Separability and Specification Tests

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1. INTRODUCTION.

In empirical studies of production and cost or of consumer demand, the use of data aggregated at least to some degree is unavoidable. In order for such aggregation to be consistent with an underlying microeconomic structure that is disaggregated, typically it will be the case that quite severe restrictions must be imposed on the model. The most important types of restriction are separability restrictions. For example in production studies, the assumption that the production technology is weakly separable can be used to justify the use of value-added measures of output in studies of productivity. In the context of consumer demand, weak separability is an important justification for the grouping of commodities into broad budget categories. Clearly an assumption with such far reaching consequences is one that it is important to test.

The literature on testing weak separability is quite extensive. Among the earliest studies were those of Berndt and Christensen (1973a,b, 1974) and Berndt and Wood (1975). In Berndt and Wood (1975), the separability of primary inputs capital and labour from intermediate inputs energy and materials was tested. The hypothesis of weak separability was rejected, leading the authors to conclude that the use of value added specifications for studies of investment and factor demand is unjustifiable. All of these studies used a flexible functional form approach in estimation, specifically the translog functional form introduced in Christensen et al. (1973). The translog functional form has been used in several other studies including Jorgenson and Lau (1975), Denny and Fuss (1977), Norsworthy and Malmquist (1983), Yuhn (1991) and Hazilla (1997). In common with the results of Berndt and Wood (1975), the balance of evidence reported in most of the literature is against the hypothesis of weak separability. However, Blackorby et al. (1977) showed that the hypothesis of weak separability requires severe restrictions on the functional form, specifically requiring the aggregator functions to be of Cobb-Douglas form and preventing the functions from being any longer flexible, meaning that they were no longer capable of giving a second order approximation to an arbitrary function in any neighbourhood of a given point. An important implication of the work of Blackorby et al. (1977) is that rejection of separability restrictions may well be due to imposition of an incorrect functional form rather than separability failing to hold.

The standard approach to separability tests is to test for the global validity of the weak separability hypothesis. In this article, it is argued that a less stringent approach may be appropriate, because provided certain additional restrictions are assumed, then a form of separability is valid locally. Thus a failure to pass a local separability test could be taken as evidence of incorrect functional form specification, rather than a failure of weak separability. Sections two and three contain discussions of the concepts of value-added production functions and primary cost functions in this context. Sections four and five discuss the principles and the practice of testing for separability, and the use of separability tests as tests of specification of the underlying functional forms. Section six is an empirical example based on the data set used by Berndt and Wood (1975); it is argued that the rejection of separability constraints reported in that article may be interpreted as evidence against the functional form used. Section seven briefly concludes the article.

2. THE PRODUCTION FUNCTION FOR VALUE ADDED.

In the context of production and cost theory, functions that are (weakly or strongly) separable play an important role in facilitating certain kinds of simplification that are often useful in economic analysis, one of the most important areas of application being the study of production functions for net output or value-added. Attempts to measure value-added are central to national income accounting and the attribution of the income generated from the sale of final products among primary factors. Monetary measurement of value-added is (conceptually) simple; at the level of an industry, deduct from the value of that industry's output the value of the goods purchased from other other industries to give a value for net output. The problems with the concept involve conversion of this monetary value to a real value and the question of whether it is meaningful to speak of a production function for real value added. Among the most insightful discussions in the literature are those of Arrow (1974), Bruno (1978) and Diewert (1978). Consider a production function $f : R^n_+ \to R_+$ (where R^n_+ denotes the non-negative orthant) giving

$$q = f(x) \tag{2.1}$$

where $x \in R^n_+$ represents a vector of factor inputs and q is gross output. Real value-added, or net output, can be measured in several ways. All definitions rely on a distinction between primary inputs and other inputs, referred to variously as material, intermediate or secondary inputs; we shall use this terminology interchangeably. Suppose there are $2 \leq m < n$ primary and $(n - m)$ intermediate inputs and without loss of generality let $x = (x_1, x_2)$ where $x_1 \in R^m_+$ and $x_2 \in R^{n-m}_+$.

It will often be useful to be able to represent net output or value added as a function of primary inputs only, writing

$$y = V(x_1) \tag{2.2}$$

where y denotes net output and V is to be interpreted as a production function. The interpretation of this equation depends on the meaning attached to net output or value-added. In practice, all definitions of real value-added compute net output from gross output using a mapping $v : R^n_+ \to R$ of the form

$$v(x_1, x_2) = h(f(x_1, x_2), x_2) = h(q, x_2) \tag{2.3}$$

where $h : R^{n-m+1}_+ \to R$ is a function and $\dfrac{\partial h}{\partial q} > 0$. If in fact value-added depends only on primary inputs then $v(x_1, x_2) = V(x_1)$ and (2.3) gives

$$V(x_1) = h(f(x_1, x_2), x_2) \tag{2.4}$$

Inverting this equation we obtain

$$f(x_1, x_2) = F(V(x_1), x_2). \tag{2.5}$$

for a function $F : R^{n-m+1}_+ \to R_+$. Thus in this case f is weakly separable in the partition $x = (x_1, x_2)$. Now in order to be able to write

$$v(x_1, x_2) = V(x_1) \tag{2.6}$$

it is necessary and sufficient that

$$\frac{\partial v}{\partial x_2}(x_1, x_2) = 0. \tag{2.7}$$

When this is the case, we may use the implicit function theorem applied to (2.7) to write $x_2 = \phi(x_1)$ over a suitable neighbourhood W of x_1 and it follows that $\forall\, x_1 \in W$

$$f(x_1, \phi(x_1)) = F(V(x_1), \phi(x_1)). \tag{2.8}$$

Consider the following example. A common way to measure real value-added is to use the single deflation method. For given input prices define nominal value-added by

$$Y = pq - w_2' x_2 \tag{2.9}$$

where w_2 is the vector of prices for the material inputs and p is the output price. Define real value-added by $y = Y/p$. Then we may take

$$v(x_1, x_2) = f(x_1, x_2) - \omega_2' x_2 \tag{2.10}$$

where ω_2 is the vector of real factor prices. Taking the input prices as given we have

$$\frac{\partial v}{\partial x_2}(x_1, x_2) = \frac{\partial f}{\partial x_2}(x_1, x_2) - \omega_2. \tag{2.11}$$

