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A_{19}/B_6: A NEW LANCZOS-TYPE ALGORITHM AND ITS IMPLEMENTATION

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ABSTRACT. Lanczos-type algorithms are mostly derived using recurrence relationships between formal orthogonal polynomials. Various recurrence relations between these polynomials can be used for this purpose. In this paper, we discuss recurrence relations A_{19} and B_6 for the choice $U_i(x) = P_i^{(1)}(x)$, where U_i is an auxiliary family of polynomials of exact degree i. This leads to new Lanczos-type algorithm A_{19}/B_6 that shows superior stability when compared to existing algorithms of the same type. This new algorithm is derived and described here. Computational results obtained with it are compared to those of the most robust algorithms of this type namely A_{12}, A_{12}^{new} A_5/B_{10} and A_8/B_{10} on the same test problems. These results are included.

Key words : Lanczos algorithm; Systems of Linear Equations; Formal Orthogonal Polynomials
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1. INTRODUCTION

In 1950, the Lanczos algorithm, [26, 13], has been introduced to calculate the eigenvalues of a matrix. However, it has later been adapted for the solution of systems of linear equations (SLEs) where it is now a well established solver. The Lanczos method is an iterative process which, in exact arithmetic, gives the exact solution in at most n number

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of steps [27], where n is the dimension of the problem. Several Lanczos-type algorithms have been designed and among them, the famous conjugate gradient algorithm of Hestenes and Stiefel [25], when the matrix is Hermitian and the bi-conjugate gradient algorithm of Fletcher [22], in the general case. In the last few decades, Lanczos-type algorithms have evolved and different variants have been derived, which can be found in [2, 3, 5, 7, 10, 11, 12, 9, 14, 23, 24, 28, 29, 30, 31, 34, 35, 17].

Lanczos-type algorithms are commonly derived using Formal Orthogonal Polynomials (FOP's), [5]. The connection between the Lanczos algorithm, [27] and orthogonal polynomials, [32] has been studied extensively in [2, 4, 5, 11, 12, 6, 8, 9, 16].

In this paper we will briefly recall recurrence relation A_{19} [17] for the choice of auxiliary polynomial $U_i(x) = x^i$, where x^i is a monic polynomial of degree i. Then we will derive expressions for the coefficients of this polynomial for a new choice of $U_i(x) = P_i^{(1)}(x)$ which was not considered before. We will also recall B_6 [1] for the same choice of $U_i(x)$. We use the new choice of A_{19} in combination of B_6 to derive a new Lanczos-type algorithm A_{19}/B_6. This algorithm is then applied to some problems considered in [17, 1, 33], and its performance is compared with that of existing algorithms of the same type namely A_{12}, A_{12}^{new} A_5/B_{10} and A_8/B_{10}, [2, 17, 33]. The paper is organized as follows. In section 2 we will explain the basic Lanczos process. In section 3 we will discuss the notion of FOPs. Relations A_{19} and B_6 are recalled in section 4. A conclusion is given in section 5.

1.1. **The Lanczos Process.** Consider the following system of linear equations,

$$A\mathbf{x} = \mathbf{b}, \tag{1}$$

The basic Lanczos approach for solving SLEs (1), can be explained as follows.

Choose $\mathbf{x}_0$ and $\mathbf{y}$, two arbitrary vectors in $\mathbb{R}^n$, such that $\mathbf{y} \neq 0$, then Lanczos process [27] consists in generating a sequence of vectors $\mathbf{x}_k \in \mathbb{R}^n$, such that

$$(\mathbf{x}_k - \mathbf{x}_0) \in F_k(A, \mathbf{r}_0) = span(\mathbf{r}_0, A\mathbf{r}_0, \ldots, A^{k-1}\mathbf{r}_0), \tag{2}$$

and

$$\mathbf{r}_k = (\mathbf{b} - A\mathbf{x}_k) \perp E_k(A^T, \mathbf{y}) = span(\mathbf{y}, A^T\mathbf{y}, \ldots, (A^T)^{k-1}\mathbf{y}), \tag{3}$$

where A^T is the transpose of matrix A.

Equation (2) implies

$$\mathbf{x}_k - \mathbf{x}_0 = -\beta_1 \mathbf{r}_0 - \beta_2 A\mathbf{r}_0 - \cdots - \beta_k A^{k-1}\mathbf{r}_0. \tag{4}$$

Multiplying both sides of (4) by A then adding and subtracting $\mathbf{b}$ on the left hand side of (4) gives

$$\mathbf{r}_k = \mathbf{r}_0 + \beta_1 A\mathbf{r}_0 + \beta_2 A^2\mathbf{r}_0 + \cdots + \beta_k A^k \mathbf{r}_0. \tag{5}$$

If we set

$$P_k(x) = 1 + \beta_1 x + ... + \beta_k x^k, \tag{6}$$

then we can write from (5)

$$\mathbf{r}_k = P_k(A)\mathbf{r}_0. \tag{7}$$

The polynomials P_k are known as residual polynomials [5]. Another interpretation of the P_k can be found in [15]. From (3), the orthogonality condition implies

$((A^T)^i\mathbf{y}, \mathbf{r}_k) = (\mathbf{y}, A^i\mathbf{r}_k) = (\mathbf{y}, A^i P_k(A)\mathbf{r}_0) = 0$, for $i = 0, ..., k-1$.

Thus, the coefficients β_1,...,β_k form a solution of the following system of linear equations

$$\begin{cases} \beta_1(\mathbf{y}, A\mathbf{r}_0) + \cdots + \beta_k(\mathbf{y}, A^k\mathbf{r}_0) = -(\mathbf{y}, \mathbf{r}_0), \\ \vdots \\ \beta_1((A^T)^{k-1}\mathbf{y}, A\mathbf{r}_0) + \cdots + \beta_k((A^T)^{k-1}\mathbf{y}, A^k\mathbf{r}_0) = -((A^T)^{k-1}\mathbf{y}, \mathbf{r}_0). \end{cases} \tag{8}$$

The scalar products involved in the above system is defined as with the first argument conjugated. If the determinant of the above system is not zero then its solution exists, and thus we can obtain $\mathbf{x}_k$ and $\mathbf{r}_k$ from (4) and (7) respectively. Obviously, in practice, solving the above system directly for increasing values of k is not viable; k is the order of the iterate in the solution process. Now we shall see how to solve the system (8) for increasing value of k, that is, if polynomials P_k can be computed recursively.

