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Approval Voting and Scoring Rules with Common Values*

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Abstract

Consider the problem of deciding a winner among three alternatives when voters have common values, but private information regarding the values of the alternatives. We compare approval voting with other scoring rules. For any finite electorate, the best equilibrium under approval voting is more efficient than either plurality rule or negative voting. If any scoring rule yields a sequence of equilibria that aggregates information in large elections, then approval voting must do so as well.

1 Model

There are three candidates $K = \{1, 2, 3\}$. There is a finite set of I voters. Each voter i can submit a ballot, or vote vector, $(c_i(1), c_i(2), c_i(3))$, where $c_i(k)$ denotes the score allocated by voter i to candidate k. The scoring rule defines the permitted ballots. As in Myerson (2002), we consider (A, B)-scoring rules, that are defined by two parameters $0 \leq A \leq B \leq 1$. For a fixed (A, B)-scoring rule, each voter can submit either a permutation of $(1, B, 0)$ or of $(1, A, 0)$. Let C denote the space of all possible ballots. Given a profile of ballots $(c_1, \ldots, c_I)$, the winner $W(c_1, \ldots, c_I)$ of the election is the candidates whose total score $\sum_I c_i(k)$ is maximal. In the case of a tie, uniform randomization is used to select among the winners.

Some specific (A, B)-scoring rules are well known. The case $(A, B) = (0, 0)$ is plurality voting, where each voter can support a single candidate. The case $(A, B) = (1, 1)$ is negative voting, where each voter can oppose a single candidate. The case $(A, B) = (0.5, 0.5)$ is the Borda count, where candidates are totally ranked and receive scores proportional to their ranks. Of particular interest is the case $(A, B) = (0, 1)$. This case is approval voting, where each voter decides a set of one or two candidates to support.

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Most of the literature on approval voting focuses on private values. We focus on the case of pure common values. Let Ω be a finite set of states of the world. The prior probability of state ω is $P(\omega)$. All voters share a common utility $U(k|\omega)$ for candidate $k = 1, 2, 3$ in state ω. We assume that there is a unique best candidate k_ω that maximizes $U(k|\omega)$ for each state of the world.

The finite set S is a set of possible signals. The conditional probability of signal $s \in S$ given ω is denoted $F(s|\omega)$. Given the state ω, each voter receives a conditionally independent signal following the distribution $F(s|\omega)$.

A strategy $\sigma : S \to C$ is a function assigning a vote vector to each signal. When $\sigma_i(s)$ is a degenerate lottery with probability one of c, we slightly abuse notation and write $\sigma_i(s) = c$. A profile $(\sigma_1, \ldots, \sigma_I)$ of strategies is *symmetric* if $\sigma_i = \sigma_j$ for all i, j. When referring to symmetric strategy profiles, we drop the subscript. The common expected utility for the strategy profile $\boldsymbol{\sigma}(\mathbf{s}) = \sigma_1(s_1), \ldots, \sigma_I(s_I)$ is

$$EU(\boldsymbol{\sigma}) = \int_\Omega \int_S \int_{\mathcal{X}^I} U\left(W(c_1, \ldots, c_I)\,|\,\omega\right)\, d\boldsymbol{\sigma}(\mathbf{s})\, dF(\mathbf{s}|\omega)\, dP(\omega).$$

Consider a sequence of symmetric strategies (σ_I). We say that the *probability of error goes to zero* if, for every ω,

$$\int_S \int_{\mathcal{X}^I} U\left(W(c_1, \ldots, c_I)\,|\,\omega\,|\,\omega\right)\, d\boldsymbol{\sigma}_I(\mathbf{s}_I)\, dF(\mathbf{s}_I|\omega) \to U(k_\omega|\omega),$$

as I goes to infinity. The probability of error goes to zero if and only if the probability of the best candidate k_ω winning the election goes to one for every state of the world.

