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Nonlinear Stochastic Systems And Controls: Lotka-Volterra Type Models, Permanence And Extinction, Optimal Harvesting Strategies, And Numerical Methods For Systems Under Partial Observations

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NONLINEAR STOCHASTIC SYSTEMS AND CONTROLS: LOTKA-VOLTERRA TYPE MODELS, PERMANENCE AND EXTINCTION, OPTIMAL HARVESTING STRATEGIES, AND NUMERICAL METHODS FOR SYSTEMS UNDER PARTIAL OBSERVATIONS

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DEDICATION

To my parents

In memory of Professor Le Van Hap, who taught me Real Analysis and Partial Differential Equations during my study at Hue University in Vietnam. I also worked with him as a teaching assistant for several courses from 2007 to 2010.

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CHAPTER 1 INTRODUCTION

This dissertation is concerned with a class of stochastic models formulated using stochastic differential equations with regime switching represented by a continuous-time Markov chain, also known as hybrid switching diffusion processes. Our motivations for studying such processes in this dissertation come from emerging and existing applications in biological systems, ecosystems, financial engineering, modeling, analysis, and control and optimization of stochastic systems under the influence of random environments, with complete or partial observations.

Chapter 2 focuses on Lotka-Volterra models given by stochastic differential equations with regime switching represented by a continuous-time Markov chain. There have been resurgent efforts in treating partially observed systems in the control and systems community. This chapter is devoted to a class of such systems that have been around for many years, but got more recent attentions owing to the new modeling perspective for complex systems with random environment. Assuming the random environment can only be observed with noise and focusing on Lotka-Volterra systems, we develop a new approach that will be of interest not only to researchers in ecology and bio-systems, but also for control theorists, operations researchers, and people who are working in system biology. Different from the existing literature, the Markov chain is hidden and can only be observed in a Gaussian white noise in our work. For such partially observed problems, we use a Wonham filter to estimate the Markov chain from the observable evolution of the given process, and convert the original system to a completely observable one. Then, we establish a number of essential biological properties of the solution including regularity and positivity, stochastic boundedness, path

continuity, asymptotic properties, permanence, and extinction. We also show how to design feedback controls to make a population system permanent or extinct when the Markov chain is only observed in white noise.

In Chapter 3, we develop optimal harvest strategies for Lotka-Volterra systems so as to establish economically, ecologically, and environmentally reasonable strategies for populations subject to the risk of extinction. It is noted that simple-minded myopic unconstrained harvesting strategies and over-harvesting could lead to detrimental effect causing local extinctions or depletion of numerous species; see the examples documented in [30]. Thus the study on the optimal harvesting strategies has significant impact on the environment, ecology, economy, and the society. To better reflect reality, a continuous-time Markov chain is used to model the random environment. The underlying systems are thus controlled regime-switching diffusions that belong to the class of singular control problems. Starting with a model having multiple species, we construct upper bounds for the value functions, prove the finiteness of the harvesting value, and derive properties of the value functions. Then we construct explicit chattering harvesting strategies and the corresponding lower bounds for the value functions by using the idea of harvesting only one species at a time. We further show that this is a reasonable candidate for the best lower bound that one can expect. Moreover, in some cases, the lower bounds provide a good approximation of the value functions.

In Chapter 4, we study optimal harvesting problems in random environments. For stochastic control problems, to find the value function and the harvesting strategy, one usually solves a so-called Hamilton-Jacobi-Bellman (HJB) equation. However, for singular control problems with regime switching, the HJB equation is in fact a coupled system of nonlinear quasi-variational inequalities. A closed-form solution is virtually impossible to obtain.

We take an alternative approach by using the Markov chain approximation methodology developed by Kushner and Dupuis [28]. In contrast to the existing literature on numerical methods for singular control problems, in the current work, we take a step towards more useful and realistic model where the Markov chain is unobservable. Although much work was devoted to to the analysis of systems in the past, there are key differences in the model that make our analysis more delicate. Using a Wonham filter, we convert the partially observed system into a fully observed controlled diffusion. Then to approximate the value function and optimal strategies, Markov chain approximation techniques are used to construct a discrete-time controlled Markov chain. Convergence of the algorithm is obtained by weak convergence method.

Finally, in Chapter 5, we provide further discussions. We summarize the central theme of the dissertation, provide further remarks, and present some future directions for future work.

CHAPTER 2 STOCHASTIC COMPETITIVE LOTKA-VOLTERRA ECOSYSTEMS UNDER PARTIAL OBSERVATIONS: FEEDBACK CONTROLS FOR PERMANENCE AND EXTINCTION

2.1 Introduction

The traditional Lotka-Volterra equations, also known as the predator-prey equations, are a pair of first-order, nonlinear, differential equations, which are frequently used to describe the dynamics of biological or ecological systems in which two species interact, one as a predator and the other as prey. Initially proposed in 1910 by Lotka in the theory of autocatalytic chemical reactions [37], the equations were used to model predator-prey interactions [38] in 1925. The rationale is that when two or more species live in proximity and share the same basic requirements, they usually compete for resources, food, habitat, or territory. In reference to the study of the systems in the literature, this work develops asymptotic analysis of Lotka-Volterra models when random environment has to be taken into consideration. In particular, we treat the case that the random environment is given by a hidden Markov chain in continuous time.

