dynamics near the leaf. Now we reach the point to study integrability properties of the system (2.1). We then consider a loop $\gamma : [0, 1] \rightarrow \mathbb{C}$ moving around the singular point in order to obtain the information about the vector field at the singular point via this loop. On the other hand, we will see the relation between a holomorphic vector field and a map obtained by using the monodromy transformation which is relative to the loop γ at neighbourhood of one of the separatrices for the vector field.

Firstly, we need some definitions to understand the concept of monodromy.

Now, we should consider a given map

$$\pi : E \rightarrow B,$$

of a space E called the total space into a space B called the base space.

A homotopy between two continuous functions $f, g : X \times [0, 1] \rightarrow B$ (where X is a given space) is a family of continuous functions $h_t : X \rightarrow E$, for $t \in [0, 1]$ such that $h_0 = f$ and $h_1 = g$, and the map $(x, t) \rightarrow h_t(x)$ is continuous from $X \times [0, 1]$ to B .

Given a space X with a map $f : X \rightarrow B$, and $f_t : X \times [0, 1] \rightarrow B$ be a given homotopy of f . A map $\tilde{f} : X \rightarrow E$ is said to be a cover f (relative π) if $f = \pi \circ \tilde{f}$.

The map π is said to have the covering homotopy property for the given space X (one says that (X, π) has homotopy lifting property) if, for every map $\tilde{f} : X \rightarrow E$ and every homotopy $f_t : X \times [0, 1] \rightarrow B$, of the map $f = \pi \circ \tilde{f} : X \rightarrow B$

there is a homotopy $\tilde{f}_t : X \times [0, 1] \rightarrow E$, of the map \tilde{f} which lifts (covers) the homotopy f_t .

A map $\pi : E \rightarrow B$ is said to be a fibering if it has the lifting homotopy property for every triangulable space X . In this situation, we say that the space E is the fibre space over a base space B with respect the projection map $\pi : E \rightarrow B$. For each $b \in B$, the subspace $\pi^{-1}(b)$ in E is called the fibre over b .

Let B be an n -punctured sphere, and let x_0 be a base point in B . We are interested in the following set of continuous functions called loops with base point x_0 .

$$\{f : X \times [0, 1] : f(0) = x_0 = f(1)\}.$$

Now, the fundamental group of Γ at x_0 is the set modulo homotopy h

$$\{f : [0, 1] \times X : f(0) = x_0 = f(1)\}/h$$

equipped with the group multiplication defined by

$$(f * g)(t) = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

With the above product, the set of all homotopy classes of loops with base point x_0 forms the fundamental group of X at x_0 and is denoted by $\pi_1(X, x_0)$.

Now we consider a non-trivial loop γ on the leaf in \mathbb{C} passing through at some point x_0 (starting point) with a small transversal Σ which is a disk at $x_0 \in \gamma$, and then we look at the first point of intersection with Σ of the leaf passing through another point in Σ near $x_0 \in \gamma$ which is obtained by going around over γ on the leaf. Moreover, we consider a family of transversal Σ_i to a leaf at each point on the loop $\gamma(t_i)$, where $t_i \in [0, 1]$. Then for any point close-by the point on loop γ

we can define a leaf, and that leaf intersect all transversal at the unique point, so one can travel over the loop $\gamma[0, 1]$ at the neighbourhood of a singular point.

In this way, to each loop γ in the fundamental group of the leaf we obtain the germ of biholomorphism of Σ which reflects the global behaviour of the trajectories near γ .

We can consider a leaf \mathcal{L}' of the SHFC foliation \mathcal{F} induced by the 1-form on \mathbb{CP}^2 associated to the vector field with p_i singular points of the foliation which lie on \mathcal{L} . Obviously, $\mathcal{L} = \mathcal{L}' \setminus p_i$ is isomorphic to an n -punctured sphere. Then we can choose a family of analytic transversal, Σ_x , passing each point x in \mathcal{L} , and choose a base point $x_0 \in \mathcal{L}$ and an analytic parametrisation z (should be also a level set of vector field at x_0) of Σ_{x_0} with $z = 0$ corresponding to the point x_0 . Regarding for each path $\gamma : [0, 1] \rightarrow \mathcal{L}$ in $\pi(\mathcal{L}, x_0)$ (we choose this loop at the x -axis, and each point on the loop is a value of the leaf which gives a fibration over the loop), a map $\mathcal{M} : \Sigma_{x_0} \rightarrow \Sigma_{x_0}$ can be defined from the neighbourhood of x_0 , $N(x_0)$, by lifting the path γ to \mathcal{L} passing $f(x_0) \in \Sigma_{x_0}$ via the transversals Σ_x , $x \in \gamma$.

By using the parameter z and fixing $z = 0$, we can identify the map \mathcal{M} with the germ of a diffeomorphism from \mathbb{C} to \mathbb{C} . The set of all such diffeomorphisms is denoted by $\text{Diff}(\mathbb{C}, 0)$.

In fact, the $\mathcal{M} : \pi(\mathcal{L}, x_0) \rightarrow \text{diff}(\mathbb{C}, 0)$ is a group homomorphism, the image of this homomorphism is called the monodromy group. Under the map \mathcal{M} , we denote the image of the path γ by \mathcal{M}_γ which depends only on the homotopy of $\gamma \subset \mathcal{L}$.

If we choose one singular point p_i (see Figure (2.4)), then the monodromy at this singular point is \mathcal{M}_γ , where γ is a loop going around the singular point p_i which does not contain any other singular point in its interior.

We note that, the germ of \mathcal{M}_γ at x_0 is determined by Σ and γ . For more details one can see Christopher and Rousseau (2004).

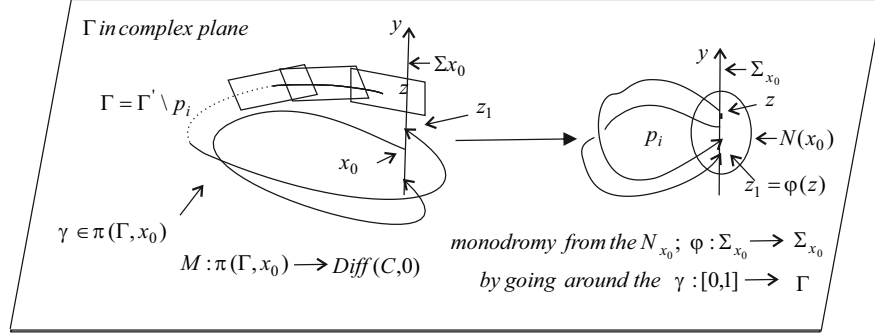


Figure 2.4: The monodromy map about a singular point

In three-dimensional vector fields we use two independent first integrals in order to obtain a monodromy map in $(\mathbb{C}^2, 0)$. Monodromy determines the type of the singular point. That is, we give a vector field and then using the parametrization of the loop $x_\theta = e^{i\theta}$, we will obtain a corresponding map. The correspondence comes from the maps at the neighbourhood of one of these separatrices. For more details one can refer to Mattei and Moussu (1980).

Now, we want to explain the application of monodromy in the 2D vector field. Let us consider a singular foliation \mathcal{F} defined in $(\mathbb{C}^2, 0)$ by the meromorphic differential equation with non-zero eigenvalues

$$\dot{x} = \lambda_1 x + f(x, y), \quad \dot{y} = \lambda_2 y + g(x, y),$$

this implies that

$$\frac{dy}{dx} = \frac{\lambda y + \tilde{g}(x, y)}{x + \tilde{f}(x, y)}, \quad (2.41)$$

where $\lambda = \frac{\lambda_2}{\lambda_1} \in \mathbb{C} \setminus \{0\}$, and $\tilde{f}, \tilde{g} \in \mathbb{C}[x, y]$ without linear terms.

The following form will be taken by the system (2.41) which defines the foliation \mathcal{F} at the x, y -coordinates

$$\frac{dy}{dx} = \frac{y}{x}(\lambda + h(x, y)), \quad \Re(\lambda) < 0, \quad h(0, 0) = 0, \quad (2.42)$$

where $\Re(\lambda)$ is the real part. If necessary, the variables would be rescaled.

Indeed, parametrizing the arc as $x = x_0 \exp(i\theta)$, $\theta \in [0, 2\pi]$ yields

$$\frac{dx}{d\theta} = i \exp(i\theta) = ix,$$

we then conclude from (2.42) the following ordinary equation

$$\frac{dy}{d\theta} = iy(\lambda + h(\theta, y)), \quad \theta \in [0, 2\pi], \quad h \in \mathbb{C}[\theta, y],$$

with the real time. A monodromy map of the above system together with the initial condition $c = y(0, c)$, is a map of the following form

$$c \rightarrow f(c), \quad y(0; c) \rightarrow y(2\pi; c), \quad \text{where} \quad y(\theta; c) = \sum_{i \geq 0} c^i \alpha_i(\theta),$$

such that $\alpha_i(0) = 0, \forall i > 1$ and $\alpha_1(0) = 1$.

However, we compare the above idea in 3D vector field χ , when χ has two independent first integrals $F_1(x, y, z) = c_1$ and $F_2(x, y, z) = c_2$, where $(c_1, c_2) \in \mathbb{R}^2$. Then, the monodromy can be determined by intersecting these two first integrals. Indeed, by parametrizing over the loop $x = \exp(i\theta)$ of the 3DDS, we get the following form

$$\frac{dx}{d\theta} = \lambda_1 \exp(i\theta), \quad \frac{dy}{d\theta} = y(\lambda_2 + q(\theta, y, z)), \quad \frac{dz}{d\theta} = z(\lambda_1 + r(\theta, y, z)), \quad (2.43)$$

where $\theta \in [0, 2\pi]$, with the initial conditions $y(0; y_0, z_0) = y_0$ and $z(0; y_0, z_0) = z_0$. Then, the monodromy map in $(\mathbb{C}^3, 0)$ is a map of the following

$$(y(0; y_0, z_0), z(0; y_0, z_0)) \rightarrow (y(2\pi; y_0, z_0), z(2\pi; y_0, z_0)),$$

where $y(\theta; y_0, z_0) = \sum_{i+j \geq 0} y_0^i z_0^j \alpha_{ij}(\theta)$, see Figure (2.5).

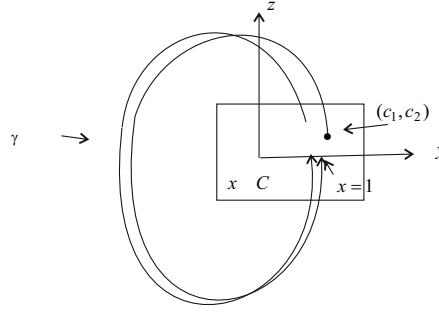


Figure 2.5: The monodromy map in 3D

For applying the monodromy map on the system (2.43) with the resonant eigenvalues satisfying the condition $\lambda + \mu + \nu = 0$. The following way can be used at the neighbourhood of the x -separatrix.

Firstly, we will choose a starting point of the first integral $F_1(x, y, z) = c_1$ at the chosen point (x_0, y_0, z_0) , we get the value of $F_1(x, y, z) = c_1$ at the transversal to the x -separatrix. Then the path surrounding the origin can be chosen of the form $x_\theta = x_0 e^{2i\pi\theta}$.

Secondly, By substituting $y_\theta = \sum_{i \geq 1} c_i(\theta) y_0^i$ with the value of c_1 as a constant into the second first integral we get a formal map with respect to y_0 . Therefore, by using Taylor expansion and simplify the result, we get a linear equation for each order of y_0 which depends on the coefficient c_i in y_θ . The monodromy map in

y -coordinate is then given by a map y_1 which is already composed with a rotation by the angle $-2\pi \frac{\lambda_2}{\lambda_1}$.

Next, the monodromy map in z -coordinate is given by substituting the map that was obtained in y -coordinate (previous step) and a power series $z_\theta = e^{2\pi i k} z_0(1 + g(x_0, y_0, z_0))$ into the first integral. We get a monodromy map in z -coordinate about x -separatrix which is given by a map z_1 already composed with a rotation by the angle $-2\pi \frac{\lambda_3}{\lambda_1}$. Consequently, we get the 2D map in normal form corresponding to the reduced normal form of the following

$$x_1 = x_0 e^{2i\pi}, \quad y_1 = y_0 e^{2i\pi \frac{\lambda_2}{\lambda_1}} (1 + f(u_0)), \quad z_1 = e^{2i\pi \frac{\lambda_3}{\lambda_1}} z_0 (1 + g(u_0)), \quad (2.44)$$

where f and g are a formal power series in u_0 , and $u_0 = x_0, y_0, z_0$ is the resonant monomial.

We can read off the terms of the reduced normal form with rank-one resonant eigenvalues from the monodromy map (2.44).

In the same way of vector field, a reduction of the map (2.44) is also needed. Then, by applying a change of coordinates, we can bring the map (2.44) into a reduced normal form for map which only contains a finite number of resonant monomials. For studying formal normal form for maps see Abate (2005), Abate and Tovenia (2005), Abate and Raissy (2013).

Chapter 3

A Sufficient Condition for Integrability of Polynomial System with Rank-One Resonant Singularities in Three-Dimensions

3.1 Introduction

In this chapter we consider the system (2.1) with rank-one resonant eigenvalues, by which we mean that the eigenvalues of the system (2.1) have exactly one independent linear dependency over \mathbb{Q} .

There are several techniques to find first integrals of polynomial systems. Basov and Romanovski (2010) considered some families of systems to find an analytic first integral of the three-dimensional system in the case of one zero eigenvalue and the other eigenvalues have negative real parts by using Darboux integrability theory. Llibre and Zhang (2012) obtained several results on the Darboux integrability of polynomial vector fields in \mathbb{R}^n or \mathbb{C}^n with $n \geq 2$. Llibre et al.

(2012) investigated formal and analytic first integrals of local analytic systems near a singular point by using the Poincaré-Dulac normal forms. They applied this method on a system which has a three-dimensional centre, that is, if all the trajectories near the singular point are stationary or closed, which is equivalent to considering the case of linearization with one zero eigenvalue. Zhang (2014) proved that if an n -dimensional polynomial differential system has $n - 1$ functionally independent Darboux Jacobian multipliers, then it has $n - 1$ functionally independent Liouvillian first integrals, and vice versa. Here, we generalize the technique from Aziz and Christopher (2012) in order to find two independent first integral of system (2.1).

This technique proves that if the system (2.1) has a Darboux-analytic first integral and a Darboux-analytic inverse Jacobi multiplier, then the system has another first integral. This contains logarithm terms if the original system has resonant monomials, see Theorem 6.

We also prove a sufficient condition for the existence of one first integral for some families of the three-dimensional Lotka-Volterra systems. These cases have been left as conjectural in Aziz and Christopher (2014), who gave a number of necessary conditions for one first integral but could not prove their sufficiency. We give the details in Section 3.3.1.

We write the system (2.1) in the following form

$$\dot{x} = \lambda x + \sum_{n \geq 2} P_n(x, y, z), \quad \dot{y} = \mu y + \sum_{n \geq 2} Q_n(x, y, z), \quad \dot{z} = \nu z + \sum_{n \geq 2} R_n(x, y, z), \quad (3.1)$$

where $\lambda, \mu, \nu \neq 0$. In this chapter, we only consider rank-one resonant eigenvalues which satisfy the condition $\lambda + \mu + \nu = 0$. This assumption simplifies the exposition and should be typical of the other cases.

This chapter is organised as follows. In Section 3.2, we describe the technique

in order to find two independent first integrals for the system (3.1). Theorem 6 explains that how one can find another first integral of the system (3.1) if it has a first integral $\varphi = x^\alpha y^\beta z^\gamma(1 + o(x, y, z))$ and an IJM, $M = x^r y^s z^t(1 + o(x, y, z))$. The second first integral is in the following form

$$\psi = \log(X)^{N_1} \sum_{\hat{I} \in E_1} c_{\hat{I}_1} K_{\hat{I}} X^{h_{\hat{I}} \delta} + \log(X)^{N_2} \sum_{\hat{I} \in E_1} c_{\hat{I}_2} K_{\hat{I}} X^{h_{\hat{I}} \delta} + \sum_{\tilde{I} \in E_2} K_{\tilde{I}} X^{1+\tilde{I}-\theta},$$

where $X = xyz(1 + o(x, y, z))$.

In Section 3.3, we assume that the system (3.1) is normalizable. That is, by a change of coordinates, we can bring the system (3.1) to a normal form

$$\dot{x} = x(\lambda + \sum_{i \geq 1} a_i u^i), \quad \dot{y} = y(\mu + \sum_{i \geq 1} b_i u^i), \quad \dot{z} = z(\nu + \sum_{i \geq 1} c_i u^i), \quad (3.2)$$

where $u = xyz$ is the resonant monomial. By applying Theorem 6 we show that the system (3.2) has two independent formal first integrals

$$\varphi = xyz, \quad \psi = \sum_{k \geq 1} \ln(x^{b_k} y^{b_k+c_k})(xyz)^k + \ln(x^\mu y^{-\lambda}).$$

$\phi = xyz$ and $\psi = \sum_{k \geq 1} \ln(x^{b_k} y^{b_k+c_k})(xyz)^k + \ln(x^\mu y^{-\lambda})$ if and only if $a_k + b_k + c_k = 0$, $k \in \mathbb{N}$, which pulled back to the original coordinates give

$$\begin{aligned} \varphi_1 &= xyz(1 + o(x, y, z)), \\ \psi_1 &= \sum_{k \geq 1} \ln(x^{b_k} y^{b_k+c_k}(1 + o(x, y, z)))(\varphi_1)^k + \ln(x^\mu y^{-\lambda}(1 + o(x, y, z))). \end{aligned}$$

as two independent first integrals of the system (3.1). We apply this to demonstrate the (formal) sufficiency of the conditions in Aziz and Christopher (2014).

3.2 Technique for integrability of 3D systems

We look for two independent first integrals of the system (3.1). We want to show that if there is a Darboux-analytic first integral and a Darboux-inverse Jacobi multiplier for the system, then a second first integral can be found (for these two type of functions, see Definition 4 and Section 2.3).

Aziz and Christopher (2012) proved that there is a second first integral for the system (3.1) at the origin if it has a first integral $\varphi^\delta = x^\alpha y^\beta z^\gamma (1 + o(x, y, z))$, $\delta = (\alpha, \beta, \gamma)$ and an IJM, $M^\theta = x^r y^s z^t (1 + o(x, y, z))$, $\theta = (r, s, t)$ such that the $\delta \times (\theta - \mathbf{1} - I) \neq 0$, for any $I = (i, j, k) \in \mathbb{Z}_{\geq 0}^3$, where $\mathbf{1} = (1, 1, 1)$. This condition forces the integrability of system.

We wish to find a second first integral in the case that there is $I = (i, j, k) \in \mathbb{Z}_{\geq 0}^3$ such that $\delta \times (\theta - \mathbf{1} - I) = 0$. In this case, there can still exist resonant terms and the first integral will not be of Darboux-analytic type.

We first prove the following proposition which will be used in Theorem 6.

Proposition 6. The polynomial system (3.1) has a first integral of the form $\phi^\delta = x^\alpha y^\beta z^\gamma$ and an inverse Jacobi Multiplier $M^\theta = x^r y^s z^t$, if and only if, it satisfies the following

1. $\lambda\alpha + \mu\beta + \nu\gamma = 0$,
2. $\alpha a_{0,j,k} = 0$, for $k + j = n$,
3. $\gamma c_{i,j,0} = 0$, for $i + j = n$,
4. $\beta b_{i,0,k} = 0$, for $k + i = n$,
5. $\alpha a_{i,j,k} + \beta b_{i-1,j+1,k} + \gamma c_{i-1,j,k+1} = 0$, for $i > 0$,
6. $\lambda(r - 1) + \mu(s - 1) + \nu(t - 1) = 0$,

$$7. (r - (i + 1))a_{i+1,j,k} + (s - (j + 1))b_{i,j+1,k} + (t - (k + 1))c_{i,j,k+1} = 0.$$

Proof. Without loss of generality, we assume that $\alpha \neq 0$. Firstly, suppose that the system (3.1) has a non-zero first integral $\phi = x^\alpha y^\beta z^\gamma$, $\delta = (\alpha, \beta, \gamma)$, then

$$\chi(\phi) = \frac{\partial \phi}{\partial x} \dot{x} + \frac{\partial \phi}{\partial y} \dot{y} + \frac{\partial \phi}{\partial z} \dot{z} = \left(\alpha \frac{\dot{x}}{x} + \beta \frac{\dot{y}}{y} + \gamma \frac{\dot{z}}{z} \right) \phi = 0,$$

which gives

$$(\alpha \lambda + \beta \mu + \gamma \nu + \alpha \frac{P_n(x, y, z)}{x} + \beta \frac{Q_n(x, y, z)}{y} + \gamma \frac{R_n(x, y, z)}{z}) \phi = 0, \quad (3.3)$$

we directly obtain the first condition $\alpha \lambda + \beta \mu + \gamma \nu = 0$. Also, from equation (3.3) we have

$$\begin{aligned} \alpha \frac{P_n}{x} + \beta \frac{Q_n}{y} + \gamma \frac{R_n}{z} &= \frac{1}{xyz} \left(\sum_{\substack{i+j+k=n \\ n \geq 2}} \alpha a_{i,j,k} x^i y^{j+1} z^{k+1} + \sum_{\substack{i+j+k=n \\ n \geq 2}} \beta b_{i,j,k} x^{i+1} y^j z^{k+1} \right. \\ &\quad \left. + \sum_{\substack{i+j+k=n \\ n \geq 2}} \gamma c_{i,j,k} x^{i+1} y^{j+1} z^k \right) = 0. \end{aligned} \quad (3.4)$$

If $i = 0$, then equation (3.4) becomes

$$\sum_{\substack{j+k=n \\ n \geq 2}} (\alpha a_{0,j,k} y^{j+1} z^{k+1} + \beta b_{0,j,k} x y^j z^{k+1} + \gamma c_{0,j,k} x y^{j+1} z^k) = 0,$$

we see that the first term of the above equation does not contain any x , then $\alpha a_{0,j,k} = 0$, for $k + j = n$. In the same way, when $j = 0$ and $k = 0$, we obtain the third and forth conditions, respectively.

When $i, j, k > 0$, we see that

$$\sum_{i>0} (\alpha a_{i,j,k} x^i y^{j+1} z^{k+1} + \beta b_{i-1,j+1,k} x^i y^{j+1} z^{k+1} + \gamma c_{i-1,j,k+1} x^i y^{j+1} z^{k+1}) = 0,$$

and hence

$$\alpha a_{i,j,k} + \beta b_{i-1,j+1,k} + \gamma c_{i-1,j,k+1} = 0.$$

Secondly, if $M = x^r y^s z^t$ is an IJM, we have $\chi(M) = M \operatorname{div}(\chi)$ and write

$$\operatorname{div}(\chi) = \lambda + \mu + \nu + i \frac{P_n(x, y, z)}{x} + j \frac{Q_n(x, y, z)}{y} + k \frac{R_n(x, y, z)}{z},$$

we see that

$$M(\chi) = \left(r\lambda + s\mu + t\nu + r \frac{P_n(x, y, z)}{x} + s \frac{Q_n(x, y, z)}{y} + t \frac{R_n(x, y, z)}{z} \right) M.$$

By subtracting the last two equations, we obtain

$$\lambda(r-1) + \mu(s-1) + \nu(t-1) = 0, \quad (r-i) \frac{P_n}{x} + (s-j) \frac{Q_n}{y} + (t-k) \frac{R_n}{z} = 0,$$

and, hence the sixth condition is obtained from the first part of the above equation.

After equating the monomials in the second part, we obtain

$$\begin{aligned} & (r-i) \frac{P_n}{x} + (s-j) \frac{Q_n}{y} + (t-k) \frac{R_n}{z} \\ &= \sum_{i>0} (r-i) a_{i,j,k} x^{i-1} y^j z^k + \sum_{j>0} (s-j) b_{i,j,k} x^i y^{j-1} z^k + \sum_{k>0} (t-k) c_{i,j,k} x^i y^j z^{k-1} \\ &= \sum_{i \geq 0} (r-(i+1)) a_{i+1,j,k} x^i y^j z^k + (s-(j+1)) b_{i,j+1,k} x^i y^j z^k \\ &+ (t-(k+1)) c_{i,j,k+1} x^i y^j z^k = 0, \end{aligned}$$

which implies that

$$(r-(i+1)) a_{i+1,j,k} + (s-(j+1)) b_{i,j+1,k} + (t-(k+1)) c_{i,j,k+1} = 0,$$

this is the last condition. □

If the system (3.1) satisfies all the conditions of the Proposition 6, then it takes the following form

$$\begin{aligned}\dot{x} &= x \left(\lambda + \sum_{\substack{i+j+k=n \\ n \geq 2}} a_{i+1,j,k} x^i y^j z^k \right) = x \left(\lambda + \tilde{P}(x, y, z) \right), \\ \dot{y} &= y \left(\mu + \sum_{\substack{i+j+k=n \\ n \geq 2}} b_{i,j+1,k} x^i y^j z^k \right) = y \left(\mu + \tilde{Q}(x, y, z) \right), \\ \dot{z} &= z \left(\nu + \sum_{\substack{i+j+k=n \\ n \geq 2}} c_{i,j,k+1} x^i y^j z^k \right) = z \left(\nu + \tilde{R}(x, y, z) \right).\end{aligned}\tag{3.5}$$

We can identify the system (3.5) with the following 2-form

$$\begin{aligned}\Omega &= x \left(\lambda + \sum_I C_{I_x} X^I \right) dy \wedge dz + y \left(\mu + \sum_I C_{I_y} X^I \right) dz \wedge dx \\ &\quad + z \left(\nu + \sum_I C_{I_z} X^I \right) dx \wedge dy,\end{aligned}\tag{3.6}$$

where $\sum_I (C_{I_x}, C_{I_y}, C_{I_z}) = \sum (a_{i+1,j,k}, b_{i,j+1,k}, c_{i,j,k+1})$.

Now, we are interesting to find another first integral of the above system. In the following theorem, we can show that the system (3.1) has two independent first integrals when the system has a Darboux-analytic first integral and an inverse Jacobi multiplier.

To simplify the notation, the multi-index $|I| = i + j + k$ and notation $X^I = x^i y^j z^k$ will be used. Let $\Delta_1 = (\frac{dy \wedge dz}{yz}, \frac{dz \wedge dx}{zx}, \frac{dx \wedge dy}{xy})$ and $\Delta_2 = (\frac{dx}{x}, \frac{dy}{y}, \frac{dz}{z})$.

