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DESCRIBING SEMIGROUPS WITH DEFINING RELATIONS OF THE FORM $xy = yz$ AND $yx = zy$ AND CONNECTIONS WITH KNOT THEORY

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Abstract. We introduce a knot semigroup as a cancellative semigroup whose defining relations are produced from crossings on a knot diagram in a way similar to the Wirtinger presentation of the knot group; to be more precise, a knot semigroup as we define it is closely related to such tools of knot theory as the 2-fold branched cyclic cover space of a knot and the involutory quandle of a knot. We describe knot semigroups of several standard classes of knot diagrams, including torus knots and torus links $T(2, n)$ and twist knots. The description includes a solution of the word problem. To produce this description, we introduce alternating sum semigroups as certain naturally defined factor semigroups of free semigroups over cyclic groups. We formulate several conjectures for future research.

1. The context and the paper plan

We consider cancellative semigroups (which we call knot semigroups) whose defining relations come in pairs of the form $xy = yz$ and $yx = zy$, where x, y, z are generators, and are 'read' from a certain natural diagram (namely, a knot diagram[1]). Our inspiration in this research comes partially from the study of right-angled Artin groups (and the corresponding semigroup-theory construction, trace monoids [1]). A right-angled Artin group is a group in which every defining relation has a form $xy = yx$. Given an undirected graph, one can define a group whose set of generators is the set of vertices of the graph, and a defining relation $xy = yx$ is introduced whenever vertices x and y are adjacent. This construction defines a natural correspondence between undirected graphs and right-angled Artin groups. Whereas knot diagrams are not as ubiquitous as graphs, they have attracted much attention of algebraists in the last century, and knot semigroups described in this paper can become a new natural way of defining semigroups corresponding to knot diagrams; we discuss this further in Section 8.

Each relation defining a knot semigroup has words of the same length on the two sides of the equality; such relations are called homogeneous and, accordingly, semigroups defined in this way are also sometimes called homogeneous; another example of homogeneous semigroups are braid semigroups (for their definition see, for example, [2]); for a brief review of more examples of classes of homogeneous semigroups see [3].

There has been a number of attempts to define conjugate elements in semigroups, generalising conjugation in groups. For recent reviews of ideas in this direction, see

[1] Note that a knot semigroup is not a knot invariant; that is, there are cases when two different diagrams of the same knot produce non-isomorphic knot semigroups. See more in Section 8.

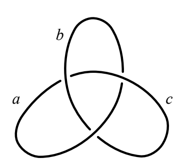
[4, 5]. In this context, knot semigroups are interesting as examples of semigroups where conjugacy is introduced explicitly: indeed, each pair of relations defining a knot semigroup states precisely that two elements x, z in the semigroup are conjugate.

Garside monoids are a well-known type of semigroups inspired by braid-theory ideas, and thin monoids are a generalisation of Garside monoids [6]. Divisibility monoids are a generalisation of trace monoids and are related to Garside monoids [7]. With the exception of some trivial cases, knot monoids are neither Garside monoids nor divisibility monoids. However, all knot monoids which we describe in this paper are thin[2].

The aim of this paper is to consider several standard types of knot diagrams and explicitly describe their knot semigroups (by the way, our description solves the word problem in the semigroups). Thus, we both produce many representative examples of knot semigroups and develop a method of studying them. Specifically, we describe knot semigroups of canonical diagrams of torus knots and torus links $T(2, n)$ and twist knots in Sections 4, 6, 7; a generalisation is suggested in Conjecture 23. The study of knot semigroups can be naturally expanded in a number of ways, including studying knot semigroups of braids. We describe one knot semigroup of a braid in Section 5, because we are naturally drawn to it by its similiarities with knot diagrams of torus knots and torus links $T(2, n)$. We formulate Conjecture 25 for future research of knot semigroups of braids. We discuss connections with knot theory in Section 8 and formulate Conjecture 24 suggesting a use of knot semigroups for detecting trivial knots.

2. Main definitions

2.1. **Knot semigroups.** We assume that the reader has at least intuitive understanding of what knots, links and braids are, and how knot diagrams represent knots; if not, see any standard knot theory textbook, for example, [8, 9, 10, 11, 12, 13].



By an arc we mean a continuous line on a knot diagram from one undercrossing to another undercrossing; for example, consider the knot diagram on the left, which we shall denote by $\mathfrak{t}_3$, and which represents a knot known as the trefoil knot. It has three arcs, denoted by a, b and c. If a knot diagram $\mathfrak{d}$ is given, we define a semigroup using this diagram, which we call the *knot semigroup* of $\mathfrak{d}$ and denote by $K\mathfrak{d}$. We assume that each arc is denoted by a letter. Then at every crossing where, say, arcs x and z form the undercrossing and arc y is the overcrossing, 'read' two defining relations $xy = yz$ and $yx = zy$. The cancellative semigroup generated by the arc letters with these defining relations is the knot semigroup of the knot diagram. For example, on diagram $\mathfrak{t}_3$ we can read relations $ab = bc$ and $ba = cb$ at the left-top crossing, relations $ac = cb$ and $ca = bc$ at the right-top crossing and relations $ba = ac$ and $ab = ca$ at the bottom crossing. Using

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[2]The property of being thin relative to knot semigroups can be reformulated as follows: if two words $u = au'$ and $v = bv'$ are equal, where a, b are letters, then there are letters c, d such that $ac = bd$ and $u = acu''$ and $v = bdv''$.

these relations, one can deduce equalities of words in $K\mathfrak{t}_3$ such as, for example, $aa = bb$; indeed, $aac = acb = bab = bbc$, hence, using cancellation, $aa = bb$.

Note that it is perfectly possible to consider a *knot monoid* instead of a knot semigroup, as a knot semigroup with an added identity element 1. Every result in this paper can be immediately reformulated for knot monoids.

Knot semigroups are convenient objects to study; in particular, this is so due to the following properties (which knot semigroups share with all homogeneous semigroups).

Proposition 1. *1) Only words of the same length can be equal in a knot semigroup.*

2) Every knot semigroup is $\mathcal{J}$-trivial.

3) Every knot semigroup has a unique set of generators.