Although $\frac{\partial v}{\partial x_2}(x_1, x_2) \neq 0$ in general, what is true is that in a competitive equilibrium

$$\frac{\partial f}{\partial x_2}(x_1, x_2) = \omega_2. \tag{2.12}$$

Thus applying the implicit function theorem to (2.12) we can solve for x_2 and write, in an appropriate neighbourhood of x_1: $x_2 = \phi(x_1, \omega_2)$, and then, everywhere in an appropriate neighbourhood of x_1, we have

$$\begin{aligned} v(x_1, x_2) &= f(x_1, \phi(x_1, \omega_2)) - \omega_2' \phi(x_1, \omega_2) && (2.13) \\ &= V(x_1; \omega_2) && (2.14) \end{aligned}$$

in which ω_2 is regarded as a parameter of the mapping $V : R_+^m \to R$. It follows at once from (2.13) and (2.14) that

$$\begin{aligned} f(x_1, x_2) &= V(x_1; \omega_2) + M(x_2; \omega_2) && (2.15) \\ &= V(x_1; \omega_2) + M(\phi(x_1, \omega_2); \omega_2) && (2.16) \end{aligned}$$

where $M(x_2; \omega_2) = \omega_2' x_2$ defines a mapping $M : R_+^{n-m} \to R$. This purely local argument can be interpreted to mean that in a suitably defined neighbourhood of a competitive equilibrium, f is not just weakly but strongly (additively) separable by the functions (V, M).

An alternative proposal for the measurement of value-added is the double-deflation method. As before, nominal value-added is taken to be

$$Y = pq - w_2' x_2 \tag{2.17}$$

In the double-deflation method, expenditure on each input is deflated by its own price, to obtain

$$y = q - e' x_2 \tag{2.18}$$

where e denotes an $((n-m) \times 1)$ vector of ones. This is exactly as for the single-deflation case with the vector ω_2 replaced by e.

Although $\frac{\partial f}{\partial x_2}(x_1, x_2) - e \neq 0$ except under very special circumstances, we can still use(2.12) and the implicit function theorem to obtain a decomposition of f as in (2.15) with slightly modified definitions of the functions V and M. What this second example brings out is that it is the use of a (locally valid) restriction like (2.12)that enables us to decompose the function f in this additively separable fashion.

These examples correspond to the definitions most relevant to the practice of national income accounting. The implications of the examples for productivity measurement are discussed in detail in Bruno (1978).

3. THE PRIMARY COST FUNCTION.

The analysis of the preceding section has a direct analogue in the context of cost functions, specifically the cost function for primary inputs. Consider whether it is possible to use a cost function for primary inputs that depends only on the prices of those inputs and not on the prices of the remaining secondary inputs. A general approach to this problem is to consider initially treating primary costs as a function p of the form

$$p(\omega_1, \omega_2, q) = h(C(\omega_1, \omega_2, q), \omega_2) \tag{3.1}$$

where C denotes the total cost function, q is gross output, and h is a function increasing in total costs. If p is independent of the prices of secondary inputs, then we have

$$p(\omega_1, \omega_2, q) = P(\omega_1, q). \tag{3.2}$$

Then by inverting (3.1) we obtain

$$C(\omega_1, \omega_2, q) = F(P(\omega_1, q), \omega_2). \tag{3.3}$$

If C can be written in the form (3.3), then C is weakly separable into a function of the primary input price vector ω_1 and the secondary price vector ω_2 (strictly in the partition of its arguments into the sets $\{\omega_1, q\}$ and $\{\omega_2\}$).

Now in practice, the obvious definition of primary costs gives

$$\begin{aligned} p(\omega_1, \omega_2, q) &= c_T - \omega_2 x_2 && (3.4)\\ &= C(\omega_1, \omega_2, q) - \omega_2 X_2(\omega_1, \omega_2, q) && (3.5)\\ &= \omega_1 X_1(\omega_1, \omega_2, q) && (3.6) \end{aligned}$$

where c_T denotes the value of total costs and X_1 and X_2 are the cost-minimizing factor demand functions for the primary and secondary inputs.

In general $\dfrac{\partial p}{\partial \omega_2}(\omega_1, \omega_2, q) \neq 0$, however for a cost minimizing firm we can use the factor demand equation

$$X_2(\omega_1, \omega_2, q) = x_2 \tag{3.7}$$

to solve for ω_2, giving, in a suitable neighbourhood of (ω_1, q), $\omega_2 = \psi(\omega_1, q, x_2)$. Substituting in (3.6) we have

$$\begin{aligned} p(\omega_1, \omega_2, q) &= \omega_1 X_1(\omega_1, \psi(\omega_1, q, x_2), q) && (3.8)\\ &= P(\omega_1, q; x_2) && (3.9) \end{aligned}$$

in which x_2 is regarded as a parameter of the function P. Using (3.5) we can now write

$$\begin{aligned} C(\omega_1, \omega_2, q) &= P(\omega_1, q; x_2) + \psi(\omega_1, q, x_2) x_2 && (3.10)\\ &= P(\omega_1, q; x_2) + M(\psi(\omega_1, q, x_2); x_2) && (3.11)\\ &= P(\omega_1, q; x_2) + M(\omega_2; x_2). && (3.12) \end{aligned}$$

This additively separable decomposition of C is locally (but not in general globally) valid in the neighbourhood of points satisfying (3.7); equation (3.7) plays the role of an additional restriction that facilitates the decomposition of C in this way.