Now, let c_i be defined as

$$c_i = ((A^T)^i\mathbf{y}, \mathbf{r}_0) = (\mathbf{y}, A^i\mathbf{r}_0), \quad \text{for} \quad i = 1, 2, \ldots,$$

and the linear functional c on the space of polynomials be given by

$$c(x^i) = c_i, \quad \text{for } i = 0, 1 \ldots, \tag{9}$$

so the system (8) can be written as

$$c(x^i P_k(x)) = 0, \quad \text{for } i = 0, 1, \ldots, k-1. \tag{10}$$

These conditions show that P_k is a polynomial of degree at most k, corresponding to the linear functional c and normalized by the condition $P_k(0) = 1$. Using normalization condition $P_k(0) = 1$, equation (6) can be written as

$$P_k(x) = 1 + xQ_{k-1}(x),$$

where $Q_{k-1} = \beta_1 + \beta_2 x + ... + \beta_k x^{k-1}$. Replacing x by A and multiplying both sides by $\mathbf{r}_0$ we get

$$P_k(A)\mathbf{r}_0 = \mathbf{r}_0 + AQ_{k-1}(A)\mathbf{r}_0.$$

Now using (7) the above relation becomes

$$\mathbf{r}_k = \mathbf{r}_0 + AQ_{k-1}(A)\mathbf{r}_0,$$

which can also be written as

$$\mathbf{b} - A\mathbf{x}_k = \mathbf{b} - A\mathbf{x}_0 + AQ_{k-1}(A)\mathbf{r}_0.$$

Simplifying and multiplying by $-A^{-1}$ on both sides of the last relation, we get

$$\mathbf{x}_k = \mathbf{x}_0 - Q_{k-1}(A)\mathbf{r}_0,$$

which shows that $\mathbf{x}_k$ can be computed from $\mathbf{r}_k$ recursively without inverting A.

1.2. **Formal Orthogonal Polynomials.** The polynomial $P_k(x)$ discussed in the previous section is defined by the following formula [5, 6, 11, 9, 12],

$$P_k(x) = \frac{\begin{vmatrix} 1 & x & \cdots & x^k \\ c_0 & c_1 & \cdots & c_k \\ \vdots & \vdots & & \vdots \\ c_{k-1} & c_k & \cdots & c_{2k-1} \end{vmatrix}}{H_k^{(1)}}, \tag{11}$$

where $H_k^{(1)}$ is called the Hankel determinant [5], which is the determinant of the system (8). This determinant has the following expression:

$$H_k^{(1)} = \begin{vmatrix} c_1 & c_2 & \cdots & c_k \\ c_2 & c_3 & \cdots & c_{k+1} \\ \vdots & \vdots & & \vdots \\ c_k & c_{k+1} & \cdots & c_{2k-1} \end{vmatrix}.$$

Clearly, P_k exists if and only if $H_k^{(1)} \neq 0$. We assume in the following sections that for all k, $H_k^{(1)} \neq 0$. If for some k, $H_k^{(1)} \approx 0$, then P_k does not exist, and breakdown occurs in the solution process. This breakdown issue is discussed elsewhere, [2, 17, 33].

Let us now define the family of orthogonal polynomials $P_k^{(1)}(x)$ corresponding to the linear functional $c^{(1)}$ where $c^{(1)}$ is define by

$$c^{(1)}(x^i) = c(x^{i+1}) = c_{i+1}, \text{ for } i = 0, 1, \ldots.$$

These polynomials are normalized by the condition that they are monic [5, 6, 11] and are given by the following formula

$$P_k^{(1)}(x) = \frac{\begin{vmatrix} c_1 & \cdots & c_{k+1} \\ \vdots & & \vdots \\ c_k & \cdots & c_{2k} \\ 1 & \cdots & x^k \end{vmatrix}}{H_k^{(1)}}. \tag{12}$$

The necessary and sufficient condition for the existence and uniqueness of $P_k^{(1)}(x)$ is that the Hankel determinant, [5, 6, 11], is different from zero, which is the same condition as for the existence of the polynomial $P_k(x)$.

2. Relations A_{19} and B_6

In the following we will recall relations A_{19} [17, 18] and B_6 [1, 2] for the choice of auxiliary polynomial $U_i(x) = x^i$, where x^i is a monic polynomial of degree i. Then we will derive expressions for the coefficients of these polynomials for the choice of $U_i(x) = P_i^{(1)}(x)$ which was not considered before for relation A_{19}. We use the new choice of A_{19} in tandem with B_6 to derive a new Lanczos-type algorithm which we call A_{19}/B_6.

2.1. Relation A_{19}.

Consider the following recurrence relation investigated in [17]

$$P_k(x) = (A_k x^2 + B_k x + C_k)P_{k-2}^{(1)}(x) + (D_k x + E_k)P_{k-1}(x), \tag{13}$$

where P_k, $P_{k-2}^{(1)}$ and P_{k-1} are polynomials of degree k, $k-2$ and $k-1$ respectively and A_k, B_k, C_k, D_k and E_k are constants to be determined using the normalization condition $\forall k, P_k(0) = 1$ and orthogonality conditions (C_1) and (C_2) given below
$\forall i = 0, 1 \cdots k-1$, $c(U_i P_k) = 0 \longrightarrow (C_1)$.

$\forall i = 0, 1, \cdots k-1$, $c^{(1)}(U_i P_k^{(1)}) = 0 \longrightarrow (C_2)$.
Using the normalization condition, equation (13) gives

$$1 = E_k + C_k P_{k-2}^{(1)}(0). \tag{14}$$

Multiplying (13) by U_i, a polynomial of exact degree i and applying 'c' on both sides we get

$$\begin{aligned} c(U_i P_k) = A_k c(x^2 U_i P_{k-2}^{(1)}) + B_k c(x U_i P_{k-2}^{(1)}) + C_k c(U_i P_{k-2}^{(1)}) \\ + D_k c(x U_i P_{k-1}) + E_k c(U_i P_{k-1}). \end{aligned} \tag{15}$$

Similarly, using (C_1), equation (15) becomes

$$\begin{aligned} A_k c(x^2 U_i P_{k-2}^{(1)}) + B_k c(x U_i P_{k-2}^{(1)}) + C_k c(U_i P_{k-2}^{(1)}) \\ + D_k c(x U_i P_{k-1}) + E_k c(U_i P_{k-1}) = 0. \end{aligned} \tag{16}$$