In general, the knot semigroup of a knot cannot be embedded into the knot group of the knot (despite what simple examples in Section 3 may suggest). The knot semigroup $K\mathfrak{t}_3$ of the diagram above provides a convenient counterexample confirming this. Assume that $K\mathfrak{t}_3$ is embedded into a group G (actually, one can prove that $K\mathfrak{t}_3$ can be embedded into a group: this follows from Theorem 3 and a remark in Subsection 2.2). In G, we can deduce $a^{-1}b = bc^{-1}$ by considering $bc = ab$ in $K\mathfrak{t}_3$ and multiplying by a^{-1} on the left and by c^{-1} on the right. Similarly, in G we have $bc^{-1} = c^{-1}a$; from these two equalities, we conclude $a^{-1}b = c^{-1}a$. Likewise, in G we have $c^{-1}b = ac^{-1}$ and $ac^{-1} = b^{-1}a$; hence, we conclude $c^{-1}b = b^{-1}a$. Therefore, $a^{-1}ba^{-1}ba^{-1}b = c^{-1}aa^{-1}ba^{-1}b = c^{-1}ba^{-1}b = b^{-1}aa^{-1}b = 1$. Thus, element $a^{-1}b \in G$ has order 3. As to the knot group of the trefoil knot, it is torsion-free because every knot group is torsion-free; see Lemma 2 in [14].

As we see from the definition, the presentation defining a knot semigroup is not a usual semigroup presentation, but a cancellative presentation. In general, we do not know yet how to produce a presentation re-defining a given knot semigroup which has been defined by a cancellative presentation[3]. Employing cancellation is natural because it ensures that the knot semigroup is preserved by the first Reidemeister move[4]. However, the knot semigroup is not preserved by neither the second nor the third Reidemeister move; see Section 8 for more details and a discussion.

2.2. **Alternating sum semigroups.** Let us introduce abstract semigroups which we will use to describe knot semigroups. Let a group G be either $\mathbb{Z}_n$ or $\mathbb{Z}$. Let $B \subseteq G$. By the alternating sum of a word $b_1b_2b_3b_4 \dots b_k \in B^+$ we shall mean the value of the expression $b_1 - b_2 + b_3 - b_4 + \cdots + (-1)^{k+1}b_k$ calculated in G. We shall say that two words $u, v \in B^+$ are in relation $\sim$ if and only if 1) the length of u is equal to the length of v; 2) the alternating sum of u (calculated in G) is equal to the alternating sum of v. It is obvious that $\sim$ is a congruence on B^+. Let us denote the factor semigroup $B^+/\!\sim$ by $AS(G, B)$ and call it an alternating sum semigroup.

Let us say that $g \in G$ is even (odd) in G if g can be represented in the form $g = 2h$ ($g = 2h+1$) for some $h \in G$. In other words, if $G = \mathbb{Z}_n$, where n is odd, then every element of G is both even in G and odd in G; if $G = \mathbb{Z}_n$, where n is even, or if

[3]It is known, see, for example, [15], that a presentation of a cancellative semigroup can be considerably more complicated than its cancellative presentation.

[4]Reidemeister moves are standard ways of transforming a knot diagram to produce other diagrams of the same knot.

FIGURE 3.1. A standard and a non-standard diagrams of the unknot

$G = \mathbb{Z}$ then an element of G is even in G (odd in G) if and only if it is even (odd) as an integer.

We shall say that two words $u, v \in B^+$ are in relation $\approx$ if and only if 1) the length of u is equal to the length of v; 2) the alternating sum of u (calculated in G) is equal to the alternating sum of v; 3) the number of entries in u which are even in G is equal to the number of entries in v which are even in G (or, equivalently, the number of entries in u which are odd in G is equal to the number of entries in v which are odd in G). To give a simple example, the two words $5 \cdot 3 \cdot 2, 15 \cdot 10 \cdot (-1) \in \mathbb{Z}^+$ are in relation $\approx$. It is obvious that $\approx$ is a congruence. Let us denote the factor semigroup $B^+/\approx$ by $SAS(G, B)$ and call it a strong alternating sum semigroup. Obviously, if $G = \mathbb{Z}_n$, where n is odd, then $SAS(G, B)$ coincides with $AS(G, B)$.

Although it is a marginal topic for this paper, it is worth noting that every alternating sum semigroup can be embedded into a group, and the following simple construction explicitly describes the smallest group containing an alternating sum semigroup $AS(G, B)$. Consider a free semigroup F over the alphabet $B \times \{1, -1\}$. Consider an equivalence on F defined as follows: two words are related if 1) the alternating sums of the first components (calculated in G) are equal; 2) the sums of the second components (calculated in $\mathbb{Z}$) are equal. It is easy to check that this is a group congruence. An embedding of the alternating sum semigroup into the factor semigroup is induced by the mapping $b \mapsto (b, 1)$ for each $b \in B$. A similar construction can also be built to demonstrate that every strong alternating sum semigroup can be embedded into a group.

3. EXAMPLES

3.1. **Trivial knot.** By $\mathbb{N}$ we denote the infinite cyclic semigroup of positive integers. The canonical diagram of the trivial knot contains one arc and no crossings, see Figure 3.1; therefore, its knot semigroup is isomorphic to $\mathbb{N}$. For comparison, the knot group of the trivial knot is isomorphic to $\mathbb{Z}$ (see, for example, Corollary 11.3 in [8]).

As a related example, consider a non-standard diagram of the trivial knot on Figure 3.1. It is not difficult to check that in its knot semigroup all generators are equal, therefore, the knot semigroup is isomorphic to $\mathbb{N}$. Indeed, from $bc = ca = gc$ it follows $b = g$. From $bf = fc = df$ it follows $b = d$. Thus, $b = d = g$. From $gg = dg = ge$ it follows $g = e$. From $gd = df = bf = gf$ it follows $d = f$. Thus, $b = d = e = f = g$. From $fc = df = gd = fd$ it follows $c = d$. Finally, from $ab = ac = cb$ it follows $a = c$. Thus, $a = b = c = d = e = f = g$.

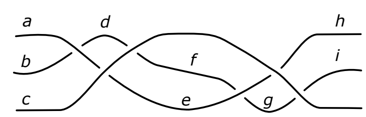
3.2. **Links.** The definition of the knot semigroup naturally generalises from diagrams of knots to diagrams of links. Here are two simple examples.

FIGURE 3.2. The trivial 2-component link and the Hopf link

The diagram of the trivial 2-component link in Figure 3.2 contains two arcs a, b and no crossings. Therefore, its knot semigroup is a free semigroup with two generators. For comparison, the knot group of this link is a free group with two generators (see Corollary 6.1.5 in [16]).

The diagram of the Hopf link in Figure 3.2 contains two arcs a, b and two crossings, each defining a single relation $ab = ba$. Hence, its knot semigroup is a free commutative semigroup with two generators. For comparison, the knot group of this link is a free Abelian group with two generators (see Example 6.2.5 in [16]).