Owing to the importance, the Lotka-Volterra models have received considerable attentions from multi-disciplinary communities such as biology, ecology, dynamic systems, and control and systems theory among others. There is a vast literature associated with the models. Along with the development of the deterministic models (see [18, 38, 62]), increasing attentions have placed on the stochastic counterpart that enables the consideration of randomly perturbed systems. As pointed out in [47, 48] (see also [12, 13]), population mod-

els should contain a multiplicative noise term, taking into account of the interaction of the ecosystem with the environment. The interaction between noise and nonlinear determinism in ecological dynamics adds an extra level of complexity and can give rise to the complex behavior of the system, which becomes very sensitive to initial conditions, various deterministic external perturbations, and to fluctuations always present in nature (see [55, 61]).

Because of the recent effort in modeling systems with both continuous dynamics and discrete events, the so-called hybrid models have gained much popularity. A trend of effort is to depict the random environment that cannot be described by stochastic differential equations using random switching processes; see for example, [34, 42, 70, 71] among others and also [44, 68] for a comprehensive treatment of switching processes.

The main issues concerning such systems include: Under what conditions, do the systems have global solutions? When will the systems be stable? Whether the systems are stochastically bounded? Whether or not the systems are stochastically permanent? Under what conditions, the species will extinct? If there is a tendency of extinction, can we find feedback controls so that this extinction be suppressed. More specifically, for $i = 1, 2, \ldots, n$, let $x_i(t)$ be the population size of the i th species in the ecosystem at time t, denote $x(t) = (x_1(t), \ldots, x_n(t))' \in \mathbb{R}^n$ (where z' denotes the transpose of z for $z \in \mathbb{R}^{l_1 \times l_2}$ with $l_1, l_2 \geq 1$). Consider a competitive Lotka-Volterra model in random environments with n species given by

$$dx(t) = \operatorname{diag}(x_1(t), \ldots, x_n(t)) \Big\{ \big[b(\alpha(t)) - A(\alpha(t))x(t) \big] dt + \Xi(\alpha) \circ dw(t) \Big\}, \tag{2.1}$$

and a constant initial condition $x(0) = x_0$. In the model, $w(\cdot) = (w_1(\cdot), \ldots, w_n(\cdot))'$ is an n-dimensional standard Brownian motion, and $b(\alpha) = (b_1(\alpha), \ldots, b_n(\alpha))'$, $A(\alpha) = (a_{ij}(\alpha))$,

$\Xi(\alpha) = \text{diag}\,(\sigma_1(\alpha), \ldots, \sigma_n(\alpha))$, $\alpha \in \mathcal{M} = \{1, \ldots, m\}$ represent the different intrinsic growth rates, the community matrices, and noise intensities in different external environments, respectively; $\alpha(t)$ is a finite state Markov chain. The above formulation is seen to be in the sense of Stratonovich integral. This form is often considered to be more suitable for environmental modeling (see [22]). For the stochastic differential equations in Stratonovich form, we refer to [56]; see also [23, Section 5.5] and [49, Chapter 3] for explanations why Stratonovich integral are more suitable for modeling in many applications.

Denote

$$\begin{aligned} &r_i(\alpha) = b_i(\alpha) + \frac{1}{2}\sigma_i^2(\alpha), \quad r(\alpha) = (r_1(\alpha), \ldots, r_n(\alpha))' \in \mathbb{R}^n, \\ &\text{diag}(x) := \text{diag}\,(x_1, \ldots, x_n)\,, \quad x = (x_1, \ldots, x_n)' \in \mathbb{R}^n. \end{aligned} \tag{2.2}$$

Then the equivalent system in Itô sense is as follows

$$dx(t) = \text{diag}\,(x(t)) \Big[r(\alpha(t)) - A(\alpha(t))x(t) \Big] dt + \text{diag}\,(x(t))\, \Xi\,(\alpha(t))\, dw(t). \tag{2.3}$$

