On the coincidence of the Prenucleolus and the Shapley Value

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Abstract

We identify a sufficient class of coalitional form games with transferable utility for which prenucleolus coincides with the Shapley value. We then apply our result to simple games and to generalized queueing games.

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1 Introduction

In cooperative game theory with transferable utilities, the Shapley value (Shapley (1953)) is a widely used solution concept. It always exists, is unique, and given by an explicit formula that pays each player his average marginal contribution. Unlike the Shapley value, the prenucleolus and the nucleolus (Schmeidler (1969)) are solutions of a minimization problem. The prenucleolus is the set of efficient payoffs that lexicographically minimizes the "dissatisfaction" of all coalitions. The nucleolus is also the solution to a similar minimization problem but differs from the prenucleolus in requiring the payoff vector to be individually rational. The prenucleolus always exists and is unique while the nucleolus is unique but exists only when the set of individually rational and efficient payoff vectors is non-empty. In this paper, we identify conditions under which the prenucleolus coincides with the Shapley value.

It is easy to show that the Shapley value and the prenucleolus coincide on all two player games. From the axiomatization of prenucleolus (Sobolev (1975)) and the Shapley value (Hart and Mas-Colell (1989)), we know that both these solution concepts satisfy efficiency and symmetry and hence they also coincide on all symmetric TU games. However, the results of Sobolev (1975) and Hart and Mas-Colell (1989) also imply that the prenucleolus and the Shapley value differ in terms of the consistency properties.[1] Therefore, for non-symmetric TU games with three or more players, there is no reason to expect that these two solution concepts to coincide.

Yet, there are applications involving non-symmetric TU games where the Shapley value and the prenucleolus do coincide. This was first demonstrated in the context of undirected graphs and hypergraphs by Deng and Papadimitriou (1994). Subsequently, van den Nouweland, Borm, Brouwers, Bruinderink and Tijs (1996) applied coalitional form games to telecommunications problems and derive a class of games for which this coincidence takes place. Chun and Hokari (2004) demonstrate the coincidence in the context of a queueing game defined by Maniquet (1999). The results of these papers

[1]See also Winter (2002).

can be summarized as saying that the class of 2-games constitute a sufficient condition for the coincidence.[2]

In this paper we identify a class of TU games, which we call PS games, for which we have this coincidence. In a PS game, each player's marginal contribution to a coalition and its complement coalition adds up to a player specific constant that does not depend on the coalition. We show that this property is sufficient to ensure the coincidence of the Shapley value and the prenucleolus. Furthermore, the class of PS games is more general than the class of 2-games in that while every 2-game is also a PS game, the converse is not true.

We then apply our result to simple games. We show that a simple game is a PS game if and only if it is either dictatorial or 'bi-dictatorial' or a 'joint dictatorship.' We also show that for a three player non-symmetric simple game, the PS property is also necessary for the coincidence. However, for non-symmetric simple games with more than three players, PS property is not necessary.

Finally, we apply our coincidence result to queueing games. We define a class of queueing games in coalitional form which we call the generalized queueing games. This class includes, as special cases, the queueing game defined by Maniquet (1999) and by Chun (2004). We then identify the sub-class of generalized queueing games that belong to the class of PS games. We refer to this class of games as *reasonable queueing games*. In particular, all queueing games that are a convex combination of Maniquet's queueing game and Chun's queueing game belongs to this class of reasonable queueing games.

The paper is organized as follows. In section 2, we set up the general model. We provide our main result on the coincidence of prenucleolus and the Shapley value in section 3 and compare it with the existing literature. In section 4, we apply our coincidence result to simple games and generalized queueing games. We conclude our analysis in section 5.

[2] The 2-games are a special case of the k-games defined by Deng and Papadimitriou (1994).

2 Preliminaries

A coalitional form game with transferable utility (or a TU game) $G = (N, v)$ consists of a finite set $N = \{1, \ldots, n\}$ of players and a function $v : 2^N \rightarrow \Re$ that associates with every coalition (or subset) S of N a real number $v(S)$. The number $v(S)$ is the worth of S which is the total payoff that is available for division among the members of S. We define $v(\emptyset) = 0$. A profile $(x_i)_{i \in N} \in \Re^N$ is said to be *efficient* if $\sum_{i \in N} x_i = v(N)$. Let $X(N, v)$ be the set of all possible efficient payoff vectors of G. An *imputation* of G is an efficient payoff profile $x = (x_1, \ldots, x_n)$ for which $x_i \geq v(i)$ for all $i \in N$. Let $I(N, v)(\subseteq X(N, v))$ be the set of all imputations of G. Let $M_i(S) = v(S \cup \{i\}) - v(S)$ be the marginal contribution of player i to the coalition S. In particular, $M_i(\emptyset) = v(i)$.

DEFINITION 2.1 The Shapley value of a game $G = (N, v)$ is defined by $\psi_i(N, v) = (1/n!) \sum_{\pi \in \Pi} M_i(P_i(\pi))$ for each $i \in N$, where Π is the set of all $n!$ orderings of N and $P_i(\pi) = \{j | \pi(j) < \pi(i)\}$.

Consider a game $G = (N, v)$. We now define two very similar solution concepts that are related to the dissatisfaction level of a coalition. To measure how unhappy a coalition S will be with a payoff vector x in G, we look at the *excess* of S with respect to x which is defined as $e(S, x) = v(S) - \sum_{i \in S} x_i$. Using $e(S, x)$ as a measure of unhappiness with respect to x, we can try to find out a payoff vector which minimizes the maximum excess. We can construct a vector $\theta(x)$ by arranging the set of 2^n (subsets of N) excesses in decreasing order. Consider any two vectors y and z. With $y <_L z$ we mean that y is lexicographically smaller that z and we say that $y \leq_L z$ to indicate that either $y <_L z$ or $y = z$.