3.3. **Braids.** It is perfectly possible to generalise the definition of a knot semigroup to braid diagrams [5]. As an example, consider the braid on the diagram below.

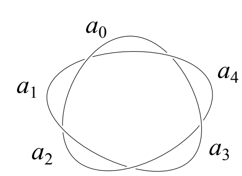


The arcs are labelled by a, b, c, d, f, g, h, i. Inspecting crossings, we read the following relations for the knot semigroup of the braid: $ba = ad$ and $ab = da$; $ac = ce$ and $ca = ec$; $dc = cf$ and $cd = fc$; $fe = eg$ and $ef = ge$; $ec = ch$ and $ce = hc$; $gc = ci$ and $cg = ic$.

The following equalities of arc letters can be deduced. From the relations, $ac = ce = hc$; hence, by cancellation, $a = h$. From the relations and since $a = h$, we have $abc = dac = dce = cfe = ceg = hcg = aic$; hence, by cancellation, $b = i$.

4. TORUS KNOTS $T(2, n)$

A torus knot $T(2, n)$ is the one whose canonical diagram, which we shall denote by $\mathfrak{t}_n$, consists of n anticlockwise half-twists of two strands connected together in a shape of a ring. For example, the diagram below is $\mathfrak{t}_5$, and the diagram of the trefoil knot in Section 2 is $\mathfrak{t}_3$. The parameter n needs to be an odd number to produce a knot; if n is even, the diagram represents not a knot, but a two-component link. In this section we describe the knot semigroup $K\mathfrak{t}_n$ of a knot diagram $\mathfrak{t}_n$ (with an odd n); the knot semigroup of a link diagram $\mathfrak{t}_n$ (with an even n) has a slightly more complicated structure, which we describe in Section 6.



It is worth mentioning briefly that a similar diagram consisting of n clockwise half-twists can be considered; it is known as a right-handed torus knot, whereas a

[5]Note that the term 'braid semigroup' already has a well established meaning, namely, it is the subsemigroup of the braid group generated by clockwise half-twists (see, for example, [2]). To avoid confusion, we do not use the words 'braid semigroup' to mean the knot semigroup of a braid.

diagram of the type $\mathfrak{t}_n$, consisting of n anticlockwise half-twists, is then referred to as a left-handed torus knot. By the definition of the defining relations, the knot semigroup of a knot diagram is isomorphic to that of its mirror image. Therefore, the knot semigroup of a right-handed torus knot is isomorphic to that of the corresponding left-handed torus knot.

In this section and in Sections 5, 6, 7 we describe knot semigroups by demonstrating that a knot semigroup is isomorphic to a certain abstract semigroup, always following the same plan, which we shall describe now. Suppose that A^+/κ is a knot semigroup, where A is the set of arcs and κ is the cancellative congruence on the free semigroup A^+ induced by the defining relations of the knot semigroup. Let $\sim$ be a congruence on B^+, where B is an alphabet of the same size as A. To establish an isomorphism between A^+/κ and $B^+/\sim$ we proceed as in the following Lemma (whose simple proof is omitted).

Lemma 2. *Consider a bijection $\phi : A \to B$. It induces an isomorphism between A^+ and B^+, which we shall also denote by ϕ. Suppose a congruence κ on A^+ and a congruence $\sim$ on B^+ are such that for each $u, v \in A^+$ if $u\ \kappa\ v$ then $\phi(u) \sim \phi(v)$. Then ϕ induces a mapping from A^+/κ to $B^+/\sim$, which we shall denote by ψ. Moreover, ψ is a homomorphism. Suppose a subset of B^+ exists, which we shall call the set of canonical words, such that in each class of $\sim$ there is exactly one canonical word and at least one word of each class of κ is mapped by ϕ to a canonical word. Then ψ is an isomorphism between A^+/κ and $B^+/\sim$.*

The following is the main result of this section.

Theorem 3. *The knot semigroup $K\mathfrak{t}_n$ of the torus knot diagram $\mathfrak{t}_n$ (where n is odd) is isomorphic to the alternating sum semigroup $AS(\mathbb{Z}_n, \mathbb{Z}_n)$.*

To prove the theorem, we shall proceed according to the plan outlined in Lemma 2. Let $A = \{a_0, a_1, \dots, a_{n-1}\}$ be the set of arcs of $\mathfrak{t}_n$. Consider a mapping ϕ from A to $\mathbb{Z}_n$ defined as $a_i \mapsto i$. It induces an isomorphism from A^+ to $\mathbb{Z}_n^+$, which we shall also denote by ϕ.

Lemma 4. *The equality $a_i a_{i+j} = a_{i+k} a_{i+j+k}$ is true in $K\mathfrak{t}_n$ (where n is odd) for all values of $i, j, k \in \mathbb{Z}_n$.*

Proof. Relations in $K\mathfrak{t}_n$ are the equalities $a_{i-1}a_i = a_i a_{i+1}$ and $a_i a_{i-1} = a_{i+1} a_i$ for all $i \in \mathbb{Z}_n$. Applying relations of the type $a_{i-1}a_i = a_i a_{i+1}$ repeatedly, we obtain $a_i a_{i+1} = a_{i+k} a_{i+1+k}$ for all values of $i, k \in \mathbb{Z}_n$. Similarly, one can obtain $a_i a_{i-1} = a_{i+k} a_{i-1+k}$ for all values of $i, k \in \mathbb{Z}_n$.

Consider $a_i a_{i+j} a_{i+j+1} = a_i a_{i+1} a_{i+2} = a_{i+2} a_{i+3} a_{i+2} = a_{i+2} a_{i+j+2} a_{i+j+1}$; hence, $a_i a_{i+j} = a_{i+2} a_{i+j+2}$. This proves the lemma for all values of $i, j \in \mathbb{Z}_n$ and $k = 2$.

Note that each value $k \in \mathbb{Z}_n$ can be represented as $k = 2p$ for some $p \in \mathbb{Z}_n$. Applying equalities of the form $a_i a_{i+j} = a_{i+2} a_{i+j+2}$ repeatedly p times, we obtain $a_i a_{i+j} = a_{i+k} a_{i+j+k}$ for every value of k. □

Notation x^t stands for x repeated t times. Canonical words in $\mathbb{Z}_n^+$ will be defined as words in which every entry (except, perhaps, the first one) is 0; that is, canonical words of length t have a form $c0^{t-1}$, where $c \in \mathbb{Z}_n$.