Theorem 6. we assume the vector field

$$\chi = x \left(\lambda + \sum_I C_{I_x} X^I \right) \partial x + y \left(\mu + \sum_I C_{I_y} X^I \right) \partial y + z \left(\nu + \sum_I C_{I_z} X^I \right) \partial z, \tag{3.7}$$

has a first integral of the form $\phi_1 = x^\alpha y^\beta z^\gamma (1 + o(x, y, z))$ with at least one of the parameters α, β and γ is not equal to zero and an inverse Jacobi multiplier

$M_1 = x^r y^s z^t (1 + o(x, y, z))$. Then it has a second first integral of the following form

$$\psi = \log(X^{N_1}) \sum_{\hat{I} \in E_1} c_{\hat{I}_1} K_{\hat{I}} X^{h_{\hat{I}} \delta} + \log(X)^{N_2} \sum_{\hat{I} \in E_1} c_{\hat{I}_2} K_{\hat{I}} X^{h_{\hat{I}} \delta} + \sum_{\tilde{I} \in E_2} K_{\tilde{I}} X^{1+\tilde{I}-\theta},$$

where the set E_1 contains all $(i, j, k) \in \hat{I}$ such that $(\theta - \hat{I} - \mathbf{1}) \times \delta = 0$, the set E_2 contains all $(i, j, k) \in \tilde{I}$ such that $(\theta - \tilde{I} - \mathbf{1}) \times \delta \neq 0$, N_1 and N_2 are constant vectors and $\mathbf{1} = (1, 1, 1)$.

Proof. Assume, without loss of generality, that $\alpha \neq 0$. After an invertible change of coordinates in the form

$$(x, y, z) \mapsto (x(1 + o(x, y, z)))^{\frac{1}{\alpha}}, y(1 + o(x, y, z))^{\frac{1}{\beta}}, z(1 + o(x, y, z))^{\frac{1}{\gamma}},$$

we can transform the function $\phi_1 = x^\alpha y^\beta z^\gamma (1 + o(x, y, z))$ to $\phi = X^\delta = x^\alpha y^\beta z^\gamma$, $\delta = (\alpha, \beta, \gamma)$, which will not change the form of the vector field (3.7) or an IJM, M_1 . Furthermore, by scaling the vector field, we can take in the factor $(1 + o(x, y, z))$ of M_1 into χ in the following way

$$\frac{\chi}{M_1} = \frac{\chi}{x^r y^s z^t (1 + o(x, y, z))} = \frac{\chi}{1 + o(x, y, z)} \frac{1}{x^r y^s z^t} = \frac{\tilde{\chi}}{M},$$

where $M = x^r y^s z^t =: X^\theta$ with $\theta = (r, s, t)$.

Now, we look for a second first integral. If for all $I = (i, j, k)$, such that $(\theta - I - \mathbf{1}) \times \delta \neq 0$, then we directly obtain the second first integral described by Aziz and Christopher (2012). That is, there is a second first integral in the following form

$$\psi = \sum_I K_I X^{I-\theta+1}, \quad \text{where } C_I = K_I(\theta - I - \mathbf{1}) \times \delta. \quad (3.8)$$

On the other hand, if there are some $I = (i, j, k)$ such that $(\theta - I - \mathbf{1}) \times \delta = 0$, then we can define two subsets E_1 and E_2 in $\mathbb{Z}_{\geq 0}^3$ in the following

- . E_1 contains all $\hat{I} = (i, j, k)$ such that $(\theta - \hat{I} - \mathbf{1}) \times \delta = 0$.
- . E_2 contains all $\tilde{I} = (i, j, k)$ such that $(\theta - \tilde{I} - \mathbf{1}) \times \delta \neq 0$.

Since we have a first integral $\phi = X^\delta = x^\alpha y^\beta z^\gamma$, $\delta = (\alpha, \beta, \gamma)$, then we obtain

$$\delta \cdot C_I = 0, \quad \text{where } C_I = (C_{xI}, C_{yI}, C_{zI}) \quad (3.9)$$

Also, since we have an IJM, $M = X^\theta$, then by using $\chi(M) = M \operatorname{div}(\chi)$ and writing $\operatorname{div}(\chi) = \sum_{I=1} (I + \mathbf{1}) C_I X^I$, where $\mathbf{1} = (1, 1, 1)$, we obtain

$$(\theta - I - \mathbf{1}) \cdot C_I = 0. \quad (3.10)$$

For all $\hat{I} \in E_1$, we have $(\theta - \hat{I} - \mathbf{1}) \times \delta = 0$, hence both vectors should be multiples each other. This implies that, there is a constant $h_{\hat{I}}$ such that $h_{\hat{I}} \delta = (\theta - \hat{I} - \mathbf{1})$. Clearly, since $\delta \neq 0$, if $\theta - \hat{I} - \mathbf{1} = 0$, then $h_{\hat{I}} = 0$.

From equation (3.9), we see that δ is orthogonal to all vectors C_I , thus all C_I lie in the same plane. Therefore, for each $C_{\hat{I}}$ there is an $N_{\hat{I}} = (n_{1\hat{I}}, n_{2\hat{I}}, n_{3\hat{I}})$ which is contained in the same plane with the vector $C_{\hat{I}}$ such that

$$C_{\hat{I}} = K_{\hat{I}} N_{\hat{I}} \times \delta. \quad (3.11)$$

Furthermore, the $N_{\hat{I}}$ can be expressed as a linear combination of two orthogonal constant vectors which span the plane orthogonal to δ , hence there exist constant vectors N_1 and N_2 (independent of \hat{I}) such that

$$N_{\hat{I}} = c_{\hat{I}_1} N_1 + c_{\hat{I}_2} N_2, \quad (3.12)$$

for some constants $c_{\hat{I}_1}$ and $c_{\hat{I}_2}$. Now for all I , the 2-form (3.6) becomes

$$\Omega = \left((\lambda + \sum_I C_{I_x}) \frac{dy \wedge dz}{yz} + (\mu + \sum_I C_{I_y}) \frac{dz \wedge dx}{zx} + (\nu + \sum_I C_{I_z}) \frac{dx \wedge dy}{xy} \right) X^{I+1},$$

where C_I includes $C_{\tilde{I}}$ and $C_{\hat{I}}$. After dividing both sides by $M = X^\theta$, the following 2-form is obtained

$$\frac{\Omega}{M} = \sum_{\hat{I} \in E_1} C_{\hat{I}} \cdot \Delta_1 X^{\hat{I}+1-\theta} + \sum_{\tilde{I} \in E_2} C_{\tilde{I}} \cdot \Delta_1 X^{\tilde{I}+1-\theta}.$$

By substituting the equations (3.11) and (3.8) into the above form

$$\begin{aligned} \frac{\Omega}{M} &= \sum_{\hat{I}} K_{\hat{I}}(N_{\hat{I}} \times \delta) \cdot \Delta_1 X^{1-\theta+\hat{I}} + \sum_{\tilde{I}} K_{\tilde{I}}(\theta - \tilde{I} - \mathbf{1}) \times \delta \cdot \Delta_1 X^{1-\theta+\tilde{I}} \\ &= \left(\sum_{\hat{I}} K_{\hat{I}}(N_{\hat{I}} \cdot \Delta_2) X^{1-\theta+\hat{I}} + \sum_{\tilde{I}} K_{\tilde{I}}((\theta - \tilde{I} - \mathbf{1}) \cdot \Delta_2) X^{1-\theta+\tilde{I}} \right) \wedge (\delta \cdot \Delta_2). \end{aligned}$$

Since, we have $h_{\hat{I}} \delta = (\theta - \hat{I} - \mathbf{1})$, and $\phi = X^\delta$, then $\frac{d\phi}{\phi} = \delta \frac{dX}{X} = \delta \left(\frac{dx}{x}, \frac{dy}{y}, \frac{dz}{z} \right) = \delta \Delta_2$, also $N_{\hat{I}} \cdot \Delta_2 = d(\log(X)^{N_{\hat{I}}})$. This implies that

$$\frac{\Omega}{M} = \left(\sum_{\hat{I}} K_{\hat{I}} d(\log X^{N_{\hat{I}}}) X^{h_{\hat{I}} \delta} + \sum_{\tilde{I}} K_{\tilde{I}} d(X^{1-\theta+\tilde{I}}) \right) \wedge \frac{d\phi}{\phi},$$

We substitute equation (3.12) to obtain

$$\frac{\Omega}{M} = \left(\sum_{\hat{I}} K_{\hat{I}} d(\log X^{c_{\hat{I}_1} N_1 + c_{\hat{I}_2} N_2}) X^{h_{\hat{I}} \delta} + \sum_{\tilde{I}} K_{\tilde{I}} d(X^{1-\theta+\tilde{I}}) \right) \wedge \frac{d\phi}{\phi},$$

thus, we obtain a formal second first integral of the following form

$$\psi = \log(X^{N_1}) \sum_{\hat{I}} c_{\hat{I}_1} K_{\hat{I}} \phi^{h_{\hat{I}}} + \log(X^{N_2}) \sum_{\hat{I}} c_{\hat{I}_2} K_{\hat{I}} \phi^{h_{\hat{I}}} + \sum_{\tilde{I}} K_{\tilde{I}} X^{1+\tilde{I}-\theta}.$$

□

Now, suppose that the system (3.6) is analytic, then we want to show that the second first integral is also analytic.

In Aziz and Christopher (2012), the authors assumed that for any I the cross product $(\theta - I - 1)$ and δ is bounded away from zero. Now, we suppose that for all \tilde{I} in E_2 the cross product $(\theta - \tilde{I} - 1)$ and δ is bounded away from zero.

When $(\theta - \tilde{I} - 1) \times \delta \neq 0$, then we can take $m > 0$ such that for all \tilde{I} in E_2

$$0 < m < |(\theta - \tilde{I} - 1) \times \delta|.$$

Since, we have $C_{\tilde{I}} = K_{\tilde{I}}((\theta - \tilde{I} - 1) \times \delta)$, then we obtain

$$|K_{\tilde{I}}| m < |C_{\tilde{I}}|, \quad \text{for all } \tilde{I}. \quad (3.13)$$

Let the series $\sum_{\tilde{I}} C_{\tilde{I}} X^{\tilde{I}}$ converges absolutely, then there is M such that

$$|C_{\tilde{I}}| \leq M R^{-|\tilde{I}|}, \quad \text{for all } |X| < R,$$

where R is the radius of convergence. From equation (3.13), we obtain

$$|K_{\tilde{I}}| m \leq M R^{-|\tilde{I}|}, \quad \text{for all } \tilde{I},$$

this implies that

$$|K_{\tilde{I}}| \leq \frac{M}{m} R^{-|\tilde{I}|}, \quad \text{for all } \tilde{I},$$

choose $\frac{M}{m} = 1$, then $|K_{\tilde{I}}|$ is also bounded by R , so $\sum_{\tilde{I}} K_{\tilde{I}} X^{\tilde{I}}$ indeed is convergent.

When $(\theta - \hat{I} - 1) \times \delta = 0$, for same \hat{I} , we have $C_{\hat{I}} = K_{\hat{I}}(N_{\hat{I}} \times \delta)$, where $N_{\hat{I}} = c_{\hat{I}_1} N_1 + c_{\hat{I}_2} N_2$, then

$$C_{\hat{I}} = K_{\hat{I}} c_{\hat{I}_1} (N_1 \times \delta) + K_{\hat{I}} c_{\hat{I}_2} (N_2 \times \delta) = K_{\hat{I}_1} V_1 + K_{\hat{I}_2} V_2, \quad (3.14)$$

where $K_{\hat{I}_1} = K_{\hat{I}} c_{\hat{I}_1}$, $K_{\hat{I}_2} = K_{\hat{I}} c_{\hat{I}_2}$ and $V_1 = N_1 \times \delta$, $V_2 = N_2 \times \delta$. We see that a non-zero vector $C_{\hat{I}}$ is a linear combination of two constant vectors V_1 and V_2 . Without loss of generality, we can assume that V_1 is the unit vector in the x -direction and V_2 is the unit vector in the y -direction. Let θ be the angle between the vector $C_{\hat{I}}$ and x -axis, hence we have the following relationship

$$\begin{aligned} K_{\hat{I}_1} V_1 &= \cos(\theta) C_{\hat{I}}, \\ K_{\hat{I}_2} V_2 &= \sin(\theta) C_{\hat{I}}, \end{aligned}$$

since both $|\cos(\theta)|, |\sin(\theta)| \leq 1$, we obtain

$$\begin{aligned} |K_{\hat{I}_1} V_1| &= |\cos(\theta)| |C_{\hat{I}}| \leq |C_{\hat{I}}|, \\ |K_{\hat{I}_2} V_2| &= |\sin(\theta)| |C_{\hat{I}}| \leq |C_{\hat{I}}|. \end{aligned}$$

When the power series $\sum_{\hat{I}} C_{\hat{I}} X^{\hat{I}}$ converges absolutely, then the both power series $\sum_{\hat{I}} c_{\hat{I}_1} K_{\hat{I}} X^{h_{\hat{I}} \delta}$ and $\sum_{\hat{I}} c_{\hat{I}_2} K_{\hat{I}} X^{h_{\hat{I}} \delta}$ are absolutely convergent, hence the second first integral indeed converges.

Theorem 6 will be applied to show the integrability of the system (3.1) in the case when $P_n = Q_n = R_n = 0$ for $n > 2$ with $\lambda, \nu > 0$ and $\mu < 0$, under the conditions in Proposition 6. In this case, the system is in the following form

$$\begin{aligned} \dot{x} &= \lambda x + \frac{\lambda c_{1,0,1}}{2\nu} x^2 + \frac{(\lambda - \mu) c_{0,1,1}}{\nu} xy + \frac{(\lambda - \nu) c_{0,0,2}}{\nu} xz \\ &\quad + a_{0,2,0} y^2 + a_{0,1,1} yz + a_{0,0,2} z^2, \\ \dot{y} &= \mu y + \frac{\mu c_{1,0,1}}{\nu} xy + \frac{\mu c_{0,1,1}}{\nu} y^2 + \frac{\mu c_{0,0,2}}{\nu} yz, \\ \dot{z} &= \nu z + c_{1,0,1} xz + c_{0,1,1} yz + c_{0,0,2} z^2. \end{aligned}$$

The above system has a first integral $\varphi = z y^{-\frac{\nu}{\mu}}$ and an IJM $M = y^{\frac{-\nu t + \lambda + \nu + \mu}{\mu}} z^t$.

Therefore, a second first integral can be found directly by using Theorem 6 which

gives the following results

- If $\lambda \neq \mu + \nu$ and $\lambda \neq 2\nu$, then A second first integral is of the following

$$\psi = y^{-\frac{-\nu t + \lambda + \nu}{\mu}} z^{1-t} \left(-x - \frac{c_{0,1,1}}{\nu} xy - \frac{c_{1,0,1}}{2\nu} x^2 - \frac{c_{0,0,2}}{\nu} xz + \frac{a_{0,1,1}}{\nu - \lambda + \mu} yz \right. \\ \left. + \frac{a_{0,2,0}}{2\mu - \lambda} y^2 + \frac{a_{0,0,2}}{2\nu - \lambda} z^2 \right).$$

- If $\lambda = \mu + \nu$, then a second first integral contains the logarithm term

$$\psi = y^{-\frac{2\mu + \nu}{\mu} + 1} \left[\frac{a_{0,1,1}}{\mu - \nu} \ln(y z^{-1}) yz - x - \frac{c_{0,1,1}}{\nu} xy - \frac{c_{1,0,1}}{2\nu} x^2 \right. \\ \left. - \frac{c_{0,0,2}}{\nu} xz - \frac{a_{0,0,2}}{\mu - \nu} z^2 + \frac{a_{0,2,0}}{\mu - \nu} y^2 \right].$$

- If $\lambda = 2\nu$, then a second first integral also contains the logarithm term

$$\psi = y^{-2\frac{\nu}{\mu}} \left[\frac{a_{0,0,2}}{\mu - \nu} \ln(y z^{-1}) z^2 - x - \frac{c_{0,1,1}}{\nu} xy - \frac{c_{1,0,1}}{2\nu} x^2 \right. \\ \left. - \frac{c_{0,0,2}}{\nu} xz + \frac{a_{0,1,1}}{\mu - \nu} yz + \frac{a_{0,2,0}}{\mu - \nu} y^2 \right].$$

3.3 Integrability and normalizability of systems in 3D

In this section, we use the relationship between the integrability and normalizability of the system (3.1) to demonstrate the sufficiency of the conditions in Aziz and Christopher (2014).

Firstly, we assume that the system (3.1) is orbitally normalizable, then by Theorem 2, the normal form (2.3) can be written in the following form

$$\dot{X} = X(\lambda + p(u)), \quad \dot{Y} = Y(\mu + q(u)), \quad \dot{Z} = Z(\nu + r(u)), \quad (3.15)$$

where $p, q, r \in \mathbb{C}[u]$, and $u = X^{n_1}Y^{n_2}Z^{n_3}$, $(n_1, n_2, n_3) \in \mathbb{Z}_{\geq 0}^3$ is the resonant monomial.

Since, we only consider the case $\lambda + \mu + \nu = 0$, then we must have $u = XYZ$. This is because by the resonant condition equation (2.2), we have

$$(\lambda, \mu, \nu) \cdot (n_1, n_2, n_3) = \lambda n_1 + \mu n_2 + \nu n_3 = \lambda(n_1 - n_3) + \mu(n_2 - n_3) = 0,$$

if $(n_1 - n_3) \neq 0$, this yields

$$\frac{\lambda}{\mu} = \frac{n_3 - n_2}{n_1 - n_3} \in \mathbb{Q},$$

which contradicts $\frac{\lambda}{\mu} \notin \mathbb{Q}$. Then $n_1 - n_3 = 0$, and $n_3 - n_2 = 0$. Consequently, $n_3 = n_2 = n_1$ and all resonant monomials are generated by $u = xyz$. Therefore, the system (3.15) becomes

$$\dot{X} = X(\lambda + \sum_{n \geq 1} a_n u^n), \quad \dot{Y} = Y(\mu + \sum_{n \geq 1} b_n u^n), \quad \dot{Z} = Z(\nu + \sum_{n \geq 1} c_n u^n), \quad (3.16)$$

where $u = XYZ$.

In the following theorem, we want to show that if we have a Darboux IJM, M , of the form $M = (xyz)^r(1 + o(x, y, z))$ with some conditions on resonant coefficients then there is a formal first integral of the form $\varphi = xyz(1 + o(x, y, z))$.

Theorem 7. Let the system (3.1) be formally orbitally normalizable with rank-one resonant eigenvalues which satisfies the condition $\lambda + \mu + \nu = 0$, then the following statements are equivalent:

- (i) The system (3.1) has formal IJM, M , of the form $M = (xyz)^r(1 + o(x, y, z))$, for $r \notin \{2, 3, \dots\}$.
- (ii) The condition $a_k + b_k + c_k = 0$ holds in the formal normal form (3.16).

(iii) The system (3.1) has formal first integral of the form $\varphi = xyz(1 + o(x, y, z))$.

Proof. Since (3.1) is orbitally normalizable, by an invertible change of coordinates

$$(X, Y, Z) = G(x, y, z) = (x + o(x, y, z), y + o(x, y, z), z + o(x, y, z)),$$

we can bring (3.1) to the system (3.16) which we identify with the following vector field

$$\chi = X(\lambda + \sum_{k \geq 1} a_k u^k) \frac{\partial}{\partial X} + Y(\mu + \sum_{k \geq 1} b_k u^k) \frac{\partial}{\partial Y} + Z(\nu + \sum_{k \geq 1} c_k u^k) \frac{\partial}{\partial Z}, \quad (3.17)$$

where $u = XYZ$ is the resonant monomial. Given $M_1 = xyz(1 + o(x, y, z))$, by Theorem 7 in Berrone and Giacomini (2003),

$$M(X, Y, Z) = (M_1 \circ G^{-1})(X, Y, Z) |J(G^{-1}(X, Y, Z))|,$$

is an IJM of the transformed system, where J is a Jacobi matrix and G is the change of coordinates of the original system. In our case, J is equal to identity at the origin corresponding to the change of coordinates G , and then an IJM of the system (3.17) becomes of the following

$$M(X, Y, Z) = (XYZ)^r (1 + o(X, Y, Z)). \quad (3.18)$$

Furthermore, after scaling we can absorb the factor $(1 + o(X, Y, Z))$ of M into the vector field, and hence we can suppose that M is of the form $(XYZ)^r$, which will not change the form of the vector field (3.17) nor the conditions in (ii). The implication (i) \Rightarrow (ii) follows because we seek an IJM M of the form $(XYZ)^r$. And we write the divergence of the system as

$$\begin{aligned} \operatorname{div}(\chi) &= \lambda + \sum_{k \geq 1} (k+1)a_k u^k + \mu + \sum_{k \geq 1} (k+1)b_k u^k + \nu + \sum_{k \geq 1} (k+1)c_k u^k \\ &= \sum_{k \geq 1} (k+1)(a_k + b_k + c_k)u^k. \end{aligned}$$

The partial derivative of M associated to the system (3.16) is

$$\begin{aligned} \chi(M) &= (r(\lambda + \sum_{k \geq 1} a_k u^k) + r(\mu + \sum_{k \geq 1} b_k u^k) + r(\nu + \sum_{k \geq 1} c_k u^k))M \\ &= (r(\lambda + \mu + \nu) + \sum_{k \geq 1} r(a_k + b_k + c_k)u^k)M \\ &= \sum_{k \geq 1} r(a_k + b_k + c_k)u^k M. \end{aligned} \tag{3.19}$$

Then M is an IJM if $\chi(M) - M \operatorname{div} = 0$. To achieve this, we have

$$\sum_{k \geq 1} r(a_k + b_k + c_k)u^k M - \sum_{k \geq 1} (k+1)(a_k + b_k + c_k)u^k M = 0,$$

this implies that

$$(a_k + b_k + c_k)(r - k - 1) = 0, \quad k = 1, 2, \dots,$$

since $r \neq 1 + k$ for all k , then we have only one solution which is $a_k + b_k + c_k = 0$. Hence $M = (XYZ)^r$ is an IJM, if $a_k + b_k + c_k = 0$ for all $k \in \mathbb{N}$. The implication (ii) \Rightarrow (i) follows because the divergence is equal to zero and from equation (3.19) we directly obtain an IJM of the form $M = (XYZ)^r$ where r is an arbitrary. The implication (iii) \Rightarrow (ii) follows because by an invertible change of coordinates, we can bring $\varphi = xyz(1 + o(x, y, z))$ into $\varphi = XYZ(1 + o(X, Y, Z))$. We can also absorb the factor $1 + o(u)$ as exactly shown above for the IJM, and hence we can suppose that the first integral is of the form $\varphi = XYZ$. This gives

$$\begin{aligned}
\chi(\varphi) &= \frac{\partial \varphi}{\partial X} \dot{X} + \frac{\partial \varphi}{\partial Y} \dot{Y} + \frac{\partial \varphi}{\partial Z} \dot{Z} \\
&= XYZ(\lambda + \sum_{k \geq 1} a_k u^k) + XYZ(\mu + \sum_{k \geq 1} b_k u^k) + XYZ(\nu + \sum_{k \geq 1} c_k u^k) \\
&= \sum_{k \geq 1} (a_k + b_k + c_k) u^{k+1},
\end{aligned}$$

hence, $\varphi = XYZ$ is the first integral if $a_k + b_k + c_k = 0$, for all $k \in \mathbb{N}$. The implication (ii) \Rightarrow (iii), it is clear. \square

Now, by using Theorem 6, we want to find another first integral of the system. Since, we have $\phi^\delta = XYZ$, $\delta = (1, 1, 1)$ and $M^\theta = (XYZ)^r$, $\theta = (r, r, r)$, we see that

$$\delta \times (\theta - I - \mathbf{1}) = (0, 0, 0), \quad \forall I = (i, i, i) \in \mathbb{Z}_{\geq 0}^3, \quad i \in \mathbb{Z}_{\geq 0},$$

then, there is $h_{I_i} \in \mathbb{R}$ such that $h_{I_i} \delta = \theta - I - \mathbf{1}$, where $\mathbf{1} = (1, 1, 1)$. Therefore, for each I , $\sum_I C_I = \sum_I (a_{I_x}, b_{I_y}, c_{I_z})$ with $C_0 = (\lambda, \mu, \nu)$ there is an $N_I = (n_{I_1}, n_{I_2}, n_{I_3})$, such that

$$C_I = K_{I_i} (N_I \times \delta), \quad (3.20)$$

where $K_{I_i} \in \mathbb{R}$. By solving the above equation, we obtain $N_0 = (\mu, -\lambda, 0)$ and for other value of I , we obtain $N_I = (b_{I_x}, -a_{I_y}, 0)$ (for example $N_1 = (b_1, -a_1, 0)$), with $K_{I_i} = 1$ for all I .

Now for all I , the 2-form Ω corresponding to the field (3.6) becomes

$$\Omega = \sum_I C_I \cdot \Delta (XYZ)^{I_i+1}.$$

where $\Delta = (\frac{dY \wedge dZ}{YZ}, \frac{dZ \wedge dX}{ZX}, \frac{dX \wedge dY}{XY})$ and $I_i = 1, 2, \dots$. After dividing both sides by $M^\theta = (XYZ)^r$, we obtain

$$\frac{\Omega}{M} = \sum_I C_I \cdot \Delta (XYZ)^{I_i+1-r},$$

we substitute equation (3.20) into the above form to obtain

$$\frac{\Omega}{M} = \sum_I K_{I_i}(N_I \times \delta) \cdot \Delta (XYZ)^{I_i+1-r},$$

this implies that

$$\frac{\Omega}{M} = \sum_I K_{I_i}(N_I \cdot \Delta_1)(XYZ)^{1-r+I_i} \wedge \delta \cdot \Delta_1,$$

where $\Delta_1 = (\frac{dX}{X}, \frac{dY}{Y}, \frac{dZ}{Z})$. Since, we have $\phi = XYZ$, then

$$\frac{d\phi}{\phi} = (\frac{dX}{X}, \frac{dY}{Y}, \frac{dZ}{Z}),$$

this gives

$$\frac{\Omega}{M} = \sum_I K_I \log(\tilde{X})^{N_I} (XYZ)^{h_{I_i}} \wedge \frac{d\phi}{\phi},$$

where $h_{I_i} = I_i + 1 - r$ and $\tilde{X}^{N_0} = X^{n_0} Y^{n_0} Z^{n_0} = X^\mu Y^{-\lambda}$ and for other value of I , $\tilde{X}^{N_I} = X^{b_{I_i}} Y^{-a_{I_i}}$. Since we have $K_{I_i} = 1$ and we can choose $r = 1$. Thus, we obtain a formal second first integral of the following form

$$\psi = \sum_{i=1} \ln(X^{b_i} Y^{-a_i})(XYZ)^i + \ln(X^\mu Y^{-\lambda}),$$

which pulled back to original coordinates gives

$$\psi = \sum_{i=1} \ln(x^{b_i} y^{b_i+c_i}(1 + o(x, y, z)))(xyz(1 + o(x, y, z)))^i + \ln(x^\mu y^{-\lambda}(1 + o(x, y, z))),$$

as the second first integral of the system (3.1).