The population model (2.3) was proposed and studied in details in [70, 71]. A question naturally arises in practice: Can we design feedback controls so that the resulting system becomes permanent or extinct if we only control a partially observed system? In particular, an important problem concerns that the Markov chain $\alpha(t)$ is unobservable. That is, at any given instance, the exact state of residency of the Markov chain is not known. Thus, we cannot see $\alpha(t)$ directly but only have noise-corrupted observation in the form of $\alpha(t)$ plus noise. Such scenarios frequently arise in the real world. Taking this fact into account, in our previous work [57], we consider the case $\Xi(\alpha) = \Xi$ being independent of α and the population process $x(t)$ represents the noisy observation-hidden Markov chain observed in white noise. We then used estimation schemes by means of the observable process $x(t)$. Distinct from

that work, here we suppose that the diffusion matrix $\Xi(\alpha)$ depends on environments. If we consider partially observed systems and use Wonham's filter similar to [7, 57], a problem arises since the filter is no longer finite dimensional. To be able to treat models in which the diffusion coefficients depend on the Markov chain, we consider (2.3) in which the Markov chain can only be observed in a Gaussian white noise. In addition, we consider the model with a control built in. Consider the controlled population system

$$dx(t) = \operatorname{diag}(x(t)) \Big[r(\alpha(t)) - A(\alpha(t))x(t) + u(t) \Big] dt + \operatorname{diag}(x(t))\, \Xi(\alpha(t))\, dw(t), \tag{2.4}$$

and

$$dy(t) = f(\alpha(t))dt + \beta(t)dB(t), \quad y(0) = 0, \tag{2.5}$$

where $\beta(\cdot) : [0, \infty) \mapsto \mathbb{R}$ is a continuously differentiable function satisfying $\inf_{t\geq 0} \beta(t) > 0$, $f : \mathcal{M} \mapsto \mathbb{R}$ is a one-to-one function, $B(t)$ is a one-dimensional standard Brownian motion being independent of $w(t)$ and $\alpha(t)$, and $u(t) = (u_1(t), \ldots, u_n(t))' \in \mathbb{R}^n$ is a feedback control.

For control problems of such partially observed systems, it is essential to converted them to completely observed ones, which can be done by using a Wonham filter. For results on the Wonham filter, we refer the reader to [63, 69]. Numerical results, including sample means and variances, assessment of approximation errors for Wonham's filter are presented in [67]. In the literature, the Wonham filters have been used widely to investigate control problems with partial observations; see [7, 14, 69] for applications in engineering science and finance, respectively. For related uses of hidden Markov chains and filtering theory in ecology and biology, we refer the readers to [15, 31] and references therein.

In contrast to the existing results, our new contributions in this chapter are as follows.

(i) We use Wonham's filter to build a stochastic competitive Lotka-Volterra ecosystem when the Markov chain is only observable in white Gaussian noise. (ii) We convert the partially observed system to a fully observed system by replacing the unknown Markovian states by their posterior probability estimates. (iii) We establish a number of essential biological properties of the solution including regularity and positivity, stochastic boundedness, path continuity, asymptotic properties, permanence, and extinction. (iv) We show how to design feedback controls to make a population system permanent or extinct when the Markov chain is only observed in white noise.

The rest of this chapter is organized as follows. Section 2.2 begins with the preliminaries and problem formulation, where Wonham's filter is introduced and the partially observed models are converted to completely observable ones. Section 2.3 is devoted to the suppression of population expression and biologically essential properties of the solution. Section 2.4 considers stochastic permanence and extinction. Feedback controls are investigated in Section 2.5 and numerical examples are provided in Section 2.6. Finally, the chapter is concluded with some concluding remarks.

2.2 Formulation

Let $\alpha(t)$ be a finite state Markov chain taking values in $\mathcal{M} = \{1, 2, \ldots, m\}$ with the generator $Q = (q_{ij}) \in \mathbb{R}^{m\times m}$. Assume throughout this chapter that both the Markov chain $\alpha(t)$ and the n-dimensional standard Brownian motion $w(t)$ are defined on a complete filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ with the filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions (i.e., it is right continuous, increasing, and $\mathcal{F}_0$ contains all the null sets). Denote by $\mathbb{1}_E$ the indicator

function of the event E, and use the following notation throughout the chapter:

$$p_k(t) := \mathbb{1}_{\{\alpha(t)=k\}}, \quad k = 1, \ldots, m,$$

$$p(t) := (p_1(t), \ldots, p_m(t))' \in \mathbb{R}^m,$$

$$\mathcal{F}_t^y := \sigma\{y(s), 0 \le s \le t\},$$

$$\varphi_k(t) := P\left(\alpha(t) = k | \mathcal{F}_t^y\right) = E[p_k(t)|\mathcal{F}_t^y], \ k = 1, \ldots, m,$$

$$\varphi(t) := (\varphi_1(t), \ldots, \varphi_m(t))' \in \mathbb{R}^m,$$

$$\hat{a}_{ij} := \max_k a_{ij}(k), \ \check{a}_{ij} := \min_k a_{ij}(k),$$

$$\hat{r}_i := \max_k r_i(k), \ \check{r}_i := \min_k r_i(k), \ \hat{\sigma}_i^2 := \max_k \sigma_i^2(k), \ \check{\sigma}_i^2 := \min_k \sigma_i^2(k),$$

$$\mathbb{R}_+^n := \{x = (x_1, \ldots, x_n)' \in \mathbb{R}^n : x_i > 0, i = 1, \ldots, n\},$$

$$S_m := \{\varphi = (\varphi_1, \ldots, \varphi_m)' \in \mathbb{R}^m : \varphi_k \ge 0, \sum_{k=1}^m \varphi_k = 1\},$$

$$\overline{\sigma}_i(\varphi) := \sum_{k=1}^m \sigma_i(k)\varphi_k, \quad \overline{r}_i(\varphi) := \sum_{k=1}^m r_i(k)\varphi_k, \ \overline{f}(\varphi) := \sum_{k=1}^m f(k)\varphi_k, \text{ for a } \varphi \in S_m.$$

