DESIGN MINING IN LEPUS3/CLASS-Z: SEARCH SPACE AND ABSTRACTION/CONCRETIZATION OPERATORS

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Abstract. LePUS3 is a specification and modelling language designed to capture the building blocks of O-O design at different levels of abstraction. We identify the set of LePUS3 specifications that agree with (are satisfied by) an O-O program (represented by a LePUS3 design model) as the search space for a host of *design mining* problems such as: reverse engineering, design recovery, design pattern detection, design pattern discovery. We show that this search space is a mathematical lattice (with relation to a particular program) and we demonstrate how it can be traversed using a set of abstraction and concretization operators.

Keywords: LePUS3, design mining

Conventions:
$\vdash$ denotes deducibility in classical logic.
$\models$ denotes satisfiability as defined in [Eden et. al 2007].
Given set $\mathcal{S}$, $|\mathcal{S}|$ stands for the size of $\mathcal{S}$.
LePUS3 constant terms:

- Lower case fixed-width characters such as `x` are reserved for *0*-dimensional constant terms (see also Definition 1)
- Capitalized fixed-width characters such as `Y` are reserved for *1*-dimensional constant terms (see also Definition 1)
- $\mathtt{x}^d$ stands for a constant term of dimension d

$\underline{Relation}$ refers to a relation, and $Relation$ refers to a relation symbol.

1 Preliminary definitions

In this section we provide or adopt from [Eden et. al 2007], [Eden et. al 2007b] all the required definitions.

Definition 1: A **design model** for LePUS3 is a finite model-theoretic structure $\mathfrak{M}=\langle\mathbb{U}_*,\mathbb{R},\mathcal{I}\rangle$ such that:

- $\mathbb{U}_*$, called the **universe** of $\mathfrak{M}$, is a finite set of entities such that $\mathbb{U}_*\triangleq\mathbb{U}_0\cup\mathbb{U}_1$ where:
  - $\mathbb{U}_0$ is a finite set of primitive entities that we call entities of dimension *0*
  - $\mathbb{U}_1\triangleq\mathcal{P}(\mathbb{U}_0)$. An entity in $\mathbb{U}_1$ is called an entity of dimension *1*
- $\mathbb{R}$ is a set of relations, including:
  - the unary relations $\underline{Class}$, $\underline{Method}$, $\underline{Signature}$, $\underline{Inheritable}$ and $\underline{Abstract}$
  - the binary relations $\underline{Inherit}$, $\underline{Member}$, $\underline{Produce}$, $\underline{Call}$, $\underline{Forward}$, $\underline{Create}$, $\underline{Return}$, $\underline{Aggregate}$ and $\underline{SignatureOf}$
- $\mathcal{I}$ is an **interpretation**[1] function as follows:
  - if c is a constant term then $\mathcal{I}(\mathrm{c})$ is an entity in $\mathbb{U}_*$
  - if c and s are constant terms, and $\mathcal{I}(\mathrm{s})\otimes\mathcal{I}(\mathrm{c})$ is defined, then $\mathcal{I}(\mathrm{s}\otimes\mathrm{c})=\mathcal{I}(\mathrm{s})\otimes\mathcal{I}(\mathrm{c})$

  if t is in the domain of $\mathcal{I}$ then $\mathcal{I}(t)$ is the interpretation of t
- $\mathfrak{M}$ fixes the interpretation of higher dimensional (non *0*-dimensional) constants

Definition 2: A LePUS3 **ground formula** is a formula in one of the following:

- a declaration in the form $t:\mathbb{CLASS}$ (or $\mathbb{SIGNATURE}$) which is shorthand for $Class(t)$ (or $Signature(t)$)
- a formula in the form $UnaryRelation(t)$ where t is a *0*-dimensional term
- a formula in the form $BinaryRelation(t_1,t_2)$ where t_1, t_2 are *0*-dimensional terms

For example, the schema presented in Table 1 contains 5 ground formulas.

Definition 3: A LePUS3 **predicate formula** is one of the following:

- a formula in the form $\textsc{All}(UnaryRelation,T)$ where $\textsc{All}$ is a predicate and T higher dimensional term

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[1] To make sure that we ignore cases where different terms have the same interpretation we shall consider in this document $\mathcal{I}$ to be a bijective function.

- a formula in the form $P(BinaryRelation, T_1, T_2)$ where P is the $\textsc{Total}$ or $\textsc{Isomorphic}$ predicate and T_1, T_2 are higher dimensional terms

For example, the schema presented in Table 1 contains 1 predicate formula.

Table 1 – A Servlet example schema

Servlet
$\mathtt{aServlet}, \mathtt{anotherServlet}, \mathtt{HTTPServlet} : \mathbb{CLASS}$ $\mathtt{JavaCollections} : \mathcal{P}(\mathbb{CLASS})$
$Inherit(\mathtt{aServlet}, \mathtt{HTTPServlet})$ $Inherit(\mathtt{anotherServlet}, \mathtt{HTTPServlet})$ $TOTAL(Member, \mathtt{aServlet}, \mathtt{JavaCollections})$

Definition 4: A LePUS3 **well-formed formula (wff)** is one of the following:

- a declaration in the form $T : \mathcal{P}(\mathbb{CLASS})$ (or $\mathcal{P}(\mathbb{SIGNATURE})$), which is a shorthand for $ALL(Class, T)$ (or $ALL(Signature, T)$)
- a ground formula
- a predicate formula

For example, the schema presented in Table 1 contains 7 wffs.

