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Power Brokers: Middlemen in Legislative Bargaining

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Abstract

We consider a model of decentralized bargaining among three parties. Parties meet one-on-one after being randomly matched, and can sell or buy votes to one another. The party with a majority of the votes can decide to implement its preferred policy or extend negotiations to capture additional rents. We provide necessary and sufficient conditions for the existence of an equilibrium in which a party acts as an intermediary, transferring resources and voting rights among parties that wouldn't negotiate directly with one another. These conditions are generic, do not require special frictions, and include 'well-behaved' (i.e., single-peaked) preference profiles.

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> "From the time he became Majority Leader, Johnson began using talk on the floor as a smoke screen for the maneuvering that was taking place in the cloakrooms, ..., as a method of stalling the Senate to give him time to work out his deals." Robert Caro in *The Years of Lyndon Johnson, Master of the Senate.*

1 Introduction

Most significant public policy choices in modern democracies are decided in legislatures and other collective bodies. From health care reform, to national defense or regulation of economic activity, enacting new policies requires mutual understanding among legislators with different political views. It also requires, more often than not, a variety of compromises and political exchanges among these legislators.

A vast majority of these political exchanges takes the form of private negotiations among legislators, which occur well before a bill is taken up for consideration. In fact, compromises among members of a legislative coalition are rarely struck publicly and simultaneously at the time when a proposal is up for a vote, but instead are constructed through a series of backroom deals carried out by *power brokers*, who act as middlemen in legislative bargaining.

Our goal in this paper is to shed light on the dynamics of decentralized legislative bargaining: on how private agreements among parties affect subsequent negotiations and policy outcomes, and on how parties' conjectures of future negotiations affect agreements in the first place. In this context, our key consideration is to explain the emergence and role of middlemen in legislative bargaining. Can some legislative actors enable deals by putting together two parties that wouldn't negotiate directly with one another? If yes, under what conditions is this possible? What do these power brokers bring to the table? Does this require a particular restriction on the frequency with which different parties interact with one another, or unorthodox restrictions on their preferences?

To address these questions, we propose a model of decentralized multilateral bargaining that captures the key dynamics of backroom deals in a parsimonious setting. There are three parties, each of which is endowed with an initial vote share, and infinitely many periods (i.e., no fixed deadline for negotiations). In each period before a policy is implemented, two parties meet one-on-one, according to a stochastic matching process, and can offer to buy or sell their votes to one another in exchange of *favors*.[1] As in Gul (1989), parties selling their votes to others relinquish their voting rights, and are excluded from further negotiations. If a party holds a majority of the votes after negotiations, it can choose to implement its preferred policy or extend negotiations for another period. Otherwise, the process of negotiations continues in the next period by default.

The main result of the paper is that a number of successful political deals occur because of the endogenous emergence of a *political broker*: a party that serves as an intermediary between two parties that wouldn't negotiate directly with one another, transferring resources and voting rights among them in an indirect trade. Moreover, we show that in order to be able to fulfill this role, the broker *must* have a stake in the policy outcome: a party who only cares about rents cannot add value to what other parties can achieve in direct negotiations. This result points to a fundamental difference between intermediaries in politics and in exchange economies (Rubinstein and Wolinsky (1987), Biglaiser (1993), Condorelli and Galeotti (2011)).

Our approach to attack this problem is to pose it as one of rationalizability of a stationary equilibrium with intermediation, or *broker equilibrium*. In particular, we ask whether for given matching parameters (discount factor and matching probabilities), there exist preference profiles for which we can support a broker equilibrium. A key advantage of this formulation is that the equilibrium conditions can be written as a set of linear inequalities $Au \leq \alpha$, where the unknowns are the payoffs $u_i(z_j)$ of party i for implementing policy z_j, and A is a matrix of matching parameters. We can then use basic duality results from convex analysis to obtain necessary and sufficient conditions for the existence of a solution to this problem. We show, in fact, that the endogenous emergence of brokers is a robust equilibrium phenomenon. The conditions under which broker equilibria exist are generic and do not require special frictions: broker equilibria can be supported with uniform matching and vanishing bargaining frictions.

The duality approach is especially productive in this setting to characterize properties of the preference profiles that admit broker equilibria. This is because any constraint on preference profiles can be expressed as a modified matrix of matching parameters A'.

[1]The exact nature of these favors depends on the application. See Baron and Diermeier (2001) for a discussion of various examples in the context of legislatures, from jobs for party stalwarts to board seats on public companies and transfers to interest groups.

As a result, establishing conditions for the existence of a broker equilibrium under some constraints on preferences boils down to finding conditions for the existence of a solution to the modified system of inequalities.

We show, in particular, that broker equilibria can be supported with 'well-behaved' preference profiles (single-peaked, even quadratic or linear loss in a unidimensional space). Thus, the existence of brokers doesn't require cycles in the majority preference. It does, however, require a minimal diversity in the space of alternatives. In fact, if the space of policies is binary, as in Philipson and Snyder (1996) or Casella, Llorente-Saguer, and Palfrey (2012), a broker equilibrium cannot exist.

Establishing the existence of a broker equilibrium could ultimately be uninteresting if middlemen did not have an impact on outcomes and/or welfare. We show, however, that this is not the case. To do this, we establish the existence of a broker equilibrium for preference profiles under which the broker equilibrium implements a different policy outcome than the one that would result in the absence of the broker, or in any equilibrium maintaining the composition of the legislature. We then show that in this situation the broker equilibrium is welfare improving, but not necessarily efficient, even as the time between trading opportunities goes to zero.

While we develop our main results in the context of a majoritarian legislature, the logic for the existence of brokers applies in a wider set of circumstances. In the paper we discuss some of these extensions. First, we consider a committee acting under unanimity rule, such as the Council of the European Union. A key distinction of this setting is that inaction becomes a possible equilibrium outcome in negotiations between two parties. We show, however, that with small modifications the analysis of brokers with unanimity rule is similar to the majoritarian case, and that our existence result extends to this environment.

Finally, we consider the possibility that an outside actor, such as the President or an interest group, acts as a broker. In fact, once granted access to negotiations, an interest group is strategically equivalent to internal members of the legislature, except for the fact that it cannot sell votes in the initial round of decentralized bargaining. Our main analysis therefore provides sufficient conditions for the existence of a broker equilibrium in this case as well. The analysis applies directly to the case of a President without veto power, and can easily be extended to allow for veto power building on the results for majoritarian and unanimous legislatures.

The rest of the paper is organized as follows. We review the literature in Section 2, and present the model in Section 3. In Section 4 we analyze the final bargaining stage, in which only two parties control voting rights, and begin our analysis of the trade-offs parties face in decentralized bargaining. In Section 5 we present our main results. In Section 6 we discuss various extensions: a majoritarian legislature with no majority party, unanimity rule, and interest groups as lobbies. Section 7 concludes.

2 Related Literature

The analysis of vote trading is fundamentally different from that of a typical exchange economy (Riker and Brams (1973), Philipson and Snyder (1996)). While voting rights per se have no impact on payoffs, they allow holders to have a say in the collective outcome. Because individuals with different preferences are willing to make different tradeoffs between policies, and between policies and favors, changes in the identity of vote holders can have an effect on all participants. As a result, participants must make conjectures about one another's behavior in order to calculate their optimal vote buying or vote trading strategies.

The prevailing approach to study decentralized buying and selling of votes in a committee is to model exchanges as occurring in a competitive market for votes (Philipson and Snyder (1996), Casella, Llorente-Saguer, and Palfrey (2012)). In these models, there are two alternatives (a majority and a minority position) and committee members have the opportunity to buy and sell votes at posted prices, which they take as given.[2] In this paper we depart from the price-taking tradition in order to capture the key dynamics of backroom deals. We assume that parties meet one-on-one, according to a probabilistic matching, and can offer to buy or sell their votes to one another at a price they negotiate. Parties are fully strategic and forward looking about the implications of their trades on

[2]Because of the interdependencies in demand, and the discontinuity in payoffs associated with simple majority rule, the standard equilibrium notion leads to nonexistence. To address this issue, Philipson and Snyder (1996) consider a semi-centralized market for votes, in which a floor manager rations any excess supply by randomly choosing suppliers among those willing to sell their vote at the posted price, while distributing all revenues generated by vote selling evenly across all members of the majority side. Casella, Llorente-Saguer, and Palfrey (2012) instead, allow for mixed (probabilistic) demands, and require only that the market clears in expectation, while ex post clearing obtains through a rationing rule.

subsequent negotiations and policy outcomes.

Our model of decentralized bargaining builds on Gul (1989), but has important differences in scope and in the model itself. Gul shows that given a condition on payoffs that guarantees that value functions are superadditive, as bargaining frictions vanish ($\delta \to 1$) there is a unique efficient equilibrium, and players' equilibrium payoffs converge to the Shapley value (under uniform matching). In our model, instead, parties interact in a majority game, and obtain status quo payoffs until a party holding a majority of the votes decides to enact a new policy. In this setup, Gul's assumption on payoffs does not necessarily hold, and therefore neither does the result on efficiency of equilibria as $\delta \to 1$.

Our paper complements several related strands of literature. First, our *decentralized bargaining* approach contrasts with the *centralized bargaining* approach of Banks and Duggan (2000) and Jackson and Moselle (2002), who build on the Baron and Ferejohn (1989) framework of bargaining over distribution. Here a proposer makes an offer of policy *and* transfers to all members of a policy coalition simultaneously. Because of this, these papers cannot consider the phenomenon of intermediation in the legislative setting.

Second, in our model, vote trading is done internally, by *members* of the committee. This complements the literature on vote buying of inside members by *outsiders* (Myerson (1993), Dixit and Londregan (1996), Groseclose and Snyder (1996), Banks (2000), Diermeier and Myerson (1999), Lizzeri and Persico (2001), Dal Bo (2007), and Dekel, Jackson, and Wolinsky (2008, 2009)). Importantly, in these models vote buyers are precluded from forming coalitions among them, or from reselling their votes to members of the committee.

Third, as in models of vote markets, bargaining over policymaking, and vote buying, we consider policy and transfers. This approach complements the literature on logrolling, where two legislators exchange their support for a bill in exchange for support in another bill (see Buchanan and Tullock (1962), Tullock (1970), Riker and Brams (1973), Bernholz (1975), and Miller (1975, 1977); see also Hortala-Vallve (2011)).

Finally, our paper is also related to the literature on intermediaries in exchange economies (see for example Rubinstein and Wolinsky (1987), Yanelle (1989), Biglaiser (1993)). The closest paper is Condorelli and Galeotti (2011). In the model, a set of agents located on a network trade a single private good, and the current owner of the good decides whether to consume or resale. In this framework, the interest lies in the characterization of which agents choose to resale, and in the payoffs they obtain. Condorelli and Galeotti show that the agents who resale in earlier periods obtain a payoff advantage over later dealers, and provide examples in which the agents who choose to resale in equilibrium are the low valuation traders who provide access to valuable areas of the network.[3] Our paper differs from these in various ways. Chief among these is that in our setup agents trade votes to determine a collective decision. Thus, when reselling, agents in our model care not only about extracting rents from individuals with high willingness to pay, but also about the policy implications of the resulting distribution of votes.

3 The Model

There are 3 parties, $i \in N = \{1, 2, 3\}$, and an infinite number of periods, $\tau = 1, 2, \ldots$. Each party $i \in N$ has an ideal policy $z_i \in X$, a discount factor δ, and is endowed with $k_i > 0$ votes. Parties participate in a process of bilateral transactions to enact a policy. Let $\mathcal{N}_\tau$ denote the set of parties holding voting rights in period τ. In each period τ in which at least two parties hold voting rights, two parties $i, j \in \mathcal{N}_\tau$ are randomly matched to negotiate with one another, and one of them is randomly selected to make an offer. We let ρ_{ij} and p_{ij} denote the probability that i and j are matched and i is selected to make an offer when $\mathcal{N}_\tau = \{i, j\}$ and $\mathcal{N}_\tau = N$ resp.

The proposer i can offer to buy or sell voting rights, or choose not to make an offer. A feasible transaction is an exchange of a party's voting rights for a numeraire, which we refer to as *favors*. If i sells its votes to j, i votes as instructed by j, and is excluded from further negotiations. We let $t_{ij}(\mathbf{k})$ denote the net transfer of favors from i to j that follows a deal when i and j are matched and i proposed to j given voting rights $\mathbf{k}$. We say that i makes a relevant offer to j when i makes an offer to j that j will accept. In any period τ in which a party i has a majority of the votes after trade ($k'_i \geq r \equiv \sum_i k_i/2$), party i can choose whether to implement its preferred policy z_i or extend negotiations. When a party chooses to implement its preferred policy, the game ends immediately and the policy z_i is implemented forever. In any period τ prior to the implementation of a new policy, the outcome is the status quo Q.

[3]See also Melo (2012), who studies middlemen in networks, extending Rubinstein and Wolinsky (1987) to this environment.

Party i's preferences are represented by the utility function

$$V_i = \sum_{\tau=0}^{\infty} \delta^\tau \left[(1-\delta)u_i(y_\tau) - t_i^\tau\right],$$

where $u_i(\cdot)$ is uniquely maximized at z_i, and we normalize $u_i(Q) = 0$ for all i. y_τ denotes the policy implemented in period τ, and t_i^τ denotes the τ period net transfer from i to others. We say that i *dominates* j ($i \gg j$) if i's willingness to pay for implementing z_i instead of z_j exceeds j's willingness to pay for implementing z_j instead of z_i; i.e., if $u_i^* - u_i(z_j) \geq u_j^* - u_j(z_i)$. Equivalently, letting $S_{ij}(y)$ denote the aggregate surplus for i and j of implementing y, i.e., $S_{ij}(y) \equiv u_i(y) + u_j(y)$, we say that $i \gg j$ if $S_{ij}(z_i) > S_{ij}(z_j)$.

An equilibrium is a Markov Perfect Equilibrium (MPE). A trading state is a pair $((i,j),\mathbf{k})$, where (i,j) denotes that i is matched with j and i is selected to propose, and $\mathbf{k}$ denotes the pre-trade allocation of voting rights. We let $W_{ij}^i(\mathbf{k}, buy)$ and $W_{ij}^i(\mathbf{k}, sell)$ denote i's equilibrium payoff from her best relevant buy and sell offers in trading state $((i,j),\mathbf{k})$, and $W_{ij}^i(\mathbf{k}, wait)$ i's equilibrium payoff from not making a relevant offer. Then $W_{ij}^i(\mathbf{k}) \equiv \max_a W_{ij}^i(\mathbf{k}, a)$, for $a \in \{buy, sell, wait\}$, denotes i's equilibrium payoff in trading state $((i,j),\mathbf{k})$. We also let $W^i(\mathbf{k}) \equiv E[W_{ij}^i(\mathbf{k})]$, where the expectation is taken over all possible realizations of matches and proposing power. Finally, because a party with a majority of the votes after trading can *choose* to implement its preferred policy or extend negotiations, we also need to consider i's post-trade equilibrium payoff after trade opportunities resulted in a vote endowment $\mathbf{k}$, which we denote by $B^i(\mathbf{k})$.

4 Preliminaries

In this section, we present two building blocks for the analysis of broker equilibria in Section 5. In Section 4.1 we analyze the final bargaining stage, in which only two parties control voting rights. The analysis of this stage is not 'off-the-shelve' because the party with a majority of the votes can *choose* to either extend negotiations or implement its preferred policy following disagreement. This feature introduces interesting differences in the analysis with respect to a standard bilateral bargaining game, where negotiations are extended by default after disagreement.In Section 4.2 we begin analyzing the trade-offs parties face in decentralized bargaining, and characterize parties' optimal actions in each decision node as a function of the continuation values.

4.1 Majority-Minority Bargaining

We begin by analyzing the final bargaining stage, in which only two parties control voting rights. Because of simple majority rule, one of these parties, i, has a majority of the votes. We call i the *majority party*, and index it by M, and $j \neq i$ the *minority party*, m. The main result of this section characterizes MPE and MPE payoffs of the majority-minority bargaining game. For simplicity of exposition, throughout this section we denote the trading nodes in which the majority and minority propose simply by M and m (instead of Mm and mM), and write the joint surplus $S_{Mm}(x)$ as $S(x)$.

Equilibrium behavior in the majority-minority bargaining game relies on two key factors. The first is the parties' relative intensity of preferences for the majority and minority policies z_M and z_m. This is standard. When $M \gg m$, total surplus is higher if the majority alternative is implemented. As a result, there is no transfer that the minority party would be willing to offer that would compensate the majority party for not implementing its preferred policy z_M. In this case, there is a MPE in which there is no trade, and the majority party implements its preferred policy.[4] When instead $m \gg M$, there are gains from trade. Whether these gains from trade are realized, and how they are distributed, depends on the parties' perception of their relative bargaining power.

The key factor here is the *option* that the majority party has to implement its preferred policy without the consent of the minority party. Differently to a standard bilateral bargaining game (where negotiations are automatically extended after disagreement), here the majority party can either reject an offer and extend negotiations, in which case $B^j(\mathbf{k}) = \delta W^j(\mathbf{k})$, *or* reject it and implement its preferred policy, in which case $B^j(\mathbf{k}) = u_j(z_M)$. The threat of implementing its preferred policy after *unsuccessful trading* (UT), however, is not always credible, and therefore not always relevant to determine how gains from trade are distributed. In fact, the majority party has incentives to implement its preferred policy after UT only if $u^*_M \geq \delta W^M(\mathbf{k})$, and otherwise prefers to extend negotiations for an additional period.[5]

[4] If in addition $u^*_M < \delta\rho_M S(z_M)$, there also exists a MPE in which m pays M so that it implements z_M immediately. In this equilibrium M offers to buy from m at a negative price (or accepts only a negative price offer), threatening m with maintaining the status quo after disagreement. This is interesting in itself, but largely irrelevant for our main argument, with the exception of the uniqueness claim in Theorem 5.4. We return to this point in the proof of this theorem.

