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Lag Length Selection for Unit Root Tests in the Presence of Nonstationary Volatility*

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Abstract

A number of recently published papers have focused on the problem of testing for a unit root in the case where the driving shocks may be unconditionally heteroskedastic. These papers have, however, assumed that the lag length in the unit root test regression is a deterministic function of the sample size, rather than data-determined, the latter being standard empirical practice. In this paper we investigate the finite sample impact of unconditional heteroskedasticity on conventional data-dependent methods of lag selection in augmented Dickey-Fuller type unit root test regressions and propose new lag selection criteria which allow for the presence of heteroskedasticity in the shocks. We show that standard lag selection methods show a tendency to over-fit the lag order under heteroskedasticity, which results in significant power losses in the (wild bootstrap implementation of the) augmented Dickey-Fuller tests under the alternative. The new lag selection criteria we propose are shown to avoid this problem yet deliver unit roots with almost identical finite sample size and power properties as the corresponding tests based on conventional lag selection methods when the shocks are homoskedastic.

Keywords: Unit root test; lag selection; information criteria; wild bootstrap; nonstationary volatility.

J.E.L. Classifications: C22; C15.

1 Introduction

Applied researchers have recently focused attention on the question of whether or not the variability in the shocks driving macroeconomic time series has changed over time; see, e.g.,

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the literature review in Busetti and Taylor (2003). The empirical evidence has suggested that time-varying behaviour, in particular a general decline, in unconditional volatility in the shocks driving macroeconomic time-series over the past twenty years or so is a relatively common phenomenon; see, *inter alia*, Kim and Nelson (1999), McConnell and Perez Quiros (2000), Van Dijk, Osborn, and Sensier (2002), Sensier and Van Dijk (2004) and references therein.[1] Sensier and Van Dijk (2004), for example, report that over 80% of the real and price variables in the Stock and Watson (1999) data-set reject the null of constant unconditional innovation variance. Empirical evidence also suggests that data are often characterized by smooth volatility changes rather than by abrupt changes (see, inter alia, Van Dijk et al., 2002).

Such, nonstationary volatility, effects can significantly impact on the size of standard unit root tests, even asymptotically, as has been shown by Cavaliere and Taylor (2007, 2008), among others. A solution to this problem is analyzed by Cavaliere and Taylor (2008, 2009b), who employ the wild bootstrap to capture the nonstationary volatility within the re-sampled data. They show that the wild bootstrap correctly reproduces the first-order limiting null distribution under nonstationary volatility, thereby allowing for the construction of asymptotically valid bootstrap tests.

The analysis in Cavaliere and Taylor (2008, 2009b) is based on the use of a lag length in the augmented Dickey-Fuller [ADF] test regression which is a deterministic function of the sample size. In practice, however, applied researchers usually base their analysis on an ADF regression where the lag order is chosen by data-dependent methods. Often this is done using standard information criteria or by sequential t-testing (using conventional critical values) on the significance on the highest lag. However, both of these approaches are misspecified in the presence of nonstationary volatility: standard information criteria are based on the assumption of constant volatility, while the limit distributions used in sequential t-testing are affected by the presence of nonstationary volatility. As such, if nonstationary volatility is present in the data, the lag length selected by the applied researcher may not be appropriate. While not necessarily invalidating the asymptotic properties of the unit root test, this may nonetheless have a significant impact on finite sample performance.

In this paper we analyze the finite sample effects of nonstationary volatility on the selection of the lag order in (bootstrap) unit root testing. Using Monte Carlo simulation methods we will show that, under certain time-varying volatility specifications, standard information criteria select too many lags and that this has a significant negative effect on the power of the resulting unit root test. As a consequence, we also propose a modification of the standard information criteria, based on the approach of Beare (2008) which re-scales the data by an estimate of the underlying volatility process. Again using Monte Carlo methods, we show that

[1] The recent financial turmoil associated with the onset of the 2008 credit crisis will undoubtedly reverse this trend and effect a corresponding rise in unconditional volatility; this, of course, reinforces the need to allow for the possibility of non-constancy in unconditional volatility.

these new criteria are considerably more robust, in terms of the lag length they select, than the standard criteria in the presence of nonstationary volatility and perform very similarly to the standard criteria when volatility is constant. We show that this results in unit root tests which display significantly more power than those based on the standard lag selection criteria under nonstationary volatility yet do not lose power relative to these tests when volatility is constant. Moreover, the sizes of the tests based on the standard and new criteria are shown to be broadly the same under both constant and nonconstant volatility environments.

The structure of the paper is as follows. In Section 2 we introduce our reference data generating process (DGP) and detail the class of heteroskedastic volatility processes under which we will work. The (wild bootstrap) unit root tests, and associated lag selection criteria with the new heteroskedasticity-robust modification thereof are discussed in Section 3. The finite-sample properties of the standard and new lag selection criteria, along with the size and power properties of the associated (wild bootstrap) unit root tests, are explored through Monte Carlo simulation in Section 4. Section 5 offers some conclusions.

2 The Heteroskedastic Model

Consider the case where we have $T+1$ observations generated according to the following data generating process (DGP),

$$y_t = x_t + \beta' z_t, \qquad t = 0, 1, \ldots, T, \tag{1a}$$

$$x_t = \rho x_{t-1} + u_t, \qquad t = 1, \ldots, T, \tag{1b}$$

$$u_t = \varepsilon_t + \sum_{j=1}^{\infty} \psi_j \varepsilon_{t-j} =: \psi(L)\varepsilon_t, \tag{1c}$$

$$\varepsilon_t = \sigma_t e_t \tag{1d}$$

with $E(x_0^2) < \infty$. Our focus in this paper is on tests for whether or not y_t contains a unit root; that is, on testing $H_0 : \rho = 1$ against $H_1 : |\rho| < 1$ in (1).

In (1a), z_t is a vector of deterministic components. As in Ng and Perron (2001) we focus on the κth-order trend function, $z_t := (1, t, \ldots, t^{\kappa})'$, with special focus on the leading cases of a constant ($\kappa = 0$) and linear trend ($\kappa = 1$). We also make the following assumptions on the shocks u_t, where $\mathcal{D} := D[0,1]$ denotes the space of right continuous with left limit (càdlàg) processes:

Assumption 1. (i) $\psi(z) \neq 0$ for all $|z| \leq 1$, and $\sum_{j=1}^{\infty} j|\psi_j| < \infty$. (ii) e_t is i.i.d. with $\mathrm{E}\, e_t = 0$, $\mathrm{E}\, e_t^2 = 1$ and $\mathrm{E}\,|e_t|^4 < \infty$. (iii) The volatility term σ_t satisfies $\sigma_t = \omega(t/T)$ for all $r \in [0,1]$, where $\omega(\cdot) \in \mathcal{D}$ is nonstochastic, twice-differentiable and strictly positive.

Remark 1. Assumption 1 corresponds to the set of conditions imposed on the shocks in

Cavaliere and Taylor (2008) and Smeekes and Taylor (2011), strengthened by the addition of condition (iii). This additional condition is required because the new heteroskedasticity-robust information criteria, which we propose in section 3 below, require a consistent estimate of the volatility process. Beare (2008) shows that (iii) suffices for this purpose when using a nonparametric kernel estimator as is required here since we have not assumed a specific parametric model for the volatility process. As the conditions in Assumption 1 are stronger than those in Cavaliere and Taylor (2008) and Smeekes and Taylor (2011), the large sample validity of the bootstrap unit root tests discussed in the next section is guaranteed. The reader is directed to Cavaliere and Taylor (2008) and Beare (2008) for further discussion of the conditions imposed by Assumption 1. Notice that Assumption 1 contains unconditional homoskedasticity as a special case.

3 Unit Root Testing and Information Criteria

3.1 Bootstrap Unit Root Tests

For the purposes of this paper we will focus our attention on wild bootstrap implementations of the augmented Dickey-Fuller (ADF) tests. We do so because of the enduring popularity of these tests with practitioners. However, we note that the analysis provided in this paper is also valid for any unit root test that requires an autoregressive lag order to be selected. The ADF t-statistic is the usual regression t-statistic of significance on γ, denoted t_γ^d in what follows, in the ADF regression

$$\Delta y_t^d = \gamma y_{t-1}^d + \sum_{j=1}^{p} \phi_{p,j} \Delta y_{t-j}^d + \varepsilon_{p,t}^d, \tag{2}$$

where $y_t^d := y_t - \hat{\beta}' z_t$ is the de-trended analogue of y_t, where the parameter estimate $\hat{\beta}$ can be obtained either by the OLS or the quasi-difference (QD) regression of y_t on z_t; see, among others, Elliott, Rothenberg, and Stock (1996). In the context of (2), p is the lag truncation order. We defer a discussion of the criteria that will be used to estimate p until sections 3.2 and 3.3.

Under nonstationary volatility, the ADF t_γ^d statistic is not asymptotically pivotal and the associated ADF test can display very large size distortions; see Cavaliere and Taylor (2008, 2009b). One solution to this problem, studied by Cavaliere and Taylor (2008, 2009b) and Smeekes and Taylor (2011) among others, is to apply the wild bootstrap principle. Cavaliere and Taylor (2008, 2009b) demonstrate the asymptotic validity of this approach, for the case of a deterministic lag length satisfying Assumption 2 below, and give simulation results which show that the method works well in finite samples. Hence, our focus in what follows will be on wild bootstrap implementations of the ADF test where data-dependent methods are used to select the lag length in (2). We now outline the wild bootstrap algorithm which we will use.

Algorithm 1.

1. Calculate $y_t^d := y_t - \hat{\beta}' z_t$, where $\hat{\beta}$ is obtained either by the OLS or QD regression of y_t on z_t.

2. Estimate by OLS the ADF regression in (2) using a lag order, q, to obtain the ADF residuals

$$\breve{\varepsilon}_{q,t}^d := \Delta y_t^d - \breve{\gamma} y_{t-1}^d - \sum_{j=1}^{q} \breve{\phi}_{q,j} \Delta y_{t-j}^d, \qquad t = 1, \ldots, T, \tag{3}$$

   by defining $y_{-1}, \ldots, y_{-q} := 0$.

3. Construct (wild) bootstrap errors ε_t^* according to the device $\varepsilon_t^* := \xi_t \breve{\varepsilon}_{q,t}^d$, where ξ_t satisfies $\mathrm{E}(\xi_t) = 0$ and $\mathrm{E}(\xi_t^2) = 1$.[2]

4. Build u_t^* recursively as $u_t^* = \sum_{j=1}^{q} \breve{\phi}_{q,j} u_{t-j}^* + \varepsilon_t^*$, using the estimated parameters $\breve{\phi}_{q,j}$ from Step 2 (initialised at $u_0^*, \ldots, u_{1-q}^* = 0$), and build y_t^* as $y_t^* = y_{t-1}^* + u_t^*$, $t = 1, ..., T$, initialised at $y_0^* = 0$.

5. Using the bootstrap sample y_t^*, apply the same method of detrending as applied to the original sample in step 1 to obtain the detrended bootstrap series $y_t^{d*} := y_t^* - \hat{\beta}^{*\prime} z_t$, where $\hat{\beta}^*$ is defined analogously as in step 1, but with the bootstrap data. Calculate the bootstrap augmented ADF statistic, denoted t_γ^{d*}, from the bootstrap analogue of the ADF regression, with lag truncation p^*,

$$\Delta y_t^{d*} = \gamma^* y_{t-1}^{d*} + \sum_{j=1}^{p^*} \phi_{p^*,j} \Delta y_{t-j}^{d*} + \varepsilon_{p^*,t}^{d*}, \qquad t = p^* + 1, \ldots, T. \tag{4}$$

6. Repeat Steps 3 to 5 N times, obtaining bootstrap test statistics, $t_{\gamma,b}^{d*}$ say, for $b = 1, \ldots, N$, and calculate the bootstrap critical value

$$cv^{d*}(\pi) := \max\{x : N^{-1} \sum_{b=1}^{N} I(t_{\gamma,b}^{d*} < x) \leq \pi\}$$

   or, equivalently, as the π-quantile of the ordered $\{t_{\gamma,b}^{d*}\}_{b=1}^{N}$ statistics. Reject the null of a unit root if $t_{\gamma,b}^{d}$ is smaller than $cv^{d*}(\pi)$, where π is the nominal level of the test. □

[2] In this paper we take ξ_t to be standard normal. Other choices are also possible, although Cavaliere and Taylor (2008, Remark 6) mention that this has almost no impact on finite sample behaviour.