For $i = 0$, equation (16) becomes $C_k c(U_0 P_{k-2}^{(1)}) = 0$. Since $c(U_0 P_{k-2}^{(1)}) \neq 0$, therefore, $C_k = 0$. Hence from (14), we get $E_k = 1$. The orthogonality condition (C_1) is true for $\forall i = 1, 2, 3, \cdots, k-4$. For $i = k-3$, equation (16) becomes $A_k c(x^2 U_{k-3} P_{k-2}^{(1)}) = 0$. This implies that $A_k = 0$ as $c(x^2 U_{k-3} P_{k-2}^{(1)}) \neq 0$. For $i = k-2$, equation (16) gives

$$B_k c^{(1)}(U_{k-2} P_{k-2}^{(1)}) + D_k c(x U_{k-2} P_{k-1}) = 0. \tag{17}$$

For $i = k-1$, equation (16) gives

$$B_k c^{(1)}(U_{k-1} P_{k-2}^{(1)}) + D_k c(x U_{k-1} P_{k-1}) = -c(U_{k-1} P_{k-1}). \tag{18}$$

If we set $a_{11} = c^{(1)}(U_{k-2} P_{k-2}^{(1)})$, $a_{12} = c(x U_{k-2} P_{k-1})$, $a_{21} = c^{(1)}(U_{k-1} P_{k-2}^{(1)})$, $a_{22} = c(x U_{k-1} P_{k-1})$, $b_1 = 0$, and $b_2 = -c(U_{k-1} P_{k-1})$ then equations (17) and (18) can be written as

$$a_{11} B_k + a_{12} D_k = 0, \tag{19}$$

$$a_{21} B_k + a_{22} D_k = b_2, \tag{20}$$

respectively. If Δ_k is the determinant of the coefficient matrix of the above system then, $\Delta_k = a_{11} a_{22} - a_{21} a_{12}$. If $\Delta_k \neq 0$ then $B_k = -\frac{b_2 a_{12}}{\Delta_k}$, $D_k = \frac{a_{11} b_2}{\Delta_k}$. Hence,

$$P_k(x) = B_k x P_{k-2}^{(1)}(x) + (D_k x + 1) P_{k-1}(x). \tag{21}$$

Let us apply the recursive formula (21) for computing polynomials P_k in order to find residuals $\mathbf{r}_k$ and the corresponding vector $\mathbf{x}_k$. For this

replace x by A and multiply both sides by $\mathbf{r}_0$. Using $\mathbf{r}_k = P_k(A)\mathbf{r}_0$ and $\mathbf{z}_k = P_k^{(1)}(A)\mathbf{r}_0$, we get

$$\mathbf{r}_k = B_k A\mathbf{z}_{k-2} + D_k A\mathbf{r}_{k-1} + \mathbf{r}_{k-1}. \tag{22}$$

Since $\mathbf{r}_k = \mathbf{b} - A\mathbf{x}_k$, (22) becomes

$$\mathbf{b} - A\mathbf{x}_k = B_k A\mathbf{z}_{k-2} + D_k A\mathbf{r}_{k-1} + \mathbf{b} - A\mathbf{x}_{k-1}.$$

Multiplying this latter equation by A^{-1} on both sides results in

$$\mathbf{x}_k = \mathbf{x}_{k-1} - B_k\mathbf{z}_{k-2} - D_k\mathbf{r}_{k-1}, \tag{23}$$

where coefficients B_k and D_k can be identified as $B_k = -\frac{b_2 a_{12}}{\Delta_k}$, $D_k = \frac{a_{11}b_2}{\Delta_k}$, respectively. So, now we choose the polynomial $U_i(x)$.

2.1.1. *Case-I: When $U_i(x) = x^i$.* In the previous section we discussed A_{19} for general auxiliary polynomial U_i. Here we recall from [17] briefly the same relation A_{19} by taking $U_i = x^i$. In [17] we have $\Delta_k = a_{11}a_{22} - a_{21}a_{12}$, where $a_{11} = c^{(1)}(x^{k-2}P_{k-2}^{(1)})$, $a_{12} = c(x^{k-1}P_{k-1})$, $a_{21} = c^{(1)}(x^{k-1}P_{k-2}^{(1)})$, $a_{22} = c(x^kP_{k-1})$, $b_1 = 0$, and $b_2 = -c(x^{k-1}P_{k-1})$. If $\Delta_k \neq 0$ then coefficients B_k and D_k appearing in (21) are as above. Since we know that

$$\begin{cases} c(x^kP_k) = ((A^T)^k\mathbf{y}, P_k(A)\mathbf{r}_0) = (\mathbf{y}_k, \mathbf{r}_k) \ \ and \\ c(x^kP_k^{(1)}) = ((A^T)^k\mathbf{y}, P_k^{(1)}(A)\mathbf{r}_0) = (\mathbf{y}_k, \mathbf{z}_k), \\ \text{with} \quad \mathbf{y}_k = A^T\mathbf{y}_{k-1}, \end{cases} \tag{24}$$

using (24), we can write $a_{11} = c^{(1)}(x^{k-2}P_{k-2}^{(1)}) = (\mathbf{y}_{k-1}, \mathbf{z}_{k-2})$,
$a_{12} = c(x^{k-1}P_{k-1}) = (\mathbf{y}_{k-1}, \mathbf{r}_{k-1})$, $a_{21} = c^{(1)}(x^{k-1}P_{k-2}^{(1)}) = (\mathbf{y}_k, \mathbf{z}_{k-2})$,
$a_{22} = c(x^kP_{k-1}) = (\mathbf{y}_k, \mathbf{r}_{k-1})$, $b_1 = 0$, and $b_2 = -c(x^{k-1}P_{k-1}) =$
$-(\mathbf{y}_{k-1}, \mathbf{r}_{k-1})$.