DEFINITION 2.2 The prenucleolus of a game $G = (N, v)$ is defined by $p\eta(N, v) = \{x \in X(N, v) \mid \theta(x) \leq_L \theta(y) \ \forall \ y \in X(N, v)\}$.

DEFINITION 2.3 The nucleolus of a game $G = (N, v)$ is defined by $\eta(N, v) = \{x \in I(N, v) \mid \theta(x) \leq_L \theta(y) \ \forall \ y \in I(N, v)\}$.

Definitions 2.2 and 2.3 differ in terms of their domain of operation. While the prenucleolus lexicographically minimizes the excess vector across the set of all efficient payoff vectors, the nucleolus does the same minimization across the set of all imputations. It can be shown, however, that the nucleolus and prenucleolus coincide for TU games with non-empty core[3] and for superadditive TU games.[4]

An equivalent way of defining the two solution concepts is in terms of objections and counterobjections. We define these as follows.

- A pair (S, y) consisting of a coalition S and an efficient vector (imputation) y is an *objection* to the efficient vector (imputation) x if $e(S, x) > e(S, y)$ (that is $\sum_{i \in S} y_i > \sum_{i \in S} x_i$).

- A coalition T is a *counterobjection* to the objection (y, S) if $e(T, y) > e(T, x)$ (that is $\sum_{i \in T} x_i > \sum_{i \in T} y_i$) and $e(T, y) \geq e(S, x)$.

DEFINITION 2.4 The prenucleolus $p\eta(N, v)$ of G is the set of all $x \in X(N, v)$ such that for every objection (S, y) to x there is a counterobjection to (S, y). Here, $y \in X(N, v)$.

DEFINITION 2.5 The nucleolus $\eta(N, v)$ of G is the set of all imputations $x \in I(N, v)$ such that for every objection (S, y) to x there is a counterobjection to (S, y). Here, $y \in I(N, v)$.

The equivalence between Definitions 2.3 and 2.5 is derived in Osborne and Rubinstein (1994). Since this result does not depend on the fact that attention is restricted to the set of imputations, it follows that Definitions 2.2 and 2.4 are also equivalent.

We end this section with two observations on the coincidence of the Shapley value and the prenucleolus. First, consider a two player game $(\{1, 2\}, v)$. A straightforward computation shows that $\psi_i(\{1, 2\}, v) = (v(\{1, 2\}) - v(j) + v(i))/2, i = 1, 2$ and that $e(\{1\}, \psi(v)) = e(\{2\}, \psi(v)) = (v(1) + v(2) -$

[3]See Peleg and Sudhölter (2003). The core of a TU game is the set of all unblocked allocations.

[4]See Moulin (1988). A TU game is superadditive if for all $S, T \subseteq N$ with $S \cap T = \emptyset$, $v(S \cup T) \geq v(S) + v(T)$.

$v(\{1,2\}))/2$. The fact that the excess vectors of the two singleton coalitions are identical along with the fact that the Shapley value is an efficient profile immediately implies that the Shapley value is also the prenucleolus.

Next, define a game $G = (N, v)$ to be **symmetric** if for all $i, j \in N$, $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N - \{i, j\}$. Again, it is straightforward to verify that for a symmetric game, $\psi_i(N, v) = p\eta_i(N, v) = \frac{v(N)}{|N|}$ for $i \in N$.

3 PS games

The set of symmetric games and two-players games discussed in the previous section are obviously a very small subset of coalitional form games. In this section, we show that the coincidence of Shapley value and the prenucleolus holds for a richer class of games which we call PS games. We also show that this class is more general than the class of 2-class games.

DEFINITION 3.6 A TU game $G = (N, v)$ satisfies the PS property if $\forall$ $i \in N$, $\exists$ $c_i \in \Re$ such that $\forall$ $S \subseteq N - \{i\}$, $M_i(S) + M_i(N - \{i\} - S) = c_i$.

We refer to any TU game satisfying the PS property as a PS game. We denote this class of TU games by $\mathbf{G}(PS)$.

THEOREM 3.1 If $G = (N, v) \in \mathbf{G}(PS)$ then $\psi(N, v) = p\eta(N, v)$.

Proof: Let $G = (N, v) \in \mathbf{G}(PS)$ and consider the efficient profile $x^* = (x_1^*, \ldots, x_n^*)$ such that $e(i, x^*) = e(N - \{i\}, x^*)$ for all $i \in N$. It is easy to verify that x^* is unique and given by $x_i^* = (v(N) - v(N - \{i\}) + v(i))/2 = c_i/2$ for all $i \in N$.
Claim 1: $\displaystyle\sum_{i \in S} x_i^* = \frac{v(N) - v(N - S) + v(S)}{2}$ for all $S \subseteq N$.
We apply induction to prove Claim 1. First note that Claim 1 is true for any singleton coalition $T = \{i\}$. We assume that Claim 1 is true for any $T \subset N$ with the property that $|T| = m < n$. To prove Claim 1, we will have to show that it holds for any $T' = T \cup \{j\}$ where $j \in N - T$. Given the vector x^* and our assumption we get

$$x_j^* + \sum_{i \in T} x_i^* = \frac{v(N) - v(N - \{j\}) + v(j)}{2} + \frac{v(N) - v(N - T) + v(T)}{2} \quad (3.1)$$

From the PS property we know that $M_j(\emptyset) + M_j(N - \{j\}) = M_j(T) + M_j(N - \{j\} - T)(\equiv c_j)$ and hence

$$v(N) - v(N - \{j\}) + v(j) = v(T') - v(T) + v(N - T) - v(N - T') \quad (3.2)$$

Substituting (3.2) in (3.1) and then simplifying it we get

$$\sum_{i \in T'} x_i^* = \frac{v(N) - v(N - T') + v(T')}{2} \quad (3.3)$$

Thus, we have established that if Claim 1 is true for any $T \subset N$ with the property that $|T| = m < n$ then Claim 1 is also true for any $T' \subset N$ with the property that $|T'| = m + 1$. This proves Claim 1.