3.2 Standard Lag Selection Criteria

While the wild bootstrap procedure outlined in Algorithm 1 takes account of any possible nonstationary volatility in the shocks without the need to parametrically model the volatility process, the presence of the lagged dependent variables in (2) is required to parametrically account for any serial correlation in the shocks. Consequently, in order to implement the ADF test, the selection of an appropriate lag length in (2), and indeed in (3) and (4), is required.

It is unrealistic to assume that the true value of p, p_0 say, in (2) is known to the practitioner, since the nature of the serial correlation in u_t cannot be reasonably assumed known. Indeed, p_0 may be infinite, as is the case, for example, if u_t is a finite-order moving average (MA) process. In such cases, it is well known, see for example Chang and Park (2002), that if the lag truncation order in (2) satisfies the following deterministic rate condition:

Assumption 2. Let $p \to \infty$ and $p = o(T^{1/3})$ as $T \to \infty$.

then, provided ε_t in (1c) is either homoskedastic or conditionally heteroskedastic (but unconditionally homoskedastic), the resulting ADF statistic, t_γ^d, will have the usual Dickey-Fuller limiting null distribution free of serial correlation nuisance parameters; as tabulated for the case of OLS detrending in Hamilton (1994, p. 763) and for QD detrending in Elliott et al. (1996, p. 825). As noted in section 3.1, Cavaliere and Taylor (2008, 2009b) demonstrate a corresponding result for the case where ε_t is unconditionally heteroskedastic; here the limiting null distribution of t_γ^d remains free of serial correlation nuisance parameters but does now depend on the form of the underlying volatility process.

As pointed out by Cavaliere and Taylor (2009a, Section 3.3), the sieve, or re-colouring, device in step 4 of Algorithm 1 is motivated purely by finite sample concerns, and q does not therefore have to increase to infinity with the sample size.[3] Also, although p^* is not required to diverge with T, we do require that $q \leq p^*$ for large T.[4] Specifically, we make the following assumptions on q and p^*:

Assumption 3. (i) Let $p^* = o(T^{1/3})$; (ii) there is a T^* such that $q \leq p^*$ for all $T > T^*$.

For a given sample size, the conditions in Assumptions 2 and 3 do not provide any practical guidance on how to select the lag length in (2), (3) and (4). A popular choice for estimating the lag length, which permits a trade-off between the size distortions that result from including too few lags and the power losses that obtain when too many lags are included, is to base it on an information criterion (see also Remark 2). This approach estimates the lag length as

[3] This differs from the approach taken by Smeekes and Taylor (2011, Assumption 5) for reasons explained in their Remark 15.

[4] Cavaliere and Taylor (2009a) assume that $q \leq p^*$ for all T but this is not necessary for the validity of the bootstrap. By allowing p^* to be smaller than q one can replicate the effect of under-fitting the lag length in the bootstrap, which may improve finite sample performance (cf. Richard, 2009).

follows

$$\hat{p} := \underset{p_{\min} \leq k \leq p_{\max}}{\arg\min} \; IC(k), \qquad IC(k) := \ln \hat{\sigma}_k^2 + k\frac{C_T}{T}, \tag{5}$$

where $\hat{\sigma}_k := (T - p_{\max})^{-1} \sum_{t=p_{\max}+1}^{T} (\hat{\varepsilon}_{k,t}^d)^2$ with $\hat{\varepsilon}_{k,t}^d$ the OLS residuals from the k-th order ADF regression for y_t^d in (2); that is, $\hat{\varepsilon}_{k,t}^d := \Delta y_t^d - \hat{\gamma} y_{t-1}^d - \sum_{j=1}^{k} \hat{\phi}_{k,j} \Delta y_{t-j}^d$, and where $p_{\min} \leq p_{\max}$ are selected such that $p_{\min}, p_{\max} \to \infty$ as $T \to \infty$ with $p_{\max}$ satisfying the rate condition in Assumption 2. In the context of (5), C_T is a penalty function that differs according to the specific information criterion to be used; for AIC $C_T := 2$, while for BIC $C_T := \ln T$. Tsay (1984) shows that for finite p, the properties of AIC and BIC in the stationary case remain the same in the presence of unit roots; that is, BIC is consistent while AIC is not (it overestimates with a positive probability). Pötscher (1989) extends these results to allow for nonconstant volatility in the errors, and finds that consistency of BIC in his general setup cannot be guaranteed (although BIC is found to be consistent in stable AR models with nonconstant volatility).

Ng and Perron (2001) propose a class of modified information criteria (MIC), motivated specifically for selecting the lag length in the ADF regression, (2). Their proposed class of information criteria can be written as

$$MIC(k) := \ln \hat{\sigma}_k^2 + k\frac{C_T + \tau_T(k)}{T},$$

where $\tau_T(k) := (\hat{\sigma}_k^2)^{-1} \hat{\gamma}^2 \sum_{p_{\max}+1}^{T} (y_{t-1}^d)^2$. The associated lag length estimate is then defined as in (5) but replacing $IC(k)$ by $MIC(k)$ in the definition of $\hat{p}$. The penalty function C_T has to be selected in the same way as for the original criteria; for example, taking $C_T := 2$ yields the modified AIC (MAIC) criterion, and taking $C_T := \ln T$ yields the modified BIC (MBIC) criterion. Although asymptotically the properties of the original criteria will be maintained, Ng and Perron (2001) show that these modified criteria yield large improvements over the standard criteria for the purpose of unit root testing, in particular if a negative moving average parameter is present in the short-run dynamics. In that case the MIC will select considerably more lags than standard criteria and thus improve the size of the corresponding unit root test. Perron and Qu (2007) propose a further modification of these criteria, by suggesting that they should always be applied to OLS rather than QD detrended data even if the unit root test itself is based on QD detrended data. This will improve the power properties of the test, in particular for alternatives further from the null.

Although nonstationary volatility may alter the large sample properties of the lag selection criteria (Pötscher, 1989), provided $p_{\min}$ and $p_{\max}$ satisfy the conditions stated above, then the limiting null distributions of t_γ^d and t_γ^{d*} will not be affected by the short-run dynamics (although they will of course be affected by the form of the volatility). Our investigation is therefore purely related to the performance of the (wild bootstrap) ADF unit root test in finite samples, since in finite samples the lag selection criteria are misspecified if the volatility process is time-varying and cannot necessarily be relied upon to yield an appropriate estimate of the required lag lengths. This is confirmed by the simulation results we will subsequently present in Section 4.

Remark 2. We focus here on lag length selection through information criteria rather than through sequential t-testing, as this approach has proven to be more popular and also more successful; sequential t-testing tends to select too many lags on average. Sequential t-testing can be adapted to the setting of nonstationary volatility by either using heteroskedasticity-robust standard errors, or applying again the wild bootstrap.

In the next subsection we will present further modifications to the standard lag order selection methods outlined above that are designed to be robust to nonstationary volatility.

3.3 Heteroskedasticity-Robust Lag Selection Criteria

In this subsection we propose a method for lag length selection based on information criteria that is designed to be robust to heteroskedasticity. Rather than modifying the information criteria themselves, we modify the series that is the input to the information criteria. We adapt the idea proposed in Beare (2008) to lag length selection; that is, we estimate the volatility nonparametrically and then re-scale the series with the estimated volatility.

To estimate the volatility nonparametrically we use the local constant, or Nadaraya-Watson, estimator also used by Beare (2008).[5] The volatility estimator at time t is then defined as

$$\hat{\sigma}_{m,t} := \sqrt{\hat{\omega}_m^2(t/T)}, \qquad \hat{\omega}_k^2(r) := \frac{\sum_{t=1}^T K\left(\frac{t/T-r}{h}\right)(\breve{\varepsilon}_{m,t}^d)^2}{\sum_{t=1}^T K\left(\frac{t/T-r}{h}\right)} \tag{6}$$

where $\{\breve{\varepsilon}_{m,t}^d\}$ are defined in (3) with a lag truncation of m, $K(\cdot)$ is a kernel function and h is a bandwidth parameter. As in Beare (2008), the following assumption is needed on the kernel $K(\cdot)$ and the bandwidth h in order to ensure that (6) consistently estimates the volatility process:

Assumption 4. (i) $K(\cdot)$ is continuously differentiable and satisfies $\int K(x)dx > 0$, $\int |xK(x)|\,dx < \infty$, and $\int |xK'(x)|\,dx < \infty$. Moreover, the Fourier transform of $K(\cdot)$, denoted $\psi(\cdot)$, satisfies $\int |x\psi(x)|\,dx < \infty$. (ii) $h \to 0$ and $Th^4 \to \infty$ as $T \to \infty$.

[5]We also considered the re-weighted local constant estimator proposed by Xu and Phillips (2011). As discussed by Xu and Phillips (2011), this estimator shares all the advantages of the local linear estimator. However, unlike the local linear estimator (but like the local constant estimator), it cannot be negative. The simulation results with this estimator were virtually identical to the results reported here with the local constant estimator and, hence, are omitted in the interests of space.

The volatility estimates from (6) are then used to re-scale the series of interest as follows:

$$\tilde{y}_t := \sum_{s=1}^{t} \frac{\Delta y_s^d}{\hat{\sigma}_{m,s}}, \qquad \tilde{y}_0 := 0. \tag{7}$$

The idea behind the re-scaling in (7) is that $\tilde{y}_t$ will be rendered (approximately) homoskedastic. The re-scaled series $\tilde{y}_t$ is then used as input to the information criteria. The corresponding re-scaled (modified) information criteria, denoted as RS(M)IC in what follows, are then calculated as

$$RSIC(k) := \ln \tilde{\sigma}_k^2 + k\frac{C_T}{T}, \qquad RSMIC(k) := \ln \tilde{\sigma}_k^2 + k\frac{C_T + \tilde{\tau}_T(k)}{T},$$

where $\tilde{\tau}_T(k) := (\tilde{\sigma}_k^2)^{-1}\tilde{\gamma}^2 \sum_{p_{\max}+1}^{T} (\tilde{y}_{t-1}^d)^2$, $\tilde{\sigma}_k = (T - p_{\max})^{-1} \sum_{t=p_{\max}+1}^{T} (\tilde{\varepsilon}_{k,t}^d)^2$, and where $\tilde{\varepsilon}_{k,t}^d$ is the OLS residual from a k-th order ADF regression on $\tilde{y}_t^d$, which is either the OLS or QD detrended analogue of $\tilde{y}_t$.

In practice one must also select a value for the lag truncation m used in the construction of the volatility estimator in (6). The choice $m = 0$ corresponds to Beare (2008), while taking $m = p_{\max}$ would also seem to be a sensible choice in the lag selection framework. In this paper we will follow Beare (2008) and set $m = 0$, but unreported simulations showed that setting $m = p_{\max}$ gave virtually identical results.[6]

4 Monte Carlo Simulations

In this section we will use Monte Carlo simulation methods to investigate the finite sample performance of the standard information criteria and their new heteroskedasticity-robust analogues developed in the previous section. Comparison is made both of the lag order selected by these criteria and of the size and power properties of the associated wild bootstrap ADF tests for a variety of homoskedastic and heteroskedastic ARMA models.