Now, since all of the above relations are only valid for $k \geq 3$, to evaluate (22) and (23) recursively, we need to evaluate $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{z}_1$ and $\mathbf{z}_2$, which are necessary, differently. These values are determined in detail in [1, 17]. They are recalled briefly here, however, for completeness, as follows.
$\mathbf{r}_1 = \mathbf{r}_0 - (\frac{c_0}{c_1})A\mathbf{r}_0$, $\mathbf{x}_1 = \mathbf{x}_0 + (\frac{c_0}{c_1})\mathbf{r}_0$ where $c_i = (\mathbf{y}, A^i\mathbf{r}_0)$,
$\mathbf{r}_2 = \mathbf{r}_0 - \alpha A\mathbf{r}_0 + \beta A^2\mathbf{r}_0$, $\mathbf{x}_2 = \mathbf{x}_0 + \alpha\mathbf{r}_0 - \beta A\mathbf{r}_0$,
$\mathbf{z}_1 = A\mathbf{r}_0 - (\frac{c_2}{c_1})\mathbf{r}_0$, $\mathbf{z}_2 = A^2\mathbf{r}_0 - \alpha_1 A\mathbf{r}_0 + \beta_1\mathbf{r}_0$.
Also $\tilde{\mathbf{z}_1} = A\tilde{\mathbf{z}_0} - (\frac{c_2}{c_1})\tilde{\mathbf{z}_0}$, $\tilde{\mathbf{z}_2} = A^2\tilde{\mathbf{z}_0} - \alpha_1 A\tilde{\mathbf{z}_0} + \beta_1\tilde{\mathbf{z}_0}$,

where $\delta = c_1c_3 - c_2^2$, $\alpha = \frac{c_0c_3 - c_1c_2}{\delta}$, $\beta = \frac{c_0c_2 - c_1^2}{\delta}$,$\delta_1 = c_1c_3 - c_2^2$, $\alpha_1 = \frac{c_1c_4 - c_2c_3}{\delta_1}$, and $\beta_1 = \frac{c_2c_4 - c_3^2}{\delta_1}$.

2.1.2. *Case-II: When* $U_i(x) = P_i^{(1)}(x)$. In this section, we derive A_{19} for a different choice of $U_i(x)$ which was not considered before. All the coefficients involved in A_{19} have completely different expressions for this new choice of $U_i(x)$ as explained below. For $U_i(x) = P_i^{(1)}(x)$ all of the above expressions will have the following form:
$a_{11} = c^{(1)}(P_{k-2}^{(1)}P_{k-2}^{(1)})$, $a_{12} = c(xP_{k-2}^{(1)}P_{k-1})$, $a_{21} = c^{(1)}(P_{k-1}^{(1)}P_{k-2}^{(1)})$, $a_{22} = c(xP_{k-1}^{(1)}P_{k-1})$, $b_1 = 0$, and $b_2 = -c(P_{k-1}^{(1)}P_{k-1})$. Using

$$\begin{cases} c^{(1)}(P_{k-1}^{(1)}P_{k-2}^{(1)}) = 0, \\ \mathbf{z}_k = P_k^{(1)}(A)\mathbf{r}_0,\ \tilde{\mathbf{z}}_k = P_k^{(1)}(A^T)\tilde{\mathbf{z}}_0,\ \mathbf{r}_k = P_k(A)\mathbf{r}_0,\ \textit{and} \\ c(U_kP_k) = (y, U_k(A)P_k(A)r_0) = (U_k(A^T)y, P_k(A)r_0) = (\tilde{z}_k, r_k) \\ [\textit{note}\ \tilde{\mathbf{z}}_0 = y], \end{cases} \tag{25}$$

we get $a_{21} = 0$, $a_{11} = c^{(1)}(P_{k-2}^{(1)}P_{k-2}^{(1)}) = (\tilde{\mathbf{z}}_{k-2}, A\mathbf{z}_{k-2})$, $a_{12} = c(xP_{k-2}^{(1)}P_{k-1})$ $= (\tilde{\mathbf{z}}_{k-2}, A\mathbf{r}_{k-1})$, $a_{22} = c(xP_{k-1}^{(1)}P_{k-1}) = (\tilde{\mathbf{z}}_{k-1}, A\mathbf{r}_{k-1})$, $b_1 = 0$, $b_2 = -c(P_{k-1}^{(1)}P_{k-1}) = -(\tilde{\mathbf{z}}_{k-1}, \mathbf{r}_{k-1})$, $\Delta_k = a_{11}a_{22}$, $B_k = -\frac{b_2a_{12}}{\Delta_k}$, and $D_k = \frac{b_2}{a_{22}}$. Hence, after evaluating the coefficients A_k, B_k, C_k, Δ_k and E_k for the choice $U_i(x) = P_i^{(1)}(x)$ equation(13) reduces to

$$P_k(x) = B_kxP_{k-2}^{(1)}(x) + (D_kx + 1)P_{k-1}(x). \tag{26}$$

Replacing x by A and multiplying both sides by $\mathbf{r}_0$ and using $\mathbf{r}_k = P_k(A)\mathbf{r}_0$, and $\mathbf{z}_k = P_k^{(1)}(A)\mathbf{r}_0$ we get

$$\mathbf{r}_k = B_kA\mathbf{z}_{k-2} + D_kA\mathbf{r}_{k-1} + \mathbf{r}_{k-1}. \tag{27}$$

Since $\mathbf{r}_k = \mathbf{b} - A\mathbf{x}_k$, (14) becomes

$$\mathbf{x}_k = \mathbf{x}_{k-1} - B_k\mathbf{z}_{k-2} - D_k\mathbf{r}_{k-1}. \tag{28}$$

2.2. **Relation** B_6**.** Consider the following recurrence relation investigated in [1]

$$P_k^{(1)}(x) = (A_kx^2 + B_kx + C_k)P_{k-2}^{(1)}(x) + (D_kx + E_k)P_{k-1}^{(1)}(x), \tag{29}$$

where $P_k^{(1)}$, $P_{k-2}^{(1)}$ and $P_{k-1}^{(1)}$ are polynomials of degree k, $k-2$ and $k-1$ respectively and A_k, B_k, C_k, D_k and E_k are constants to be determined as already discussed in[1, 2] for the choices x^i, $P_k(x)$, $P_k^{(1)}(x)$. We discuss

and recall B_6 for general auxiliary polynomial U_i, a polynomial of exact degree i. Multiply (29) by U_i and apply $c^{(1)}$ on both sides to get

$$\begin{aligned} c^{(1)}(U_iP_k^{(1)}) = A_kc^{(1)}(x^2U_iP_{k-2}^{(1)}) + B_kc^{(1)}(xU_iP_{k-2}^{(1)}) \\ +C_kc^{(1)}(U_iP_{k-2}^{(1)}) + D_kc^{(1)}(xU_iP_{k-1}^{(1)}) + E_kc^{(1)}(U_iP_{k-1}^{(1)}). \end{aligned} \tag{30}$$

Using (C_2) we get

$$\begin{aligned} A_kc^{(1)}(x^2U_iP_{k-2}^{(1)}) + B_kc^{(1)}(xU_iP_{k-2}^{(1)}) + C_kc^{(1)}(U_iP_{k-2}^{(1)}) \\ +D_kc^{(1)}(xU_iP_{k-1}^{(1)}) + E_kc^{(1)}(U_iP_{k-1}^{(1)}) = 0. \end{aligned} \tag{31}$$

