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Generalised Fourier Transform and Perturbations to Soliton Equations

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Abstract

A brief survey of the theory of soliton perturbations is presented. The focus is on the usefulness of the so-called Generalised Fourier Transform (GFT). This is a method that involves expansions over the complete basis of "squared solutions" of the spectral problem, associated to the soliton equation. The Inverse Scattering Transform for the corresponding hierarchy of soliton equations can be viewed as a GFT where the expansions of the solutions have generalised Fourier coefficients given by the scattering data.

The GFT provides a natural setting for the analysis of small perturbations to an integrable equation: starting from a purely soliton solution one can 'modify' the soliton parameters such as to incorporate the changes caused by the perturbation.

As illustrative examples the perturbed equations of the KdV hierarchy, in particular the Ostrovsky equation, followed by the perturbation theory for the Camassa-Holm hierarchy are presented.

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1 Introduction

Integrable equations are widely used as model equations in various problems. The integrability concept originates from the fact that these equations are in some sense exactly solvable, e.g. by the inverse scattering method (ISM), and exhibit global regular solutions. This feature is very important for applications, where in general analytical results (first integrals, particular solutions) are preferable to numerical computations, which are not only long and costly, but also intrinsically subject to numerical error. In a hydrodynamic context, even though water waves are expected to be unstable in general, they do exhibit certain stability properties in physical regimes where integrable model equations are accurate approximations for the evolution of the free surface water wave cf. [1, 21].

There are situations however where the model equation is not integrable, but is somehow close to an integrable equation, i.e. can be considered as a perturbation of an integrable equation. In such case it is still possible to obtain approximate analytical solutions.

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There are several approaches treating the perturbations of integrable equations. One possibility is to consider expanding the solutions of the perturbed nonlinear equation around the corresponding unperturbed solution and to determine the corrections due to perturbations. In other words, one represents the solutions $\tilde{u}(x,t)$ in the form:

$$\tilde{u}(x,t) = u(x,t) + \Delta u(x,t),$$

where $u(x,t)$ is the solution of the corresponding unperturbed nonlinear evolutionary equation and $\Delta u(x,t)$ is a perturbation. The strength of the perturbation is measured by a parameter ϵ, $\Delta u(x,t) = \mathcal{O}(\epsilon)$. By *small (weak) perturbation* one means $0 < \epsilon \ll 1$. Such perturbations can be studied *directly* in the configuration (coordinate) space, while the effect of the perturbations on the corresponding scattering data can be studied in the *spectral space* (usually the complex plane of the spectral parameter) of the associated spectral problem.

For a *direct* study of soliton perturbations, one can use the multi-scale expansion method [29, 30], introducing multiple scales, i.e. transforming the independent time variable t into several variables t_n, $(n = 0, 1, 2, \dots)$ by

$$t_n = \epsilon^n t, \qquad n = 0, 1, 2, \dots,$$

where each $t_n$is an order of ϵ smaller than the previous time t_{n-1}. Then, the time-derivative are replaced by the expansion (the so-called "derivative expansion") with respect to the multiple scales:

$$\partial_t = \sum_{n=0}^{\infty} \epsilon^n \partial_{t_n}.$$

The dependent variable is expanded in an asymptotic series

$$u(x,t) = \sum_{n=0}^{\infty} \epsilon^n u_n(x,t).$$

These expressions are substituted back into the equation, giving a sequence of equations for $u_n(x,t)$, corresponding to each order of ϵ (each time scale $t_n = \epsilon t$). Solving the system of equations for $u_n(x,t)$, one has to ensure that there are no singularities in the solutions (i.e. that the solutions do not blow up in time, etc.). This may lead to some additional conditions on the functions $u_n(x,t)$ (or on the parameters in them), known as *secular conditions.*

Several authors had used various versions of the direct approach in the study of soliton perturbations: D. J. Kaup [66] had used a similar approach for the perturbed sine-Gordon equation. Keener and McLaughlin [69] had proposed a direct approach by obtaining the appropriate Green functions for the nonlinear Schrodinger and sine-Gordon equations. For a comprehensive review of the direct perturbation theory see e.g. [29, 44] and the references therein.

In the spectral space, the study of the soliton perturbations is based on the perturbations of the scattering data, associated to the spectral problem. Such methods are used by a number of authors, for studying perturbations of various nonlinear evolutionary equations, like the sin-Gordon equation [65], the nonlinear Schrödinger equation [70, 38, 37] and of course, the KdV equation which is discussed in details in the following sections. The method is based on expanding the 'potential' (i.e. the dependent variable) $u(x,t)$ of the associated spectral problem over the complete set of "squared solutions", which are eigenfunctions of the corresponding recursion operator.

The squared eigenfunctions of the spectral problem associated to an integrable equation represent a complete basis of functions, which helps to describe the Inverse Scattering Transform for the corresponding hierarchy as a Generalised Fourier transform (GFT). The Fourier modes for the GFT are the Scattering data. Thus all the fundamental properties of an integrable equation such as the integrals of motion, the description of the equations of the whole hierarchy and their Hamiltonian structures can be naturally expressed making use of the completeness relation for the squared eigenfunctions and the properties of the recursion operator.

The GFT also provides a natural setting for the analysis of small perturbations to an integrable equation. The leading idea is that starting from a purely soliton solution of a certain integrable equation one can 'modify' the soliton parameters such as to incorporate the changes caused by the perturbation. There is a contribution to the equations for the scattering data that comes from the GFT-expansion of the perturbation.

In this review article we illustrate these ideas with several examples. Firstly we consider the equations of the KdV hierarchy and the KdV perturbed version – the Ostrovsky equation. Then we present the perturbation theory for the Camassa-Holm hierarchy.