We first recall some results on Wonham's filter. As we mentioned in Section 2.1, the Markov chain $\alpha(t)$ is observed through (2.5). It was proved in [63] that the posterior probability $\varphi(\cdot)$ satisfies the following system of stochastic differential equations

$$\begin{cases} d\varphi_j(t) = \Big[\sum_{k=1}^m q_{kj}\varphi_k(t) - \beta^{-2}(t)\big(f(j) - \overline{f}\left(\varphi(t)\right)\big)\overline{f}\left(\varphi(t)\right)\varphi_j(t)\Big]dt \\ \qquad\qquad + \beta^{-2}(t)\big(f(j) - \overline{f}\left(\varphi(t)\right)\big)\varphi_j(t)dy(t), \quad j = 1, \ldots, m, \\ \varphi_j(0) = \varphi_j^0, \quad j = 1, \ldots, m, \end{cases} \tag{2.6}$$

where $\varphi^0 = (\varphi_1^0, \ldots, \varphi_m^0)' \in \mathbb{R}^m$ is the initial distribution of $\alpha(t)$. Let

$$d\overline{w}(t) = \beta^{-1}(t)\big(dy(t) - \overline{f}\left(\varphi(t)\right)dt\big), \quad \overline{w}(0) = 0,$$

be the one dimensional innovation process. Then the first m equations in (2.6) can be rewrit-

ten as

$$d\varphi_j(t) = \sum_{k=1}^{m} q_{kj}\varphi_k(t)dt + \beta^{-1}(t)\big(f(j) - \overline{f}\,(\varphi(t))\,\big)\varphi_j(t)d\overline{w}(t), \quad j = 1, \ldots, m, \tag{2.7}$$

which are easier to work with in the subsequent analysis. Equivalently,

$$d\varphi(t) = Q'\varphi(t)dt + \beta^{-1}(t)C(t)\varphi(t)d\overline{w}(t), \tag{2.8}$$

where $C(t) = \mathrm{diag}\,(f(1), \ldots, f(m)) - \overline{f}\,(\varphi(t))\,I_m$, and I_m is the $m \times m$ identity matrix. Note that system (2.4) can be written as

$$dx(t) = \mathrm{diag}(x(t))\Big[\sum_{k=1}^{m} p_k(t)\Big(r(k) - A(k)x(t)\Big) + u(t)\Big]dt + \mathrm{diag}(x(t))\sum_{k=1}^{m} p_k(t)\Xi(k)dw(t). \tag{2.9}$$

The solution of (2.8) is the well-known Wonham filter $\varphi(t)$. which is an estimate of the hidden state $p(t)$. Replace $p(t)$ by $\varphi(t)$ in (2.9), we arrive at

$$dx(t) = \mathrm{diag}(x(t))\Big[\sum_{k=1}^{m} \varphi_k(t)\Big(r(k) - A(k)x(t)\Big) + u(t)\Big]dt + \mathrm{diag}(x(t))\sum_{k=1}^{m} \varphi_k(t)\Xi(k)dw(t). \tag{2.10}$$

In component-wise form, system (2.10) becomes

$$dx_i(t) = x_i(t)\Big[\sum_{k=1}^{m}\Big(r_i(k) - \sum_{j=1}^{n} a_{ij}(k)x_j(t)\Big)\varphi_k(t) + u_i(t)\Big]dt + x_i(t)\sum_{k=1}^{m}\sigma_i(k)\varphi_k(t)dw_i(t). \tag{2.11}$$

Hence (2.8) and (2.10) form a competitive Lotka-Volterra ecosystem with complete observations.

We assume the following standing assumptions.

(A) For $i, j = 1, \ldots, n$ with $i \neq j$, $\min_k a_{ii}(k) > 0$, $\min_k a_{ij}(k) \geq 0$.

(B) The feedback control is $u(t) := u(x(t), \varphi(t))$, where $u(x, \varphi)$ is locally Lipschitz in (x, φ).