The orthogonality condition (C_2) is true for $\forall i = 0, 1, 2, \cdots k-5$.
For $i = k-4$, we get $A_kc^{(1)}(x^2U_{k-4}P_{k-2}^{(1)}) = 0$, which implies that $A_k = 0$ as $c^{(1)}(x^2U_{k-4}P_{k-2}^{(1)}) \neq 0$. But $P_k^{(1)}$ is a monic polynomial of degree k; so $D_k = 1$. For $i = k-3$, we get $B_kc^{(1)}(xU_{k-3}P_{k-2}^{(1)}) = 0$. Since $c^{(1)}(xU_{k-3}P_{k-2}^{(1)}) \neq 0$, $B_k = 0$.
For $i = k-2$, we have $c^{(1)}(xU_{k-2}P_{k-1}^{(1)}) + C_kc^{(1)}(U_{k-2}P_{k-2}^{(1)}) = 0$ which implies that

$$C_k = -\frac{c^{(1)}(xU_{k-2}P_{k-1}^{(1)})}{c^{(1)}(U_{k-2}P_{k-2}^{(1)})}. \tag{32}$$

For $i = k-1$, we get

$$c^{(1)}(xU_{k-1}P_{k-1}^{(1)}) + C_kc^{(1)}(U_{k-1}P_{k-2}^{(1)}) + E_kc^{(1)}(U_{k-1}P_{k-1}^{(1)}) = 0.$$

Since $c^{(1)}(U_{k-1}P_{k-1}^{(1)}) \neq 0$,

$$E_k = \frac{-c^{(1)}(xU_{k-1}P_{k-1}^{(1)}) - C_kc^{(1)}(U_{k-1}P_{k-2}^{(1)})}{c^{(1)}(U_{k-1}P_{k-1}^{(1)})}. \tag{33}$$

Hence (29) becomes

$$P_k^{(1)}(x) = C_kP_{k-2}^{(1)}(x) + (x + E_k)P_{k-1}^{(1)}(x),$$

where

$$C_k = -\frac{c^{(1)}(xU_{k-2}P_{k-1}^{(1)})}{c^{(1)}(U_{k-2}P_{k-2}^{(1)})},$$

and

$$E_k = \frac{-c^{(1)}(xU_{k-1}P_{k-1}^{(1)}) - C_kc^{(1)}(U_{k-1}P_{k-2}^{(1)})}{c^{(1)}(U_{k-1}P_{k-1}^{(1)})}.$$

2.2.1. *Relation* B_6 *when* $U_i(x) = P_i^{(1)}(x)$. In this case, (32) and (33) become $C_k = -\frac{c(x^2P_{k-2}^{(1)}P_{k-1}^{(1)})}{c(xP_{k-2}^{(1)}P_{k-2}^{(1)})}$, and
$E_k = \frac{-c^{(1)}(xP_{k-1}^{(1)}P_{k-1}^{(1)})-C_kc^{(1)}(P_{k-1}^{(1)}P_{k-2}^{(1)})}{c^{(1)}(P_{k-1}^{(1)}P_{k-1}^{(1)})}$, *respectively*. Using

$$\begin{cases} c^{(1)}(P_{k-1}^{(1)}P_{k-2}^{(1)}) = 0, \\ \mathbf{z}_k = P_k^{(1)}(A)\mathbf{r}_0,\ \tilde{\mathbf{z}}_k = P_k^{(1)}(A^T)\tilde{\mathbf{z}}_0,\ \mathbf{r}_k = P_k(A)\mathbf{r}_0,\ \textit{and} \\ c(U_kP_k) = (y, U_k(A)P_k(A)r_0) = (U_k(A^T)y, P_k(A)r_0) = (\tilde{z}_k, r_k), \\ [\textit{note } \tilde{\mathbf{z}}_0 = y], \end{cases} \tag{34}$$

we get $C_k = -\frac{c(x^2P_{k-2}^{(1)}P_{k-1}^{(1)})}{c(xP_{k-2}^{(1)}P_{k-2}^{(1)})} = -\frac{(\tilde{\mathbf{z}}_{k-2}, A^2\mathbf{z}_{k-1})}{(\tilde{\mathbf{z}}_{k-2}, A\mathbf{z}_{k-2})}$, and
$E_k = -\frac{c(x^2P_{k-1}^{(1)}P_{k-1}^{(1)})}{c(xP_{k-1}^{(1)}P_{k-1}^{(1)})} = -\frac{(\tilde{\mathbf{z}}_{k-1}, A^2\mathbf{z}_{k-1})}{(\tilde{\mathbf{z}}_{k-1}, A\mathbf{z}_{k-1})}$.
Hence after evaluating the coefficients A_k, B_k, C_k, D_k and E_k for the choice $U_i(x) = P_i^{(1)}(x)$ equation (29) reduces to

$$P_k^{(1)}(x) = C_kP_{k-2}^{(1)}(x) + (x + E_k)P_{k-1}^{(1)}(x). \tag{35}$$

Replacing x by A, multiplying by $\mathbf{r}_0$ and using $\mathbf{z}_k = P_k^{(1)}(A)\mathbf{r}_0$ we get

$$\mathbf{z}_k = C_k\mathbf{z}_{k-2} + A\mathbf{z}_{k-1} + E_k\mathbf{z}_{k-1}.$$

Replacing x by A^T and multiply by $\tilde{\mathbf{z}}_0 = \mathbf{y}$ and using $\tilde{\mathbf{z}}_k = P_k^{(1)}(A^T)\tilde{\mathbf{z}}_0$, we get

$$\tilde{\mathbf{z}}_k = C_k\tilde{\mathbf{z}}_{k-2} + A^T\tilde{\mathbf{z}}_{k-1} + E_k\tilde{\mathbf{z}}_{k-1}.$$

2.3. **Algorithm** A_{19}/B_6**.** We now consider the combination of A_{19} and B_6 for the choice $U_i(x) = P_i^{(1)}(x)$ which, as said earlier, was never considered before. The new algorithm is called A_{19}/B_6 and its pseudo-code is given below as Algorithm 1.

Algorithm 1 Lanczos-type Algorithm A_{19}/B_6.