[5] This is similar to bargaining games with *outside options* (see for example Muthoo (1999)). However, in bargaining games with outside options it is assumed that the party receiving the offer can reject the offer

This off-equilibrium-path choice has important consequences for equilibrium behavior and the distribution of rents in majority-minority bargaining. Consider the problem of the majority party when it has an opportunity to propose. M can buy or sell votes to m, generating payoffs

$$W_M^M(\mathbf{k}, sell) = S(z_m) - B^m(\mathbf{k}) \quad \text{and} \quad W_M^M(\mathbf{k}, buy) = S(z_M) - B^m(\mathbf{k}),$$

or it can choose not to make m a relevant offer, yielding $W_M^M(\mathbf{k}, wait) = B^M(\mathbf{k})$. The key here is that M's payoffs for waiting and trading votes depend on the reservation values $B^M(\mathbf{k})$ and $B^m(\mathbf{k})$, which in turn depend on whether M prefers to implement its preferred policy or extend negotiations after UT.

Given equilibrium beliefs about play after UT and the proposer's payoffs associated with each choice at the proposing stage, we can characterize parties' optimal actions in each decision node as a function of the continuation values. Using these results, wecharacterize equilibria of the majority-minority bargaining game.

Proposition 4.1 *Consider a Majority-Minority Bargaining Game starting in period τ_0, and suppose $m \gg M$. Then there exists a MPE in which the minority party buys the votes of the majority party independently of who has the opportunity to propose and implements its preferred policy; i.e., $y_\tau = z_m$ for all $\tau \geq \tau_0$. Moreover,*

1. *If $u_M^* \leq \delta\rho_M S(z_m)$, the majority party extends negotiations after UT. Here $W^l(\mathbf{k}) = \rho_l S(z_m)$ for $l = m, M$ and $u_M^* \leq \delta W^M(\mathbf{k})$.*

2. *If $u_M^* \geq \delta\rho_M S(z_m)$ and $(1-\delta)u_M^* \geq \delta\rho_M [S(z_m) - S(z_M)]$, the majority party implements z_M after UT. Here $W^l(\mathbf{k}) = u_l(z_M) + \rho_l(S(z_m) - S(z_M))$ for $l = m, M$, and $u_M^* \geq \delta W^M(\mathbf{k})$.*

3. *If neither of these conditions hold, there is no MPE in pure strategies. In equilibrium, the majority party implements its preferred policy after UT with probability*

$$\alpha^* = \frac{(1-\delta)}{\delta\rho_M}\left(\frac{u_M^* - \delta\rho_M S(z_m)}{\delta S(z_m) - S(z_M)}\right)$$

Here $\delta W^M(\mathbf{k}) = u_M^*$ and $\delta W^m(\mathbf{k}) = \delta S(z_m) - u_M^*$.

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and take her outside option. This counterbalances the proposal power of the other party. In our game, instead, it is only the majority party who can implement its preferred policy after UT, independently of whether it is the proposer or the receiver of the offer. This difference in the sequence introduces relatively large changes in the equilibrium of the game.

As part three of the Proposition makes clear, establishing existence of equilibrium in the majority-minority bargaining game requires using mixed strategies. This is because the majority's option to extend negotiations or implement its preferred policy after UT creates a discontinuity in payoffs that leads to nonexistence of a MPE in pure strategies.[6] Mixing after UT smoothes out this discontinuity in equilibrium payoffs and restores existence.

A critical implication of Proposition 4.1 is that conjectures of equilibrium play after disagreement are fundamental for the analysis of equilibria with intermediaries. If the joint payoff of implementing the minority policy is not large enough, or the majority can't appropriate a large fraction of this surplus ($\delta\rho_M S(z_m) < u_M^*$), the majority implements its preferred policy after UT with positive probability, and in equilibrium $u_M^* \geq \delta W^M(\mathbf{k})$. But in this case the majority-minority bargaining node wouldn't be reached in the first place. This is because the decision problem of the majority party after UT in bilateral bargaining is equivalent to its decision problem after acquiring the majority in decentralized bargaining. It follows that for an equilibrium with intermediaries to exist it is essential that the broker's relevant threat after disagreement in bilateral bargaining is *not* to implement its preferred policy after UT, and hence that

$$u_M^* \leq \delta\rho_M S(z_m). \tag{1}$$

4.2 Trade-Offs in Decentralized Bargaining

In this section, we discuss the basic trade-offs parties face in decentralized bargaining, and characterize parties' optimal actions in each decision node as a function of the continuation values. To do this we exploit the fact that in each node, only one player moves at a time. We can therefore solve for the proposer's best response by comparing the payoff she would obtain with the best relevant buy and sell offers, and with the payoff resulting from not making a relevant offer at all (either waiting for new meetings to occur, or possibly to implement its preferred policy if she already has a majority).

[6] To see why this is the case, suppose that in equilibrium the majority party implements its preferred policy after UT. This generates a payoff for the majority party of $W^M = u_M^* + \rho_M[S(z_m) - S(z_M)]$. But the majority party has an incentive to implement its preferred policy after UT if and only if $u_M^* \geq \delta W^M(\mathbf{k})$. Thus, this is a credible threat in equilibrium if and only if $(1-\delta)u_M^* \geq \delta\rho_M\left[S(z_m) - S(z_M)\right]$. Suppose instead that the majority party extends negotiations after UT. Then $W^M = \rho_M S(z_m)$, and M has an incentive to extend negotiations after UT if and only if $u_M^* \leq \delta\rho_M S(z_m)$. When neither of these conditions hold, there is no MPE in pure strategies.

In fact, as we show in Lemma 8.2 (in the Appendix), in order to characterize negotiations between i and j it is enough to analyze the problem of party i when it is its turn to propose to j. Since the proposer can appropriate the joint expected surplus net of the parter's post-trade reservation value, then for any $i, j \in N$ and $\mathbf{k}$,

$$W^j_{ji}(\mathbf{k}, buy) + B^i(\mathbf{k}) = W^i_{ij}(\mathbf{k}, sell) + B^j(\mathbf{k}).$$

From this it follows that i prefers selling to buying in $((i, j), \mathbf{k})$ if and only if j prefers buying to selling in $((j, i), \mathbf{k})$. Similarly, i prefers selling to extending negotiations in $((i, j), \mathbf{k})$ if and only if j prefers buying to extending negotiations in $((j, i), \mathbf{k})$.

Best responses in decentralized bargaining depend of course on the initial distribution of voting rights. With majority rule, however, only two classes of vote allocations are strategically relevant: either one of the parties has a majority of the votes – and can therefore implement its preferred policy if it chooses to – or no party controls a majority. In our main analysis we will assume that one party (party 1) has a majority of the votes, and that the two other parties, 2 and 3, control minority shares. We do this because this initial allocation of voting rights appear intuitively to be less conducive to the existence of brokers. We study the case of no majority party in Section 6.1.

Our first result characterizes the equilibrium payoffs of the proposer for waiting, or making a relevant buy or sell offer when the majority party 1 meets minority party j. We know already that $W^1_{1j}(\mathbf{k}, wait) = B^1(\mathbf{k})$, where $B^i(\mathbf{k}) = u_i(z_1)$ for $i = 1, j$ if $u^*_1 \geq \delta W^1(\mathbf{k})$, and otherwise $B^i(\mathbf{k}) = \delta W^i(\mathbf{k})$. Proposition 4.2 characterizes 1's equilibrium payoffs for selling and buying votes. By Lemma 8.2, it is enough to focus on 1's payoff in the proposal node $1j$.

Proposition 4.2 *Suppose party 1 meets with party j, and party h is unmatched. If there is a trade (either a sell or a buy), call $s \in \{1, j\}$ the seller, and $b \in \{1, j\}$ the buyer, and let $\mathbf{k}'_{bh}$ denote the resulting vote endowment. Then*

$$W^1_{1j}(\mathbf{k}, trade) = \begin{cases} \Pi(sb, h) - B^j(\mathbf{k}) & \text{if } h \gg b \text{ and } u^*_b < \delta \rho_{bh} S_{bh}(z_h), \\ S_{1j}(z_b) - B^j(\mathbf{k}) & \text{otherwise,} \end{cases}$$

where $\Pi(sb, u) \equiv \delta[W^s(\mathbf{k}'_{bu}) + W^b(\mathbf{k}'_{bu})] = \delta[u_s(z_u) + \rho_{bu} S_{bh}(z_u)]$ is the aggregate discounted value of a buyer b and seller s when b resales to an ultimate buyer u.

The condition that $h \gg b$ and $u_b^* < \delta\rho_{bh}S_{bh}(z_h)$ determines whether the buyer in the trade between 1 and j will go on to sell to the remaining party, or instead just buy to implement its preferred policy. Suppose for example that the condition holds when 1 sells to 3; i.e., $2 \gg 3$ and $u_3^* < \delta\rho_{32}S_{32}(z_2)$. Then 1's payoff from selling to 3 is the joint discounted equilibrium payoff for 1 and 3 of putting 3 in a position to sell the majority to 2, $\Pi(13,2)$, net of party 3's reservation value $B^3(\mathbf{k})$. When instead this condition doesn't hold, the value for 1 of selling to 3 is instead given by the joint surplus for 1 and 3 of implementing z_3 net of party 3's reservation value $B^3(\mathbf{k})$.

Our second result characterizes the payoffs of the proposer when the two minority parties meet in decentralized bargaining. The result is similar to that of Proposition 4.2, with the exception that the relevant aggregate surplus for buyer (b) and seller (s) in case of further trades is not given by what b can obtain by resaling to an ultimate buyer (u), $\Pi(sb,u)$, but instead by what the buyer can obtain by consolidating the minority vote and then buying again from an ultimate seller (ℓ), $\tilde{\Pi}(sb,\ell)$.

Proposition 4.3 *Suppose party* 2 *meets with party* 3*. If there is a trade (either sell or buy), call* $s \in \{2,3\}$ *the seller, and* $b \in \{2,3\}$ *the buyer, and let* $\mathbf{k}'_{b1}$ *denote the resulting vote endowment. Then for* $i,j \in \{2,3\}$,

$$W_{23}^i(\mathbf{k}, trade) = \begin{cases} \tilde{\Pi}(sb,1) - B^j(\mathbf{k}) & \text{if } b \gg 1 \text{ and } u_1^* < \delta\rho_{1b}S_{1b}(z_b), \\ S_{23}(z_1) - B^j(\mathbf{k}) & \text{otherwise.} \end{cases}$$

where $\tilde{\Pi}(sb,\ell) \equiv \delta[W^s(\mathbf{k}'_{b\ell}) + W^b(\mathbf{k}'_{b\ell})] = \delta[u_s(z_b) + \rho_{b\ell}S_{b\ell}(z_b)]$ *is the aggregate discounted value of a buyer* b *and a seller* s *when* b *buys again from an ultimate seller* ℓ*.*

Consider for example 2's payoffs from making 3 a relevant buy offer when it is its turn to propose. When $2 \gg 1$ and $u_1^* < \delta\rho_{12}S_{12}(z_2)$, the payoff for 2 of making 3 a relevant buy offer is the joint discounted equilibrium payoff for 2 and 3 of getting 2 to buy votes from 1, $\tilde{\Pi}(32,1)$, minus 3's reservation value $B^3(\mathbf{k})$. When instead $2 \gg 1$, or $1 \gg 2$ but $u_1^* \geq \delta\rho_{12}S_{12}(z_2)$, 1 implements z_1 immediately after 2 consolidates 2 and 3's votes, so the payoff for 2 of making 3 a relevant buy offer is simply their joint surplus of inducing 1 to implement z_1 immediately, net of 3's reservation value $B^3(\mathbf{k})$.

Propositions 4.2-4.3 can be used to check the consistency of any proposed equilibrium. In Section 5 we use these results to study the existence of an equilibrium with intermediaries.

5 Main Results

In this section, we establish our main results on the existence of intermediaries in decentralized bargaining (Theorems 5.4 and 5.11). We show that under generic conditions on matching probabilities there exists a compact set of preference profiles that admit an equilibrium with intermediaries in decentralized bargaining. We then discuss the implications of this result for welfare and policy outcomes in Section 5.2, and characterize conditions on preference profiles under which equilibria with brokers can and cannot arise in Section 5.3.

In section 5.1, we establish conditions for the existence of an equilibrium with intermediaries under two types of constraints: (1) a restriction on strategies, and (2) a restriction on preferences. Together, they assure that on the equilibrium path, trade only occurs through the broker (by (1)), and that relative to the equilibrium of legislative bargaining *without* the broker, the presence of the broker changes outcomes (by (2)).

Part 1 is our definition of a *broker equilibrium* (BE). Note that because party 1 has a majority of the votes and without loss of generality $2 \gg 3$, in any equilibrium with an intermediary party 3 brokers a deal between 1 and 2 that implements 2's preferred policy. This still leaves negotiations between 1 and 2 and between 2 and 3 unspecified. To assure that the broker is not merely replicating indirectly a trade that would also occur directly, we require that in a broker equilibrium parties 1 and 2 do not trade when they meet. We also ask that the trade enabled by the broker occurs on the equilibrium path independently of the realization of meetings that occur. This requires that party 2 does not trade with 1 or 3 in decentralized bargaining, and that 1 extends negotiations after this occurs.[7]

The second constraint, which we maintain until section 5.4, is a restriction on preferences. Since one of the minority parties will dominate the other, it is without loss of generality to assume that $2 \gg 3$. In Sections 5.1 - 5.3, moreover, we also assume that $1 \gg 2$ and $1 \gg 3$. These conditions imply that in bilateral bargaining with 2 or 3, party 1 would implement its preferred policy without trading (see Proposition 4.1). Thus, whenever it exists, a broker enables a trade that would otherwise not occur, causing a change in policy

[7]We emphasize here that establishing the existence of a broker equilibrium therefore establishes the existence of an equilibrium with intermediaries when we do not include these additional requirements on equilibrium behavior. Thus, the set of parameters under which a broker equilibrium can be supported is a subset of the set of parameters under which any equilibrium with intermediaries can be supported.

outcomes. In Section 5.4 we extend Theorem 5.4 to all dominance relations.

5.1 Intermediaries in Legislative Bargaining

Implementing a BE in decentralized trading introduces equilibrium incentive constraints. Two are straightforward, and independent of the dominance relations among parties. First, we require that party 1 extends negotiations after UT in decentralized trading (after not trading with 2, or after 2 and 3 do not trade)

$$u_1^* \leq \delta W^1(\mathbf{k}) \tag{2}$$

Second, we need 3 to extend negotiations after buying votes from 1, in order to broker a deal with 2. As we discussed in Section 4.1 above, this is in fact the same strategic problem faced by party 3 after UT in the majority-minority bargaining game with 2 (with $M = 3$ and $m = 2$). Thus after 3 acquires the majority from 1 in decentralized trading, it will extend negotiations if and only if $2 \gg 3$ (as we are assuming throughout) *and* [8]

$$u_3^* \leq \delta \rho_{32} S_{23}(z_2) \tag{3}$$

The remaining constraints come from parties best responses in each decision node. First, we require that whenever 1 and 3 meet, 1 sells its votes to 3. Given $2 \gg 3$ and $u_3^* \leq \delta \rho_{32} S_{23}(z_2)$, we know that if 3 were to buy 1's votes, it would go on to broker a deal with 2. Thus from Proposition 4.2, $W_{13}^1(\mathbf{k}; \mathit{sell}) = \Pi(13, 2) - \delta W^3(\mathbf{k})$. If instead 1 were to buy 3's votes, $W_{13}^1(\mathbf{k}; \mathit{buy}) = S_{13}(z_1) - \delta W^3(\mathbf{k})$, because with $1 \gg 2$ there are no gains from trade between 1 and 2 in majority-minority bargaining. Thus 1 prefers selling to buying iff

$$\Pi(13, 2) \geq S_{13}(z_1), \tag{4}$$

and prefers selling to extending negotiations if and only if

$$\Pi(13, 2) \geq \delta[W^1(\mathbf{k}) + W^3(\mathbf{k})]. \tag{5}$$

[8] In some cases the party who carries or acquires a majority in decentralized trading will be indifferent between extending negotiations and implementing its preferred policy. To assure the robustness of our results, we brake this indifference against an equilibrium with brokers, and assume hereafter that in any such case the party with a majority of the votes implements its preferred policy.

Second, when 1 and 2 meet, we ask that they do not trade. Now, given that $1 \gg 3$ and $2 \gg 3$, neither 1 nor 2 has a further gain from trade with party 3 in bilateral trading, and thus from Proposition 4.2, $W_{12}^1(\mathbf{k}; buy) = S_{12}(z_1) - \delta W^2(\mathbf{k})$ and $W_{12}^1(\mathbf{k}; sell) = S_{12}(z_2) - \delta W^2(\mathbf{k})$. And since $1 \gg 2$, it follows that 1 would rather buy from 2 than sell to 2. Moreover, 1 prefers extending negotiations than making 2 a relevant buy offer if and only if 1 and 2's aggregate discounted continuation value is larger than their joint payoff of implementing z_1, i.e.,

$$S_{12}(z_1) \leq \delta[W^1(\mathbf{k}) + W^2(\mathbf{k})]. \tag{6}$$

Finally, we also need that parties 2 and 3 do not reach an agreement to trade when they meet. But given $1 \gg 3$ and $1 \gg 2$, the analysis is very similar to the one above, for an uneventful meeting between 1 and 2. Here *party 1* implements its preferred policy immediately after 2 sells to 3 *or* 3 sells to 2, but extends negotiations if 2 and 3 fail to reach an agreement (Proposition 4.3). Then $W_{23}^2(\mathbf{k}; sell) = W_{23}^2(\mathbf{k}; buy) = S_{23}(z_1) - \delta W^3(\mathbf{k})$, and 2 prefers not to make a relevant offer in $((2,3), \mathbf{k})$ than to sell or buy from 3 if and only if

$$S_{23}(z_1) \leq \delta[W^2(\mathbf{k}) + W^3(\mathbf{k})]. \tag{7}$$

Conditions (2)-(7) are necessary and sufficient for a BE, given the continuation values. Continuation values, in turn, are determined by equilibrium strategies, independently of the dominance relation or incentive compatibility constraints. They can be easily computed for an equilibrium with brokers.[9]

Lemma 5.1 *Consider a MPE in which party 3 brokers a deal between 1 and 2. Then*

$$W^2(\mathbf{k}) = \frac{\delta(p_{13} + p_{31})\rho_{23} S_{23}(z_2)}{(1-\delta) + \delta(p_{13} + p_{31})},$$

$$W^1(\mathbf{k}) = \frac{p_{13}\Pi(13,2)}{(1-\delta) + \delta(p_{13} + p_{31})} \quad \textit{and} \quad W^3(\mathbf{k}) = \frac{p_{31}\Pi(13,2)}{(1-\delta) + \delta(p_{13} + p_{31})}$$

Substituting the values from Lemma 5.1 in the equilibrium conditions (2)-(7), we have the following result. For this, and the remainder of the paper, it is useful to define

$$\upsilon \equiv (1-\delta) + \delta(p_{13} + p_{31}), \quad \theta \equiv p_{13} + p_{31}\rho_{23}, \quad \mu \equiv p_{31} + p_{13}\rho_{23}$$

[9] The reader might notice that $W^1(\mathbf{k})/(W^1(\mathbf{k}) + W^3(\mathbf{k})) = p_{13}/(p_{13} + p_{31})$, which is equivalent to the equilibrium payoffs of the unique equilibrium in the standard bilateral bargaining game.