Definition 5: A **LePUS3 specification** is a finite set of LePUS3 wffs.

Definition 6: A ground formula is satisfied by design model $\mathfrak{M}$ under the following conditions:

- $\mathfrak{M} \vDash UnaryRelation(t)$ if and only if $\mathcal{I}(t) \in \underline{UnaryRelation}$
- $\mathfrak{M} \vDash BinaryRelation(t_1, t_2)$ if and only if one of the following conditions hold:
  - $\langle \mathcal{I}(t_1), \mathcal{I}(t_2) \rangle \in \underline{BinaryRelation}$
  - **Subtyping:** There exists some class of dimension 0 $\underline{\mathtt{subcls}}$ in $\mathbb{U}_*$ such that $\langle \mathcal{I}(t_1), \underline{\mathtt{subcls}} \rangle \in \underline{BinaryRelation}$ and $\langle \underline{\mathtt{subcls}}, \mathcal{I}(t_2) \rangle \in \underline{Inherit}^+$

Definition 7: An $\textsc{All}$ predicate formula of the form $\textsc{All}(UnaryRelation, T)$ is satisfied by design model $\mathfrak{M}$ if and only if for each entity $\underline{e}$ in $\mathcal{I}(T_1)$: $\mathfrak{M} \vDash \underline{UnaryRelation}(\mathtt{e})$

Definition 8: A $\textit{TOTAL}$ predicate formula of the form $\textit{TOTAL}(BinaryRelation, T_1, T_2)$ is satisfied by design model $\mathfrak{M}$ if and only if for each entity $\underline{e}_1$ in $\mathcal{I}(T_1)$ that is not an abstract method, there exists some $\underline{e}_2$ entity in $\mathcal{I}(T_2)$ such that $\mathfrak{M} \vDash BinaryRelation(e_1, e_2)$

Definition 9: An $\textit{ISOMORPHIC}$ predicate formula in the form $\textit{ISOMORPHIC}$ $(BinaryRelation, T_1, T_2)$ is satisfied by design model $\mathfrak{M}$ if and only if there exists pair $\langle \underline{e}_1, \underline{e}_2 \rangle$ where $\underline{e}_1 \in \mathcal{I}(T_1)$ and $\underline{e}_2 \in \mathcal{I}(T_2)$ such that:

- $\mathfrak{M} \vDash BinaryRelation(e_1, e_2)$ unless $\underline{e}_1$, $\underline{e}_2$ are abstract and
- $\mathfrak{M} \vDash \textit{ISOMORPHIC}(BinaryRelation, T_1 - e_1, T_2 - e_2)$ unless both $T_1 - e_1$ and $T_2 - e_2$ are empty

where $\mathcal{I}(T - e) = \mathcal{I}(T) - \mathcal{I}(e)$

2 Search Space

In this section we introduce LePUS3 *bottom* and *top specifications* with relation to a design model $\mathfrak{M}$ (that satisfies them). We establish the conditions under which a specification is *in normal form* and show that the *set of specifications* and *set of specifications in normal form* (with relation to a design model $\mathfrak{M}$ that satisfies them) are lattice stuctures.

Definition 10: Given specifications Φ, Ψ and design model $\mathfrak{M}$ we write $\Phi \vdash_{\mathfrak{M}} \Psi$ if and only if:

- $\Phi \vdash \Psi$ given $\mathfrak{M}$
- $\mathfrak{M} \vDash \Phi$ implies $\mathfrak{M} \vDash \Psi$

For example given the schema in Table 1, there is no way to prove `Servlet` $\vdash$ `Servlet2` using some syntactic proof theory and in the general case it would not be satisfied by any model for LePUS3. However, given a particular design model $\mathfrak{M}$ that satisfies both `Servlet` and `Servlet2` we can prove that `Servlet` $\vdash_{\mathfrak{M}}$ `Servlet2` if we consider that:

$Inherit(\texttt{aServlet}, \texttt{HTTPServlet}) \wedge Inherit(\texttt{anotherServlet}, \texttt{HTTPServlet}) \vdash_{\mathfrak{M}}$
$Hiearachy(\texttt{Servlets})$

As from that specific design model $\mathfrak{M}$ we know that:

$\mathcal{I}(\texttt{Servlets}) = \{\mathcal{I}(\texttt{aServlet}), \mathcal{I}(\texttt{anotherServlet}), \mathcal{I}(\texttt{HTTPServlet})\}$

Table 2 – Another Servlet example schema

```
 ┌──── Servlet2 ─────────────────
 │ Servlets : HIERARCY
 │ JavaCollections : P(CLASS)
 ├───────────────────────────────
 │
 └───────────────────────────────
```

Definition 11: Given specifications Φ, Ψ and design model $\mathfrak{M}$ we say that Φ is equivalent to Ψ written as $\Phi \equiv_{\mathfrak{M}} \Psi$ if and only if $\Phi \vdash_{\mathfrak{M}} \Psi$ and $\Psi \vdash_{\mathfrak{M}} \Phi$.

Proposition 1: For any design model $\mathfrak{M}$, $\vdash_{\mathfrak{M}}$ is a partial order relation as $\vdash_{\mathfrak{M}}$ is:

- Reflexive, that is $\Psi \vdash_{\mathfrak{M}} \Psi$
- Anti-symmetric, that is if $\Psi \vdash_{\mathfrak{M}} \Phi$ and $\Phi \vdash_{\mathfrak{M}} \Psi$ then $\Psi \equiv_{\mathfrak{M}} \Phi$
- Transitive, that is if $\Psi \vdash_{\mathfrak{M}} \Phi$ and $\Phi \vdash_{\mathfrak{M}} \Omega$ then $\Psi \vdash_{\mathfrak{M}} \Omega$

Definition 12: $Spec(\mathfrak{M})$ is the set of all LePUS3 specifications that $\mathfrak{M}$ satisfies.