Moreover, for $i, j = 1, \ldots, n$, there are constants c_i , d_i, $\lambda_{ij} \geq 0$, $\rho_{ij} \geq 0$, $\rho_{ii} > 0$, and $\lambda_{ii} > 0$ such that for any tuple $(x, \varphi) \in \mathbb{R}^n_+ \times S_m$,

$$d_i - \sum_{j=1}^{n} \rho_{ij} x_j \leq u_i(x, \varphi) + \sum_{k=1}^{m} \Big(r_i(k) - \sum_{j=1}^{n} a_{ij}(k) x_j(t) \Big) \varphi_k(t) \leq c_i - \sum_{j=1}^{n} \lambda_{ij} x_j. \quad (2.12)$$

Observe that assumptions (A) and (B) ensure that our original population system is competitive and this property still holds in the controlled system (2.11). For convenience, let us combine (2.8) and (2.9) to obtain the following system

$$\begin{cases} dx(t) = \operatorname{diag}(x(t)) \Big[\sum_{k=1}^{m} \varphi_k(t) \big(r(k) - A(k) x(t) \big) + u(t) \Big] dt \\ \qquad\qquad + \operatorname{diag}(x(t)) \sum_{k=1}^{m} \varphi_k(t) \Xi(k) dw(t), \\ d\varphi(t) = Q' \varphi(t) dt + \beta^{-1}(t) C(t) \varphi(t) d\overline{w}(t). \end{cases} \quad (2.13)$$

Let $\mathcal{A}(x, \varphi, t)$ denote the diffusion matrix of the population model (2.13). Then

$$\begin{aligned} \mathcal{A}(x, \varphi, t) &= \begin{pmatrix} \operatorname{diag}(x) \sum_{k=1}^{m} \varphi_k \Xi(k) & 0 \\ 0 & \beta^{-1}(t) C \varphi \end{pmatrix} \begin{pmatrix} \operatorname{diag}(x) \sum_{k=1}^{m} \varphi_k \Xi(k) & 0 \\ 0 & \beta^{-1}(t) \varphi' C \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix}, \end{aligned}$$

where

$$\mathcal{A}_1 = \operatorname{diag}\{x_1^2 \left(\overline{\sigma}_1(\varphi)\right)^2, \ldots, x_n^2 \left(\overline{\sigma}_n(\varphi)\right)^2\} \text{ and } \mathcal{A}_2 = \beta^{-2}(t) C \varphi (C\varphi)'.$$

Remark 2.1. Note that $\varphi(t)$ is the probability vector conditioned on the observation $\sigma\{y(s) : 0 \leq s \leq t\}$, for each $k \in \mathcal{M}$ and each $t \geq 0$, $\varphi_k(t) \geq 0$ and $\sum_{k=1}^{m} \varphi_k(t) = 1$, i.e., $\varphi(t) \in S_m$. We will use this property of $\varphi(t)$ frequently.

For $x = (x_1, \ldots, x_n)' \in \mathbb{R}^n_+$, its norm is denoted by $|x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$. The operator associated with (2.13) is defined as follows: For a sufficiently smooth real-valued function

$h : \mathbb{R}^n_+ \times \mathbb{R}^m \mapsto \mathbb{R}$ being independent of φ, let

$$\mathcal{L}h(x,\varphi) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} x_i \Big[\sum_{k=1}^{m} \Big(r_i(k) - \sum_{j=1}^{n} a_{ij}(k) x_j \Big) \varphi_k + u_i \Big] + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} x_i^2 \left(\overline{\sigma}_i(\varphi) \right)^2 . \quad (2.14)$$

Now the study of (2.4) can be carried out by investigating system (2.13). Throughout the chapter, we use K, K_κ to denote a generic positive constant whose exact value may be different in different appearances where K_κ indicates that the constant depends on a quantity κ to be specified later.

2.3 Properties of the Solution

Theorem 2.2. *Assume* (A) *and* (B) *hold. Then for any initial condition* $(x(0), \varphi(0)) \in \mathbb{R}^n_+ \times S_m$, *there is a unique solution* $(x(t), \varphi(t))$ *to* (2.13) *on* $t \geq 0$ *such that* $x(t)$ *remains in* $\mathbb{R}^n_+$ *almost surely, i.e.,* $P\{x(t) \in \mathbb{R}^n_+ : \text{ for all } t \geq 0\} = 1$.

Proof. Since the coefficients of (2.13) are locally Lipschitz, there is a unique local solution $(x(t), \varphi(t))'$ on $t \in [0, \zeta)$, where ζ is the explosion time (see [42, Theorem A.2]). Let l_0 be a sufficiently large positive integer such that every component of $x(0)$ is contained in the interval $\left(\frac{1}{l_0}, l_0\right)$. For each $l \geq l_0$, we define

$$\tau_l := \inf \left\{ t \geq 0 : x_i(t) \notin \left(\frac{1}{l}, l \right), \text{ for some } i = 1, .., n \right\} . \quad (2.15)$$

Clearly the sequence $\{\tau_l\}$ is monotonically increasing. Let $\tau_\infty := \lim_{l \to \infty} \tau_l$. Then $\tau_\infty \leq \zeta$. It suffices to show that $\tau_\infty = \infty$ w.p.1. If this were false, there would exist a $T > 0$ and $\varepsilon > 0$ such that $P\{\tau_\infty \leq T\} > \varepsilon$. Therefore we can find some $l_1 \geq l_0$ such that

$$P\{\tau_l \leq T\} > \varepsilon, \quad \text{for all } l \geq l_1. \quad (2.16)$$