4.1 The Monte Carlo Design

In the simulation study we use the following DGP:

$$y_t = x_t + \beta' z_t, \qquad t = 0, 1, \ldots, T, \tag{8a}$$
$$x_t = \rho_T x_{t-1} + u_t, \qquad t = 1, \ldots, T, \tag{8b}$$
$$u_t = \phi_1 u_{t-1} + \phi_2 u_{t-2} + \phi_3 u_{t-3} + \varepsilon_t + \theta\varepsilon_{t-1} \tag{8c}$$
$$\varepsilon_t = \sigma_t e_t, \qquad e_t \sim i.i.d.\ N(0,1), \tag{8d}$$

[6]Similarly, it is possible to use the residuals which are obtained when imposing the unit root null hypothesis. Unreported simulation results indicated that the results do not change in this case either.

for the local-to-unity setting where $\rho_T = 1 - c/T$, such that $c = 0$ corresponds to the unit root null hypothesis and $c > 0$ to local alternatives. Without loss of generality we set $\beta = 0$ and $x_0 = 0$.

We report results for the combinations of the AR and MA parameters ϕ_1, ϕ_2, ϕ_3 and θ in (8c) given in Table 1. Table 1 also reports for each model the true value, p_0, of the associated lag augmentation in (2). These ARMA parameters allow for a range of different dynamic models, ranging from near I(2) data (models 8 and, to a lesser extent, 5 and 10, with $\rho_T = 1$) to near over-differenced data (model 11 with $\rho_T = 1$). The range of models is very similar to that considered by Ng and Perron (2005) and allows both finite AR (of orders 1,2 and 3) and MA(1) models.

Insert Table 1 about here

Results are reported for following two models of volatility models:

1. Single break in volatility: $\sigma_t^2 = \sigma_0^2 + (\sigma_1^2 - \sigma_0^2) I(t > \lfloor \tau T \rfloor)$, where we set $\sigma_0 = 0$. Defining $\delta = \sigma_0/\sigma_1$, we consider parameters $\delta = 1/3, 3$ and $\tau = 0.2, 0.8$.

2. Stochastic volatility: $\sigma_t^2 = \omega^2(t/T)$ where $\omega^2(s) = \sigma_0^2 \exp(\nu J_c(s))$. Again we set $\sigma_0 = 1$, and we consider parameters $c = 0, 10$ and $\nu = 4, 9$.

Notice that neither of these volatility models are formally allowed under the assumptions needed on the kernel estimation, although both are allowed for the wild bootstrap unit root tests (see Cavaliere and Taylor, 2009b). We still chose these models of volatility as they are popular choices in the literature and appear to describe empirically observed patterns well. Moreover, good performance by the new lag selection criteria for models such as these which fall outside the class of models they are intended for can be argued to reinforce their potential. Also observe that the homoskedastic case is contained in the first model when $\delta = 1$, and in the second model when $\nu = 0$.

In this analysis we present results only for the MAIC criterion of Ng and Perron (2001) and the heteroskedasticity-robust analogue thereof, RSMAIC, from section 3.3. We do so because MAIC is the most popular and successful criterion used in unit root testing. However, a summary of the corresponding results for other popular lag selection methods is given at the end of this section. In the context of the MAIC and RSMAIC criteria the minimum lag length, $p_{\min}$ was set to zero throughout, while the maximum lag length was set to $p_{\max} = A(T/100)^{1/4}$, with the choice of the constant A specified in the subsections which follow. We report results for the sample sizes $T = 150$ and $T = 250$.[7] Throughout this section we will only report results for the specification where a constant is included in z_t in (8a). Results for the constant and

[7] For smaller sample sizes the differences between the regular and re-scaled IC are not so noticeable. This is most likely at least partly caused by the fact that the maximum lag lengths are considerably smaller for such sample sizes.

trend case are very similar, and are available on request. As recommended by Perron and Qu (2007), we apply the information criteria to OLS detrended data. As mentioned before, the volatility estimator used in the RSMAIC is $\hat{\sigma}_{0,t}$, with the kernel $K(\cdot)$ taken as the Gaussian kernel and the bandwidth set equal to $h = 0.1$.[8]

4.2 Selected Lag Lengths

We first focus on the lag lengths selected by the standard MAIC and the new heteroskedasticity-robust RSMAIC criteria. As part of our analysis we vary the maximum lag length, $p_{\max}$, by considering results for both $A = 6$ and $A = 12$. In large samples and for the (low-order) autoregressive models the lag selection should not be significantly affected by changing the upper bound. If, however, a criterion is seriously affected by the choice of $p_{\max}$ then this provides clear evidence that the criterion is not selecting the lag length appropriately for the sample sizes considered. All results are based on 5000 simulations.

Insert Table 2 about here

Table 2 reports the average (taken across the Monte Carlo replications) selected lag lengths obtained under homoskedasticity. It can be seen from these results that the MAIC and RSMAIC criteria perform very similarly to one another here for all of the AR and MA models considered. These results suggest that the re-scaling approach used in calculating the RSMAIC criterion does not fundamentally change its properties from those of the MAIC criterion under homoskedasticity, which is a necessary condition to apply it successfully. It can also be seen that for the AR models considered, other things being equal, changing the maximum lag length (through the choice of the constant A) has only a minor impact on the average lag length selected for both criteria, as expected.

Insert Tables 3-6 about here

Tables 3 to 6 present the corresponding results for the case of a single break in the volatility. From these results we can see that both the direction and timing of the break have a considerable impact on the lag length selected by the standard MAIC criterion. In particular, while late negative or early positive breaks do not appear to have a significant impact on the lag length selected by MAIC, the effect of either a late positive or early negative break is, on the other hand, substantial. For these volatility models MAIC selects considerably higher lag lengths lags than it does under homoskedasticity. This effect can be seen for all of the ARMA models considered. Moreover, in these cases changing the maximum lag length now has a major impact on the performance of MAIC, which is again a clear indication that the standard MAIC criterion selects too many lags, "pushing" up against the upper bound as

[8]Different specifications again gave very similar results.

a result. In contrast, the RSMAIC criterion appears to select roughly the same number of lags in the single break models as it does under homoskedasticity; there only appears to be a minimal increase in some of the cases considered. Also, RSMAIC is far less affected by varying $p_{\max}$ than MAIC is, which again confirms the robustness of the lag length selected by RSMAIC to the single break model of volatility.

Insert Tables 7-10 about here

Tables 7 to 10 present the average selected lags under stochastic volatility. Here from a comparison with the results in Table 2 it can again be seen that the standard MAIC criterion selects a higher lag length on average than it does under homoskedasticity, most notably when $c = 0$. We now also see an increase in the average lag length selected by RSMAIC, although it still selects a considerably lower average lag length than MAIC. Hence, even though RSMAIC is affected to some degree by stochastic volatility, it remains considerably more reliable than MAIC in this setting.

To summarise, our simulation results have shown that lag length selection by MAIC is affected by the presence of nonstationary volatility in the errors. As such it cannot be reliably used to select an appropriate lag length for a unit root test in this setting. The simulation results also show that RSMAIC appears to be significantly more robust to the presence of nonstationary volatility, while its performance under homoskedasticity is almost identical to MAIC. In the context of unit root testing, it is arguably the performance of the unit root test for which lag orders are selected, rather than the actual selected lag order, which is of primary importance. If the lag selection has no effect on the size or power properties of the resulting unit root test, then there is no problem in using a potentially misspecified method such as MAIC. Therefore we will now investigate the impact of nonstationary volatility on the finite sample size and power properties of the wild bootstrap ADF unit root test, when the lag length in the ADF regression has been selected by either MAIC or RSMAIC.

4.3 Rejection Frequencies of Bootstrap Unit Root Tests

In this subsection we investigate the performance of the wild bootstrap ADF unit root test from Algorithm 1, using QD detrending, and where the lag truncation order in the original ADF regression (2), the sieve regression (3), and the bootstrap ADF regression (4), were selected by either MAIC or RSMAIC, using the same tuning parameters as outlined in section 4.1, with results reported for $A = 12$. All results in this subsection are based on 5000 simulations and 199 bootstrap replications.

Insert Tables 11-13 about here

We first report, in Tables 11 to 13, the size properties of the wild bootstrap ADF tests based on MAIC and RSMAIC lag selection for the same set of ARMA and volatility models as were used in the previous subsection. Sizes for MAIC and RSMAIC seem to be comparable across the different models; both give sizes close to the nominal level of 5% except for model 11 (which has a large negative MA parameter), where there is some oversize (of roughly the same degree) seen for both methods. Overall it does not appear that the choice between using MAIC or RSMAIC when choosing the lag length has a significant impact on the size of the resulting unit root test, regardless of whether the errors are homoskedastic or heteroskedastic.

We next present finite sample local power curves for the bootstrap ADF tests. In order to keep the number of graphs manageable, we need to make a selection of the ARMA models considered. To this end we report results for the i.i.d. model (model 1), the AR(1) model with $\phi_1 = 0.5$ (model 4) and the MA(1) model with $\theta = -0.5$ (model 12). We consider the same type of volatility models as before but focus on the cases where MAIC is most affected by the volatility process. The simulation results in Section 4.2 showed that for the volatility model with a single break, MAIC was most affected by a late positive or early negative break, while for the stochastic volatility model, MAIC was most affected if a unit root was present in the volatility ($c = 0$). These cases together with the benchmark of homoskedasticity will therefore be considered in the power analysis.

Insert Figure 1 about here

In Figure 1 we first present the finite sample local power curves of the wild bootstrap ADF tests based on MAIC and RSMAIC lag selection for the homoskedastic model. In the homoskedastic case the power of the tests using MAIC and RSMAIC are almost identical to one another, which is again as expected given the results from section 4.2. This shows that the power losses incurred by using the RSMAIC criterion to select the lag length when in fact the MAIC criterion is correctly specified are negligible even for $T = 150$.

Inserts Figures 2-3 about here

Figures 2 and 3 give the corresponding local power curves for the single break in variance model with a late positive break and an early negative break, respectively.[9] For these single break models, the bootstrap ADF test based on the use of RSMAIC is clearly more powerful than the corresponding test based on MAIC. This is a direct consequence of the results reported in section 4.2 which showed that the MAIC criterion significantly over-fits the lag order relative to the RSMAIC criterion for these designs. It is clear that in these cases there are considerable finite sample power gains available by using RSMAIC. Moreover, the power

[9] Notice that the local power curves for the single break models are quite different from the corresponding local power curves seen in Figure 1 under homoskedasticity, even for $T = 250$. This is not an effect of the lag order selection method but rather a consequence of the result that if nonstationary volatility is present, then the limiting distributions of the ADF statistic, t^d_γ, under both the null hypothesis and local alternatives, and hence the asymptotic local power function of the associated bootstrap test, are functions of the underlying volatility process (cf. Cavaliere and Taylor, 2008, p. 8).

differences between using MAIC and RSMAIC lag selection even increase slightly between $T = 150$ and $T = 250$, which appears to be related to the associated increase in the maximum lag length, $p_{\max}$, between the two sample sizes.

Insert Figures 4-5 about here

Figures 4 and 5 graph the finite sample local power curves for the stochastic volatility models with $c = 0$ and $v = 4, 9$. While the bootstrap ADF test based on RSMAIC is still more powerful than the corresponding test based on MAIC, the difference between the two is now rather smaller than was seen for the single break in volatility models. This is to be expected from the results on the average lag length selected by these two criteria in section 4.2, which showed that RSMAIC has a tendency to over-fit the lag length in this case, although not to the same extent as is seen with MAIC. While the gains of using RSMAIC may be smaller for the stochastic volatility case, it is nonetheless important to note that there is never a loss in power when using RSMAIC rather than MAIC to select the lag length.