2 Basic facts for the inverse scattering method for the KdV hierarchy

2.1 Direct scattering transform and scattering data

The spectral problem for the equations of the KdV hierarchy is [82, 48]

$$-\Psi_{xx} + u(x)\Psi = k^2\Psi, \tag{1}$$

in which the real-valued potential $u(x)$ is taken for simplicity to be a function of Schwartz-type: $u(x) \in \mathcal{S}(\mathbb{R})$, $k \in \mathbb{C}$ is spectral parameter. The continuous spectrum under these conditions corresponds to real k. The discrete spectrum consists of finitely many points $k_n = i\kappa_n$, $n = 1, \dots, N$ where κ_n is real.

The Jost solutions for (1) are as follows: $f^+(x,k)$ and $\bar{f}^+(x,\bar{k})$ are fixed by their asymptotic when $x \to \infty$ for all real $k \neq 0$ [82]:

$$\lim_{x\to\infty} e^{-ikx} f^+(x,k) = 1, \tag{2}$$

$f^-(x,k)$ and$\bar{f}^-(x,\bar{k})$ fixed by their asymptotic when $x \to -\infty$ for all real $k \neq 0$:

$$\lim_{x\to-\infty} e^{ikx} f^-(x,k) = 1, \tag{3}$$

Since $u(x)$ is real then

$$\bar{f}^+(x,\bar{k}) = f^+(x,-k), \qquad \text{and} \qquad \bar{f}^-(x,\bar{k}) = f^-(x,-k). \tag{4}$$

In particular, for real $k \neq 0$ we have:

$$\bar{f}^\pm(x,k) = f^\pm(x,-k), \tag{5}$$

and the vectors of the two bases are related [4]:

$$f^-(x,k) = a(k)f^+(x,-k) + b(k)f^+(x,k), \qquad \text{Im } k = 0. \tag{6}$$

The coefficient $a(k)$ allows analytic extension in the upper half of the complex k-plane [82] and

$$\bar{a}(\bar{k}) = a(-k), \qquad \bar{b}(\bar{k}) = b(-k), \qquad |a(k)|^2 - |b(k)|^2 = 1. \tag{7}$$

The quantities $\mathcal{R}^\pm(k) = b(\pm k)/a(k)$ are known as reflection coefficients (to the right with superscript $(+)$ and to the left with superscript $(-)$ respectively). It is sufficient to know $\mathcal{R}^\pm(k)$ only on the half line $k > 0$, since from (7): $\mathcal{R}^\pm(-k) = \bar{\mathcal{R}}^\pm(k)$ and also (7)

$$|a(k)|^2 = (1 - |\mathcal{R}^\pm(k)|^2)^{-1}, \tag{8}$$

Furthermore $\mathcal{R}^\pm(k)$ uniquely determines $a(k)$ [82]. At the points κ_n of the discrete spectrum, $a(k)$ has simple zeroes i.e.:

$$a(k) = (k - i\kappa_n)\dot{a}_n + \frac{1}{2}(k - i\kappa_n)^2\ddot{a}_n + \cdots, \tag{9}$$

The dot stands for a derivative with respect to k and $\dot{a}_n \equiv \dot{a}(i\kappa_n)$, $\ddot{a}_n \equiv \ddot{a}(i\kappa_n)$, etc. The following dispersion relation holds:

$$a(k) = \exp\left(-\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln(1 - |\mathcal{R}^\pm(k')|^2)}{k' - k}\mathrm{d}k'\right)\prod_{j=1}^{N} \frac{k - i\kappa_j}{k + i\kappa_j}. \tag{10}$$

At the points of the discrete spectrum f^- and f^+ are linearly dependent:

$$f^-(x, i\kappa_n) = b_n f^+(x, i\kappa_n). \tag{11}$$

In other words, the discrete spectrum is simple, there is only one (real) linearly independent eigenfunction, corresponding to each eigenvalue $i\kappa_n$, say

$$f_n^-(x) \equiv f^-(x, i\kappa_n) \tag{12}$$

From (12) and (2), (3) it follows that $f_n^-(x)$ falls off exponentially for $x \to \pm\infty$, which allows one to show that $f_n(x)$ is square integrable. Moreover, for compactly supported potentials $u(x)$ (cf. (11) and (6))

$$b_n = b(i\kappa_n), \qquad b(-i\kappa_n) = -\frac{1}{b_n}. \tag{13}$$

One can argue [82], that the results from this case can be extended to Schwarz-class potentials by an appropriate limiting procedure.
The asymptotic of f_n^-, according to (5), (2), (11) is

$$f_n^-(x) = e^{\kappa_n x} + o(e^{\kappa_n x}), \qquad x \to -\infty; \tag{14}$$
$$f_n^-(x) = b_n e^{-\kappa_n x} + o(e^{-\kappa_n x}), \qquad x \to \infty. \tag{15}$$

[4] According to the notations used in [82] $f^+(x,k) \equiv \bar{\psi}(x,\bar{k})$, $f^-(x,k) \equiv \varphi(x,k)$.

The sign of b_n obviously depends on the number of the zeroes of f_n^-. Suppose that $0 < \kappa_1 < \kappa_2 < \ldots < \kappa_N$. Then from the oscillation theorem for the Sturm-Liouville problem [3], f_n^- has exactly $n-1$ zeroes. Therefore

$$b_n = (-1)^{n-1}|b_n|. \tag{16}$$

The sets

$$\mathcal{S}^{\pm} \equiv \{\mathcal{R}^{\pm}(k) \quad (k>0), \quad \kappa_n, \quad R_n^{\pm} \equiv \frac{b_n^{\pm 1}}{i\dot{a}_n}, \quad n=1,\ldots N\} \tag{17}$$

are called scattering data. Clearly, due to (10) each set – $\mathcal{S}^+$ or $\mathcal{S}^-$ of scattering data uniquely determines the other one and also the potential $u(x)$ [82, 48, 88].