Corollary 1: $Spec(\mathfrak{M})$ is a partially ordered set with relation to $\vdash_{\mathfrak{M}}$.

Corollary 2: Given specifications Φ, Ψ if $\Phi \vdash_{\mathfrak{M}} \Psi$ then Φ, Ψ are in $Spec(\mathfrak{M})$.

Definition 13: A specification Φ is **in normal form** if and only if:

- Φ contains only ground formulas
- There exist no distinct ground formulas ψ, ϕ in Φ such that $\psi \vdash \phi$

2.1 Bottom and Top LePUS3 Specifications

Definition 14: A **bottom specification** $\perp_{\mathfrak{M}}$ with relation to a design model $\mathfrak{M}$ is a specification such that:

- $\perp_{\mathfrak{M}}$ is in normal form
- for any specification Φ, $\perp_{\mathfrak{M}} \vdash_{\mathfrak{M}} \Phi$

Definition 15: Let us call $Max_{\mathfrak{M}}$ a specification with relation to design model $\mathfrak{M}$ that is created by considering all tuples t in all relations in $\mathbb{R}$ such that:

$\forall t \in \bigcup_{\underline{\mathcal{R}} \in \mathbb{R}} \underline{\mathcal{R}}$

1) If $t \in \underline{Class}$ ($t \in \underline{Signature}$) then there exists exactly one *0*-dimensional constant term $\mathtt{t}$ of type $\mathbb{CLASS}$ ($\mathbb{SIGNATURE}$) in $Max_{\mathfrak{M}}$ such that $\mathcal{I}(\mathtt{t})$ is t

2) If $t \in \underline{Method}$ then there exists exactly one *0*-dimensional constant $\mathtt{c}$ of type $\mathbb{CLASS}$ and a *0*-dimension constant $\mathtt{s}$ of type $\mathbb{SIGNATURE}$ in $Max_{\mathfrak{M}}$ such that $(t,\mathcal{I}(\mathtt{s})) \in \underline{SignatureOf}$, $(t,\mathcal{I}(\mathtt{c})) \in \underline{Member}$ and $s \otimes c$ is a superimposition expression in at least one wff in $Max_{\mathfrak{M}}$

3) If $t \in \underline{Abstract}$ then there exists a *0*-dimensional constant term $\mathtt{t}$ in $Max_{\mathfrak{M}}$ such $\mathcal{I}(\mathtt{t})$ is t and $Abstract(\mathtt{t})$ is a wff in $Max_{\mathfrak{M}}$

4) If $t \in \underline{R}$, and $\underline{R}$ is one of the following: $\underline{Member}$, $\underline{Inherit}$, $\underline{Create}$, $\underline{Call}$, $\underline{Produce}$, $\underline{Return}$, $\underline{Forward}$ then t is a pair in the form (t_1,t_2) such that there exist *0*-dimensional constant terms $\mathtt{t}_1$, $\mathtt{t}_2$ in $Max_{\mathfrak{M}}$, $\mathcal{I}(\mathtt{t}_1)$ is t_1, $\mathcal{I}(\mathtt{t}_2)$ is t_2 and $R(\mathtt{t}_1,\mathtt{t}_2)$ is a wff in $Max_{\mathfrak{M}}$

Proposition 2: For any design model $\mathfrak{M}$, $Max_{\mathfrak{M}}$ is a bottom specification ($\perp_{\mathfrak{M}}$).

Proof

From Definition 15 we know that $Max_{\mathfrak{M}}$ contains all ground formulas that are satisfied by design model $\mathfrak{M}$. As it contains only ground formulas, it is in normal form (Definition 13). And as it contains all possible ground formulas that $\mathfrak{M}$ satisfies (Definition 6) it is a bottom specification.

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Proposition 3: For any design model $\mathfrak{M}$, there is one bottom specification ($\perp_{\mathfrak{M}}$).

Proof

Since LePUS3 specification are sets of formulas, there is only one bottom specification that contains all and only ground formulas that $\mathfrak{M}$ satisfies (Definition 6).

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Corollary 3: For any design model $\mathfrak{M}$ and respective bottom specification $\perp_{\mathfrak{M}}$, $\mathfrak{M} \vDash \perp_{\mathfrak{M}}$ (and $\perp_{\mathfrak{M}}$ is in $Spec(\mathfrak{M})$).

Definition 16: A **top specification** $\top_{\mathfrak{M}}$ with relation to a design model $\mathfrak{M}$ is a specification such that:

- $\top_{\mathfrak{M}}$ is in normal form
- for any specification Φ, $\Phi \vdash_{\mathfrak{M}} \top_{\mathfrak{M}}$

Definition 17: Let us call Min the specification which is the empty set: $Min=\{\}$.

Corollary 4: For any design model $\mathfrak{M}$, Min is a top specification ($\top_{\mathfrak{M}}$).

Corollary 5: For any design model $\mathfrak{M}$, there is one bottom specification $\top_{\mathfrak{M}}$.