We can summarise the results in this subsection by observing that lag order selection based on MAIC has a negative impact on the finite sample power of the resulting wild bootstrap ADF unit root test if nonstationary volatility is present, with the extent of this effect depending on the specific volatility model. Based on our results, we recommend the use of the RSMAIC lag selection criterion for selecting the lag length in the context of ADF unit root testing, given its greater degree of robustness to nonstationary volatility than the standard MAIC lag selection criterion, and the resulting higher finite sample power which is achievable when using RSMAIC over MAIC. These power gains are most strongly seen for single break in volatility models. Moreover, under homoskedasticity we found almost no differences in power between the unit root tests which use RSMAIC and MAIC to select the lag order. Under all of the volatility and ARMA models considered the finite sample size properties of the unit root tests based on MAIC and RSMAIC were virtually identical. As such we believe it provides a reliable practical alternative to MAIC.

We conclude this section by noting that the conclusions drawn above concerning wild bootstrap ADF tests based on the MAIC lag selection method and its re-scaled analogue, RSMAIC, all carry through qualitatively to the corresponding ADF tests based other information criteria such as AIC and BIC (where the re-scaling in computing their heteroskedasticity-robust analogues is done identically). We also considered sequential t-tests for specifying the lag truncation order, as in Ng and Perron (1995), comparing their standard approach with modifications thereof based on either the use of White (1980) heteroskedasticity-robust standard errors or the wild bootstrap. Simulations indicated that sequential t-testing is affected by nonstationary volatility in much the same way as the information criteria reported here. Using White standard errors helps to alleviate the problems, but does not erase them. Wild bootstrap ADF tests using lag selection based on wild bootstrap sequential t-tests, like the tests based on the RSMAIC method, achieve higher power than the tests based on the standard sequential t-tests but have the considerable drawback that they take a very long time to compute. Moreover, we found them to be generally inferior than the tests based on RSMAIC, and so we do not report these results in detail. They are, however, available on request.

5 Conclusion

In this paper we have investigated the effect of nonstationary volatility on lag length selection in the context of unit root testing. We have also proposed a modification of the popular information criteria used for lag length selection, designed to be robust against nonstationary volatility. The modification consisted of re-scaling the data by a nonparametric estimate of the volatility process before computing the information criterion of interest.

Focusing on the popular MAIC criterion, we found that nonstationary volatility can have a significant impact on lag length selection in finite samples. Simulations were presented which showed that for several volatility models the lag order was over-fitted, with the selected lag length being highly dependent on the maximum lag length allowed in certain cases. Our proposed re-scaled MAIC, labeled RSMAIC, criterion did not demonstrate this feature and was shown to be robust to nonstationary volatility, most notably to the presence of a break in volatility. Moreover, the RSMAIC criterion was shown to perform almost identically to the MAIC criterion in terms of the lag order selected under homoskedasticity.

We then investigated the relative behaviour of the wild bootstrap ADF unit root tests obtained for these two different lag selection criteria. It was found that using MAIC in the presence of nonstationary volatility leads to a loss of finite sample power in the associated unit root test, caused by the tendency of MAIC to fit significantly more lags than RSMAIC. This despite the fact that size properties of the unit root tests based on MAIC and RSMAIC lag selection were shown to be broadly comparable. Moreover, under homoskedasticity no significant losses in power were observed for the unit root tests based on RSMAIC relative to those based on MAIC.

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Table 1: ARMA models considered

Model	p_0	β_1	β_2	β_3	θ
1	0	0.00	0.00	0.00	0.00
2	1	-0.80	0.00	0.00	0.00
3	1	-0.50	0.00	0.00	0.00
4	1	0.50	0.00	0.00	0.00
5	1	0.80	0.00	0.00	0.00
6	2	0.40	0.20	0.00	0.00
7	2	1.10	-0.35	0.00	0.00
8	2	1.30	-0.35	0.00	0.00
9	3	0.30	0.20	0.10	0.00
10	3	0.10	0.20	0.30	0.00
11	∞	0.00	0.00	0.00	-0.80
12	∞	0.00	0.00	0.00	-0.50
13	∞	0.00	0.00	0.00	0.50
14	∞	0.00	0.00	0.00	0.80

Table 2: Average lags selected; homoskedasticity

		$A=6$				$A=12$			
		$T=150$		$T=250$		$T=150$		$T=250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	0.57	0.54	0.67	0.69	0.76	0.69	0.85	0.82
2		1.59	1.63	1.65	1.69	1.85	1.95	1.90	1.98
3		1.57	1.54	1.61	1.60	1.82	1.79	1.84	1.81
4		1.57	1.62	1.62	1.75	1.79	1.86	1.82	1.99
5		1.55	1.70	1.62	1.86	1.76	2.03	1.83	2.25
6		2.20	2.23	2.49	2.56	2.47	2.52	2.76	2.89
7		2.49	2.58	2.56	2.64	2.80	2.96	2.82	3.06
8		2.50	2.79	2.56	2.96	2.78	3.50	2.82	3.81
9		2.56	2.53	2.94	2.98	2.83	2.85	3.20	3.36
10		3.34	3.35	3.49	3.55	3.66	3.72	3.82	3.94
11		5.21	5.21	6.21	6.22	7.54	7.56	8.62	8.61
12		3.00	2.98	3.40	3.38	3.37	3.32	3.75	3.67
13		2.70	2.69	3.18	3.17	3.02	3.00	3.47	3.52
14		4.82	4.79	5.75	5.74	5.94	5.91	7.20	7.23
1	7	0.75	0.71	0.76	0.71	0.96	0.91	1.01	0.93
2		1.85	1.92	1.86	1.89	2.33	2.43	2.18	2.27
3		1.73	1.72	1.77	1.75	2.06	2.05	2.07	2.05
4		1.64	1.61	1.71	1.68	1.92	1.83	1.97	1.89
5		1.63	1.61	1.68	1.69	1.94	1.90	1.94	1.95
6		2.03	1.97	2.43	2.39	2.37	2.25	2.72	2.65
7		2.59	2.59	2.66	2.65	2.97	2.94	2.98	2.96
8		2.54	2.62	2.63	2.75	2.88	3.02	2.93	3.16
9		2.13	2.08	2.66	2.63	2.40	2.31	3.02	2.95
10		3.15	3.07	3.54	3.52	3.60	3.42	3.93	3.84
11		5.06	5.05	6.34	6.34	8.35	8.34	9.97	10.01
12		3.45	3.43	3.86	3.86	4.12	4.09	4.40	4.40
13		2.67	2.62	3.05	3.02	3.03	2.96	3.42	3.33
14		4.69	4.66	5.47	5.46	5.74	5.66	7.04	6.98

Table 3: Average lags selected; single break: $\sigma_0/\sigma_1 = 1/3$ and $\tau = 0.2$

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	0.82	0.70	0.95	0.84	1.08	0.86	1.29	1.08
2		1.76	1.75	1.86	1.80	2.12	2.11	2.24	2.14
3		1.76	1.66	1.89	1.77	2.11	1.91	2.28	2.07
4		1.74	1.70	1.86	1.81	2.05	1.98	2.25	2.14
5		1.76	1.81	1.88	1.92	2.10	2.20	2.23	2.36
6		2.33	2.30	2.68	2.66	2.73	2.63	3.17	3.11
7		2.64	2.64	2.72	2.74	3.05	3.06	3.15	3.22
8		2.63	2.79	2.81	2.99	3.02	3.59	3.26	3.80
9		2.68	2.61	3.09	3.04	3.13	3.00	3.57	3.52
10		3.43	3.40	3.66	3.64	3.89	3.81	4.24	4.17
11		5.17	5.17	6.15	6.16	7.64	7.53	8.74	8.57
12		3.08	2.98	3.53	3.44	3.59	3.36	4.11	3.88
13		2.83	2.76	3.29	3.22	3.29	3.14	3.78	3.62
14		4.84	4.80	5.73	5.72	6.17	5.98	7.42	7.29
1	7	0.93	0.78	1.09	0.92	1.27	1.02	1.53	1.20
2		1.98	1.99	2.10	2.03	2.54	2.49	2.67	2.55
3		1.92	1.84	2.04	1.90	2.42	2.24	2.54	2.28
4		1.86	1.73	1.89	1.78	2.26	2.05	2.40	2.15
5		1.78	1.72	1.99	1.91	2.19	2.02	2.46	2.29
6		2.15	2.01	2.62	2.50	2.59	2.34	3.15	2.88
7		2.75	2.67	2.84	2.75	3.22	3.08	3.44	3.24
8		2.68	2.70	2.84	2.86	3.18	3.18	3.38	3.49
9		2.17	2.04	2.84	2.71	2.55	2.31	3.41	3.12
10		3.13	2.96	3.67	3.58	3.60	3.28	4.30	4.03
11		4.99	4.98	6.28	6.29	8.31	8.20	10.03	10.00
12		3.47	3.42	3.96	3.90	4.33	4.13	4.77	4.54
13		2.79	2.66	3.22	3.07	3.34	3.06	3.87	3.54
14		4.70	4.64	5.48	5.43	5.89	5.62	7.29	7.02

Table 4: Average lags selected; single break: $\delta = 1/3$ and $\tau = 0.8$

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	2.37	0.71	2.99	0.84	4.75	0.91	6.31	1.04
2		2.92	1.73	3.53	1.81	5.24	2.13	6.75	2.16
3		2.89	1.61	3.52	1.74	5.31	1.88	6.95	2.04
4		2.92	1.69	3.50	1.82	5.33	1.97	6.80	2.19
5		2.90	1.88	3.49	2.04	5.25	2.34	6.80	2.63
6		3.18	2.28	3.84	2.66	5.50	2.62	7.06	3.15
7		3.40	2.67	4.03	2.80	5.65	3.11	7.33	3.32
8		3.51	2.99	4.05	3.22	5.98	4.06	7.45	4.42
9		3.32	2.57	4.09	3.04	5.70	2.95	7.41	3.49
10		3.82	3.34	4.54	3.64	6.24	3.81	7.79	4.19
11		4.98	5.19	6.04	6.20	8.39	7.47	10.26	8.70
12		3.64	3.02	4.34	3.44	6.03	3.45	7.67	3.85
13		3.42	2.70	4.23	3.21	5.75	3.06	7.55	3.64
14		4.78	4.75	5.76	5.70	7.59	5.87	9.51	7.24
1	7	2.22	0.82	2.83	0.91	4.37	1.11	5.81	1.18
2		2.88	2.10	3.35	2.06	5.09	2.76	6.36	2.60
3		2.82	1.88	3.36	1.87	5.04	2.32	6.44	2.28
4		2.75	1.71	3.33	1.86	4.92	2.03	6.32	2.22
5		2.80	1.73	3.36	1.86	4.89	2.05	6.45	2.25
6		2.86	2.01	3.63	2.50	5.03	2.33	6.64	2.88
7		3.49	2.67	3.96	2.78	5.65	3.16	7.04	3.19
8		3.43	2.76	3.97	2.95	5.64	3.34	6.93	3.61
9		2.86	2.11	3.76	2.74	5.12	2.45	6.79	3.14
10		3.44	2.99	4.36	3.58	5.55	3.39	7.41	4.03
11		4.70	5.06	6.02	6.32	8.34	8.32	10.51	10.10
12		3.80	3.51	4.55	3.90	6.09	4.24	7.70	4.65
13		3.37	2.66	4.05	3.08	5.44	3.09	7.17	3.59
14		4.63	4.62	5.54	5.49	7.19	5.74	9.16	7.10