In the special case of constant returns to scale, which was assumed by Berndt and Wood (1975), we can work with the unit cost function $c : R^n_+ \to R$ and write

$$c(\omega_1, \omega_2)q = \pi(\omega_1, q; x_2)q + M(\omega_2; x_2) \tag{3.13}$$

where π can be interpreted as unit primary costs (which vary with the output level because of the fixed factor x_2).

4. SEPARABILITY AND FUNCTIONAL FORM SPECIFICATION.

In the previous section it was shown that for any production function $f : R^n_+ \to R$ we can construct functions $V : R^m_+ \to R$ and $M : R^{n-m}_+ \to R$ such that in a suitable local neighbourhood characterized by the condition that

$$\frac{\partial f}{\partial x_2}(x_1, x_2) = \omega_2 \tag{4.1}$$

f can be additively decomposed by two functions (V, M). These functions are parameterized by ω_2 and given explicitly by

$$V(x_1; \omega_2) = f(x_1, \phi(x_1, \omega_2)) - \omega_2' \phi(x_1, \omega_2)) \tag{4.2}$$

and

$$M(x_2; \omega_2) = \omega_2' x_2. \tag{4.3}$$

and where

$$x_2 = \phi(x_1, \omega_2). \tag{4.4}$$

Now condition (4.1) is satisfied by input values $x = (x_1, x_2)$ that will be chosen by competitive firms facing the (real) price vector ω_2 for material inputs. What this means is that if a researcher models input data as generated by a competitive market then that researcher is assuming that at given input prices ω_2 the production function f may be additively decomposed by (V, M) over a set that includes the data points as

$$f(x_1, x_2) = V(x_1; \omega_2) + M(x_2; \omega_2) \tag{4.5}$$

that is

$$f(x_1, \phi(x_1, \omega_2)) = V(x_1; \omega_2) + \omega_2' \phi(x_1, \omega_2). \tag{4.6}$$

Equation(4.5) in effect says that locally f is not only weakly but strongly separable.[1] Any proposed functional form for f that does not satisfy this criterion of being locally additively separable in this way is simply inadmissible. Furthermore, this makes clear the need to pay careful attention in the discussion of separability tests to two crucial aspects. One of these is the subset of the domain over which a function is (strongly or weakly) separable. The other aspect is the separating functions ((V, M) in the above discussion). Thus, it is not possible to reject the hypothesis that there exist functions V and M such that a production function f is locally additively separable by (V, M) when (4.1) is satisfied—we have shown how to construct such functions.

Suppose a researcher proposes a specific functional form for a (single deflation) real value-added production function V. If this is a valid functional form then equation (4.6) must be satisfied in a neighbourhood for which (4.1) holds. Differentiating both sides of (4.6) gives

$$\frac{\partial f}{\partial x_1}(x_1, \phi(x_1, \omega_2)) + \frac{\partial \phi}{\partial x_1}(x_1, \omega_2)\frac{\partial f}{\partial x_2}(x_1, \phi(x_1, \omega_2)) = \frac{\partial V}{\partial x_1}(x_1, \omega_2) + \frac{\partial \phi}{\partial x_1}(x_1)\omega_2. \tag{4.7}$$

Using

$$\frac{\partial f}{\partial x_2}(x_1, \phi(x_1, \omega_2)) = \omega_2 \tag{4.8}$$

(4.7) reduces to the condition that

$$\frac{\partial V}{\partial x_1}(x_1; \omega_2) = \frac{\partial f}{\partial x_1}(x_1, \phi(x_1, \omega_2)). \tag{4.9}$$

Note that we have included the parameter ω_2 in the definition of V (until now we have been considering $V : R_+^m \to R$). With a slight abuse of notation we can regard V as a function of x_1 and ω_2, that is as a mapping $V : R_+^n \to R$; With this interpretation, $V(x_1, \omega_2)$ is simply the (real) variable profit function;

[1] Note that the usual definitions of weak and strong separability refer to global, not local properties of f; our usage here is not standard.

the properties of V are discussed in detail in Diewert (1974) and (specifically interpreted as a real value-added function) in Diewert (1978). Condition (4.9) can be interpreted as a necessary and sufficient condition that f be locally strongly separable by (V, M). If V is a valid specification for the functional form, then the specification must satisfy (4.9). The conclusion is that if we test the specification in (4.9) for some chosen functional form V and if the specification is rejected, then V is not a valid specification for the functional form.

The discussion above has been couched in terms of the production function f and an associated value-added function V. Precisely analogous results hold for a total cost function C and associated primary cost function P. Details of the results for the cost function approach will be discussed in section six.

In a sense this turns the methodological approach of Berndt and Wood (1975), Norsworthy and Malmquist (1983), Yuhn (1991) and others on its head. In those papers failure to pass a (weak) separability test was taken as evidence against the existence of a real value-added production function. In the approach we have been outlining here, a real value added function is shown to exist (over some part of the domain) and failure of a specific functional form to pass a (strong) separability test is evidence that the proposed functional form is inadmissible. The possibility that so-called separability tests are in fact tests of the functional forms used in the investigation is exactly what was postulated in the Blackorby et al. (1977) critique of these tests.