Consider a non-negative integer valued parameter $\pi(w)$ of a word w in $\mathbb{Z}_n^+$, which we shall call the defect of w, defined as the largest number $d > 1$ such that the entry in w at position d is not 0; otherwise, if such position d does not exists, $\pi(w)$ is 0.

Lemma 5. *A word in $\mathbb{Z}^+$ is canonical if and only if its defect is* 0.

Proof. The result follows obviously from the form of canonical words. □

Lemma 6. *Let u be a word in A^+. Unless the defect of $\phi(u)$ is* 0, *there is a word v in A^+ such that $v = u$ in $K\mathfrak{t}_n$ and the defect of $\phi(v)$ is less than the defect of $\phi(u)$.*

Proof. Suppose the defect of $\phi(u)$ is $d > 0$. Then by the definition of $\pi(\phi(u))$ the last non-zero entry q in $\phi(u)$ stands at position $d > 1$. To be more specific, $u = u'a_pa_qu''$, where $u' \in A^+$, word u'' consists only of letters a_0, and $p, q \in \mathbb{Z}_n$, with $q \neq 0$. Define $v = u'a_{p-q}a_0u''$; it is easy to see that $v = u$ in $K\mathfrak{t}_n$ and the defect of $\phi(v)$ is less than the defect of $\phi(u)$. □

Corollary 7. *Every word in A^+ is equal in $K\mathfrak{t}_n$ to a word in A^+ which is mapped by ϕ to a word with defect* 0.

Proof of the theorem. Relations in $K\mathfrak{t}_n$ are equalities $a_{i-1}a_i = a_ia_{i+1}$ and $a_ia_{i-1} = a_{i+1}a_i$ for all $i \in \mathbb{Z}_n$. It is obvious that for each relation $u = v$ the words $\phi(u)$ and $\phi(v)$ have the same length and the same alternating sum. Therefore, by Lemma 20, ϕ induces a homomorphism $\psi : K\mathfrak{t}_n \to \mathbb{Z}_n^+/\!\sim$.

Suppose two canonical words are $\sim$-equivalent; then they have a form c_10^{t-1} and c_20^{t-1} (where t is the length of the words). Since these two words coincide at every position (except, perhaps, the first one) and have the same alternating sum, we conclude $c_1 = c_2$. Thus, each class of $\sim$ contains at most one canonical word.

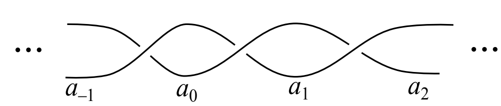
Consider a word $w \in \mathbb{Z}_n^+$ which has a length t and an alternating sum s. Then the canonical word $s0^{t-1}$ is $\sim$-equivalent to w. Thus, each class of $\sim$ contains at least one canonical word.

To prove that each word in A^+ is equal in $K\mathfrak{t}_n$ to a word mapped by ϕ to a canonical word, it is sufficient to refer to Corollary 7 and Lemma 5.

Now the result follows from Lemma 2. □

5. An infinite braid

In this section we study an object which can be seen informally as a 'limit case' when one considers $\mathfrak{t}_n$ as n tends to infinity. Namely, we describe the knot semigroup of a diagram of a two-strand braid consisting of infinitely many anticlockwise half-twists; let us denote it by $\mathfrak{B}$. A fragment of $\mathfrak{B}$ is shown below.



Since $\mathfrak{B}$ looks locally like a part of a diagram $\mathfrak{t}_n$, one can ask whether $K\mathfrak{B}$ is not similar to the alternating sum semigroups used in Section 4 to describe knot semigroups of torus knots $K\mathfrak{t}_n$; as we shall see, the description is similar, and finding it will prepare us for describing knot semigroups of torus links $K\mathfrak{t}_n$ (for even n) in Section 6.

Diagram $\mathfrak{B}$ is an interesting object to consider because it contains infinitely many arcs (denoted on the diagram by a_i) and infinitely many crossings. Some combinatorial techniques of knot theory do not work for such objects (which are referred to in knot theory as 'wild knots'). However, nothing stops us studying the knot semigroup $K\mathfrak{B}$ of $\mathfrak{B}$; the only new feature we shall notice is that $K\mathfrak{B}$ is infinitely generated.

Proceeding according to the plan outlined in Lemma 2, let $A = \{\dots, a_{-1}, a_0, a_1, \dots\}$ be the set of arcs of $\mathfrak{B}$. Consider a mapping ϕ from A to $\mathbb{Z}$ defined as $a_i \mapsto i$. It induces a isomorphism from A^+ to $\mathbb{Z}^+$, which we shall also denote by ϕ.

Lemma 8. *The equality $a_i a_{i+j} = a_{i+k} a_{i+j+k}$ is true in $K\mathfrak{B}$ for all values of i, j, k such that 1) j is odd or 2) k is even.*

Proof. Relations in $K\mathfrak{B}$ are equalities $a_{i-1}a_i = a_i a_{i+1}$ and $a_i a_{i-1} = a_{i+1} a_i$ for all $i \in \mathbb{Z}$. Applying relations of the type $a_{i-1}a_i = a_i a_{i+1}$ repeatedly, we obtain $a_i a_{i+1} = a_{i+k} a_{i+1+k}$ for all values of i, k. Similarly, one can obtain $a_i a_{i-1} = a_{i+k} a_{i-1+k}$ for all values of i, k.

Let us prove 2). For the case $k = 2$, consider $a_i a_{i+j} a_{i+j+1} = a_i a_{i+1} a_{i+2} = a_{i+2} a_{i+3} a_{i+2} = a_{i+2} a_{i+j+2} a_{i+j+1}$; hence, $a_i a_{i+j} = a_{i+2} a_{i+j+2}$. Applying equalities of this form repeatedly, we obtain $a_i a_{i+j} = a_{i+k} a_{i+j+k}$ for all even values of k. (In particular, note that $a_i a_i = a_{i+k} a_{i+k}$; we shall use this presently.)

To prove 1), assume that j is odd and consider $a_i a_{i+j} a_{i+j+1} = a_i a_{i+1} a_{i+2} = a_{i+1} a_{i+2} a_{i+2} = a_{i+1} a_{i+j+1} a_{i+j}$; hence, $a_i a_{i+j} = a_{i+1} a_{i+j+1}$. Applying equalities of this form repeatedly, we obtain $a_i a_{i+j} = a_{i+k} a_{i+j+k}$. □

Introduce canonical words in $\mathbb{Z}^+$ having the following form: for each word length t, canonical words are either $c0^{t-1}$, where c is an even integer, or $c1^{t-m-1}0^m$, where $0 \le m \le t-1$ and c is an odd integer.