Table 5: Average lags selected; single break: $\delta = 3$ and $\tau = 0.2$

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	2.38	0.65	3.07	0.76	4.84	0.86	6.46	0.97
2		2.93	1.76	3.53	1.78	5.23	2.13	6.82	2.18
3		2.95	1.60	3.50	1.70	5.29	1.88	6.87	1.97
4		2.95	1.63	3.53	1.76	5.33	1.91	6.95	2.09
5		2.89	1.76	3.58	1.93	5.42	2.19	6.89	2.36
6		3.15	2.27	3.90	2.61	5.60	2.62	7.26	3.01
7		3.45	2.60	3.99	2.73	5.88	3.02	7.26	3.18
8		3.42	2.81	4.01	3.06	5.88	3.66	7.33	4.00
9		3.33	2.55	4.10	2.98	5.67	2.94	7.33	3.40
10		3.87	3.35	4.59	3.62	6.24	3.77	7.91	4.10
11		4.99	5.22	6.10	6.24	8.50	7.78	10.41	8.93
12		3.67	3.04	4.35	3.46	6.15	3.42	7.71	3.83
13		3.52	2.77	4.24	3.18	5.86	3.05	7.61	3.57
14		4.89	4.84	5.83	5.75	7.86	6.02	9.71	7.30
1	7	2.25	0.81	2.88	0.85	4.42	1.01	6.14	1.19
2		2.83	1.98	3.45	2.04	5.11	2.50	6.60	2.52
3		2.83	1.79	3.39	1.83	5.01	2.16	6.47	2.17
4		2.79	1.70	3.42	1.79	4.88	1.95	6.64	2.13
5		2.84	1.70	3.41	1.83	5.03	2.06	6.55	2.15
6		2.94	2.09	3.71	2.53	5.05	2.41	6.78	2.86
7		3.41	2.64	3.94	2.73	5.64	3.08	7.07	3.16
8		3.43	2.70	3.95	2.83	5.59	3.27	6.96	3.45
9		2.96	2.29	3.77	2.79	5.17	2.58	6.94	3.17
10		3.63	3.18	4.46	3.56	5.83	3.59	7.57	3.96
11		4.82	5.21	6.08	6.33	8.57	8.49	10.57	9.85
12		3.82	3.42	4.59	3.82	6.02	4.04	7.74	4.40
13		3.38	2.70	4.10	3.14	5.53	3.06	7.39	3.56
14		4.78	4.78	5.59	5.59	7.45	5.96	9.44	7.27

Table 6: Average lags selected; single break: $\delta = 3$ and $\tau = 0.8$

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	0.87	0.71	1.00	0.82	1.31	0.93	1.43	1.05
2		1.81	1.75	1.92	1.83	2.27	2.13	2.41	2.21
3		1.76	1.64	1.87	1.74	2.23	1.95	2.34	1.99
4		1.75	1.69	1.87	1.80	2.21	1.97	2.38	2.15
5		1.77	1.83	1.88	1.99	2.19	2.18	2.41	2.50
6		2.34	2.30	2.67	2.64	2.80	2.62	3.20	3.02
7		2.64	2.66	2.77	2.79	3.17	3.09	3.32	3.23
8		2.62	2.83	2.76	3.03	3.12	3.53	3.34	3.91
9		2.68	2.62	3.12	3.09	3.11	2.94	3.72	3.58
10		3.43	3.40	3.65	3.62	4.03	3.82	4.25	4.12
11		5.19	5.21	6.19	6.21	7.64	7.64	8.90	8.79
12		3.10	3.04	3.55	3.48	3.71	3.44	4.21	3.92
13		2.83	2.73	3.29	3.20	3.38	3.07	3.88	3.62
14		4.85	4.83	5.78	5.74	6.25	6.04	7.56	7.38
1	7	0.93	0.71	1.04	0.81	1.39	0.94	1.50	1.05
2		1.99	1.93	2.01	1.93	2.56	2.42	2.62	2.39
3		1.90	1.74	1.99	1.80	2.43	2.13	2.51	2.12
4		1.85	1.63	1.99	1.79	2.27	1.90	2.46	2.03
5		1.83	1.66	1.96	1.80	2.32	1.96	2.50	2.11
6		2.12	1.99	2.64	2.48	2.62	2.29	3.24	2.86
7		2.74	2.60	2.82	2.70	3.31	3.01	3.42	3.08
8		2.70	2.67	2.83	2.82	3.25	3.15	3.44	3.36
9		2.25	2.15	2.81	2.70	2.74	2.44	3.33	3.01
10		3.21	3.11	3.69	3.56	3.82	3.48	4.35	3.94
11		5.05	5.06	6.27	6.30	8.24	8.23	10.04	10.01
12		3.47	3.39	3.92	3.82	4.29	4.02	4.74	4.43
13		2.79	2.65	3.22	3.08	3.40	3.01	3.89	3.48
14		4.71	4.67	5.54	5.52	5.93	5.63	7.45	7.11

Table 7: Average lags selected; stochastic volatility: $c = 0$ and $v = 4$

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	2.27	0.98	2.96	1.21	4.28	1.32	5.98	1.66
2		2.86	2.04	3.44	2.26	4.90	2.63	6.27	2.94
3		2.81	1.89	3.48	2.13	4.80	2.30	6.38	2.69
4		2.78	1.92	3.50	2.22	4.90	2.30	6.38	2.80
5		2.83	2.08	3.47	2.43	4.93	2.71	6.37	3.31
6		3.06	2.42	3.89	2.90	5.17	2.92	6.90	3.61
7		3.39	2.83	3.97	3.12	5.44	3.40	6.95	3.92
8		3.42	3.10	4.00	3.46	5.55	4.12	7.07	4.99
9		3.34	2.74	4.09	3.25	5.34	3.23	6.98	3.98
10		3.86	3.45	4.51	3.86	5.95	4.05	7.57	4.73
11		4.93	5.14	6.02	6.14	8.21	7.67	10.02	8.90
12		3.63	3.13	4.35	3.63	5.72	3.66	7.33	4.29
13		3.50	2.88	4.26	3.44	5.53	3.35	7.25	4.12
14		4.81	4.80	5.81	5.75	7.50	6.07	9.38	7.57
1	7	2.15	1.06	2.85	1.29	4.12	1.42	5.56	1.75
2		2.81	2.21	3.37	2.40	4.80	2.97	6.22	3.30
3		2.77	1.98	3.33	2.16	4.74	2.48	6.12	2.79
4		2.79	1.88	3.36	2.09	4.67	2.25	6.17	2.55
5		2.77	1.95	3.36	2.21	4.68	2.43	6.02	2.79
6		2.86	2.20	3.64	2.73	4.84	2.63	6.45	3.32
7		3.38	2.79	3.92	3.00	5.35	3.31	6.66	3.63
8		3.38	2.89	3.99	3.20	5.32	3.70	6.73	4.23
9		2.90	2.33	3.88	3.01	4.81	2.70	6.63	3.60
10		3.53	3.15	4.38	3.72	5.51	3.61	7.17	4.33
11		4.70	5.04	6.02	6.27	8.30	8.27	10.42	9.95
12		3.79	3.47	4.50	3.96	5.84	4.24	7.29	4.85
13		3.30	2.79	4.10	3.30	5.27	3.29	6.85	3.93
14		4.75	4.71	5.54	5.55	7.14	5.89	9.02	7.34

Table 8: Average lags selected; stochastic volatility: $c = 10$ and $v = 4$

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	1.23	0.81	1.63	1.00	1.75	1.02	2.41	1.25
2		2.10	1.81	2.39	1.97	2.69	2.18	3.22	2.33
3		2.08	1.69	2.39	1.88	2.71	1.96	3.19	2.17
4		2.04	1.76	2.30	1.96	2.67	2.03	3.17	2.31
5		2.05	1.93	2.35	2.14	2.59	2.28	3.15	2.71
6		2.51	2.33	2.98	2.75	3.10	2.64	3.79	3.19
7		2.83	2.67	3.11	2.92	3.45	3.10	3.98	3.41
8		2.82	2.95	3.16	3.24	3.46	3.78	4.12	4.31
9		2.81	2.62	3.32	3.12	3.44	2.98	4.21	3.58
10		3.51	3.38	3.92	3.72	4.17	3.78	4.90	4.24
11		5.12	5.18	6.15	6.18	7.71	7.55	8.96	8.64
12		3.21	2.99	3.74	3.48	3.87	3.31	4.72	3.89
13		2.96	2.73	3.52	3.24	3.53	3.01	4.49	3.65
14		4.84	4.83	5.78	5.74	6.27	5.93	7.80	7.27
1	7	1.32	0.86	1.60	0.99	1.92	1.05	2.44	1.29
2		2.17	2.03	2.43	2.10	2.92	2.56	3.41	2.65
3		2.13	1.83	2.37	1.96	2.85	2.22	3.23	2.30
4		2.10	1.73	2.41	1.93	2.72	1.99	3.27	2.26
5		2.09	1.76	2.36	1.94	2.65	2.07	3.19	2.28
6		2.24	2.02	2.87	2.53	2.88	2.31	3.80	2.91
7		2.91	2.70	3.18	2.86	3.58	3.08	4.11	3.34
8		2.83	2.72	3.14	2.96	3.46	3.22	4.09	3.59
9		2.37	2.14	3.10	2.81	3.01	2.43	4.02	3.17
10		3.25	3.07	3.90	3.63	3.94	3.38	4.87	4.01
11		4.97	5.06	6.22	6.31	8.19	8.20	10.07	9.97
12		3.56	3.45	4.07	3.88	4.58	4.15	5.19	4.50
13		2.90	2.67	3.43	3.15	3.61	3.01	4.47	3.58
14		4.67	4.64	5.50	5.47	6.00	5.64	7.55	7.02

Table 9: Average lags selected; stochastic volatility: $c = 0$ and $v = 9$

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	3.33	2.19	4.44	3.06	6.41	3.29	8.87	5.01
2		3.58	2.99	4.64	3.72	6.61	4.43	9.18	6.12
3		3.66	2.83	4.68	3.60	6.64	4.13	9.15	5.75
4		3.63	2.81	4.66	3.59	6.69	4.11	9.23	5.72
5		3.72	3.01	4.75	3.86	6.78	4.40	9.32	6.31
6		3.72	3.07	4.84	3.96	6.78	4.34	9.29	6.14
7		3.95	3.40	4.94	4.20	7.13	4.81	9.46	6.47
8		4.06	3.69	5.00	4.53	7.27	5.61	9.76	7.55
9		3.84	3.24	4.92	4.12	6.95	4.52	9.48	6.34
10		4.18	3.77	5.12	4.54	7.25	5.08	9.54	6.77
11		4.54	4.93	5.72	6.02	8.14	7.84	10.45	9.65
12		3.97	3.53	4.97	4.35	7.07	4.92	9.35	6.53
13		3.95	3.36	4.92	4.14	6.94	4.55	9.36	6.25
14		4.82	4.82	5.81	5.76	8.33	6.74	10.42	8.57
1	7	3.19	2.23	4.34	3.05	6.13	3.48	8.64	5.12
2		3.47	3.08	4.46	3.83	6.39	4.78	8.82	6.42
3		3.54	2.88	4.50	3.57	6.47	4.26	8.81	5.77
4		3.50	2.71	4.50	3.48	6.41	3.88	8.83	5.59
5		3.61	2.86	4.59	3.62	6.54	4.19	9.09	5.77
6		3.54	2.91	4.63	3.80	6.56	4.11	9.01	5.83
7		3.89	3.39	4.77	4.08	6.81	4.68	9.07	6.21
8		4.02	3.54	4.90	4.31	7.10	5.29	9.38	7.06
9		3.56	3.02	4.67	3.96	6.61	4.26	8.99	5.97
10		3.91	3.56	4.99	4.43	6.87	4.78	9.31	6.53
11		4.37	4.93	5.70	6.08	8.11	8.21	10.62	10.21
12		3.98	3.69	4.90	4.43	6.93	5.26	9.07	6.71
13		3.79	3.33	4.81	4.10	6.74	4.55	9.11	6.21
14		4.75	4.78	5.61	5.66	7.97	6.64	10.32	8.56