2.2 Generalised Fourier Transform

The recursion operator for the KdV hiererchy is

$$L_{\pm} = -\frac{1}{4}\partial^2 + u(x) - \frac{1}{2}\int_{\pm\infty}^{x} \mathrm{d}\tilde{x} u'(\tilde{x}) \cdot . \tag{18}$$

The eigenfunctions of the recursion operator are the squared eigenfunctions of the spectral problem:

$$F^{\pm}(x,k) \equiv (f^{\pm}(x,k))^2, \qquad F_n^{\pm}(x) \equiv F(x, i\kappa_n), \tag{19}$$

where $F_n^{\pm}(x)$ are related to the discrete spectrum $k_n = i\kappa_n$. Using (1) and the asymptotics (2), (3) one can show that

$$L_{\pm}F^{\pm}(x,k) = k^2 F^{\pm}(x,k) \qquad L_{\pm}F_n^{\pm}(x) = k_n^2 F_n^{\pm}(x). \tag{20}$$

The Generalised Fourier expansion can be formulated as follows:

Theorem 2.1 *Suppose that f^+ and f^- are not linearly dependent at $x=0$. For each function $g(x) \in \mathcal{S}(\mathbb{R})$ the following expansion formulas hold:*

$$g(x) = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty} \tilde{g}^{\pm}(k) F_x^{\pm}(x,k)\mathrm{d}k \mp \sum_{j=1}^{N}\left(g_{1,j}^{\pm}\dot{F}_{j,x}^{\pm}(x) + g_{2,j}^{\pm}F_{j,x}^{\pm}(x)\right),$$

where $\dot{F}_j^{\pm}(x) \equiv [\frac{\partial}{\partial k}F^{\pm}(x,k)]_{k=k_j}$ and the Fourier coefficients are

$$\begin{aligned}
\tilde{g}^{\pm}(k) &= \frac{1}{ka^2(k)}\left(g, F^{\mp}\right), \quad \text{where} \quad (g,F) \equiv \int_{-\infty}^{\infty} g(x)F(x)\mathrm{d}x, \\
g_{1,j}^{\pm} &= \frac{1}{k_j\dot{a}_j^2}\left(g, F_j^{\mp}\right), \\
g_{2,j}^{\pm} &= \frac{1}{k_j\dot{a}_j^2}\left[\left(g, \dot{F}_j^{\mp}\right) - \left(\frac{1}{k_j} + \frac{\ddot{a}_j}{\dot{a}_j}\right)\left(g, F_j^{\mp}\right)\right].
\end{aligned}$$

Proof: The details of the derivation can be found e.g. in [33, 48]. ■

In particular one can expand the potential $u(x)$, the coefficients are given through the scattering data [33, 48]:

$$u(x) = \pm\frac{2}{\pi i}\int_{-\infty}^{\infty} k\mathcal{R}^{\pm}(k)F^{\pm}(x,k)\mathrm{d}k + \sum_{j=1}^{N} 4ik_j R_j^{\pm} F_j^{\pm}(x). \tag{21}$$

The variation $\delta u(x)$ under the assumption that the number of the discrete eigenvalues is conserved is

$$\delta u(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \delta\mathcal{R}^{\pm}(k)F_x^{\pm}(x,k)\mathrm{d}k \pm 2\sum_{j=1}^{N}\left[R_j^{\pm}\delta k_j \dot{F}_{j,x}^{\pm}(x) + \delta R_j^{\pm} F_{j,x}^{\pm}\right]. \tag{22}$$

An important subclass of variations are due to the time-evolution of u, i.e. effectively we consider a one-parametric family of spectral problems, allowing a dependence of an additional parameter t (time). Then $\delta u(x,t) = u_t\delta t + Q((\delta t)^2)$, etc. The equations of the KdV hierarchy can be written as

$$u_t + \partial_x\Omega(L_{\pm})u(x,t) = 0, \tag{23}$$

where $\Omega(k^2)$ is a rational function specifying the dispersion law of the equation. The substitution of (22) and (21) in (23) due to (20) gives a system of trivial linear ordinary differential equations for the scattering data:

$$\begin{aligned} \mathcal{R}_t^{\pm} \pm 2ik\Omega(k^2)\mathcal{R}^{\pm} &= 0, & (24)\\ R_{j,t}^{\pm} \pm 2ik_j\Omega(k_j^2)R_j^{\pm} &= 0, & (25)\\ k_{j,t} &= 0. & (26) \end{aligned}$$

The KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{27}$$

can be obtained for $\Omega(k^2) = -4k^2$.

Once the scattering data are determined from (24) – (26) one can recover the solution from (22). Thus the Inverse Scattering Transform can be viewed as a GFT.

3 Perturbations of the equations of the KdV hierarchy

Let us now consider a perturbed equation from the KdV hierarchy, i.e. an equation of the form

$$u_t + \partial_x\Omega(L_{\pm})u(x,t) = P[u], \tag{28}$$

where $P[u]$ is a small perturbation, which is also assumed in the Schwartz-class. The perturbed equation is, in general, non-integrable. One can expand $P[u]$ according to

Theorem 2.1 and u_t and u according to (22) and (21). Their substitution in (28) now apparently leads to a modification of (24) – (26) as follows:

$$\begin{aligned} \mathcal{R}_t^{\pm} \pm 2ik\Omega(k^2)\mathcal{R}^{\pm} &= \pm\frac{(P,F^{\mp})}{2ka^2(k)}, &(29)\\ R_{j,t}^{\pm} \pm 2ik_j\Omega(k_j^2)R_j^{\pm} &= -\frac{1}{2k_j\dot{a}_j^2}\left[(P,\dot{F}_j^{\mp}) - \left(\frac{1}{k_j}+\frac{\ddot{a}_j}{\dot{a}_j}\right)(P,F_j^{\mp})\right], &(30)\\ k_{j,t} &= -\frac{(P,F_j^{\mp})}{2k_j\dot{a}_j^2R_j^{\pm}}. &(31)\end{aligned}$$

Note that due to (31) as a result of the perturbation the discrete eigenvalues are time-dependent. Another feature is the contribution from the continuous spectrum: even if one starts with a pure soliton solution of the unperturbed equation ($\mathcal{R}^{\pm}(k,0)=0$) then, in general $\mathcal{R}^{\pm}(k,t)\neq 0$ due to (29).