Consider two non-negative integer valued parameters of a word w in $\mathbb{Z}^+$:

- Let $\pi_1(w)$ be the largest number p such that the entry in w at position p is even and the entry in w at position $p+1$ is odd; otherwise, if such position p does not exists, $\pi_1(w)$ is 0.
- Let $\pi_2(w)$ be the largest number $q > 1$ such that the entry in w at position q is neither 0 nor 1; otherwise, if such position q does not exists, $\pi_2(w)$ is 0.

By the defect of a word $u \in \mathbb{Z}^+$ we mean the pair $(\pi_1(u), \pi_2(u))$. The defects are assumed to be ordered by the lexicographic order.

Lemma 9. *A word in $\mathbb{Z}^+$ is canonical if and only if its defect is $(0, 0)$.*

Proof. The result follows obviously from the form of canonical words. □

Lemma 10. *Let u be a word in A^+. Unless the defect of $\phi(u)$ is $(0, 0)$, there is a word v in A^+ such that $v = u$ in $K\mathfrak{B}$ and the defect of $\phi(v)$ is less than the defect of $\phi(u)$.*

Proof. Suppose the defect of $\phi(u)$ is (p, q). Suppose $p > 0$. Note that in the notation of Lemma 8, the entries in u at positions p and $p+1$ are $a_i a_{i+j}$, where j is odd (because i is even and $i + j$ is odd). That is, $u = u' a_i a_{i+j} u''$, where $u', u'' \in A^+$, and all letters in u'' have even indices. Define $v = u' a_{i+1} a_{i+j+1} u''$; it is easy to see that $v = u$ in $K\mathfrak{B}$. In $\phi(v)$ the entry at position $p+1$ is even, and all entries in at positions greater than $p+1$ are also even. Therefore, $\pi_1(\phi(v))$ is at most $p-1$, hence, it is less than $\pi_1(\phi(u))$.

Now suppose $p = 0$ and $q > 0$. Then at some position $q > 1$ in u there is an entry a_d which is neither a_0 nor a_1. To be more specific, $u = u' a_c a_d u''$, where $u', u'' \in A^+$ and all letters in u'' are either a_0 or a_1. If d is even, define $v = u' a_{c-d} a_0 u''$. If d is odd, define $v = u' a_{c-d+1} a_1 u''$. In either case, it is easy to see that $v = u$ in $K\mathfrak{B}$. Note that the entry at position $q-1$ or q in $\phi(v)$ is even (odd) if and only if the entry at this position in $\phi(v)$ is even (odd); therefore, $\pi_1(\phi(v))$ is equal to $\pi_1(\phi(u))$, that is, $\pi_1(\phi(v)) = 0$. All entries in v at positions equal to or greater than q are either a_0 or a_1. Thus, $\pi_2(\phi(v))$ is at most $q-1$, therefore, it is less than $\pi_2(\phi(u))$. □

Corollary 11. *Every word in A^+ is equal in $K\mathfrak{B}$ to a word in A^+ which is mapped by ϕ to a word with defect $(0,0)$.*

Theorem 12. *The knot semigroup $K\mathfrak{B}$ of $\mathfrak{B}$ is isomorphic to $SAS(\mathbb{Z}, \mathbb{Z})$.*

Proof. Relations in $K\mathfrak{B}$ are equalities $a_{i-1}a_i = a_i a_{i+1}$ and $a_i a_{i-1} = a_{i+1} a_i$ for all $i \in \mathbb{Z}$. It is obvious that for each relation $u = v$ the words $\phi(u)$ and $\phi(v)$ have the same length and the same alternating sum. Since $i-1$ and $i+1$ are either both odd or both even, $\phi(u)$ and $\phi(v)$ have the same number of odd entries. Therefore, by Lemma 20, ϕ induces a homomorphism $\psi : K\mathfrak{B} \to \mathbb{Z}^+/\approx$.

Suppose two canonical words are $\approx$-equivalent. Since they have the same length and the same number of odd entries, the two words have a form either $c_1 0^{t-1}$ and $c_2 0^{t-1}$ (where c_1 and c_2 are even) or $c_1 1^{t-m-1} 0^m$ and $c_2 1^{t-m-1} 0^m$ (where c_1 and c_2 are odd). In either case, since these two words coincide at every position (except, perhaps, the first one) and have the same alternating sum, we conclude $p_1 = p_2$. Thus, each class of $\approx$ contains at most one canonical word.

Consider a word $w \in \mathbb{Z}^+$ which has a length t, an alternating sum s and contains d odd entries. If $d = 0$ then, obviously, s is even and then the word $s0^{t-1}$ is canonical and $\approx$-equivalent to w. Suppose d is positive and odd; then, obviously, s is odd and then the word $s1^{d-1}0^{t-d}$ is canonical and $\approx$-equivalent to w. Suppose d is positive and even; then, obviously, $s+1$ is odd and then the word $(s+1)1^{d-1}0^{t-d}$ is canonical and $\approx$-equivalent to u. Thus, each class of $\approx$ contains at least one canonical word.

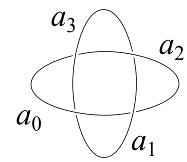
To prove that each word in A^+ is equal in $K\mathfrak{B}$ to a word mapped by ϕ to a canonical word, it is sufficient to refer to Corollary 11 and Lemma 9.

Now the result follows from Lemma 2. □

6. Torus links $T(2, n)$

The aim of this section is to describe knot semigroups of torus links $T(2, n)$ and to propose a formulation unifying this result with that describing knot semigroups of torus knots $T(2, n)$. The diagram below shows an example of a torus link, namely, $\mathfrak{t}_4$.

Theorem 13. *The knot semigroup $K\mathfrak{t}_n$ of the torus link diagram $\mathfrak{t}_n$ (where n is even) is isomorphic to the strong alternating sum semigroup $SAS(\mathbb{Z}_n, \mathbb{Z}_n)$.*

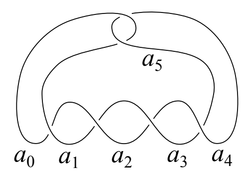


We skip the proof of this theorem because it almost literally repeats the arguments in Section 5, with $\mathbb{Z}_n$ substituted for $\mathbb{Z}$ where necessary.