Corollary 6: For any design model $\mathfrak{M} \vDash \top_{\mathfrak{M}}$ (and $\top_{\mathfrak{M}}$ is in $Spec(\mathfrak{M})$).

2.2 Normal Forms of LePUS3 Specifications

Given a specification in normal form, we examine its properties and establish when a specification is the normal form of another (Definition 18).

Corollary 7: For any specifications Φ, Φ' such that $\Phi' \subseteq \Phi$, if Φ is in normal form then Φ' is in normal form.

Proposition 4: Given specifications Φ, $\bot_{\mathfrak{M}}$ and design model $\mathfrak{M}$ such that $\mathfrak{M} \vDash \Phi$, Φ is in normal form if and only if $\Phi \subseteq \bot_{\mathfrak{M}}$.

Proof

If Φ is in normal form *then* $\Phi \subseteq \bot_{\mathfrak{M}}$.

As $\mathfrak{M} \vDash \Phi$, we know that there exists a specification $\bot_{\mathfrak{M}}$ (Definition 14) such that one of the following is true:

- $\Phi=\bot_{\mathfrak{M}}$ as $\bot_{\mathfrak{M}}$ is in normal form (Definition 14)
- $\Phi\neq\bot_{\mathfrak{M}}$. We know that $\bot_{\mathfrak{M}}$ contains all ground formulas that $\mathfrak{M}$ satisfies (Proposition 2). As $\bot_{\mathfrak{M}}$ is in normal form, for all ground formulas ψ in $\bot_{\mathfrak{M}}$ there does not exist ground formula ϕ in $\bot_{\mathfrak{M}}$ such that $\psi \vdash \phi$ (Definition 13). But also $\mathfrak{M} \vDash \Phi$, thus $\mathfrak{M}$ satisfies every ground formula in Φ (Definition 6), which means that every ground formula in Φ is also in $\bot_{\mathfrak{M}}$. That is $\Phi \subset \bot_{\mathfrak{M}}$

If $\Phi \subseteq \bot_{\mathfrak{M}}$ *then* Ψ is in normal form.

It follows from (Corollary 7) that Φ is in normal form as $\bot_{\mathfrak{M}}$ is in normal form (Proposition 2).

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Proposition 5: Given specifications Φ, Ψ in normal form and design model $\mathfrak{M}$ that satisfies Φ, Ψ then $\Psi \subseteq \Phi$ if and only if $\Phi \vdash_{\mathfrak{M}} \Psi$.

Proof

If $\Psi \subseteq \Phi$ *then* $\Phi \vdash_{\mathfrak{M}} \Psi$.

Let $\Phi=\{\phi_1 \ldots \phi_n\}$ and $\Psi=\{\phi_x \ldots \phi_y\}$ with $1 \leq x \leq y \leq n$.

We know that $\mathfrak{M} \vDash \Phi$ which means that $\mathfrak{M}$ satisfies every formula in it.

Starting from the premise $\phi_1 \wedge \ldots \wedge \phi_n$ which is satisfied by $\mathfrak{M}$ and applying and-elimination we get:

$$\frac{\phi_1 \wedge \ldots \wedge \phi_n}{\phi_1 \wedge \ldots \wedge \phi_{n-1}} \wedge e_n$$

$$\frac{\phi_1 \wedge \ldots \wedge \phi_{n-1}}{\ldots} \wedge e_{n-1}$$

$\phi_x \wedge \ldots \wedge \phi_y$ which is Ψ

If $\Phi \vdash_{\mathfrak{M}} \Psi$ *then* $\Psi \subseteq \Phi$.

Since $\mathfrak{M} \vDash \Phi$, there exists a bottom specification $\perp_{\mathfrak{M}}$ such that $\Phi \subseteq \perp_{\mathfrak{M}}$ and $\Psi \subseteq \perp_{\mathfrak{M}}$. (Proposition 4). From Definition 13 we know that for all ground formulas ψ in $\perp_{\mathfrak{M}}$ there does not exist specification ϕ such that $\psi \vdash \phi$. Thus if $\Phi \vdash_{\mathfrak{M}} \Psi$ it means that every wff in Ψ is also in Φ.

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Corollary 8: There are no specifications Φ, Φ' in normal form and design model $\mathfrak{M}$ such that $\Phi' \subset \Phi$ and $\Phi' \vdash_{\mathfrak{M}} \Phi$.

Definition 18: Let Ψ, Φ be specifications and $\mathfrak{M}$ a design model such that $\mathfrak{M} \vDash \Phi$. We will say that Φ is **the normal form of** Ψ with relation to design model $\mathfrak{M}$ if and only if:

- Φ is in normal form
- $\Phi \vdash_{\mathfrak{M}} \Psi$
- There is no Φ' in normal form, such that $\Phi \vdash_{\mathfrak{M}} \Phi' \vdash_{\mathfrak{M}} \Psi$

Proposition 6: Given specifications Φ, Ψ, their respective normal forms Φ', Ψ' and design model $\mathfrak{M}$, if $\Phi \vdash_{\mathfrak{M}} \Psi$ then $\Phi' \vdash_{\mathfrak{M}} \Psi'$.