For practical applications it is easier to work with an equation for b_n instead of (30). Such an equation can be obtained as follows. We notice that

$$\begin{aligned} R_{n,t}^{+} &= \left(\frac{b_n}{i\dot{a}_n}\right)_t = \frac{1}{i\dot{a}_n}b_{n,t} + b_n\left(\frac{1}{i\dot{a}_n}\right)_t,\\ R_{n,t}^{-} &= \left(\frac{1}{ib_n\dot{a}_n}\right)_t = -\frac{1}{i\dot{a}_nb_n^2}b_{n,t} + \frac{1}{b_n}\left(\frac{1}{i\dot{a}_n}\right)_t,\end{aligned}$$

thus $b_{n,t} = \frac{i\dot{a}_n}{2}(R_{n,t}^{+} - b_n^2R_{n,t}^{-})$. Then using (30) and the fact that $F_n^{-} = b_n^2F_n^{+}$, cf. (11) we have

$$b_{n,t} + 2ik_n\Omega(k_n^2)b_n = \frac{i}{4k_n\dot{a}_n}\left(P, b_n^2\dot{F}_n^{+} - \dot{F}_n^{-}\right). \tag{32}$$

As an example let us consider the adiabatic perturbation of the one-soliton solution of the KdV equation. The one-soliton solution is

$$u_s(x,t) = -2\kappa_1^2\mathrm{sech}^2 z, \qquad z=\kappa_1(x-\xi), \qquad \xi = 4\kappa_1^2t+\xi_0. \tag{33}$$

The eigenfunctions are

$$f^{\pm}(x,k) = \frac{e^{\pm ikx}(k\pm i\kappa_1\tanh z)}{k+i\kappa_1}, \quad a(k) = \frac{k-i\kappa_1}{k+i\kappa_1}, \quad b_1 = e^{2\kappa_1\xi}. \tag{34}$$

The perturbed solution is

$$u(x,t) = -2\kappa_1^2\mathrm{sech}^2 z + u_r(x,t), \qquad z = \kappa_1(t)[x-\xi(t)]. \tag{35}$$

Here $u_r(x,t)$ is the contribution from the continuous spectrum (radiation).
From (31) we have

$$\kappa_{1,t} = -\frac{1}{4\kappa_1}\int_{-\infty}^{\infty} P[u_s(z)]\mathrm{sech}^2 z\mathrm{d}z. \tag{36}$$

Writing $b_1 = e^{2\kappa_1(t)\xi(t)}$ and using (36) and (32) we obtain

$$\xi_t = 4\kappa_1^2 - \frac{1}{4\kappa_1^3}\int_{-\infty}^{\infty} P[u_s(z)]\left(z + \frac{1}{2}\sinh 2z\right)\mathrm{sech}^2 z \mathrm{d}z. \tag{37}$$

For the reflection coefficient (29) gives

$$\mathcal{R}_t^+ - 8ik^3\mathcal{R}^+ \quad = \quad \frac{ie^{-2ik\xi}}{2k\kappa_1}\int_{-\infty}^{\infty} P[u_s(z)]e^{-2ikz/\kappa_1}(k - i\kappa_1 \tanh z)^2 \mathrm{d}z. \tag{38}$$

then according to [59] using approximations in Gelfand-Levitan-Marchenko equation one can obtain

$$u_r(x,t) = \frac{\kappa_1}{\pi}\frac{\mathrm{d}}{\mathrm{d}z}\int_{-\infty}^{\infty}\mathcal{R}^+(k)e^{2ik\xi + 2ikz/\kappa_1}\left(\frac{k + i\kappa_1 \tanh z}{k + i\kappa_1}\right)^2 \mathrm{d}k. \tag{39}$$

Alternatively, from (21) it follows that

$$\begin{aligned} u_r(x,t) &= \frac{2}{\pi i}\int_{-\infty}^{\infty} k\mathcal{R}^+(k)F^+(x,k)\mathrm{d}k \\ &= \frac{2}{\pi i}\int_{-\infty}^{\infty} k\mathcal{R}^+(k)e^{2ik\xi + 2ikz/\kappa_1}\left(\frac{k + i\kappa_1 \tanh z}{k + i\kappa_1}\right)^2 \mathrm{d}k. \end{aligned} \tag{40}$$

Both formulae give an approximation of the radiation component since the z- derivative of $\tanh z$ can be neglected [76].

The perturbation results for the Zakharov-Shabat (ZS) type spectral problems have been obtained firstly in [63] and for KdV in [59]. As it has been explained, the perturbation theory is based on the completeness relations for the squared eigenfunctions. For the Sturm-Liouville spectral problem such relations apparently have been studied as early as in 1946 [4] and then by other authors, e.g. [71, 48]. The completeness relation for the eigenfunctions of the ZS spectral problem is derived in [64] and generalisations are studied further in [36, 38, 39, 41, 87], see also [40].