In Theorem 7, we have given the proof in the formal case, in which we mean that we only use the formal power series in the change of coordinates.

3.3.1 The conjecture of Aziz and Christopher

In this section, we apply Theorem 7 to solve the sufficiency for the existence of one first integral of the three-dimensional Lotka-Volterra system. This problem have been left as conjectural in Aziz and Christopher (2014).

Consider the 3D Lotka-Volterra system

$$\dot{x} = x(\lambda + by + cz), \quad \dot{y} = y(\mu + dx + ey + fz), \quad \dot{z} = z(\nu + gx + hy). \quad (3.21)$$

We seek conditions for one first integral of the system (3.21). Some necessary conditions were found by Aziz and Christopher (2014). But they could not prove sufficiency in every case and left the remaining cases as conjectural. Here we prove the sufficiency of these conditions for the existence of one first integral.

The necessary conditions for the existence of one first integral were found in Aziz and Christopher (2014) by looking for a first integral of the form $\phi_1 = xyz(1 + o(x, y, z))$.

The following conditions are given by checking the resonant coefficients in $\phi_1 = xyz(1 + o(x, y, z))$ which are the necessary conditions for the existence of one first integral of the system (3.21). The three cases left as conjectural, where

1. $f = d = 0$,
2. $e = 0$,
3. $f + c = g + d = 0$.

These conditions relate to the numbers (4'), (5') and (6'), respectively of Theorem 4 in Aziz and Christopher (2014).

Using Theorem 7, we search for an IJM of the form $M = (xyz)^r(1 + o(x, y, z))$.

We write the divergence of the system (3.21) as

$$\operatorname{div} = (d + g)x + (b + e + h)y + (c + f)z.$$

Then, we have the following IJMs corresponding to the cases (1), (2) and (3) respectively,

1. $M = xyz \left(1 + \frac{ey}{\mu}\right)$, corresponding to the condition $f = d = 0$,
2. $M = xyz$, corresponding to the condition $e = 0$,
3. $M = (xyz)^{\frac{b+2e+h}{b+e+h}}$, corresponding to the condition $f + c = g + d = 0$, such that $e + b + h \neq 0$.

All these cases are directly consequences of Theorem 7. Hence, there is the formal first integral of the system which is of the form $\phi = xyz(1 + o(x, y, z))$.

Chapter 4

Normalizability, Integrability and Monodromy Map of Rank-One Resonant Singularities in 3D

4.1 Introduction

In this chapter, we continue the investigation of the system (3.1) with rank-one resonant eigenvalues satisfying the condition $\lambda + \mu + \nu = 0$.

In Chapter 3, by using the formal normal form method, we found two independent first integrals of the system (3.1) under some conditions on the resonant coefficients. One of these first integrals is of Darboux-analytic type and the other contains a logarithmic term allowing for non-zero resonant terms in the formal normal form. Now, we look at the general case, by which we mean that we intend to find two independent first integrals of the system (3.1) in the generic case. One of these first integrals will still contain a logarithmic term. We give the proof in the formal case. When results hold in the analytic case, this will be mentioned in the text.

We write the system (3.1) in the following form

$$\dot{x} = \lambda x + \sum_{n \geq 2} P_n(x, y, z), \quad \dot{y} = \mu y + \sum_{n \geq 2} Q_n(x, y, z), \quad \dot{z} = \nu z + \sum_{n \geq 2} R_n(x, y, z), \quad (4.1)$$

where $\lambda + \mu + \nu = 0$.

As we mentioned in the previous chapter, the system (4.1) can be brought formally into the orbitally normalizable system (3.16) which we can write in the following form

$$\dot{x} = x(\lambda + \sum_{k \geq 1} a_k u^k), \quad \dot{y} = y(\mu + \sum_{k \geq 1} b_k u^k), \quad \dot{z} = z(\nu + \sum_{k \geq 1} c_k u^k), \quad (4.2)$$

where $u = xyz$ is the resonant monomial.

Since we work with orbitally normalizability, we can divide the system (4.2) by $1 + \frac{1}{\lambda}(\sum_k a_k u^k)$ so that, without loss of generality, the system (4.2) can be taken to be in the following form

$$\dot{x} = \lambda x, \quad \dot{y} = y(\mu + \sum_{k \geq 1} \tilde{b}_k u^k), \quad \dot{z} = z(\nu + \sum_{k \geq 1} \tilde{c}_k u^k), \quad (4.3)$$

where $u = xyz$ is the resonant monomial. The sum of the resonant coefficients becomes

$$\tilde{b}_k + \tilde{c}_k = a_k + b_k + c_k, \quad k = 1, 2, \dots,$$

where $\tilde{b}_k + \tilde{c}_k$ are the corresponding terms of the system (4.3) to $a_k + b_k + c_k$ in (4.2).

For convenience, we write the system (4.3) as

$$\dot{x} = \lambda x, \quad \dot{y} = y(\mu + Q(u)), \quad \dot{z} = z(\nu + R(u)), \quad (4.4)$$

where $Q(u)$ and $R(u)$ are formal power series in u : $Q(u) = b_1u + b_2u^2 + \dots$ and $R(u) = c_1u + c_2u^2 + \dots$.

In 2D, it is known that when the two-dimensional system

$$\dot{x} = x, \quad \dot{y} = -\frac{p}{q}y + y h(x, y), \quad \text{where } h(x, y) \in \mathbb{C}[x, y],$$

is normalizable but not integrable, then the above system can be brought by an analytic change of coordinates $Y = y \varphi(x, y)$, $\varphi(0) \neq 0$, into the following form

$$\dot{x} = x, \quad \dot{Y} = -\frac{p}{q}Y(1 + \psi(u)), \quad (4.5)$$

where $\psi = u^k - au^{2k} + o(u^{3k})$ is an analytic function in $u = x^p Y^q$.

In this chapter, we consider when we can further simplify the normal form (4.4) by an analytic change of coordinates, in order to obtain a reduced normal form (RNFS) (see (4.7)). In this way, two independent first integrals can be found by using Theorem 6 for this RNFS which are of the following forms

$$H_1 = x^\alpha y^\beta z^\gamma, \quad H_2 = \frac{1}{xyz} + \epsilon_1 \ln x + \epsilon_2 \ln y.$$

We give the details of this in Section 4.2.

In Section 4.3, we introduce the monodromy map of the reduced normal form by using the above two independent first integrals, and relate this to the corresponding normal form for 2D maps. That is for 2D maps, we can also find formal normal forms only containing resonant monomials, and in this case the corresponding map has eigenvalues whose product is equal to unity based on the condition $\lambda + \mu + \nu = 0$ of the system (4.4).

In a similar way to the case of vector fields, we can also find a change of coordinates to reduce this map into a reduced map. By which we mean that we

can also apply a further transformation to reduce the 2D map obtained by using the monodromy into a reduced map containing only finite number of resonant monomials (see Section 4.3.1).

4.2 Orbitally normalizable system

The aim of this section is to obtain a reduced normal form for the system (4.4) containing only a finite number of resonant monomials.

In 3D, a suitable reduced normal form is not easy to find. We have tried out and examined several different types of normal form seeing which is the best form for the reduced normal form of the system (4.4).

In the 2D systems, it is known that a further change of coordinates will bring the system (4.5) into the reduced normal form

$$\dot{X} = X, \quad \dot{Y} = -\frac{p}{q}Y(1 + \frac{U^k}{1 + aU^k}),$$

where $U = X^p Y^q$. For more information one can refer to Christopher et al. (2003).

In analogy with 2D case, we seek an analytic change of coordinates

$$(x, Y, Z) = (x, y e^{\varphi(u)}, z e^{\psi(u)}), \quad (4.6)$$

to bring the system (4.4) into the reduced normal form

$$\dot{x} = \lambda x, \quad \dot{Y} = Y(\mu + \frac{bU}{1 + aU}), \quad \dot{Z} = Z(\nu + \frac{cU}{1 + aU}), \quad (4.7)$$

where $U = xYZ$, and $\varphi(u) = \sum_{j \geq 1} f_j u^j$, $\psi(u) = \sum_{j \geq 1} g_j u^j$.

The system (4.4) which arises from the system (4.2) has two independent first integrals under some conditions (see Chapter 3). In the following theorem, we want to show that system (4.4) can be reduced to the system (4.7) in order to

be able to find two independent first integrals of the reduced normal form of the generic case. From the system (4.4), where $u = xyz$, we have

$$\begin{aligned} \dot{u} &= u^2(b_1 + c_1 + (b_2 + c_2)u + \cdots) = (b_1 + c_1)u^2(1 + \frac{(b_2 + c_2)u}{b_1 + c_1} + \cdots) \\ &= \hat{a}_1 u^2(1 + \frac{\hat{a}_2 u}{\hat{a}_1} + \frac{\hat{a}_3 u^2}{\hat{a}_1} + \cdots) =: G(u), \end{aligned} \quad (4.8)$$

where we define $\hat{a}_k = b_k + c_k$, $k = 1, 2, \dots$, and $\hat{a}_1 \neq 0$.

We summarize all results in this section in the following theorem.

Theorem 8. Given an orbitally normalizable system (4.4) with rank-one resonant eigenvalues satisfying the condition $\lambda + \mu + \nu = 0$.

- If $b_i + c_i = 0$, for $i = 1, 2, \dots, k-1$, and $b_k + c_k \neq 0$, $c_i, b_i \neq 0$ for $i = 1, 2, \dots$, then by an analytic change of coordinates (4.6), we can bring the system (4.4) to a reduce normal form

$$\dot{x} = \lambda x, \quad \dot{Y} = Y(\mu + \frac{F(U) + bU^k}{1 + aU^k}), \quad \dot{Z} = Z(\nu + \frac{-F(U) + cU^k}{1 + aU^k}),$$

for some $a, b, c \in \mathbb{C}$, where $F(U) = b_l U^l + h_{l+1} U^{l+1} + \cdots + h_{k-1} U^{k-1}$, and b_l is the first non-zero term of b_i , (if $c_i = b_i = 0$, $i = 1, 2, \dots, k-1$, then $F(U) \equiv 0$).

- If there are $l, k \in \mathbb{N}$ with $b_k, c_l \neq 0$ for $k > l$, and $b_1 = \cdots = b_{k-1} = c_1 = \cdots = c_{l-1} = 0$, then by an analytic change of coordinates (4.6), we can bring the system (4.4) to a reduce normal form

$$\dot{x} = \lambda x, \quad \dot{Y} = \mu Y, \quad \dot{Z} = Z(\nu + \frac{cU^l}{1 + aU^l}),$$

for some $a, c \in \mathbb{C}$.

Now, we split the proof of Theorem 8 into several cases. Firstly, we want to prove the case $k = 1$.

Theorem 9. Given an orbitally normalizable system (4.4) with rank-one resonant eigenvalues satisfying the condition $\lambda + \mu + \nu = 0$. If $b_1, c_1 \neq 0$ and $c_1 + b_1 \neq 0$, then by an analytic change of coordinates (4.6), we can bring the system (4.4) to a reduce normal form (4.7) for some $a, b, c \in \mathbb{C}$.

Proof. Firstly, we take the derivative of $Y = ye^\varphi$ and $Z = ze^\psi$ to obtain

$$\dot{Y} = \dot{y}e^\varphi + ye^\varphi\varphi'\dot{u}, \quad \dot{Z} = \dot{z}e^\psi + ze^\psi\psi'\dot{u}, \quad (4.9)$$

by substituting the system (4.7) into the first equation in (4.9) (\dot{Y}), gives

$$Y\left(\mu + \frac{bU}{1+aU}\right) = \dot{y}e^\varphi + ye^\varphi\varphi'\dot{u},$$

substituting the system (4.4) and (4.8), respectively, at each step, we obtain

$$ye^\varphi\left(\mu + \frac{bU}{1+aU}\right) = y(\mu + Q(u))e^\varphi + ye^\varphi\varphi'G(u),$$

dividing both sides by ye^φ , yields

$$\frac{bU}{1+aU} = Q(u) + G(u)\varphi'. \quad (4.10)$$

In the same way, by using a change of coordinates $Z = ze^\psi$, and substituting the systems (4.4), (4.7) and (4.8) into the second equation in (4.9) (\dot{Z}), respectively, at each step, we directly obtain

$$\frac{cU}{1+aU} = R(u) + G(u)\psi', \quad (4.11)$$

where $\varphi' = \sum_{j \geq 1} j f_j u^{j-1}$ and $\psi' = \sum_{j \geq 1} j g_j u^{j-1}$. Writing $U = xYZ = xyz e^{\varphi(u)+\psi(u)} = u e^{K(u)}$, where $K(u) = \varphi + \psi = \sum_{j \geq 1} \xi_j u^j$. Equation (4.10) gives

$$\begin{aligned} b u e^{K(u)}(1 + a u e^{K(u)} + \dots) &= b_1 u + b_2 u^2 + \dots + (\hat{a}_1 u^2 + \dots)(f_1 + 2f_2 u + \dots) \\ &= b_1 u + b_2 u^2 + \dots + (f_1 \hat{a}_1 u^2 + \dots) = b_1 u(1 + \dots), \end{aligned}$$

yields

$$\begin{aligned} (b - b_1)u + ((-a + \xi_1)b - \hat{a}_1 f_1 - b_2)u^2 + (\hat{a}^2 h - b_3(-2a\xi_1 + \xi_2)b - 2f_2 \hat{a}_1 \\ - \hat{a}_2 f_1 + \frac{b\xi_1^2}{2})u^3 + \dots = 0. \end{aligned} \tag{4.12}$$

Hence, from the first and second terms in u , we obtain

$$b = b_1, \quad \xi_1 b_1 - f_1 \hat{a}_1 = a b_1 + b_2. \tag{4.13}$$

From (4.11), we have

$$c u e^{K(u)}(1 + a e^{K(u)} + \dots) = c_1 u(1 + \frac{c_2 + \hat{a}_1 g_1}{c_1} u + \dots),$$

yields

$$\begin{aligned} (c - c_1)u + ((-a + \xi_1)c - \hat{a}_1 g_1 - c_2)u^2 + (\hat{a}^2 c - c_3 + (-2a\xi_1 + \xi_2)c - 2\hat{a}_1 g_2 \\ - \hat{a}_2 g_1 + \frac{c\xi_1^2}{2})u^3 + \dots = 0, \end{aligned} \tag{4.14}$$

which gives

$$c = c_1, \quad \xi_1 c_1 - g_1 \hat{a}_1 = a c_1 + c_2. \tag{4.15}$$

We sum the two equations (4.12) and (4.14), and substituting the values of b and

c , to obtain

$$\xi_1(b_1 + c_1) - \xi_1 \hat{a}_1 = a(\hat{a}_1 + c_1) + b_2 + c_2,$$

and hence $a = -\frac{\hat{a}_2}{\hat{a}_1}$. To find f_1 we can choose one of the equations (4.13) or (4.15), to give

$$f_1 = -\frac{a b_1 + b_2}{c_1} + \frac{b_1}{c_1} g_1.$$

We can find further coefficients f_i in $\varphi(u)$. By substituting equation (4.13) with $a = -\frac{\hat{a}_2}{\hat{a}_1}$ into (4.12), the coefficient of u^3 term in equation (4.12) can be solved with respect to f_2 which gives

$$f_2 = \frac{c_3 + b_1 g_2}{c_1 + \hat{a}_1} + \frac{b_1 c_2^2}{2c_1^2 (c_1 + \hat{a}_1)} - \frac{\hat{a}_2 c_2 + \hat{a}_1 \hat{a}_3}{\hat{a}_1 (c_1 + \hat{a}_1)} + \frac{\hat{a}_2^2}{2\hat{a}_1^2} + \frac{\hat{a}_1 b_1 g_1 (\hat{a}_1 g_1 + 2c_2)}{2c_1^2 (c_1 + \hat{a}_1)}. \quad (4.16)$$

In the same way, we can continue to solve equation (4.12), and the coefficients f_i can be determined term by term. In more detail, we want to show that equation (4.12) is solvable for each individual term. We first need to find the power series $K(u) = \sum_{i \geq 1} \xi_i u^i$. We can sum the two equations (4.12) and (4.14), and considering the Taylor expansion with substituting the values of c and b to obtain

$$\begin{aligned} & (\hat{a}^2 \hat{a}_1 - \hat{a}_1 \xi_2 - \hat{a}_3 + \frac{\hat{a}_1 \xi_1^2}{2} - \xi_1 (2a \hat{a}_1 + \hat{a}_2)) u^2 + (\xi_3 + \frac{\hat{a}_4}{2\hat{a}_1} \\ & + \frac{\hat{a}_2^3}{2\hat{a}_1^3} + \frac{2\xi_2 \hat{a}_2}{\hat{a}_1} + \frac{(\hat{a}_1^2 \xi_2 - \hat{a}_3 \hat{a}_1 + 3\hat{a}_2^2) \xi_1}{-2\hat{a}_1^2} + \frac{\xi_1^2 (\hat{a}_1 \xi_1 - 12\hat{a}_2)}{-12\hat{a}_1}) u^3 + \dots = 0. \end{aligned} \quad (4.17)$$

This gives from u^2 and u^3 terms, respectively,

$$\begin{aligned} \hat{a}_1 \xi_2 &= \hat{a}_3 - \hat{a}^2 \hat{a}_1 - \frac{\hat{a}_1 \xi_1^2}{2} - \xi_1 (2a \hat{a}_1 + \hat{a}_2), \\ 2\hat{a}_1 \xi_3 &= -\hat{a}_4 - \frac{\hat{a}_2^3}{\hat{a}_1^2} - 4\xi_2 \hat{a}_2 + \frac{(\hat{a}_1^2 \xi_2 - \hat{a}_3 \hat{a}_1 + 3\hat{a}_2^2) \xi_1}{\hat{a}_1} + \frac{\xi_1^2 (\hat{a}_1 \xi_1 - 12\hat{a}_2)}{6}. \end{aligned}$$

To find for further terms, we continue to compare powers of u in equation (4.12)

and (4.14) which are determined term by term. Now, we want to show that there is an analytic $K(u)$ in the change of coordinate $U = ue^{K(u)}$. Following Kostov (1984), we want to show that the one-dimensional field $G(u)\frac{d}{du}$ has an analytic change of coordinate, which brings the system (4.8) into

$$\dot{u} = \hat{a}_1 u^2 (1 + \tilde{a} u + o(u^2)), \quad (4.18)$$

where $\tilde{a} = \frac{\hat{a}_2}{\hat{a}_1}$ is a formal orbital invariant. Thus, we need to solve

$$\frac{dU}{du} = \frac{(c+b)U^2(1+aU)^{-1}}{(b_1+c_1)u^2(1+\tilde{a}u+o(u^2))}.$$

Since we have $b = b_1$ and $c_1 = c$, then this gives

$$\frac{(1+aU)dU}{U^2} = \frac{du}{u^2(1+\tilde{a}u+o(u^2))}.$$

This implies that

$$\left(\frac{1}{U^2} + \frac{a}{U}\right)dU = \left(\frac{1-\tilde{a}u}{u^2} + \frac{\tilde{a}^2+o(u)}{1+\tilde{a}u+o(u^2)}\right)du.$$

By integrating, we obtain

$$\frac{-1}{U} + a \ln U = \frac{-1}{u} - \tilde{a} \ln u + o(u^0),$$

and using $U = ue^K$, gives

$$\frac{1}{u}(1 - e^{-K}) + (a + \tilde{a}) \ln(u) + aK - o(u^0) = 0.$$

Since, we have $a = -\frac{\hat{a}_2}{\hat{a}_1}$ and $\tilde{a} = \frac{\hat{a}_2}{\hat{a}_1}$, and then multiplying by u , gives

$$F(u, K(u)) = 1 - e^{-K} + auK - o(u^1) = 0.$$

Solving the above equation is equivalent to solving $F(u, K(u)) = 0$, where $K(u) = e^{\varphi(u)+\psi(u)}$. Clearly, $F(0, 0) = 0$, and

$$\frac{\partial F}{\partial K} = e^{-K} + au,$$

which implies that $\frac{\partial F}{\partial K}(0, 0) = 1$, thus giving $K = K(u)$ as an analytic solution to $F(u, K(u)) = 0$ by the implicit function theorem.

Now, we want to show that each equations (4.12) and (4.14) has a solution for each individual linear term with respect to the power series $\psi(u)$ and $\varphi(u)$ via the change of coordinate $U = ue^{K(u)}$.

Since, from (4.10) we have

$$\frac{b e^{K(u)}}{1 + a u e^{K(u)}} - Q(u) = G(u)\varphi',$$

by integrating with respect to u , and considering the Taylor expansion, we obtain

$$\varphi = \int \left(\frac{b e^{K(u)}}{1 + a u e^{K(u)}} - Q(u) \right) G(u)^{-1} du = \int \left(\frac{b_1 \xi_1}{\hat{a}_1} - \frac{b_2}{\hat{a}_1} - \frac{b_1 \hat{a}_2}{\hat{a}_1^2} + o(u) \right) du.$$

It is clear that the above integral exists and is analytic in giving the change of coordinate $Y = y e^{\varphi(u)}$. Consequently, the change of Z -coordinate also exists by $\psi(u) = K(u) - \varphi(u)$.

Therefore, by solving (4.12) and (4.14), respectively, we obtain power series

$$\begin{aligned} \varphi(u) = & \left(\frac{b_1}{c_1} g_1 + \frac{c_2 b_1 - b_2 c_1}{\hat{a}_1 c_1} \right) u + \left(\frac{\hat{a}_1 (\hat{a}_1 - c_1) g_1^2}{2 c_1^2} + \frac{c_2 (\hat{a}_1 - c_1) g_1}{c_1^2} \right. \\ & \left. + \frac{\hat{a}_3 c_1^3 + ((-2 \hat{a}_3 + c_3) \hat{a}_1 + \hat{a}_2 (\hat{a}_2 - c_2)) c_1^2 - \hat{a}_1 c_1 c_2^2 + \hat{a}_1^2 c_2^2}{2 c_1^2 \hat{a}_1^2} \right) u^2 + \dots, \end{aligned}$$

$$\psi(u) = g_1 u + \left(\frac{c_2 g_1}{c_1} + \frac{\hat{a}_1 g_1^2}{2c_1} - \frac{\hat{a}_1 c_1 c_3 - \hat{a}_1 c_2^2 - \hat{a}_2 c_1 c_2 + c_1^2 \hat{a}_3}{2c_1 \hat{a}_1^2} \right) u^2 + \dots$$

□

In the following theorem, if one of the conditions in Theorem 9 does not hold (i.e., $b_1 = 0$, $c_1 = 0$ or $b_1 + c_1 = 0$), we can not use the RNFS (4.7) for the reduction of the normal form (4.4). We then seek an another RNFS depending on vanishing the resonant coefficients b_i , c_i and $b_i + c_i$ in the normal form (4.4).

Thus, if the situation is not generic, we consider the following possibilities in turn. In Theorem 10, we suppose $b_1 + c_1 = 0$, and $b_1, c_1, b_2 + c_2 \neq 0$. In Theorem 11, we suppose $b_i + c_i = 0$, for $i = 1, 2, \dots, k-1$, $1 < k \in \mathbb{N}$, and $b_k + c_k \neq 0$, and $b_i, c_i \neq 0$ for all $i \in \mathbb{N}$. In Theorem 12, we suppose $b_1 = 0$ and $c_1, b_2 \neq 0$. In Theorem 13, we suppose $b_i = 0$, $i = 2, 3, \dots, k-1$ and $c_1, b_k \neq 0$. In Theorem 14, we suppose $b_1 = b_2 = \dots = b_{k-1} = c_1 = c_2 = \dots = c_{l-1} = 0$ and $b_k, c_l \neq 0$.

Theorem 10. Given an orbitally normalizable system (4.4) with rank-one resonant eigenvalues satisfying the condition $\lambda + \mu + \nu = 0$. Assume $c_1 + b_1 = 0$, $c_2 + b_2 \neq 0$ and $c_1, b_1 \neq 0$, then by an analytic change of coordinates (4.6), we can bring the system (4.4) to a reduce normal form

$$\dot{x} = \lambda x, \quad \dot{Y} = Y \left(\mu + \frac{bU + \tilde{b}U^2}{1 + aU^2} \right), \quad \dot{Z} = Z \left(\nu + \frac{cU + \tilde{c}U^2}{1 + aU^2} \right), \quad (4.19)$$

for some $a, b, c \in \mathbb{C}$.