Lemma 5.2 *Suppose* $(1 \gg 2 \gg 3, 1 \gg 3)$*. There exists a BE if and only if there are payoffs* $u_i(z_j) \in \mathbb{R}$ *for all* $i, j \in N$ *such that the following system of linear inequalities is satisfied:*

$$\upsilon u_1^* - \delta^2 p_{13} u_1(z_2) - \delta^2 p_{13}\rho_{32} u_2^* - \delta^2 p_{13}\rho_{32} u_3(z_2) \le 0 \quad \text{(2b)}$$

$$-\delta\rho_{32} u_2^* - \delta\rho_{32} u_3(z_2) + u_3^* \le 0 \quad \text{(3b)}$$

$$u_1^* + u_3(z_1) - \delta u_1(z_2) - \delta\rho_{32} u_2^* - \delta\rho_{32} u_3(z_2) \le 0 \quad \text{(4b)}$$

$$-u_1(z_2) - \rho_{32} u_2^* - \rho_{32} u_3(z_2) \le 0 \quad \text{(5b)}$$

$$\upsilon u_1^* + \upsilon u_2(z_1) - \delta^2 p_{13} u_1(z_2) - \delta^2 \theta u_2^* - \delta^2 \theta u_3(z_2) \le 0 \quad \text{(6b)}$$

$$\upsilon u_2(z_1) + \upsilon u_3(z_1) - \delta^2 p_{31} u_1(z_2) - \delta^2 \mu u_2^* - \delta^2 \mu u_3(z_2) \le 0, \quad \text{(7b)}$$

Since this is a homogeneous system, we know that it has a solution when all parties are indifferent between all alternatives, i.e., $u_i(z_j) = 0$ for all $i, j \in N$. This situation, however, is not too interesting. We want to know if there can be a BE when each party has a strict preference for its own ideal policy. Formally, we ask that $u \in \mathcal{U}$, where

$$\mathcal{U} \equiv \{u \in \mathbb{R}^9 : -u_i^* < 0, -u_i^* + u_i(z_j) < 0 \; \forall i = 1, 2, 3, j \neq i\}$$

The equilibrium conditions (2)-(7) together with the requirement that $u \in \mathcal{U}$, and the dominance relations $(1 \gg 2 \gg 3, 1 \gg 3)$, still form a system of linear inequalities in the unknowns $u_i(z_j)$, which can be written as $Au \le \alpha$, for a matrix of coefficients A, where

$$u^T = \begin{pmatrix} u_1^* & u_2(z_1) & u_3(z_1) & u_1(z_2) & u_2^* & u_3(z_2) & u_1(z_3) & u_2(z_3) & u_3^* \end{pmatrix},$$

$\alpha^T \equiv (\mathbf{0}_9, -\mathbf{b}_9)$, and A is an $m \times 9$ matrix, whose elements are functions of the *matching parameters* $G \equiv (p, \rho, \delta) \in \mathcal{G} \equiv \Delta^6 \times (\Delta^2)^3 \times [0, 1]$ (here Δ^n is the unit n-simplex). Thus, proving that there exists $u \in \mathcal{U}$ that admits a BE boils down to proving that the system of linear inequalities $Au \le \alpha$ has a solution. At this point, the following result is useful:

Lemma 5.3 (Rockafellar (1970), Theorem 22.1) *Let* $a_i \in \mathbf{R}^n$*, with elements* $a_{i1}, \ldots, a_{in}$ *and* $\alpha_i \in \mathbf{R}$ *for* $i = 1, \ldots, m$*. Then one and only one of the following alternatives holds:*

1. *There exists a vector* $u \in \mathbf{R}^n$*, with elements* $u_1, \ldots, u_n$*, such that*

$$\sum_{j=1}^{n} a_{ij} u_j \le \alpha_i \quad \text{for all } i = 1, \ldots, m$$

2. *There exist non-negative real numbers $\lambda_1, \ldots, \lambda_m$ such that*

$$\sum_{i=1}^{m} \lambda_i a_{ij} = 0 \quad \textit{for all } j = 1, \ldots, n \textit{ and} \quad \sum_{i=1}^{m} \lambda_i \alpha_i < 0.$$

Using Lemma 5.3, we can prove our first main result. We show that there exist preference profiles that admit a BE if and only if

$$(1 - \delta\rho_{32})\upsilon \leq \delta^2\theta, \tag{8}$$

Furthermore, the result is *generic*, in the sense that small changes in the parameters of the game do not affect the existence of a broker equilibrium. To state this result we define $\mathcal{P} \equiv \{G \in \mathcal{G} : \text{condition (8) holds}\}$.

Theorem 5.4 *Suppose* $(1 \gg 2 \gg 3, 1 \gg 3)$, *and* $k_1 \geq r$.

1. *Take $G \in \mathcal{G}$. There exists a compact set $C_G \subset \mathcal{U}$ such that for any $u \in C_G$, the legislative bargaining game with parameters (G, u) admits a broker equilibrium if and only if $G \in \mathcal{P}$.*

2. *For every $G \in int\mathcal{P}$ there is an open subset $P \subset \mathcal{P}$ containing G, and an open subset $U \subset \mathcal{U}$, such that for any $(G', u) \in P \times U$, (G', u) admits a broker equilibrium.*

3. *Whenever a broker equilibrium exists, it is the unique equilibrium in which z_2 is the policy outcome, and there is no equilibrium in which z_3 is the policy outcome.*

Note that increasing ρ_{32} relaxes (8), and thus expands the conditions under which there is a BE. The intuition is as follows. Increasing the likelihood that the broker has agenda setting power in majority-minority bargaining has the direct effect of increasing its bargaining power, and therefore the share of the surplus it can obtain when negotiating with the ultimate buyer. As a result, party 3 is now more inclined to negotiate with party 2 instead of implementing its preferred policy after obtaining 1's votes, in line with equilibrium.[10] Increasing p_{13} or p_{31}, on the other hand, has both a distributive and an income effect.

[10] Increasing ρ_{32} also has indirect effects on bargaining incentives in decentralized trading. In particular, since 3 is now more able to extract surplus from 2, reaching the majority minority stage is not as desirable for party 2 (7b-8b). Condition (8) therefore says that the tightening of the constraints (7b) and (8b) never overpowers the loosening of the constraints (3-6).

While raising p_{13} (p_{31}) directly increases the effectiveness of the brokerage and equilibrium payoffs, it also improves (resp, deteriorates) the bargaining position of the majority party vis a vis the broker. Thus, changes in p_{13} and p_{31} can in general increase or reduce the space for brokers. Simple calculus shows, however, that increasing p_{31} tightens condition (8) and increasing p_{13} relaxes (8) whenever this is satisfied. Thus, if for some configuration of parameters the set of preference profiles admitting a BE is nonempty, this will also be the case after reducing p_{31} or increasing p_{13}.

Reducing bargaining frictions, on the other hand, unambiguously expands the conditions under which there is a BE. In fact, increasing δ not only relaxes condition (8) but also weakly relaxes each of the equilibrium constraints (2)-(7). It follows that reducing bargaining frictions increases the set of preference profiles for which there is a BE. In addition, since condition (8) is always satisfied in the limit as $\delta \to 1$, we have that as bargaining frictions vanish, there is always a preference profile for which there is a BE. This shows that the existence of broker equilibria does not require that 1 and 2 have few opportunities to trade.

5.2 Outcomes and Welfare

As we discussed earlier, the dominance relation ($1 \gg 2 \gg 3, 1 \gg 3$) is particularly interesting because in this case party 1 would not trade with 2 if party 3 were not present, and would not trade with 3 if 2 were not present. Thus, the presence of the broker unambiguously changes policy outcomes vis a vis a two party legislature. Furthermore, we have constructed BE so that 1 and 2 do not trade directly when they meet. Thus *in this equilibrium* the broker is creating a trade that would not occur without him. In addition, the second part of Theorem 5.4 establishes that *there is no equilibrium* in which 1 and 2 trade directly. We conclude that the broker is creating a trade that would not have occurred without it, in this or any other equilibrium.

The fact that brokers enable transactions that wouldn't have occurred in their absence does not imply that parties 1 and 2 are better off with than without brokers. Note that the values of the game in which only parties 1 and 2 are present are given by $\hat{W}^1(\mathbf{k}) = u_1^*$ and $\hat{W}^2(\mathbf{k}) = u_2(z_1)$. Since in an equilibrium with brokers $u_1^* < \delta W^1(\mathbf{k})$, it is immediate to verify that the majority party benefits from the existence of the broker. However, from

Lemma 5.1,

$$\Delta W^2(\mathbf{k}) = \frac{\delta\,(p_{13} + p_{31})\,\rho_{23} S_{23}(z_2)}{[(1-\delta) + \delta(p_{13} + p_{31})]} - u_2\,(z_1)\,,$$

which in general can be positive or negative. Thus the ultimate buyer might prefer that no trades were set in motion in the first place.

In fact, the broker equilibrium is generally not efficient, even as frictions vanish. This follows immediately with $\delta < 1$ because while efficiency requires that every meeting ends in agreement, in equilibrium the majority party 1 and the ultimate buyer 2 only trade through the broker (and do not trade directly when they meet). Now, in the limit with $\delta \to 1$, it is still possible that equilibrium payoffs approach efficiency. We show however that this is not the case generically.[11] Note that from Lemma 5.1, parties' aggregate welfare in an equilibrium with brokers is given by

$$\sum W^i(\mathbf{k}) = \frac{\delta(p_{13} + p_{31})[u_1(z_2) + u_2^* + u_3(z_2)]}{(1-\delta) + \delta(p_{13} + p_{31})}$$

so that $\sum W^i(\mathbf{k}) \to u_1(z_2) + u_2^* + u_3(z_2)$ as $\delta \to 1$. It is then enough to show that there is a preference profile $u \in \mathcal{U}$ with the property that $\sum_i u_i(z_2) < \sum_i u_i(z_3)$ admitting an equilibrium with brokers with $\delta \to 1$. It can be verified that this happens for example with the preferences of Table 1, given uniform matching.

Party/Policy	z_1	z_2	z_3	Q
1	10	-485	0	0
2	505	1000	990	0
3	10	10	20	0

Table 1: A Preference Profile admitting an Inefficient Equilibrium with Brokers with uniform matching and $\delta \to 1$.

On the other hand, broker equilibria are always *welfare improving* (for all δ):

[11] This result contrasts the second result in Gul (1989), which establishes efficiency for $\delta \to 1$. It should be pointed out, however, that the two games have important differences. First, Gul requires strict superadditivity on the payoffs, a property that our majority game doesn't satisfy (it is superadditive but not strictly superadditive). Second, Gul's efficiency result requires superadditivity of the continuation values (his condition VA). This property is not satisfied in our game either. As a result, efficiency is not guaranteed, even in the limit as $\delta \to 1$.

Remark 5.5 *Suppose $k_1 \geq r$ and $(1 \gg 2 \gg 3, 2 \gg 3)$. In any equilibrium with brokers,*

$$\sum_i [W^i(\mathbf{k}) - u_i(z_1)] > 0$$

5.3 The Role of the Broker

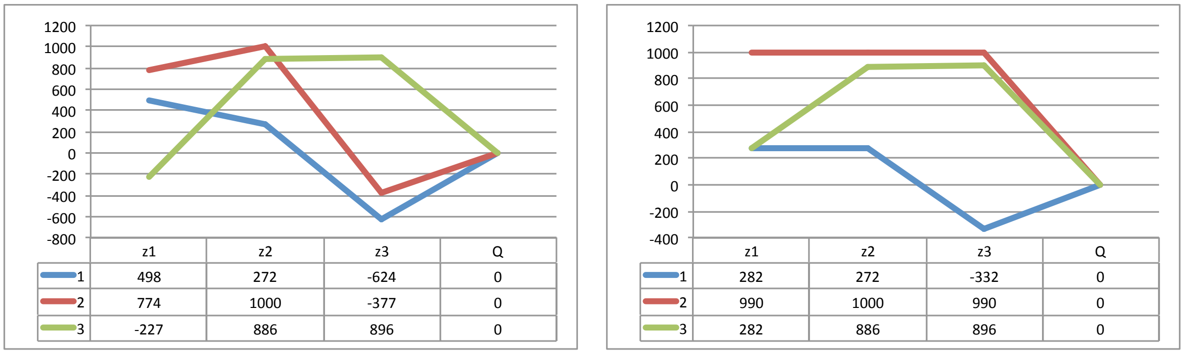
Theorem 5.4 shows that under a broad set of conditions, there exists an equilibrium with brokers. The theorem, however, says nothing about the preference profiles under which brokers can emerge in equilibrium. Thus, while we know that when (8) is satisfied the set of preference profiles that admit an equilibrium with brokers is nonempty (Theorem 5.4), it is still possible that these preference profiles are in some sense 'exceptional', and not likely to arise in applications. We show that this is not the case. We also establish properties of the broker and the environment under which brokers emerge in equilibrium.[12]

We begin with an example. Figure 1 illustrates two preference profiles $u, u' \in \mathcal{U}$ for which there is an equilibrium with brokers (for the examples we assume $\delta = 0.95$ and uniform matching probabilities). These examples are interesting because they illustrate very intuitively the two ways in which an equilibrium with brokers comes about.

In the profile plotted in the left panel, u, the party of the winning policy is willing to pay a 'large' amount to move away from the status quo. Moreover, party 3 is both (i) willing to trade with 2 in bilateral bargaining, and (ii) able to credibly use the status quo as an outside option in these negotiations. Thus, by using 3 as a broker, 1 and 3 can *lock in* party 2 in a situation that allows them to extract these rents. In the profile u' depicted in the right panel, instead, there is an equilibrium with brokers even if party 2 doesn't have strong preferences regarding z_3, or about the status quo. What is financing the trade in this case is the broker's willingness to avoid her least preferred alternative in exchange for her second best.

The contrast between these two cases poses an interesting question: can a party act as a broker if it only cares about rents? Our next result shows that this cannot be the case:

[12] To obtain these results, we further exploit the duality results from convex analysis, transforming restrictions on preference profiles into a modified matrix A' of matching parameters, and obtaining conditions for existence of a solution to the underlying system of inequalities following the same steps as in Theorem 5.4. The various results in this section, therefore, illustrate the power of the technique. The particular algebraic derivations in each case, however, are not interesting per se, and are relegated to an online appendix.



u $\qquad\qquad\qquad\qquad$ u'

Figure 1: Two preference profiles $u, u' \in \mathcal{U}$ s.t. $1 \gg 2 \gg 3$ and $1 \gg 3$ for which there is an equilibrium with brokers (uniform matching probs and $\delta = 0.95$).

in order to have an equilibrium with brokers, the party acting as broker *must* have a stake in the policy outcome. In other words, for all $u \in \mathcal{U}$ such that $u_3(z_j) = u_3(Q) = 0$ for $j = 1, 2, 3$, and any matching parameters G, the game (G, u) does not admit an equilibrium with brokers. In fact, the broker must strictly prefer z_2 to both the status quo and the preferred policy of the party whose votes it buys in decentralized trading.

Proposition 5.6 *Suppose* $(1 \gg 2 \gg 3, 1 \gg 3)$*, and* $k_1 \geq r$*. If the (potential) broker weakly prefers either the status quo or the policy of the initial majority to that of the ultimate buyer* ($u_3(z_2) \leq u_3(Q)$*, or* $u_3(z_2) \leq u_3(z_1)$)*, there does not exist a broker equilibrium.*

The first order implication of Proposition 5.6 is that the party acting as broker must have a stake in the policy outcome. The result, however, goes well beyond this. Since the broker must prefer implementing z_2 to the status quo, it increases the aggregate surplus of implementing z_2 *relative to inaction*. In addition, Proposition 5.6 shows that the broker must also prefer implementing z_2 to z_1. As a result, whenever there is a broker equilibrium, the broker must also increase the aggregate surplus of implementing z_2 *relative to the majority policy*. The next remark shows that this additional value is at least partly appropriated by the majority party, ultimately inducing 1 and 2 not to trade directly.

Remark 5.7 *Suppose* $(1 \gg 2 \gg 3, 1 \gg 3)$*, and* $k_1 \geq r$*. In a broker equilibrium, the net expected transfer to party* 3 *is negative.*

One should not conclude from this that the broker is the only source of compensation to the majority party. In fact, the strategic environment must be such that the broker can extract sufficient rents from the ultimate buyer, putting in motion a *chain* of rent extraction. To illustrate this, we show that in a binary policy environment ($X = \{Q, A\}$), as in Philipson and Snyder (1996) and Casella, Llorente-Saguer, and Palfrey (2012), a broker equilibrium cannot exist. In fact, the model with three alternatives is the minimal structure for which brokers can arise.[13] This minimal diversity of the space of alternatives is in fact needed for the broker to be able to extract enough rents from the ultimate buyer.