Since $SAS(\mathbb{Z}_n, \mathbb{Z}_n) = AS(\mathbb{Z}_n, \mathbb{Z}_n)$ for odd values of n, one can combine the results of Theorems 3 and 13 in one statement:

Corollary 14. *The knot semigroup $K\mathfrak{t}_n$ of the diagram $\mathfrak{t}_n$ for every positive n is isomorphic to the strong alternating sum semigroup $SAS(\mathbb{Z}_n, \mathbb{Z}_n)$.*

7. Twist knots



A twist knot is the one whose canonical diagram, which we shall denote by $\mathfrak{tw}_n$, consists of n clockwise half-twists of two strands (shown on the diagram as horizontal) and 2 anticlockwise half-twists of two strands (shown on the diagram as vertical); the definition can be found, for instance, in Exercise E9.6 in [10]. For example, the diagram below is $\mathfrak{tw}_4$. In this section we describe knot semigroups of diagrams $\mathfrak{tw}_n$; they are isomorphic to alternating sum semigroups of some special type, as the following result shows. Denote the set $\{0, 1, \dots, n+1\}$ by $[n+2]$.

Theorem 15. *The knot semigroup $K\mathfrak{tw}_n$ of the twist knot diagram $\mathfrak{tw}_n$ is isomorphic to the alternating sum semigroup $AS(\mathbb{Z}_{2n+1}, [n+2])$.*

It is interesting to note that according to this theorem, $K\mathfrak{tw}_1$ is isomorphic to $K\mathfrak{t}_3$. This is not surprising, because diagrams $\mathfrak{tw}_1$ and $\mathfrak{t}_3$ represent the same knot, namely, the trefoil knot.

To prove the theorem, we proceed once again according to the plan in Lemma 2. Let $A = \{a_0, a_1, \dots, a_{n+1}\}$ be the set of arcs of $\mathfrak{tw}_n$. Consider a mapping ϕ from A to $[n+2]$ defined as $a_i \mapsto i$. It induces an isomorphism from A^+ to $[n+2]^+$, which we shall also denote by ϕ.

Lemma 16. *The equality $a_i a_{i+j} = a_{i+k} a_{i+j+k}$ is true in $K\mathfrak{tw}_n$ for all values of i, j, k such that $0 \le i \le i+j \le i+j+k \le n+1$.*

Proof. Relations in $K\mathfrak{tw}_n$ are, on the one hand, the equalities $a_{i-1}a_i = a_i a_{i+1}$ and $a_i a_{i-1} = a_{i+1} a_i$ for all $i = 1, 2, \dots, n$ (from the crossings at the bottom of the diagram) and, on the other hand, two equalities $a_0 a_{n+1} = a_{n+1} a_1$ and $a_{n+1} a_0 = a_1 a_{n+1}$ and two equalities $a_n a_0 = a_0 a_{n+1}$ and $a_0 a_n = a_{n+1} a_0$ (from the crossings at the top of the diagram). Using the former type of relations and the same proof as in the first two paragraphs of the proof of Lemma 4 or Lemma 8, one can show that $a_i a_{i+j} = a_{i+k} a_{i+j+k}$ for all even k.

Now we shall prove that $a_0 a_j = a_1 a_{j+1}$ for all values of $j = 0, 1, \dots, n$. Let j be odd. Consider $a_0 a_0 a_j = a_{j-1} a_{j-1} a_j = a_{j-1} a_j a_{j+1} = a_0 a_1 a_{j+1}$; hence, $a_0 a_j = a_1 a_{j+1}$. Consider $a_{n+1} a_0 a_0 = a_0 a_n a_0 = a_0 a_0 a_{n+1} = a_0 a_{n+1} a_1 = a_{n+1} a_1 a_1$; hence, $a_0 a_0 = a_1 a_1$. Let j be even and positive. Consider $a_0 a_0 a_j = a_1 a_1 a_j = a_{j-1} a_{j-1} a_j = a_{j-1} a_j a_{j+1} = a_{j-2} a_{j-1} a_{j+1} = a_0 a_1 a_{j+1}$; hence, $a_0 a_j = a_1 a_{j+1}$.

Now suppose k is odd. If i is even we have $a_i a_{i+j} = a_0 a_j = a_1 a_{j+1} = a_{i+1} a_{i+j+1} = a_{(i+1)+(k-1)} a_{(i+j+1)+(k-1)} = a_{i+k} a_{i+j+k}$. If i is odd we have $a_i a_{i+j} = a_1 a_{j+1} = a_0 a_j = a_{i+1} a_{i+j+1} = a_{(i+1)+(k-1)} a_{(i+j+1)+(k-1)} = a_{i+k} a_{i+j+k}$. □

Canonical words in $[n+2]^+$ will be defined as words of the form 000^{t-2} or $c00^{t-2}$ or $0c0^{t-2}$, where $t \geq 2$ is the length of the word and $c \in \{1, 2, \dots, n\}$. It is worth noting that canonical words of length 1 can only be of the first two kinds, thus, they are $0, 1, \dots, n+1$. (Let us comment informally that in an alternating sum semigroup $AS(n, B)$ with a 'sufficiently large' B we would expect, for almost every t, the set of words of length t to split into exactly n classes of the congruence $\sim$. This is why in $AS(2n+1, [n+2])$ we have exactly $2n+1$ canonical words of length t for each $t \geq 2$.)

Consider a non-negative integer valued parameter $\pi(w)$ of a word w in $[n+2]^+$, which is 0 if the first two entries in w are 00 or $c0$ or $0c$ for some $c \in \{1, 2, \dots, n\}$, and which is 1 otherwise. Define the defect of a word $w = b_1 b_2 b_3 ... b_t$ in $[n+2]^+$ as a word $\pi(w) b_3 ... b_t$. Defects are assumed to be ordered antilexicographically (that is, by the right-to-left dictionary order).

Lemma 17. *A word in $[n+2]^+$ is canonical if and only if its defect is a word consisting of* 0*s.*

Proof. The result follows obviously from the form of canonical words. □

Lemma 18. *Let u be a word in A^+. Unless the defect of $\phi(u)$ is a word consisting of* 0*s, there is a word v in A^+ such that $v = u$ in $K\mathfrak{tw}_n$ and the defect of $\phi(v)$ is less than the defect of $\phi(u)$.*

Proof. Suppose the defect of $\phi(u)$ has a non-zero entry at a position which is not the first one. This means that at some position $d \geq 3$ there is a non-zero entry r in $\phi(u)$. To be more specific, $u = u' a_p a_q a_r u''$, where $u', u'' \in A^+$ and $p, q, r \in [n+2]$, with $r \neq 0$. If $q \neq 0$, define $v = u' a_p a_{q-1} a_{r-1} u''$. Now suppose $q = 0$. If $p \neq n+1$ note that $a_p a_0 a_r = a_{p+1} a_1 a_r = a_{p+1} a_0 a_{r-1}$ and define $v = u' a_{p+1} a_0 a_{r-1} u''$. If $p = n+1$ note that $a_{n+1} a_0 a_r = a_1 a_{n+1} a_r = a_1 a_n a_{r-1}$ and define $v = u' a_1 a_n a_{r-1} u''$. In each case, it is easy to see that $v = u$ in $K\mathfrak{tw}_n$ and the defect of $\phi(v)$ is less than the defect of $\phi(u)$.