Claim 2: $x^* = \left(\frac{c_1}{2}, \dots, \frac{c_n}{2}\right) = p\eta(N, v)$.

Using Claim 1 one can verify that for all $S \subseteq N$ and $S \neq \emptyset$

$$e(S, x^*) = e(N - S, x^*) = \frac{v(S) + v(N - S) - v(N)}{2} \quad (3.4)$$

From (3.4) it follows that the profile $x^* \in X(N, v)$ has the property that for every objection (S, y) to x^*, the coalition $T = N - S$ is a counterobjection to (S, y). To see this, consider any objection (S, y) to x^*. This means that $\sum_{i \in S} y_i > \sum_{i \in S} x_i^*$ and hence $e(N - S, x^*) < e(N - S, y)$. To show that $T = N - S$ is a valid counterobjection to (S, y) we will now show that $e(N-S, y) > e(S, x^*)$. Observe that $e(N-S, y) = v(N-S) - v(N) + \sum_{i \in S} y_i > v(N-S) - v(N) + \sum_{i \in S} x_i^* = e(S, x^*)$. The last equality follows from Claim 1 and condition (3.4). Hence from Definition 2.4 it follows that $x^* = p\eta(N, v)$.

Claim 3: $x^* = \left(\frac{c_1}{2}, \dots, \frac{c_n}{2}\right) = \psi(N, v)$.

To prove this claim choose any ordering $\pi = (\pi_1, \dots, \pi_n) \in \Pi$ of the agents in N and its dual $\pi' = \underline{n+1} - \pi = (n + 1 - \pi_1, \dots, n + 1 - \pi_n) \in \Pi$. From PS property it follows that $M_i(P_i(\pi)) + M_i(P_i(\pi')) = c_i$, since $P_i(\pi') = N - P_i(\pi) - \{i\}$. Using the definition of the Shapley value we get $\psi_i(N, v) = \frac{1}{n!}\left(\frac{c_i n!}{2}\right) = \frac{c_i}{2}$ for all $i \in N$. Therefore, $x^* = \psi(N, v)$. ■

REMARK 3.1 The following two observations are immediate consequences of Theorem 3.1: (1) an additive TU game G is a PS game with $c_i = 2v(i)$

for all $i \in N$ and (2) a zero-sum game is a PS game only if it is additive.

REMARK 3.2 Note that the PS property, while strong, is different from convexity. A game $G = (N, v)$ is **convex** if and only the marginal contribution of a player to a coalition is monotone nondecreasing with respect to set theoretic inclusion. The following examples shows that the PS property and convexity are independent.

EXAMPLE 3.1 Let $G = (\{1,2,3\}, v)$ with $v(1) = v(3) = v(13) = 1$, $v(2) = 0$, $v(23) = 2$, $v(12) = 3$ and $v(123) = 4$. This game is a PS game with $c_1 = 3, c_2 = 3, c_3 = 2$. However, this game is not convex since $v(123) - v(23) = 2 < 3 = v(12) - v(2)$. To see that convexity does not imply the PS property, let $G = (\{1,2,3\}, w)$ with $w(1) = w(2) = w(3) = 0, w(12) = w(13) = 0.5, w(23) = 1, w(123) = 3$. This game is convex but it does not satisfy the PS property for player 1 because $w(12) - w(2) + w(13) - w(3) = 1 \neq 2 = w(123) - w(23) + w(1) - w(\emptyset)$.

3.1 The class of k games

The following definition is due to Deng and Papadimitriou (1994).

DEFINITION 3.7 A coalitional form game $G^k = (N, v)$ is a k-game if

$$v(S) = \begin{cases} 0 & \text{if } |S| < k \\ \sum\limits_{T \subseteq S, |T| = k} v(T) & \text{otherwise} \end{cases}$$

Observe that a 1-game is simply an additive game and hence it is a PS game. In different applications of cooperative game theory, the 2-game has led to the coincidence of prenucleolus and the Shapley value.[5] The following proposition establishes that every 2-game is a PS game.

PROPOSITION 3.1 A 2-game $G = (N, v)$ is a PS game.

[5] See Brown and Housman (1988), Chun and Hokari (2005), Deng and Papadimitriou (1994) and Nouweland, Borm, Brouwers, Bruinderink and Tijs (1996)).

Proof: It follows from the definition of a 2-game that for all $i \in N$ and all $S \subseteq N - \{i\}$,

$$M_i(S) + M_i(N - S - \{i\}) = \sum_{j \in S} v(ij) + \sum_{j \in N-S-\{i\}} v(ij) = \sum_{j \in N-\{i\}} v(ij).$$

This shows that a 2-game is a PS game with $c_i = \sum_{j \in N-\{i\}} v(ij)$ for all $i \in N$. ■

The converse of Proposition 3.1 is not true. In particular, the TU game $(\{1,2,3\}, v)$ in Example 3.1 is a PS game but not a 2-game.

4 Applications

In this section, we consider some applications of cooperative game theory where our main result can be applied.

4.1 Simple Games

Simple games—where the value of a coalition is either zero or one—are games used mainly to describe parliaments, councils and committees. They occur in many applications of game theory to political science.[6] The Shapley value in the context of simple games is a measure of the "power" of individual players and is better known as the Shapley-Shubik index.[7] Peleg and Sudhölter (2003) have studied the nucleolus in the context of simple games. Before analyzing the PS property in this context, we provide some relevant definitions.

DEFINITION 4.8 A game $G = (N, v)$ is a **simple game** if (i) $v(S) \in \{0, 1\}$ for all $S \subseteq N$, (ii) $v(N) = 1$ and (iii) if $v(S) = 1$ and $S \subset T$ then $v(T) = 1$.

DEFINITION 4.9 The simple game $G^s = (N, v)$ is **dictatorial** if there exists a player $i \in N$ such that for all $S \subseteq N$, $v(S) = 1$ if and only if $i \in S$.