A smaller and newer literature studies information aggregation for multiple candidates with common or interdependent values. A theme in these studies is that approval voting outperforms other institutions. However, this comparison is currently understood only for restricted classes of environments. Goertz and Maniquet (2009) consider a class of environments where voters are indifferent between the two inferior candidates. Within this class, approval voting is the only simple scoring rule that admits an informationally efficient limit equilibrium.[1] Bouton and Castanheira (2010) consider a class of environments where a majority of voters prefer two candidates to a third minority candidate, but have incomplete information about which of the two candidates is better. Within this class, approval voting yields a unique limit equilibrium that efficiently aggregates information, while plurality rule can have multiple equilibria. Both papers work with Poisson population uncertainty, which allows for a cleaner construction of limit equilibrium.

Our analysis, which makes no assumptions either on the information or on the utility, applies to arbitrary environments. However, we do not attempt to explicitly construct limit equilibria. Instead, we argue indirectly by adapting an insight due to McLennan (1998): in a common value election, any strategy that maximizes utility is an equilibrium. While our hypotheses are more general than those in the literature, our conclusions are also less sharp. We cannot speak to the

[1] A scoring rule is simple if $B = 0$. For example, negative voting is not a simple scoring rule.

uniqueness of equilibrium nor to its characterization.

2 Results

Our first result asserts that the maximal equilibrium utility under approval voting is greater than or equal to the maximal equilibrium utility under approval voting or under negative voting.

Proposition 1. *If σ^* is a symmetric equilibrium for plurality rule or for negative voting, then there exists a symmetric equilibrium ρ^* such that $U(\rho^*) \geq U(\sigma^*)$.*

Proof. First take the case where σ^* is a symmetric equilibrium of plurality rule. Then define the strategy profile ρ for approval voting by $\rho(s) = \sigma^*(s)$.[2] Then the expected utility of ρ is identical to the expected utility of σ^*. Since U is continuous and the space of symmetric strategies is compact, there exists a symmetric strategy profile ρ^* that maximizes the common expected utility U for approval voting among all symmetric strategy profiles. By Theorem 2 of McLennan (1998), ρ^* is an equilibrium for approval voting. By its construction, $U(\rho^*) \geq U(\rho) = U(\sigma)$.

The argument for σ^* as a symmetric equilibrium of negative voting is identical. □

While the reasoning for Proposition 1 is mathematically straightforward, to our knowledge it has not been previously observed. Specifically, any ballot that can be submitted under plurality rule can also be submitted under approval voting. So the expected utility under plurality rule can be replicated under approval voting, by having each voter approve the singleton set corresponding to the candidate that she would support under plurality rule. Of course, approval voting also allows other ballots that support two candidates. However, the observation of McLennan (1998) is that the best equilibrium in a game of common values must be as good as any strategy profile. So the best equilibrium under approval voting must be at least as efficient as any equilibrium under plurality rule. The argument makes transparent the connection between approval voting and plurality rule: the fundamental reason that approval voting outperforms plurality rule (or negative voting) is that approval voting allows a larger set of feasible ballots. While similar arguments for approval voting are suggested in private value environments, its flexibility has direct force in common value environments.

Our second finding is that the best asymptotic equilibrium under approval voting is at least as good as the best asymptotic equilibrium under any interior (A, B)-scoring rule where $0 < A \leq B < 1$. Unlike plurality rule or negative voting, the strategies under general (A, B)-scoring rules cannot be exactly replicated by approval voting. When $A \neq B$, a general (A, B)-scoring rule allows for twelve distinct vote vectors, while approval voting allows only for six. However, the outcome of the (A, B)-scoring rule can be approximated through appropriate randomization of the ballots under approval voting. Specifically, the expected vote counts can be maintained under approval voting. This approximation becomes asymptotically precise, so the limit outcome under a general scoring rule can be replicated by approval voting.

[2] To be pedantic, we identify $\sigma^*(s)$ within the strictly large strategy simplex for approval voting.

Proposition 2. *Suppose $A, B \in (0,1)$. If there exists a sequence (σ_I) of symmetric strategies for the (A, B) scoring rule that takes the error probability to zero, then there exists a sequence (ρ_I^*) of symmetric equilibria for approval voting that takes the error probability to zero.*

Proof. For any symmetric strategy profile σ of the (A, B) scoring rule, define the symmetric strategy profile ρ of approval voting with permutations of the following:

$$[\rho(s)](1,1,0) = \sum_{X=A,B} X \{[\sigma(s)](1,X,0) + [\sigma(s)](X,1,0)\} \tag{1}$$

$$[\rho(s)](1,0,0) = \sum_{X=A,B} (1-X) \{[\sigma(s)](1,X,0) + [\sigma(s)](1,0,X)\} \tag{2}$$