Proposition 5.8 *Suppose* $(1 \gg 2 \gg 3, 1 \gg 3)$*,* $k_1 \geq r$*. If there is an equilibrium with brokers, then for any* $i, j \in N$*,* $z_i \neq z_j$*. Therefore* $|X| \geq 3$*. Moreover, if* $\rho_{23} < \delta/(1+\delta)$*, then* $z_i \neq Q$ $\forall i \in N$*, and* $|X| = 4$*.*

To see why this is the case, note that by Proposition 5.6, party 3 must strictly prefer z_2 to Q and z_1. Thus, if there is a broker equilibrium in the binary policy setting, it must be that $z_3 = z_2 \neq Q = z_1$. But then since $z_3 = z_2$, party 3 would be unable to extract sufficient rents from 2 in bilateral bargaining to dissuade him from implementing his preferred policy immediately after buying 1's votes. In addition, Proposition 5.8 also establishes that if $\rho_{23} < \delta/(1+\delta)$, all policies must be different than the status quo, and thus $|X| \geq 4$. This implies, in particular, that when ρ_{23} is small enough there cannot exist a broker equilibrium if the status quo already reflects the preferred position of the majority party.[14]

Proposition 5.8 shows that a minimal diversity in the space of alternatives is necessary to create a bargaining situation in which the broker can profitably carry out the deal. This

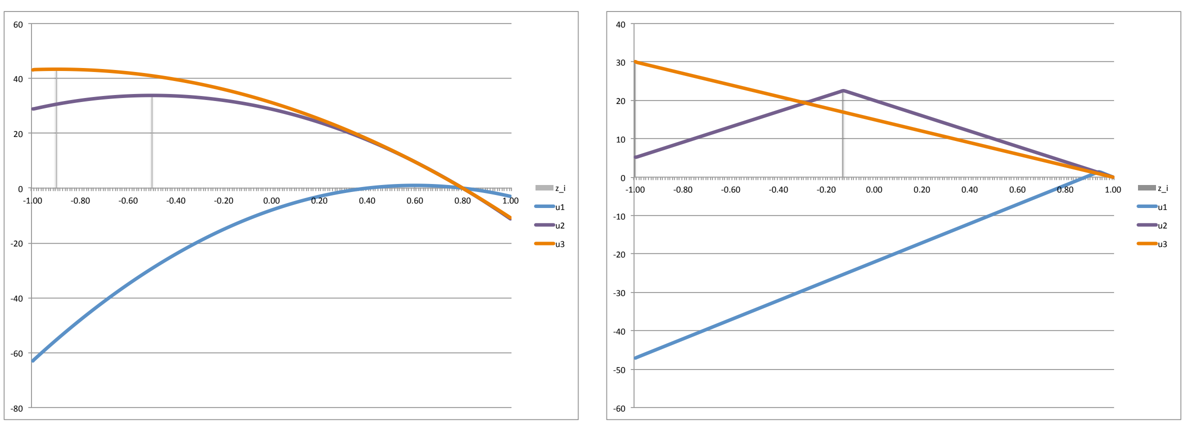
[13] Proposition 5.6 implies that at least the broker must perceive a difference between z_2 and z_1. However, this still allows $z_1 = Q$, and is silent about z_3 and z_2, because it *assumes* that $u_3^* > u_3(z_2)$.

[14] The intuition for this result hinges on the incentives for 2 not to make a relevant offer to 1 or 3 in decentralized bargaining. Because when ρ_{23} is small, party 2's expected surplus out of the majority-minority bargaining stage with 3 is small as well, party 2 is willing to pay more to (or demand less from) parties 1 and 3 in decentralized bargaining in order to implement z_1 immediately. Thus, in order for 2 to be willing to comply with the equilibrium path of play, the cost for 2 of buying from 1 or 3 when ρ_{23} becomes smaller should increase. Attaining this without affecting other incentive constraints requires reducing $u_2(z_1) - u_2(Q)$, but this presupposes that $z_1 \neq Q$.

suggests the question of whether conventional restrictions on preferences profiles would make the existence of brokers impossible. Does the existence of brokers relies in some way on *pathological* preference profiles? A way to implement this question is to study whether brokers can exist with a standard notion of 'well-behaved' preferences, as for example the class of single-peaked preference profiles, $\mathcal{U}^{SP}$. This is what we do in the next result.

Proposition 5.9 *Suppose* $(1 \gg 2 \gg 3, 1 \gg 3)$, *and* $k_1 \geq r$. *Then if* $(1-\delta\rho_{32})\upsilon < \min\{\delta^2\theta, \delta^2\mu\}$, *there exists a compact set* $C \subset \mathcal{U}^{SP}$ *such that all* $u \in C$ *admit a broker equilibrium.*

In fact, linear and quadratic payoffs are also admissible. Figure 5.3 illustrates this with two examples. Given the normalization $u_i(Q) = 0$, i's payoff function is $u_i(x) = -\beta_i(x - z_i)^2 + \beta_i(Q - z_i)^2$ in the case of the quadratic utility function, and similarly for the linear payoffs. Note that $i \gg j$ iff $\beta_i > \beta_j$. Thus, in these examples, $\beta_2 > \beta_3$. This must be the case, because the ultimate buyer has to dominate the broker so that there is a final transaction. In addition, per Proposition 5.6, $u_3(z_2) \geq u_3(z_1)$ and $u_3(z_2) \geq u_3(Q)$, which in this context implies that the broker must be closer to z_2 than to z_1 or the status quo. Thus, if for example $z_2 < z_1 < Q$ as in the figure, the broker cannot be more right-winged than the majority party.



u $\qquad\qquad$ u'

Figure 2: Quadratic and linear utility functions admitting an equilibrium with brokers. Here $\rho_{23} = 0.2$, and p is uniform. $\delta = 0.95$. $\beta_1 = 25, \beta_2 = 20, \beta_3 = 15$.

5.4 Broker Equilibria in All Dominance Relations

As we stated in Section 5.1, the trade-offs that parties face when negotiating with one another in decentralized bargaining depend on their equilibrium beliefs about the outcome of negotiations following each possible trade, on and off the path of play. And while a BE is uniquely determined in the benchmark dominance relation, in general multiple continuations are possible off the equilibrium path for different preferences and matching parameters. Using Propositions 4.2 and 4.3, our next result characterizes all broker equilibria. The values $W^i(\mathbf{k})$ are still determined by Lemma 5.1.

Proposition 5.10 *Suppose $k_1 \geq r$. Suppose furthermore that $u_3^* \leq \delta\rho_{32}S_{23}(z_2)$ and $u_1^* \leq \delta W^1(\mathbf{k})$. The following conditions characterize all broker equilibria.*

1. *3 buys 1's votes iff $\Pi(13,2) \geq \delta[W^1(\mathbf{k}) + W^3(\mathbf{k})]$ and either (i) $\Pi(13,2) \geq \Pi(31,2)$, if [$2 \gg 1$ and $u_1^* \leq \delta\rho_{12}S_{12}(z_2)$], or (ii) $\Pi(13,2) \geq S_{13}(z_1)$ otherwise.*
2. *If 1 extends negotiations after buying from 2 to sell its votes to 3 ($3 \gg 1$ and $u_1^* < \delta\rho_{13}S_{13}(z_3)$), parties 1 and 2 would not trade after meeting iff either $\Pi(21,3) \leq S_{12}(z_2) \leq \delta[W^1(\mathbf{k}) + W^2(\mathbf{k})]$ or $S_{12}(z_2) \leq \Pi(21,3) \leq \delta[W^1(\mathbf{k}) + W^2(\mathbf{k})]$. If party 1 implements z_1 after buying from 2, iff $\delta[W^1(\mathbf{k}) + W^2(\mathbf{k})] \geq \max\{S_{12}(z_1), S_{12}(z_2)\}$.*
3. *Let $i, j, \in \{2,3\}$, $i \neq j$. If 1 extends negotiations after i sells to j (if $j \gg 1$ and $u_1^* < \delta\rho_{1j}S_{1j}(z_j)$), parties 2 and 3 would not trade with one another iff $\tilde{\Pi}(ij,1) \leq \delta[W^2(\mathbf{k}) + W^3(\mathbf{k})]$. If instead 1 implements z_1 after i sells to j, iff $S_{ij}(z_1) \leq \delta[W^2(\mathbf{k}) + W^3(\mathbf{k})]$.*

Note that equilibrium conditions (4)-(7) in section 5.1 come as a direct corollary of this result: given $1 \gg 2$, part 1 implies (4) and (5), given $1 \gg 3$, part 2 implies (6), and given $1 \gg 2 \gg 3$, part 3 implies (7). Note also that for *this* dominance relation, all equilibrium continuations are uniquely determined. As a result, substituting the values from Lemma 5.1, the system (2)-(7) completely characterizes the set of parameters for which a broker equilibrium exists.

In general, however, off-path continuations can vary for different parameters, even fixing the dominance relation.[15] As a result, changing parameters can change the equilibrium

[15] To illustrate this, consider a meeting between parties 1 and 3. In a broker equilibrium, we require that 1 sells its votes to 3, who then brokers a deal with party 2. But suppose 1 deviates, and makes 3 a

strategy profile in a BE. While this can appear daunting, in fact the set of all possible BE can be described by the combination of the conditions in Proposition 5.10, together with the conditions $u_3^* \leq \delta\rho_{32}S_{23}(z_2)$ and $u_1^* \leq \delta W^1(\mathbf{k})$. Using Proposition 5.10 we can establish our second main result. This extends Theorem 5.4 to all dominace relations.

Theorem 5.11 *Suppose $k_1 \geq r$. Then for any given dominance relation there is a compact set of matching probabilities and preference profiles that admit an equilibrium with brokers.*

To prove the result, we first show that if there is a preference profile that admits brokers when $(1 \gg 2 \gg 3, 1 \gg 3)$, there is one that admits brokers with $(1 \gg 2 \gg 3, 3 \gg 1)$ (Proposition 9.2). Thus, the sufficient condition for brokers in Theorem 5.4 is also sufficient for brokers whenever $1 \gg 2 \gg 3$. We then extend the existence result to the remaining dominance relations, and provide alternative conditions under which a similar result holds when $(2 \gg 3 \gg 1, 2 \gg 1)$ (Proposition 9.3) and $(2 \gg 1 \gg 3, 2 \gg 3)$ (Proposition 9.4). These proofs follow the logic of Theorem 5.4, and are therefore relegated to the online appendix.

We had shown before, in Theorem 5.4, that under generic conditions on the matching parameters (eq. (8)), there are preference profiles that admit a BE. Theorem 5.11 shows that this is true for *any given* dominance relation. It follows that while the prevailing dominance relation among parties can shape the characteristics of BE, and the parameters under which any given broker strategy profile can be supported as an equilibrium, it doesn't determine the existence of BE in the first place.

6 Extensions and Applications

In the analysis so far, we focused on a majoritarian legislature where one of the parties is initially endowed with a majority of the votes. The logic for the existence of brokers, however, applies in a wider set of circumstances. In this section we consider some of these

relevant buy offer. If $1 \gg 2$, as in the benchmark dominance, there are no gains from trade between 1 and 2, and 1 would implement its preferred policy after buying from 3. But if $2 \gg 1$ *and* $u_1^* \leq \delta\rho_{12}S_{12}(z_2)$, 1 would sell to 2 in bilateral bargaining. Because 1 would have a larger "buy" payoff in the second case, it is harder to dissuade him not to trade with 3 in the first place.

extensions. We begin by considering a majoritarian legislature in which no party initially has a majority of the votes. We then consider a committee operating under unanimity rule, such as the Council of the European Union. Finally, we consider the possibility that interest groups (actors not formally endowed with voting power) act as brokers. This includes the President as a special case, if the President doesn't have veto power. We then consider the case of a President with veto. Throughout, we focus on the benchmark dominance relation.

6.1 A Majoritarian Legislature with No Majority Party

Up to now we focused on the case in which one of the parties in the legislature has a majority of the votes. However, in 45% of the seat distributions in presidential democracies and 57% of seat distributions in parliamentary democracies, no party controlled a majority of seats in the legislature (Cheibub, Przeworski, and Saiegh (2004)). In these cases, either minority parties form relatively stable policy coalitions, or policy compromises are attained on a case-by-case basis.

Surprisingly enough, the absence of a majority party doesn't seem to affect *legislative success*, at least as measured by the proportion of government bills turned into law. In fact, Cheibub, Przeworski, and Saiegh (2004) report that single-party minority governments are at least as successful as legislative coalitions (even majority coalitions). This points to successful bargaining among parties even in minority situations. In this section we investigate within our model whether the absence of a party holding the majority of votes affect the conditions making the emergence of brokers possible. We focus again for simplicity on the benchmark dominance relation ($1 \gg 2 \gg 3, 1 \gg 3$).

There are two fundamental differences in the incentive compatibility constraints needed to support a given profile of trades between the two settings. The first is *whether* negotiations are extended by default after UT (no majority party) or whether this is endogenous (a majority party exists). The second is *which party* decides to extend negotiations or implement its preferred policy after each particular trade. When party 1 has a majority, any trade involving party 1 resolves in the buyer having a majority of the votes, but any trade between the two minority parties leaves the majority unchanged. When no party has a majority, on the other hand, any trade between any two parties resolves in the buyer having a majority of the votes, independently of the initial distribution of voting rights.[16]

It follows that any node involving the majority party 1 remains unchanged, except for the fact that we do not need to induce party 1 to extend negotiations in UT after meeting with party 2 (we lose the constraint (2)). The fundamental difference with the majority party case is that the party that emerges with the majority after trade in the node $((2,3), \mathbf{k})$ can choose between extending negotiations or implementing his preferred policy right away, just as the buyer in a 1-2 or a 1-3 meeting in the majority party case. As a result, the decisions of the proposer in 2-3 trading are now also governed by the payoffs in Lemma 4.2. In effect, Lemma 4.2 now applies symmetrically to all matchings. Moreover, because no party has a majority in UT, then $W^i_{ij}(\mathbf{k}, wait) = B^i(\mathbf{k}) = \delta W^i(\mathbf{k})$ for $i = 2, 3$. Thus when 2 and 3 are matched and trade, with $i, j \in \{2, 3\}$ and denoting the seller by $s \in \{2, 3\}$ and the buyer by $b \in \{2, 3\}$,

$$W^i_{ij}(\mathbf{k}, trade) = \begin{cases} \Pi(sb, 1) - \delta W^j(\mathbf{k}) & \text{if } 1 \gg b \text{ and } u^*_b < \delta \rho_{b1} S_{b1}(z_1), \\ S_{ij}(z_b) - \delta W^j(\mathbf{k}) & \text{otherwise,} \end{cases}$$

Using the previous result, and a logic similar to the one in Theorem 5.4, Proposition 6.1 provides sufficient conditions for the existence of brokers when no party holds a majority.

Proposition 6.1 *Suppose* $(1 \gg 2 \gg 3, 1 \gg 3)$ *and no party has a majority of the votes. Then there exists a set* $(U, P) \subset \mathcal{U} \times \mathcal{G}$*, with nonempty interior, such that if* $(u, G) \in (U, P)$*, there exists a broker equilibrium.*

While a full analysis of this case is beyond the scope of this paper, Proposition 6.1 establishes that the presence of a majority party is not necessary for the emergence of brokers in equilibrium.

6.2 Unanimity Rule

Here we consider a committee operating under unanimity rule, such as the Council of the European Union. Consider then a *small Europe*, with Germany (1), Spain (2) and France

[16] There is a third difference once we lift the focus off a particular profile of trades. Even keeping the dominance relations fixed, there are now multiple parties that can act as brokers. Here we focus on the same brokerage as before, in which 3 brokers a deal between 1 and 2 to implement z_2, for this is the only equilibrium trade that can implement z_2.

(3) represented in the Council. Suppose Spain favors a fiscal bailout (z_2), Germany a fiscal tightening (z_1), and France monetary easing (z_3). Assume as before that ($1 \gg 2 \gg 3, 1 \gg 3$). This implies that Germany faces the steepest policy loss vis a vis Spain and France, and that Spain cares more about the difference between bailout and monetary easing than France. Can France broker a deal between Spain and Germany to carry out a bailout?

The analysis of brokers in unanimity differs from our analysis under majority rule in that no party can implement its preferred policy after UT in either decentralized or bilateral bargaining, or after a single trade in decentralized bargaining. This feature changes the negotiation tradeoffs at key points in the game.First, differently than in the case of a majoritarian legislature, inaction is now a possible equilibrium outcome in the "majority-minority" bargaining game. In fact, $S(z_m) \geq 0$ is necessary and sufficient for gridlock not to occur. We state this result without proof.

Remark 6.2 *Consider the "majority-minority" bilateral bargaining game among parties i and j under unanimity rule at time τ_0, and suppose $i \gg j$. Then $y_\tau = Q$ for all $\tau \geq \tau_0$ if $S_{ij}(z_i) \leq 0$, and $y_\tau = z_i$ for all $\tau \geq \tau_0$ if $S_{ij}(z_i) > 0$.*

It follows that for an equilibrium in which France (3) brokers a deal to bailout Spain (2) to exist, we must have $S_{23}(z_2) > 0$; i.e., France and Spain must jointly gain from bailing out Spain, relative to inaction.[17] Incentives in decentralized trading change accordingly. Consider for example negotiations between Spain (2) and France (3) in decentralized trading, and suppose for concreteness that $S_{13}(z_1) \geq 0$ and $S_{12}(z_1) < 0$. Because $S_{13}(z_1) \geq 0$, if France were to buy out Spain, it would then sell its votes to Germany in bilateral bargaining, and thus $\Pi(23, 1) = \delta[u_2(z_1) + \rho_{31} S_{31}(z_1)]$, as in the benchmark case. But because $S_{12}(z_1) < 0$ (and therefore also $S_{12}(z_2) < 0$, since $1 \gg 2$), bilateral bargaining between Germany and Spain would end in disagreement, and thus $\Pi(32, 1) = 0$. Thus, the fact that Germany and Spain cannot agree on an alternative to the status quo diminishes the incentives for France to sell its votes to Spain in decentralized trading.

With these modifications, the analysis of brokers with unanimity rule is similar to the majoritarian case. Within the benchmark dominance relation, in fact, the conditions

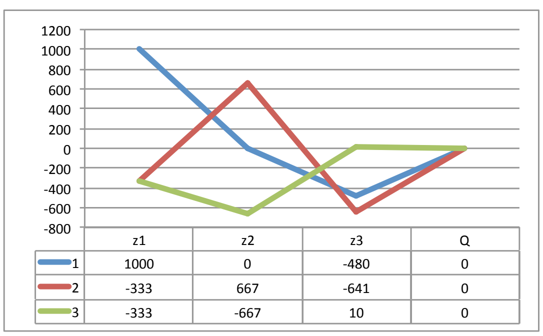
[17]In fact, given the benchmark dominance, in any equilibrium in which $y_{\tau_0} = z_2$, then $S_{23}(z_2) > 0$.