Choose $\mathbf{x}_0$ and $\mathbf{y}$ such that $\mathbf{y} \neq 0$.
Set $\mathbf{r}_0 = \mathbf{b} - A\mathbf{x}_0$, $\tilde{\mathbf{z}}_0 = \mathbf{y}$,
$\mathbf{z}_0 = \mathbf{r}_0$.
$\mathbf{p} = A\mathbf{r}_0$, $\mathbf{p}_1 = A\mathbf{p}$, $\mathbf{p}_2 = A\mathbf{p}_1$, $\mathbf{p}_3 = A\mathbf{p}_2$,
$c_0 = (\mathbf{y}, \mathbf{r}_0)$, $c_1 = (\mathbf{y}, \mathbf{p})$, $c_2 = (\mathbf{y}, \mathbf{p}_1)$,
$c_3 = (\mathbf{y}, \mathbf{p}_2)$, $c_4 = (\mathbf{y}, \mathbf{p}_3)$, $\delta = c_1c_3 - c_2^2$,
$\alpha = \frac{c_0c_3 - c_1c_2}{\delta}$, $\beta = \frac{c_0c_2 - c_1^2}{\delta}$, $\delta_1 = c_1c_3 - c_2^2$,
$\alpha_1 = \frac{c_1c_4 - c_2c_3}{\delta_1}$, $\beta_1 = \frac{c_2c_4 - c_3^2}{\delta_1}$,
$\mathbf{r}_1 = \mathbf{r}_0 - (\frac{c_0}{c_1})\mathbf{p}$, $\mathbf{x}_1 = \mathbf{x}_0 + (\frac{c_0}{c_1})\mathbf{r}_0$,
$\mathbf{r}_2 = \mathbf{r}_0 - \alpha\mathbf{p} + \beta\mathbf{p}_1$,
$\mathbf{x}_2 = \mathbf{x}_0 + \alpha\mathbf{r}_0 - \beta\mathbf{p}$,
$\mathbf{z}_1 = \mathbf{p} - (\frac{c_2}{c_1})\mathbf{r}_0$, $\mathbf{z}_2 = \mathbf{p}_1 - \alpha_1\mathbf{p} + \beta_1\mathbf{r}_0$,
$\mathbf{y}_1 = A^T\mathbf{y}$, $\mathbf{y}_2 = A^T\mathbf{y}_1$,
$\tilde{\mathbf{z}}_1 = \mathbf{y}_1 - (\frac{c_2}{c_1})\tilde{\mathbf{z}}_0$,
$\tilde{\mathbf{z}}_2 = \mathbf{y}_2 - \alpha_1\mathbf{y}_1 + \beta_1\tilde{\mathbf{z}}_0$,
for k=3,4,... **do**
$q_1 = Ar_{k-1}$, $q_2 = Az_{k-1}$, $q_3 = Aq_2$, $q_4 = Az_{k-2}$, $s = A^T\tilde{\mathbf{z}}_{k-1}$
$a_{11} = c^{(1)}(P^{(1)}_{k-2}P^{(1)}_{k-2}) = (\tilde{\mathbf{z}}_{k-2}, q_4)$
$a_{12} = c(xP^{(1)}_{k-2}P_{k-1}) = (\tilde{\mathbf{z}}_{k-2}, q_1)$
$a_{22} = c(xP^{(1)}_{k-1}P_{k-1}) = (\tilde{\mathbf{z}}_{k-1}, q_1)$
$b_2 = -c(P^{(1)}_{k-1}P_{k-1}) = -(\tilde{\mathbf{z}}_{k-1}, \mathbf{r}_{k-1})$
$\Delta_k = a_{11}a_{22}$,
if $\Delta_k \leq \epsilon$
print "ghost-type breakdown"
stop.
end if
$B_k = -\frac{b_2a_{12}}{\Delta_k}$, $D_k = \frac{b_2}{a_{22}}$
$\mathbf{r}_k = B_kq_4 + D_kq_1 + \mathbf{r}_{k-1}$
$\mathbf{x}_k = \mathbf{x}_{k-1} - B_k\mathbf{z}_{k-2} - D_k\mathbf{r}_{k-1}$
if $||\mathbf{r}_k|| > \epsilon$ **then**
$C_k = -\frac{c(x^2P^{(1)}_{k-2}P^{(1)}_{k-1})}{c(xP^{(1)}_{k-2}P^{(1)}_{k-2})} = -\frac{(\tilde{\mathbf{z}}_{k-2}, q_3)}{(\tilde{\mathbf{z}}_{k-2}, q_4)}$
$E_k = -\frac{c(x^2P^{(1)}_{k-1}P^{(1)}_{k-1})}{c(xP^{(1)}_{k-1}P^{(1)}_{k-1})} = -\frac{(\tilde{\mathbf{z}}_{k-1}, q_3)}{(\tilde{\mathbf{z}}_{k-1}, q_2)}$
$\mathbf{z}_k = C_k\mathbf{z}_{k-2} + q_2 + E_k\mathbf{z}_{k-1}$
$\tilde{\mathbf{z}}_k = C_k\tilde{\mathbf{z}}_{k-2} + s + E_k\tilde{\mathbf{z}}_{k-1}$
else
$\mathbf{x} = \mathbf{x}_k$
stop
end if
end for

Algorithm 1 has been implemented in Matlab and tested on the following problem which was considered in [17, 1, 33, 21]. This problem arises in the 5-point discretisation of the operator $\frac{-d^2}{dx^2} - \frac{d^2}{dy^2} + \gamma\frac{d}{dx}$ on a rectangular region, [17, 1]. Comparative results on instances of the problem $A\mathbf{x} = \mathbf{b}$ ranging from dimension 10 to 900 for parameter δ taking value 0.0 and 0.2 and for the tolerance $\epsilon = 10^{-05}$ are recorded in Tables 1 and 2.

$$A = \begin{pmatrix} B & -I & \cdots & \cdots & 0 \\ -I & B & -I & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & -I & B & -I \\ 0 & \cdots & \cdots & -I & B \end{pmatrix},$$

with

$$B = \begin{pmatrix} 4 & \alpha & \cdots & \cdots & 0 \\ \beta & 4 & \alpha & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \beta & 4 & \alpha \\ 0 & \cdots & & \beta & 4 \end{pmatrix},$$

and $\alpha = -1 + \delta$, $\beta = -1 - \delta$. When $\delta = 0$, the matrix of coefficients A is symmetric and the problem is easy to solve. For $\delta = 0.2$ the matrix A is non-symmetric and the problem is comparatively harder as the region is not a regular mesh. The dimension of the matrix $B = 10$. The right hand side $\mathbf{b}$ is taken to be $\mathbf{b} = A\mathbf{x}$, where $\mathbf{x} = (1, 1, \ldots, 1)^T$, is the solution of the system.