5. TESTING SEPARABILITY.

In this section we briefly outline an approach to testing for the validity of value-added specifications for a production function. A standard approach is to apply to the production function a variation of the technique used in Berndt and Wood (1975) in the context of cost functions. A flexible functional form, say $f : R^n_+ \to R$, usually a translog form, is proposed for the production function and primary and material inputs are distinguished. Weak separability of primary from material inputs is defined to mean that the marginal rate of substitution between any pair of primary inputs is independent of the quantity of any material input used, that is

$$\frac{\partial}{\partial x_{2k}} \left(\frac{\dfrac{\partial f}{\partial x_{1i}}(x)}{\dfrac{\partial f}{\partial x_{1j}}(x)} \right) = 0 \quad \forall\, i, j, k \tag{5.1}$$

everywhere, where the x_{1i} and x_{1j} are primary inputs and the x_{2k} are material inputs. As is well known, this means that there are real valued functions F and G such that (at least in a neighbourhood of every point in R^n_+)

$$f(x_1, x_2) = F(G(x_1), x_2). \tag{5.2}$$

For a good account of separability theory based on this approach, see Geary and Morishima (1973). In the case of the translog form we have

$$\ln f(x) = \sum_{i=1}^{n} \alpha_i \ln x_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} \ln x_i \ln x_j \tag{5.3}$$

where $\beta_{ij} = \beta_{ji}$. Condition (5.1) is (see Blackorby et al. (1977))

$$\alpha_j \beta_{ik} - \alpha_i \beta_{jk} + \sum_{s=1}^{n} (\beta_{js}\beta_{ik} - \beta_{is}\beta_{jk}) \ln x_s = 0 \quad \forall\, i, j, k \tag{5.4}$$

or, in the partitioning $x = (x_1, x_2)$

$$\begin{aligned} \alpha_j \beta_{ik} - \alpha_i \beta_{jk} &+ \sum_{p=1}^{m} (\beta_{jp}\beta_{ik} - \beta_{ip}\beta_{jk}) \ln x_{1p} \\ &+ \sum_{s=1}^{n-m} (\beta_{j,m+s}\beta_{ik} - \beta_{i,m+s}\beta_{jk}) \ln x_{2s} = 0 \quad \forall\, i, j, k \end{aligned} \tag{5.5}$$

Since f is (globally) weakly separable if and only if (5.4), and therefore (5.5), is true everywhere in R^n_+, it follows that f is weakly separable if and only if for all primary inputs i and j and secondary inputs k,

$$\alpha_j \beta_{ik} - \alpha_i \beta_{jk} = 0 \tag{5.6}$$

$$(\beta_{js}\beta_{ik} - \beta_{is}\beta_{jk}) = 0 \quad \forall\, s = 1, \ldots, n. \tag{5.7}$$

In the standard approach the restrictions in (5.6) and (5.7) are tested. A failure to accept these restrictions is taken as evidence against the existence of a real value-added function.

It should by now be clear that accepting the restrictions in (5.6) and (5.7) is in one sense too stringent a requirement to impose on f and in another sense is not sufficiently stringent. It is too stringent because it requires that (5.4) be true for every $x = (x_1, x_2) \in R^n_+$. In the previous analysis we have shown that when (4.1) holds so that $x_2 = \phi(x_1, \omega_2)$, this additional restriction enables us locally to decompose f in the additive form given in (4.5). Thus if the additional restriction is assumed to apply to the data points, one approach to the problem would be to test the restrictions in (5.4) at each data point. If any of these restrictions were rejected at one or more data points, this would indicate that the chosen f is not weakly separable even locally and hence there cannot exist functions (V, M) such that the decomposition (4.5) is valid. However, the conclusion to be drawn from the rejection is not that a value-added specification is impossible; instead the conclusion must be that the chosen functional form for f is inadmissable for the representation of the data.

Even if this approach is followed, a test of (5.4) over a subset of R^n_+ for which a real value-added function exists is in another sense not a sufficiently stringent test of any proposed functional form—for the simple reason that it is a test of a necessary condition for weak separability of f over the subset, not a test of strong separability, which is what is required by the value added specification.

A better procedure would be to test the condition in (4.9). If competition is assumed, $\dfrac{\partial f}{\partial x_1}(x) = \omega_1$, where $\omega_1 = (\omega_{11}, \ldots, \omega_{1m})$ is the vector of real prices of the primary inputs and (4.9) can be written as a system of share equations

$$s_{1i} = \frac{\partial \ln V}{\partial \ln x_{1i}}(x_1, \omega_2) \quad i = 1, \ldots, m \tag{5.8}$$

where

$$s_{1i} = \frac{\omega_{1i} x_{1i}}{y} \quad i = 1, \ldots, m \tag{5.9}$$

in which

$$y = q - \omega_2' x_2 = V(x_1, \omega_2) \tag{5.10}$$

is (measured) real value-added.

For the purposes of econometric testing of the proposed specification, the equations in (5.8) can be augmented in two ways. One approach is to use the fact that (see (4.2))

$$\frac{\partial \ln V}{\partial \ln \omega_2}(x_1, \omega_2) = -\phi(x_1, \omega_2) = -x_2 \tag{5.11}$$

which gives the additional equations

$$s_{2j} = -\frac{\partial \ln V}{\partial \ln \omega_{2j}}(x_1, \omega_2) \quad j = 1, \ldots, n-m \tag{5.12}$$

where

$$s_{2j} = \frac{\omega_{2j} x_{2j}}{y} \quad i = 1, \ldots, m. \tag{5.13}$$

Joint estimation of (5.8) and (5.12) can then provide a standard way to test the specification of the functional form V.