Example: *Ostrovsky equation.* This equation has the form [83]:

$$u_t + u_{xxx} - 6uu_x = \gamma\partial^{-1}u, \tag{41}$$

where ∂^{-1} is an operator such that $(\partial^{-1}u)_x \equiv u$, in general not uniquely determined. The Ostrovsky equation can be viewed as a time-dependent nonlocal perturbation of the KdV equation (27). Here γ is a constant parameter. The equation is often called the Rotation-Modified Korteweg-de Vries equation. It describes gravity waves propagating down a channel under the influence of Coriolis force. In essence, u in the equation can be regarded as the fluid velocity in the $x-$direction. The physical parameter γ measures the effect of the Earth's rotation. More details about the Ostrovsky equation can be found e.g. in [83, 7, 79, 85].

In the perturbation theory $\gamma \ll 1$ plays the role of a small parameter. In order to ensure that the perturbation is decaying fast enough at $x = \pm\infty$ we take the one-soliton KdV solution in the form

$$u_s = 2\kappa_1^2/\sinh^2 z \tag{42}$$

which can be obtained formally from (33) for $\kappa_1\xi_0 = \pi i/2$. It is not continuous at $z=0$ but decays fast enough at $x=\pm\infty$. Using the fact that

$$\frac{\mathrm{d}}{\mathrm{d}z}\coth z = -\frac{1}{\sinh^2 z} + 2\delta(z) = -\frac{1}{\sinh^2 z} + \frac{\mathrm{d}}{\mathrm{d}z}[\theta(z) - \theta(-z)] \tag{43}$$

[34] we obtain

$$P[u_s] = \gamma\partial^{-1}u_s = -2\gamma\kappa_1[\coth z - \theta(z) + \theta(-z)], \tag{44}$$

which is an odd function of z and then (36) gives $\kappa_{1,t} = 0$. Thus the amplitude of the soliton does not change under this perturbation. The computation of (37) gives a correction to the velocity of the soliton:

$$\xi_t = 4\kappa_1^2 + \frac{\pi^2\gamma}{8\kappa_1^2}.$$

4 Conservation laws and perturbed soliton equations

It is well known [88, 82] that the KdV equation is a completely integrable Hamiltonian system and possesses infinitely-many integrals of motion. These integrals can be constructed from the scattering coefficients $a(k)$ of the associated spectral problem (1) and are polynomials of the dependent variable $u(x,t)$ and its x-derivatives:

$$I_n = \int_{-\infty}^{\infty} P_n(u, u_x, u_{xx}, \dots)\,\mathrm{d}x, \tag{45}$$

where P_n is a polynomial with respect to u and its derivatives. Since $a(k)$ is time-independent, it can be viewed as generating functional of integrals of motion a_k [82]:

$$\ln a(k) = \sum_{s=0}^{\infty} \frac{I_{s+1}}{(2\mathrm{i}k)^s}. \tag{46}$$

Skipping the details (see e.g. [82]), we provide here only the list of the first few integrals of motion:

$$I_1 = -\frac{1}{2}\int_{-\infty}^{\infty} u(x)\,\mathrm{d}x; \tag{47}$$

$$I_2 = -\frac{1}{2}\int_{-\infty}^{\infty} u(x)^2\,\mathrm{d}x; \tag{48}$$

$$I_3 = -\frac{1}{2}\int_{-\infty}^{\infty} \left(u_x^2(x) + 2u^3(x)\right)\,\mathrm{d}x; \tag{49}$$

$$I_4 = -\frac{1}{2}\int_{-\infty}^{\infty} \left(u_{xx}^2 - 5u^2u_{xx} + 5u^4\right)\,\mathrm{d}x; \tag{50}$$

The KdV equation (27) can be written as a Hamiltonian system

$$u_t = \frac{\partial}{\partial x}\frac{\delta H}{\delta u(x)}, \tag{51}$$

where the symbol $\delta/\delta u$ denotes variational derivative. Moreover, (51) can be further represented in its Hamiltonian form with a Hamiltonian H:

$$u_t = \{u, H\}. \tag{52}$$

where $H = I_3$ (49). The Poisson bracket is defined as

$$\{F, G\} \equiv \int \frac{\delta F}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta G}{\delta u(x)} \mathrm{d}x. \tag{53}$$

The first three integrals of motion (47)–(49) have the same interpretation for all members of KdV hirarchy: The first one, I_1 is related to the algebraic structure of the Poisson bracket (53): it follows from the presence of the operator $\partial/\partial x$ in the Poisson brackets. The integral (48) has a meaning of a momentum. It is related to the translation invariance of the Hamiltonian. Since $H[u(x+\varepsilon)] - H[u(x)] \equiv 0$, the expansion of $\int(H[u(x+\varepsilon)] - H[u(x)])\mathrm{d}x$ in ε about $\varepsilon = 0$ gives (note that $u(x) = \delta P/\delta u$)

$$0 = \int \frac{\delta H}{\delta u(x)} \frac{\partial u}{\partial x} \mathrm{d}x = \int \frac{\delta H}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta P}{\delta u(x)} \mathrm{d}x \equiv \{P, H\} = P_t,$$

Consider now the perturbed KdV equation (28).

We will seek the integrals of motion for the perturbed equation $\tilde{I}_k$ in the form $\tilde{I}_k = I_k + \Delta I_k$, $k = 1, 2, \ldots$. Here ΔI_k can be viewed as a correction to the integrals of motion of the unperturbed equation (27) coming from the perturbations $P[u]$. After a direct integration of (28), one gets:

$$\Delta I_1 = \int_{-\infty}^{\infty} P[u] \, \mathrm{d}x. \tag{54}$$

Then, multiplying (27) by $u(x,t)$, and integrating leads to:

$$\Delta I_2 = 2 \int_{-\infty}^{\infty} u P[u] \, \mathrm{d}x, \tag{55}$$

and so on.