Proof. We suppose that $b_1 + c_1 = 0$, then $b_1 = -c_1$ and \dot{u} becomes

$$\dot{u} = \hat{a}_2 u^3 \left(1 + \frac{\hat{a}_3}{\hat{a}_2} u + \frac{\hat{a}_4}{\hat{a}_2} u^2 + \dots \right) =: G(u). \quad (4.20)$$

where $\hat{a}_k = b_k + c_k$ for $k = 2, 3, \dots$, and $\hat{a}_2 \neq 0$. By the same way that we showed in Theorem 9, we can substitute the systems (4.4), (4.19) and (4.20) into

the change of Y -coordinate, $\dot{Y} = \dot{y}e^\varphi + ye^\varphi\varphi'\dot{u}$, to obtain

$$\frac{bU + \tilde{b}U^2}{1 + aU^2} = Q(u) + G(u)\varphi', \quad (4.21)$$

where $U = ue^{K(u)}$ and $K(u) = \varphi + \psi = \sum_{j \geq 1} \xi_j u^j$. Equation (4.21) gives

$$bue^{\varphi+\psi} + \tilde{b}u^2e^{2(\varphi+\psi)} - (1 + au^2e^{2(\varphi+\psi)})(Q(u) + G(u)\varphi') = 0. \quad (4.22)$$

Moreover, using the Taylor expansion to simplify equation (4.21), we obtain

$$\begin{aligned} (b - b_1)u + (\tilde{b} - c_1\xi_1 - b_2)u^2 + \left(\left(\frac{-\xi_1^2}{2} - \xi_2\right)c_1 + 2\xi_1\tilde{b} + ac_1 - b_3 - \hat{a}_2f_1\right)u^3 \\ + \dots = 0. \end{aligned} \quad (4.23)$$

By solving the first and second terms in u , we obtain

$$b = b_1 = -c_1, \quad \tilde{b} = c_1\xi_1 + b_2. \quad (4.24)$$

Also, in the same way, we can substitute the systems (4.4), (4.19) and (4.20) into the change of coordinate, $Z = ze^\psi$, to obtain

$$\frac{cU + \tilde{c}U^2}{1 + aU^2} = R(u) + G(u)\psi', \quad (4.25)$$

which yields,

$$cue^{\varphi+\psi} + \tilde{c}ue^{2(\varphi+\psi)} - (1 + au^2e^{2(\varphi+\psi)})(R(u) + G(u)\psi') = 0. \quad (4.26)$$

We consider the Taylor expansion and simplifying equation (4.26) to obtain

$$(c - c_1)u + (\tilde{c} + c_1\xi_1 - c_2)u^2 + ((\frac{\xi_1^2}{2} + \xi_2)c_1 + 2\tilde{c}\xi_1 - ac_1 - g_1\hat{a}_2 - c_3)u^3 + \dots = 0. \quad (4.27)$$

From the first and second terms in u , gives

$$c = c_1, \quad \tilde{c} = -c_1\xi_1 + c_2. \quad (4.28)$$

From sum of the equations (4.24) and (4.28) to obtain her, we obtain

$$\tilde{c} + \tilde{b} = c_1\xi_1 + c_2 + b_1\xi_1 + b_2 = c_2 + b_2 = \hat{a}_2 \neq 0.$$

Now, we first need to find a . We sum the equations (4.23) and (4.27), considering the Taylor expansion, and substituting the values of c, b, \tilde{b} and \tilde{c} , to obtain

$$(\tilde{c} + \tilde{b} - \hat{a}_2)u^2 + (\hat{a}_2\xi_1 - \hat{a}_3)u^3 + (2\xi_1^2\hat{a}_2 - a\hat{a}_2 - \hat{a}_3\xi_1 + \hat{a}_4)u^4 + \dots = 0. \quad (4.29)$$

The terms in u^3 and u^4 , respectively, give

$$\xi_1 = \frac{\hat{a}_3}{\hat{a}_2}, \quad a = 2\xi_1^2 - \frac{\hat{a}_3}{\hat{a}_2}\xi_1 - \frac{\hat{a}_4}{\hat{a}_2}, \quad (4.30)$$

and hence,

$$a = (\frac{\hat{a}_3}{\hat{a}_2})^2 - \frac{\hat{a}_4}{\hat{a}_2}.$$

Thus, it is necessary to find the value of a in the process before seeking for the value of the each power series φ and ψ .

From equation (4.29), we have

$$f_1 = -g_1 + \frac{\hat{a}_3}{\hat{a}_2}. \quad (4.31)$$

Now, we want to find $K(u) = \sum_{i \geq 1} \xi_i u^i$. We substitute the values of b, \tilde{b}, a into (4.29) to obtain

$$(\hat{a}_2 g_1 - \xi_2 c_1 - c_3 + \frac{4 \hat{a}_2 \hat{a}_3 c_2 + 2 \hat{a}_2 \hat{a}_4 c_1 - 5 \hat{a}_3^2 c_1}{2 \hat{a}_2^2}) u^5 + \dots = 0. \quad (4.32)$$

This gives

$$c_1 \xi_2 = \hat{a}_2 g_1 - c_3 + \frac{4 \hat{a}_2 \hat{a}_3 c_2 + 2 \hat{a}_2 \hat{a}_4 c_1 - 5 \hat{a}_3^2 c_1}{2 \hat{a}_2^2}.$$

In the same way, from the u^6 term, we obtain

$$\hat{a}_2 \xi_3 = 2 \hat{a}_3 \xi_2 - \hat{a}_5 - 3 \frac{\hat{a}_3 \hat{a}_4}{\hat{a}_2} + \frac{8 \hat{a}_3^3}{3 \hat{a}_2^2}.$$

To find for further terms, we can continue to find the coefficients ξ_i from equation (4.29) which is determined term by term. In more detail, we want to show that there is an analytic $K(u)$ in the change of coordinate $U = u e^{K(u)}$. Since, from the system (4.8), we have

$$\dot{u} = \hat{a}_2 u^3 \left(1 + \frac{\hat{a}_3}{\hat{a}_2} u + \frac{\hat{a}_4}{\hat{a}_2} u^2 + o(u^3) \right).$$

Thus, we need to solve

$$\frac{dU}{du} = \frac{(\tilde{c} + \tilde{b}) U^3 (1 + a U^2)^{-1}}{\hat{a}_2 u^3 \left(1 + \frac{\hat{a}_3}{\hat{a}_2} u + \frac{\hat{a}_4}{\hat{a}_2} u^2 + o(u^3) \right)}.$$

Since we have $\tilde{b} + \tilde{c} = b_2 + c_2 = \hat{a}_2$, this gives

$$\frac{(1 + a U^2) dU}{U^3} = \frac{du}{u^3 \left(1 + \frac{\hat{a}_3}{\hat{a}_2} u + \frac{\hat{a}_4}{\hat{a}_2} u^2 + o(u^3) \right)},$$

which simplifies to

$$\left(\frac{1}{U^3} + \frac{a}{U}\right) dU = \left(\frac{1 - \frac{\hat{a}_3}{\hat{a}_2} u + \left(\left(\frac{\hat{a}_3}{\hat{a}_2}\right)^2 - \frac{\hat{a}_4}{\hat{a}_2}\right) u^2}{u^3} + \frac{-\frac{\hat{a}_3}{\hat{a}_2}^3 + o(u^2)}{1 + \frac{\hat{a}_3}{\hat{a}_2} u + \frac{\hat{a}_4}{\hat{a}_2} u^2 + o(u^3)} \right) du.$$

Since we have $a = \left(\frac{\hat{a}_3}{\hat{a}_2}\right)^2 - \frac{\hat{a}_4}{\hat{a}_2}$, and by integrating on the above equation we obtain

$$\frac{-2}{U^2} + a \ln U = \frac{-2}{u^2} + \frac{\hat{a}_3}{\hat{a}_2} \frac{1}{u} + a \ln u + o(u^0),$$

this implies that

$$\frac{2}{u^2}(1 - e^{-2K}) + a \ln(u e^K) - a \ln u - \frac{\hat{a}_3}{\hat{a}_2} \frac{1}{u} - o(u^0) = 0,$$

multiplying by u^2 , gives

$$F(u, K(u)) = 2(1 - e^{-2K}) + au^2 K(u) - \tilde{a}u - o(u^2) = 0.$$

Solving the above equation is equivalent to solving $F(u, K(u)) = 0$, where $K(u) = e^{\varphi(u)+\psi(u)}$. Clearly, $F(0, 0) = 0$, and

$$\frac{\partial F}{\partial K} = 4e^{-2K} + au^2,$$

which implies that $\frac{\partial F}{\partial K}(0, 0) = 4$, thus giving $K = K(u)$ as an analytic solution to $F(u, K(u)) = 0$ by the implicit function theorem.

Also, we want to show that (by the same way shown in Theorem 9) each individual equations (4.22) and (4.26) has a solution with respect the power series $\varphi(u)$ and $\psi(u)$, respectively. Since, from equation (4.22) we have

$$\left(\frac{bue^{K(u)} + \tilde{b}u^2e^{2K(u)}}{1 + au^2e^{2K(u)}} - Q(u) \right) G(u)^{-1} = \varphi',$$

by integrating with respect to u and considering the Taylor expansion, and sub-

stituting the values of $a, b, c, \tilde{b}, \tilde{c}$ and f_1 we obtain

$$\begin{aligned}\varphi &= \int \left(\frac{b_1 u e^{K(u)} + \tilde{b} u^2 e^{2K(u)}}{(1 + a u^2 e^{2K(u)})} - Q(u) \right) G(u)^{-1} du \\ &= \int \left(\frac{-g_1 \hat{a}_2 + \hat{a}_3}{\hat{a}_2} + o(u) \right) du,\end{aligned}$$

which is analytic in u and gives the change of coordinate $Y = y e^{\varphi(u)}$. Consequently, the change of Z -coordinate also exists by $\psi(u) = K(u) - \varphi(u)$.

Therefore, by solving the equations (4.23) and (4.27), respectively, we obtain the power series

$$\begin{aligned}\varphi(u) &= -\frac{\hat{a}_2 g_1 - \hat{a}_3}{\hat{a}_2} u + \left(\frac{2 \hat{a}_2 c_3 + c_1 c_4 - 2 c_2 c_3}{2 \hat{a}_2 c_1} - \frac{(4 \hat{a}_2 c_2 + c_1 c_3) \hat{a}_3 + 4 \hat{a}_3 c_2^2}{2 \hat{a}_2^2 c_1} \right. \\ &\quad \left. - \frac{(2 \hat{a}_2 - c_2) \hat{a}_4 + \hat{a}_5 c_1}{2 \hat{a}_2^2} + \frac{(5 \hat{a}_2 - 4 c_2) \hat{a}_3^2}{2 \hat{a}_2^3} - \frac{2 \hat{a}_3 c_1 (\hat{a}_2 \hat{a}_4 - \hat{a}_3^2)}{\hat{a}_2^4} + \frac{g_1 (\hat{a}_2 - c_2)}{c_1} \right) u^2 \\ &\quad + \dots, \\ \psi(u) &= g_1 u + \left(\frac{c_2 \hat{a}_4 \hat{a}_2 + \hat{a}_5 c_1 \hat{a}_2 - 4 c_2 a_3^2 - 4 \hat{a}_4 c_1 \hat{a}_3}{2 \hat{a}_2^3} \right. \\ &\quad \left. - \frac{\hat{a}_2^3 c_1 c_4 - 2 \hat{a}_2^3 c_2 c_3 - \hat{a}_2^2 a_3 c_1 c_3 + 4 \hat{a}_2^2 \hat{a}_3 c_2^2 + 4 \hat{a}_3^3 c_1^2}{2 \hat{a}_2^4 c_1} + \frac{c_2 g_1}{c_1} \right) u^2 + \dots,\end{aligned}$$

□

However, in the above case, from equation (4.24) ($\tilde{b} = c_1 \xi_1 + b_2$), if we choose that $f_1 = -\frac{b_2}{c_1} - g_1$, this gives $\tilde{b} = 0$. Then, by substituting f_1 and $\tilde{b} = 0$ into (4.23), we obtain

$$\begin{aligned}&\left(\frac{(\hat{a}_2 - c_2)(\hat{a}_2 + c_2)}{2 c_1} - \frac{c_1 (\hat{a}_2 \hat{a}_4 - \hat{a}_3^2)}{\hat{a}_2^2} + \hat{a}_2 g_1 - c_1 \xi_2 - b_3 \right) u^2 + \left(\frac{\hat{a}_3 (\hat{a}_2 - c_2)}{c_1} \right. \\ &\quad \left. + \frac{(\hat{a}_2 - c_2) (\hat{a}_2^2 (\hat{a}_2 - c_2)^2 + 18 (\hat{a}_2 \hat{a}_4 - v_3^2) c_1^2)}{6 c_1^2 \hat{a}_2^2} - (\hat{a}_2 + c_2) f_2 + (\hat{a}_2 - c_2) g_2 \right. \\ &\quad \left. - c_1 \xi_3 - b_4 + g_1 \hat{a}_3 \right) u^3 + \dots = 0,\end{aligned}\tag{4.33}$$

solving the above equation with respect to f_i , gives

$$\left\{ \begin{array}{l} f_2 = -\frac{-\hat{a}_2^4 + 2\hat{a}_2^2\hat{a}_3c_1 + 2\hat{a}_2\hat{a}_4c_1^2 - 2\hat{a}_3^2c_1^2}{2c_1^2\hat{a}_2^2} + \frac{c_3}{c_1} - \frac{c_2^2}{2c_1^2} + \frac{\hat{a}_2}{c_1}g_1 - g_2, \\ f_3 = -\frac{1}{3c_1^3\hat{a}_2^2}(\hat{a}_2^3c_1c_2g_1 - 3\hat{a}_2^2\hat{a}_3c_1^2g_1 + 3\hat{a}_2^2c_1c_2c_3 + 6\hat{a}_2\hat{a}_4c_1^2c_2 + 3g_3c_1^3\hat{a}_2^2 \\ + 3\hat{a}_2^4c_1g_1 + 3\hat{a}_2^3c_1c_3 - 6\hat{a}_3^2c_1^2c_2 - 9\hat{a}_2^2\hat{a}_4c_1^2 + 12\hat{a}_2\hat{a}_3^2c_1^2 - 6\hat{a}_2^3\hat{a}_3c_1 \\ - 3c_4c_1^2\hat{a}_2^2 - 6g_2c_1^2\hat{a}_2^3 - \hat{a}_2^2c_2^3 - 3\hat{a}_2^3c_2^2 + 3\hat{a}_2^4c_2 + \hat{a}_2^5), \\ \vdots \\ f_n = \dots, \end{array} \right.$$

substituting f_i into equation (4.27), we obtain

$$\begin{aligned} & (\tilde{c} - \hat{a}_2)u - \left(\frac{\hat{a}_2^2 - \hat{a}_2c_2 + c_1\hat{a}_3}{c_1}\right)u^2 + \left(\frac{(2\hat{a}_2^2 - 2\hat{a}_2c_2 - c_1\hat{a}_3)(\hat{a}_2^2 - \hat{a}_2c_2 + c_1\hat{a}_3)}{\hat{a}_2c_1^2}\right)u^3 \\ & + \dots = 0. \end{aligned} \tag{4.34}$$

We note that the second term in u does not contain g_i , then we reach a point now we can not solve equation (4.34) to find the power series $\psi = g_1u + \dots$, in which we mean that we can not find the change of coordinates $U = ue^{K(u)}$ when $\tilde{b} = 0$. By which we mean that we do not have a freedom choice for chosen the value of f_1 . Therefore, it is necessary to find the value of f_1 in the end of process.

Theorem 11. Given an orbitally normalizable system (4.4) with rank-one resonance satisfying the condition $\lambda + \mu + \nu = 0$. If $b_l + c_l = 0$ for $i = 1, 2, \dots, k-1$ and $b_k + c_k \neq 0$. Then, by an analytic change of coordinates (4.6), we can bring the system (4.4) to a reduce normal form

$$\dot{x} = \lambda x, \quad \dot{Y} = Y\left(\mu + \frac{F(U) + h_k U^k}{1 + a U^k}\right), \quad \dot{Z} = Z\left(\nu + \frac{-F(U) + w_k U^k}{1 + a U^k}\right), \tag{4.35}$$

where $F(U) = b_l U^l + h_{l+1} U^{l+1} + \dots + h_{k-1} U^{k-1}$, and b_l is first non-zero term of b_i .

Proof. Since $b_{k-1} + c_{k-1} = 0$, then $b_i = -c_i$ for $i = 1, 2, \dots, k-1$, and \dot{u} becomes

$$\dot{u} = u(Q(u) + R(u)) = \hat{a}_k u^{k+1} \left(1 + \sum_{j \geq 1} \frac{\hat{a}_{k+j}}{\hat{a}_k} u^j\right) =: G(u), \quad i = 1, 2, \dots \quad (4.36)$$

where $\hat{a}_k = b_k + c_k$ for $k = 2, 3, \dots$.

Firstly, By the same way that we showed in Theorem 9, we can substitute the systems (4.4), (4.35) and (4.36) into the change of Y -coordinate and Z -coordinates, respectively, to obtain

$$\frac{F(U) + h_k U^k}{1 + a U^k} = Q(u) + G(u)\varphi', \quad (4.37)$$

and

$$\frac{-F(U) + w_k U^k}{1 + a U^k} = R(u) + G(u)\psi', \quad (4.38)$$

where $U = ue^{K(u)}$, and $K(u) = \varphi(u) + \psi(u) = \sum_{j \geq 1} (f_j + g_j)u^j = \sum_{j \geq 1} \xi_j u^j$.

Now, we need to find the value of a . We sum (4.37) and (4.38) to obtain

$$\frac{(h_k + w_k) U^k}{1 + a U^k} = Q(u) + R(u) + G(u) K'. \quad (4.39)$$

Since, from the system (4.36), we have

$$\dot{u} = \hat{a}_k u^{k+1} \left(1 + \frac{\hat{a}_{k+1}}{\hat{a}_k} u + \frac{\hat{a}_{k+2}}{\hat{a}_k} u^2 + o(u^{k+3})\right).$$

Thus, we need to solve

$$\frac{dU}{du} = \frac{(h_k + w_k) U^{k+1} (1 + a U^k)^{-1}}{\hat{a}_k u^{k+1} \left(1 + \frac{\hat{a}_{k+1}}{\hat{a}_k} u + \frac{\hat{a}_{k+2}}{\hat{a}_k} u^2 + o(u^{k+3})\right)}.$$

Since, we have $h_k + w_k = b_k + c_k = \hat{a}_k$, this gives

$$\frac{(1 + a U^k) dU}{U^{k+1}} = \frac{du}{u^{k+1} \left(1 + \frac{\hat{a}_{k+1}}{\hat{a}_k} u + \frac{\hat{a}_{k+2}}{\hat{a}_k} u^2 + o(u^3) \right)},$$

which simplifies to

$$\left(\frac{1}{U^{k+1}} + \frac{a}{U} \right) dU = \left(\frac{1 + \hat{a}_1 u + \cdots + \hat{a}_{k-1} u^{k-1} + a u^k}{u^{k+1}} + \frac{\left(\frac{\hat{a}_{k+1}}{\hat{a}_k} \right)^{k+1} + o(u)}{1 + \frac{\hat{a}_{k+1}}{\hat{a}_{k+2}} u + o(u^2)} \right) du, \quad (4.40)$$

where

$$\left\{ \begin{array}{l} \hat{a}_1 = -\frac{\hat{a}_{k+1}}{\hat{a}_k}, \\ \hat{a}_2 = -\left(\frac{\hat{a}_{k+2}}{\hat{a}_k} + \hat{a}_1 \frac{\hat{a}_{k+1}}{\hat{a}_k} \right), \\ \vdots \\ \hat{a}_j = -\left(\frac{\hat{a}_{k+j}}{\hat{a}_k} + \hat{a}_1 \frac{\hat{a}_{k+j-1}}{\hat{a}_k} + \cdots + \hat{a}_{j-1} \frac{\hat{a}_{k+1}}{\hat{a}_k} \right), \\ a = -\left(\frac{\hat{a}_{k+j+1}}{\hat{a}_k} + \hat{a}_1 \frac{\hat{a}_{k+j}}{\hat{a}_k} + \cdots + \hat{a}_j \frac{\hat{a}_{k+1}}{\hat{a}_k} \right), \end{array} \right. \quad j = 1, 2, \dots, k-1, k > 2.$$

by determining the above, we can find the value of a . Now, by integrating on equation (4.40), we obtain

$$\frac{-k}{U^k} + a \ln U = \frac{-k}{u^k} + \sum_{i \geq 1}^j \frac{\hat{a}_i}{-k+i} u^{-k+i} + a \ln u + o(u^0),$$

this implies that

$$\frac{k}{u^k} (1 - e^{-k K(u)}) + a \ln(u e^K) - a \ln u - \sum_{i \geq 1}^j \frac{\hat{a}_i}{-k+i} u^{-k+i} - o(u^0) = 0,$$

multiplying by u^k , gives

$$F(u, K(u)) = k(1 - e^{-k K(u)}) + a u^k K(u) - \sum_{i \geq 1}^j \frac{\hat{a}_i}{-k+i} u^i - o(u^k) = 0.$$

Solving the above equation is equivalent to solving $F(u, K(u)) = 0$. However,

since $F(0, 0) = k$, and

$$\frac{\partial F}{\partial K} = k^2 e^{-kK(u)} + au^k,$$

this implies that

$$\frac{\partial F}{\partial K}(0, 0) = k^2 \neq 0,$$

which gives $K(u)$ as an analytic solution to $F(u, K(u)) = 0$ by the implicit function theorem.

Also, each equation (4.37) and (4.38) is solvable with respect to power series φ and ψ , respectively, by integrating with respect to u as shown in Theorem 9. For example, when $k = 3$, we obtain the two power series $\varphi = f_1 u + f_2 u^2 + \dots$ and $\psi = g_1 u + g_2 u^2 + \dots$, given by

$$\begin{aligned} \varphi &= (-g_1 + \frac{\hat{a}_5}{4\hat{a}_5})u + (-g_2 + \frac{\hat{a}_7}{3\hat{a}_5} - \frac{17}{96} \frac{\hat{a}_6^2}{\hat{a}_5^2})u^2 + \dots, \\ \psi(u) &= g_1 u + g_2 u^2 + g_3 u^3 + \left(\frac{1}{\hat{a}_5^9 c_1^3} (S_1 g_1 + S_2 g_2 + S_3 g_3) + S_4 \right) u^4 + \dots, \end{aligned}$$

where $S_i(c_1, c_2, c_3, \hat{a}_5, \hat{a}_6, \hat{a}_7)$ for $i = 1, 2, 3, 4$ are polynomials. □

In the above case, we do not have freedom choice for chosen the value of f_i for $i = 1, 2, \dots, k-1$ in order to find the change of Y -coordinate, because from equation (4.37) we have

$$F(U) + h_k U^k - (1 + aU^k)(Q(u) + G(u)\varphi') = 0.$$

We use the Taylor expansion of the above equation until the term u^{k-1} to give

$$\begin{aligned} &(c_1 \xi_1 - c_2 - h_2)u^2 + ((f_2 + g_2)c_1 - h_3 - c_3 - 2h_2 \xi_1 + \frac{c_1 \xi_1^2}{2})u^3 + (c_1 \xi_2 \xi_1 \\ &+ \frac{c_1 \xi_1^3}{6} - 2(\xi_1^2 + \xi_2)h_2 - 3\xi_1 h_3 + \xi_3 c_1 - h_4 - c_4)u^4 + \dots = 0, \end{aligned}$$

this implies that

$$\left\{ \begin{array}{l} h_2 = c_1 \xi_1 - c_2, \\ h_3 = \frac{c_1 \xi_1^2}{2} + c_1 \xi_2 - 2 h_2 \xi_1 - c_3, \\ \vdots \\ h_{k-1} = \dots, \end{array} \right.$$

from the values $h_1 = c_1 \xi_1 - c_2$, if we choose for example $f_1 = -g_1 + \frac{c_2}{c_1}$, and then substituting the value of f_1 in equation (4.38). We obtain a term which does not contain g_i . Thus, later on we can not find the change of coordinates (it obtains contradiction which was showed in Theorem 10). Again, we must find the value of f_i or g_i in the end of the process.

However, if $c_k + b_k = 0, \forall k$, we obtain the following normal form

$$\dot{x} = \lambda x, \quad \dot{Y} = Y(\mu + F(U)), \quad \dot{Z} = Z(\nu - F(U)),$$

which is equivalent to the system (4.4) and then the condition $b_k + c_k = 0, \forall k$ corresponding to system (4.4) is equal to the condition $\hat{a}_k + b_k + c_k = 0$ corresponding to the system (4.3). In fact, we have already given this type of normal form in the previous chapter.

In the following case, we assume that $b_i = c_i = 0$, for $i = 1, 2, \dots, k-1$ we clearly have $b_i + c_i = 0$, in this case we have the following theorem

Corollary 1. Given an orbitally normalizable system (4.4) with rank-one resonance satisfying the condition $\lambda + \mu + \nu = 0$. If $b_i, c_i = 0$ for $i = 1, 2, \dots, k-1$ and $b_k, c_k \neq 0$. Then by an analytic change of coordinates (4.6), we can bring the system (4.4) to a reduce normal form

$$\dot{x} = \lambda x, \quad \dot{Y} = Y\left(\mu + \frac{b U^k}{1 + a U^k}\right), \quad \dot{Z} = Z\left(\nu + \frac{c U^k}{1 + a U^k}\right), \quad (4.41)$$

Proof. Its proof is similar to Theorem 11, by putting only $F(U) = 0$. \square

The following case is not the general case, but will help in the understanding of the general case.

Theorem 12. Given an orbitally normalizable system (4.4) with rank-one resonant eigenvalues satisfying the condition $\lambda + \mu + \nu = 0$. Assume $b_1 = 0$ and $c_1, b_2 \neq 0$, then by an analytic change of coordinates (4.6), we can bring the system (4.4) to a reduce normal form

$$\dot{x} = \lambda x, \quad \dot{Y} = \mu Y, \quad \dot{Z} = Z \left(\nu + \frac{cU}{1 + aU} \right), \quad (4.42)$$

for some $a, c \in \mathbb{C}$.

Proof. We suppose $c_1 \neq 0$, then from (4.8) \dot{u} becomes

$$\dot{u} = c_1 u^2 \left(1 + \frac{\hat{a}_2}{c_1} u^2 + \frac{\hat{a}_3}{c_1} u^3 + \cdots \right) =: G(u). \quad (4.43)$$

By the same way that we showed in Theorem 9, we can substitute (4.4), (4.42) and (4.43) into the change of Y -coordinate and Z -coordinates, respectively, to obtain

$$Q(u) + \varphi' G(u) = (b_2 u + \cdots + (f_1 + 2f_2 u + \cdots)(c_1 u^2 + \hat{a} u^3 + \cdots)) = 0, \quad (4.44)$$

and

$$\frac{cu e^{\varphi(u) + \psi(u)}}{1 + au e^{\varphi(u) + \psi(u)}} = R(u) + \psi' G(u). \quad (4.45)$$

We consider the Taylor expansion, and substituting $f_1 = -\frac{b_2}{c_1}$. The equations (4.44) and (4.45) become

$$\left(\frac{\hat{a}_2 b_2}{c_1} - 2c_1 f_2 - b_3\right)u^3 + \left(-2\hat{a}_2 f_2 + \frac{\hat{a}_3 b_2}{c_1} - 3c_1 f_3 - b_4\right)u^4 + \dots = 0, \quad (4.46)$$

and

$$\begin{aligned} & (c - c_1)u + \left(-\frac{cb_2}{c_1} - ca - c_2\right)u^2 + \left(\frac{cb_2^2}{2c_1^2} + \frac{c(b_2 a_2 - c_2)a}{c_1}\right. \\ & + \left(ca + \frac{cb_2}{c_1}\right)a - c_3 + cf_2 + \left(-\frac{cb_2}{c_1} - 2ca - \hat{a}_2\right)g_1 + (c - 2c_1)g_2 + \frac{cg_1^2}{2}u^3 \\ & + \dots = 0. \end{aligned} \quad (4.47)$$

We solve the first and second terms of equations (4.46) and (4.47) in u , respectively, we obtain

$$\begin{aligned} c = c_1, \quad a = -\frac{\hat{a}_2}{c_1}, \quad f_2 &= \frac{\hat{a}_2^2 - \hat{a}_2 c_2 - \hat{a}_3 c_1 + c_1 c_3}{2c_1^2}, \\ g_2 &= \frac{\hat{a}_2 c_2 - \hat{a}_3 c_1 - c_1 c_3 + c_2^2}{2c_1^2} + \frac{c_2 g_1}{c_1} + \frac{g_1^2}{2}. \end{aligned}$$

And hence

$$\xi_2 = f_2 + g_2 = \frac{\hat{a}_2^2 - 2\hat{a}_3 c_1 + c_2^2}{2c_1^2} + \frac{c_2 g_1}{c_1} + \frac{g_1^2}{2}.$$

To find for further terms, we can continue to compare powers of u in the equations (4.46) and (4.47). We can find the coefficients ξ_i which is determined term by term.