An alternative approach is to augment 5.8 by the equations

$$\frac{\partial f}{\partial x_2}(x_1, x_2) = \omega_2 \tag{5.14}$$

which can be represented as share equations by

$$\sigma_{2j} = \frac{\partial \ln f}{\partial \ln x_{2j}}(x_1, x_2) \quad j = 1, \ldots, n-m \tag{5.15}$$

where

$$\sigma_{2j} = \frac{\omega_{2j} x_{2j}}{q}. \tag{5.16}$$

Joint estimation of 5.8 and (5.15) allows for a direct test of the coherence between functional forms proposed for f and V. This makes obvious that separability tests are specification tests. If the specification is rejected for some chosen functional forms f and V, for the production function and the variable profit function, then f cannot be additively decomposed by (V, M) as in (4.5) and hence f or V is data inadmissible. Clearly it would be sensible to choose as candidates for V members of the family of flexible functional forms; for example, $V(x_1, \omega_2)$ translog in the arguments (x_1, ω_2).

6. AN EMPIRICAL EXAMPLE.

As an illustration of the ideas contained in the preceding sections, we consider now an empirical example. The illustration we use is based on the cost function and the data and model used in the seminal work of Berndt and Wood (1975), which has been an important influence in subsequent studies of the separability issue and the use of value added in studies of production.

We showed in section two that, using the condition in (3.7), which implies locally that we can write $\omega_2 = \psi(\omega_1, q, x_2)$, and which will certainly hold when inputs are chosen in accordance with cost minimization, then locally the cost function can be decomposed as

$$C(\omega_1, \omega_2, q) = P(\omega_1, q; x_2) + M(\omega_2; x_2). \tag{6.1}$$

where the primary cost function is

$$P(\omega_1, q; x_2) = \omega_1 X_1(\omega_1, \psi(\omega_1, q, x_2), q) \tag{6.2}$$

and the intermediate or material costs are

$$\begin{aligned} M(\omega_2; x_2) &= \omega_2 x_2 \qquad (6.3) \\ &= \psi(\omega_1, q, x_2) x_2. \qquad (6.4) \end{aligned}$$

In the special case of constant returns to scale we can simplify further and use (3.13).

$$c(\omega_1, \omega_2) q = \pi(\omega_1, q; x_2) q + M(\omega_2; x_2) \tag{6.5}$$

Here x_2 is regarded as a parameter of the functions $P : R_+^{m+1} \to R$ and $M : R_+^{n-m} \to R$ which may be interpreted respectively as primary and material cost functions (or alternatively as the variable cost function associated with fixed inputs x_2, and the associated fixed costs); π can be interpreted as unit primary costs.

In Berndt and Wood (1975) capital (K) and labour (L) were treated as primary and energy (E) and materials (M) as intermediate inputs. Perfect competition in product and factor markets was assumed as well as constant returns to scale. The parameters of a translog unit cost function were estimated by estimating a system of share equations of the form

$$\sigma_i = \alpha_i + \sum_j \gamma_{ij} \ln p_j \quad i, j = K, L, E, M \tag{6.6}$$

where σ_i denotes the share of the ith input in total cost and p_j the price of the jth input. Symmetry and price homogeneity were imposed in the estimation, which was accomplished by using the iterated three stage least squares (I3SLS) method. Weak separability was tested by testing the restrictions (which Berndt and Wood refer to as the non-linear separability restrictions)

$$\alpha_K/\alpha_L = \gamma_{KK}/\gamma_{KL} = \gamma_{KL}/\gamma_{LL} = \gamma_{KE}/\gamma_{LE} = \gamma_{KM}/\gamma_{LM} \tag{6.7}$$

The restrictions in (6.7) are the equivalent for the model being considered of the restrictions (5.5) described in the general discussion in section 5 above. The restrictions (6.7) were rejected by the data and Berndt and Wood concluded that the value-added specification is unreliable for the purposes of investment and factor demand studies.

Since Berndt and Wood (1975) assumed perfect competition they effectively assumed that at each observation the condition (3.7) is met so that $\omega_2 = \psi(\omega_1, q, x_2)$ in an appropriate local neighbourhood. Hence by assumption their cost function can locally be decomposed as in (6.1). It was argued in section 5 above that a test for weak separability (globally in R_+^{n+1}) is an inappropriate procedure and that it is preferable to test only a necessary condition for local weak separability at each of the data points.

When the translog functional form is assumed, weak separability at each data point may be tested by testing the following restrictions at each data point:

$$\begin{aligned} &\alpha_L\gamma_{KE} - \alpha_K\gamma_{LE} + (\gamma_{LK}\gamma_{KE} - \gamma_{KK}\gamma_{LE}) \ln p_K \\ &\quad + (\gamma_{LL}\gamma_{KE} - \gamma_{KL}\gamma_{LE}) \ln p_L + (\gamma_{LM}\gamma_{KE} - \gamma_{KM}\gamma_{LE}) \ln p_M = 0 \end{aligned} \tag{6.8}$$

$$\begin{aligned} &\alpha_L\gamma_{KM} - \alpha_K\gamma_{LM} + (\gamma_{LK}\gamma_{KM} - \gamma_{KK}\gamma_{LM}) \ln p_K \\ &\quad + (\gamma_{LL}\gamma_{KM} - \gamma_{KL}\gamma_{LM}) \ln p_L + (\gamma_{LE}\gamma_{KM} - \gamma_{KE}\gamma_{LM}) \ln p_E = 0 \end{aligned} \tag{6.9}$$