[6] See Shapley (1962) and Curiel (1996).
[7] See Shapley and Shubik (1954).

DEFINITION 4.10 The simple game $G^s = (N, v)$ is **bi-dictatorial** if there exist two distinct players $i, j \in N$ such that $v(S) = 1$ if and only if $\{i, j\} \cap S \neq \emptyset$.

DEFINITION 4.11 The simple game $G^s = (N, v)$ is a **joint dictatorship** if there exists two distinct players $i, j \in N$ such that $v(S) = 1$ if and only if $\{i, j\} \subset S$.

We shall need the following terminology in proving the main result of this section. Let $G = (N, v)$ be a simple game. A coalition S is a *swing* for player i if $v(S \cup \{i\}) = 1$ and $v(T) = 0$ for all $T \subseteq S$. Player i is a *null player* if $M_i(S) = 0$ for all $S \subseteq N - \{i\}$.

PROPOSITION 4.2 The simple game $G^s = (N, v)$ is a PS game *if and only if* it is either dictatorial or bi-dictatorial or a joint dictatorship.

Proof: (**Necessity**) In a simple game, $M_i(S) = 1$ if S is a swing coalition for i and zero otherwise. Hence, $M_i(S) \in \{0, 1\}$ for all $i \in N$. Therefore, a simple game G^s is a PS game only if for all $i \in N$ and for all $S \subseteq N - \{i\}$, $c_i = M_i(S) + M_i(N - S - \{i\}) \in \{0, 1, 2\}$. Moreover, for a simple game satisfying the PS property, Theorem 3.1 implies that the Shapley Value and the prenucleolus are given by $\psi_i(N, v) = p\eta_i(N, v) = c_i/2$ for all $i \in N$. Since both solutions are efficient, we have $\sum_{k \in N} \psi_k(N, v) = \sum_{k \in N} p\eta_k(N, v) = \sum_{k \in N} c_k/2 = 1$. Therefore, given $c_i \in \{0, 1, 2\}$ for all $i \in N$ and $\sum_{k \in N} c_k = 2$, we have two possibilities.

Case 1 The simple game $G^s(N, v)$ is a PS game such that there exists $i \in N$ with $c_i = 2$ and $c_k = 0$ for all $k \in N - \{i\}$.

Case 2 The simple game $G^s(N, v)$ is a PS game such that there exists $i, j \in N$ with $c_i = c_j = 1$ and $c_k = 0$ for all $k \in N - \{i, j\}$.

Case 1: Since $c_i = M_i(S) + M_i(N - S - \{i\}) = 2$ for all $S \subseteq N - \{i\}$, this implies that $M_i(S) = 1$ for all $S \subseteq N - \{i\}$ and hence $v(S \cup \{i\}) = 1$ and $v(S) = 0$ for all $S \subseteq N - \{i\}$. Hence, the simple game is dictatorial.

Case 2: We will show that the simple game must be either bi-dictatorial or a joint dictatorship. We first prove the following claim.

Claim: If $c_i = c_j = 1$ and $c_k = 0$ for all $k \in N - \{i, j\}$, then $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N - \{i, j\}$.

We prove this claim by contradiction. Suppose that there exists $S \subseteq N - \{i, j\}$ such that $v(S \cup \{i\}) \neq v(S \cup \{j\})$. Without loss of generality let

$$v(S \cup \{i\}) = 1 > v(S \cup \{j\}) = 0. \tag{4.5}$$

Since $c_i = c_j = 1$, it follows that

$$M_i(S) + M_i(N - S - \{i\}) = M_j(S) + M_j(N - S - \{j\}) = 1. \tag{4.6}$$

Since $v(S \cup \{j\}) = 0$, the monotonoicity of a simple game implies that $v(S) = 0$.[8] Hence, by (4.5), $M_i(S) = 1$ and $M_j(S) = 0$. By (4.6), it follows that $M_j(N - S - \{j\}) = 1 > M_i(N - S - \{i\}) = 0$. Therefore, $v(N - S) = v(N - S - \{i\}) = 1 > v(N - S - \{j\}) = 0$. Using (4.5) and (4.6) again, we get $v(S \cup \{i\}) = 1$, $v(S^c \cup \{i\}) = 0$, $v(S \cup \{j\}) = 0$ and $v(S^c \cup \{j\}) = 1$ where $S^c = N - S - \{i, j\}$. Since $c_k = 0$ for all $k \in N - \{i, j\}$, all such players are null players. Adding the null players in S^c to $S \cup \{i\}$, we get $v(S^c \cup S \cup \{i\}) = v(S \cup \{i\})$. Hence, $v(N - \{j\}) = 1$. On the other hand, adding the null players of S to $S^c \cup \{i\}$, we get $v(S \cup S^c \cup \{i\}) = v(S^c \cup \{i\})$ which implies $v(N - \{j\}) = 0$. This is a contradiction and proves the claim.

From the above claim it follows that players i and j are symmetric. Now consider player i. Since $c_i = M_i(j) + M_i(N - \{i, j\}) = 1$, we have two possibilities.

2a) $M_i(j) = 0$ and $M_i(N - \{i, j\}) = 1$

2b) $M_i(j) = 1$ and $M_i(N - \{i, j\}) = 0$

If 2a) is true, then $v(N - \{j\}) = 1$. Using symmetry of players i and j we get $v(N - \{i\}) = 1$. Moreover, all players $k \in N - \{i, j\}$ are null players (since $c_k = 0$) and therefore, by the monotonicity property of the simple game, $v(T - \{i\}) = v(T - \{j\}) = 1$ for all $T \subseteq N$ such that $i, j \in T$. Thus, for $T = \{i, j\}$ we get $v(i) = v(j) = 1$. Monotonicity of the simple game gives

[8] See Definition 4.8(iii).

that for $l \in \{i, j\}$, $v(S \cup \{l\}) = 1 > v(S)$ for all $S \subseteq N - \{i, j\}$ and $M_i(j) = 0$ gives $v(ij) = v(i) = v(j) = 1$. So $\{i, j\}$ are the only dictators.