In more detail, we want to show that $K(u)$ in the change of coordinate $U = ue^{K(u)}$ exists and is analytic. From the system (4.43), we see that

$$\dot{u} = c_1 u^2 (1 + \tilde{a}u + o(u^2)),$$

where $\tilde{a} = \frac{\hat{a}_2}{c_1}$ is a formal orbital invariant. Since we have $c = c_1$, thus we need to solve

$$\frac{dU}{du} = \frac{cU^2(1 + aU)^{-1}}{c_1 u^2(1 + \tilde{a}u + o(u^2))},$$

which simplifies to

$$\left(\frac{1}{U^2} + \frac{a}{U} \right) dU = \left(\frac{1 - \tilde{a}u}{u^2} + \frac{\tilde{a}^2 + o(u)}{1 + \tilde{a}u + o(u^2)} \right) du,$$

Since we have $a = \frac{-\hat{a}_2}{c_1} = -\tilde{a}$, and by integrating, we obtain

$$\frac{-1}{U} - \tilde{a} \ln U = \frac{-1}{u} - \tilde{a} \ln u + o(u^0),$$

and using $U = ue^{K(u)}$, gives

$$\frac{1}{u}(1 - e^{-K}) - \tilde{a}K - (u^0) = 0,$$

multiplying by u , gives

$$F(u, K(u)) = 1 - e^{-K} - \tilde{a}uK - o(u) = 0.$$

Solving the above equation is equivalent to solving $F(u, K(u)) = 0$. Clearly $F(0, 0) = 0$, and

$$\frac{\partial F}{\partial K} = e^{-K} - \tilde{a}u,$$

this implies that $\frac{\partial F}{\partial K}(0, 0) = 1$, thus giving $K(u)$ as an analytic solution to $F(u, K(u)) = 0$ by the implicit function theorem.

Also, we want to show that (by the same way shown in Theorem 9) each individual equations (4.46) and (4.47) has solution for each individual term with respect to φ and ψ , respectively. From equation (4.45), we see that

$$\psi' = \left(\frac{c u e^{K(u)}}{1 + a u e^{K(u)^2}} - R(u) \right) G(u)^{-1},$$

by integrating with respect to u and considering the Taylor expansion, and sub-

stituting the values of c and a , we obtain

$$\psi = \int \left(\frac{c_1 g_1 + b_2}{c_1} + o(u) \right) du = \int (g_1 + o(u)) du,$$

which is analytic in u and gives the change of coordinate $Z = y e^{\varphi(u)}$. Consequently, the change of Z -coordinate also exists by $\varphi(u) = K(u) - \psi(u)$. \square

In the following, we want to prove the case when k and l , have different values. We have already considered $k = 2$, $l = 1$, and now adapt this for $k > 3$ and $l = 1$, before talking the general case. This is because this case contain some different features than the case in Theorem 12.

Theorem 13. Given an orbitally normalizable system (4.4) with rank-one resonant eigenvalues satisfying the condition $\lambda + \mu + \nu = 0$. Assume $b_i = 0$ for $i = 1, 2, \dots, k-1$ for $k \geq 2$ and $c_1, b_k \neq 0$, then by an analytic change of coordinates (4.6), we can bring the system (4.4) to a reduce normal form

$$\dot{x} = \lambda x, \quad \dot{Y} = \mu Y, \quad \dot{Z} = Z \left(\nu + \frac{cU}{1 + aU} \right), \quad (4.48)$$

for some $a, c \in \mathbb{C}$.

Proof. For $k = 2$, is the same as Theorem 12. Let $k = 3$, then \dot{u} becomes

$$\dot{u} = u(Q(u) + R(u)) = c_1 u^2 + \dots + c_{k-1} u^{k-1} + \hat{a}_k u^k + \dots =: G(u). \quad (4.49)$$

After substituting (4.4), (4.48) and (4.49) into the change of Y -coordinate and Z -coordinates, respectively, we obtain

$$Q(u) + \varphi' G(u) = 0, \quad (4.50)$$

and

$$\frac{c u e^{\varphi(u)+\psi(u)}}{(1 + a u e^{\varphi(u)+\psi(u)})} = R(u) + \psi' G(u). \quad (4.51)$$

where $U = u e^{K(u)}$, and $K(u) = \varphi(u) + \psi(u) = \sum_{j \geq 1} (f_j + g_j) u^j = \sum_{j \geq 1} \xi_j u^j$.

We want to show that these equations can be solved for same values a and c . Now, we need to find the value of c . We sum (4.57) and (4.58) to obtain

$$\frac{c U}{1 + a U} = Q(u) + R(u) + G(u) K'. \quad (4.52)$$

Let $\tilde{G}(u) = Q(u) + R(u)$, this gives

$$\tilde{G}(u) = \frac{G(u)}{u} = u \left(1 + \frac{c_2}{c_1} u + \cdots + \frac{c_{k-1}}{c_1} u^{k-1} + \frac{\hat{a}_k}{c_1} u^k + \cdots \right),$$

and using $U = u e^{K(u)}$, equation (4.52) gives

$$\begin{aligned} c U (1 + a U)^{-1} &= \tilde{G}(u) (1 + u K') \\ c u e^{K(u)} (1 + a u e^{K(u)})^{-1} &= c_1 u \left(1 + \frac{c_2}{c_1} u + \cdots + \frac{c_{k-1}}{c_1} u^{k-1} + \frac{\hat{a}_k}{c_1} u^k + \cdots \right) (1 + u K'), \end{aligned}$$

and hence $c = c_1$.

We consider the Taylor expansion. Equations (4.50) and (4.51) become

$$\begin{aligned} &- c_2 u^3 f_1 + (-2 c_2 f_2 - c_3 f_1 - b_4) u^4 + (-\hat{a}_4 f_1 - 3 c_2 f_3 - 2 c_3 f_2 - b_5) u^5 \\ &+ \cdots = 0, \end{aligned} \quad (4.53)$$

and

$$\begin{aligned} &((-a + f_1) c_1 - c_2) u^2 + \left(\left(\frac{g_1^2}{2} + (f_1 - 2a) g_1 - g_2 + \frac{f_1^2}{2} + \right. \right. \\ &\left. \left. a^2 + f_2 - 2 t f_1 \right) c_1 - c_2 g_1 - c_3 \right) u^3 + \cdots = 0. \end{aligned} \quad (4.54)$$

Solving the terms in u of the equations (4.53) and (4.54) in u , respectively, we

obtain

$$a = -\frac{c_2}{c_1}, \quad f_1 = 0, \quad f_2 = 0, \quad f_3 = \frac{c_4 - \hat{a}_4}{3c_1},$$

$$g_1 = \frac{-c_2 + \sqrt{2c_1^2 g_2 + 2c_1 c_3 - c_2^2}}{c_1}.$$

In more detail, we want to show that $K(u) = \sum_{i \geq 1} \xi_i u^i$ in the change of coordinate $U = ue^{K(u)}$ exists and is analytic. From the system (4.49), we see that

$$\dot{u} = c_1 u^2 \left(1 + \tilde{a} u + \frac{\hat{a}_3}{c_1} u^2 + o(u^2)\right),$$

where $\tilde{a} = \frac{c_2}{c_1}$ is a formal orbital invariant. Since we have $c = c_1$, then we need to solve

$$\frac{dU}{du} = \frac{c U^2 (1 + a U)^{-1}}{c_1 u^2 (1 + \frac{c_2}{c_1} u + \dots + o(u^2))}.$$

This gives

$$\frac{(1 + a U) dU}{U^2} = \frac{du}{u^2 (1 + \frac{c_2}{c_1} u + o(u^2))}.$$

which simplifies to

$$\left(\frac{1}{U^2} + \frac{a}{U}\right) dU = \left(\frac{1 - \frac{c_2}{c_1} u}{u^2} - \frac{\frac{c_2}{c_1} + o(u)}{1 + \frac{c_2}{c_1} u + o(u^2)}\right) du.$$

Since we have $a = -\frac{c_2}{c_1}$, and by integrating gives

$$\frac{-1}{U} + a \ln U = \frac{-1}{u} - \frac{c_2}{c_1} \ln u + o(u^0),$$

and using $U = ue^{K(u)}$, gives

$$\frac{1}{u} (1 - e^{-K}) + a K - o(u^0) = 0,$$

multiplying by u , gives

$$F(u, K(u)) = 1 - e^{-K} + auK + o(u) = 0.$$

Solving the above equation is equivalent to solving $F(u, K(u)) = 0$, where $K(u) = \varphi + \psi$. Clearly, $F(0, 0) = 0$, and

$$\frac{\partial F}{\partial K} = e^{-K} + au,$$

this implies that

$$\frac{\partial F}{\partial K}(0, 0) = 1,$$

thus giving $K = K(u)$ as an analytic solution to $F(u, K(u)) = 0$ by the implicit function theorem.

Also, we want to show that each equations (4.50) and (4.51) has solution for each individual term with respect to φ and ψ , respectively. From equation (4.51), we see that

$$\psi' = \left(\frac{c u e^{K(u)}}{(1 + a u e^{K(u)})} - R(u) \right) G(u)^{-1},$$

by integrating with respect to u and considering the Taylor expansion with substituting $c = c_1$ and $a = -\frac{c_2}{2c_1}$, we obtain

$$\psi(u) = \int \left(-\frac{c_2(c_1 t - c_1 f_1 - g_1 c_1 + c_2)}{c_1^2} + o(u) \right) du.$$

It is clear that the above integral exists and is analytic in giving the change of coordinate $Z = y e^{\varphi(u)}$. Consequently, the change of Y -coordinate also exists by $\psi(u) = K(u) - \varphi(u)$.

For $i = k - 1$, by which we mean that $b_1, \dots, b_{k-1} = 0$ and $c_1, b_k, \dots \neq 0$, such that $k > 3$. To prove this case, we only need the power series $\varphi(u) = \sum_{j \geq k-1} f_j u^j$, and the rest is the same as the above prove, where in this case $f_{k-1} = -\frac{b_k}{(k-1)c_1}$. \square

Theorem 14. Given an orbitally normalizable system (4.4) with rank-one resonance satisfying the condition $\lambda + \mu + \nu = 0$. If there are $l, k \in \mathbb{N}$ for $k > l$ with $b_k, c_l \neq 0$ and $c_1 = c_2 = \dots = c_{l-1} = b_1 = b_2 = \dots = c_{k-1} = 0$. Then, by an analytic change of coordinates (4.6), we can bring the system (4.4) to a reduce normal form

$$\dot{x} = \lambda x, \quad \dot{Y} = \mu Y, \quad \dot{Z} = Z \left(\nu + \frac{c U^l}{1 + a U^l} \right), \quad (4.55)$$

Proof. Since $k > l$, then there is $i \in \mathbb{N}$ such that $k = l + i$. Here \dot{u} becomes

$$\begin{aligned} \dot{u} &= u(Q(u) + R(u)) = c_l u^{l+1} \left(1 + \frac{c_{l+1}}{c_l} u + \dots + \frac{\hat{a}_k}{c_l} u^{k-l-1} + \frac{\hat{a}_{k+1}}{c_l} u^{k-l} + \dots \right) \\ &= c_l u^{l+1} \left(1 + \sum_l^{l+i-1} \frac{c_{l+1} u^l}{c_l} + \sum_k \frac{\hat{a}_k u^k}{c_l} \right) =: G(u). \end{aligned} \quad (4.56)$$

where $\hat{a}_k = b_k + c_k$ for $k = 3, 4, \dots$, and $Q(u) + R(u) = \frac{G(u)}{u} =: \tilde{G}(u)$

By the same way that we showed in Theorem 9, we can substitute the systems (4.4), (4.55) and (4.56) into the change of Y -coordinate and Z -coordinates, respectively, to obtain

$$Q(u) + G(u)\varphi' = 0, \quad (4.57)$$

and

$$\frac{c U^l}{1 + a U^l} = R(u) + G(u)\psi', \quad (4.58)$$

where $U = u e^{K(u)}$, and $K(u) = \varphi(u) + \psi(u) = \sum_{j \geq 1} (f_j + g_j) u^j = \sum_{j \geq 1} \xi_j u^j$. We want to show that these equations can be solved for same values a and c .

Now, we need to find the value of c . We sum (4.57) and (4.58) to obtain

$$\frac{c U^l}{1 + a U^l} = Q(u) + R(u) + G(u) K', \quad (4.59)$$

and using $U = u e^{K(u)}$, this gives

$$c U^l (1 + a U^l)^{-1} = \tilde{G}(u)(1 + u K')$$

$$c u^l e^{l K(u)} (1 + a u^l e^{l K(u)})^{-1} = c_l u^l \left(1 + \sum_l^{l+i-1} \frac{c_{l+1} u^l}{c_l} + \sum_k \frac{\hat{a}_k u^k}{c_l} \right) (1 + u K'),$$

and hence $c = c_l$.

To find the value of a , we need to solve the

$$\frac{dU}{du} = \frac{c U^{l+1} (1 + a U^l)^{-1}}{c_l u^{l+1} \left(1 + \frac{c_{l+1}}{c_l} u + \dots + \frac{\hat{a}_k}{c_l} u^{k-l-1} + o(u^{k-l}) \right)}.$$

This gives

$$\frac{1 + a U^l dU}{U^{l+1}} = \frac{du}{u^{l+1} \left(1 + \frac{c_{l+1}}{c_l} u + o(u^2) \right)}.$$

which simplifies to

$$\left(\frac{1}{U^{l+1}} + \frac{a}{U} \right) dU = \left(\frac{1 + \sum_{i \geq 1}^l \hat{a}_i u^i}{u^{l+1}} + \frac{a^k + o(u)}{1 + \frac{c_{l+1}}{c_l} u^l + o(u^2)} \right) du. \quad (4.60)$$

where \hat{a}_i are polynomials of the variables $\frac{c_{l+1}}{c_l}, \frac{c_{l+2}}{c_l}, \dots, \frac{\hat{a}_k}{c_l}$ as follows $\hat{a}_1 = -\frac{c_{l+1}}{c_l}$, $\hat{a}_2 = \frac{-c_{l+2} \hat{a}_1}{c_l} - \frac{c_{l+1}}{c_l}$, and $\hat{a}_3 = \frac{-c_{l+1} \hat{a}_2}{c_l} - \frac{c_{l+2} \hat{a}_1}{c_l} - \frac{c_{l+3}}{c_l}$. In particular, we must have $a = a_l$.

Using $U = u e^K$, equation (4.60) becomes

$$\frac{l}{u} (1 - e^{-l K(u)}) + (a - a_l) \ln(u e^K) - \sum_{i \geq 1}^j \frac{a_i u^{-l-1+i}}{-l + i - 1} - o(u^0) = 0,$$

multiplying by u^l , gives

$$F(u, K(u)) = l(1 - e^{-l K(u)}) - \sum_{i \geq 1}^j \frac{a_i u^{-1+i}}{-l + i - 1} - o(u^l) = 0.$$

Solving the above equation is equivalent to solving $F(u, K(u)) = 0$. However, since $F(0, 0) = 0$, and

$$\frac{\partial F}{\partial K} = l^2 e^{-l K(u)},$$

this implies that $\frac{\partial F}{\partial K}(0,0) = l^2 \neq 0$, which gives $K(u)$ as an analytic solution to $F(u, K(u)) = 0$ by the implicit function theorem.

Also, each equation (4.57) and (4.58) is solvable with respect to power series φ and ψ , respectively. From equation (4.58) and (4.56) we have

$$\varphi' = \frac{c u^l e^{lK(u)} (1 + a u^i e^{lK(u)})^{-1} - c_l u^l (1 + \frac{c_{l+1} u^1}{c_l} + \dots)}{c_l u^{l+1} (1 + \frac{c_{l+1} u}{c_l} + \dots)},$$

this gives

$$\varphi' = \frac{e^{lK(u)} (1 + a u e^{K(u)})^{-1} - (1 + \frac{c_{l+1} u^1}{c_l} + \dots)}{u (1 + \frac{c_{l+1} u}{c_l} + \dots)},$$

We consider the Taylor expansion, and simplifying to remove the first term for the above equation which gives

$$\varphi' = P(a, c, f_i, g_i) + o(u), \quad (4.61)$$

where P is a polynomial for parameters a, c and f_l, g_l , where f_l, g_l are coefficients in φ, ψ . By integrating with respect to u on (4.61), we obtain

$$\varphi = \int (P(a, c, f_i, g_i) + o(u)) du.$$

It is clear that the above integral exists and is analytic in giving the change of coordinate $Y = y e^{\varphi(u)}$. Consequently, the change of Z -coordinate also exists by $\varphi(u) = K(u) - \psi(u)$. \square

Now, we want to show that the system (4.7) with conditions $b_1, c_1 \neq 0$ and $b_1 + c_1 \neq 0$ has two independent first integrals. Firstly, by multiplying the system

(4.7) by $1 + aU$, we obtain the following system

$$\begin{aligned}\dot{x} &= x(\lambda + \lambda a U), \\ \dot{Y} &= Y(\mu + (\mu a + b) U), \\ \dot{Z} &= Z(\nu + (\nu a + c) U),\end{aligned}\tag{4.62}$$

which are analytically equivalent. The above system corresponds to putting the 2-form into reduced normal form.

Corollary 2. Assume $b_1, c_1, b_1 + c_1 \neq 0$, then the system (4.62) with rank-one eigenvalues has one Darboux-analytic first integral and one explicit first integral given in terms of logarithm function.

Proof. System (4.62) had the following Darboux factors and cofactors:

$$\begin{aligned}G_1(x, Y, Z) &= x, & K_1(x, Y, Z) &= (a\lambda U + \lambda), \\ G_2(x, Y, Z) &= Y, & K_2(x, Y, Z) &= (\mu + (\mu a + b) U), \\ G_3(x, Y, Z) &= Z, & K_3(x, Y, Z) &= (\nu + (\nu a + c) U),\end{aligned}$$

by the Darboux method, we have the following first integral

$$H_1(x, Y, Z) = x^\alpha Y^\beta Z,$$

where $\alpha = \frac{c\mu - b\nu}{b\lambda}$ and $\beta = \frac{-c}{b}$. We write the divergence of the system (4.62) as

$$\text{div} = 2(b + c)U,$$

and seek for an IJM, $M = x^r Y^s Z^t$. Then, we want to solve the equation

$$\chi(M) = M \text{div}(\chi).$$

This gives

$$(r\lambda + s\mu + t\nu + (r\lambda a + s\mu a + sb + tva + tc)U)M = 2M(b+c)U,$$

which gives

$$r\lambda + s\mu + t\nu = 0,$$

and

$$(r\lambda + s\mu + t\nu)a + (s-2)b + (t-2)c = 0,$$

we can choose $r = s = t = 2$. Hence, $M = U^2$. By using Theorem 6, the system has another first integral which is of the following

$$H_2 = \frac{1}{U} + \epsilon_1 \ln x + \epsilon_2 \ln Y,$$

where $\epsilon_1 = \frac{(a\mu+b)(b+c)}{b\lambda}$ and $\epsilon_2 = -\frac{a(b+c)}{b}$. □

The more general cases in Theorem 8 also have explicit first integrals, but we do not obtain a first integral of Darboux-analytic type. So, here we have only mentioned the above generic case.

4.2.1 Normalizability and orbital normalizability of critical points

In this section, we want to try to identify normalizability of the 3D system from its first integrals, we give the details for this in Theorem 15. Also we want to find different criteria for bringing an orbital normalizable system to normalizable system see Theorem 16.

At the first, we give some results on orbitally normalizable systems in the form (4.1). We consider only the generic case $b_1, c_1, b_1 + c_1 \neq 0$, and we look at the

relationship between orbitally normalizable and normalizability for the system (4.1) under these conditions.

After multiplying the system (4.7) by $1 + au$, and consider the case $b_1, c_1, b_1 + c_1 \neq 0$. Then, by Theorem 9 via an analytic change of coordinates, an orbitally normalizable system (4.1) can be brought into the following form

$$\begin{aligned}\dot{x} &= x(\lambda + \lambda a u)h(x, y, z), \\ \dot{y} &= y(\mu + (\mu a + b)u)h(x, y, z), \\ \dot{z} &= z(\nu + (\nu a + c)u)h(x, y, z),\end{aligned}\tag{4.63}$$

where $h(x, y, z)$ is an analytic and $h(0, 0, 0) = 1$. Furthermore, the system (4.63) has two independent first integral of the form

$$H_1 = x^\alpha y^\beta z, \quad H_2 = \frac{1}{xyz} + \epsilon_1 \ln x + \epsilon_2 \ln y,\tag{4.64}$$

where $\alpha = \frac{c\mu - b\nu}{b\lambda}$, $\beta = \frac{-c}{b}$, $\epsilon_1 = \frac{(a\mu + b)(b + c)}{b\lambda}$ and $\epsilon_2 = -\frac{a(b + c)}{b}$.

To prove the existence of these first integrals, we note that the system (4.63) has the following Darboux factors and cofactors,

$$\begin{aligned}G_1(x, y, z) &= x, & K_1(x, y, z) &= (a\lambda u + \lambda) h(x, y, z), \\ G_2(x, y, z) &= y, & K_2(x, y, z) &= (\mu + (\mu a + b)u) h(x, y, z), \\ G_3(x, y, z) &= z, & K_3(x, y, z) &= (\nu + (\nu a + c)u) h(x, y, z).\end{aligned}$$

By the Darboux method, we have the following first integral

$$H_1(x, y, z) = x^\alpha y^\beta z^\gamma,$$

where $\alpha = \frac{c\mu - b\nu}{b\lambda}$, $\beta = \frac{-c}{b}$ and $\gamma = 1$. We write the divergence of the system

(4.63) as

$$\operatorname{div} = 2 (b + c) u h(x, y, z),$$

then taking $M = x^r y^s z^t$, and using $\chi(M) = M \operatorname{div}(\chi)$, we see that

$$(r\lambda + s\mu + t\nu + (r\lambda a + s\mu a + sb + t\nu a + tc)u)M h(x, y, z) = 2 M (b + c) u h(x, y, z).$$

which gives

$$r\lambda + s\mu + t\nu = 0, \quad \text{and} \quad (r\lambda + s\mu + t\nu)a + (s - 2)b + (t - 2)c = 0.$$

Taking $r = s = t = 2$, we obtain $M = u^2$. Then, by using Theorem 6, we can find a second first integral which is of the following form

$$H_2 = \frac{1}{u} + \epsilon_1 \ln x + \epsilon_2 \ln y,$$

where $\epsilon_1 = \frac{(a\mu+b)(b+c)}{b\lambda}$ and $\epsilon_2 = -\frac{a(b+c)}{b}$.

The second first integral will here be more convenient to work in the form $\frac{1}{H_2}$ which is of the form:

$$H_2 = \frac{u}{1 + \epsilon_1 u \ln x + \epsilon_2 u \ln y}.$$

Theorem 15. Assume that $\alpha, \beta, \epsilon_1, \epsilon_2 \neq 0$. If the system (4.1) with rank-one resonant eigenvalues satisfying the condition $\lambda + \mu + \nu = 0$ has two independent first integrals of the following form

$$\begin{aligned} H_1(x, y, z) &= x^\alpha y^\beta z h(x, y, z), \\ H_2(x, y, z) &= \frac{u}{F(x, y, z) + \epsilon_1 u \ln x + \epsilon_2 u \ln y}, \end{aligned} \tag{4.65}$$

where $u = xyz$ and $F(0), h(0) = 1$ are analytic functions, then the system (4.1)

is orbitally normalizable.

Proof. We assume that $b_1, c_1, b_1 + c_1 \neq 0$, then we consider an analytic transformation $(x, Y, Z) = (x, y e^{\frac{\varphi}{\beta}}, z e^{\psi})$ to bring (4.65) into the first integrals (4.64), respectively. The first one becomes

$$\tilde{H}_1(x, Y, Z) = x^\alpha Y^\beta Z \tilde{h}(x, Y, Z) e^{\frac{-\varphi}{\beta} - \psi} = x^\alpha Y^\beta Z \tilde{h}(x, Y, Z) e^{K(x, Y, Z)}. \quad (4.66)$$

$\tilde{h}(x, Y, Z) = 1 + o(x, Y, Z)$ and $K = -\frac{\varphi(x, Y, Z)}{\beta} - \psi(x, Y, Z)$. Now we want to show that $\tilde{h}(x, Y, Z) e^{K(x, Y, Z)} = 1$.

Firstly, we want to prove that there is an analytic $K(x, Y, Z)$ in the change of coordinates. That is, we want to solve

$$G(x, Y, Z, K) = h(x, Y, Z) e^K - 1 = 0.$$

Clearly,

$$G(0) = h(0) e^{K(0)} - 1 = 1 - 1 = 0,$$

and

$$\frac{\partial G}{\partial K} = h e^K,$$

this implies that

$$\frac{\partial G}{\partial K}(0) = 1,$$

thus giving K as the analytic solution to $G(x, Y, Z, K) = 0$ by the implicit function theorem.