$S_{12}(z_1) \geq\leq 0$ and $S_{13}(z_1) \geq\leq 0$ define four possible cases, and Table 6.2 summarizes the relevant equilibrium conditions in decentralized trading (in addition to $S_{23}(z_2) > 0$).

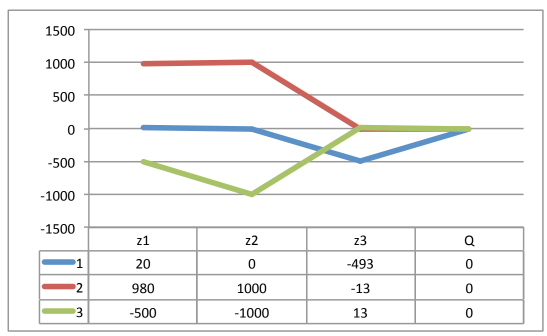
				$S_{12}(z_1) > 0$ $S_{13}(z_1) > 0$	$S_{12}(z_1) > 0$ $S_{13}(z_1) < 0$	$S_{12}(z_1) < 0$ $S_{13}(z_1) > 0$	$S_{12}(z_1) < 0$ $S_{13}(z_1) < 0$
1-3	sell $\gtrsim$ NT	$\Pi(13,2)$	$>$	$\delta(W^1+W^3)$	$\delta(W^1+W^3)$	$\delta(W^1+W^3)$	$\delta(W^1+W^3)$
	sell $\gtrsim$ buy			$\tilde{\Pi}(31,2)$	$\tilde{\Pi}(31,2)$	0	0
1-2	NT $\gtrsim$ sell	$\delta(W^1+W^2)$	$>$	$\tilde{\Pi}(12,3)$	$\tilde{\Pi}(12,3)$	$\tilde{\Pi}(12,3)$	$\tilde{\Pi}(12,3)$
	NT $\gtrsim$ buy			$\tilde{\Pi}(21,3)$	0	$\tilde{\Pi}(21,3)$	0
2-3	NT $\gtrsim$ sell	$\delta(W^2+W^3)$	$>$	$\tilde{\Pi}(23,1)$	0	$\tilde{\Pi}(23,1)$	0
	NT $\gtrsim$ buy			$\tilde{\Pi}(32,1)$	$\tilde{\Pi}(32,1)$	0	0

Table 2: Incentive Constraints in Decentralized Trading: Unanimity Rule

While a full analysis of the unanimity case is beyond the scope of this paper, two points are worth mentioning. First, as the reader might suspect, it is possible to support a broker equilibrium when $S_{12}(z_1) < 0$ and $S_{13}(z_1) < 0$, for in this case the broker is the only path to realize gains from trade. However, we can show that a result similar to Theorem 5.4 holds for all possible configurations of these bilateral surpluses.



$u : S_{12}(z_1) > 0, S_{13}(z_1) > 0$



$u' : S_{12}(z_1) > 0, S_{13}(z_1) < 0$

Figure 3: Two preference profiles admitting an equilibrium with brokers with unanimity rule, given uniform matching and $\delta = 0.95$

Figure 6.2 illustrates two examples of preference profiles that admit a broker equilibrium, when $S_{12}(z_1) > 0$ and $S_{13}(z_1) > 0$ (left) and $S_{12}(z_1) > 0$ and $S_{13}(z_1) < 0$ (right). The figure highlights two interesting results. First, note that with unanimity the broker does not have to prefer bailout to inaction, as it would in the majoritarian setting (Proposition 5.6). Moreover, the equilibrium with brokers is not necessarily welfare improving, as it is in the majoritarian setting (Remark 5.5). The examples suggest that the decision rule has important effects on the properties of equilibrium outcomes. More work is needed to fully assess the nature of these effects.

6.3 The President as a Broker

Up to this point, we have maintained the assumption that all actors in the model are members of the legislature, and thus endowed with voting rights. However, nothing in the model prevents the possibility that an outside party (say an interest group) plays the role of the broker, if allowed to participate in backroom deals. In fact, given access, an interest group is strategically equivalent to internal members, except that it cannot sell votes in the initial round of decentralized bargaining. But since in equilibrium the broker only sells votes after buying votes, our main analysis provides sufficient conditions for the existence of an equilibrium with brokers where the interest group perform the role of broker or ultimate buyer.

While this logic applies to any interest group, in the United States the role of outside broker is often played by the President, who can break an impasse between the majority and minority parties in Congress. A well documented example is that of President Clinton's involvement in the passage of the NAFTA treaty, which the Bush administration had negotiated with Canada and Mexico, and had the support of a majority of Republicans in the House. According to President Clinton (Clinton (2004), pg. 546), "Al (Gore) and I had called or seen two hundred members of Congress... We also had to make deals on a wide array of issues; the lobbying effort for NAFTA looked even more like sausage making than the budget fight had."

The analysis of the President as a broker follows immediately as a special case if we assume that the President does not have veto power. Since the President of the United States does have veto power, however, our previous example does not squarely fall within the majoritarian bargaining model. The analysis, however, can easily be adapted to include this institutional difference, building on our results for majoritarian and unanimous legislatures.[18]

[18] The analysis of the legislature with a President holding veto power falls in between the cases of the majoritarian and unanimous legislatures. The majority party 1 can form a winning coalition by negotiating with either the President or the minority party (together they can override the President's veto). The minority party and the President, however, do not constitute a winning coalition. Because of this, the equilibrium of bilateral bargaining between 1 and 2 is characterized by Proposition 4.1 if the

7 Conclusion

Reaching agreements in collective bodies requires a variety of compromises and political exchanges. The last fifty years have witnessed copious amounts of progress in explaining four key types of these political transactions. The literature on logrolling focuses on bilateral agreements among legislators, in which legislators exchange their support for a bill in exchange for support for another bill. The literature on centralized bargaining focuses on how potential competition among proposers shapes policy outcomes and distribution of rents. The literature on vote buying studies how external lobbies, aware of the competition of one another, design the pattern of offers to legislators in order to obtain the support of a winning coalition. A fourth body of work studies decentralized buying and selling of votes in a competitive market for votes.

The vast progress made notwithstanding, key questions remain. A crucial consideration is that a large number of these political exchanges take the form of private negotiations among legislators, which occur well before a bill is taken up for consideration. In fact, legislative coalitions are often constructed through a series of backroom deals, carried out by power brokers. Explaining the emergence and role of power brokers, and how legislators reach extended political compromises, therefore requires (i) a model of decentralized (as opposed to centralized) bargaining, (ii) where members of the committee (as opposed to outsiders) can buy and sell votes, (iii) at prices that they negotiate with one another (instead of taking prices as given) aware of further policy and rent repercussions, (iv) in a setting that allows a gradual process of coalition formation (as opposed to final bilateral agreements). In this paper, we propose such a model.

The main result of the paper shows that under a wide range of circumstances, successful political deals occur because of the endogenous emergence of a *power broker*: a party that serves as an intermediary between two parties that wouldn't negotiate directly with one another, transferring resources and voting rights among them in an indirect trade. Furthermore, the broker can contribute to implement a different policy outcome from that which would emerge in its absence, or in any other equilibrium keeping fixed the composition of the legislature. We also show that in contrast with intermediaries in

minority party 3 sold its votes to 1 in decentralized bargaining, but by Remark 6.2 (from the analysis of unanimity rule) if 3 sold its votes to the President (2). This difference in the end nodes transpires to the rest of the path of play. Incentives for trading in nodes 1-3 and 1-2 are exactly as in the majoritarian case, but 2-3 is governed by the logic under unanimity.

exchange economies, political brokers must have a stake in the policy outcome.

The analysis of legislative brokers can be extended to decision rules other than simple majority. We illustrated this by discussing briefly the case of unanimity and veto power. The theoretical framework is also useful to understand other features of the process by which legislators reach political compromises. Two issues that deserve more attention are the conditions leading to gridlock, and a further characterization of the set of equilibrium outcomes, both in general and in specialized settings (e.g., single-peaked preferences).

The three party model that we studied in this paper has the minimal structure required to study the emergence of middlemen in legislative bargaining. However, some interesting extensions would naturally require expanding the model to incorporate $n > 3$ legislators. This is the case, for example, of endogenous party formation in the legislature, or more generally endogenous leadership in committees. With more than three players, the particular definition of a broker equilibrium that we used in this paper would need to be amended slightly to reflect the fact that more than one player could act as a broker (either in competition with one another, or as part of a chain of trades). Thus we would not necessarily require that any particular broker trades on the equilibrium path with probability one, but that *some* broker does. We leave these extensions for future research.

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8 Appendix

Lemma 8.1 *For any* $\mathbf{k}$*, any* i, j*, and any number of players remaining,*

$$W_{ji}^{j}(\mathbf{k}, buy) = W_{ij}^{i}(\mathbf{k}, sell) + B^{j}(\mathbf{k}) - B^{i}(\mathbf{k})$$

Proof of Lemma 8.1. First, note that $W_{ij}^{i}(\mathbf{k}, sell) = B^{i}(\mathbf{k}') - t_{ij}(\mathbf{k})$, where $\mathbf{k}'$ is the post trade vote allocation. By sequential rationality, j accepts the offer iff $B^{j}(\mathbf{k}') + t_{ij}(\mathbf{k}) \geq B^{j}(\mathbf{k})$. Then in equilibrium $-t_{ij}(\mathbf{k}) = B^{j}(\mathbf{k}') - B^{j}(\mathbf{k})$, and $W_{ij}^{i}(\mathbf{k}, sell) = B^{i}(\mathbf{k}') + B^{j}(\mathbf{k}') - B^{j}(\mathbf{k})$. Next, note that $W_{ji}^{j}(\mathbf{k}, buy) = B^{j}(\mathbf{k}') - t_{ji}(\mathbf{k})$. By sequential rationality, i accepts j's offer iff $B^{i}(\mathbf{k}') + t_{ji}(\mathbf{k}) \geq B^{i}(\mathbf{k})$. Then in equilibrium $t_{ji}(\mathbf{k}) = B^{i}(\mathbf{k}) - B^{i}(\mathbf{k}')$, and $W_{ji}^{j}(\mathbf{k}, buy) = B^{j}(\mathbf{k}') + B^{i}(\mathbf{k}') - B^{i}(\mathbf{k})$. Subtracting yields the result. ■

Proof of Proposition 4.1. Consider the problem of the majority party when it has an opportunity to propose. M can, first of all, choose not to make a relevant offer (wait), guaranteeing its post trade continuation value $W_{M}^{M}(\mathbf{k}, wait) = B^{M}(\mathbf{k})$. The post-trade continuation values $B^{M}(\mathbf{k})$ and $B^{m}(\mathbf{k})$ depend on whether M prefers to implement its preferred policy or extend negotiations after UT: if $u_{M}^{*} \geq \delta W^{M}(\mathbf{k})$, M prefers to implement z_M and $B^{M}(\mathbf{k}) = u_{M}^{*}$, $B^{m}(\mathbf{k}) = u_{m}(z_{M})$, and if $u_{M}^{*} < \delta W^{M}(\mathbf{k})$, then M prefers to extend negotiations after UT, so $B^{M}(\mathbf{k}) = \delta W^{M}(\mathbf{k})$ and $B^{m}(\mathbf{k}) = \delta W^{m}(\mathbf{k})$.

But M can also exchange policy for rents by trading with m. If M makes a relevant sell offer to m, the minority will then implement z_m, so M gets a payoff $W_{M}^{M}(\mathbf{k}, sell) = u_{M}(z_{m}) - t_{M}^{S}(\mathbf{k})$. For the minority party to accept the offer $W_{M}^{m}(\mathbf{k}, sell) = u_{m}^{*} + t_{M}^{S}(\mathbf{k}) \geq B^{m}(\mathbf{k})$. Thus in equilibrium a relevant sell offer has a transfer $-t_{M}^{S}(\mathbf{k}) = u_{m}^{*} - B^{m}(\mathbf{k})$, and

$$W_{M}^{M}(\mathbf{k}; sell) = S(z_{m}) - B^{m}(\mathbf{k})$$

Similarly, if M makes a relevant buy offer, it obtains a payoff $W_{M}^{M}(\mathbf{k}, buy) = u_{M}^{*} - t_{M}^{B}(\mathbf{k})$. For the minority to accept this offer, $W_{M}^{m}(\mathbf{k}, buy) = u_{m}(z_{M}) + t_{M}^{B}(\mathbf{k}) \geq B^{m}(\mathbf{k})$, so in equilibrium $t_{M}^{B}(\mathbf{k}) = B^{m}(\mathbf{k}) - u_{m}(z_{M})$, and

$$W_{M}^{M}(\mathbf{k}, buy) = S(z_{M}) - B^{m}(\mathbf{k})$$

We can now establish the majority party's best response when it is its turn to propose, given the equilibrium continuation values. Note that selling dominates buying if and only if $m \gg M$ (which we are assuming here by hypothesis), and implementing z_M dominates extending negotiations after UT if and only if $u_M^* \geq \delta W^M(\mathbf{k})$. Thus, if $u_M^* \geq \delta W^M(\mathbf{k})$, M either implements z_M or makes m a relevant sell offer. But here selling dominates implementing z_M because $W_M^M(\mathbf{k}; \mathit{sell}) = u_M(z_m) + u_m^* - u_m(z_M) \geq u_M^* \Leftrightarrow m \gg M$. When instead $u_M^* \leq \delta W^M(\mathbf{k})$, M either waits or makes m a relevant sell offer. Since in this case the majority party extends negotiations in the event that m rejects an offer, $B^M(\mathbf{k}) = \delta W^M(\mathbf{k})$ and $B^m(\mathbf{k}) = \delta W^m(\mathbf{k})$, and M sells to m if and only if $S(z_m) \geq \delta\left[W^M(\mathbf{k}) + W^m(\mathbf{k})\right]$. To summarize, when the majority party has the opportunity to propose and $m \gg M$, M sells if either (i) $u_M^* \geq \delta W^M(\mathbf{k})$ or (ii) $u_M^* \leq \delta W^M(\mathbf{k})$ and $S(z_m) \geq \delta\left[W^M(\mathbf{k}) + W^m(\mathbf{k})\right]$, and otherwise waits and extends negotiations.

Establishing the best response of the minority party in the proposal node $((m, M), \mathbf{k})$ for fixed continuation values is considerably easier because of Lemma 8.1. The minority party m can buy, sell or wait. If it waits, it gets $W_m^m(\mathbf{k}, \mathit{wait}) = B^m(\mathbf{k})$, and by Lemma 8.1, $W_m^m(\mathbf{k}, \mathit{buy}) = S(z_m) - B^M(\mathbf{k})$ and $W_m^m(\mathbf{k}, \mathit{sell}) = S(z_M) - B^M(\mathbf{k})$. Thus, given $m \gg M$, m either waits or makes M a relevant buy offer. If $u_M^* \geq \delta W^M(\mathbf{k})$, the majority party implements z_M after an unsuccessful trade, and buying dominates waiting for m, since $W_m^m(\mathbf{k}; \mathit{buy}) \geq W_m^m(\mathbf{k}; \mathit{wait}) \Leftrightarrow m \gg M$. If instead $u_M^* \leq \delta W^M(\mathbf{k})$, the minority party prefers to trade if and only if $S(z_m) \geq \delta\left[W^M(\mathbf{k}) + W^m(\mathbf{k})\right]$.

Proving the statements in the proposition now only requires to check the consistency of these "short-run" best responses when values are determined endogenously.

1. Consider first part 1 of the Proposition. Suppose that in equilibrium (i) $u_M^* < \delta W^M(\mathbf{k})$ and (ii) $S(z_m) \geq \delta[W^M(\mathbf{k}) + W^m(\mathbf{k})]$. We have established that in this case m buys from M in both trading nodes $((M, m), \mathbf{k})$ and $((m, M), \mathbf{k})$ and implements z_m, while the majority party extends negotiations after UT. Then $W_M^M(\mathbf{k}) = S(z_m) - \delta W^m(\mathbf{k})$, $W_m^m(\mathbf{k}) = S(z_m) - \delta W^M(\mathbf{k})$, $W_M^m(\mathbf{k}) = \delta W^m(\mathbf{k})$ and $W_m^M(\mathbf{k}) = \delta W^M(\mathbf{k})$. Thus

$$W^j(\mathbf{k}) = \delta W^j(\mathbf{k}) + \rho_j\left\{S(z_m) - \delta[W^M(\mathbf{k}) + W^m(\mathbf{k})]\right\} \quad \text{for } j = m, M,$$

and therefore $W^M(\mathbf{k}) = \rho_M S(z_m)$ and $W^m(\mathbf{k}) = \rho_m S(z_m)$. Substituting these values in $u_M^* < \delta W^M(\mathbf{k})$ to check for consistency gives $u_M^* \leq \delta\rho_M S(z_m)$. Substituting in $S(z_m) \geq \delta[W^M(\mathbf{k}) + W^m(\mathbf{k})]$ gives $S(z_m) \geq 0$, which is implied by $u_M^* \leq \delta\rho_M S(z_m)$.

2. Consider next part 2. Suppose that in equilibrium $u_M^* \geq \delta W^M(\mathbf{k})$. Then m buys from M in both trading nodes and implements z_m, while the majority party implements its

preferred policy after UT. Then $W^\ell_\ell(\mathbf{k}) = S(z_m) - u_j(z_M)$ for $\ell = m, M$, $W^M_m(\mathbf{k}) = u^*_M$, and $W^m_M(\mathbf{k}) = u_m(z_M)$. Thus

$$W^l(\mathbf{k}) = u_l(z_M) + \rho_l(S(z_m) - S(z_M)) \text{ for } l = m, M$$

Substituting back in $u^*_M \geq \delta W^M(\mathbf{k})$ for consistency gives $(1-\delta)u^*_M \geq \delta\rho_M\,[S(z_m) - S(z_M)]$.