TABLE 1. A_{19}/B_6 versus A_{12}, A_{12}^{new}, A_5/B_{10} and A_8/B_{10} for problems of different dimensions when $\delta = 0$

	A_5/B_{10}		A_8/B_{10}		A_{12}		A_{12}^{new}		A_{19}/B_6	
n	$\|\|r_k\|\|$	t(sec)	$\|\|r_k\|\|$	t(sec)	$\|\|r_k\|\|$	t(sec)	$\|\|r_k\|\|$	t(sec)	$\|\|r_k\|\|$	t(sec)
10	$2.2940E^{-13}$	0.010854	$1.7704E^{-13}$	0.010376	$4.9623E^{-13}$	0.048924	$2.9118E^{-13}$	0.020067	$6.8468E^{-13}$	0.008025
20	$2.5256E^{-14}$	0.011083	$1.7489E^{-13}$	0.010333	$1.7536E^{-13}$	0.048976	$2.4453E^{-15}$	0.020012	$6.1935E^{-07}$	0.008509
30	$3.9026E^{-09}$	0.011525	$4.9472E^{-09}$	0.010885	$5.4705E^{-08}$	0.049626	$2.5346E^{-10}$	0.021378	$8.5523E^{-06}$	0.009085
40	$1.4770E^{-10}$	0.011533	$8.4658E^{-10}$	0.011027	$1.4776E^{-08}$	0.049785	$3.6924E^{-11}$	0.021413	$8.1290E^{-06}$	0.010240
50	$1.9959E^{-06}$	0.012044	$1.3598E^{-06}$	0.011429	$4.7994E^{-06}$	0.051143	$1.2732E^{-06}$	0.022533	$9.2021E^{-06}$	0.010890
60	$9.1910E^{-06}$	0.012473	$3.7470E^{-06}$	0.011487	$5.0010E^{-06}$	0.051385	$2.3592E^{-06}$	0.022561	$3.1422E^{-06}$	0.010661
70	$4.9035E^{-06}$	0.013022	$4.2579E^{-06}$	0.012160	$1.3781E^{-06}$	0.052743	$5.1279E^{-07}$	0.023865	$4.5622E^{-06}$	0.011104
80	$4.4311E^{-06}$	0.013973	$7.7199E^{-06}$	0.013356	$7.5581E^{-06}$	0.052522	$3.5448E^{-06}$	0.023640	$7.2687E^{-06}$	0.011604
90	NaN		$9.2478E^{-06}$	0.019182	$6.2686E^{-06}$	0.055501	$4.4189E^{-06}$	0.024360	$3.4159E^{-06}$	0.012699
100	$1.1889E^{-06}$	0.013331	$3.1695E^{-06}$	0.012546	$8.9530E^{-07}$	0.052106	$2.3809E^{-07}$	0.023172	$5.8577E^{-06}$	0.012053
200	NaN		NaN		NaN		$8.2198E^{-06}$	0.041164	$5.8982E^{-06}$	0.026923
300	NaN		NaN		NaN		$7.3650E^{-06}$	0.083002	$3.6483E^{-06}$	0.055710
400	NaN		NaN		NaN		$8.2378E^{-06}$	0.121768	$8.9684E^{-06}$	0.130579
500	NaN		NaN		NaN		$9.8283E^{-06}$	0.990127	$7.5932E^{-06}$	0.260776
600	NaN		NaN		NaN		$9.8207E^{-06}$	1.574158	$9.8773E^{-06}$	0.466735
700	NaN		NaN		NaN		$9.4625E^{-06}$	2.854015	$9.7121E^{-06}$	0.720233
800	NaN		NaN		NaN		$9.1387E^{-06}$	4.476857	$8.6149E^{-06}$	1.254427
900	NaN		NaN		NaN		NaN		$7.4319E^{-06}$	4.066442

The results show that for $\delta = 0$ algorithm A_{19}/B_6 solved the given problems for dimensions up to 900 while the existing three algorithms namely A_5/B_{10}, A_8/B_{10} and A_{12} failed on systems of dimension $n > 100$.

TABLE 2. A_{19}/B_6 versus A_{12}, A_{12}^{new}, A_5/B_{10} and A_8/B_{10} for problems of different dimensions when $\delta = 0.2$

	A_5/B_{10}		A_8/B_{10}		A_{12}		A_{12}^{new}		A_{19}/B_6	
n	$\|\|r_k\|\|$	t(sec)	$\|\|r_k\|\|$	t(sec)	$\|\|r_k\|\|$	t(sec)	$\|\|r_k\|\|$	t(sec)	$\|\|r_k\|\|$	t(sec)
10	$9.8216E^{-10}$	0.017858	$2.3567E^{-10}$	0.010808	$2.0583E^{-08}$	0.049232	$4.7266E^{-08}$	0.021279	$7.5451E^{-06}$	0.017863
20	$4.1778E^{-11}$	0.011341	$5.8526E^{-11}$	0.010930	$6.3915E^{-10}$	0.049101	$5.9892E^{-10}$	0.021293	$3.7108E^{-06}$	0.020886
30	$2.6438E^{-06}$	0.012366	$5.9072E^{-06}$	0.012117	$5.9403E^{-06}$	0.050672	$6.8627E^{-06}$	0.022881	$9.2819E^{-06}$	0.021169
40	NaN		NaN		$7.6080E^{-06}$	0.051437	$7.8688E^{-06}$	0.023776	$6.8914E^{-06}$	0.023724
50	NaN		NaN		$8.8143E^{-06}$	0.066431	$5.0100E^{-06}$	0.028116	$7.2611E^{-06}$	0.022813
60	NaN		NaN		NaN		$2.6424E^{-06}$	0.024839	$5.9941E^{-06}$	0.025928
70	NaN		NaN		NaN		$9.9853E^{-06}$	0.237743	$9.3422E^{-06}$	0.024190
80	NaN		NaN		NaN		NaN		$3.8215E^{-06}$	0.025606
90	NaN		NaN		NaN		NaN		$7.7488E^{-06}$	0.037380
100	NaN		NaN		NaN		NaN		$7.7186E^{-06}$	0.045345
200	NaN		NaN		NaN		NaN		$3.5339E^{-06}$	0.052006
300	NaN		NaN		NaN		NaN		$6.5440E^{-06}$	0.136388
400	NaN		NaN		NaN		NaN		$7.2847E^{-06}$	0.259785
500	NaN		NaN		NaN		NaN		$2.8823E^{-06}$	0.278282
600	NaN		NaN		NaN		NaN		$8.9049E^{-06}$	0.445157

As can be seen in Table 2, for $\delta = 0.2$, algorithm A_{19}/B_6 solved the given problems for dimensions up to 500 while algorithms A_5/B_{10}, and A_8/B_{10} failed for $n = 40$ and A_{12}, A_{12}^{new} failed for $n = 60$ and above.
If we decrease the tolerance ϵ from 10^{-05} to 10^{-013} the numerical results are strongly in favor of A_{19}/B_6 which is clear from Table 3 and Table 4.