Proof

From our premise we know that $\Phi \vdash_{\mathfrak{M}} \Psi$ (*1*) and from Definition 18 we know that $\Phi' \vdash_{\mathfrak{M}} \Phi$ (*2*) and $\Psi' \vdash_{\mathfrak{M}} \Psi$ (*3*). Since Φ' and Ψ' are sets of ground formulas (Definition 13), one of the following is true:

- $\Phi' \cap \Psi' = \{\}$. In this case, from (*1*), (*2*) we can conclude: $\Phi' \vdash_{\mathfrak{M}} \Phi \vdash_{\mathfrak{M}} \Psi$ which means that $\Phi' \vdash_{\mathfrak{M}} \Psi$. Since Φ' and Ψ' are in normal form, given Definition 13, they should have at least one ground formula in common which is not true as it violates our assumption
- $\Phi' \cap \Psi' \neq \{\}$. In this case one of the following is true about Φ', Ψ' :
  - $\Phi' = \Psi'$. In this case $\Phi' \vdash_{\mathfrak{M}} \Psi'$ as relation $\vdash_{\mathfrak{M}}$ is reflexive (Proposition 1)
  - $\Phi' \subset \Psi'$. From Proposition 5 we know that $\Psi' \vdash_{\mathfrak{M}} \Phi'$. From (*2*) we can conclude $\Psi' \vdash_{\mathfrak{M}} \Phi' \vdash_{\mathfrak{M}} \Phi$ and from (*1*): $\Psi' \vdash_{\mathfrak{M}} \Phi' \vdash_{\mathfrak{M}} \Phi \vdash_{\mathfrak{M}} \Psi$. From Definition 18, we can conclude that Φ' would be the normal form of Ψ which is not true
  - $\Psi' \subset \Phi'$

We conclude that $\Psi' \subseteq \Phi'$ which given Proposition 5 means that $\Phi' \vdash_{\mathfrak{M}} \Psi'$.

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Corollary 9: Given specifications Φ, Ψ, their respective normal forms Φ', Ψ' and design model $\mathfrak{M}$, if $\Phi \vdash_{\mathfrak{M}} \Psi$ then $\Psi' \subseteq \Phi'$.

2.3 Lattice Structures

Given the *set of specification in normal form* (Definition 19) (with relation to a design model $\mathfrak{M}$) and the *set of specifications* (with relation to a design model $\mathfrak{M}$), we show that each set is a mathematical lattice. For this reason we provide definitions of upper (lower) bound, supremum (infimum) and lattice that are based on the definitions found in [Burris & Sankappanavar 1981] and [Manzano 1999].

Definition 19: $Norm(\mathfrak{M})$ is the set of all LePUS3 specifications in normal form that $\mathfrak{M}$ satisfies.

Corollary 10: $Norm\ (\mathfrak{M})$ is a partially ordered set with relation to $\vdash_{\mathfrak{M}}$.

Corollary 11: $Norm\ (\mathfrak{M})$ is a subset of $Spec(\mathfrak{M})$.

Corollary 12: $\bot_{\mathfrak{M}}$ is in $Norm(\mathfrak{M})$.

Corollary 13: $\top_{\mathfrak{M}}$ is in $Norm(\mathfrak{M})$.

Definition 20: Let $\mathcal{A}$, $\mathcal{B}$ be sets such that $\mathcal{A} \subseteq \mathcal{B}$ and $\preceq$ a partial order relation on $\mathcal{B}$. An element b in $\mathcal{B}$ is an **upper bound** for $\mathcal{A}$ if for all a in $\mathcal{A}$ $a \preceq b$. An element b in $\mathcal{B}$ is a **lower bound** for $\mathcal{A}$ if for all a in $\mathcal{A}$ $b \preceq a$.

Definition 21: Let $\mathcal{A}$, $\mathcal{B}$ be sets such that $\mathcal{A} \subseteq \mathcal{B}$ and $\preceq$ a partial order relation on $\mathcal{B}$. An element b in $\mathcal{B}$, is the **least upper bound** of $\mathcal{A}$ if b is an upper bound of $\mathcal{A}$ and for all x that are upper bounds of $\mathcal{A}$ $b \preceq x$. If such b exists it is called the **supremum** of $\mathcal{A}$ or $Sup(\mathcal{A})$. An element b in $\mathcal{B}$ is the **greatest lower bound** of $\mathcal{A}$ if b is a lower bound of $\mathcal{A}$ and for all x that are lower bounds of $\mathcal{A}$ $x \preceq b$. If such b exists it is called the **infimum** of $\mathcal{A}$ or $Inf(\mathcal{A})$.

Definition 22: A partially ordered set $\mathcal{L}$ is a **lattice** if for all x, y in $\mathcal{L}$ both $Sup(\{x,y\})$ and $Inf(\{x,y\})$ exist (in $\mathcal{L}$).

Proposition 7: $\langle Norm(\mathfrak{M}), \vdash_{\mathfrak{M}} \rangle$ is a lattice.

Proof

For all specifications Ψ, Φ in $Norm(\mathfrak{M})$, $\{\Psi, \Phi\}$ is a subset of $Norm(\mathfrak{M})$. We know that $Norm(\mathfrak{M})$ is a partially ordered set (Corollary 10). Let us assume that $Inf(\{\Psi,\Phi\})=\Gamma$ exists and is in $Norm(\mathfrak{M})$.

If Γ is a lower bound (Definition 20) then: $\Gamma \vdash_{\mathfrak{M}} \Psi$ (*1*) and $\Gamma \vdash_{\mathfrak{M}} \Phi$ (*2*) for all Ψ, Φ.

Since Ψ, Φ and Γ are in normal form:

From Proposition 5 and (*1*) we know that: $\Psi \subseteq \Gamma$ (*3*)

From Proposition 5 and (*2*) we know that: $\Phi \subseteq \Gamma$ (*4*)

In order for Γ to be the greatest lower bound (Definition 21) given (*3*), (*4*) it needs to be $\Gamma=\Phi \cup \Psi$.