If 2b) is true, then $v(ij) = 1 > v(j) = 0$. Using the fact that all players $k \in N - \{i, j\}$ are null players and the symmetry of i and j we get $v(S \cup \{i\}) = v(S \cup \{j\}) = 0$ for all $S \subseteq N - \{i, j\}$. Using $v(ij) = 1$ and monotonicity of simple games we get $v(T) = 1$ for all $T \subseteq N$ such that $i, j \in T$. So i and j are joint dictators.

(**Sufficiency**) If a simple game is dictatorial, then for the dictator i, $M_i(S) = 1$ for all $S \subseteq N - \{i\}$ and hence $M_i(S) + M_i(N - S - \{i\}) = 2 = c_i$. Moreover, since all other players are null players, for all $k \in N - \{i\}$, $M_k(S) = 0$ and hence $M_k(S) + M_i(N - S - \{k\}) = 0 = c_k$ for all $S \in N - \{k\}$. Therefore, if a simple game is dictatorial then it is a PS game.

If a simple game is bi-dictatorial, then players i and j are the two dictators and all other players in $N - \{i, j\}$ are null players. Consider player i. Note that for player i, $M_i(S) = 0$ if $j \in S$ and $M_i(S) = 1$ if $j \notin S$. Thus, player i is a swing for a coalition S if $j \notin S$ and player i is not a swing for a coalition S if $j \in S$. Therefore, for all $S \subseteq N - \{i\}$, $M_i(S) + M_i(N - S - \{i\}) = 1$ since either (a) $j \in S \Leftrightarrow j \notin N - S - \{i\}$ or (b) $j \notin S \Leftrightarrow j \in N - S - \{i\}$. A same sort of reasoning holds for player j. Any player $k \in N - \{i, j\}$ is a null player, we get $c_k = 0$ for all $k \in N - \{i, j\}$. Therefore, if a simple game is bi-dictatorial then it is a PS game.

Now consider a simple game which is a joint dictatorship with i and j the joint dictators and all other players in $N - \{i, j\}$ are null players. For player i, $M_i(S) = 0$ if $j \notin S$ and $M_i(S) = 1$ if $j \in S$. Thus, player i is a swing for a coalition S if $j \in S$ and player i is not a swing for a coalition S if $j \notin S$. Therefore, for all $S \subseteq N - \{i\}$, $M_i(S) + M_i(N - S - \{i\}) = 1$ since either (a) $j \in S \Leftrightarrow j \notin N - S - \{i\}$ or (b) $j \notin S \Leftrightarrow j \in N - S - \{i\}$. A similar reasoning holds for player j. Since any player $k \in N - \{i, j\}$ is a null player, we get $c_k = 0$ for all $k \in N - \{i, j\}$. This shows that a joint dictatorship is a PS game and concludes the proof of the theorem. ∎

We conclude this section with two observations.

1. *For a non-symmetric three player simple game, PS property is* **necessary** *for the coincidence.* There can be exactly two types of non-

symmetric three player simple games that are not PS games. They are (a) $\bar{G}^s = (\{i,j,k\}, v)$ with $v(j) = v(k) = 0$, $v(i) = v(ij) = v(ik) = v(jk) = v(ijk) = 1$ and (b) $\tilde{G}^s = (\{i,j,k\}, v)$ with $v(i) = v(j) = v(k) = v(jk) = 0$, $v(ij) = v(ik) = v(ijk) = 1$. For $\bar{G}^s$, $\psi(N,v) = (\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) \neq p\eta(N,v) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. For $\tilde{G}^s$, $\psi(N,v) = (\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) \neq p\eta(N,v) = (1,0,0)$. Thus, for non-symmetric three player games that are not PS games, $\psi(N,v) \neq p\eta(N,v)$.

2. *For a non-symmetric simple game with more than three players, PS property is* **not necessary** *for the coincidence.* Consider the TU game $G^s = (\{1,2,3,4\}, v)$ where $v(1) = v(2) = v(3) = v(4) = v(13) = v(14) = v(23) = v(24) = v(34) = 0$, $v(12) = 1$, $v(S) = 1$ for all $S \subseteq \{1,2,3,4\}$ such that $|S| \geq 3$. G^s is not a PS game (since $M_1(\emptyset) + M_1(234) < M_1(2) + M_1(34)$) and yet $\psi(N,v) = p\eta(N,v) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$.

4.2 Generalized queueing games

Let $N = \{1, \ldots, n\}$ be the set of agents. Each agent wants to consume a service provided by a server. It is assumed that agents can be served only sequentially and that serving any agent occupies a unit of time. Each agent is identified with a waiting cost $\theta_i \in \Re_+$ which is her disutility of waiting in the queue. If agent i occupies the σ_ith position in the queue, then her cost is $-(\sigma_i - 1)\theta_i$. Let $\theta_N = (\theta_1, \ldots, \theta_n)$ be a profile of waiting costs of all the agents. Given a profile of waiting cost $\theta_N = (\theta_1, \ldots, \theta_n) \in \Re^n_+$ and a non-empty set $S (\subseteq N)$ of agents with associated waiting costs $\theta_S = (\theta_i)_{i \in S}$, a queue $\sigma^*(\theta_S) = \{\sigma^*_i(\theta_S)\}_{i \in S} \in \tau(|S|)$ is said to be *S-efficient* if it minimizes the aggregate waiting costs for the set of S agents, that is $\sigma^*(\theta_S) \in \operatorname{argmin}_{\sigma \in \tau(|S|)} \sum_{i \in S} (\sigma_i - 1)\theta_i$ where $\tau(|S|)$ is the set of all possible permutations of the integers $\{1, \ldots, |S|\}$. The queue $\sigma^*(\theta_N)$ is said to be an *efficient* queue for the set of N agents.