These equations are the equivalent for this model of equations (5.5) above (several of the restrictions in (5.5) are redundant). For the ith data point, denote the Wald test statistic for the restrictions by W_i. For each i the W_i are χ_2^2 random variables under the null hypothesis that the restrictions in (6.8) and (6.9) are true. If for any i the null hypothesis is rejected, then the proposed functional form violates a necessary condition for local weak separability implicit in the underlying model. Thus a reasonable decision rule is to reject the proposed functional form if $\max_i W_i > c_\alpha$ where c_α is a critical value chosen to control the significance level for the test at 100α per cent. Although the W_i are not independent, by the Bonferroni inequality (see Hochberg and Tamhane (1987)),

$$P(\max_i W_i \leq c_\alpha) = P(\cap_i (W_i \leq c_\alpha)) \tag{6.10}$$

$$\geq 1 - \sum_i P(W_i > c_\alpha) \tag{6.11}$$

$$= 1 - T\delta \tag{6.12}$$

where T is the sample size and where for each i

$$\delta = P(W_i > c_\alpha) = P(\chi_2^2 > c_\alpha) \tag{6.13}$$

It follows that if we choose c_α so that $\delta = \alpha/T$ then the proposed decision rule has a significance level of at most 100α per cent.

The test proposed was applied to the data of Berndt and Wood (1975). Three sets of estimates of the parameters of the share equations in (6.6) were obtained using the I3SLS method. These were, respectively, unrestricted, restricted by price homogeneity and fully restricted by homogeneity and symmetry. For each set of parameter estimates the value of the largest Wald test statistic for the hypothesis was computed. The results are reported in Table 1. The Bonferroni method described above gives for $\alpha =$ .10, .05, and .01 the critical values 11.04, 12.43 and 15.65 respectively. Although the unrestricted parameter estimates and those restricted only by homogeneity do not lead to rejection of the null hypothesis, the fully restricted parameter estimates generate a clear rejection at a significance level of at most 1 per cent. These fully restricted parameter estimates are the ones reported by Berndt and Wood and the ones that correspond completely to the underlying economic model used.

Whereas the conclusion drawn by Berndt and Wood was that their cost function was not weakly separable and therefore that a value-added specification is invalid, the interpretation to be placed on the above result is that the fully restricted parameter estimates fail to pass the test of being locally weakly separable and hence that the associated functional form is misspecified.

We turn now to the issue of local strong separability at each data point. So long as the data satisfy the assumptions of the competitive model, then a valid cost function must satisfy (6.1). From (6.1), (6.2) and (6.4)

$$P(\omega_1, q, x_2) = C(\omega_1, \psi(\omega_1, q, x_2), q) - \psi(\omega_1, q, x_2)x_2 \tag{6.14}$$

from which it is easy to see that locally the following conditions must be satisfied by the functions P and C:

$$\begin{aligned}
\frac{\partial P}{\partial \omega_1}(\omega_1, q, x_2) &= \frac{\partial C}{\partial \omega_1}(\omega_1, \psi(\omega_1, q, x_2), q) && (6.15)\\
\frac{\partial P}{\partial q}(\omega_1, q, x_2) &= \frac{\partial C}{\partial q}(\omega_1, \psi(\omega_1, q, x_2), q) && (6.16)
\end{aligned}$$

where we have used $\partial C/\partial \omega_2 = X_2$ and (3.7).

The equations (6.15) and (6.16) can be used as the basis for a specification test for a proposed pair of functional forms C and P. Failure to satisfy the conditions in (6.15) and (6.16) could be due to poor specification of the total cost function C or of the primary cost function P. If we augment (6.15) and (6.16) by

$$x_2 = \frac{\partial C}{\partial \omega_2}(\omega_1, \omega_2, q) \tag{6.17}$$

and test the joint specification, we can directly test the coherence between functional forms proposed for C and P.

A test was carried out by specifying a translog functional form for both C and P. Constant returns to scale was not assumed; in Berndt (1991) the original data from Berndt and Wood (1975) is augmented by data for gross output in U.S. Manufacturing and this was used in the analysis. In the equations estimated, trend terms were included to allow for technical progress. The equations (6.15), (6.16) and (6.17) have a convenient representation in terms of various factor shares. Using Shephard's Lemma and the equality of product price and marginal cost $\partial C/\partial q$ under competition we easily obtain

$$s_i = \frac{\partial \ln P}{\partial \ln \omega_{1i}} \quad i = 1, \ldots, m \tag{6.18}$$

$$s_q = \frac{\partial \ln P}{\partial \ln q} \tag{6.19}$$

$$\sigma_j = \frac{\partial \ln C}{\partial \ln \omega_{2j}} \quad j = 1, \ldots, n-m \tag{6.20}$$

where the s_i are the shares of the primary factors in the primary cost $\sum_{i=1}^{m} \omega_{1i} x_{1i}$, the σ_j are the shares of the intermediate factors in the total cost $\sum_{i=1}^{m} \omega_{1i} x_{1i} + \sum_{j=1}^{n-m} \omega_{2j} x_{2j}$ and s_q is the ratio of the value of gross output pq to the primary cost of production. Since $\sum_{i=1}^{m} s_i = 1$, the primary share equations are not independent and one must be dropped in estimation. The equations estimated were

$$s_K = \delta_K + \theta_K t + \beta_{KK} \ln p_K + \beta_{KL} \ln p_L + \beta_{KE} \ln E + \beta_{KM} \ln M + \beta_{Kq} \ln q \tag{6.21}$$

$$s_q = \delta_q + \theta_q t + \beta_{qK} \ln p_K + \beta_{qL} \ln p_L + \beta_{qE} \ln E + \beta_{qM} \ln M + \beta_{qq} \ln q \tag{6.22}$$

$$\sigma_E = \alpha_E + \lambda_E t + \gamma_{EK} \ln p_K + \gamma_{EL} \ln p_L + \gamma_{EE} \ln p_E + \gamma_{EM} \ln p_M + \gamma_{Eq} \ln q \tag{6.23}$$

$$\sigma_M = \alpha_M + \lambda_M t + \gamma_{MK} \ln p_K + \gamma_{ML} \ln p_L + \gamma_{ME} \ln p_E + \gamma_{MM} \ln p_M + \gamma_{Mq} \ln q \tag{6.24}$$