But $\Phi \cup \Psi$ is in normal form (Definition 13), as Φ, Ψ are and $\Phi \cup \Psi$ is a subset of $\bot_{\mathfrak{M}}$ (Proposition 4). Since there is exactly one subset of $\bot_{\mathfrak{M}}$ that contains all and only ground formulas in $\Phi \cup \Psi$ then Γ exists and is in $Norm(\mathfrak{M})$.

Symmetrically we can show that for any two specifications in $Norm(\mathfrak{M})$, $Sup(Norm(\mathfrak{M}))$ is $\top_{\mathfrak{M}}$.

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Proposition 8: $\langle Spec(\mathfrak{M}),\vdash_{\mathfrak{M}}\rangle$ is a lattice.

Proof

For all specifications Ψ, Φ in $Spec(\mathfrak{M})$, $\{\Psi, \Phi\}$ is a subset of $Spec(\mathfrak{M})$. $Spec(\mathfrak{M})$ is a partially ordered set (Corollary 10). Let us assume that $Inf(\{\Psi,\Phi\})=\Gamma$ exists and is in $Spec(\mathfrak{M})$.

If Γ is a lower bound (Definition 20) then $\Gamma \vdash_{\mathfrak{M}} \Psi$ (*1*) and $\Gamma \vdash_{\mathfrak{M}} \Phi$ (*2*).
Given Corollary 2, if such Γ exists it will be in $Spec(\mathfrak{M})$.
If Γ is the least upper bound (Definition 21) there should not exist another upper bound Δ such that $\Gamma \vdash_{\mathfrak{M}} \Delta \vdash_{\mathfrak{M}} \Psi$ (*3*) and $\Gamma \vdash_{\mathfrak{M}} \Delta \vdash_{\mathfrak{M}} \Phi$ (*4*).

Let Γ', Δ', Φ', Ψ' be the normal forms of Γ, Δ, Φ, Ψ respectively.
Given Definition 18 if Γ exists, then there also exists specification Γ' in normal form such that $\Gamma' \vdash_{\mathfrak{M}} \Gamma$.

If both Γ' and Δ' exist:
From Proposition 6 and (*3*) we know that $\Gamma' \vdash_{\mathfrak{M}} \Delta' \vdash_{\mathfrak{M}} \Psi'$ (*5*)
From Proposition 6 and (*4*) we know that $\Gamma' \vdash_{\mathfrak{M}} \Delta' \vdash_{\mathfrak{M}} \Phi'$ (*6*)
From Corollary 9 and (*5*) we know that $\Psi' \subseteq \Delta' \subseteq \Gamma'$ (*7*)
From Corollary 9 and (*6*) we know that $\Phi' \subseteq \Delta' \subseteq \Gamma'$ (*8*)
Therefore, to prove that Γ is $Inf(\{\Psi,\Phi\})$ it is enough to show that Γ' exists and Δ' does not (unless $\Gamma' = \Delta'$).

From Proposition 6 and (*1*) we know that: $\Gamma' \vdash_{\mathfrak{M}} \Psi'$ (*9*)
From Proposition 6 and (*2*) we know that: $\Gamma' \vdash_{\mathfrak{M}} \Phi'$ (*10*)
From Corollary 9 and (*9*) we know that: $\Psi' \subseteq \Gamma'$ (*11*)
From Corollary 9 and (*10*) we know that: $\Phi' \subseteq \Gamma'$ (*12*)
From (*11*), (*12*) we conclude that Γ' should be: $\Gamma'=\Phi' \cup \Psi'$ so that there does not exist Δ' such that (*7*), (*8*) are true.

But there is exactly one subset of $\perp_{\mathfrak{M}}$ that contains all and only ground formulas in $\Phi' \cup \Psi'$, therefore Γ' exists and is in $Spec(\mathfrak{M})$ and so does Γ.

Symmetrically we can show that any two specifications in $Norm(\mathfrak{M})$, $Sup(Norm(\mathfrak{M}))$ is$\top_{\mathfrak{M}}$.

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3 Operators

Given a design model $\mathfrak{M}$ and the set of specifications $Spec(\mathfrak{M})$ that $\mathfrak{M}$ satisfies, which is a lattice structure, we show how it is possible to traverse it by making *steps* (Definition 25) from one specification (node in the lattice) to another. Each step is performed by the application of an *operator* (Definition 26). Operators are divided into two sets: the *abstraction* and the *concretization* operators and are outlined in Table 3.

Definition 23: Let $\mathcal{S}PEC$ be the set of all LePUS3 specifications.

Definition 24: **Verbosity** of a specification Ψ written as $Verbosity(\Psi)$ is a function

$$Verbosity : \mathcal{S}PEC \rightarrow \mathbb{N}$$

such that values in its range calculate as the sum of the number of constant terms in Ψ and the number of wffs in Ψ.

Definition 25: Given $\mathfrak{M}$, we say that the transition from specification Ψ to Φ is an **abstraction step**, if the following conditions hold:

- $\mathfrak{M} \models \Psi$
- $\Psi \vdash_{\mathfrak{M}} \Phi$
- $Verbosity(\Psi) \geq Verbosity(\Phi)$

Remark: The transition from Φ to Ψ would be a **concretization step.**

Corollary 14: The transition from Ψ to Φ is an abstraction step if and only if the normal forms $\Psi' \subseteq \Phi'$.