Consider the Liapunov function $V(x, \varphi) = \sum_{i=1}^{n} \big(x_i - 1 - \ln x_i \big)$, $\quad (x, \varphi) \in \mathbb{R}^n_+ \times S_m$. Then

$V(x,\varphi) \geq 0$ for every $(x,\varphi) \in \mathbb{R}^n_+ \times S_m$. Using (2.10), detail computations lead to

$$\begin{aligned}\mathcal{L}V(x,\varphi) &= \sum_{i=1}^n (x_i - 1)\Big[\sum_{k=1}^m \Big(r_i(k) - \sum_{j=1}^n a_{ij}(k)x_j\Big)\varphi_k + u_i\Big] + \frac{1}{2}\sum_{i=1}^n (\overline{\sigma}_i(\varphi))^2 \\ &\leq \sum_{i=1}^n x_i\Big(c_i - \sum_{j=1}^n \lambda_{ij}x_j\Big) - d_i + \sum_{j=1}^n \rho_{ij}x_j + \frac{1}{2}\sum_{i=1}^n (\overline{\sigma}_i(\varphi))^2 \\ &\leq \sum_{i=1}^n \Big[-\lambda_{ii}x_i^2 + \Big(c_i + \sum_{j=1}^n \rho_{ji}\Big)x_i + \frac{\hat{\sigma}_i^2}{2} - d_i\Big].\end{aligned} \tag{2.17}$$

Hence $\mathcal{L}V(x,\varphi)$ is bounded above by some constant $K > 0$. By Dynkin's formla, we obtain

$$EV\left(x(\tau_l \wedge T), \varphi(\tau_l \wedge T)\right) - EV\left(x(0), \varphi(0)\right) = E\int_0^{\tau_l \wedge T} \mathcal{L}V\left(x(s), \varphi(s)\right) ds \leq KT.$$

It follows that

$$\begin{aligned}KT + V\left(x(0), \varphi(0)\right) &\geq EV\left(x(\tau_l \wedge T), \varphi(\tau_l \wedge T)\right)) \\ &\geq E\left[V\left(x(\tau_l), \varphi(\tau_l)\right) 1\!\!1_{\{\tau_l \leq T\}}\right].\end{aligned} \tag{2.18}$$

Note that for each $\omega \in \{\tau_l \leq T\}$, there is some i such that $x_i(\tau_l(\omega)) \geq l$ or $x_i(\tau_l(\omega)) \leq \frac{1}{l}$. Hence the properties of the function $V(\cdot,\cdot)$ give us

$$V\left(x(\tau_l), \varphi(\tau_l)\right)(\omega) \geq (l^\gamma - 1 - \gamma \ln l) \wedge \left(\frac{1}{l^\gamma} - 1 + \gamma \ln l\right). \tag{2.19}$$

In view of (2.16) and (2.19), we get from (2.18) that

$$KT + V\left(x(0), \varphi(0)\right) \geq \varepsilon\left[(l^\gamma - 1 - \gamma \ln l) \wedge \left(\frac{1}{l^\gamma} - 1 + \gamma \ln l\right)\right].$$

This leads to a contradiction as $l \to \infty$. Therefore, $\tau_\infty = \infty$ a.s. $\square$

Next we shall show that solutions of the converted completely observable system (2.13) has such properties as finite moments, path continuity, and positive recurrence. These properties are important from the biological point of view.

Theorem 2.3. *Assume that* (A) *and* (B) *are satisfied. Then the following statements hold.*

(a) *For any* $\kappa > 0$,

$$\sup_{t\geq 0} E\big[|x(t)|^{\kappa}\big] < \infty.$$

(b) *For any* $\kappa > 0$,

$$\limsup_{t\to\infty} E\big[|x(t)|^{\kappa}\big] \leq K_{\kappa} < \infty,$$

where the constant K_{κ} *is independent of* $(x(0), \varphi(0))$.

(c) *The process* $x(t)$ *is stochastically bounded, i.e., for any* $\varepsilon > 0$, *there is a constant* $H = H(\varepsilon)$ *such that for any* $(x(0), \varphi(0))$, *we have*

$$\limsup_{t\to\infty} P\{|x(t)| \leq H\} \geq 1 - \varepsilon.$$

(d) *The process* $(x(t), \varphi(t))'$ *is a Markov process having continuous sample paths a.s.*

Proof. (a) Let

$$V(x, \varphi) = \sum_{i=1}^{n} x_i^{\kappa}, \quad (x, \varphi) \in \mathbb{R}_+^n \times S_m.$$