Without loss of generality, fix a state ω such that $k_\omega = 1$. We now prove that the limit probability that candidate 1 wins the election goes to one, conditional on the state ω. All probabilities and expectations hereon are conditional on ω. Define the random variable

$$\delta_{Ii} = \sigma_{Ii} \cdot (1,0,0) - \sigma_{Ii} \cdot (0,1,0),$$

where the "$\cdot$" operation denotes the dot product. The random variable δ_{Ii} is the *difference* in the scores given to candidate 1 and candidate 2 by voter i when playing the strategy σ_I in the (A, B)-scoring rule. Similarly, let

$$\Delta_{Ii} = \rho_{Ii} \cdot (1,0,0) - \rho_{Ii} \cdot (0,1,0),$$

i.e. the difference in the scores of candidate 1 and candidate 2 when playing the strategy ρ_I under approval voting.

We first show that the expectation of the scores differences are identical under σ and ρ.

Lemma 1. $\mathbf{E}(\delta_{Ii}) = \mathbf{E}(\Delta_{Ii})$.

Proof. We will show that the expectations conditional on a fixed signal are equal: $\mathbf{E}(\delta_{Ii}|s_i = s) = \mathbf{E}(\Delta_{Ii}|s_i = s)$ for all signals s_i. Then the unconditional expectations are also equal.

First, computing the expectation for the (A, B)-scoring rule:

$$\begin{aligned}
\mathbf{E}(\delta_{Ii}|s_i = s) = & \sum_{X=A,B} \{[\sigma_I(s)](1,0,X) - [\sigma_I(s)](0,1,X)\} \\
& + \sum_{X=A,B} X \{[\sigma_I(s)](X,0,1) - [\sigma_I(s)](0,X,1)\} \\
& + \sum_{X=A,B} (1-X) \{[\sigma_I(s)](1,X,0) - [\sigma_I(s)](X,1,0)\}
\end{aligned}$$

Next, using the construction of ρ_I in (1) and (2) for approval voting:

$$\begin{aligned}\mathbf{E}(\Delta_{Ii}|s_i = s) &= [\rho_I(s)](1,0,0) + [\rho_I(s)](1,0,1) - [\rho_I(s)](0,1,0) - [\rho_I(s)](0,1,1)] \\ &= \sum_{X=A,B} (1-X)\left\{[\sigma(s)](1,X,0) + [\sigma(s)](1,0,X)\right\} \\ &\quad + \sum_{X=A,B} X\left\{[\sigma(s)](1,0,X) + [\sigma(s)](X,0,1)\right\} \\ &\quad - \sum_{X=A,B} (1-X)\left\{[\sigma(s)](X,1,0) + [\sigma(s)](0,1,X)\right\} \\ &\quad - \sum_{X=A,B} X\left\{[\sigma(s)](0,1,X) + [\sigma(s)](0,X,1)\right\}\end{aligned}$$

The right hand sides are equal to each other. $\square$

Consider the case where $\lim \mathbf{V}(\delta_{Ii}) \to 0$ for sufficiently large I. Then δ_{Ii} converges to a point mass on some constant E for large I. However, because (A, B) are interior, 0 is not part of the support of δ_{Ii} because it is impossible for a ballot to provide equal scores to candidates 1 and 2. Therefore, since the probability that $\sum_{Ii} \delta_{Ii} > 0$ goes to one, it must be the case that the point mass is located at a strictly positive point. This point defines the expectation $\lim \mathbf{E}(\delta_{Ii}) = E > 0$ for large enough I. By Lemma 1, it is also the case that $\lim \mathbf{E}(\Delta_{Ii}) = E > 0$ for large enough I. By the weak law of large numbers for triangular arrays, the probability that $\sum_{i=1}^{I} \Delta_{Ii}/I > 0$ goes to one. Hence the probability $\sum_{i=1}^{I} \Delta_{Ii} > 0$ also goes to one.