Table 10: Average lags selected; stochastic volatility: $c = 10$ and $v = 9$

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	2.51	1.33	3.46	2.00	3.80	1.59	5.66	2.55
2		2.97	2.31	3.82	2.87	4.31	2.75	6.27	3.71
3		2.94	2.08	3.81	2.68	4.24	2.41	6.22	3.28
4		2.93	2.14	3.88	2.75	4.20	2.48	6.24	3.40
5		3.00	2.37	3.85	3.06	4.38	2.85	6.25	4.09
6		3.17	2.55	4.08	3.28	4.41	2.91	6.46	4.05
7		3.48	2.99	4.27	3.54	4.79	3.47	6.77	4.49
8		3.48	3.26	4.32	3.96	4.78	4.15	6.80	5.53
9		3.31	2.75	4.27	3.56	4.61	3.09	6.65	4.34
10		3.85	3.49	4.68	4.09	5.20	3.89	7.06	4.95
11		5.01	5.14	6.05	6.12	7.88	7.44	9.58	8.77
12		3.62	3.15	4.52	3.81	5.03	3.61	6.86	4.55
13		3.47	2.95	4.37	3.62	4.76	3.32	6.76	4.38
14		4.86	4.80	5.80	5.72	6.85	5.91	8.95	7.53
1	7	2.26	1.26	3.21	1.87	3.42	1.52	5.49	2.40
2		2.91	2.37	3.70	2.92	4.25	2.90	5.92	3.75
3		2.79	2.14	3.60	2.59	4.04	2.50	5.86	3.24
4		2.80	2.01	3.67	2.56	4.05	2.29	5.95	3.11
5		2.89	2.18	3.75	2.77	4.18	2.56	6.05	3.51
6		2.87	2.25	3.87	3.02	4.02	2.50	6.19	3.69
7		3.37	2.90	4.11	3.33	4.62	3.29	6.36	4.09
8		3.46	3.04	4.25	3.68	4.84	3.66	6.53	4.80
9		2.87	2.36	3.99	3.22	4.13	2.63	6.40	3.92
10		3.53	3.19	4.53	3.90	4.79	3.54	6.84	4.60
11		4.66	5.01	5.95	6.19	7.87	8.07	10.16	9.74
12		3.75	3.50	4.52	4.04	5.12	4.08	6.85	4.91
13		3.33	2.79	4.23	3.49	4.54	3.12	6.53	4.17
14		4.67	4.63	5.52	5.50	6.46	5.55	8.55	7.21

Table 11: Size; homoskedasticity

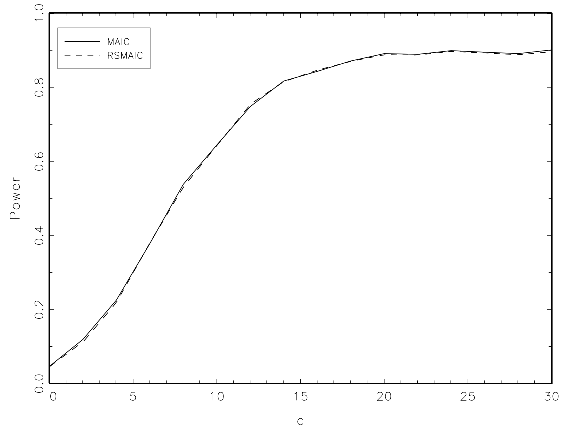
	$T = 150$		$T = 250$	
Model	MAIC	RSMAIC	MAIC	RSMAIC
1	0.046	0.045	0.053	0.051
2	0.047	0.041	0.047	0.045
3	0.044	0.044	0.050	0.049
4	0.049	0.057	0.053	0.052
5	0.051	0.051	0.051	0.050
6	0.043	0.042	0.049	0.047
7	0.056	0.051	0.049	0.048
8	0.051	0.052	0.047	0.046
9	0.041	0.035	0.040	0.039
10	0.054	0.052	0.053	0.053
11	0.101	0.098	0.089	0.090
12	0.041	0.059	0.056	0.054
13	0.049	0.046	0.048	0.046
14	0.041	0.040	0.039	0.040

Table 12: Size; single break

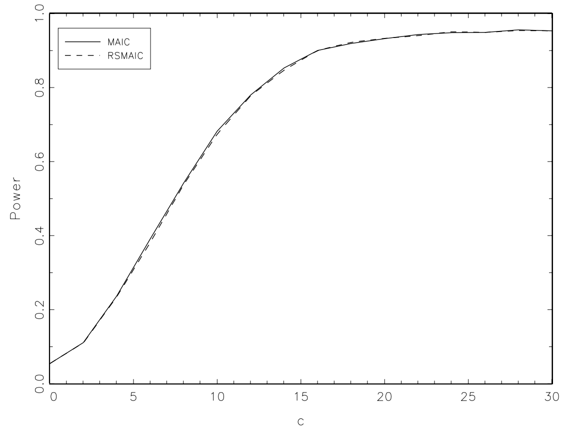
		$\delta = 1/3$				$\delta = 3$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	τ	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0.2	0.056	0.052	0.052	0.049	0.052	0.053	0.046	0.053
2		0.046	0.047	0.052	0.048	0.045	0.050	0.039	0.045
3		0.049	0.046	0.050	0.047	0.045	0.058	0.044	0.051
4		0.050	0.049	0.052	0.053	0.041	0.063	0.055	0.053
5		0.054	0.051	0.051	0.050	0.047	0.054	0.054	0.059
6		0.041	0.038	0.045	0.042	0.051	0.053	0.051	0.052
7		0.053	0.050	0.052	0.052	0.052	0.053	0.054	0.056
8		0.052	0.049	0.052	0.051	0.042	0.047	0.052	0.056
9		0.036	0.034	0.045	0.042	0.045	0.041	0.050	0.052
10		0.049	0.044	0.049	0.048	0.048	0.053	0.058	0.061
11		0.103	0.102	0.089	0.078	0.098	0.106	0.059	0.071
12		0.067	0.061	0.056	0.053	0.054	0.061	0.046	0.061
13		0.041	0.038	0.045	0.042	0.054	0.053	0.052	0.054
14		0.040	0.039	0.043	0.036	0.057	0.060	0.056	0.057
1	0.8	0.048	0.074	0.051	0.051	0.047	0.045	0.047	0.047
2		0.046	0.049	0.051	0.053	0.046	0.045	0.050	0.049
3		0.050	0.050	0.047	0.055	0.045	0.042	0.052	0.052
4		0.054	0.067	0.052	0.061	0.048	0.047	0.054	0.052
5		0.049	0.051	0.046	0.049	0.050	0.047	0.053	0.052
6		0.051	0.053	0.050	0.054	0.045	0.044	0.047	0.049
7		0.052	0.060	0.047	0.054	0.057	0.054	0.047	0.048
8		0.038	0.041	0.051	0.054	0.046	0.048	0.049	0.049
9		0.040	0.043	0.041	0.045	0.040	0.039	0.047	0.048
10		0.047	0.054	0.049	0.051	0.052	0.049	0.050	0.049
11		0.118	0.117	0.087	0.098	0.096	0.093	0.079	0.077
12		0.075	0.085	0.053	0.062	0.064	0.061	0.055	0.057
13		0.048	0.046	0.050	0.048	0.040	0.043	0.043	0.043
14		0.046	0.053	0.044	0.047	0.048	0.047	0.052	0.048

Table 13: Size; stochastic volatility

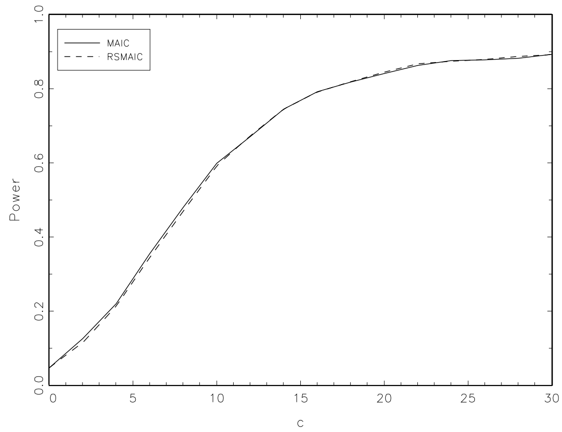
		$c = 0$				$c = 10$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	v	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	4	0.050	0.049	0.051	0.052	0.048	0.045	0.053	0.051
2		0.046	0.047	0.046	0.047	0.052	0.048	0.046	0.045
3		0.056	0.054	0.051	0.049	0.051	0.048	0.045	0.046
4		0.048	0.055	0.052	0.055	0.050	0.049	0.051	0.050
5		0.054	0.059	0.057	0.059	0.049	0.049	0.044	0.042
6		0.048	0.050	0.051	0.057	0.044	0.047	0.050	0.046
7		0.050	0.050	0.047	0.048	0.048	0.047	0.050	0.047
8		0.048	0.052	0.047	0.050	0.047	0.045	0.046	0.044
9		0.046	0.043	0.050	0.046	0.038	0.036	0.043	0.044
10		0.048	0.048	0.055	0.053	0.046	0.048	0.052	0.055
11		0.117	0.111	0.083	0.084	0.106	0.106	0.086	0.085
12		0.065	0.067	0.056	0.060	0.059	0.062	0.065	0.069
13		0.050	0.050	0.049	0.048	0.046	0.044	0.044	0.046
14		0.049	0.048	0.050	0.051	0.039	0.040	0.050	0.046
1	9	0.057	0.057	0.051	0.050	0.048	0.050	0.046	0.050
2		0.057	0.055	0.050	0.051	0.049	0.049	0.044	0.044
3		0.051	0.052	0.052	0.054	0.053	0.049	0.048	0.049
4		0.050	0.053	0.053	0.059	0.044	0.051	0.043	0.048
5		0.052	0.050	0.042	0.050	0.045	0.049	0.046	0.046
6		0.048	0.045	0.050	0.054	0.042	0.046	0.051	0.054
7		0.052	0.048	0.055	0.061	0.049	0.049	0.047	0.049
8		0.044	0.049	0.046	0.047	0.041	0.042	0.040	0.041
9		0.038	0.046	0.044	0.046	0.039	0.043	0.042	0.048
10		0.043	0.052	0.049	0.056	0.047	0.048	0.043	0.046
11		0.136	0.106	0.088	0.084	0.107	0.099	0.084	0.084
12		0.065	0.059	0.049	0.056	0.060	0.061	0.054	0.056
13		0.049	0.054	0.046	0.045	0.049	0.048	0.047	0.050
14		0.050	0.052	0.053	0.055	0.045	0.041	0.049	0.049