3. Finally, suppose that in equilibrium (i) $u^*_M < \delta W^M(\mathbf{k})$ and (ii) $S(z_m) < \delta[W^M(\mathbf{k}) + W^m(\mathbf{k})]$. Then M does not make a relevant offer in $((M,m),\mathbf{k})$ and m does not make a relevant offer in $((m,M),\mathbf{k})$, after which M extends negotiations. As a result, agreement is never reached, and therefore $W^M(\mathbf{k}) = 0$ and $W^m(\mathbf{k}) = 0$. Substituting in (i), we get $u^*_M < 0$, which is impossible as long as $z_M \neq Q$. This shows that (i) there does not exist a MPE in which M and m do not trade in $((M,m),\mathbf{k})$ or $((m,M),\mathbf{k})$ and M extends negotiations. It also shows that the equilibria in points 1 and 2 above are the only possible pure strategy MPE of the majority-minority bargaining game with $m \gg M$.

4. We now show that if (i) $u^*_M > \delta\rho_M S(z_m)$ and (ii) $(1-\delta)u^*_M < \delta\rho_M\,[S(z_m) - S(z_M)]$, there exists a MPE in which M sells to m when they meet and M implements z_1 after UT with probability $\alpha^* \in (0,1)$, and extends negotiations with probability $1-\alpha^*$, with α^* as defined in the Proposition. Suppose then that M sells to m in both trading nodes, and after UT M implements z_M with probability α, and extends negotiations with probability $(1-\alpha)$. Consider $((M,m),\mathbf{k})$. From indifference after UT,

$$u^*_M = \delta W^M(\mathbf{k}) \tag{9}$$

For M to prefer selling than implementing z_M (or waiting),

$$W^M_M(\mathbf{k}, sell) = u_M(z_m) + u^*_m - B^m(\mathbf{k}) \geq u^*_M \tag{10}$$

Given the randomization after UT, $B^M(\mathbf{k}) = u^*_M$, and

$$\begin{aligned} B^m(\mathbf{k}) &= \alpha u_m(z_M) + (1-\alpha)\delta W^m(\mathbf{k}) \\ &= u_m(z_M) - (1-\alpha)u_m(z_M) + \delta(1-\alpha)W^m(\mathbf{k}). \end{aligned} \tag{11}$$

Then $W^m_M(\mathbf{k}) = B^m(\mathbf{k})$ (given above) and $W^M_M(\mathbf{k}) = W^M_M(\mathbf{k}, sell)$ is

$$\begin{aligned} W^M_M(\mathbf{k}) &= u_M(z_m) + u^*_m - \alpha u_m(z_M) - (1-\alpha)\delta W^m(\mathbf{k}) \\ &= u^*_M + (S(z_m) - S(z_M)) - \delta(1-\alpha)W^m(\mathbf{k}) + (1-\alpha)u_m(z_M). \end{aligned} \tag{12}$$

Next, consider $((m, M), \mathbf{k})$. In equilibrium, m will make M an offer that leaves her indifferent. Then $W_m^M(\mathbf{k}) = B^M(\mathbf{k}) = u_M^*$, and $W_m^m(\mathbf{k}) = W_m^m(\mathbf{k}, buy) = S(z_m) - u_M^*$. By $m \gg M$, m prefers buying than selling. It prefers buying than waiting if $W_m^m(\mathbf{k}, buy) = S(z_m) - u_M^* \geq W_m^m(\mathbf{k}, wait) = B^m(\mathbf{k})$, which is given by (11). Recalling that $W_m^m(\mathbf{k}) = S(z_m) - u_M^*$ and $W_M^m(\mathbf{k}) = B^m(\mathbf{k})$ then,

$$\begin{aligned} W_M^m(\mathbf{k}) &= u_m(z_M) + (1-\alpha)\left(\delta W^m(\mathbf{k}) - u_m(z_M)\right) \\ W_m^m(\mathbf{k}) &= u_m(z_M) + S(z_m) - S(z_M) \end{aligned}$$

which imply that

$$W^m(\mathbf{k}) = u_m(z_M) + \frac{\rho_m\left(S(z_m) - S(z_M)\right) - (1-\delta)(1-\alpha)\rho_M u_m(z_M)}{1 - \delta\rho_M(1-\alpha)} \tag{13}$$

From (12), (13), and given $W_m^M(\mathbf{k}) = u_M^*$, it follows that

$$W^M(\mathbf{k}) = u_M^* + \rho_M\left(S(z_m) - S(z_M)\right) + (1-\alpha)(1-\delta)\rho_M u_m(z_M) - (1-\alpha)\delta\rho_M\left(W^m(\mathbf{k}) - u_m(z_M)\right) \tag{14}$$

and using $W^m(\mathbf{k})$, and the fact that M has to be indifferent between implementing z_M and extending negotiations after UT, so that (9) must hold, we get

$$W^M(\mathbf{k}) - u_M^* = \rho_M \frac{1 - \delta(1-\alpha)\left(S(z_m) - S(z_M)\right)}{1 - \delta\rho_M(1-\alpha)} + \frac{(1-\alpha)(1-\delta)\rho_M u_m(z_M)}{1 - \delta\rho_M(1-\alpha)} \tag{15}$$

Substituting $W^M(\mathbf{k})$ and simplifying

$$\alpha^* = \frac{(1-\delta)}{\delta\rho_M}\left(\frac{u_M^* - \delta\rho_M S(z_m)}{\delta S(z_m) - S(z_M)}\right) \tag{16}$$

Note that $u_M^* \geq \delta\rho_M S(z_m)$ and $(1-\delta)\,u_M^* < \delta\rho_M\left(S(z_m) - S(z_M)\right)$ together imply that $\delta S(z_m) > S(z_M)$, so $\alpha^* \geq 0$. On the other hand, note that $\alpha^* < 1$ iff $(1-\delta)\,u_M^* < \delta\rho_M\left(S(z_m) - S(z_M)\right)$. It follows that $\alpha^* \in [0,1)$ when $u_M^* \geq \delta\rho_M S(z_m)$ and $(1-\delta)\,u_M^* < \delta\rho_M\left(S(z_m) - S(z_M)\right)$. Finally, note that from (9), evaluating (14) at α^* gives $\delta W^M(\mathbf{k}) = u_M^*$. Adding (13) and (15) gives $u_m^* + u_M(z_m) = W^M(\mathbf{k}) + W^m(\mathbf{k})$. Then from here $\delta W^m(\mathbf{k}) = \delta S(z_m) - u_M^*$. ■

Lemma 8.2 *Suppose that the initial vote endowment profile at the beginning of a trading round is* $\mathbf{k}$*, and let* $\mathbf{k}'_{im}$ *denote the vote endowment profile after* i *buys from* j *in that round,* *and is left with minority party m. Then (0.a) $W^i_{ij}(\mathbf{k}; buy) = B^i(\mathbf{k}'_{im}) + B^j(\mathbf{k}'_{im}) - B^j(\mathbf{k})$, (0.b) $W^i_{ij}(\mathbf{k}; sell) = B^i(\mathbf{k}'_{jm}) + B^j(\mathbf{k}'_{jm}) - B^j(\mathbf{k})$, and (0.c) $W^i_{ij}(\mathbf{k}; wait) = B^i(\mathbf{k})$, and thus:*

1. $W^j_{ji}(\mathbf{k}, buy) + B^i(\mathbf{k}) = W^i_{ij}(\mathbf{k}, sell) + B^j(\mathbf{k})$.
2. $W^i_{ij}(\mathbf{k}; sell) \geq W^i_{ij}(\mathbf{k}; buy)$ *if and only if* $B^i(\mathbf{k}'_{jm}) + B^j(\mathbf{k}'_{jm}) \geq B^i(\mathbf{k}'_{im}) + B^j(\mathbf{k}'_{im})$.
3. $W^i_{ij}(\mathbf{k}; sell) \geq W^i_{ij}(\mathbf{k}; buy)$ *iff* $W^j_{ji}(\mathbf{k}; buy) \geq W^j_{ji}(\mathbf{k}; sell)$.
4. $W^i_{ij}(\mathbf{k}; sell) \geq W^i_{ij}(\mathbf{k}; wait)$ *iff* $W^j_{ji}(\mathbf{k}; buy) \geq W^j_{ji}(\mathbf{k}; wait)$.

Proof of Lemma 8.2. To establish this result, note that $W^i_{ij}(\mathbf{k}; buy) = B^i(\mathbf{k}'_{im}) - t^{buy}_{ij}(\mathbf{k})$. For j to accept, $B^j(\mathbf{k}'_{im}) + t^{buy}_{ij}(\mathbf{k}) \geq B^j(\mathbf{k})$. Then in equilibrium $t^{buy}_{ij}(\mathbf{k}) = B^j(\mathbf{k}) - B^j(\mathbf{k}'_{im})$. Substituting, $W^i_{ij}(\mathbf{k}; buy) = B^i(\mathbf{k}'_{im}) + B^j(\mathbf{k}'_{im}) - B^j(\mathbf{k})$. Similarly, $W^i_{ij}(\mathbf{k}; sell) = B^i(\mathbf{k}'_{jm}) - t^{sell}_{ij}(\mathbf{k})$, and for j to accept, $B^j(\mathbf{k}'_{jm}) + t^{sell}_{ij}(\mathbf{k}) \geq B^j(\mathbf{k})$, so in equilibrium $t^{sell}_{ij}(\mathbf{k}) = B^j(\mathbf{k}) - B^j(\mathbf{k}'_{jm})$. Substituting, $W^i_{ij}(\mathbf{k}; sell) = B^i(\mathbf{k}'_{jm}) + B^j(\mathbf{k}'_{jm}) - B^j(\mathbf{k})$. This establishes part 0. Parts 1 and 2 follow immediately from 0. Part 3 follows from 2, and part 4 follows from 1. ■

Proof of Proposition 4.2. Consider first the state $((1,2), \mathbf{k})$, for $\mathbf{k}$ such that $k_1 \geq r$, $k_2 > 0$, $k_3 > 0$. Party 1 can extend negotiations (wait), make a relevant sell offer to 2, make a relevant buy offer to 2, or implement z_1. First note that party 1 can choose not to make a relevant offer and implement z_1, getting u^*_1. Also $W^1_{12}(\mathbf{k}; wait) = B^1(\mathbf{k})$. Now consider $W^1_{12}(\mathbf{k}; sell)$. With $2 \gg 3$, in the next period (with endowment $\mathbf{k}''$), 2 would implement z_2 immediately with no trade. Anticipating this, 2 would implement z_2 immediately after buying 1's votes. Then $W^1_{12}(\mathbf{k}; sell) = u_1(z_2) - t_{12}(\mathbf{k})$. And for 2 to be willing to buy, it must be that $u^*_2 + t_{12}(\mathbf{k}) \geq B^2(\mathbf{k})$. Thus in equilibrium $-t_{12}(\mathbf{k}) = u^*_2 - B^2(\mathbf{k})$, and then

$$W^1_{12}(\mathbf{k}; sell) = S_{12}(z_2) - B^2(\mathbf{k})$$

The value for 1 of buying 2's votes depends on whether 1 would trade with 3 in majority-minority bargaining or not. From Lemma 8.2 for $(M, m) = (1, 3)$, $W^1_{12}(\mathbf{k}; buy) = B^1(\mathbf{k}'_{13}) + B^2(\mathbf{k}'_{13}) - B^2(\mathbf{k})$. The continuation values follow from Proposition 4.1 for $(M, m) = (1, 3)$. If $3 \gg 1$ and $u^*_1 < \delta\rho_{13}S_{13}(z_3)$, the majority party 1 extends negotiations after trade with 2, and goes on to sell its votes to 3 in the next trading period. Then $B^1(\mathbf{k}'_{13}) = \delta W^1(\mathbf{k}'_{13}) =$

$\delta\rho_{13}S_{13}(z_3)$, and $B^2(\mathbf{k}'_{13}) = \delta W^2(\mathbf{k}'_{13}) = \delta u_2(z_3)$. If instead either $1 \gg 3$, or $3 \gg 1$ and $u_1^* \geq \delta\rho_{13}S_{13}(z_3)$, party 1 implements z_1 after trading with 2. Then $B^1(\mathbf{k}'_{13}) = u_1^*$, $B^2(\mathbf{k}'_{13}) = u_2(z_1)$. Thus

$$W_{12}^1(\mathbf{k}; buy) = \begin{cases} \Pi(21,3) - B^2(\mathbf{k}) & \text{if } 3 \gg 1 \text{ and } u_1^* < \delta\rho_{13}S_{13}(z_3) \\ S_{12}(z_1) - B^2(\mathbf{k}) & \text{otherwise.} \end{cases}$$

Next, consider the trading state $((1,3),\mathbf{k})$. As in $((1,2),\mathbf{k})$, party 1 can extend negotiations (wait), make a relevant sell offer, make a relevant buy offer, or implement z_1. As before, $W_{13}^1(\mathbf{k}; gov) = u_1^*$ and $W_{13}^1(\mathbf{k}; wait) = B^1(\mathbf{k})$. Because 1 implements z_1 after UT if $u_1^* \geq \delta W^1(\mathbf{k})$, in this case $B^1(\mathbf{k}) = u_1^*$ and $B^3(\mathbf{k}) = u_3(z_1)$. If instead $u_1^* \leq \delta W^1(\mathbf{k})$, 1 extends negotiations after UT, so $B^1(\mathbf{k}) = \delta W^1(\mathbf{k})$ and $B^3(\mathbf{k}) = \delta W^3(\mathbf{k})$. Consider 1 selling votes to 3. From Lemma 8.2, $W_{13}^1(\mathbf{k}; sell) = B^1(\mathbf{k}'_{32}) + B^3(\mathbf{k}'_{32}) - B^3(\mathbf{k})$. The continuation values are determined from Proposition 4.1 for $(M,m) = (3,2)$. If $u_3^* < \delta\rho_{32}S_{23}(z_2)$, 3 extends negotiations after trade with 1, and goes on to sell votes to 2 in the next trading period. Then $B^3(\mathbf{k}'_{32}) = \delta W^3(\mathbf{k}'_{32}) = \delta\rho_{32}S_{23}(z_2)$, $B^1(\mathbf{k}'_{32}) = \delta W^1(\mathbf{k}'_{32}) = \delta u_1(z_2)$. If instead $u_3^* \geq \delta\rho_{32}S_{23}(z_2)$, party 3 implements z_3 immediately after trade with 1, so that $B^3(\mathbf{k}'_{32}) = u_3^*$, and $B^1(\mathbf{k}'_{32}) = u_1(z_3)$. Thus

$$W_{13}^1(\mathbf{k}; sell) = \begin{cases} \Pi(13,2) - B^3(\mathbf{k}) & \text{if } 2 \gg 3 \text{ and } u_3^* < \delta\rho_{32}S_{23}(z_2) \\ S_{13}(z_3) - B^3(\mathbf{k}) & \text{otherwise.} \end{cases}$$

Finally, consider 1 buying votes from 3. Because $1 \gg 2$ and $1 \gg 3$, analyzing the case in which 1 buys from 3 is equivalent to the case in which 1 buys from 2, changing the role of 2 and 3. Therefore

$$W_{13}^1(\mathbf{k}; buy) = \begin{cases} \Pi(31,2) - B^3(\mathbf{k}) & \text{if } 2 \gg 1 \text{ and } u_1^* < \delta\rho_{12}S_{12}(z_2) \\ S_{13}(z_1) - B^3(\mathbf{k}) & \text{otherwise.} \end{cases}$$

■

Proof of Proposition 4.3. Consider first the state $((2,3),\mathbf{k})$. Party 2 can extend negotiations (wait), make a relevant sell offer, or make a relevant buy offer. As before, $W_{23}^2(\mathbf{k}; wait) = B^2(\mathbf{k})$. Because 1 implements z_1 after UT if $u_1^* \geq \delta W^1(\mathbf{k})$, in this case $B^i(\mathbf{k}) = u_i(z_1)$ for $i = 2,3$, and if instead $u_1^* \leq \delta W^1(\mathbf{k})$, 1 extends negotiations after UT, so $B^i(\mathbf{k}) = \delta W^i(\mathbf{k})$ for $i = 2,3$.

Consider 2 selling votes to 3. From Lemma 8.2, $W^2_{23}(\mathbf{k};sell) = B^2(\mathbf{k}'_{13}) + B^3(\mathbf{k}'_{13}) - B^3(\mathbf{k})$. The continuation values are determined from Proposition 4.1 for $(i,j) = (1,3)$. If $3 \gg 1$ and $u^*_1 < \delta\rho_{13}S_{13}(z_3)$, 1 extends negotiations after 2 sells to 3, and goes on to sell votes to 3 in the next trading period. Then $B^2(\mathbf{k}'_{13}) = \delta u_2(z_3)$, $B^3(\mathbf{k}'_{13}) = \delta W^3(\mathbf{k}'_{13}) = \delta\rho_{31}S_{13}(z_3)$, and $W^2_{23}(\mathbf{k};sell) = \tilde{\Pi}(23,1) - B^3(\mathbf{k})$. If instead either $1 \gg 3$, or $3 \gg 1$ and $u^*_1 \geq \delta\rho_{13}S_{13}(z_3)$, party 1 implements z_1 immediately after 2 sells to 3. Then $B^2(\mathbf{k}'_{13}) = u_2(z_1)$, $B^3(\mathbf{k}'_{13}) = u_3(z_1)$, and $W^2_{23}(\mathbf{k};sell) = S_{23}(z_1) - B^3(\mathbf{k})$.

Next consider 2 buying votes from 3. From Lemma 8.2, $W^2_{23}(\mathbf{k};buy) = B^2(\mathbf{k}'_{32}) + B^3(\mathbf{k}'_{32}) - B^3(\mathbf{k})$. The relevant continuation values follow from Proposition 4.1 for $(i,j) = (1,2)$. If $2 \gg 1$ and $u^*_1 < \delta\rho_{12}S_{12}(z_2)$, party 1 extends negotiations after 2 buys from 3, and goes on to sell votes to 2 in the next trading period. Then $B^2(\mathbf{k}'_{32}) = \delta W^2(\mathbf{k}'_{12}) = \delta\rho_{21}S_{12}(z_2)$, $B^3(\mathbf{k}'_{32}) = \delta u_3(z_2)$, and $W^2_{23}(\mathbf{k};buy) = \tilde{\Pi}(32,1) - B^3(\mathbf{k})$. If instead either $1 \gg 2$, or $2 \gg 1$ and $u^*_1 \geq \delta\rho_{12}S_{12}(z_2)$, party 1 implements z_1 immediately after 2 buys from 3. Then $B^2(\mathbf{k}'_{32}) = u_2(z_1)$, $B^3(\mathbf{k}'_{32}) = u_3(z_1)$, and $W^2_{23}(\mathbf{k};buy) = S_{23}(z_1) - B^3(\mathbf{k})$.