There are different ways of modeling queueing situation as a coalitional form TU game. Maniquet (1999) has one way of defining a queueing game in coalitional form. He defines $v(S) = -\sum_{i \in S} (\sigma^*_i(\theta_S) - 1)\theta_i$ for all $S \subseteq N$. This is an optimistic approach to the queueing problem because a coalition

S thinks that its members will be served first in the queue (that is, before the agents of coalition $N - S$). Hence, they agree to the S-efficient queue amongst themselves. The other approach is the pessimistic approach of Chun (2004) where each coalition thinks that its members will be served only after all the members of their complement coalition has been served. So for Chun's game, $v(S) = -\sum_{i \in S}(n - |S| + \sigma_i^*(\theta_S) - 1)\theta_i$ for all $S \subseteq N$.

DEFINITION 4.12 A TU game $G^Q = (N, v)$ is a **generalized queueing game** if given any $\theta_N = (\theta_1, \ldots, \theta_n) \in \Re^n_+$,

1. $v(S) = \sum\limits_{i \in S} a\left(|S|, \sigma_i^*(\theta_S)\right)\theta_i$ for all $S \subseteq N$ with $S \neq \emptyset$,
2. $\forall\, S \subset N$, $|S| > 1$, $\{a\left(|S|, k+1\right) < a\left(|S|, k\right) \leq 0\}\; \forall\, k \in \{1, \ldots, |S|-1\}$.
3. $a\left(|N|, \sigma_i^*(\theta_N)\right) = -(\sigma_i^*(\theta_N) - 1)$ for all $i \in N$ and
4. $a(1, 1) \in [-(n-1), 0]$.

This is a very general way of representing a queueing situation as a coalitional form game. The first condition simply gives us the worth of a coalition. The second restriction simply guarantees that in a coalition the agent served earlier incur lower waiting cost than agents served later. The third restriction follows from efficiency condition for the grand coalition. The fourth restriction that $a(1, 1) \in [-(n-1), 0]$ is reasonable in the sense that the best thing that can happen to a single agent is that the agent gets first position in the queue implying $a(1, 1)$ must be at most zero and the worst thing that can happen to a singleton coalition is to get the last queue position which means that $a(1, 1)$ must be weakly greater than $-(n-1)$.

Observe that if $a\left(|S|, \sigma_i^*(\theta_S)\right) = -(\sigma^*(\theta_S) - 1)$ for all $S \subseteq N$ then we have Maniquet's queueing game (see Maniquet (1999)) and if $a\left(|S|, \sigma_i^*(\theta_S)\right) = -(n - |S| + \sigma^*(\theta_S) - 1)$ for all $S \subseteq N$ then we have Chun's queueing game (see Chun (2004)). Using this general specification we try to identify the sub-class of generalized queueing games that are PS games.

DEFINITION 4.13 A generalized queueing game G^Q is said to be a **rea-**

sonable queueing game if for all non-empty sets $S \subseteq N$,

$$a\left(|S|, \sigma_i^*(\theta_S)\right) = -\left\{\delta(n - |S|) + (\sigma_i^*(\theta_S) - 1)\right\} \tag{4.7}$$

where $\sigma_i^*(\theta_S) \in \{1, \ldots, |S|\}$ and $\delta = -\frac{a(1,1)}{(n-1)} \in [0, 1]$.

We can interpret condition (4.7) in the following way. In our generalized queueing game, $a(|S|, \sigma_i^*(\theta_S))$ measures the externality imposed by all agents (that is, agents from the set $N - \{i\}$) on an agent $i \in S$ whose queue position is $\sigma_i^*(\theta_S)$ in the *S-efficient* queue. For a reasonable queueing game, $a(|S|, \sigma_i^*(\theta_S))$ has two components. The component $(\sigma_i^*(\theta_S)) - 1)$ captures the externality imposed upon $i \in S$ by her group members. The term $\delta(n - |S|)$ captures the externality imposed on agent $i \in S$ by outsiders (that is, agents in $N - S$). It shows that the externality imposed by the outsiders (that is, $N - S$) is proportional to the number of outsiders.

PROPOSITION 4.3 A generalized queueing game G^Q is a PS game *if and only if* it is a reasonable queueing game.

Proof: For PS property it is necessary that for all $i \in N$ and for all $S \subseteq N - \{i\}$, $MC_i(S) + MC_i(N - \{i\} - S) = c_i$, where c_i is a number independent of S. For a generalized queueing problem this means that given any $\theta_N = (\theta_1, \ldots, \theta_n) \in \Re_+^n$ and any $i \in N$,

$$\alpha_i(S)\theta_i + \sum_{j \in S} \beta_j(S)\theta_j + \sum_{l \in N - S} \gamma_l(S)\theta_l = c_i \tag{4.8}$$

for all $S \subseteq N - \{i\}$. Here $\alpha_i(S)$, $\beta_j(S)$ and $\gamma_l(S)$ have the following expressions:

(1) $\alpha_i(S) = a\left(|S| + 1, \sigma_i^*(\theta_{S \cup \{i\}})\right) + a\left(|N - S|, \sigma_i^*(\theta_{N-S})\right)$,

(2) $\beta_j(S) = a\left(|S| + 1, \sigma_j^*(\theta_{S \cup \{i\}})\right) - a\left(|S|, \sigma_j^*(\theta_S)\right)$ and

(3) $\gamma_l(S) = a\left(|N - S|, \sigma_l^*(\theta_{N-S})\right) - a\left(|N - S| - 1, \sigma_l^*(\theta_{N-\{i\}-S})\right)$.