Now suppose the defect of $\phi(u)$ has a non-zero entry only at the first position. Let $u = a_p a_q u'$, where $u' \in A^+$ and $p, q \in [n+2]$. If both $p \neq 0$ and $q \neq 0$, let $m = \min(p, q)$, note that $a_p a_q = a_{p-m} a_{q-m}$ and define $v = a_{p-m} a_{q-m} u'$. If $p = 0$ and $q = n+1$, recall that by one of the defining relations $a_0 a_{n+1} = a_n a_0$ and define $v = a_n a_0 u'$. Likewise, if $p = n+1$ and $q = 0$, define $v = a_0 a_n u'$. In each case, it is easy to see that $v = u$ in $K\mathfrak{tw}_n$ and the defect of $\phi(v)$ is a word consisting of 0s. □

Corollary 19. *Every word in A^+ is equal in $K\mathfrak{tw}_n$ to a word in A^+ which is mapped by ϕ to a word with a defect consisting of* 0*s.*

Proof of the theorem. Relations in $K\mathfrak{tw}_n$ are listed in the proof of Lemma 16. It is obvious that for each relation $u = v$ the words $\phi(u)$ and $\phi(v)$ have the same length and the same alternating sum (calculated in $\mathbb{Z}_{2n+1}$). Therefore, by Lemma 20, ϕ induces a homomorphism $\psi : K\mathfrak{t}_n \to [n+2]^+/\sim$.

Consider two canonical words which are $\sim$-equivalent. If their alternating sums are both 0 then, since a canonical word can have at most one non-zero entry, both words consists only of 0s and, therefore, are equal. Suppose two canonical words share the same non-zero alternating sum. First consider the case when they have the non-zero entry at the same positions, for example, $c_1 0 0^{t-2}$ and $c_2 0 0^{t-2}$ (where t is the length of the words). Since these two words coincide at every position (except, perhaps, one) and have the same alternating sum, we conclude $c_1 = c_2$. Now suppose two canonical words have the non-zero entry at different positions, for example, $c_1 0 0^{t-2}$ and $0 c_2 0^{t-2}$. Because the two words have the same alternating sum, $c_1 = -c_2$ in $\mathbb{Z}_{2n+1}$; since both $c_1, c_2 \in \{1, 2, \ldots, n\}$, this case is impossible. Thus, each class of $\sim$ contains at most one canonical word.

Consider a word $w \in [n+2]^+$ which has a length t and an alternating sum s. If $s \in [n+2]$, the canonical word $s 0 0^{t-2}$ is $\sim$-equivalent to w. Otherwise, let $q = -s$, with the inverse calculated in $\mathbb{Z}_{2n+1}$; the canonical word $0 q 0^{t-2}$ is $\sim$-equivalent to w. Thus, each class of $\sim$ contains at least one canonical word.

To prove that each word in A^+ is equal in $K\mathfrak{tw}_n$ to a word mapped by ϕ to a canonical word, it is sufficient to refer to Corollary 19 and Lemma 17.

Now the result follows from Lemma 2. $\square$

8. Context in knot theory and conjectures

8.1. **Groups and quandles.** Among instruments of knot theory, there are some which consider a knot as an oriented curve, that is, are defined in the context of a specific prescribed direction of travel along the curve, and some others which do not require such an orientation. The Wirtinger presentation (and, likewise, the Dehn presentation) of the knot group (see, for example, Section 6.11 in [12] or Chapter 11 in [8]) is a well-known example of the former kind. The fundamental groups of branched cyclic cover spaces of knots are an example of the latter kind. They are used in one of formulations of a deep result in low-dimensional topology, the so-called Smith conjecture (see page 12 in the main book on the subject [17]). Specifically, one partial case of the Smith conjecture can be formulated [18] as stating that if the fundamental group of the 2-fold branched cyclic cover space of a knot is trivial then the knot is a trivial knot (the converse statement is obviously true). That is, one can use these groups to 'untangle' trivial knots. The fundamental group of the 2-fold branched cyclic cover space of a knot is related to the knot group: indeed, it is isomorphic to the factor group of the knot group produced by assuming that the square of each meridian[6] is 1. This implies that at each crossing in a knot diagram where the Wirtinger presentation produces one relation, say, $ab = bc$, the other group will have two relations, $ab = bc$ and $ba = cb$. These are the same relations as in the definition of a knot semigroup.

Another important pair of algebraic constructions are the quandle of an (oriented) knot and the involutory quandle of an (unoriented) knot. Quandles are powerful knot invariants (for an introduction to quandles, see [19] or [11]). It is possible to reconstruct the knot group from the knot's quandle [20]. Involutory quandles have been used to distinguish knots [20]. It is possible to reconstruct the fundamental group of the 2-fold branched cyclic cover space of a knot from the knot's involutory quandle [18]. The involutory quandle of a knot is trivial if and only if the knot

[6] A meridian is an element of the knot group corresponding to one arc.

is trivial; thus, involutory quandles can be used to 'untangle' trivial knots [18]. However, applying these constructions to individual examples of knot diagrams can be hard: as it is said in [21], 'unfortunately, it is often very difficult to decide upon the triviality' of the involutory quandle of a given knot (in order to check whether the knot is trivial). Or, as it is said in [22], 'Quandles ... are very difficult to work with. The problem of recognizing whether there exists an isomorphism between two quandles given by presentations is apparently not easier than the problem of recognition of isomorphism between groups'.

The knot semigroups introduced in this paper are closely related to the fundamental groups of the 2-fold branched cyclic cover spaces of knots and to involutory quandles of knots. In particular, one can prove that the fundamental group of the 2-fold branched cyclic cover space of a knot is a factor semigroup of the knot semigroup produced by assuming that the square of each generator is 1.

Because each knot semigroup has a unique set of generators (see Proposition 1), one may hope that knot semigroups can be easier to work with than knot groups or knot quandles. In particular, 'untangling' trivial knots using knot semigroups (see Conjecture 24 below) may be easier than with other constructions.