The expressions for party 3's payoffs in $((3,1),\mathbf{k})$ as a function of the continuation values now follow immediately using Lemma 8.2. ■

Proof of Lemma 5.1. Consider $W^2(\mathbf{k})$. Note that in all states ij other than 13 or 31, $W^2_{ij}(\mathbf{k}) = \delta W^2(\mathbf{k})$, and $W^2_{13}(\mathbf{k}) = W^2_{31}(\mathbf{k}) = \delta W^2(\mathbf{k}'_{32}) = \delta\rho_{23}S_{23}(z_2)$. Then

$$W^2(\mathbf{k}) = \frac{\delta(p_{13} + p_{31})\rho_{23}S_{23}(z_2)}{(1-\delta) + \delta(p_{13} + p_{31})}$$

Now consider $W^1(\mathbf{k})$. Note that in all states ij other than 13 (including 31) $W^1_{ij}(\mathbf{k}) = \delta W^1(\mathbf{k})$, and $W^1_{13}(\mathbf{k}) = \Pi(13,2) - \delta W^3(\mathbf{k})$. Then

$$W^1(\mathbf{k}) = \frac{p_{13}[\Pi(13,2) - \delta W^3(\mathbf{k})]}{1 - \delta(1 - p_{13})} \tag{17}$$

Now consider $W^3(\mathbf{k})$. Note that in all states ij other than 31 (including 13), $W^3_{ij}(\mathbf{k}) = \delta W^3(\mathbf{k})$, and $W^3_{31}(\mathbf{k}) = W^1_{13}(\mathbf{k}) + B^3(\mathbf{k}) - B^1(\mathbf{k})$, so $W^3_{31}(\mathbf{k}) = \Pi(13,2) - \delta W^1(\mathbf{k})$. Then

$$W^3(\mathbf{k}) = \frac{p_{31}[\Pi(13,2) - \delta W^1(\mathbf{k})]}{1 - \delta(1 - p_{31})} \tag{18}$$

Solving the system (17)-(18) gives

$$W^1(\mathbf{k}) = \frac{p_{13}\Pi(13,2)}{(1-\delta) + \delta(p_{13} + p_{31})} \quad \text{and} \quad W^3(\mathbf{k}) = \frac{p_{31}\Pi(13,2)}{(1-\delta) + \delta(p_{13} + p_{31})}$$

■

Proof of Theorem 5.4, part 1. The equilibrium conditions (2)-(7) together with the requirement that $u \in \mathcal{U}$, and the dominance relations $1 \gg 2$, $2 \gg 3$, and $1 \gg 3$ form a system of linear inequalities in the unknowns $u_i(z_j)$, which can be written as $Au \leq \alpha$, where $\alpha^T \equiv (\mathbf{0}_9, -\mathbf{b}_9)$,

$$u^T = \begin{pmatrix} u_1^* & u_2(z_1) & u_3(z_1) & u_1(z_2) & u_2^* & u_3(z_2) & u_1(z_3) & u_2(z_3) & u_3^* \end{pmatrix},$$

and

$$A = \begin{pmatrix}
\upsilon & \upsilon & 0 & -\delta^2 p_{13} & -\delta^2\theta & -\delta^2\theta & 0 & 0 & 0 \\
0 & \upsilon & \upsilon & -\delta^2 p_{31} & -\delta^2\mu & -\delta^2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\delta\rho_{32} & -\delta\rho_{32} & 0 & 0 & 1 \\
1 & 0 & 1 & -\delta & -\delta\rho_{32} & -\delta\rho_{32} & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -\rho_{32} & -\rho_{32} & 0 & 0 & 0 \\
\upsilon & 0 & 0 & -\delta^2 p_{13} & -\delta^2 p_{13}\rho_{32} & -\delta^2 p_{13}\rho_{32} & 0 & 0 & 0 \\
-1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1
\end{pmatrix}$$

(The rows in the matrix correspond to the inequalities in the text in the following order. The first six rows are inequalities (6b), (7b), (3b), (4b), (5b), (2b). The next three rows are the dominance order, and the last nine rows guarantee that for all $i, j \in N$, $u_i^* > u_i(z_j)$ and $u_i^* > 0 = u_i(Q)$.)

It follows from Lemma 5.3 that our original system of inequalities does not have a solution if there exists a $\lambda \geq 0$ such that:

$$\upsilon\lambda_1 + \lambda_4 + \upsilon\lambda_6 - \lambda_7 - \lambda_9 - \lambda_{10} - \lambda_{11} - \lambda_{12} = 0$$

$$\upsilon\lambda_1 + \upsilon\lambda_2 - \lambda_7 + \lambda_{14} = 0$$

$$\upsilon\lambda_2 + \lambda_4 - \lambda_9 + \lambda_{17} = 0 \tag{19}$$

$$-\delta^2 p_{13}\lambda_1 - \delta^2 p_{31}\lambda_2 - \delta\lambda_4 - \lambda_5 - \delta^2 p_{13}\lambda_6 + \lambda_7 + \lambda_{11} = 0$$

$$-\delta^2\theta\lambda_1 - \delta^2\mu\lambda_2 - \delta\rho_{32}\lambda_3 - \delta\rho_{32}\lambda_4 - \rho_{32}\lambda_5 - \delta^2 p_{13}\rho_{32}\lambda_6 + \lambda_7 - \lambda_8 - \lambda_{13} - \lambda_{14} - \lambda_{15} = 0$$

$$-\delta^2\theta\lambda_1 - \delta^2\mu\lambda_2 - \delta\rho_{32}\lambda_3 - \delta\rho_{32}\lambda_4 - \rho_{32}\lambda_5 - \delta^2 p_{13}\rho_{32}\lambda_6 - \lambda_8 + \lambda_{18} = 0 \tag{20}$$

$$\lambda_9 + \lambda_{12} = 0 \tag{21}$$

$$\lambda_8 + \lambda_{15} = 0 \tag{22}$$

$$\lambda_3 + \lambda_8 + \lambda_9 - \lambda_{16} - \lambda_{17} - \lambda_{18} = 0$$

and

$$\sum_{i=10}^{18} \lambda_i > 0 \tag{23}$$

From (21), $\lambda_9 = \lambda_{12} = 0$, from (22), $\lambda_8 = \lambda_{15} = 0$, and from (19) and $\lambda_9 = 0$, $\lambda_2 = \lambda_4 = \lambda_{17} = 0$. After substituting, we can further obtain $\lambda_3 = \lambda_{16} + \lambda_{18} \geq 0$, and $\lambda_7 = \upsilon\lambda_1 + \lambda_{14} \geq 0$. Substituting, the dual system becomes

$$\upsilon\lambda_6 - \lambda_{14} - \lambda_{10} - \lambda_{11} = 0 \tag{24}$$

$$[\upsilon - \delta^2 p_{13}]\lambda_1 - \lambda_5 - \delta^2 p_{13}\lambda_6 + \lambda_{14} + \lambda_{11} = 0$$

$$[\upsilon - \delta^2\theta]\lambda_1 - \delta\rho_{32}\lambda_{16} - \delta\rho_{32}\lambda_{18} - \rho_{32}\lambda_5 - \delta^2 p_{13}\rho_{32}\lambda_6 - \lambda_{13} = 0$$

$$-\delta^2\theta\lambda_1 - \rho_{32}\lambda_5 - \delta^2 p_{13}\rho_{32}\lambda_6 - \delta\rho_{32}\lambda_{16} + [1 - \delta\rho_{32}]\lambda_{18} = 0 \tag{25}$$

and

$$\lambda_{10} + \lambda_{11} + \lambda_{13} + \lambda_{14} + \lambda_{16} + \lambda_{18} > 0$$

From (24), $\upsilon\lambda_6 = \lambda_{10} + \lambda_{11} + \lambda_{14} \geq 0$, and from (25), $[1 - \delta\rho_{32}]\lambda_{18} = \delta^2\theta\lambda_1 + \rho_{32}\lambda_5 + \delta^2 p_{13}\rho_{32}\lambda_6 + \delta\rho_{32}\lambda_{16} \geq 0$. Substituting, and simplifying, the dual system is

$$[\upsilon - \delta^2 p_{13}]\lambda_1 - \lambda_5 - \frac{\delta^2 p_{13}}{\upsilon}\lambda_{10} + \left[\frac{\upsilon - \delta^2 p_{13}}{\upsilon}\right]\lambda_{11} + \left[\frac{\upsilon - \delta^2 p_{13}}{\upsilon}\right]\lambda_{14} = 0 \tag{26}$$

$$\left[(1 - \delta\rho_{32})\upsilon - \delta^2\theta\right]\lambda_1 = \rho_{32}\lambda_5 + (1 - \delta\rho_{32})\lambda_{13} + \delta\rho_{32}\lambda_{16} + \frac{\delta^2 p_{13}\rho_{32}}{\upsilon}(\lambda_{10} + \lambda_{11} + \lambda_{14}) \tag{27}$$

and

$$\delta^2\theta\lambda_1 + \rho_{32}\lambda_5 + \delta^2 p_{13}\rho_{32}\lambda_6 + (1-\delta\rho_{32})[\lambda_{10}+\lambda_{11}+\lambda_{13}+\lambda_{14}] + \lambda_{16} > 0 \tag{28}$$

Since all the coefficients on the RHS of (27) are positive, a necessary and sufficient condition for the solution of the dual system is that the coefficient of λ_1 is positive as well, i.e., $(1-\delta\rho_{32})\upsilon - \delta^2\theta > 0$. Therefore $(1-\delta\rho_{32})\upsilon - \delta^2\theta \leq 0$ is a necessary and sufficient condition for the existence of a solution to the primal. ■

The proof of the second part of Theorem 5.4 uses the following Lemma.

Lemma 8.3 *If $(p,\rho,\delta) \in \mathcal{P}$ then there is an open subset $U \subset \mathcal{U}$ such that for every $u \in U$, the legislative bargaining game with parameters (G,u) admits an equilibrium with brokers for the matching probabilities (p,ρ,δ).*

Proof of Lemma 8.3. Denote the dominance relation under preference profile u by $\gg_u$. We say that $i \gg_u j$ is *stronger* than $i \gg_{u'} j$ if $u_i^* + u_j(z_i) - (u_i(z_j) + u_j^*) > u_i^{*\prime} + u_j'(z_i) - (u_i'(z_j) + u_j^{*\prime})$. Fix some $(p,\rho,\delta) \in \mathcal{P}$. Theorem (5.4) implies that there is some $u_{(p,\rho,\delta)} \in \mathcal{U}$ such that $(2b)-(7b)$ and the dominance conditions hold at least with equality. We now show that we can construct an open subset $U_{u_{(p,\rho,\delta)}} \subset \mathcal{U}$ around $u_{(p,\rho,\delta)}$ such that the legislative bargaining game $((p,\rho,\delta), u')$ also admits a broker equilibrium for every $u' \in U_{u_{(p,\rho,\delta)}}$.

Take any pair $(p,\rho,\delta) \in \mathcal{P}$ and any $u_{(p,\rho,\delta)} \in \mathcal{U}$ such that $\left((p,\rho,\delta), u_{(p,\rho,\delta)}\right)$ admits a broker equilibrium. First note that reducing $u_1(z_3)$ by $\widetilde{\eta}_1 > 0$ makes the dominance relation $1 \gg 3$ stronger and does not affect any of the conditions in $(2b)-(7b)$. Note that by adding $\eta_3 > 0$ to $u_3(z_2)$ all conditions in $(2b)-(7b)$ hold with strict inequality and the dominance relation $2 \gg 3$ becomes stronger. Moreover, since $u_{(p,\rho,\delta)} \in \mathcal{U}$, then $u_3(z_2) < u_3^*$, and we can choose η_3 sufficiently small in order to still remain in $\mathcal{U}$. Note now that reducing $u_1(z_2)$ makes the dominance relation $1 \gg 2$ stronger but makes all conditions $(2b)-(7b)$ but $(3b)$, tighter. Let

$$\Delta_{\overline{\eta}} = \left\{\eta = (\widetilde{\eta}_1, \eta_1, \eta_3) \in (0,\overline{\eta})^3 : \min\left\{\rho_{32}, \frac{\theta}{p_{13}}, \frac{\mu}{p_{31}}\right\}\eta_3 > \eta_1\right\}$$

for $\overline{\eta} > 0$, and define for every $\eta \in \Delta_{\overline{\eta}}$ the vector

$$(\Delta u_\eta)^T = \begin{pmatrix} 0 & 0 & 0 & \eta_1 & 0 & \eta_3 & -\widetilde{\eta}_1 & 0 & 0 \end{pmatrix},$$

Define $u'_\eta = u_{(p,\rho,\delta)} + \Delta u_\eta$, and note that for sufficiently small η, $u'_\eta \in \mathcal{U}$. Let $U_{\overline{\eta},u_{(p,\rho,\delta)}} = \left\{u'_\eta \in \mathcal{R}^9 : u'_\eta = u_{(p,\rho,\delta)} + \Delta u_\eta \text{ and } \eta \in \Delta_{\overline{\eta}}\right\}$. Note that since $\Delta_{\overline{\eta}}$ is open $U_{\overline{\eta},u}$ is also open. It is easy to see that for sufficiently small $\overline{\eta} > 0$, if $u'' \in U_{\overline{\eta},u_{(p,\rho,\delta)}}$ then for (p,ρ,δ) the preference profile u'' verifies all conditions in $(2b)-(7b)$, the dominance relations are stronger under u'' than under $u_{(p,\rho,\delta)}$ and $u'' \in \mathcal{U}$. Therefore for every $u'' \in U_{\overline{\eta},u_{(p,\rho,\delta)}}$ the legislative bargaining $((p,\rho,\delta), u'')$ admits a broker equilibrium. ■

Proof of Theorem 5.4, part 2. By Lemma 8.3, for every $(p,\rho,\delta) \in int\mathcal{P}$ there is an open subset $U_{(p,\rho,\delta)} \subset \mathcal{U}$ such that for every $u \in U_{(p,\rho,\delta)}$, the legislative bargaining game with parameters $((p,\rho,\delta), u)$ admits a broker equilibrium. We need to show that if $u \in U_{(p,\rho,\delta)}$ there is an open ball around (p,ρ,δ) such that $u \in U_{(p,\rho,\delta)}$ still admits a broker equilibrium for every element in that open ball around (p,ρ,δ). We will do this in a way that does not depend on the particular $u \in U_{(p,\rho,\delta)}$ that we chose, thus establishing the result for all element of $U_{(p,\rho,\delta)}$. Let $(p,\rho,\delta) \in int\mathcal{P}$, and construct (p',ρ',δ') in the following way: add $\eta_{13} > 0$ to p_{13}, $-\eta_{13}$ to p_{31}, $\eta_{23} > 0$ to ρ_{23} and $\eta_\delta > 0$ to δ. Since $(p,\rho,\delta) \in int\mathcal{P}$, it follows that $(p',\rho',\delta') \in int\mathcal{P}$ for sufficiently small $(\eta_{13},\eta_{23},\eta_\delta)$. Since every $u \in U_{(p,\rho,\delta)}$ verifies $(2b)-(7b)$ with strict inequality for (p,ρ,δ), we must have that for sufficiently small $(\eta_{13},\eta_{23},\eta_\delta)$, $u \in U_{(p,\rho,\delta)}$ verifies $(2b)-(7b)$ with strict inequality for (p',ρ',δ'). Let $\eta \in [0,\xi]^3$ for $\xi > 0$, and, with some abuse of notation, let $(p',\rho',\delta')_\eta \equiv (p,\rho,\delta) + \eta$. Note that for every $u \in U_{(p,\rho,\delta)}$ there is some $\overline{\eta}_u = (\overline{\eta}_{13},\overline{\eta}_{23},\overline{\eta}_\delta)$ such that if $\eta < \overline{\eta}_u$, $u \in U_{(p,\rho,\delta)}$ admits a broker equilibrium for $(p',\rho',\delta')_\eta$ and conditions $(2b)-(7b)$ hold with strict inequality. Define $\underline{\eta} \equiv \min_{u \in U_{(p,\rho,\delta)}} \overline{\eta}_u$, and note that since $\overline{\eta}_u > 0$ for all $u \in U_{(p,\rho,\delta)}$, then $\underline{\eta} > 0$. Next define

$$P_{(p,\rho,\delta)} = \left\{(p',\rho',\delta')_\eta \in int\mathcal{P} : \eta < \underline{\eta}\right\},$$

and note that by definition of $\underline{\eta}$ we have that every $u \in U_{(p,\rho,\delta)}$ admits a broker equilibrium for every $(p',\rho',\delta') \in P_{(p,\rho,\delta)}$. ■

Proof of Theorem 5.4, part 3. We need to show that whenever there exists a BE, there is no other equilibrium implementing z_2, and no equilibrium implementing z_3. Consider z_2 first. If party 2 doesn't acquire the majority from 3, it must acquire the majority directly from party 1. We will show that this is not possible. First, note that because $1 \gg 2$, there is no equilibrium in which 2 acquires the majority from 1 in the majority minority bargaining game and implements z_2. Thus, this could only happen in decentralized bargaining (when $k_3 > 0$).