As an illustrative example we will take again the Ostrovsky equation (41). Due to the concrete choice of the perturbation in the right hand side of (41), the integrals in (54) and (55) vanish, so the perturbations do not contribute to these integrals: the first two integrals of motion are the same as for the KdV equation. The nontrivial contributions of perturbations to the integrals of motion in the Hamiltonian I_3 are:

$$\Delta I_3 = \frac{\gamma}{2} \int_{-\infty}^{\infty} (\partial^{-1} u)^2 \, \mathrm{d}x. \tag{56}$$

Note also, that there is no second Hamiltonian formulation for the Ostrovsky equation, compatible with the one given above, i.e. the equation is not bi-Hamiltonian – indeed (41) is not completely integrable for $\gamma \neq 0$, [7].

5 Perturbations to the equations of the Camassa-Holm hierarchy

Closely related to the KdV hierarchy is the hierarchy of the Camassa-Holm (CH) equation [6]. This equation has the form

$$u_t - u_{xxt} + 2\omega u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{57}$$

where ω is a real constant. It is integrable with a Lax pair [6]

$$\Psi_{xx} = \Big(\frac{1}{4} + \lambda(m+\omega)\Big)\Psi \tag{58}$$

$$\Psi_t = \Big(\frac{1}{2\lambda} - u\Big)\Psi_x + \frac{u_x}{2}\Psi + \gamma\Psi \tag{59}$$

where $m \equiv u - u_{xx}$ and γ is an arbitrary constant.

Both CH and KdV equations appeared initially as models of the propagation of two-dimensional shallow water waves over a flat bottom. More about the physical relevance of the CH equation can be found e.g. in [6, 54, 55, 31, 32, 51, 21, 49, 50]. The paper [75] suggests that KdV and CH might be relevant to the modelling of tsunami waves (see also the discussion in [18]).

While all smooth data yield solutions of the KdV equation existing for all times, certain smooth initial data for CH lead to global solutions and others to breaking waves: the solution remains bounded but its slope becomes unbounded in finite time (see [13, 8, 5]). The solitary waves of KdV are smooth solitons, while the solitary waves of CH, which are also solitons, are smooth if $\omega > 0$ [6, 55] and peaked (called "peakons" and representing weak solutions) if $\omega = 0$ [6, 14, 2, 23, 77]. Both solitary wave forms for CH are stable [26, 24, 27].

It could be pointed out that the peakons appear also as travelling wave solutions of greatest height (for the governing equations for water waves), cf. [11, 12, 86].

In geometric context, the CH equation arises as a geodesic equation on the diffeomorphism group (if $\omega = 0$) [8, 19, 20, 74] and on the Bott-Virasoro group (if $\omega > 0$) [81].

CH equation also allows for solutions with compactly supported $m(x,t)$, [10], however $u(x,t)$ looses instantly its compact support, whether $\omega \neq 0$ [42] or $\omega = 0$ [78].

The problem of perturbation of the CH equation arises when one deals with model equations that are in general non-integrable but close to the CH equation. A perturbation could appear for example when one takes into account the viscosity effect [84]. Another possible scenario comes from the so-called 'b-equation' [28, 47]

$$m_t + b\omega u_x + bmu_x + m_x u = 0.$$

The b-equation generalizes the CH equation and is integrable only for $b = 2$ (when it coincides with the CH) and $b = 3$ (then known as Degasperis-Procesi equation) [80, 47, 52]. Qualitatively, the DP equation exhibits most of the features of the CH equation, e.g. the infinite propagation speed for DP was established in [43]. In [75] it is suggested that DP (as well as CH) might be relevant to the modelling of tsunami waves (see also the discussion in [18]).

The hydrodynamic relevance of the b-equation is discussed e.g. in [56, 51]. Therefore, the solutions of the b-equation for values of b close to $b = 2$ can be analyzed in the framework of the CH-perturbation theory. We can represent the equation as a CH perturbation

$$m_t + 2\omega u_x + 2mu_x + m_x u = (2-b)(\omega u_x + mu_x) \equiv P[u],$$

for a small parameter $\epsilon = b - 2$.

5.1 Inverse scattering and generalised Fourier transform for the CH spectral problem

The CH spectral problem (58) can be handled in a way, similar to the one, already outlined. For simplicity we consider the case where m is a Schwartz class function, $\omega > 0$ and $m(x,0) + \omega > 0$. Let is introduce the notation $q(x,t) = m(x,t) + \omega$. Then one can show that $q(x,t) > 0$ for all t [9]. Let $k^2 = -\frac{1}{4} - \lambda\omega$, i.e.

$$\lambda(k) = -\frac{1}{\omega}\Big(k^2 + \frac{1}{4}\Big). \tag{60}$$

The spectrum of the problem (58) under these conditions is described in [9]. The continuous spectrum in terms of k corresponds to k – real. The discrete spectrum consists of finitely many points $k_n = i\kappa_n$, $n = 1, \ldots, N$ where κ_n is real and $0 < \kappa_n < 1/2$. The continuous spectrum vanishes for $\omega = 0$, [22].