Table 3a – Abstraction operators		Table 3b – Concretization operators	
Aggregation	... ⇒	Enumeration	... ⇐
Union	... ⇒	Partition	... ⇐
Hierarchy to Set	⇒	Set to Hierarchy	⇐
Collapse to Hierarchy	Inherit ⇒	Hierarchy Expansion	Inherit ⇐
Hierarchies Union	... ⇒	Partition to Hierarchies	... ⇐
To Top	$\top_{\mathfrak{M}}$	To Bottom	$\bot_{\mathfrak{M}}$
Elimination	⇒	Introduction	⇐

Definition 26: An **operator** $\mathcal{O}(\{t_1 \ldots t_n\}, \Psi)$ takes a set of constant terms $\{t_1 \ldots t_n\}$ and specification Ψ, and produces $(\{t_1' \ldots t_m'\}, \Phi)$ that is: a set of constant terms $\{t_1' \ldots t_m'\}$ and specification Φ, such that the following conditions hold:

- All $t_1, \ldots t_n$, are in Ψ
- All $t_1', \ldots t_n'$ are in Φ

All conditions in

- Definition 25 hold.

The set of operators is symmetric. If $\mathcal{O}$ is an abstraction operator that makes a transition from Ψ to Φ then there exists a concretization operator $\mathcal{O}'$ that makes a transition from Φ to Ψ and vice versa.

3.1 Concretization Operators

3.1.1 Enumeration

$$(\{\mathtt{T}\}, \Psi) \rightarrow (\{\mathtt{t}_1 \ldots \mathtt{t}_n\}, \Phi)$$

Pre-conditions:

- $\mathtt{T}$ is a term of type $\mathbb{CLASS}$ or $\mathbb{SIGNATURE}$

Post-conditions:

- Terms $\mathtt{t}_1 \ldots \mathtt{t}_n$ are all of the same type as $\mathtt{T}$ in Φ
- $\mathcal{I}(\mathtt{T}) = \{\mathcal{I}(\mathtt{t}_1) \ldots \mathcal{I}(\mathtt{t}_n)\}$

3.1.2 Partition

$$(\{\mathtt{T}\}, \Psi) \rightarrow (\{\mathtt{T}_1 \ldots \mathtt{T}_n\}, \Phi)$$

Pre-conditions:

- $\mathtt{T}$ is a term of type $\mathbb{CLASS}$ or $\mathbb{SIGNATURE}$
- $|\mathcal{I}(\mathtt{T})| \geq 2$

Post-conditions:

- Terms $\mathtt{T}_1 \ldots \mathtt{T}_n$ all of the same type as $\mathtt{T}$ in Φ
- $\mathcal{I}(\mathtt{T}) = \mathcal{I}(\mathtt{T}_1) \cup \ldots \cup \mathcal{I}(\mathtt{T}_n)$
- For at least $n-1$ terms $\mathtt{T}_i$, $1 \leq i \leq n$ introduced there exists at least one formula of the following forms with that term that is satisfied by $\mathfrak{M}$:
  - $\textit{Total}(\textit{BinaryRelation}, \mathtt{x}^d, \mathtt{T}_i)$
  - $\textit{Isomorphic}(\textit{BinaryRelation}, \mathtt{x}^d, \mathtt{T}_i)$
  - $\textit{Total}(\textit{BinaryRelation}, \mathtt{T}_i, \mathtt{x}^d)$
  - $\textit{Isomorphic}(\textit{BinaryRelation}, \mathtt{T}_i, \mathtt{x}^d)$
  - $\textit{All}(\textit{BinaryRelation}, \mathtt{T}_i)$
  - $\textit{Method}(\mathtt{x}^d \otimes \mathtt{T}_i)$

where $\mathtt{x}^d$ is some term in Φ

3.1.3 Set to Hierarchy

$$(\{\mathtt{C}\}, \Psi) \rightarrow (\{\mathtt{H}\}, \Phi)$$

Pre-conditions:

- $\mathtt{C}$ is a term of type $\mathbb{CLASS}$ in Ψ
- $Hierarchy(\mathtt{C})$ is satisfied by $\mathfrak{M}$

Post-conditions:

- $\mathtt{H}$ is a term of type $\mathbb{HIERARCHY}$ in Φ

3.1.4 Hierarchy Expansion

$$(\{\mathtt{H}\}, \Psi) \rightarrow (\{\mathtt{C}^d, \mathtt{r}\}, \Phi)$$

Such that : if $|\mathcal{I}(\mathtt{H})| > 2$ then $d=1$
if $|\mathcal{I}(\mathtt{H})| = 2$ then $d=0$

Pre-conditions:

- $\mathtt{H}$ is a term of type $\mathbb{HIERARCHY}$ in Ψ

Post-conditions:

- $\mathtt{C}^d$ is a term of type $\mathbb{CLASS}$ in Φ
- $\mathcal{I}(\mathtt{H}) = \{\mathcal{I}(\mathtt{r})\} \cup \mathcal{I}(\mathtt{C}^d)$
- $\textit{TOTAL}(Inherit, \mathtt{C}^d, \mathtt{r})$ in Φ is satisfied by $\mathfrak{M}$

3.1.5 Partition to Hierarchies

$$(\{\mathtt{C}\}, \Psi) \rightarrow (\{\mathtt{H}_1 \ldots \mathtt{H}_n\}, \Phi)$$

Pre-conditions:

- $\mathtt{C}$ is a term of type $\mathbb{CLASS}$ in Ψ

Post-conditions:

- All terms $\mathtt{h}_i$ $1 \leq i \leq n$ introduced are of type $\mathbb{HIERARCHY}$ in Φ
- $\mathcal{I}(\mathtt{C}) = \mathcal{I}(\mathtt{H}_1) \cup \ldots \cup \mathcal{I}(\mathtt{H}_n)$

3.1.6 To bottom

$$(\{\}, \Psi) \rightarrow (\{\}, \perp_{\mathfrak{M}})$$

3.1.7 Introduction

$$(\{\}, \Psi) \rightarrow (\{\mathtt{t}_1^{\,d} \ldots \mathtt{t}_n^{\,d}\}, \Phi)$$

Such that : $0 \leq d \leq 1$

Post-conditions:

- $\mathtt{t}_1^{\,d} \ldots \mathtt{t}_n^{\,d}$ are terms of any type in Φ

3.2 Abstraction operators

3.2.1 Aggregation

$$(\{\mathtt{t}_1 \ldots \mathtt{t}_n\}, \Psi) \rightarrow (\{\mathtt{T}\}, \Phi)$$

Pre-conditions:

- Terms $\mathtt{t}_1 \ldots \mathtt{t}_n$ are all of type $\mathbb{CLASS}$ or $\mathbb{SIGNATURE}$ in Ψ

Post-conditions:

- $\mathtt{T}$ is a term of the same type as $\mathtt{t}_1 \ldots \mathtt{t}_n$
- $\mathcal{I}(\mathtt{T}) = \{\mathcal{I}(\mathtt{t}_1) \ldots \mathcal{I}(\mathtt{t}_n)\}$

3.2.2 Union

$$(\{\mathtt{T}_1 \ldots \mathtt{T}_n\}, \Psi) \rightarrow (\mathtt{T}, \Phi)$$

Pre-conditions:

- Terms $\mathtt{T}_1 \ldots \mathtt{T}_n$ are all of type $\mathbb{CLASS}$ or $\mathbb{SIGNATURE}$ in Ψ
- $n \geq 2$

Post-conditions:

- $\mathtt{T}$ is a term of the same type as $\mathtt{T}_1 \ldots \mathtt{T}_n$
- $\mathcal{I}(\mathtt{T}) = \mathcal{I}(\mathtt{T}_1) \cup \ldots \cup \mathcal{I}(\mathtt{T}_n)$
- There exists at least one formula with $\mathtt{T}$ of the flowing forms that is satisfied by $\mathfrak{M}$:
  - $\textsc{Total}(BinaryRelation, \mathtt{x}^d, \mathtt{T})$
  - $\textsc{Isomorphic}(BinaryRelation, \mathtt{x}^d, \mathtt{T})$
  - $\textsc{Total}(BinaryRelation, \mathtt{T}, \mathtt{x}^d)$
  - $\textsc{Isomorphic}(BinaryRelation, \mathtt{T}, \mathtt{x}^d)$
  - $\textsc{All}(BinaryRelation, \mathtt{T})$
  - $Method(\mathtt{x}^d \otimes \mathtt{T})$

where $\mathtt{x}^d$ is some term in Φ

3.2.3 Hierarchy to Set

$$(\{\mathtt{H}\}, \Psi) \rightarrow (\{\mathtt{C}\}, \Phi)$$

Pre-conditions:

- $\mathtt{H}$ is a term of type $\mathbb{HIERARCHY}$ in Ψ

Post-conditions:

- $\mathtt{C}$ is a term of type $\mathbb{CLASS}$ in Φ

3.2.4 Collapse to Hierarchy

$$(\{\mathtt{C}^d, \mathtt{r}\}, \Psi) \rightarrow (\mathtt{H}, \Phi)$$

Such that : $0 \leq d \leq 1$

Pre-conditions:

- $\mathtt{C}^d$ is a term of type $\mathbb{CLASS}$ in Ψ
- $\textsc{Total}(Inherit, \mathtt{C}^d, \mathtt{r})$ in Ψ is satisfied by $\mathfrak{M}$

Post-conditions:

- $\mathtt{H}$ is a term of type $\mathbb{HIERARCHY}$ in Φ
- $\mathcal{I}(\mathtt{H}) = \{\mathcal{I}(\mathtt{r})\} \cup \mathcal{I}(\mathtt{C}^d)$

3.2.5 Hierarchies Union

$$(\{\mathtt{H}_1 \ldots \mathtt{H}_n\}, \Psi) \rightarrow (\mathtt{C}, \Phi)$$

Pre-conditions:

- All terms $\mathtt{h}i$, $1 \leq i \leq n$ introduced are of type $\mathbb{HIERARCHY}$ in Ψ

Post-conditions:

- $\mathtt{C}$ is a term of type $\mathbb{CLASS}$ in Φ
- $\mathcal{I}(\mathtt{C}) = \mathcal{I}(\mathtt{H}_1) \cup \ldots \cup \mathcal{I}(\mathtt{H}_n)$

3.2.6 To Top

$$(\{\}, \Psi) \rightarrow (\{\}, \top_{\mathfrak{M}})$$

3.2.7 Elimination

$$(\{t_1^d \ldots t_n^d\}, \Psi) \rightarrow (\{\ \}, \Phi)$$

Such that : $0 \leq d \leq 1$

Pre-conditions:

- $t_1^d \ldots t_n^d$ are terms of any type in Ψ

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