So, without loss of generality, suppose $\lim \mathbf{V}(\delta_{Ii}) > 0$ for all I, i. First, note that:

$$\mathbf{Pr}\left(\sum_{i=1}^{I} \delta_{Ii} \leq 0\right) = \mathbf{Pr}\left(\frac{\sum_{i=1}^{I} \delta_{Ii} - I\mathbf{E}(\delta_{Ii})}{\sqrt{I\mathbf{V}(\delta_{Ii})}} \leq -\frac{I\mathbf{E}(\delta_{Ii})}{\sqrt{I\mathbf{V}(\delta_{Ii})}}\right)$$

By the Central Limit Theorem for triangular arrays:

$$\left|\mathbf{Pr}\left(\frac{\sum_{i=1}^{I} \delta_{Ii} - I\mathbf{E}(\delta_{Ii})}{\sqrt{I\mathbf{V}(\delta_{Ii})}} \leq -\frac{I\mathbf{E}(\delta_{Ii})}{\sqrt{I\mathbf{V}(\delta_{Ii})}}\right) - \Phi\left(-\frac{I\mathbf{E}(\delta_{Ii})}{\sqrt{I\mathbf{V}(\delta_{Ii})}}\right)\right| \to 0$$

Since $\mathbf{Pr}\left(\sum_{i=1}^{I} \delta_{Ii} \leq 0\right)$ must go to zero by the assumed efficiency of σ_I, the triangle inequality implies that $\Phi\left(-\frac{I\mathbf{E}(\delta_{Ii})}{\sqrt{I\mathbf{V}(\delta_{Ii})}}\right)$ must go to zero. Hence:

$$\sqrt{I}\frac{\mathbf{E}(\delta_{Ii})}{\sqrt{\mathbf{V}(\delta_{Ii})}} \to \infty.$$

Then

$$\sqrt{I}\frac{\mathbf{E}(\Delta_{Ii})}{\sqrt{\mathbf{V}(\Delta_{Ii})}} = \sqrt{I}\frac{\mathbf{E}(\delta_{Ii})}{\sqrt{\mathbf{V}(\Delta_{Ii})}}, \text{ by Lemma 1}$$
$$= \sqrt{I}\frac{\mathbf{E}(\delta_{Ii})}{\sqrt{\mathbf{V}(\delta_{Ii})}} \times \sqrt{\frac{\mathbf{V}(\delta_{Ii})}{\mathbf{V}(\Delta_{Ii})}}.$$

The second factor $\sqrt{\frac{\mathbf{V}(\delta_{Ii})}{\mathbf{V}(\Delta_{Ii})}}$ is strictly positive. It is uniformly bounded away from zero, since $\lim \mathbf{V}(\delta_{Ii}) > 0$ and $\mathbf{V}(\Delta_{Ii})$ is uniformly bounded Δ_{Ii} takes values in a bounded set $[-1, 1]$. Since the first factor $\frac{\mathbf{E}(\delta_{Ii})}{\sqrt{\mathbf{V}(\delta_{Ii})}}$ goes to infinity, so does the product:

$$\sqrt{I}\frac{\mathbf{E}(\Delta_{Ii})}{\sqrt{\mathbf{V}(\Delta_{Ii})}} \to \infty.$$

Hence $\Phi\left(-\sqrt{I}\frac{\mathbf{E}(\Delta_{Ii})}{\sqrt{\mathbf{V}(\Delta_{Ii})}}\right) \to 0$. Applying the Central Limit Theorem for triangular arrays and the triangle inequality as before, we have that $\mathbf{Pr}\left(\sum_{i=1}^{I} \Delta_{Ii} \leq 0\right)$ goes to zero.

Since the probability that $\mathbf{Pr}\left(\sum_{i=1}^{I} \Delta_{Ii} \leq 0\right)$ goes to zero, the probability that the total score for candidate 1 is strictly larger than the total score for candidate 2 must go to one. Similarly, the probability that the total score for candidate 1 is strictly larger than the total score for candidate 3 also goes to one. The intersection event is that candidate 1 wins the election, and the probability of this event goes to one.

By Theorem 2 of McLennan (1998), there must exist a sequence of symmetric equilibrium ρ^*_{Ii} that provides weakly more common utility than ρ_{Ii} for each fixed population I. But since ρ_{Ii} takes the probability of error to zero, then ρ^*_{Ii} must also take the probability of error to zero. □

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