All results (2) – (17) remain formally the same with the exception of (10) which now has the form [17, 15, 16]

$$a(k) = \exp\left(-i\alpha k - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln(1 - |\mathcal{R}^{\pm}(k')|^2)}{k' - k}\mathrm{d}k'\right)\prod_{j=1}^{N}\frac{k - i\kappa_j}{k + i\kappa_j}. \tag{61}$$

where

$$\begin{aligned}\alpha &= \int_{-\infty}^{\infty}\Big(\sqrt{\frac{q(x)}{\omega}} - 1\Big)\mathrm{d}x \\ &= \sum_{n=1}^{N}\ln\Big(\frac{1+2\kappa_n}{1-2\kappa_n}\Big)^2 + \frac{4}{\pi}\int_0^{\infty}\frac{\ln(1 - |\mathcal{R}^{+}(\widetilde{k})|^2)}{4\widetilde{k}^2 + 1}\mathrm{d}\widetilde{k}\end{aligned}$$

is one of the CH integrals of motion (Casimir).

With the asymptotics of the Jost solutions and (58) one can show that

$$L_{\pm}F^{\pm}(x,k) = \frac{1}{\lambda}F^{\pm}(x,k) \qquad L_{\pm}F_n^{\pm}(x) = \frac{1}{\lambda_n}F_n^{\pm}(x), \tag{62}$$

where $\lambda_n = \lambda(i\kappa_n)$; $F^{\pm}$ are again the squares of the Jost solutions as in (19) and

$$L_{\pm} = (\partial^2 - 1)^{-1}\Big[4q(x) - 2\int_{\pm\infty}^{x}\mathrm{d}y\, m'(y)\Big]\cdot \tag{63}$$

is the recursion operator. The inverse of this operator is also well defined.

The completeness relation for the eigenfunctions of the recursion operator is [16]

$$\frac{\omega}{\sqrt{q(x)q(y)}}\theta(x-y) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F^{-}(x,k)F^{+}(y,k)}{ka^2(k)}\mathrm{d}k$$
$$+\sum_{n=1}^{N}\frac{1}{i\kappa_n\dot{a}_n^2}\Big[\dot{F}_n^{-}(x)F_n^{+}(y)+F_n^{-}(x)\dot{F}_n^{+}(y)-\Big(\frac{1}{i\kappa_n}+\frac{\ddot{a}_n}{\dot{a}_n}\Big)F_n^{-}(x)F_n^{+}(y)\Big]. \tag{64}$$

Therefore $F^{\pm}$, $F_n^{\pm}$ and $\dot{F}_n^{\pm}$ can be considered as 'generalised' exponents. Like in the KdV case it is possible to expand $m(x)$ and its variation over the aforementioned basis, or rather the quantities that are determined by $m(x)$ and $\delta m(x)$, [16]:

$$\omega\Big(\sqrt{\frac{\omega}{q(x)}}-1\Big) = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{2k\mathcal{R}^{\pm}(k)}{\lambda(k)}F^{\pm}(x,k)\mathrm{d}k+\sum_{n=1}^{N}\frac{2\kappa_n}{\lambda_n}R_n^{\pm}F_n^{\pm}(x); \tag{65}$$

$$\frac{\omega}{\sqrt{q(x)}}\int_{\pm\infty}^{x}\delta\sqrt{q(y)}\ \ \mathrm{d}y = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i}{\lambda(k)}\delta\mathcal{R}^{\pm}(k)F^{\pm}(x,k)\mathrm{d}k$$
$$\pm\quad\sum_{n=1}^{N}\Big[\frac{1}{\lambda_n}(\delta R_n^{\pm}-R_n^{\pm}\delta\lambda_n)F_n^{\pm}(x)+\frac{R_n^{\pm}}{i\lambda_n}\delta\kappa_n\dot{F}_n^{\pm}(x)\Big] \tag{66}$$

The expansion coefficients as expected are given by the scattering data and their variations. This makes evident the interpretation of the ISM as a generalized Fourier transform. Now it is straightforward to describe the hierarchy of Camassa-Holm equations. To every choice of the function $\Omega(z)$, known also as the dispersion law we can put into correspondence the nonlinear evolution equation (NLEE) that belongs to the Camassa-Holm hierarchy:

$$\frac{2}{\sqrt{q}}\int_{\pm\infty}^{x}(\sqrt{q})_t\mathrm{d}y+\Omega(L_{\pm})\Big(\sqrt{\frac{\omega}{q}}-1\Big)=0. \tag{67}$$

An equivalent form of the equation is

$$q_t+2q\tilde{u}_x+q_x\tilde{u}=0, \qquad \tilde{u}=\frac{1}{2}\Omega(L_{\pm})\Big(\sqrt{\frac{\omega}{q}}-1\Big). \tag{68}$$

The choice $\Omega(z)=z$ leads to $\tilde{u}=u$ and thus to the CH equation (57). Other choices of the dispersion law and the corresponding equations of the Camassa-Holm hierarchy are discussed in [16, 53].

By virtue of the expansions (65) and (66) the NLEE (67) is equivalent to the following linear evolution equations for the scattering data:

$$\mathcal{R}_t^{\pm}(k)\mp ik\Omega(\lambda^{-1})\mathcal{R}^{\pm}(k)=0, \tag{69}$$
$$R_{n,t}^{\pm}\pm\kappa_n\Omega(\lambda_n^{-1})R_n^{\pm}=0, \tag{70}$$
$$\kappa_{n,t}=0. \tag{71}$$

The time-evolution of the scattering data for the CH equation (57) can be computed from the above formulae for $\Omega(z)=z$, see also [17, 15].