We first argue that $\alpha_i(S)$ must be independent of S. Consider any state $\theta_N \in \Re^n_+$ with the property that $\theta_j \neq \theta_l$ for all $j, l \in N$ with $j \neq l$, any agent $i \in N$ and any two distinct sets S_1 and S_2 such that $S_1 \subset S_2 \subseteq N - \{i\}$. Using condition (4.8) for S_1 and for S_2 separately and then taking their difference we get

$$(\alpha_i(S_2) - \alpha_i(S_1))\theta_i + \sum_{j \neq i} z(S_2, S_1)\theta_j = 0 \tag{4.9}$$

Now consider another state $\theta'_N \in \Re^n_+$ with the property that $\theta'_j = \theta_j$ for all $j \in N - \{i\}$, $\theta'_i \neq \theta_i$ and $\sigma^*(\theta'_N) = \sigma^*(\theta_N)$. Therefore, the state θ'_N is constructed from the state θ_N by perturbing the waiting cost of agent i in such a way that the efficient queue remains unchanged under both the states. Like in the earlier case, consider the sets S_1 and S_2. Using condition (4.8) for S_1 and for S_2 separately and then taking their difference we get

$$(\alpha_i(S_2) - \alpha_i(S_1))\theta'_i + \sum_{j \neq i} z(S_2, S_1)\theta_j = 0 \tag{4.10}$$

By subtracting (4.9) from (4.10) we get

$$(\alpha_i(S_2) - \alpha_i(S_1))(\theta'_i - \theta_i) = 0 \tag{4.11}$$

Condition (4.11) implies that $\alpha_i(S_1) = \alpha_i(S_2)$ since $\theta'_i \neq \theta_i$. Since the selection of S_1 and S_2 was arbitrary, it follows that for any $i \in N$,

(a) $\alpha_i(S) = \alpha_i$ for all $S \subseteq N - \{i\}$.

Observe that given any $\theta_N = (\theta_1, \ldots, \theta_n)$ and any $i \in N$ with queue position $\sigma^*_i(\theta_N)$,

(b) if $\sigma^*_i(\theta_{S \cup \{i\}}) = y$ then $\sigma^*_i(\theta_{N-S}) = \sigma^*(\theta_N) + 1 - y$.

From (1), (a) and (b) we get the following: Given any queue position $\sigma^*_i(\theta_N) = q \in \{1, \ldots, n\}$, for all $x \in \{1, \ldots, n\}$ and for all $y \in \{1, \ldots, min[q, x]\}$

$$a(x, y) + a(n + 1 - x, q + 1 - y) = -(q - 1) + a(1, 1) \tag{4.12}$$

Consider a coalition S such that $i, j \notin S$, $x = |S| + 1$, $\sigma^*_i(\theta_N) = q < \sigma^*_j(\theta_N) = q + 1$ and $\sigma_l(\theta_N) > \sigma_j(\theta_N)$ for all $l \in S$. Using condition (4.12) for agents i and j separately we get

$$a(x,1) + a(n+1-x,q) = -(q-1) + a(1,1) \tag{4.13}$$

$$a(x,1) + a(n+1-x,q+1) = -q + a(1,1) \tag{4.14}$$

Subtracting (4.14) from (4.13) we get

$$a(n+1-x,q) - a(n+1-x,q+1) = 1 \tag{4.15}$$

Solving condition (4.15) we get for all $q \leq r \equiv n+1-x$,

$$a(r,q) = a(r,1) - (q-1) \tag{4.16}$$

Using condition (4.16) in the generalized queueing problem we get for all $S \subseteq N$,

$$v(S) = \sum_{i \in S} \left[a(|S|,1) - (\sigma_i^*(\theta_S) - 1)\right] \theta_i \tag{4.17}$$

Using observation (b) in condition (4.12) for $q = 1$ we get for all $x \in \{1, \ldots, n-1\}$,

$$a(x,1) + a(n+1-x,1) = a(1,1) \tag{4.18}$$

Now consider the term $\beta_j(S) = a\left(|S|+1, \sigma_j^*(\theta_{S \cup \{i\}})\right) - a\left(|S|, \sigma_j^*(\theta_S)\right)$ in condition (4.8). We argue that for $|N| \geq 3$, $\beta_j(S)$ term is independent of S for any $j \in S \subseteq N - \{i\}$ such that $\sigma_j^*(\theta_N) < \sigma_i^*(\theta_N)$. Consider any state $\theta_N \in \Re_+^n$ with the property that $\theta_l \neq \theta_k$ for all $k, l \in N$ with $k \neq l$ and consider $j \in T_1 \subset T_2 \subseteq N - \{i\}$. By applying condition (4.8) for the sets T_1 and T_2 separately and then taking the difference we get

$$(\beta_j(T_2) - \beta_j(T_1))\theta_j + \sum_{l \in N - \{i,j\}} \hat{z}(T_2, T_1)\theta_l = 0 \tag{4.19}$$

Observe that in (4.19), condition (a) guarantees that the left hand side is independent of θ_i. Now consider another state $\theta'_N \in \Re_+^n$ with the property that $\theta'_l = \theta_l$ for all $l \in N - \{j\}$, $\theta'_j \neq \theta_j$ and $\sigma^*(\theta'_N) = \sigma^*(\theta_N)$. Therefore, the state θ'_N is constructed from the state θ_N by perturbing the waiting cost of agent j in such a way that the efficient queue remains unchanged under both the states. Like in the earlier case, consider the sets T_1 and T_2. Using condition (4.8) for T_1 and for T_2 separately and then taking their difference we get

$$(\beta_j(T_2) - \beta_j(T_1))\theta'_j + \sum_{l \in N-\{i,j\}} \hat{z}(T_2, T_1)\theta_l = 0 \tag{4.20}$$

By subtracting (4.19) from (4.20) we get

$$(\beta_j(T_2) - \beta_j(T_1))(\theta'_j - \theta_j) = 0 \tag{4.21}$$

Condition (4.21) gives that

(c) $\forall \; j \in T_1 \subset T_2 \subseteq N - \{i\}$ such that $\sigma^*_j(\theta_N) < \sigma^*_i(\theta_N)$, $\beta_j(T_2) = \beta_j(T_1)$ since $\theta'_j \neq \theta_j$.