By using the same change of coordinates $u = xyz = U e^K$, where $U = xYZ$, the second first integral becomes

$$\tilde{H}_2 = \frac{U e^K}{F(x, Y, Z) + \epsilon_1 U e^K \ln(x) + \epsilon_2 U e^K \ln(Y e^{-\frac{\varphi}{\alpha}})},$$

which simplifies to

$$\tilde{H}_2 = \frac{U}{F(x, Y, Z) e^{-K} - \epsilon_2 U \frac{\varphi}{\alpha} + \epsilon_1 U \ln(x) + \epsilon_2 U \ln(Y)}.$$

To bring up the above form into the second first integral (4.64), we can bring it to the following form

$$\tilde{H}_2 = \frac{U}{1 + \epsilon_1 U \ln(x) + \epsilon_2 U \ln(Y)}. \quad (4.67)$$

Since, we have an analytic power series $K = -\frac{\varphi(x, Y, Z)}{\beta} - \psi(x, Y, Z)$, we only need to solve the following equation

$$G_1(x, Y, Z, \varphi, \psi) = F(x, Y, Z) e^{-\frac{\varphi}{\beta} - \psi} - \epsilon_2 U \frac{\varphi}{\beta} - 1 = 0.$$

Clearly,

$$G_1(0) = F(0) e^{K(0)} - 0 - 1 = 0,$$

and

$$\frac{\partial G_1}{\partial \varphi} = \frac{-1}{\beta} F e^{-\frac{\varphi}{\beta} - \psi} - \frac{\epsilon_2}{\beta} U,$$

this implies that

$$\frac{\partial G_1}{\partial \varphi}(0) = \frac{-1}{\beta} \neq 0,$$

thus giving φ as the analytic solution to $G_1(x, Y, Z, \varphi, \psi) = 0$ by the implicit function theorem. Therefore, the change of Y -coordinate exists and is analytic. Consequently, the change of Z -coordinate also exists by $\psi = -(K + \frac{\varphi}{\beta})$.

Therefore, if we have the two first integrals (4.67) and (4.66), we should have a system like (4.64). \square

In the case of 2D system, it is known that an orbitally normalizable system

can be brought to the following form

$$\begin{aligned}\dot{x} &= x(1 + au^k)h(x, y), \\ \dot{y} &= -\frac{p}{q}y(1 + (a-1)u^k)h(x, y).\end{aligned}$$

Christopher et al. (2003) considered that an orbital normalizable system is normalizable to the resonant model

$$\begin{aligned}\dot{X} &= X(1 + aU^k), \\ \dot{Y} &= -\frac{p}{q}Y(1 + (a-1)U^k),\end{aligned}$$

where $U = X^pY^q$ is the resonant monomial. The solving of this problem, it was given by relative exactness in 1-form

$$\Omega = \frac{p}{q}y(1 + (a-1)u^k) dx + x(1 + au^k) dy.$$

Here, we generalize the result in Theorem 9 to normalizable system, in which we mean that the conditions $b, c, b + c \neq 0$ hold in the system (4.63). More precisely, we study when the orbitally normalizable system (4.1) is normalizable to the reduced normal form

$$\dot{X} = X(\lambda + \lambda a U), \quad \dot{Y} = Y(\mu + (\mu a + b) U), \quad \dot{Z} = Z(\nu + (\nu a + c) U). \quad (4.68)$$

By Theorem 15, we can start with the orbitally normalizable system (4.63) which has two independent first integrals $H_{1,2}$ with an IJM, M ,

$$H_1(x, y, z) = x^\alpha y^\beta z, \quad H_2(x, y, z) = x^{\epsilon_1} y^{\epsilon_2} e^{\frac{1}{u}}, \quad M = (xyz)^2, \quad (4.69)$$

where $\alpha = \frac{c\mu - b\nu}{b\lambda}$, $\beta = \frac{c}{b}$, $\epsilon_1 = \frac{(a\mu + b)(b+c)}{b\lambda}$ and $\epsilon_2 = -\frac{a(b+c)}{b}$ such that $b + c \neq 0$ and

$b, c \neq 0$. We seek when (4.63) can be put to the normal form (4.68) such that both have the same formal invariant a, b, c and $b + c \neq 0$. Here $u = xyz$ and $U = XYZ$.

We identify the system (4.63) with the following the 2-form

$$\Omega = x(\lambda + \lambda a u) dy \wedge dz + y(\mu + (\mu a + b) u) dz \wedge dx + z(\nu + (\nu a + c) u) dx \wedge dy, \quad (4.70)$$

which gives

$$dt = \frac{dx}{x(\lambda + \lambda a u) h(x, y, z)}, \quad dt_{norm} = \frac{dx}{x(\lambda + \lambda a u)}, \quad (4.71)$$

we see that

$$\eta = dt - dt_{norm} = \frac{dx}{x(\lambda + \lambda a u)} \frac{1 - h(x, y, z)}{h(x, y, z)} = \frac{o(x, y, z) dx}{\lambda x},$$

and hence

$$\eta \wedge \Omega = \frac{1 - h}{h} dx \wedge dy \wedge dz. \quad (4.72)$$

Theorem 16. Assume $b + c \neq 0$, the system (4.68) is formally normalizable to the system (4.63) if and only if there is a germ of formal power series $g(x, y, z)$ vanishing at the origin, such that

$$((b + c)\eta - dg) \wedge \Omega = 0. \quad (4.73)$$

(If $b + c = 0$, we need an another reduced normal form instead of (4.68) which was described in previous section)

Proof. Firstly, we assume that the system (4.68) is formally normalizable. A change of coordinates preserving the orbital normal form preserves the invariant

coordinate axes, so should be in the following form

$$X = x m(x, y, z), \quad Y = y s(x, y, z), \quad Z = z r(x, y, z), \quad (4.74)$$

where $m, s, r \in \mathbb{R}[[x, y, z]]$ and $m(0), r(0), s(0) = 1$. We require that the two first integrals H_1 and H_2 in (4.69) must be preserved, that is the following relations must hold

$$\begin{aligned} H_1(x, y, z) &= \tilde{H}_1(X(x, y, z), Y(x, y, z), Z(x, y, z)), \\ H_2(x, y, z) &= \tilde{H}_2(X(x, y, z), Y(x, y, z), Z(x, y, z)). \end{aligned} \quad (4.75)$$

In fact, the above two functions in X, Y, Z -coordinates are also first integrals of the system (4.68), so the quotient of the two first integrals

$$\begin{aligned} \frac{\tilde{H}_1(X(x, y, z), Y(x, y, z), Z(x, y, z))}{H_1(x, y, z)} &= \frac{X^\alpha Y^\beta Z}{x^\alpha y^\beta z} = K_1(x, y, z), \\ \frac{\tilde{H}_2(X(x, y, z), Y(x, y, z), Z(x, y, z))}{H_2(x, y, z)} &= \frac{X^{\epsilon_1} Y^{\epsilon_2}}{x^{\epsilon_1} y^{\epsilon_2}} e^{\frac{1}{v} - \frac{1}{u}} = K_2(x, y, z) e^{\frac{1}{v} - \frac{1}{u}}, \end{aligned} \quad (4.76)$$

are also two independent first integrals of (4.63).

Since, we have the change of coordinates $X = x e^\varphi$ and $Y = y e^\psi$, then $X^\alpha = x^\alpha e^{\alpha\varphi}$, $Y^\beta = y^\beta e^{\beta\psi}$. Therefore, $\frac{\tilde{H}_1}{H_1} = e^{\alpha\varphi + \beta\psi}$ is analytic. Hence, K_1 should be analytic, and it is of the following form

$$K_1 = m^\alpha s^\beta r = 1 + o(x, y, z). \quad (4.77)$$

Also, we see that

$$\frac{X^{\epsilon_1} Y^{\epsilon_2}}{x^{\epsilon_1} y^{\epsilon_2}} = e^{\epsilon_1\varphi + \epsilon_2\psi},$$

then K_2 is analytic as well, and it is of the following form

$$K_2 = m^{\epsilon_1} s^{\epsilon_2} = 1 + o(x, y, z). \quad (4.78)$$

Putting

$$g(x, y, z) = -\frac{1}{U} + \frac{1}{u} = \frac{1}{u} \frac{(msr - 1)}{msr} = \frac{1}{u} (1 + \tilde{o}(x, y, z)), \quad (4.79)$$

where $1 + \tilde{o}(x, y, z) = \frac{msr-1}{msr} = \frac{(1+o(x,y,z)-1)}{1+o(x,y,z)}$.

If $g(x, y, z)$ is analytic, then $K_2 e^{g(x,y,z)}$ is analytic. Therefore, by taking the logarithm of H_2 in (4.76), we obtain a meromorphic first integral of the system (4.63). However, the system (4.63) is non-integrable because of existence of resonant term. This implies that any meromorphic first integral of (4.63) is trivial.

Moreover, by taking logarithm of $\frac{\tilde{H}_2}{H_2}$, we give

$$\begin{aligned} \epsilon_1 \log m + \epsilon_2 \log s + \frac{1}{u} \left(\frac{1}{msr} - 1 \right) &= \frac{1}{u} \left(u(\epsilon_1 \log m + \epsilon_2 \log s) + \frac{1}{msr} - 1 \right) \\ &= \frac{\tilde{h}(x, y, z)}{u}, \end{aligned}$$

where $\tilde{h}(0) = 0$. This implies that $\frac{\tilde{h}(x,y,z)}{u}$ is also a first integral of the system. We assume that there is a term in $\tilde{h}(x, y, z)$ which is not divisible by u . Then, we have

$$\chi\left(\frac{\tilde{h}(x, y, z)}{u}\right) = 0,$$

where χ is the vector field corresponding to the system (4.63). This gives that

$$\chi(\tilde{h}(x, y, z)) = (L_x + L_y + L_z) \tilde{h}(x, y, z),$$

where L_x , L_y and L_z are cofactors for the system (4.63). By simplifying, we obtain

$$\begin{aligned} \chi(\tilde{h}(x, y, z)) &= (\lambda + \lambda a u + \mu + (\mu a + b) u + \nu + (\nu a + c) u) h(x, y, z) \tilde{h}(x, y, z) \\ &= (b + c) u h(x, y, z) \tilde{h}(x, y, z). \end{aligned} \quad (4.80)$$

Let $msr = 1 + k(x, y, z)$, then $\frac{1-msr}{msr} = \frac{-k(x,y,z)}{1+k(x,y,z)}$. We see that

$$\begin{aligned}
 \chi(\tilde{h}) = & \left(yz(\epsilon_1 \log m + \epsilon_2 \log s) + u\left(\frac{\epsilon_1 m_x}{m} + \frac{\epsilon_2 s_x}{s}\right) + \frac{-k_x}{(1+k)^2} \right) x(\lambda + \lambda a u)h \\
 & + \left(xz(\epsilon_1 \log m + \epsilon_2 \log s) + u\left(\frac{\epsilon_1 m_y}{m} + \frac{\epsilon_2 s_y}{s}\right) + \frac{-k_y}{(1+k)^2} \right) y(\mu + (\mu a + b)u)h \\
 & + \left(xy(\epsilon_1 \log m + \epsilon_2 \log s) + u\left(\frac{\epsilon_1 m_z}{m} + \frac{\epsilon_2 s_z}{s}\right) + \frac{-k_z}{(1+k)^2} \right) z(\nu + (\nu a + c)u)h.
 \end{aligned} \tag{4.81}$$

It must have that $\frac{-k_y y\mu - k_z z\nu}{(1+k)^2}$ divisible by x . We see that

$$\frac{-k_y y\mu - k_z z\nu}{(1+k)^2} = \frac{-\mu y f'_1 - \nu z f'_2 + x\tilde{f}(x, y, z)}{(1+k)^2},$$

which gives that $-\mu y f'_1 - \nu z f'_2 = 0$. And then $\frac{-k_x}{(1+k)^2}$ should be divisible by yz , $\frac{-k_y}{(1+k)^2}$ should be divisible by xz and $\frac{-k_z}{(1+k)^2}$ should be divisible by xy . Therefore, we must have a function of the following

$$\begin{aligned}
 \frac{-k_x}{(1+k)^2} &= yz(k_1(x, y, z)), \\
 \frac{-k_y}{(1+k)^2} &= xz(k_2(x, y, z)), \\
 \frac{-k_z}{(1+k)^2} &= xy(k_3(x, y, z)),
 \end{aligned}$$

where $k_1 k_2, k_3(0) = 0$, by integrating, gives

$$\begin{aligned}
 \frac{-1}{(1+k)} &= xyz(\tilde{k}_1(x, y, z)) + f_1(y, z), \\
 \frac{-1}{(1+k)} &= xyz(\tilde{k}_2(x, y, z)) + f_2(x, z), \\
 \frac{-1}{(1+k)} &= xyz(\tilde{k}_3(x, y, z)) + f_3(y, z),
 \end{aligned}$$

where $\tilde{k}_1, \tilde{k}_2, \tilde{k}_3(0) = 0$, which gives that

$$xyz(\tilde{k}_1(x, y, z)) + f_1(y, z) - xyz(\tilde{k}_2(x, y, z)) - f_2(x, z) = 0,$$

yields

$$xyz(\tilde{k}_{1,2}(x, y, z)) + f_1(y, z) - f_2(x, z) = 0,$$

where $\tilde{k}_{1,2}(0) = 0$, it must have $f_1(y, z) - f_2(x, z)$ divisible by yz , then we have one possibility, is that, $f_1(y, z) = f_2(x, z)$, In the same way of other cases, we obtain $f_1, f_2, f_3 = 0$. Thus we should have

$$\frac{-1}{(1+k)} = xyz(\tilde{k}(x, y, z))$$

where $\tilde{k}(0) = 0$. Since, we have $m s r = 1 + k(x, y, z)$ this gives

$$\frac{1}{m s r} = -xyz(\tilde{k}(x, y, z)),$$

this implies that $\tilde{h}(x, y, z)$ should be divisible by u , and hence $e^{g(x,y,z)}$ is analytic.

We note that (4.68) and (4.63) give

$$\dot{u} = (b+c)u^2h(x, y, z), \quad \dot{U} = (b+c)U^2.$$

From (4.79) ($g(x, y, z) = -\frac{1}{U} + \frac{1}{u}$) via the above vector fields with the change of coordinates (4.74), the power series $g(x, y, z)$ should satisfies

$$\dot{g} = \frac{(b+c)U^2}{U^2} - \frac{(b+c)u^2 h(x, y, z)}{u^2} = (b+c)(1-h). \quad (4.82)$$

Here, by using the orbitally normalizable system (4.63), we have

$$\begin{aligned} \dot{g} &= \frac{\partial g}{\partial x} \dot{x} + \frac{\partial g}{\partial y} \dot{y} + \frac{\partial g}{\partial z} \dot{z} \\ &= \frac{\partial g}{\partial x} x(\lambda + \lambda a u)h + \frac{\partial g}{\partial y} y(\mu + (\mu a + b)u)h + \frac{\partial g}{\partial z} z(\nu + (\nu a + c)u)h. \end{aligned} \quad (4.83)$$

We see that

$$\begin{aligned}
 dg \wedge \Omega &= \left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz \right) \wedge \Omega \\
 &= \frac{\partial g}{\partial x} x(\lambda + \lambda a u) + \frac{\partial g}{\partial y} y(\mu + (\mu a + b) u) + \frac{\partial g}{\partial z} z(\nu + (\nu a + c) u),
 \end{aligned} \tag{4.84}$$

then, directly we obtain

$$dg \wedge \Omega = \frac{\dot{g}}{h} = \frac{(c + b)(1 - h)}{h}.$$

Therefore, the above form is equivalent to (4.72). Hence, there is a germ $g(x, y, z)$ to normalize the system.

Conversely, if there exists a germ, $g(x, y, z)$, satisfies equation (4.79) then we want to show that the system is formally normalizable.

To achieve this, we want to find the values of m , s and r in the change of coordinates (4.74). To motivate the construction, we set $K_1 = K_2 = 1$ in the equations (4.77) and (4.78), respectively. Then, equation (4.77) gives

$$m^\alpha s^\beta r = 1, \tag{4.85}$$

and equation (4.76) gives

$$m^{\epsilon_1} s^{\epsilon_2} e^{g(x, y, z)} = 1. \tag{4.86}$$

Relation (4.78) gives $ug(x, y, z) = 1 - \frac{1}{msr}$, this implies that

$$msr = (1 - ug(x, y, z))^{-1}. \tag{4.87}$$

From (4.85) we have

$$r = m^{-\alpha} s^{-\beta}. \tag{4.88}$$

We substitute (4.88) into (4.87) to give

$$m^{1-\alpha} s^{1-\beta} = (1 - ug(x, y, z))^{-1},$$

this gives

$$m = s^{\frac{\beta-1}{1-\alpha}} (1 - ug(x, y, z))^{\frac{-1}{1-\alpha}}. \quad (4.89)$$

Putting (4.89) into (4.86) ($m^{\epsilon_1} s^{\epsilon_2} = e^{-g(x, y, z)}$) which allows to find s , we see that

$$s^{\frac{(\beta-1)\epsilon_1}{1-\alpha} + \epsilon_2} = (1 - ug(x, y, z))^{\frac{\epsilon_1}{1-\alpha}} e^{-g(x, y, z)},$$

which gives

$$s = (1 - ug(x, y, z))^{\frac{\epsilon_1}{\epsilon_1(\beta-1) + \epsilon_2(1-\alpha)}} e^{-\frac{1-\alpha}{\epsilon_1(\beta-1) + \epsilon_2(1-\alpha)} g(x, y, z)}.$$

And, putting this in (4.89), this allows us to find m , which is

$$m = \left((1 - ug(x, y, z))^{\frac{\epsilon_1}{\epsilon_1(\beta-1) + \epsilon_2(1-\alpha)}} e^{-\frac{1-\alpha}{\epsilon_1(\beta-1) + \epsilon_2(1-\alpha)} g(x, y, z)} \right)^{\frac{\beta-1}{1-\alpha}} (1 - ug(x, y, z))^{\frac{-1}{1-\alpha}},$$

which simplifies to

$$m = (1 - ug(x, y, z))^{\frac{-\epsilon_2}{\epsilon_1(\beta-1) + \epsilon_2(1-\alpha)}} e^{-\frac{\beta-1}{\epsilon_1(\beta-1) + \epsilon_2(1-\alpha)} g(x, y, z)}.$$

Substituting m and s in (4.88), directly we obtain

$$r = (1 - ug(x, y, z))^{\frac{\alpha\epsilon_2 - \epsilon_1\beta}{\epsilon_1(\beta-1) + \epsilon_2(1-\alpha)}} e^{\frac{\beta-\alpha}{\epsilon_1(\beta-1) + \epsilon_2(1-\alpha)} g(x, y, z)},$$

thus we directly obtain the change of coordinates (4.74) of the following

$$\begin{aligned} X &= x (1 - ug(x, y, z))^{\frac{-\epsilon_2}{\epsilon_1(\beta-1)+\epsilon_2(1-\alpha)}} e^{-\frac{\beta-1}{\epsilon_1(\beta-1)+\epsilon_2(1-\alpha)} g(x, y, z)}, \\ Y &= y (1 - ug(x, y, z))^{\frac{\epsilon_1}{\epsilon_1(\beta-1)+\epsilon_2(1-\alpha)}} e^{-\frac{1-\alpha}{\epsilon_1(\beta-1)+\epsilon_2(1-\alpha)} g(x, y, z)}, \\ Z &= z (1 - ug(x, y, z))^{\frac{\alpha\epsilon_2-\epsilon_1\beta}{\epsilon_1(\beta-1)+\epsilon_2(1-\alpha)}} e^{\frac{\beta-\alpha}{\epsilon_1(\beta-1)+\epsilon_2(1-\alpha)} g(x, y, z)}. \end{aligned}$$

□

4.3 Using monodromy map

In this section, we introduce the monodromy map in the neighbourhood of the x -separatrix for the reduced normal form (4.7) with $b, c, b + c \neq 0$

$$\dot{x} = \lambda x, \quad \dot{y} = y \left(\mu + \frac{bu}{1+au} \right), \quad \dot{z} = z \left(\nu + \frac{cu}{1+au} \right),$$

by using the following two independent first integrals

$$\varphi(x, y, z) = x^\alpha y^\beta z, \quad \psi(x, y, z) = \frac{1}{xyz} + \epsilon_1 \ln x + \epsilon_2 \ln y,$$

where $\alpha = \frac{c\mu-b\nu}{b\lambda}$, $\beta = \frac{-c}{b}$ and $\epsilon_1 = \frac{(a\mu+b)(b+c)}{b\lambda}$, $\epsilon_2 = -\frac{a(b+c)}{b}$. This will then be related to the corresponding normal form for two-dimensional maps.

To find the monodromy map we need to consider the trajectory of the system near a closed loop in the neighbourhood of the x -separatrix. To achieve this, we take a transversal to the x -separatrix at every point of the loop and we look at the trajectories which are close to the loop, and hence intersects the trajectory.

Monodromy does not depend up to conjugation on the homotopy class of the loop if the base point is fixed, and hence a closed loop can be chosen to be of the form $x_\theta = x_0 e^{i\theta}$, $\theta \in [0, 2\pi]$ which starts at a base point $(x_0, 0, 0)$ on the x -separatrix.

We extract z from the first integral $\varphi(x, y, z) = k_1$ at the starting point

(x_0, y_0, z_0) where k_1 is constant to obtain

$$\varphi(x, y, z) = x^\alpha y^\beta z \Rightarrow z = k_1 x^{-\alpha} y^{-\beta}. \quad (4.90)$$

When $x = x_0$ and $y = y_0$, then $k_1 = z_0 x_0^\alpha y_0^\beta$. Therefore, we substitute equation (4.90) into the second first integral ($\psi(x, y, z) = k_2$) to obtain

$$\frac{1}{k_1} x^{\alpha-1} y^{\beta-1} + \epsilon_1 \ln x + \epsilon_2 \ln y = k_2. \quad (4.91)$$

Secondly, we can substitute a power series $y_\theta = \sum_{i \geq 1} c_i(\theta) y_0^i$ into equation (4.91) which gives

$$\frac{1}{k_1} (x_0 e^{2i\pi\theta})^{\alpha-1} y_\theta^{\beta-1} + \epsilon_1 \ln(x_0 e^{2i\pi\theta}) + \epsilon_2 \ln y_\theta = \frac{1}{k_1} x_0^{\alpha-1} y_0^{\beta-1} + \epsilon_1 \ln x_0 + \epsilon_2 \ln y_0,$$

this implies that

$$\frac{1}{k_1} x_0^{\alpha-1} y_\theta^{\beta-1} (e^{2i\pi\theta} - 1) + \epsilon_1 2i\pi\theta + \epsilon_2 \ln(y_\theta - y_0) = 0.$$

Since $k_0 = z_0 x_0^\alpha y_0^\beta$, this yields

$$\frac{1}{z_0 x_0^\alpha y_0^\beta} x_0^{\alpha-1} y_\theta^{\beta-1} (e^{2i\pi\theta} - 1) + \epsilon_1 2i\pi\theta + \epsilon_2 \ln(y_\theta - y_0) = 0,$$

by simplifying gives

$$\frac{1}{x_0 y_0 z_0} ((c_1 + c_2 y_0 + \dots)^{\beta-1} e^{2i\pi\theta(\alpha-1)} - 1) + \epsilon_1 2i\pi\theta + \epsilon_2 \ln(c_1 + c_2 y_0 + \dots) = 0,$$

we consider the Taylor expansion to obtain

$$\begin{aligned} & \frac{c_1^{\beta-1} e^{2i\pi\theta(\alpha-1)} - 1}{x_0 y_0 z_0} + \left(\frac{c_1^{\beta-1} c_2 e^{2i\pi\theta(\alpha-1)}}{c_1 x_0 z_0} + \epsilon_2 \ln(c_1) + 2i\pi\theta n_1 \right) + \left(\frac{\epsilon_2 c_2}{2c_1} \right. \\ & \left. + \frac{c_1^{\beta-1} (2c_3 c_1 + c_2^2 (\beta-2)) (\beta-1) e^{2i\pi\theta(\alpha-1)}}{c_1^2 x_0 z_0} \right) y_0 + \dots = 0. \end{aligned}$$

Since, we have the value of α , β , ϵ_1 and ϵ_2 , then by solving term by term for each term in y_0 with respect to the coefficient c_i in y_θ , we obtain

$$\begin{cases} c_1 = e^{2i\pi\theta\frac{\mu}{\lambda}}, \\ c_2 = \frac{2i\pi\theta b}{\lambda} x_0 z_0, \\ c_3 = -\frac{2\theta\pi b(ia\lambda + 2\pi b\theta + \pi c\theta)}{\lambda^2} (x_0 z_0)^2, \\ \vdots, \end{cases}$$

which gives

$$y_\theta = y_0 e^{2i\pi\theta\frac{\mu}{\lambda}} \left(1 + \frac{2i\pi\theta b}{\lambda} x_0 y_0 z_0 - \frac{2\theta\pi b(ia\lambda + 2\pi b\theta + \pi c\theta)}{\lambda^2} (x_0 y_0 z_0)^2 + \dots \right).$$

Thus the monodromy map of y -coordinate is given by a map y_1 ($\theta = 1$) which is of the form

$$y_1 = y_0 e^{2i\pi\frac{\mu}{\lambda}} \left(1 + \frac{2i\pi b}{\lambda} x_0 z_0 y_0 - \frac{2\theta\pi b(ia\lambda + 2\pi b + \pi c)}{\lambda^2} (x_0 y_0 z_0)^2 + \dots \right), \quad (4.92)$$

where c_i are functions of the parameters a, b and c .