TABLE 3. A_{19}/B_6 versus A_{12}, A_{12}^{new}, A_5/B_{10} and A_8/B_{10} for problems of different dimensions when $\delta = 0$

	A_5/B_{10}		A_8/B_{10}		A_{12}		A_{12}^{new}		A_{19}/B_6	
n	$\|\|r_k\|\|$	t(sec)	$\|\|r_k\|\|$	t(sec)	$\|\|r_k\|\|$	t(sec)	$\|\|r_k\|\|$	t(sec)	$\|\|r_k\|\|$	t(sec)
10	$4.5196e^{-015}$	0.010969	$4.4052e^{-014}$	0.010418	$3.4389e^{-014}$	0.048920	$2.9118e^{-015}$	0.020013	$9.4527e^{-014}$	0.019143
20	$2.5256e^{-014}$	0.010665	$1.6217e^{-014}$	0.010309	$4.4538e^{-014}$	0.048298	$2.4453e^{-015}$	0.019989	$8.2368e^{-014}$	0.019618
30	NaN		NaN		NaN		$9.2583e^{-014}$	0.030612	$3.2875e^{-014}$	0.023630
40	NaN		NaN		NaN		$8.4447e^{-014}$	0.027055	$2.2496e^{-014}$	0.027095
50	NaN		NaN		NaN		$4.4856e^{-014}$	0.057526	$1.5384e^{-014}$	0.027677
60	NaN		NaN		NaN		NaN		$3.9895e^{-014}$	0.025539
70	NaN		NaN		NaN		$9.0212e^{-014}$	0.047472	$2.0157e^{-014}$	0.027536
80	NaN		NaN		NaN		NaN		$7.3023e^{-014}$	0.028718
90	NaN		NaN		NaN		$4.5477e^{-014}$	0.072918	$6.4488e^{-014}$	0.032048
100	NaN		NaN		NaN		$6.8764e^{-014}$	0.193226	$4.8541e^{-014}$	0.030108
200	NaN		NaN		NaN		$7.6191e^{-014}$	0.359243	$4.8439e^{-014}$	0.076182
300	NaN		NaN		NaN		$4.3659e^{-014}$	1.027307	$6.8328e^{-014}$	0.227479
400	NaN		NaN		NaN		$9.7388e^{-014}$	3.493297	$7.2821e^{-014}$	0.517246
500	NaN		NaN		NaN		$9.5239e^{-014}$	10.783390	$6.3551e^{-014}$	1.951565

The results show that for $\epsilon = 10^{-013}$ and $\delta = 0$ algorithms A_{19}/B_6 and A_{12}^{new} solved the given problem for dimensions up to 500 while A_5/B_{10}, A_8/B_{10} and A_{12} failed for $n = 30$ and above.

TABLE 4. A_{19}/B_6 versus A_{12}, A_{12}^{new}, A_5/B_{10} and A_8/B_{10} for problems of different dimensions when $\delta = 0.2$

	A_5/B_{10}		A_8/B_{10}		A_{12}		A_{12}^{new}		A_{19}/B_6	
n	$\|\|r_k\|\|$	t(sec)	$\|\|r_k\|\|$	t(sec)	$\|\|r_k\|\|$	t(sec)	$\|\|r_k\|\|$	t(sec)	$\|\|r_k\|\|$	t(sec)
10	$1.4521E^{-14}$	0.012071	$4.7905E^{-14}$	0.011465	$2.9806E^{-14}$	0.050647	NaN		$2.4294E^{-14}$	0.020081
20	NaN		NaN		NaN		NaN		$6.0869E^{-14}$	0.028402
30	NaN		NaN		NaN		NaN		$5.1766E^{-14}$	0.028676
40	NaN		NaN		NaN		NaN		$5.1502E^{-14}$	0.032685
50	NaN		NaN		NaN		NaN		$8.9242E^{-14}$	0.031879
60	NaN		NaN		NaN		NaN		$1.9212E^{-14}$	0.037306
70	NaN		NaN		NaN		NaN		$5.5211E^{-14}$	0.048051
80	NaN		NaN		NaN		NaN		$9.8420E^{-14}$	0.049704
90	NaN		NaN		NaN		NaN		$5.0930E^{-14}$	0.061245
100	NaN		NaN		NaN		NaN		$9.0537E^{-14}$	0.069353
200	NaN		NaN		NaN		NaN		$1.0460E^{-14}$	0.126791

Again, for $\epsilon = 10^{-013}$ and $\delta = 0.2$ algorithm A_{19}/B_6 solved the given problems up to dimension 200 while the other algorithms failed for $n = 10$ and above. The obvious reason is breakdown, [19, 20]. Since all these algorithms consist of recursively computing P_k and $P_k^{(1)}$, which involves the calculation of some scalar products appearing as denominators and numerators of the coefficient of the recurrence relationships, when any of the denominators become very small, as small as, for instance 2.3879×10^{-014}, breakdown occurs and the algorithms fail. This breakdown issue is being investigated further and any finding will be reported in forthcoming papers.

According to [1, 17, 21, 33], algorithms A_{12}, A_{12}^{new}, A_5/B_{10} and A_8/B_{10} are considered as the most robust Lanczos-type algorithms. We have now compared our new algorithm with these algorithms on a standard problem considered in this paper and elsewhere. Our results show that algorithm A_{19}/B_6 is faster through out, and more robust overall.

3. Conclusion

In this paper, we derived the recurrence relation A_{19} [17] and recalled B_6 [1] both using the general auxilliary polynomial $U_i(x)$. We used A_{19} in tandem with B_6 to derive a new Lanczos-type algorithm A_{19}/B_6. This new algorithm has been applied to a number of instances of some standard test problem considered in [17, 1, 33] and elsewhere. The performance of this algorithm is compared to that of existing and well established algorithms of the same type namely, A_{12}, A_{12}^{new}, A_5/B_{10} and A_8/B_{10}, [2, 17, 33]. Numerical results are strongly in favour of the new algorithm A_{19}/B_6.

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