(d) Observe that if $\sigma^*_j(\theta_N) < \sigma^*_i(\theta_N)$ then for all $S \subseteq N - \{i\}$ such that $j \in S$, $\sigma^*_j(\theta_{S\cup\{i\}}) = \sigma^*_j(\theta_S)$ and hence $\beta_j(S) = a(|S|+1, \sigma^*_j(\theta_S)) - a(|S|, \sigma^*_j(\theta_S))$.

From observations (c) and (d) we get for all $x \in \{1, \dots, n-2\}$ and for all $p \in \{1, \dots, x\}$,

$$a(x+1, p) - a(x, p) = \bar{e} \tag{4.22}$$

Using (4.16) in (4.22) we get

$$a(x+1, 1) - a(x, 1) = \bar{e} \tag{4.23}$$

From (4.23) we get $x \in \{1, \dots, n-2\}$

$$a(x, 1) = (x-1)\bar{e} + a(1, 1) \tag{4.24}$$

From (4.18) and (4.24) we get

$$a(x, 1) = \left(\frac{n-x}{n-1}\right) a(1, 1) \tag{4.25}$$

By substituting condition (4.25) in (4.17) we get

$$v(S) = \sum_{i \in S} \left[\left(\frac{n - |S|}{n - 1} \right) a(1,1) - (\sigma_i^*(\theta_S) - 1) \right] \theta_i \qquad (4.26)$$

Substituting $\frac{a(1,1)}{n-1} = -\delta$ in (4.26) we get the result.

To prove the sufficiency part, note that for a realistic queueing game we have for all $i \in N$ and for all $S \subseteq N - \{i\}$,

$$MC_i(S) + MC_i(N - S - \{i\}) = -a(1,1) \sum_{j \in N - \{i\}} \theta_j - (\sigma_i^*(\theta_N) - 1)\theta_i - \sum_{s \in P_i^c(\sigma^*(\theta_N))} \theta_s \quad (4.27)$$

where $P_i^c(\sigma^*(\theta_N)) = \{s \in N - \{i\} \mid \sigma_i^*(\theta_N) < \sigma_s^*(\theta_N)\}$. Observe that the right hand side of (4.27) is independent of S. ■

Proposition 4.3 establishes that for a reasonable queueing game, the Shapley value will coincide with prenucleolus. Observe, that if in a reasonable queueing game, $\delta = 0$ then we have Maniquet's optimistic queueing game which gives no weightage to the players outside the coalition. If $\delta = 1$ we have Chun's pessimistic queueing game that gives full weightage to the players outsider the coalition. Therefore, our reasonable class of queueing games includes all queueing games that are a convex combination of the optimistic and the pessimistic queueing games and all these games are PS games.

5 Conclusion

The chief contribution of this paper has been to shed further light on the coincidence of the Shapley value and the prenucleolus. As noted earlier, these solutions are motivated by very different concerns and as such, there is no reason to expect them to coincide. We have extended previous work (Deng and Papadimitriou (1994), Nouweland *et al* (1996) and Chun and Hokari (2004)) on this coincidence by providing a more general sufficiency condition. In addition, we have used this sufficiency condition to identify the subclasses of simple games and queueing games where this coincidence holds. A remaining agenda is to identify a *necessary* condition for this coincidence, at least in the context of simple games and queueing games.

References

[1] Brown, D. and Housman, D. (1988). "Cooperative Games on Weighted Graphs", Internal Report, Worcester Polytechnic Institute, Worcester.

[2] Chun, Y. (2004), "A note on Maniquet's characterization of the Shapley value in queueing problems." Working paper, Rochester University.

[3] Chun, Y. and Hokari, T. (2004), "On the coincidence of the Shapley value and the nucleolus in queueing problems."

[4] Curiel, I. (1996), "Cooperative Game Theory and Applications." Kluwer Academic Publishers.

[5] Deng, X. and Papadimitriou, C. H. (1994), "On the complexity of cooperative solution concepts." *Mathematics of Operations Research* **19,** 257-266.

[6] Hart, S. and Mas-Colell, A. (1989), "Potential value and Consistency." *Econometrica* **57,** 589-614.

[7] Maniquet, F. (1999), "A characterization of the Shapley value in queueing problems." *Journal of Economic Theory* **109,** 90-103.

[8] Moulin, H. (1988), "Axioms of cooperative decision making." Cambridge University Press.

[9] Nouweland, A. van den, Brouwers, W. van G., Bruinderink, P. and Tijs, S. (1996), "A game theoretic approach to problems in telecommunication." *Management Science* **42,** 294-303.

[10] Osborne, M. J. and Rubinstein, A. (1994), "A Course in Game Theory." The MIT Press.

[11] Peleg, B. and Sudhölter, P. (2003), "Introduction to the Theory of Cooperative Games." Kluwer Academic Publishers.

[12] Schmeidler, D. (1969), "The nucleolus of a characteristic function game." *Siam Journal of Applied Mathematics* **17,** 1163-1170.

[13] Shapley, L. S. (1953), "A value for n-person games." In: Contributions to the Theory of Games II (H. Kuhn and A. W. Tucker edited). Princeton University Press, Princeton, 307-317.

[14] Shapley, L. S. (1962), "Simple games: An outline of the descriptive theory." *Behavioral Science* **7,** 59-66.

[15] Shapley, L. S. and Shubik, M. (1954), "A method of evaluating the distribution of power in committee system." *American Political Science Review* **48,** 787-792.

[16] Sobolev, A. I. (1975), "The characterization of optimality principles in cooperative games by functional equations." *Mathematical Methods in Social Sciences* **6,** 94-151 (Russian).

[17] Winter, E. (2002), "The Shapley value." *Handbook of Game Theory with Economic Applications* **3,** 2025-2054.