Using (2.14), detail computations give us that

$$\begin{aligned}
\mathcal{L}V(x, \varphi) &= \sum_{i=1}^{n} \kappa x_i^{\kappa} \Bigg[\sum_{k=1}^{m} \Big(r_i(k) - \sum_{j=1}^{n} a_{ij}(k) x_j \Big) \varphi_k + u_i \Bigg] + \frac{1}{2} \sum_{i=1}^{n} \left(\overline{\sigma}_i(\varphi)\right)^2 \kappa(\kappa - 1) x_i^{\kappa} \\
&= \kappa \sum_{i=1}^{n} x_i^{\kappa} \Bigg[\sum_{k=1}^{m} \Big(r_i(k) - \sum_{j=1}^{n} a_{ij}(k) x_j \Big) \varphi_k + u_i + \frac{1}{2}(\kappa - 1) \left(\overline{\sigma}_i(\varphi)\right)^2 \Bigg] \\
&\leq \kappa \sum_{i=1}^{n} x_i^{\kappa} \left[c_i - \lambda_{ii} x_i + \frac{|\kappa - 1| \hat{\sigma}_i^2}{2} \right].
\end{aligned} \tag{2.20}$$

Hence $\mathcal{L}V(x, \varphi)$ is bounded above by some constant $K > 0$. Note that the boundedness of

$$\kappa \sum_{i=1}^{n} x_i^{\kappa} \left[\frac{1}{\kappa} + c_i + \frac{|\kappa - 1| \hat{\sigma}_i^2}{2} - \lambda_{ii} x_i \right],$$

by a positive constant can be obtained by studying the range of the above function in the variable $x_i \in (0,\infty), i=1,\dots,n$. Applying Itô's formula to the function $e^t V(x,\varphi)$, we obtain

$$E\left[e^{t\wedge\tau_l}\sum_{i=1}^n x_i^\kappa(t\wedge\tau_l)\right] = E\sum_{i=1}^n x_i^\kappa(0) + \int_0^{t\wedge\tau_l} e^s[V+\mathcal{L}V]\,(x(s),\varphi(s))\,ds$$

where τ_l is the stopping time defined by (2.15). Using (2.20), we have

$$\begin{aligned} E\left[e^{t\wedge\tau_l}\sum_{i=1}^n x_i^\kappa(t\wedge\tau_l)\right] &\le \sum_{i=1}^n x_i^\kappa(0) + E\int_0^{t\wedge\tau_l} \kappa e^s \sum_{i=1}^n x_i^\kappa\left[\frac{1}{\kappa}+c_i+\frac{|\kappa-1|\hat{\sigma}_i^2}{2}-\lambda_{ii}x_i\right]ds \\ &\le \sum_{i=1}^n x_i^\kappa(0) + E\int_0^{t\wedge\tau_l} e^s K_\kappa ds \\ &\le \sum_{i=1}^n x_i^\kappa(0) + K_\kappa(e^t-1). \end{aligned} \tag{2.21}$$

Letting $k\to\infty$ in (2.21), by virtue of Fatou's lemma,

$$E\big[e^t\sum_{i=1}^n x_i^\kappa(t)\big] \le \sum_{i=1}^n x_i^\kappa(0) + K_\kappa(e^t-1),$$

i.e.,

$$E\left[\sum_{i=1}^n x_i^\kappa(t)\right] \le e^{-t}\sum_{i=1}^n x_i^\kappa(0) + K_\kappa(1-e^{-t}).$$

Note that for $x=(x_1,\dots,x_n)'\in\mathbb{R}_+^n$, $|x|^\kappa \le \big(\sqrt{n}\max_i x_i\big)^\kappa \le n^{\kappa/2}\sum_{i=1}^n x_i^\kappa$, then

$$E\big[|x(t)|^\kappa\big] \le n^{\kappa/2}\left[e^{-t}\sum_{i=1}^n x_i^\kappa(0) + K_\kappa(1-e^{-t})\right]. \tag{2.22}$$

The desired inequality is easily obtained.

(b) follows from (2.22).

(c) follows from (b) with $\kappa=2$ and Tchebyshev's inequality.

(d) Both the drift and the diffusion coefficient given in (2.13) satisfy the linear growth and Lipschitz condition in every bounded open set in $\mathbb{R}^{n+m}$. By [23, Theorem 3.5], it suffices to

show that there is a nonnegative, twice continuously differentiable function $V : \mathbb{R}^n_+ \times S_m \to \mathbb{R}$ such that $\inf_{|x|\geq R, \varphi\in S_m} V(x,\varphi) \to \infty$ as $R \to \infty$ and that there is an $c > 0$ satisfying $\mathcal{L}V \leq cV$. To this end, take $V(x,\varphi) = \sum_{i=1}^{n} x_i^2$. Then $\inf_{|x|\geq R, \varphi\in S_m} V(x,\varphi) \to \infty$ as $R \to \infty$ and by the calculation and estimation in (2.20) with $\kappa = 2$, we have

$$\mathcal{L}V(x,\varphi) \leq \max_i \left(2c_i + \hat{\sigma}_i^2\right) V(x,\varphi).$$

This completes the proof. □

Theorem 2.4. *Assume* (A) *and* (B) *hold. Then the following assertions hold.*

(a) $\limsup_{t\to\infty} \dfrac{\ln|x(t)|}{\ln t} \leq 1$ a.s.,

(b) $\limsup_{t\to\infty} \dfrac{\ln|x(t)|}{t} \leq 0$ a.s.