5.2 Perturbation theory for the CH hierarchy

Let us start with a perturbed equation of the CH hierarchy of the form

$$q_t + 2q\tilde{u}_x + q_x\tilde{u} = P[u], \qquad \tilde{u} = \frac{1}{2}\Omega(L_\pm)\Big(\sqrt{\frac{\omega}{q}} - 1\Big), \tag{72}$$

where again, $P[u]$ is a small perturbation, by assumption in the Schwartz-class. It is useful to write (72) in the form

$$\frac{2}{\sqrt{q}}\int_{\pm\infty}^{x}(\sqrt{q})_t \mathrm{d}y + \Omega(L_\pm)\Big(\sqrt{\frac{\omega}{q}} - 1\Big) = \frac{1}{\sqrt{q}}\int_{\pm\infty}^{x}\frac{P(y)}{\sqrt{q(y)}}\mathrm{d}y. \tag{73}$$

With the completeness relation (64) one can deduce the gereralised Fourier expansion for expressions, like the one on the right-hand side of (72)

Theorem 5.1 *Assuming that f^+ and f^- are not linearly dependent at $x = 0$ and $g(x) \in \mathcal{S}(\mathbb{R})$, the following expansion formulas hold:*

$$\begin{aligned} \frac{\omega}{\sqrt{q}}\int_{\pm\infty}^{x}\frac{g(y)}{\sqrt{q(y)}}dy = &\pm \frac{1}{2\pi i}\int_{-\infty}^{\infty}\tilde{g}^\pm(k)F_x^\pm(x,k)\mathrm{d}k \\ &\mp \sum_{j=1}^{N}\Big(g_{1,j}^\pm \dot{F}_{j,x}^\pm(x) + g_{2,j}^\pm F_{j,x}^\pm(x)\Big), \end{aligned} \tag{74}$$

where $\dot{F}_j^\pm(x) \equiv [\frac{\partial}{\partial k}F^\pm(x,k)]_{k=k_j}$ and the Fourier coefficients are

$$\begin{aligned} \tilde{g}^\pm(k) &= \frac{1}{ka^2(k)}\left(g, F^\mp\right), \\ g_{1,j}^\pm &= \frac{1}{k_j\dot{a}_j^2}\left(g, F_j^\mp\right), \\ g_{2,j}^\pm &= \frac{1}{k_j\dot{a}_j^2}\left[\left(g, \dot{F}_j^\mp\right) - \left(\frac{1}{k_j} + \frac{\ddot{a}_j}{\dot{a}_j}\right)\left(g, F_j^\mp\right)\right]. \end{aligned}$$

The substitution of the expansions (74) for $P[u]$, (65) and (66) into the perturbed equation (73) gives the following expressions for the modified scattering data:

$$\mathcal{R}_t^\pm \mp ik\Omega(1/\lambda)\mathcal{R}^\pm = \mp\frac{i\lambda(P, F^\mp)}{2ka^2(k)}, \tag{75}$$

$$k_{j,t} = \frac{\lambda_j(P, F_j^\mp)}{2k_j\dot{a}_j^2 R_j^\pm} \tag{76}$$

$$\begin{aligned} R_{j,t}^\pm - R_j^\mp\lambda_{j,t} &\pm \kappa_j\Omega(1/\lambda_j)R_j^\pm \\ &= -\frac{\lambda_j}{2k_j\dot{a}_j^2}\left[(P, \dot{F}_j^\mp) - \left(\frac{1}{k_j} + \frac{\ddot{a}_j}{\dot{a}_j}\right)(P, F_j^\mp)\right], \end{aligned} \tag{77}$$

From (77) we obtain the following for the coefficient b_j:

$$b_{j,t} + \kappa_j\Omega(1/\lambda_j)b_j = -\frac{\lambda_j}{4\kappa_j\dot{a}_j}\left(P, b_j^2\dot{F}_j^+ - \dot{F}_j^-\right).$$

The 'perturbed' solution for the hierarchy in the adiabatic approximation can be recovered from the following expansion for $\tilde{u}(x)$ with the 'modified' scattering data keeping the unperturbed 'generalised' exponents:

$$\tilde{u}(x) = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{k\Omega(1/\lambda(k))}{\omega\lambda(k)}\mathcal{R}^{\pm}(k)F^{\pm}(x,k)\mathrm{d}k + \sum_{n=1}^{N}\frac{\kappa_n\Omega(1/\lambda_n)}{\omega\lambda_n}R_n^{\pm}F_n^{\pm}(x).$$

This formula follows from the second part of (68) and (65). Note that for the CH equation (57) $\tilde{u} \equiv u$.

6 Discussions

We have presented a review of some aspects of the perturbation theory for integrable equations using as main examples the KdV and CH hierarchies.

In our derivations we used completeness relations that are valid only given the assumption that the Jost solutions f^+ and f^- are linearly independent at $x = 0$. The case when this condition is not satisfied is quite exceptional, however this is exactly the case when one has purely soliton solution [33, 48]. Then one has to take into account a nontrivial contribution from the scattering data at $k = 0$ [46] and some of the presented results require modification. E.g. (37) should be [58]

$$\xi_t = 4\kappa_1^2 - \frac{1}{4\kappa_1^3}\int_{-\infty}^{\infty} P[u_s(z)](z\operatorname{sech}^2 z + \tanh z + \tanh^2 z)\mathrm{d}z. \tag{78}$$

In the presented example with the Ostrovsky equation

$$\int_{-\infty}^{\infty} P[u_s(z)]\tanh^2 z\mathrm{d}z = 0 \tag{79}$$

since $P(z)$ is an odd function and the additional term does not contribute. The meaning of the condition (79) is that no shelf is formed behind the soliton [58, 46]. The presence of shelf for KdV equation is observed e.g. under the perturbation $P[u] = \epsilon u$ [58, 76].

The corrections to the conservation laws due to perturbations have been used in studies of the effects of the disturbance on the initial soliton [58, 65], or as a correctness check of results obtained otherwise [67].

The evaluation of the perturbation terms for the CH hierarchy could be technically difficult due to the complicated form of the CH multisoliton solutions [9, 15]. However the limit $\omega \to 0$ leads to the relatively simple peakon solutions. Therefore, using the presented general formulae one should be able to access the perturbations of the peakon parameters.

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