It is not hard to show that the primary cost function P is homogeneous of degree 1 in p_K and p_L. Since the total cost function is homogeneous of degree 1 in p_K, p_L, p_E and p_M the parameters in (6.21) to (6.24) are constrained by

$$\beta_{KK} + \beta_{KL} = 0 \tag{6.25}$$

$$\beta_{qK} + \beta_{qL} = 0 \tag{6.26}$$

$$\gamma_{EK} + \gamma_{EL} + \gamma_{EE} + \gamma_{EM} = 0 \tag{6.27}$$

$$\gamma_{MK} + \gamma_{ML} + \gamma_{ME} + \gamma_{MM} = 0 \tag{6.28}$$

Symmetry of cross partial derivatives yields the additional restrictions

$$\beta_{Kq} = \beta_{qK} \tag{6.29}$$

$$\gamma_{EM} = \gamma_{ME} \tag{6.30}$$

Table 2 reports estimates of the equations in (6.21) to (6.24). The parameter estimates reported are constrained to satisfy homogeneity and symmetry. The estimates were obtained using the I3SLS method with the same instrument set as used in Berndt and Wood (1975). Note that constant returns to scale has not been imposed (a test for constant returns to scale easily rejects the null hypothesis). The equations seem to provide a reasonable fit, and the Durbin-Watson statistics and Ljung-Box Q statistic do not indicate problems with autocorrelation. A comparison with the results reported in Berndt and Wood shows that the parameter estimates are similar to those obtained by Berndt and Wood for the σ_E equation, but differ substantially for the σ_M equation. However, tests for homogeneity and symmetry indicate some problems of coherence between the proposed primary and total cost functional forms. The first column of Table 4 reports test statistics for the theoretical homogeneity and symmetry restrictions. The homogeneity restrictions in (6.25) to (6.28) were tested equation by equation. The test statistics thus each have Chi-squared distributions with one degree of freedom. Note that since the equations have different included variables the results of the test on one equation are not independent of the other equations being estimated. In each case the test rejects the null hypothesis of homogeneity. The test statistic for symmetry (Chi-squared with two degrees of freedom) was obtained by ignoring this rejection and simply including homogeneity as part of the maintained hypothesis. Even under this favourable assumption, the symmetry hypothesis is barely accepted at the 5% level of significance and is rejected at the 10% level. These results indicate a lack of coherence in the assumption that both the total cost function and the primary cost function are translog in form. The lack of coherence may be due to incorrect specification of either or both of these two functions. We have already found evidence of misspecification for the total cost function. Let us now examine the primary cost function.

From (6.1), (6.2) and (6.4) it is easy to see that

$$\frac{\partial P}{\partial x_2} = -\psi(\omega_1, q, x_2) = -\omega_2 \tag{6.31}$$

It is then easy to deduce the following set of share equations for the primary cost function.

$$s_i = \frac{\partial \ln P}{\partial \ln \omega_{1i}} \quad i = 1, \ldots, m \tag{6.32}$$

$$s_q = \frac{\partial \ln P}{\partial \ln q} \tag{6.33}$$

$$-s_j = \frac{\partial \ln P}{\partial \ln x_{2j}} \quad j = 1, \ldots, n-m \tag{6.34}$$

where as before the s_i are shares of primary factors in the primary cost and now the s_j represent ratios of material inputs to primary costs. Note that $\sum_{i=1}^{m} s_i = 1$, but the material input "shares" s_j are not so constrained. If the primary cost function is assumed to be of the translog form, then the KLEM model gives for estimation the set of "share" equations

$$s_K = \delta_K + \theta_K t + \beta_{KK} \ln p_K + \beta_{KL} \ln p_L + \beta_{KE} \ln E + \beta_{KM} \ln M + \beta_{Kq} \ln q \tag{6.35}$$

$$s_q = \delta_q + \theta_q t + \beta_{qK} \ln p_K + \beta_{qL} \ln p_L + \beta_{qE} \ln E + \beta_{qM} \ln M + \beta_{qq} \ln q \tag{6.36}$$

$$s_E = \delta_E + \theta_E t + \beta_{EK} \ln p_K + \beta_{EL} \ln p_L + \beta_{EE} \ln E + \beta_{EM} \ln M + \beta_{Eq} \ln q \tag{6.37}$$

$$s_M = \delta_M + \theta_M t + \beta_{MK} \ln p_K + \beta_{ML} \ln p_L + \beta_{ME} \ln E + \beta_{MM} \ln M + \beta_{Mq} \ln q \tag{6.38}$$