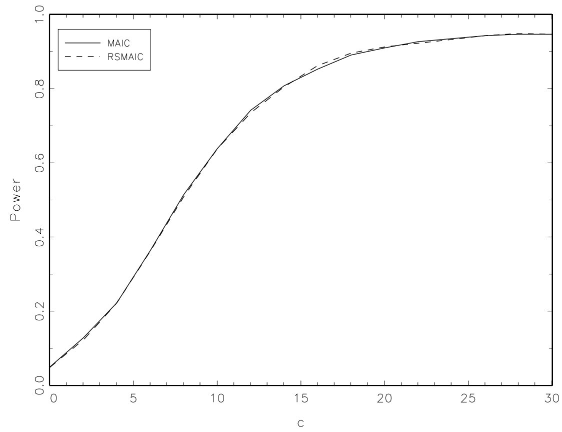
(a) $T = 150$, ARMA model 1



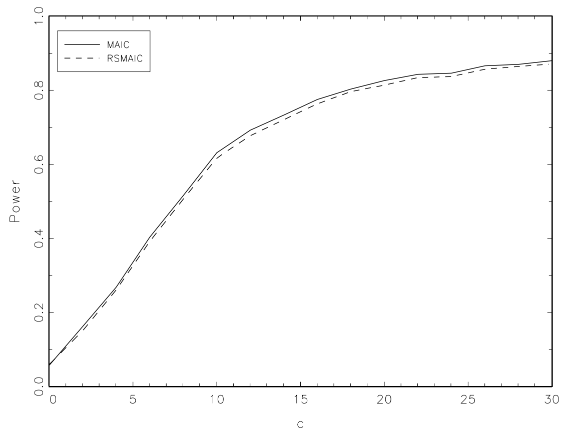
(b) $T = 250$, ARMA model 1



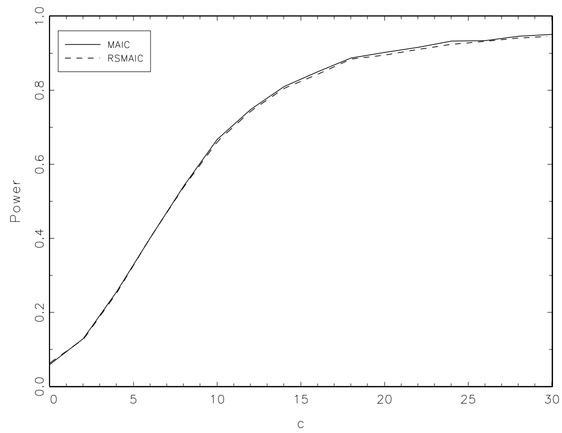
(c) $T = 150$, ARMA model 4



(d) $T = 250$, ARMA model 4

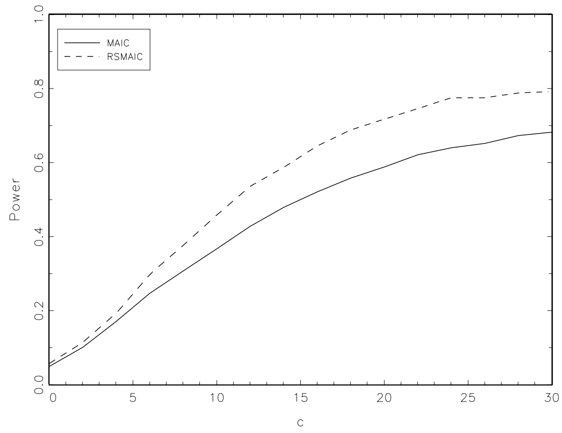


(e) $T = 150$, ARMA model 12

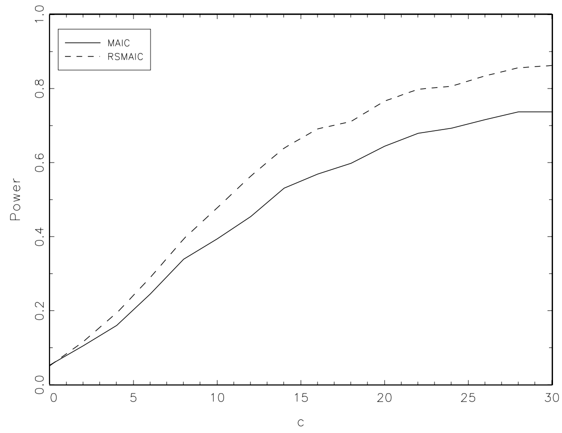


(f) $T = 250$, ARMA model 12

Figure 1: Power ADF-GLS test; constant, homoskedasticity



(a) $T = 150$, ARMA model 1



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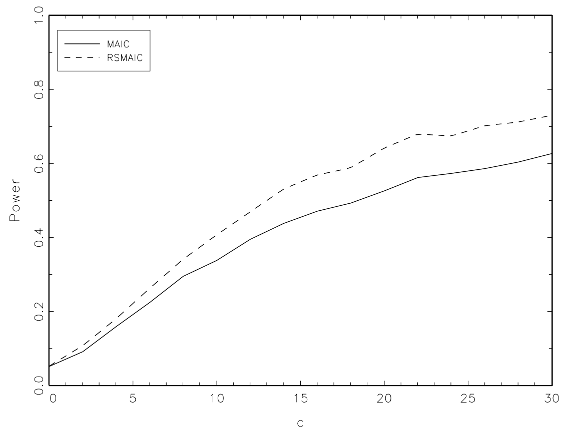
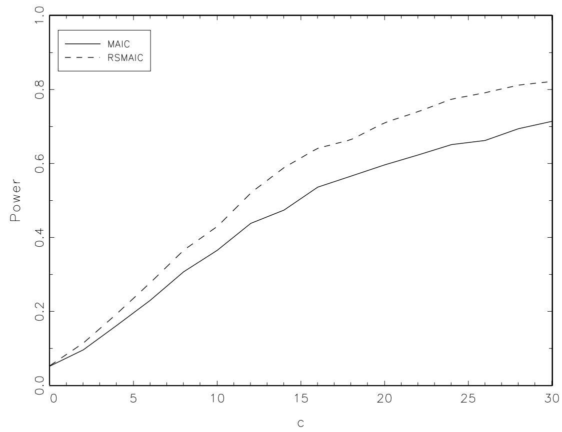
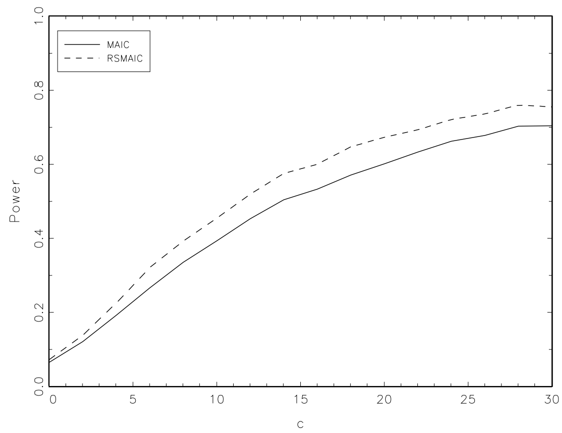


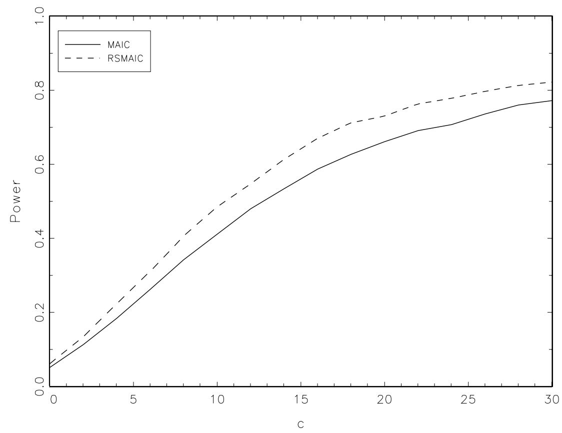
(c) $T = 150$, ARMA model 4



(d) $T = 250$, ARMA model 4

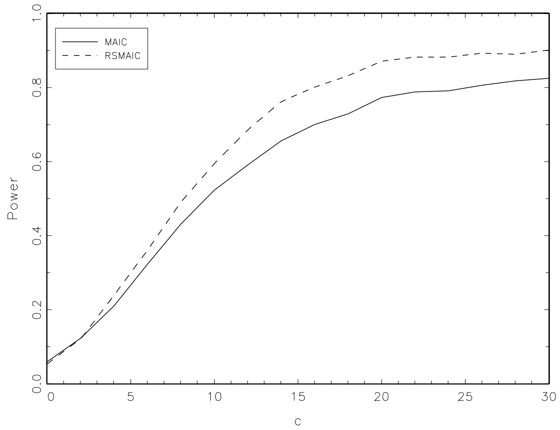


(e) $T = 150$, ARMA model 12

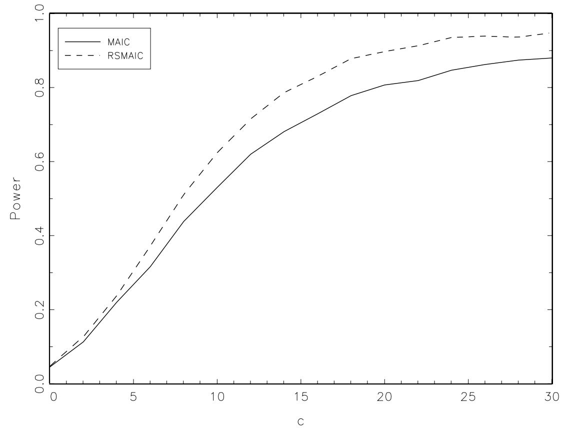


(f) $T = 250$, ARMA model 12

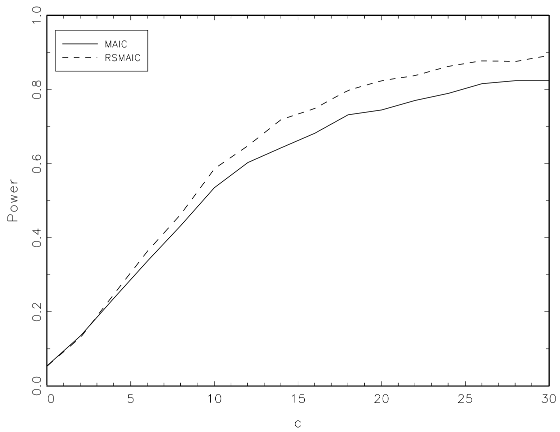
Figure 2: Power ADF-GLS test; single break: $\delta = 1/3$, $\tau = 0.8$