So suppose this is the case. There are two possible continuations. First, suppose that 2 implements z_2 immediately after buying 1 out. Note that this cannot be an equilibrium, for in this case equilibrium payoffs for 1 and 2 are the same as if 3 were not present, and then $1 \gg 2$ again rules out 1 selling to 2. It must be then that 2 extends negotiations after buying 1 out in order to negotiate with 3 in bilateral bargaining, and is then able to both implement z_2 and obtain a transfer from 3 (allowing him to make 1 a higher offer in the first place).[19]

Let $\mathbf{k}'_{ij}$ denote the state in majority-bargaining between i and j where i has the majority of the votes. Since $2 \gg 3$, we know that 2 prefers to make a buy offer instead of a sell offer, in which case the transfer is determined by $u_3(z_2) + t_{23}(\mathbf{k}'_{23}) = \delta W^3(\mathbf{k}'_{23})$ yielding the continuation values $W^2_{23}(\mathbf{k}'_{23}) = S_{23}(z_2) - \delta W^3(\mathbf{k}'_{23})$ and $W^3_{23}(\mathbf{k}'_{23}) = \delta W^3(\mathbf{k}'_{23})$. On the other hand, 3 will make a sell offer when it is her turn to propose (see Lemma 8.2) which implies that the transfer is determined by $u^*_2 + t_{32}(\mathbf{k}'_{23}) = \delta W^2(\mathbf{k}'_{23})$. Thus $W^2_{32}(\mathbf{k}'_{23}) = \delta W^2(\mathbf{k}'_{23})$ and $W^3_{32}(\mathbf{k}'_{23}) = S_{23}(z_2) - \delta W^2(\mathbf{k}'_{23})$, and therefore

$$W^2(\mathbf{k}'_{23}) = \rho_{23} S_{23}(z_2) \quad \text{and} \quad W^3(\mathbf{k}'_{23}) = (1 - \rho_{23}) S_{23}(z_2)$$

For this to be an equilibrium, we need to verify $u^*_2 \leq \delta W^2(\mathbf{k}'_{23}) = \rho_{23} S_{23}(z_2)$, so that 2 wants to extend negotiations after buying from 1 in decentralized trading. Equivalently, we need $u^*_2 \leq \frac{\delta\rho_{23}}{1-\delta\rho_{23}} u_3(z_2)$. Using this result in $(3b)$ we have

$$u^*_3 \leq \frac{\delta\rho_{32}}{1-\delta\rho_{23}} u_3(z_2) \quad \Rightarrow \quad u^*_3 - u_3(z_2) \leq \frac{\delta - 1}{1-\delta\rho_{23}} u_3(z_2) \leq 0 \quad \Rightarrow \quad u \notin \mathcal{U}$$

Next consider z_3. In order for z_3 to be implemented it must be that 3 acquires the majority. There are four ways in which this can happen: 1) the majority-minority bargaining stage is reached by 1 and 3 (it does not matter who bought 2's votes), 2) the majority-minority bargaining stage is reached by 2 and 3 and 2 has the majority of the votes, 3) the majority-minority bargaining stage is reached by 2 and 3 and 3 has the majority of the votes, and 4) the majority-minority bargaining stage is not reached and 3 buys the votes from 1 in decentralized bargaining. In the first case, since $1 \gg 3$, we have that z_3 cannot be

[19] As discussed in footnote 4, this outcome cannot be directly ruled out in bilateral bargaining because a majority party 2 can under some conditions charge 3 to implement z_2 by threatening her to keep the status quo after UT.

implemented because 3 does not have enough resources to pay 1. The same is true in case 2 but now with 2 instead of 1. Note that in order to reach the majority-minority bargaining stage it must be that 3 prefers to move onto that stage after buying all the votes from 1. Therefore it must be that $u_3^* \leq \delta W^3 (\mathbf{k}'_{32})$. Note that if there is some other equilibria it must be that 3 is charging 2 to implement z_3. Therefore the "buy" offer from 3 is such that $u_2(z_3) + t_{32}(\mathbf{k}'_{32}) = \delta W^2(\mathbf{k}'_{32})$, which implies that the payoffs are given by

$$W_{32}^3(\mathbf{k}_{32}) = u_3^* + u_2(z_3) - \delta W^2(\mathbf{k}'_{32}) \text{ and } W_{32}^2(\mathbf{k}'_{32}) = \delta W^2(\mathbf{k}'_{32})$$

Note that this implies that $W_{32}^3(\mathbf{k}'_{32}) + W_{32}^2(\mathbf{k}'_{32}) = S_{23}(z_3) < S_{23}(z_2)$ so there is another offer that 3 can make that will leave him better off. In fact 3 is better off by making a sell offer as presented in Proposition 4.1. Finally let's focus in the case where 3 bought the votes from 1 and implements z_3 after decentralized bargaining. It is easy to see that in decentralized bargaining stage 3 does not have enough resources to pay 1 since $1 \gg 3$ so z_3 cannot be implemented in this way either. ■

Proof of Remark 5.5. First note that $(2b)$ implies that

$$\begin{aligned} \upsilon u_1^* - \delta^2 p_{13}\rho_{23}u_1(z_2) &\leq \delta^2 p_{13}\rho_{32}(u_1(z_2) + u_2^* + u_3(z_2)) \\ 0 &< \left(\upsilon - \delta^2 p_{13}\rho_{23}\right) u_1^* \leq \delta^2 p_{13}\rho_{32}(u_1(z_2) + u_2^* + u_3(z_2)) \end{aligned}$$

Now adding $(2b)$ and $(7b)$ we obtain

$$\upsilon\left(u_1^* + u_2(z_1) + u_3(z_1)\right) \leq \delta^2 (p_{31} + p_{13})(u_1(z_2) + u_2^* + u_3(z_2))$$

and since $0 < u_1(z_2) + u_2^* + u_3(z_2)$ it follows that

$$u_1^* + u_2(z_1) + u_3(z_1) < \frac{\delta(p_{31} + p_{13})}{\upsilon}(u_1(z_2) + u_2^* + u_3(z_2)) = \sum_i W^i(\mathbf{k})$$

■

Proof of Remark 5.7. Let $\mathbf{k}'_{32} = \mathbf{k} + (-k_1, 0, +k_1)$ denote the vote allocation after 1 sold the votes to 3. Recall that on the equilibrium path, there are only two potential transfers in the decentralized bargaining stage: $t_{13}(\mathbf{k})$ and $t_{31}(\mathbf{k})$, characterized by $t_{ji}(\mathbf{k}) = B^i(\mathbf{k}) - B^i(\mathbf{k}'_{32})$. Since in equilibrium 1 extends negotiations after UT, then $B^i(\mathbf{k}) = \delta W^i(\mathbf{k})$ while $B^1(\mathbf{k}'_{32}) = \delta u_1(z_2)$ and $B^3(\mathbf{k}'_{32}) = \delta\rho_{32}S_{32}(z_2)$. Therefore, we have that

$t_{31}(\mathbf{k}) = \delta W^1(\mathbf{k}) - \delta u_1(z_2)$ and $t_{13}(\mathbf{k}) = \delta W^3(\mathbf{k}) - \delta\rho_{32}S_{32}(z_2)$. Using Lemma (5.1) for $W^1(\mathbf{k})$ and $W^3(\mathbf{k})$, then

$$t_{31}(\mathbf{k}) = \frac{\delta p_{13}}{\upsilon}\Pi(13,2) - \delta u_1(z_2) \quad \text{and} \quad t_{13}(\mathbf{k}) = \frac{\delta p_{31}}{\upsilon}\Pi(13,2) - \delta\rho_{32}S_{32}(z_2),$$

In the majority-minority bargaining stage, on the other hand, we have $t_{32}(\mathbf{k}'_{32}) = \rho_{23}S_{32}(z_2) - u_2^*$ and $t_{23}(\mathbf{k}'_{32}) = \rho_{32}S_{32}(z_2) - u_3(z_2)$, and thus the expected transfer from 2 to 3 in majority-minority bargaining is given by

$$ET_{23}(\mathbf{k}') = \rho_{23}t_{23}(\mathbf{k}'_{32}) - \rho_{32}t_{32}(\mathbf{k}'_{32}) = \rho_{32}S_{23}(z_2) - u_3(z_2)$$

In order to calculate the expected transfer to 3 we use a recursive representation given by

$$\begin{aligned} ET(\mathbf{k}) &= (1 - p_{13} - p_{31})\,\delta ET(\mathbf{k}) + p_{13}\left(t_{13}(\mathbf{k}'_{32}) + \delta ET_{23}(\mathbf{k}'_{32})\right) + p_{31}\left(-t_{31}(\mathbf{k}'_{32}) + \delta ET_{23}(\mathbf{k}'_{32})\right) \\ &= \frac{ET_{13}(\mathbf{k}) + \delta(p_{13} + p_{31})\,ET_{23}(\mathbf{k}'_{32})}{\upsilon} \\ &= \frac{\delta}{\upsilon}\left[(p_{31} + p_{13})\,S(z_2) - (p_{13}u_1(z_2) + \theta S_{23}(z_2))\right] \end{aligned}$$

Using that $(6b)$ implies $\frac{\upsilon}{\delta^2}S_{12}(z_1) \leq p_{13}u_1(z_2) + \theta S_{23}(z_2)$ we have that

$$\begin{aligned} ET(\mathbf{k}) &\leq \delta\frac{(p_{31} + p_{13})\,S_{12}(z_2) - \frac{\upsilon}{\delta^2}S_{12}(z_1)}{\upsilon} \\ &\leq \delta\frac{(p_{31} + p_{13}) - \frac{\upsilon}{\delta^2}}{\upsilon}S_{12}(z_1) < 0 \end{aligned}$$

where the second line follows from $S_{12}(z_1) \geq S_{12}(z_2)$. ■

Proof of Proposition 5.10. **Part 1**. By Lemma 8.2, it is enough to analyze party 1's decisions in node $((1,3),\mathbf{k})$. First note that since $u_3^* \leq \delta\rho_{32}S_{23}(z_2)$, party 3 extends negotiations after buying from 1. Then from Lemma 4.2, we have $W_{13}^1(\mathbf{k}; sell) = \Pi(13,2) - B^3(\mathbf{k})$. Moreover, by hypothesis, $u_1^* \leq \delta W^1(\mathbf{k})$, so that party 1 extends negotiations after UT with three parties. Then $B^3(\mathbf{k}) = \delta W^3(\mathbf{k})$ and $W_{13}^1(\mathbf{k}; sell) = \Pi(13,2) - \delta W^3(\mathbf{k})$. Party 1's payoff for buying votes from 3 depends on whether party 1 would implement z_1 immediately or extend negotiations after buying from 3 (off the equilibrium path). If 1 implements z_1 after buying from 3 (either $1 \gg 2$, or $2 \gg 1$ and $u_1^* > \delta\rho_{12}S_{12}(z_2)$), then from Lemma 4.2, $W_{13}^1(\mathbf{k}; buy) = S_{13}(z_1) - \delta W^3(\mathbf{k})$. Then 1 prefers selling to buying iff $S_{13}(z_1) \leq \Pi(13,2)$, and prefers selling to extending negotiations iff $\delta[W^1(\mathbf{k}) + W^3(\mathbf{k})] \leq \Pi(13,2)$. If instead party 1 extends negotiations after buying from 3 ($2 \gg 1$ and $u_1^* \leq$

$\delta\rho_{12}S_{12}(z_2))$, then from Lemma 4.2, $W^1_{13}(\mathbf{k}; buy) = \Pi(31,2) - \delta W^3(\mathbf{k})$. Then 1 prefers selling to buying iff $\Pi(31,2) \leq \Pi(13,2)$, and selling to extending negotiations iff $\delta[W^1(\mathbf{k}) + W^3(\mathbf{k})] \leq \Pi(13,2)$. This proves part (1) of the Proposition.

Part 2. By Lemma 8.2, it is enough to analyze party 1's decisions in node $((1,2),\mathbf{k})$. By hypothesis $u_1^* \leq \delta W^1(\mathbf{k})$, and therefore in any equilibrium party 1 extends negotiations after UT in decentralized trading. Since $2 \gg 3$ (wlog), if 2 were to buy from 1, it would immediately implement z_2. We have two cases, depending on whether 1 would choose to (i.a) extend negotiations or (ii.b) implement its preferred policy after buying from 2.

(a) Suppose party 1 extends negotiations after buying from 2, and goes on to sell votes to 3 in the next trading period ($3 \gg 1$ and $u_1^* < \delta\rho_{13}S_{13}(z_3)$). Then $W^1_{12}(\mathbf{k}, gov) = u_1^* < W^1_{12}(\mathbf{k}, wait) = \delta W^1_{12}(\mathbf{k})$, $W^1_{12}(\mathbf{k}, sell) = u_1(z_2) + u_2^* - \delta W^2(\mathbf{k})$, and $W^1_{12}(\mathbf{k}, buy) = \Pi(21,3) - \delta W^2(\mathbf{k})$. Thus 1 prefers selling than buying iff

$$S_{12}(z_2) \geq \Pi(21,3), \tag{29}$$

prefers selling to extending negotiations if

$$S_{12}(z_2) \geq \delta[W^1(\mathbf{k}) + W^2(\mathbf{k})], \tag{30}$$

and prefers buying than extending negotiations if

$$\Pi(21,3) \geq \delta[W^1(\mathbf{k}) + W^2(\mathbf{k})]. \tag{31}$$

It follows that if (29) and (30), 1 sells to 2, who then implements z_2. And if (31) and not(29), 1 buys votes from 2, and goes on to sell votes to 3 in the next trading period. If, however, either (29) and not(30), or not(29) and not(31), 1 does not make a relevant offer and extends negotiations.

(b) Suppose party 1 implements its preferred policy immediately after buying from 2 (either $1 \gg 3$, or $3 \gg 1$ and $u_1^* > \delta\rho_{13}S_{13}(z_3)$). From (i) $W^1_{12}(\mathbf{k}; buy) = S_{12}(z_1) - \delta W^2(\mathbf{k})$. Moreover, from (ii), $B^1(\mathbf{k}) = \delta W^1(\mathbf{k})$ and $B^2(\mathbf{k}) = \delta W^2(\mathbf{k})$, and therefore $W^1_{12}(\mathbf{k}; sell) = S_{12}(z_2) - \delta W^2(\mathbf{k})$. Then 1 prefers selling than buying iff $2 \gg 1$, prefers selling to extending negotiations if (30), and buying to extending negotiations if

$$S_{12}(z_1) \geq \delta[W^1(\mathbf{k}) + W^2(\mathbf{k})]. \tag{32}$$

It follows that if $2 \gg 1$ and (30), 1 sells to 2 in $((1,2),\mathbf{k})$, who then implements z_2. And if $1 \gg 2$ and (32), 2 pays 1 in $((1,2),\mathbf{k})$ to implement z_1 immediately. If, however, either

$2 \gg 1$ and not(30), or $1 \gg 2$ and not(32), 1 doesn't make a relevant offer in $((1,2),\mathbf{k})$ and extends negotiations. This concludes the proof of part (2) of the Proposition.

Part 3. By Lemma 8.2, it is enough to analyze party 2's decisions in node $((2,3),\mathbf{k})$. We organize the analysis in four cases, depending on whether party 1 (i.a) implements z_1 or (i.b) extends negotiations after 2 buys from 3 (off the equilibrium path), and on whether party 1 (ii.a) implements z_1 or (ii.b) extends negotiations after 3 buys from 2 (off the equilibrium path). In all cases we require that 1 extends negotiations after UT in decentralized trading; i.e., $u_1^* \leq \delta W^1(\mathbf{k})$.

(a) Suppose (i) party 1 implements z_1 immediately after 2 sells to 3 (either $1 \gg 3$, or $3 \gg 1$ and $u_1^* > \delta\rho_{13}S_{13}(z_3)$), and (ii) party 1 implements z_1 immediately after 2 buys from 3 (either $1 \gg 2$, or $2 \gg 1$ and $u_1^* > \delta\rho_{12}S_{12}(z_2)$). Then from Lemma 4.3, $W_{23}^2(\mathbf{k}; sell) = S_{23}(z_1) - \delta W^3(\mathbf{k})$ and $W_{23}^2(\mathbf{k}; buy) = S_{23}(z_1) - \delta W^3(\mathbf{k})$. Thus 2 is indifferent between selling or buying, and prefers not to make a relevant offer in $((2,3),\mathbf{k})$ than to sell if and only if (7), or $S_{23}(z_1) \leq \delta[W^2(\mathbf{k}) + W^3(\mathbf{k})]$.

(b) Suppose (i) party 1 implements z_1 immediately after 2 sells to 3 (either $1 \gg 3$, or $3 \gg 1$ and $u_1^* > \delta\rho_{13}S_{13}(z_3)$), and (ii) party 1 extends negotiations after 2 buys from 3 ($2 \gg 1$ and $u_1^* \leq \delta\rho_{12}S_{12}(z_2)$). Then from Lemma 4.3, $W_{23}^2(\mathbf{k}; sell) = S_{23}(z_1) - \delta W^3(\mathbf{k})$ and $W_{23}^2(\mathbf{k}; buy) = \tilde{\Pi}(32,1) - \delta W^3(\mathbf{k})$. Then 2 prefers waiting to selling iff (7), and prefers waiting to buying iff

$$\tilde{\Pi}(32,1) \leq \delta[W^2(\mathbf{k}) + W^3(\mathbf{k})]. \tag{33}$$

(c) Suppose (i) party 1 extends negotiations after 2 sells to 3, and goes on to sell votes to 3 in the next trading period ($3 \gg 1$ and $u_1^* \leq \delta\rho_{13}S_{13}(z_3)$), and (ii) party 1 implements z_1 immediately after 2 buys from 3 (either $1 \gg 2$, or $2 \gg 1$ and $u_1^* > \delta\rho_{12}S_{12}(z_2)$). Then from Lemma 4.3, $W_{23}^2(\mathbf{k}; sell) = \tilde{\Pi}(23,1) - \delta W^3(\mathbf{k})$ and $W_{23}^2(\mathbf{k}; buy) = S_{23}(z_1) - \delta W^3(\mathbf{k})$. Then 2 prefers to wait than to buy iff (7), and to wait than to sell iff

$$\tilde{\Pi}(23,1) \leq \delta[W^2(\mathbf{k}) + W^3(\mathbf{k})]. \tag{34}$$

(d) Suppose (i) party 1 extends negotiations after 2 sells to 3, and goes on to sell votes to 3 in the next trading period ($3 \gg 1$ and $u_1^* \leq \delta\rho_{13}S_{13}(z_3)$), and (ii) party 1 extends negotiations after 2 buys from 3 ($2 \gg 1$ and $u_1^* \leq \delta\rho_{12}S_{12}(z_2)$). Then from Lemma 4.3, $W_{23}^2(\mathbf{k}; sell) = \tilde{\Pi}(23,1) - \delta W^3(\mathbf{k})$ and $W_{23}^2(\mathbf{k}; buy) = \tilde{\Pi}(32,1) - \delta W^3(\mathbf{k})]$. Then 2 prefers waiting to selling iff (34), and prefers waiting to buying iff (33). ■