Next, we assume that the monodromy map of z -coordinate is given by substituting the power series $z_\theta = e^{2i\pi\theta k} z_0 (1 + g(x_0, y_0, z_0, \theta))$ with equation (4.91) into the first integral to obtain

$$k_1 = (e^{i2\pi\theta} x_0)^\alpha (y_0 e^{\frac{2i\pi\theta\mu}{\lambda}} (1 + c_2 y_0 + \dots))^\beta e^{2i\pi\theta k} z_0 (1 + g(x_0, y_0, z_0, \theta)),$$

this implies that

$$x_0^\alpha y_0^\beta z_0 = x_0^\alpha y_0^\beta (1 + c_2 y_0 + \dots)^\beta z_0 (1 + g(x_0, y_0, z_0)) e^{i2\pi\theta\alpha} e^{\frac{2i\pi\theta\beta\mu}{\lambda}} e^{2i\pi\theta k},$$

letting $k + \alpha + \frac{\beta\mu}{\lambda} = 0$, then $k = -\frac{b\lambda+b\mu+c\mu}{b\lambda} + \frac{c\mu}{b\lambda} = -\frac{\lambda+\mu}{\lambda} = \frac{\nu}{\lambda}$, this gives

$$(1 + c_2 y_0 + \dots)^\beta (1 + g(x_0, y_0, z_0, \theta)) = 1,$$

which simplifies to give

$$g(x_0, y_0, z_0, \theta) = (1 + c_2 y_0 + \dots)^{-\beta} - 1.$$

We consider the Taylor expansion to obtain

$$g(x_0, y_0, z_0, \theta) = \frac{2i\pi\theta c}{\lambda} (x_0 y_0 z_0) - \frac{2c\pi\theta (ia\lambda + \pi\theta(b+2c))}{\lambda^2} (x_0 y_0 z_0)^2 + \dots.$$

Therefore, we have the monodromy map in z -coordinate which is given by a map z_1 of the following

$$z_1 = e^{2i\pi\frac{\nu}{\lambda}} z_0 \left(1 + \frac{-2i\pi c}{\lambda} (x_0 y_0 z_0) - \frac{2c\pi (ia\lambda + \pi(b+2c))}{\lambda^2} (x_0 y_0 z_0)^2 + \dots \right).$$

Thus, we obtain the 2D map corresponding to the RNFS (4.7) of the following

$$\begin{aligned} x_1 &= e^{2i\pi} x_0, \\ y_1 &= e^{2i\pi\frac{\mu}{\lambda}} y_0 \left(1 + \frac{2i\pi b}{\lambda} u_0 - \frac{2\pi b (ia\lambda + \pi(2b+c))}{\lambda^2} u_0^2 + \dots \right), \\ z_1 &= e^{2i\pi\frac{\nu}{\lambda}} z_0 \left(1 + \frac{2i\pi c}{\lambda} u_0 - \frac{2c\pi (ia\lambda + \pi(b+2c))}{\lambda^2} u_0^2 + \dots \right), \end{aligned} \quad (4.93)$$

where $u_0 = x_0 z_0 y_0$ is the resonant monomial. Moreover, each the resonant coefficient of this map is a polynomial of the parameters a, b and c . Hence, we can read off the terms of the normal form (4.7) from the monodromy map (4.93).

As a consequence, from the above technique we see the relation between the reduced normal form for system and the monodromy map which also gives the map in normal form.

We started with an orbital normalizable system, and then the monodromy map is clearly a normalizable map, so we can state the following theorem.

Proposition 7. If the system (4.1) is orbitally normalizable with rank-one resonant eigenvalues, then monodromy map of the any separatrix is normalizable.

In the similar way to the case of vector field, by using a further change of coordinates we can bring the map (4.93) into a reduced normal form.

Now, to simplify calculations, we work in the two variables Y and Z . Then we can rewrite the maps (4.93) in the following form

$$F = \tilde{\mu}y_0(1 + b_1u_0 + b_2u_0^2 + \dots), \quad G = \tilde{\nu}z_0(1 + c_1u_0 + c_2u_0^2 + \dots), \quad (4.94)$$

where $b_1 = \frac{2i\pi b}{\lambda}$, $c_1 = \frac{2i\pi c}{\lambda}$, $b_2 = b_1(i a + b_1 - \frac{i}{2}c_1)$, $c_2 = b_1(i a + c_1 - \frac{i}{2}b_1)$, $\tilde{\mu} = e^{2i\pi\frac{\mu}{\lambda}}$, $\tilde{\nu} = e^{2i\pi\frac{\nu}{\lambda}}$, and $u_0 = y_0z_0$ is the resonant monomial.

Since, we have $\lambda + \mu + \nu = 0$, then

$$\frac{\mu}{\lambda} + \frac{\nu}{\lambda} = -1,$$

and hence

$$\tilde{\mu} \tilde{\nu} = e^{2i\pi(\frac{\mu}{\lambda} + \frac{\nu}{\lambda})} = e^{2i\pi(-1)} = 1,$$

showing that the eigenvalues have modulus equal to unity at the origin. Thus, the map (4.94) has rank-one resonant eigenvalues, in which we mean that the product of the eigenvalues have logarithm of one independent linear dependency over \mathbb{Q} .

4.3.1 Reduction of normal form for map

In this section, we seek an invertible change of coordinates $(Y, Z) = \Omega(y_0, z_0)$ to bring the map (4.94) into a reduced map $(F_1, G_1) = \Omega^{-1} \circ (F, G) \circ \Omega$ only containing a finite number of resonant terms. Through this way, we only need the non-zero parameters a, b and c in the RNFS (4.7) to obtain a reduce map, in order to make a conclusion about the relation between the reduced normal form and the normalizable system.

Theorem 17. Assume a, b, c and $b+c$ are not equal to zero, then by an invertible change of coordinates we can bring the map (4.94) into a reduced map

$$F_1 = \tilde{\mu} y(1 + k_1(u)), \quad G_1 = \tilde{\nu} z(1 + k_2(u)), \quad (4.95)$$

where $k_1 = b_1 u_0 + b_2 u_0^2$, $k_2 = c_1 u_0 + c_2 u_0^2$ and b_1, b_2, c_1, c_2 are the same as the original map (4.94).

Proof. We seek an analytic change of coordinates in the following

$$\Omega : (y, z) \rightarrow (Y, Z) = (y e^{\phi(u)}, z e^{\psi(u)}), \quad (4.96)$$

where $\phi(u) = f_1 u + f_2 u^2 + \dots$, $\psi(u) = g_1 u + g_2 u^2 + \dots$, to bring the map (4.94) into the reduced map (4.95). To achieve this, we only need to prove that

$$\Omega^{-1} \circ (F, G) \circ \Omega = (F_1, G_1),$$

has an analytic solution. The above equation implies that

$$(F(\Omega), G(\Omega)) = \Omega(F_1, G_1),$$

this yields,

$$(F(Y, Z), G(Y, Z)) = (Y(F_1, G_1), Z(F_1, G_1)),$$

we expand out the above equation to obtain

$$F(Y, Z) - Y(F_1, G_1) = 0, \quad \text{and} \quad G(Y, Z) - Z(F_1, G_1) = 0. \quad (4.97)$$

We consider the Taylor expansion and By substitute equations (4.94), (4.95) and (4.96) into (4.97) to obtain

$$\begin{aligned} F(Y, Z) - Y(F_1, G_1) &= \tilde{\mu}Y(1 + b_1YZ + \dots) - \tilde{\mu}y(1 + b_1u + b_2u^2)(e^{\varphi(F_1, G_1)}) \\ &= \tilde{\mu}ye^{\varphi}(1 + b_1ue^{\varphi+\psi} + b_2u^2e^{2(\varphi+\psi)} + \dots). \end{aligned} \quad (4.98)$$

This implies that

$$\begin{aligned} &\tilde{\mu}y \left(1 + (b_1 + f_1)u + (b_1^2 + f_2 - (a - \frac{c_1}{2} - 2f_1 - g_1)b_1)u^2 + \dots \right) \\ &- \tilde{\mu}y \left(1 + (f_1 + b_1)u + (b_1^2 + (2f_1 - a + \frac{c_1}{2})b_1 + f_2 + c_1f_1)u^2 + \dots \right) = 0. \end{aligned}$$

By the same way, we obtain

$$\begin{aligned} G(Y, Z) - Z(F_1, G_1) &= \tilde{\nu}Y(1 + c_1YZ + \dots) - \tilde{\nu}z(1 + c_1u + c_2u^2)(e^{\psi(F_1, G_1)}) \\ &= \tilde{\nu}z(1 + (c_1 + g_1)u + (c_1^2 + g_2 - (a - f_1 - 2g_1 - \frac{b_1}{2})c_1)u^2 \\ &+ \dots). \end{aligned} \quad (4.99)$$

This implies that

$$\begin{aligned} &\tilde{\nu}z \left(1 + (c_1 + g_1)u + (c_1^2 + g_2 - (a - f_1 - 2g_1 - \frac{b_1}{2})c_1)u^2 + \dots \right) \\ &- \tilde{\nu}z \left(1 + (g_1 + c_1)u + (c_1^2 + (2g_1 - a + \frac{b_1}{2})c_1 + g_2 + b_1g_1)u^2 + \dots \right) = 0. \end{aligned}$$

Since we have $\tilde{\mu} = \frac{1}{\tilde{\nu}}$, and simplifying (4.98) and (4.99), this gives

$$\begin{aligned} \tilde{\mu} x((b_1 g_1 - c_1 f_1)u^2 - (f_1 b_1 a + 2 g_1 b_1 a - f_1 c_1 a - 2 g_1 b_1^2 + 2 b_1 f_1 c_1 - g_1 b_1 c_1 \\ - b_1 f_1^2 - 2 b_1 f_1 g_1 + f_1 c_1^2 + b_1 f_2 - b_1 g_2 + 2 f_2 c_1 - b_3)u^3 + \dots) = 0, \end{aligned} \quad (4.100)$$

and

$$\begin{aligned} \tilde{\nu} z(b_1 g_1 - f_1 c_1)u^2 - (b_1 a g_1 - 2 c_1 a f_1 - c_1 a g_1 - b_1^2 g_1 + b_1 c_1 f_1 - 2 b_1 c_1 g_1 + 2 c_1^2 f_1 \\ + 2 c_1 f_1 g_1 + c_1 g_1^2 - 2 b_1 g_2 + f_2 c_1 - c_1 g_2 + c_3)u^3 + \dots = 0. \end{aligned} \quad (4.101)$$

It is clear that each resonant coefficient contains at least one coefficient of the power series $\phi = f_1 u + f_2 u^2 + \dots$ and $\psi = g_1 u + g_2 u^2 + \dots$, respectively. Then, the coefficients f_i and g_i are determined by solving of term by term, we have

$$\begin{aligned} (b_1 g_1 - c_1 f_1) &= 0, \\ (f_1 b_1 a + 2 g_1 b_1 a - f_1 c_1 a - 2 g_1 b_1^2 + 2 b_1 f_1 c_1 - g_1 b_1 c_1 - b_1 f_1^2 - 2 b_1 f_1 g_1 \\ + f_1 c_1^2 + b_1 f_2 - b_1 g_2 + 2 f_2 c_1 - b_3) &= 0, \\ \vdots, \\ (b_1 g_1 - f_1 c_1) &= 0, \\ (b_1 a g_1 - 2 c_1 a f_1 - c_1 a g_1 - b_1^2 g_1 + b_1 c_1 f_1 - 2 b_1 c_1 g_1 + 2 c_1^2 f_1 + 2 c_1 f_1 g_1 \\ + c_1 g_1^2 - 2 b_1 g_2 + f_2 c_1 - c_1 g_2 + c_3) &= 0, \\ \vdots, \end{aligned} \quad (4.102)$$

this gives

$$\left\{ \begin{array}{l} f_1 = \frac{b_1 g_1}{c_1}, \\ f_2 = \frac{(2b_3 + c_3)b_1 + b_3 c_1}{2(b_1 + c_1)^2} - \frac{g_1 a b_1}{c_1} + \frac{g_1^2 (2b_1 + c_1)b_1}{2c_1^2}, \\ \vdots, \\ g_1 = g_1, \\ g_2 = \frac{(b_3 + 2c_3)c_1 + b_1 c_3}{2(b_1 + c_1)^2} - g_1 a + \frac{g_1^2 (b_1 + 2c_1)}{2c_1}, \\ \vdots. \end{array} \right.$$

In more detail, we want to show that each equation (4.100) and (4.101) has a solution for each individual term of the power series $\psi(u)$ and $\varphi(u)$. Firstly, we consider the Taylor expansion on (4.100), and then the coefficient for each individual term in u gives an equation with respect to f_i . Thus, (4.100) can be determined term by term. For example, if we choose a coefficient of the term u^k , we see that we have an equation of f_k with f_i for $i = 1, 2, \dots, k-1$, and then we solve this term in u^k with respect to f_k . Therefore, we can continue to solve other term has a lower degree than u^k , and so on we find all f_i in equation (4.100). Moreover, we substitute f_i in (4.101) and consider the Taylor expansion to find g_i in the each individual term in u^k by the same way. Then, equation (4.101) is solvable. Consequently, by using the power series below with the change of coordinates (4.96)

$$\begin{aligned} \varphi &= \left(\frac{b_1 g_1}{c_1}\right)u + \left(\frac{(2b_3 + c_3)b_1 + b_3 c_1}{2(b_1 + c_1)^2} - \frac{g_1 a b_1}{c_1} + \frac{g_1^2 (2b_1 + c_1)b_1}{2c_1^2}\right)u^2 + \dots, \\ \psi &= g_1 u + \left(\frac{(b_3 + 2c_3)c_1 + b_1 c_3}{2(b_1 + c_1)^2} - g_1 a + \frac{g_1^2 (b_1 + 2c_1)}{2c_1}\right)u^2 + \dots, \end{aligned}$$

the two polynomial maps (4.94) and (4.95) are formally conjugated. \square

4.4 Summary for Chapter 4

We have started with an orbital normalizable system with rank-one resonant eigenvalues which satisfies the condition $\lambda + \mu + \nu = 0$ which is one in the following form

$$\dot{x} = \lambda x + \sum_{n \geq 2} P_n(x, y, z), \quad \dot{y} = \mu y + \sum_{n \geq 2} Q_n(x, y, z), \quad \dot{z} = \nu z + \sum_{n \geq 2} R_n(x, y, z).$$

By an invertible change of coordinated $(X, Y, Z) = (x, y, z)$, we bring the above system to the following normal form

$$\dot{X} = X(\lambda + \sum_{k \geq 1} a_k u^k), \quad \dot{Y} = Y(\mu + \sum_{k \geq 1} b_k u^k), \quad \dot{Z} = Z(\nu + \sum_{k \geq 1} c_k u^k),$$

where $u = XYZ$ is the resonant monomial. After dividing the above system by $1 + \frac{1}{\lambda} \sum_{k \geq 1} a_k u^k$, without loss of generality, and to simplify calculations, we have had the following form

$$\dot{x} = \lambda x, \quad \dot{y} = y(\mu + \sum_{k \geq 1} b_k u^k), \quad \dot{z} = z(\nu + \sum_{k \geq 1} c_k u^k),$$

where $u = xyz$ is the resonant monomial. We only considered the generic case, in which we assume that $b_1, c_1, b_1 + c_1 \neq 0$. By using a further change of coordinates

$$(x, Y, Z) = (x, y e^{\varphi(u)}, z e^{\psi(u)}),$$

we could bring the above system into the reduced normal form

$$\dot{x} = \lambda x, \quad \dot{Y} = Y \left(\mu + \frac{bU}{1 + aU} \right), \quad \dot{Z} = Z \left(\nu + \frac{cU}{1 + aU} \right).$$

where $U = xYZ$, and $\varphi(u) = \sum_{j \geq 1} f_j u^j$, $\psi(u) = \sum_{j \geq 1} g_j u^j$. The above reduced normal form has the following first integrals

$$H_1(x, Y, Z) = x^\alpha Y^\beta Z, \quad H_2 = \frac{1}{U} + \epsilon_1 \ln x + \epsilon_2 \ln Y.$$

Furthermore, we have introduced the monodromy map in the neighbourhood of the one of the separatrices (we choose x -separatrix) by using the above first integrals to obtain formal normal form for 2D maps which is one in the following form

$$\begin{aligned} x_1 &= e^{2i\pi} x_0, \\ y_1 &= e^{2i\pi \frac{\mu}{\lambda}} y_0 \left(1 + \frac{2i\pi b}{\lambda} u_0 - \frac{2\pi b (ia\lambda + \pi (2b + c))}{\lambda^2} u_0^2 + \dots \right), \\ z_1 &= e^{2i\pi \frac{\nu}{\lambda}} z_0 \left(1 + \frac{2i\pi c}{\lambda} u_0 - \frac{2c\pi (ia\lambda + \pi (b + 2c))}{\lambda^2} u_0^2 + \dots \right), \end{aligned}$$

where $u_0 = x_0 z_0 y_0$ is the resonant monomial.

The above map is determined by the three invariant parameters a , b and c which appear in the reduced normal form for the system.

In the same way to the case of vector fields, by an invertible change of coordinates we could also reduce formally the above map into a reduced map

$$\begin{aligned} x_1 &= e^{2i\pi\lambda} x_0, \\ y_1 &= e^{2i\pi\mu} y_0 \left(1 + \frac{2i\pi b}{\lambda} u_0 - \frac{2\pi b (ia\lambda + \pi (2b + c))}{\lambda^2} u_0^2 \right), \\ z_1 &= e^{2i\pi\nu} z_0 \left(1 + \frac{-2i\pi c}{\lambda} u_0 + \frac{2c\pi (ia\lambda + \pi (b + 2c))}{\lambda^2} u_0^2 \right), \end{aligned}$$

which only contains the three invariant parameters a , b and c , and $u_0 = x_0 z_0 y_0$ is the resonant monomial. We can read off the terms of the reduced normal form for the system from the reduced map. In which we mean that in three dimensional system there is a relationship between the reduced normal form for the system and the reduced map.

We conclude this chapter by suggesting possible work in this area.

- Extend Theorem 17 to show analytical equivalence for a suitable choice of reduced normal form for map.
- Investigate the formal monodromy maps in the other cases of reduced normal forms in Theorem 8.

Chapter 5

Rank-Two Resonant Singularity in Three Dimensions

5.1 Introduction

In this chapter, we consider the 3D generalised Lotka-Volterra system (1.2) with resonant eigenvalues $(\lambda : \mu : \nu)$ as

$$\begin{aligned}\dot{x} &= x \left(\lambda + \sum_{\substack{i+j+k=n \\ n \geq 2}} a_{i,j,k} x^i y^j z^k \right) = x (\lambda + P(x, y, z)), \\ \dot{y} &= y \left(\mu + \sum_{\substack{i+j+k=n \\ n \geq 2}} b_{i,j,k} x^i y^j z^k \right) = y (\mu + Q(x, y, z)), \\ \dot{z} &= z \left(\nu + \sum_{\substack{i+j+k=n \\ n \geq 2}} c_{i,j,k} x^i y^j z^k \right) = z (\nu + R(x, y, z)),\end{aligned}\tag{5.1}$$

where $P, Q, R \in \mathbb{C}[x, y, z]$.

We are interested in studying the integrability of the origin when the singular point of the system (5.1) has rank-two resonant eigenvalues λ, μ and ν which lie in the Siegel domain. After a possible scaling of time, we can suppose that

$\lambda, \mu, \nu \in \mathbb{Z} \setminus \{0\}$ and $\text{g.c.d.}(\lambda, \mu, \nu) = 1$, and that the eigenvalues must not all have the same sign. Without loss of generality, we can take $\lambda, \nu > 0$ and $\mu < 0$.

To show this, we take two different linear dependencies of the eigenvalues

$$a\lambda + b\mu + c\nu = 0, \quad a'\lambda + b'\mu + c'\nu = 0,$$

and solve the above equations simultaneously to obtain

$$\mu(a b' - a' b) + \nu(a c' - c a') = 0,$$

setting $a b' - a' b = k_1$ and $a c' - c a' = k_2$, yields

$$\mu k_1 + \nu k_2 = 0,$$

where $k_1, k_2 \in \mathbb{Z} \setminus \{0\}$. If one of them is equal to zero, then one of the eigenvalues is also equal to zero which contradicts the assumption that the eigenvalues are non-zero. Also, if both k_1 and k_2 are equal to zero, then the two linear dependencies are not independent which is also a contradiction. Therefore,

$$\frac{\nu}{\mu} = \frac{k_1}{k_2} \in \mathbb{Q},$$

and in the same way, we obtain $\frac{\lambda}{\mu} \in \mathbb{Q}$. Thus after a possible scaling of time we can assume that $\lambda, \mu, \nu \in \mathbb{Z} \setminus \{0\}$.

In the following theorem, we prove that the normal form of the system (5.1) with rank-two resonant eigenvalues can be generated by two independent resonant monomials under some conditions.

Theorem 18. Suppose that the system (5.1) is normalizable at the origin with rank-two resonant eigenvalues $C = (\lambda, \mu, \nu)$, $\lambda, \mu, \nu \in \mathbb{Z} \setminus \{0\}$, such that one of

the following holds

1. If $\mu = -1$, then any resonant monomial u in normal form can be expressed as

$$u^D = x^r y^s z^t = (x y^\lambda)^\alpha (y^\nu z)^\beta \quad \text{where } \alpha, \beta \in \mathbb{Z}_{\geq 0}.$$

2. If $\lambda = -n\mu$ (or, $\nu = -n\mu$) for $n \in \mathbb{N}$, then any resonant monomial u in normal form can be expressed as

$$u^D = x^r y^s z^t = (x y^n)^\alpha (y^\nu z^{-\mu})^\beta \quad \text{where } \alpha, \beta \in \mathbb{Z}_{\geq 0}.$$

or

$$u^D = x^r y^s z^t = (x^{-\mu} y^\lambda)^\alpha (y z^n)^\beta \quad \text{where } \alpha, \beta \in \mathbb{Z}_{\geq 0}.$$

That is, there are two monomials $X^A = x^a y^b z^c$ and $X^B = x^{a'} y^{b'} z^{c'}$ such that all other terms can be constructed in the normal form from these. If the following condition holds

3. $\lambda + \mu + \nu = 0$.

Then, any resonant monomial u in normal form can be expressed as

$$u = x^r y^s z^t = (x y x)^\alpha (y^\nu z^{-\mu})^\beta, \quad \alpha, \beta \in \mathbb{Z}.$$

Proof. We bring the system (5.1) to the normal form

$$\begin{aligned} \dot{X} &= X(\lambda + P(X, Y, Z)), \\ \dot{Y} &= Y(\mu + Q(X, Y, Z)), \\ \dot{Z} &= Z(\nu + R(X, Y, Z)), \end{aligned} \tag{5.2}$$

where $P, Q, R \in \mathbb{C}[X, Y, Z]$ only contain the resonant monomials. Let

$$M = \{D = (r, s, t) \in \mathbb{Z}^3 : D \cdot C = 0\},$$

Now, we want to choose A and B with respect the eigenvalues λ, μ and ν of the following.

1. If $\mu = -1$, we can choose $A = (1, \lambda, 0)$ and $B = (0, \nu, 1)$. Then, for any $D = (r, s, t) \in M$ we have,

$$r\lambda + (-1)s + t\nu = 0,$$

this implies that $s = r\lambda + t\nu$, we see that

$$\begin{aligned} D = (r, s, t) &= (r, r\lambda + t\nu, t) = \alpha(1, \lambda, 0) + \beta(0, \nu, 1) \\ &= (\alpha, \alpha\lambda + \beta\nu, \beta), \end{aligned}$$

this gives

$$D = (r, s, t) = rA + tB.$$

Hence, any resonant monomial is of this form

$$u^D = x^r y^s z^t = (x y^\lambda)^\alpha (y^\nu z)^\beta = (v)^r (v')^t.$$

2. If $\lambda = -n\mu$, we can choose $A = (1, n, 0)$ and $B = (0, \nu, -\mu)$ such that $\gcd(\mu, \nu) = 1$. Then, any $D = (r, s, t) \in M$ we have

$$r(-n\mu) + s\mu + t\nu = 0,$$

this implies that

$$\mu(-r n + s) + t\nu = 0.$$

Since, μ is not divisible by ν , then it should be divisible by t . Let $t = \mu \tilde{t}$, since $t \geq 0$ and $\mu < 0$, then $\tilde{t} \leq 0$. From the above equation, we obtain

$$\mu(-r n + s + \tilde{t}\nu) = 0,$$

this gives $s = r n - \tilde{t}\nu$, and we see that

$$D = (r, s, t) = (r, r n - \tilde{t}\nu, \mu \tilde{t}) = \alpha(1, n, 0) + \beta(0, \nu, -\mu)$$

this gives

$$D = (r, s, t) = rA - \tilde{t}B.$$

Hence, any resonant monomial in the normal form is of this form

$$u = x^r y^s z^t = (x y^n)^\alpha (y^\nu z^{-\mu})^\beta = (v)^r (v')^{-\tilde{t}}.$$

3. If $\lambda + \mu + \nu = 0$, we can choose $A = (1, 1, 1)$ and $B = (0, \nu, -\mu)$ such that $\gcd(\mu, \nu) = 1$. Then, for any $D = (r, s, t) \in M$, we have

$$r\lambda + s\mu + t\nu = 0.$$

Let,

$$r\lambda + r\mu + r\nu = 0.$$

We subtract the last two equations to obtain

$$(s - r)\mu + (t - r)\nu = 0.$$

Since, $\gcd(\mu, \nu) = 1$, then we should have

$$(s - r) = k\nu, \text{ and } (t - r) = -k\mu, \text{ for } k \in \mathbb{Z},$$

this implies that

$$s = r + k\nu, \quad \text{and} \quad t = r - k\mu,$$

Now, for any $D = (r, s, t) \in M$, we can express it as the following

$$D = (r, s, t) = (r, r + k\nu, r - k\mu) = \alpha(1, 1, 1) + \beta(0, \nu, -\mu),$$

this gives

$$D = (r, s, t) = rA - kB.$$

Hence, any resonant monomial is of this form

$$u = x^r y^s z^t = (xyz)^\alpha (y^\nu z^{-\mu})^\beta = (v)^r (v')^{-k}.$$

Consequently, the system (5.2) directly becomes

$$\dot{X} = X(\lambda + h_1(v, v')), \quad \dot{Y} = Y(\mu + h_2(v, v')), \quad \dot{Z} = Z(\nu + h_3(v, v')),$$

where $h_i \in \mathbb{C}[v, v']$ for $i = 1, 2, 3$, and v, v' are resonant monomials as defined above. □

At this moment, we are unable to prove that $-k \in \mathbb{Z}_{\geq 0}$.

In the following example, we show that we can not choose any $A, B \in \mathbb{Z}$ such that $\alpha, \beta \in \mathbb{Z}_{\geq 0}$.

For example consider the case $C = (3 : -4 : 1)$ -resonance. Then, the normal

form corresponding to the this case is in the form

$$\begin{aligned}\dot{x} &= x(3 + a_1xyz + a_2y^1x^4 + a_3x^4y^3 + \cdots), \\ \dot{y} &= y(-4 + b_1xyz + b_2y^1x^4 + b_3x^4y^3 + \cdots), \\ \dot{z} &= z(1 + c_1xyz + c_2y^1x^4 + c_3x^4y^3 + \cdots).\end{aligned}\tag{5.3}$$

If we choose $A = (1, 1, 1)$ and $B = (0, 1, 4)$. Clearly, the vectors A and B are linearly independent. Let $D = (5, 4, 1)$, we see that $(5, 4, 1) \cdot (3, -4, 1) = 0$, and

$$(5, 4, 1) = \alpha(1, 1, 1) + \beta(0, 1, 4),$$

we get $\alpha = 5, \beta = -1 \in \mathbb{Z}$, which contradicts with the values of $\alpha, \beta \in \mathbb{Z}_{\geq 0}$.