Since the primary cost function is homogeneous of degree one in p_K and p_L the parameters of (6.35) to (6.38) satisfy

$$\beta_{KK} + \beta_{KL} = 0 \tag{6.39}$$

$$\beta_{qK} + \beta_{qL} = 0 \tag{6.40}$$

$$\beta_{EK} + \beta_{EL} = 0 \tag{6.41}$$

$$\beta_{MK} + \beta_{ML} = 0 \tag{6.42}$$

Symmetry of cross-partial derivatives yields the additional constraints

$$\beta_{Kq} = \beta_{qK} \tag{6.43}$$

$$\beta_{KE} = -\beta_{EK} \tag{6.44}$$

$$\beta_{KM} = -\beta_{MK} \tag{6.45}$$

$$\beta_{qE} = -\beta_{Eq} \tag{6.46}$$

$$\beta_{qM} = -\beta_{Mq} \tag{6.47}$$

$$\beta_{EM} = \beta_{ME} \tag{6.48}$$

Parameter estimates from the estimation of (6.35) to (6.38) are reported in Table 3; these estimates are fully restricted by homogeneity and symmetry. Two equations are common to (6.21) to (6.24) and (6.35) to (6.38), those for s_K and s_q. For the s_K equation, the two sets of parameter estimates are quite similar but for the s_q equation the two sets of estimates differ quite substantially. The R^2 values reported in Table 3 indicate that a translog primary cost function fits the data more closely than the equations reported in Table 2. Although the Durbin-Watson statistics are further from 2, they all lie within the inconclusive range of critical values given in Savin and White (1977). A test of the validity of the homogeneity and symmetry restrictions was carried out in exactly the same way as for the equations in (6.21) to (6.24). The results are reported in the second column of Table 4. By contrast with the results for equations (6.21) to (6.24), in all but one case the homogeneity restrictions are accepted at the 5significance level. When homogeneity is imposed as part of the maintained hypothesis, the symmetry restrictions are also easily accepted (note that the number of degrees of freedom for the symmetry test statistic is different between columns two and three of Table 4).

The results reported in Tables 2, 3 and 4 indicate that the assumption that both the total cost function and the primary cost function are translog functional forms is a misspecification. The results also suggest that the misspecification can more plausibly be attributed to incorrect specification of the functional form for total cost than to the functional form for primary cost. Taken together with the test for local weak separability reported at the beginning of this section they strongly suggest that the translog form for the total cost function reported in Berndt and Wood (1975) is misspecified.

It is important to emphasize once more that the evidence for misspecification of the translog form for the total cost function is the failure of that form to pass tests for local weak and strong separability in the local neighbourhood of each data point. Furthermore, the evidence reported here indicates that a value-added specification actually seems to conform more closely with the data of Berndt and Wood than their proposed gross output specification.

7. CONCLUSIONS.

In much of the recent literature, the use of value-added measures of output has been criticized on the grounds that the underlying production technology fails to pass separability tests. In this article we have argued that most of the criticism is ill-founded because the separability tests that have been performed have tested for global separability properties. We have shown that under standard assumptions, such as competitive markets, production and cost functions are, in a well defined sense, (locally) separable over certain subsets of their domains. Although tests for global separability are inappropriate, we can nonetheless use these local separability tests as tests for the validity of proposed functional forms. We conclude that when properly implemented the value-added framework is quite innocuous.

TABLE 1

BONFERRONI TEST STATISTICS

Unrestricted	9.12
Homogeneity	10.15
Homogeneity and Symmetry	41.05

TABLE 2

COMBINED PRIMARY/TOTAL COST SHARES

Equation:	s_K	s_q	σ_E	σ_M
Constant	.6476	-.1022	.1785	.2490
	(6.59)	(-0.06)	(10.69)	(2.00)
Trend	.0036	-.0291	.0013	-.0043
	(3.07)	(-2.19)	(5.98)	(-3.84)
$\ln p_K$	.0804	-.0133	-.0096	.0102
	(7.77)	(-0.19)	(-5.08)	(1.14)
$\ln p_L$	-.0804	.0133	-.0182	-.0081
	(-7.77)	(0.19)	(-2.03)	(-0.28)
$\ln p_E$	-	-	.0250	.0028
			(4.56)	(0.30)
$\ln p_M$	-	-	.0028	-.0049
			(0.30)	(-0.13)
$\ln q$	-.0133	2.0870	-.0258	.0764
	(-0.19)	(4.41)	(-8.15)	(3.20)
$\ln E$	.0357	-.6919	-	-
	(0.93)	(-2.60)		
$\ln M$	-.0995	-1.2501	-	-
	(-1.32)	(-3.11)		
R^2	.78	.65	.93	.68
$D.W.$	1.35	2.12	2.04	2.05
Q	20.50	6.89	13.65	5.27
	(.059)	(.865)	(.324)	(.948)

TABLE 3

ESTIMATES OF PRIMARY COST FUNCTION

Equation:	s_K	s_q	s_E	s_M
Constant	.5929	-.6383	.4915	-.5051
	(7.20)	(-0.58)	(9.14)	(-0.55)
Trend	.0030	-.0522	-.0005	-.0351
	(4.62)	(-6.51)	(-0.87)	(-5.18)
$\ln p_K$	.0769	-.0764	-.0407	.0202
	(8.51)	(-1.27)	(-7.93)	(0.32)
$\ln p_L$	-.0769	.0764	.0407	-.0202
	(-8.51)	(1.27)	(7.93)	(0.32)
$\ln q$	-.0764	-1.9928	-.0826	-3.0611
	(-1.27)	(-2.47)	(-2.08)	(-3.96)
$\ln E$	.0407	.0826	.0382	.0026
	(7.93)	(2.08)	(2.25)	(0.06)
$\ln M$	-.0202	3.0611	.0026	3.9545
	(-0.32)	(3.96)	(0.06)	(5.01)
R^2	.80	.81	.96	.81
$D.W.$	1.37	1.45	1.36	1.62
Q	23.13	12.25	13.72	12.31
	(.027)	(.426)	(.319)	(.422)

TABLE 4

TESTS STATISTICS FOR HOMOGENEITY AND SYMMETRY

Restriction	Equations (88) to (91)	Equations (102) to (105)
Homogeneity:		
s_K equation	8.65	11.68
s_q equation	4.16	2.41
s_E equation	-	1.35
s_M equation	-	1.73
σ_E equation	3.84	-
σ_M equation	4.01	-
Symmetry	5.60	7.29

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