It must be said that the elements and the multiplication in the semigroups studied in this paper have nothing in common with constructions which define an associative product of knots and thus turn the set of all knots into a semigroup. Such constructions include the connected sum of knots (see [16] or [9] or other textbooks, in which this operation may be also called the sum or the product) and generalisations of the product in the braid group (for an introduction to tangles, see [13, 23] or the original paper [24]; for examples of applying semigroups to study tangles, see [25, 26]).

8.2. **Knot invariants.** We want to present an example demonstrating that knot semigroups are not knot invariants. The simplest example of the kind is not a knot, but a link. Compare the standard diagram of the trivial 2-component link on Figure 3.2 and its non-standard diagram $\mathfrak{l}$ below.

Lemma 20. *Let $\sim$ be the smallest cancellative congruence on a free semigroup A^+ containing a set of pairs of words $\{(u_\lambda, v_\lambda) | \lambda \in \Lambda\}$, where $u_\lambda, v_\lambda \in A^+$. Let $\phi : A^+ \to S$ be a homomorphism from A^+ to a cancellative semigroup S. Suppose $\phi(u_\lambda) = \phi(v_\lambda)$ for each $\lambda \in \Lambda$. Then $\sim \subseteq \operatorname{Ker} \phi$.*

Proof. Since a subsemigroup of a cancellative semigroup is cancellative, $\operatorname{Im} \phi$ is a cancellative semigroup; therefore, $\operatorname{Ker} \phi$ is a cancellative congruence on A^+. By the hypothesis, $\operatorname{Ker} \phi$ contains all pairs (u_λ, v_λ). At the same time, $\sim$ is the smallest cancellative congruence on A^+ containing all pairs (u_λ, v_λ). Therefore, $\sim \subseteq \operatorname{Ker} \phi$. □

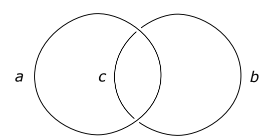
Proposition 21. *In the knot semigroup $K\mathfrak{l}$ of $\mathfrak{l}$ words a, b and c are pairwise distinct.*

Proof. The plan of the proof is as follows. Consider two parameters of a word over the alphabet $\{a, b, c\}$: 1) the number of as in the word; 2) the number of bs preceded by an even number of as minus the number of bs preceded by an odd number of as plus the number of cs preceded by an odd number of as minus the number of cs preceded by an even number of as. Whenever words are equal in Kɩ they have the same values of both parameters. Therefore, words a, b and c are not equal in Kɩ.

Now let us present the argument in detail. Consider a semigroup S whose set of elements is $(\{0\} \cup \mathbb{N}) \times \mathbb{Z}$ with multiplication $(p, q)(r, s) = (p + r, q + (-1)^p s)$. It is easy to check that this multiplication is associative, and that S is cancellative. Consider a homomorphism ϕ from the free semigroup $\{a, b, c\}^+$ to S induced by the mapping defined on the generators $a \mapsto (1, 0)$, $b \mapsto (0, 1)$, $c \mapsto (0, -1)$.

The diagram ɩ has two crossings, each defining two relations $ab = ca$ and $ba = ac$. It is easy to check that $\phi(ab) = (1, -1)$ and $\phi(bc) = (1, -1)$. Similarly, $\phi(ba) = (1, 1)$ and $\phi(ac) = (1, 1)$. By Lemma 20, from this it follows that if two words u, v are equal in Kɩ then $\phi(u) = \phi(v)$.

Now recall that $\phi(a) = (1, 0)$, $\phi(b) = (0, 1)$ and $\phi(c) = (0, -1)$. Since the values of ϕ on a, b, c are pairwise distinct, we conclude that a, b, c are pairwise distinct in Kɩ. □

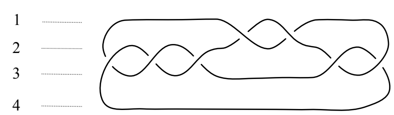


Let S be a semigroup without an identity element. By saying that $s \in S$ is an indecomposable element we mean that s cannot be represented as a product $s = ab$ for any $a, b \in S$. The following result is obvious.

Lemma 22. *In a knot semigroup the only indecomposable elements are those that can be expressed by one-letter words.*

By Proposition 21, semigroup Kɩ has three indecomposable elements, whereas the knot semigroup of the standard diagram, as discussed in Subsection 3.2, has two (a and b). Therefore, the two knot semigroups are not isomorphic. Therefore, knot semigroups are not knot invariants.

Although knot semigroups are not knot invariants, it is possible to consider knot invariants based on knot semigroups; for instance, the family of knot semigroups of all diagrams of a knot is, obviously, an invariant of a knot.



8.3. **Rational knots.** We hope that the results of Sections 4 and 7 can be generalised. Both torus knots and twist knots are rational knots. There are several elegant definitions of the class of rational knots (see, for example, Chapter 12 in [10] and [23]) and, accordingly, several types of diagrams which can be associated with them. For our purposes, a canonical diagram of a rational knot is the one which is based on a 4-strand braid in which every crossing is either a clockwise half-twist of strands in positions 1 and 2 or an anticlockwise half-twist of strands in positions 2 and 3 (as in the example shown on the diagram); this type of diagram is described, for example, in Proposition 12.13 in [10] and on page 187 in [13]. Canonical diagrams of torus knots and twist knots can be transformed into this type of diagrams in a way which does not affect knot semigroups. These and other examples allow us to formulate the following hypothesis for future research.

Conjecture 23. *The knot semigroup of a canonical diagram of a rational knot is isomorphic to an alternating sum semigroup.*

8.4. **Other conjectures.** Thanks to examples such as in Subsection 3.1, and due to the similarity of knot semigroups to constructions in [18, 17], the following hypothesis seems justified.

Conjecture 24. *A knot diagram has the knot semigroup isomorphic to* $\mathbb{N}$ *if and only if it is a diagram of the trivial knot.*

Denote the endpoints of an n-strand braid on the left by $L_1, L_2, \dots, L_n$ and on the right by $R_1, R_2, \dots, R_n$, in both cases counting from the top to the bottom. By $a(L_i)$ or $a(R_i)$ denote the letter which is the label of the arc incidental with L_i or R_i.

Thanks to examples such as in Subsection 3.3, and due to the existence of a similar group-based construction (see, for example, page 32 in [27]), the following hypothesis seems justified.

Conjecture 25. *A braid diagram* $\mathfrak{b}$ *represents a trivial braid if and only if the equalities* $a(L_i) = a(R_i)$ *are true in* $K\mathfrak{b}$ *for all* i.

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