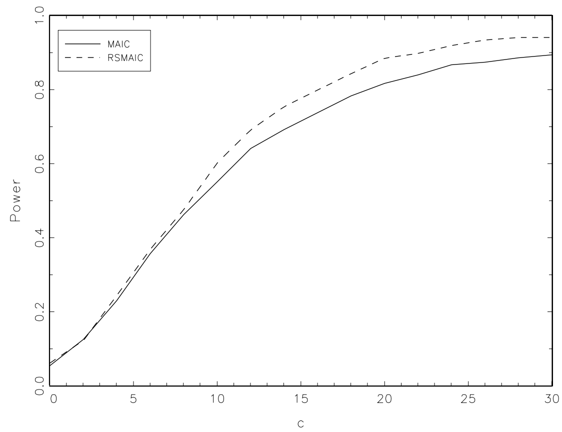
(a) $T = 150$, ARMA model 1



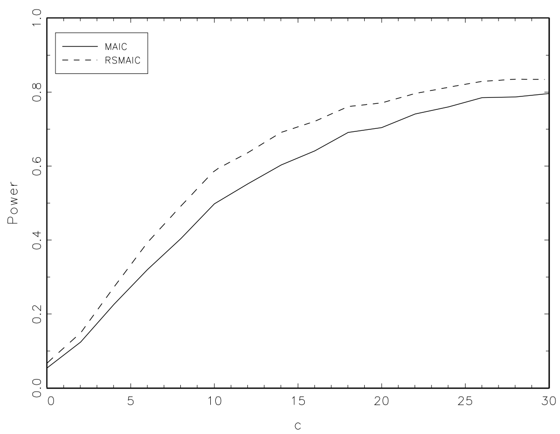
(b) $T = 250$, ARMA model 1



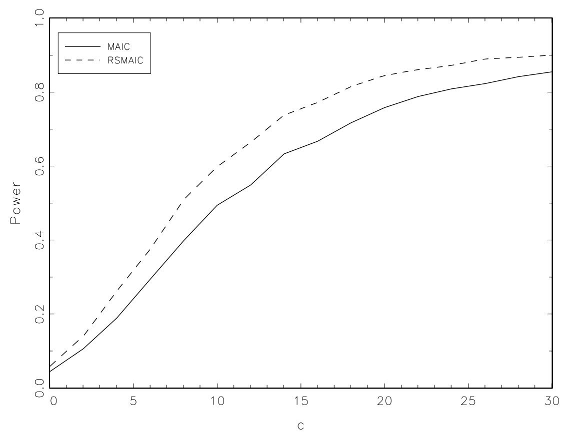
(c) $T = 150$, ARMA model 4



(d) $T = 250$, ARMA model 4

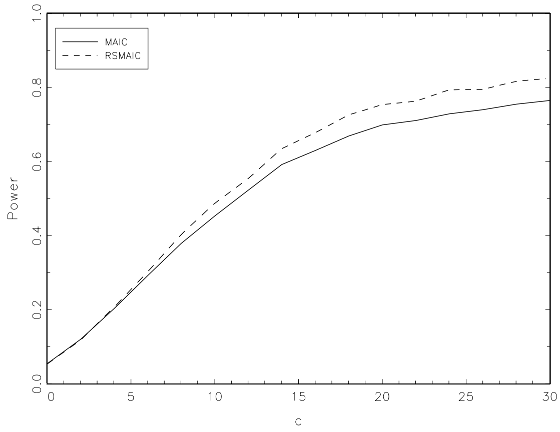


(e) $T = 150$, ARMA model 12

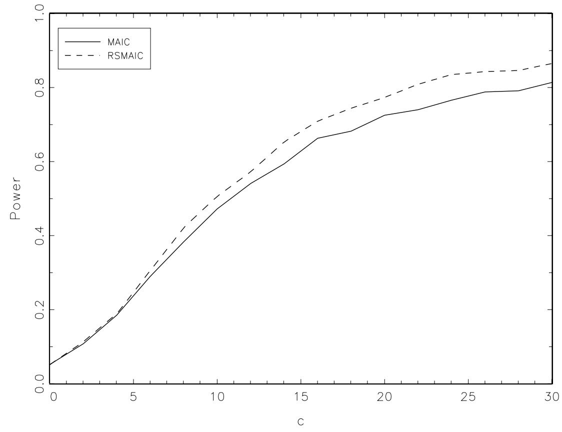


(f) $T = 250$, ARMA model 12

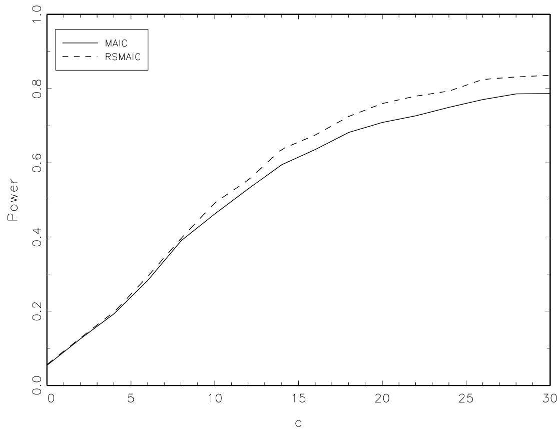
Figure 3: Power ADF-GLS test; single break: $\delta = 3$, $\tau = 0.2$



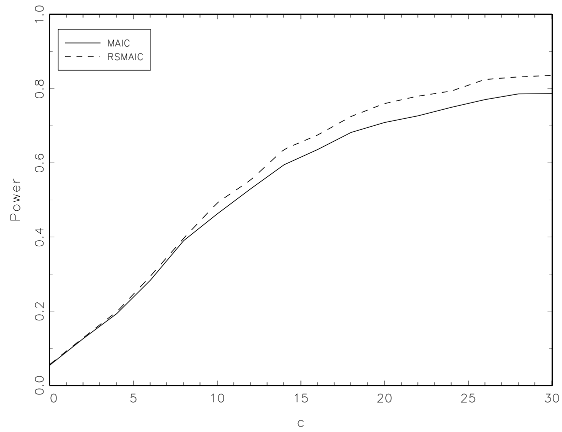
(a) $T = 150$, ARMA model 1



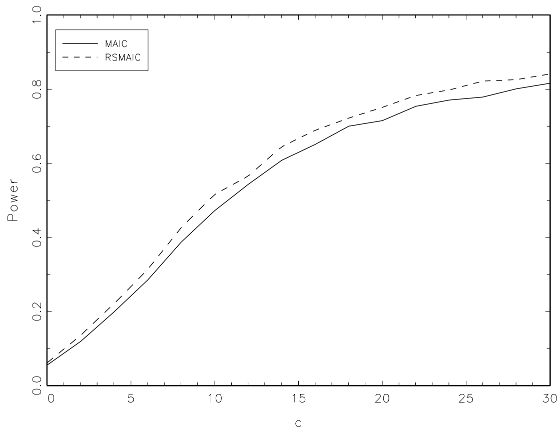
(b) $T = 250$, ARMA model 1



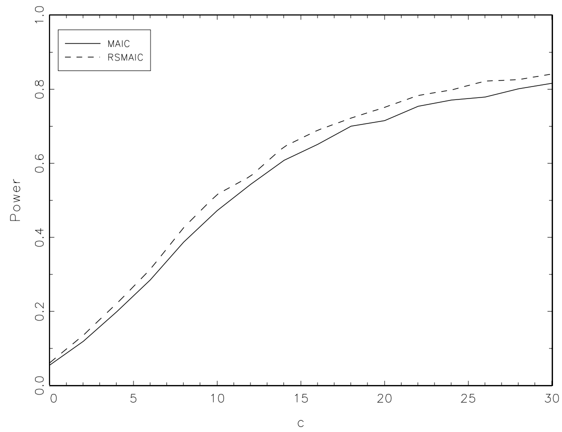
(c) $T = 150$, ARMA model 4



(d) $T = 250$, ARMA model 4

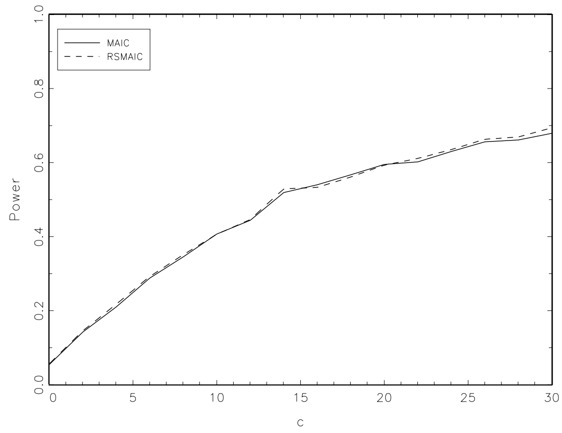


(e) $T = 150$, ARMA model 12

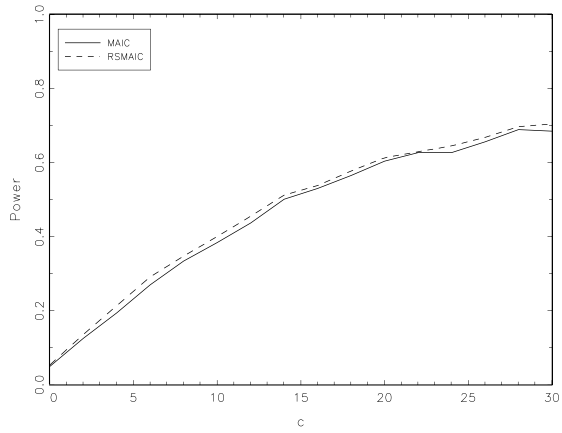


(f) $T = 250$, ARMA model 12

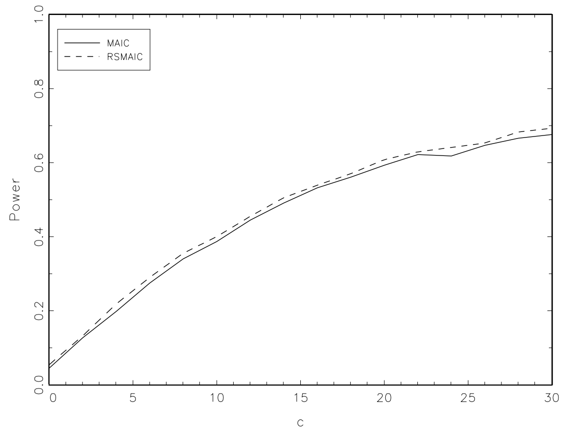
Figure 4: Power ADF-GLS test; stochastic volatility: $c = 0$, $v = 4$



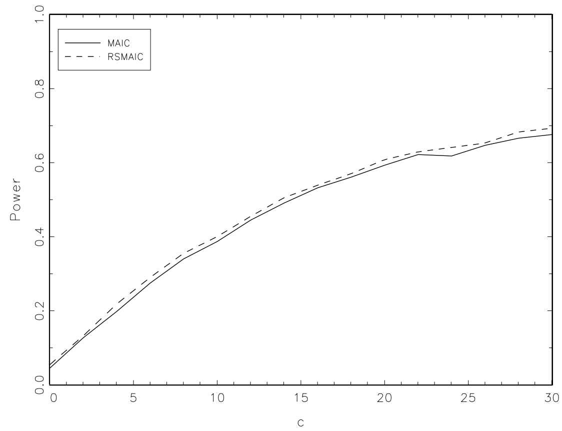
(a) $T = 150$, ARMA model 1



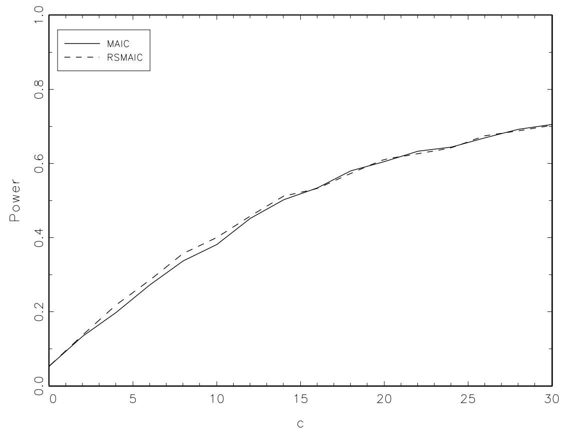
(b) $T = 250$, ARMA model 1



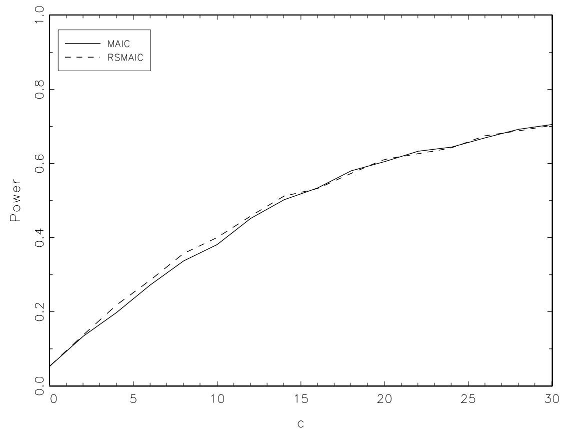
(c) $T = 150$, ARMA model 4



(d) $T = 250$, ARMA model 4



(e) $T = 150$, ARMA model 12



(f) $T = 250$, ARMA model 12

Figure 5: Power ADF-GLS test; stochastic volatility: $c = 0$, $v = 9$