Given the eigenvalues λ, μ and ν satisfies the first two conditions in Theorem 18, then we can rewrite the system (5.2) of the form

$$\begin{aligned}\dot{x} &= x(\lambda + \sum_{n \geq 1} P_n(u, v)) = x(\lambda + P(u, v)), \\ \dot{y} &= y(\mu + \sum_{n \geq 1} Q_n(u, v)) = y(\mu + Q(u, v)), \\ \dot{z} &= z(\nu + \sum_{n \geq 1} R_n(u, v)) = z(\nu + R(u, v)),\end{aligned}\tag{5.4}$$

where $P, Q, R \in \mathbb{C}[u, v]$, $u = X^A = x^a y^b z^c$ and $v = X^B = x^{a'} y^{b'} z^{c'}$, as shown in Theorem 18. Taking the derivative on u and v , respectively, we get

$$\dot{u} = u(\sum_{n \geq 1} H_n(u, v)) = u(H(u, v)), \quad \dot{v} = v(\sum_{n \geq 1} K_n(u, v)) = (K(u, v)),$$

where $H(u, v), K(u, v) \in \mathbb{C}[u, v]$ which depend on the eigenvalues. By which we mean that we can reduce the 3D system into 2D system.

However, the normal form (5.4) still contains an infinite number of resonant

monomials. We tried to find finite reduced normal form, but we could not find a suitable form associating to the system (5.4) at the moment. This situation is therefore much more complex than the rank-one case and so we simplify the investigation by truncating the 3D system to a 3D homogeneous cubic system as a first step to understanding the general case.

Here, we study the integrability and normalizability for the cubic system with $(1, -1, 1)$ -resonant eigenvalues. By using the Darboux method we can find a first integral to the cubic polynomial system. In order to find another first integral which is independent from the first one, we then extract one variable from the first integral to reduce the cubic system from 3D into 2D, then a second first integral can be found for the reduced system.

5.2 Integrability of the cubic polynomial systems in 3D

We firstly consider the 3D system:

$$\begin{aligned}\dot{x} &= x + \sum_{\substack{i+j+k=n \\ n \geq 2}} a_{i,j,k} x^i y^j z^k = x + P(x, y, z), \\ \dot{y} &= -y + \sum_{\substack{i+j+k=n \\ n \geq 2}} b_{i,j,k} x^i y^j z^k = -y + Q(x, y, z), \\ \dot{z} &= z + \sum_{\substack{i+j+k=n \\ n \geq 2}} c_{i,j,k} x^i y^j z^k = z + R(x, y, z),\end{aligned}\tag{5.5}$$

where $P, Q, R \in \mathbb{C}[x, y, z]$ without linear term.

We suppose the above system is normalizable at the origin, then there is an invertible change of coordinates, which transforms the system (5.5) into the following normal form

$$\begin{aligned}\dot{x} &= x + a_1 x^2 y + a_2 x y z + a_3 y z^2 + \cdots, \\ \dot{y} &= -y + b_1 x y^2 + b_2 y^2 z + \cdots, \\ \dot{z} &= z + c_1 x y z + c_2 y z^2 + c_3 x^2 y + \cdots.\end{aligned}$$

Now, we are going to consider the cubic terms in the normal form, by which we mean that we will truncate the above system to the terms which have the power three. Therefore, we consider the system

$$\begin{aligned}\dot{x} &= x + a_1 x^2 y + a_2 x y z + a_3 y z^2, \\ \dot{y} &= -y + b_1 x y^2 + b_2 y^2 z, \\ \dot{z} &= z + c_1 x y z + c_2 y z^2 + c_3 x^2 y.\end{aligned}\tag{5.6}$$

Now, we want to use an analytic change of coordinates in order to remove some resonant monomials. System (5.6) has the same eigenvalues for both x , z -coordinates, then we can use the following analytic linear change of coordinates

$$\tilde{x} = d x + f z, \quad \tilde{z} = g x + h z,\tag{5.7}$$

where $d, f, g, h \in \mathbb{R} \setminus \{0\}$ such that $dg - fh \neq 0$. The inverse transformation is

$$x = \frac{h\tilde{x} - f\tilde{z}}{dh - gf}, \quad z = \frac{-g\tilde{x} + d\tilde{z}}{dh - gf}.\tag{5.8}$$

We first take the derivative of equation (5.7), and substitute equation (5.8) into the system (5.6) to obtain

$$\begin{aligned}\dot{\tilde{x}} &= \tilde{x} + \tilde{a}_1 \tilde{x}^2 y + \tilde{a}_2 \tilde{x} y \tilde{z} + \tilde{a}_3 y \tilde{z}^2, \\ \dot{y} &= -y + \tilde{b}_1 \tilde{x} y^2 + \tilde{b}_2 y^2 \tilde{z}, \\ \dot{\tilde{z}} &= \tilde{z} + \tilde{c}_1 \tilde{x} y \tilde{z} + \tilde{c}_2 \tilde{z}^2 y + \tilde{c}_3 \tilde{x}^2 y,\end{aligned}\tag{5.9}$$

where

$$\tilde{a}_1 = \frac{((-a_2gh + a_1h^2 + a_3g^2)d + f(g^2c_2 - c_1gh + c_3h^2))}{(dh - gf)^2},$$

$$\tilde{c}_2 = \frac{((-da_2f + a_1f^2 + a_3d^2)g + h(d^2c_2 - fc_1d + c_3f^2))}{(dh - gf)^2}.$$

$$\tilde{b}_1 = \frac{(hb_1 - gb_2)}{dh - gf}, \quad \tilde{b}_2 = \frac{(b_2d - b_1f)}{dh - gf}.$$

$$\tilde{a}_2 = \frac{((a_2h - 2ga_3)d^2 + f((a_2 - 2c_2)g + h(4a_1 - 2c_1))d + f^2(gc_1 - 2hc_3))}{(dh - gf)^2},$$

$$\tilde{c}_1 = \frac{((a_2f - 2da_3)g^2 + h((a_2 - 2c_2)d + f(4a_1 - 2c_1))g + h^2(c_1d - 2c_3f))}{(dh - gf)^2}.$$

$$\tilde{a}_3 = \frac{(a_3d^3 - f(a_2 - c_2)d^2 + f^2(a_1 - c_1)d + c_3f^3)}{(dh - gf)^2}, \quad (5.10)$$

$$\tilde{c}_3 = \frac{(a_3g^3 - h(a_2 - c_2)g^2 + h^2(a_1 - c_1)g + c_3h^3)}{(dh - gf)^2}. \quad (5.11)$$

We see that the equations (5.10) and (5.11) are cubic homogeneous polynomials of the variables d, f and g, h , respectively, with the same coefficients. To remove these resonant monomials, we can only choose two different solutions of the equation (5.10) or (5.11).

To simplify calculations, let $\hat{a}_2 = a_2 - c_2 \neq 0$ and $\hat{a}_1 = a_1 - c_1 \neq 0$ to get

$$\tilde{a}_3 = \frac{a_3d^3 - f d^2 \hat{a}_2 + f^2 d \hat{a}_1 + c_3f^3}{(dh - gf)^2} = \frac{(d + \rho f)^3}{dh - gf},$$

$$\tilde{c}_3 = \frac{a_3g^3 - h g^2 \hat{a}_2 + h^2 g \hat{a}_1 + c_3 h^3}{(dh - gf)^2} = \frac{(h + \rho g)^3}{dh - gf}.$$

We have the following solutions of the equation (5.10)

$$\rho_1 = \frac{P}{6c_3} + \frac{2(\hat{a}_1^2 - 3\hat{a}_2c_3)}{c_3P} + \frac{\hat{a}_1}{3c_3},$$

$$\rho_{2,3} = -\frac{P}{2c_3} - \frac{\hat{a}_1^2 - 3\hat{a}_2c_3}{3c_3P} + \frac{\hat{a}_1}{3c_3} \pm i \frac{\sqrt{3}}{2} \left(\frac{P}{6c_3} - \frac{2(\hat{a}_1^2 - 3\hat{a}_2c_3)}{3c_3P} \right),$$

where $P = 4 \left(\frac{(27a_3c_3^2 - \hat{a}_1(2\hat{a}_1^2 - 9\hat{a}_2c_3))^2 - 4(\hat{a}_1^2 - 3\hat{a}_2c_3)^3}{9c_3^2} \right)^{\frac{1}{2}} - 108a_3c_3^2 + 4\hat{a}_1(2\hat{a}_1^2 - 9\hat{a}_2c_3).$

If we choose ρ_1 and ρ_2 , we then directly get $\tilde{a}_3 = \tilde{c}_3 = 0$. Thus, by using the linear transformation (5.7), the system (5.9) becomes the following form

$$\dot{\tilde{x}} = \tilde{x}(1 + \tilde{a}_1 \tilde{x}y + \tilde{a}_2 y\tilde{z}), \quad \dot{\tilde{y}} = \tilde{y}(-1 + \tilde{b}_1 \tilde{x}y + \tilde{b}_2 y\tilde{z}), \quad \dot{\tilde{z}} = \tilde{z}(1 + \tilde{c}_1 \tilde{x}y + \tilde{c}_2 \tilde{z}y),$$

To simplify calculations, we can rewrite the above system of the following form

$$\dot{x} = x(1 + a_1 xy + a_2 yz), \quad \dot{y} = y(-1 + b_1 xy + b_2 yz), \quad \dot{z} = z(1 + c_1 xy + c_2 yz). \quad (5.12)$$

Theorem 19. System (5.6) with rank-two resonant eigenvalues has two explicit independent first integrals given in terms of hypergeometric functions.

Proof. We start at the system (5.6). By an analytic linear change of coordinates (5.7) we can bring the system (5.6) to the system (5.12).

The system (5.12) has the following Darboux factors and cofactors

$$\begin{aligned} G_1(x, y, z) &= x, & K_1(x, y, z) &= 1 + a_1 xy + a_2 yz, \\ G_2(x, y, z) &= y, & K_2(x, y, z) &= -1 + b_1 xy + b_2 yz, \\ G_3(x, y, z) &= z, & K_3(x, y, z) &= 1 + c_1 xy + c_2 yz, \\ G_4(x, y, z) &= 1 + \frac{z(a_2 - c_2)}{x(a_1 - c_1)}, & K_4(x, y, z) &= yz(c_2 - a_2), \end{aligned}$$

Then, we get the following generalized Darboux first integral

$$\varphi = y x^{1-\epsilon} z^\epsilon \left(1 + k \frac{z}{x}\right)^\alpha, \quad (5.13)$$

where $\alpha = \frac{a_1 b_2 + a_1 c_2 - a_2 b_1 - a_2 c_1 + b_1 c_2 - b_2 c_1}{(a_2 - c_2)(a_1 - c_1)}$, $\epsilon = \frac{a_1 + b_1}{a_1 - c_1}$ and $k = \frac{a_2 - c_2}{a_1 - c_1}$, such that $a_2 - c_2 \neq 0$ and $a_1 - c_1 \neq 0$.

We extract the variable x from equation (5.13), and let $w = \frac{z}{x}$ to obtain

$$x = F(y, w, \varphi) = \varphi y^{-1} w^{-\epsilon} (1 + kw)^{-\alpha}. \quad (5.14)$$

We take the derivative on $w = \frac{z}{x}$, and after substituting the above equation, the system (5.12) becomes

$$\dot{w} = xyw(c_1 - a_1)(1 + kw), \quad \dot{y} = xy^2\left(-\frac{1}{xy} + b_1 + b_2w\right),$$

dividing by $\frac{1}{xy^2w(1+kw)}$, gives

$$\dot{w} = \frac{c_1 - a_1}{y}, \quad \dot{y} = -\frac{w^{\epsilon-1}(kw + 1)^{\alpha-1}}{\varphi} + \frac{b_2w + b_1}{w(kw + 1)}. \quad (5.15)$$

The above system has the following first integral

$$\begin{aligned} \psi(w, y) &= (c_1 - a_1) \ln(y) - b_1 \ln(w) + \left(b_1 - \frac{b_2}{k}\right) \ln(kw + 1) \\ &\quad + (\epsilon \varphi)^{-1} w^\epsilon {}_2F_1(\epsilon, -\alpha + 1; 1 + \epsilon; -kw), \end{aligned}$$

where ${}_2F_1$ is the hypergeometric function

- **If** $a_1 = c_1$. Using the Darboux method with the exponential factor, the system(5.12) has first integrals of the form

$$\varphi = x^{1+\alpha} y z^{-\alpha} e^{\frac{kx}{z}} = xyw^\alpha e^{kw},$$

where $\alpha = \frac{a_2+b_2}{c_2-a_2}$ and $k = \frac{b_1+c_1}{c_2-a_2}$, such that $c_2 - a_2 \neq 0$, and $w = \frac{x}{z}$. In the same way, by taking the derivative on $w = \frac{z}{x}$, and substituting $x = \varphi y^{-1} w^{-\alpha} e^{-kw}$, the system (5.12) becomes

$$\dot{w} = (a_2 - c_2) \varphi w^{-\alpha} e^{-kw}, \quad \dot{y} = (b_1 + b_2 w^{-1} - \varphi w^\alpha e^{kw}) kw.$$

We can directly have another first integral of the following

$$\begin{aligned} \psi &= (a_2 - c_2) \ln(y) - b_1 \frac{x}{z} - b_2 \ln\left(\frac{x}{z}\right) + \varphi^{-1} k^{-1} w^\alpha e^{kw} \\ &\quad - \varphi^{-1} k^{-1} (w)^\alpha {}_1F_1(\alpha; \alpha + 1; kw). \end{aligned} \quad (5.16)$$

- **If** $a_2 = c_2$, then the system (5.12) has two first integrals of the form

$$\varphi = x^{1-\alpha} y z^\alpha e^{\frac{kz}{x}},$$

$$\psi = (c_1 - a_1) \ln(y) - b_1 \ln\left(\frac{z}{x}\right) - b_2 \frac{z}{x} + \varphi^{-1} \alpha^{-1} w^\alpha (-kw)^\alpha {}_1F_1(\alpha; \alpha + 1; kw),$$

where $\alpha = \frac{a_1+b_1}{a_1-c_1}$, $k = \frac{b_2+c_2}{a_1-c_1}$, such that $c_1 - a_1 \neq 0$.

- **If** $a_1 = c_1$ and $a_2 = c_2$, then in the same way the system (5.12) has two first integrals of the form:

$$\varphi = x^{-1} z, \quad \psi = \frac{1}{xy} - \left((b_1 + c_1) + (b_2 + c_2) \frac{z}{x} \right) \ln(y) + \left(b_1 + b_2 \frac{z}{x} \right) \ln(xy).$$

□

Open Question for Further study:

We introduced the monodromy method in the 3D vector field with rank-one resonant eigenvalues by using two independent first integrals. One of these first integrals contains logarithm terms. System (5.12) has the two independent first integrals. One of these first integrals contains the hypergeometric function. This result clearly needs a further study. For example, can the second first integral (5.16) be used to find the monodromy map in this case in order to make some conclusions about the relation between integrability and normalizability of the system (5.12) with rank-two resonant eigenvalues.

Chapter 6

Contribution

In the thesis, we address the question of normalizability, integrability and monodromy maps of singularities for the analytic system

$$\dot{x} = \lambda x + \sum_{n \geq 2} P_n(x, y, z), \quad \dot{y} = \mu y + \sum_{n \geq 2} Q_n(x, y, z), \quad \dot{z} = \nu z + \sum_{n \geq 2} R_n(x, y, z). \quad (6.1)$$

Firstly, we consider rank-one resonant eigenvalues (λ, μ, ν) which satisfy the condition $\lambda + \mu + \nu = 0$.

We at first extend a technique which was described by Aziz and Christopher (2012) in order to show integrability for 3D vector fields. We then use this new technique (see Theorem 6) in order to find two independent explicit first integrals of the system (6.1).

We also used the normal form method to find the sufficient conditions for existence of one first integral of a 3D vector field. This is detailed in Section 3.3. We apply this idea to demonstrate the (formal) sufficiency of the conditions for the existence of one analytic first integral in Aziz and Christopher (2014). In their paper a number of necessary conditions for the existence of one first integral for 3D Lotka-Volterra systems was found. The sufficiency of these conditions was left as

conjectural. We apply our method to show formal sufficiency of these conditions.

Secondly, we consider when the system (6.1) is an orbitally normalizable system. Then, by an analytic change of coordinates, we can bring the system (6.1) to the following normal form after scaling

$$\dot{X} = X(\lambda + \sum_{k \geq 1} a_k u^k), \quad \dot{Y} = Y(\mu + \sum_{k \geq 1} b_k u^k), \quad \dot{Z} = Z(\nu + \sum_{k \geq 1} c_k u^k),$$

where $u = XYZ$ is the resonant monomial. Since, we work with orbitally normalizability, without loss of generality, we can divide the above system by $F(u) = 1 + \frac{1}{\lambda} \sum_{k \geq 1} a_k u^k$. Thus, we obtain the system

$$\dot{x} = \lambda x, \quad \dot{y} = z(\mu + \sum_{k \geq 1} b_k u^k), \quad \dot{z} = z(\nu + \sum_{k \geq 1} c_k u^k).$$

We seek a further analytic change of coordinates to bring the system into one of the following reduced normal forms:

$$\dot{x} = \lambda x, \quad \dot{Y} = Y(\mu + \frac{F(U) + bU^k}{1 + aU^k}), \quad \dot{Z} = Z(\nu + \frac{-F(U) + cU^k}{1 + aU^k}),$$

$$\dot{x} = \lambda x, \quad \dot{Y} = \mu Y, \quad \dot{Z} = Z(\nu + \frac{cU^l}{1 + aU^l}),$$

which depend on the first non-zero resonant monomial in the original system, where $u = xyz$ and $U = xYZ$. This is detailed in Theorem 8. This idea in the case of 2D systems was addressed by Christopher et al. (2003).

From now on, we only consider the case $k = 1$ (the generic case). Then, we have the following reduced normal form

$$\dot{x} = \lambda x, \quad \dot{Y} = Y(\mu + \frac{bU}{1 + aU}), \quad \dot{Z} = Z(\nu + \frac{cU}{1 + aU}), \quad (6.2)$$

where $U = xYZ$. Using the technique in Section 3.2, the system (6.2) has one Darboux-analytic first integral and one explicit first integral of the following form

$$H_1(x, Y, Z) = x^\alpha Y^\beta Z, \quad H_2(x, Y, Z) = \frac{1}{u} + \epsilon_1 \ln x + \epsilon_2 \ln Y. \quad (6.3)$$

The more general cases in Theorem 8 also have explicit first integrals, but we do not obtain a first integral of Darboux-analytic type, so, we have only mentioned the generic case here.

Thirdly, we have studied different criteria for bringing an orbital normalizable system to normalizable system. After multiplying the reduced normal form by $1 + aU$, and assuming that the system (6.1) is orbitally normalizable, then we can bring the system formally into the following form, now taking account of time

$$\begin{aligned} \dot{x} &= x(\lambda + \lambda a u) h(x, y, z), \\ \dot{y} &= y(\mu + (\mu a + b) u) h(x, y, z), \\ \dot{z} &= z(\nu + (\nu a + c) u) h(x, y, z), \end{aligned}$$

where $h(x, y, z)$ is an analytic and $h(0, 0, 0) = 1$. This is detailed in Theorem 16. We showed that the above model is analytically normalizable to the system

$$\dot{x} = x(\lambda + \lambda a U), \quad \dot{Y} = Y(\mu + (\mu a + b) U), \quad \dot{Z} = Z(\nu + (\nu a + c) U),$$

if and only if there is an analytic series $g(x, y, z)$ vanishing at the origin such that

$$((b + c)\eta - dg) \wedge \Omega = 0.$$

This idea generalizes a theorem in Christopher et al. (2004).

Furthermore, we introduce the monodromy map in the neighbourhood of the

x -separatrix for the system (6.2) normal for by the two first integrals. To find the monodromy map, we need to consider the trajectory of the system near a closed loop $x_\theta = x_0 e^{i2\pi\theta}$ at the x -separatrix. To achieve this, we take a transversal to the x -separatrix at each point of the loop starting at a base point, and we look at the trajectories which are close to the loop, and hence intersects the trajectory.

We at first extract Z from the first integral $H_1(x, Y, Z) = k_1$ at the starting point (x_0, Y_0, Z_0) ($k_1 = x_0^\alpha Y_0^\beta Z_0$), and then we substitute H_1 and $Y_\theta = \sum_{i \geq 1} c_i(\theta) Y_0^i$ into the second first integral ($H_2(x, Y, Z) = k_2$) to obtain

$$\frac{1}{x_0^\alpha Y_0^\beta Z_0} x_0^{\alpha-1} Y_\theta^{\beta-1} (e^{2i\pi\theta} - 1) + \epsilon_1 2i\pi\theta + \epsilon_2 \ln(Y_\theta - Y_0) = 0.$$

Solving term by term in Y_0 , we obtain

$$Y_\theta = Y_0 e^{2i\pi\theta \frac{\mu}{\lambda}} \left(1 + \frac{2i\pi\theta b}{\lambda} x_0 Y_0 Z_0 - \frac{2\theta\pi b (ia\lambda + 2\pi b\theta + \pi c\theta)}{\lambda^2} (x_0 Y_0 Z_0)^2 + \dots \right).$$

Next, we assume that the monodromy map of Z -coordinate is given by substituting the power series $Z_\theta = e^{2i\pi\theta \frac{\nu}{\lambda}} Z_0 (1 + g(x_0, Y_0, Z_0, \theta))$ with Y_θ into first integral $H_1 = k_1$. After considering the Taylor expansion to find $g(x_0, Y_0, Z_0, \theta)$, we obtain

$$g(x_0, Y_0, Z_0, \theta) = \frac{2i\pi\theta c}{\lambda} (x_0 Y_0 Z_0) - \frac{2c\pi\theta (ia\lambda + \pi\theta(b+2c))}{\lambda^2} (x_0 Y_0 Z_0)^2 + \dots$$

Then, the monodromy map in Y, Z -coordinate is given by a map $Y_{\theta=1}$ and $Z_{\theta=1}$, respectively. Hence, we obtain the 2D map in normal form

$$\begin{aligned} x_1 &= e^{2i\pi} x_0, \\ Y_1 &= e^{2i\pi \frac{\mu}{\lambda}} Y_0 \left(1 + \frac{2i\pi b}{\lambda} u_0 - \frac{2\pi b (ia\lambda + \pi(2b+c))}{\lambda^2} u_0^2 + \dots \right), \\ Z_1 &= e^{2i\pi \frac{\nu}{\lambda}} Z_0 \left(1 + \frac{2i\pi c}{\lambda} u_0 - \frac{2c\pi (ia\lambda + \pi(b+2c))}{\lambda^2} u_0^2 + \dots \right), \end{aligned}$$

where $u_0 = x_0 Y_0 Z_0$. This map is the monodromy map corresponding to the system (6.2). This is detailed in Section 4.3.

In the same way to the case of vector fields, by an invertible change of coordinates, we can bring the monodromy map formally into the finite reduced map

$$\begin{aligned} x_1 &= e^{2i\pi} x_0, \\ Y_1 &= e^{2i\pi \frac{\mu}{\lambda}} Y_0 \left(1 + \frac{2i\pi b}{\lambda} u_0 - \frac{2\pi b (ia\lambda + \pi (2b + c))}{\lambda^2} u_0^2 \right), \\ Z_1 &= e^{2i\pi \frac{\nu}{\lambda}} Z_0 \left(1 + \frac{2i\pi c}{\lambda} u_0 - \frac{2c\pi (ia\lambda + \pi (b + 2c))}{\lambda^2} u_0^2 \right), \end{aligned}$$

where $u_0 = x_0 Y_0 Z_0$. We can read off the terms of the system (6.3) from the above reduced map. Thus, we can relate the reduced map and the corresponding system. In the case of 2D vector fields, this idea was addressed by Christopher and Rousseau (2004).

Finally, we consider the case of rank-two resonant eigenvalues (λ, μ, ν) for the system

$$\dot{x} = x(\lambda + \sum_{n \geq 1} P_n(x, y, z)), \quad \dot{y} = y(\mu + \sum_{n \geq 1} Q_n(x, y, z)), \quad \dot{z} = z(\nu + \sum_{n \geq 1} R_n(x, y, z)).$$

After a possible scaling of time, we assume that $\lambda, \mu, \nu \in \mathbb{Z}$, and $g.c.d(\lambda, \mu, \nu) = 1$. We proved that if the eigenvalues of the above system with rank-two resonance satisfy one of the following conditions

$$\mu = -1, \quad \lambda = -n\mu \ (\nu = -n\mu) \ n \in \mathbb{N}, \quad \text{or} \quad \lambda + \mu + \nu = 0,$$

then the normal form for the system can be generated by two independent resonant monomials. The following normal form corresponding to the above system is therefore

$$\dot{x} = x(\lambda + \sum_{n \geq 1} P_n(u, v)), \quad \dot{y} = y(\mu + \sum_{n \geq 1} Q_n(u, v)), \quad \dot{z} = z(\nu + \sum_{n \geq 1} R_n(u, v)),$$

where $P_n, Q_n, R_n \in \mathbb{C}[u, v]$. This details are given in Theorem 18.

We have tried to find a finite reduced normal form for the above system, but we could not at the moment. Thus, we have simplified our investigation by only considering the case $(1, -1, 1)$ -resonant eigenvalues, and truncating the normal to terms which have the power three or less. Thus, we consider the following system

$$\begin{aligned}\dot{x} &= x + a_1 x^2 y + a_2 x y z + a_3 y z^2, \\ \dot{y} &= -y + b_1 x y^2 + b_2 y^2 z, \\ \dot{z} &= z + c_1 x y z + c_2 z^2 y + c_3 x^2 y.\end{aligned}$$

The above system has two explicit first integrals of the following form

$$\varphi(x, y, z) = y x^{1-\epsilon} z^\epsilon \left(1 + \frac{kz}{x}\right)^\alpha,$$

and

$$\begin{aligned}\psi(x, y, z) &= (c_1 - a_1) \ln(y) - b_1 \ln\left(\frac{z}{x}\right) + \left(b_1 - \frac{b_2}{k}\right) \ln\left(\frac{kz}{x} + 1\right) \\ &\quad + (\epsilon \varphi)^{-1} \frac{z^\epsilon}{x} {}_2F_1(\epsilon, -\alpha + 1; 1 + \epsilon; \frac{-kz}{x}),\end{aligned}$$

where ${}_2F_1$ is the hypergeometric function. This is detailed in Theorem 19.

The application of the monodromy in this case using hypergeometric function in 3D vector field is much harder than the case of rank-one resonance. We have left this problem as an interesting topic for further investigation.

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