Proof. (a) The proof is a modification of that of [71]. We only give a sketch of the outline and omit some technical details. Define

$$V(t,x,\varphi) = e^t \ln(|x|), \quad (t,x,\varphi) \in [0,\infty) \times \mathbb{R}^n_+ \times S_m.$$

By Itô's formula, we obtain

$$\begin{aligned} & e^t \ln(|x(t)|) - \ln(|x(0)|) \\ &= \int_0^t e^s \sum_{i=1}^n \frac{x_i^2(s)}{|x(s)|^2} \Bigg[\sum_{k=1}^m \Big(r_i(k) - \sum_{j=1}^n a_{ij}(k) x_j(s) \Big) \varphi_k(s) + u_i(s) \Bigg] \\ &\quad + \frac{1}{2}\Big(1 - \frac{2x_i^2(s)}{|x(s)|^2}\Big) \left(\overline{\sigma}_i(\varphi(s))\right)^2 \Bigg] ds + \int_0^t e^s \ln\big(|x(s)|\big) ds \\ &\quad + \sum_{i=1}^n \int_0^t e^s . \frac{x_i^2(s)}{|x(s)|^2} \overline{\sigma}_i(\varphi(s))\, dw_i(s). \end{aligned} \tag{2.23}$$

Using the argument as in [71, Theorem 3.3], there exist a $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ satisfying that for any positive constants $\gamma > 0$, $\theta > 1$, $\varepsilon \in (0,1)$ and any $\omega \in \Omega_0$, there is a positive

integer $\xi = \xi_0(\omega)$ such that $\xi \geq \xi_0(\omega)$ and $t \in [0, \xi\gamma]$ imply

$$\sum_{i=1}^{n} \int_0^t e^s \frac{x_i^2(s)}{|x(s)|^2} \overline{\sigma}_i\left(\varphi(s)\right) dw_i(s) \leq \frac{n\varepsilon e^{-\xi\gamma}}{2} \sum_{i=1}^{n} \int_0^t e^{2s} \frac{x_i^4(s)}{|x(s)|^4} \hat{\sigma}_i^2 ds + \frac{\theta e^{\xi\gamma} \ln(\xi)}{\varepsilon}. \tag{2.24}$$

It follows from (2.23) and (2.24) that

$$\begin{aligned}
& e^t \ln\left(|x(t)|\right) - \ln\left(|x(0)|\right) \\
& \leq \int_0^t e^s \sum_{i=1}^{n} \frac{x_i^2(s)}{|x(s)|^2} \Bigg[\sum_{k=1}^{m} \Big(r_i(k) - \sum_{j=1}^{n} a_{ij}(k) x_j(s) \Big) \varphi_k(s) + u_i(s) \\
& \quad + \frac{1}{2} \Big(1 - \frac{2x_i^2(s)}{|x(s)|^2} \Big) \left(\overline{\sigma}_i\left(\varphi(s)\right)\right)^2 \Bigg] ds + \int_0^t e^s \ln\big(|x(s)|\big) ds \\
& \quad + \frac{n\varepsilon e^{-\xi\gamma}}{2} \sum_{i=1}^{n} \int_0^t e^{2s} \frac{x_i^4(s)}{|x(s)|^4} \hat{\sigma}_i^2 ds + \frac{\theta e^{\xi\gamma} \ln(\xi)}{\varepsilon}. \\
& \leq \int_0^t e^s \Bigg[\ln|x(s)| + \sum_{i=1}^{n} \frac{x_i^2(s)}{|x(s)|^2} \Big[c_i - \lambda_{ii} x_i(s) + \frac{1}{2} \hat{\sigma}_i^2 \Big] \\
& \quad + \sum_{i=1}^{n} \frac{n\varepsilon e^{s-\xi\gamma}}{2} \frac{x_i^4(s)}{|x(s)|^4} \hat{\sigma}_i^2 - \sum_{i=1}^{n} \frac{x_i^4(s)}{|x(s)|^4} \check{\sigma}_i^2 \Bigg] ds + \frac{\theta e^{\xi\gamma} \ln(\xi)}{\varepsilon}.
\end{aligned} \tag{2.25}$$

Note that for any $t \in [0, \xi\gamma]$, $s \in [0, t]$, and $(x, \varphi) \in \mathbb{R}_+^n \times S_m$, there exist $K_i > 0$ for $i = 1, 2, 3$ such that

$$\begin{aligned}
& \ln\left(|x|\right) + \sum_{i=1}^{n} \frac{x_i^2}{|x|^2} \Big[c_i - \lambda_{ii} x_i + \frac{1}{2} \hat{\sigma}_i^2 \Big] + \sum_{i=1}^{n} \frac{n\varepsilon e^{s-\xi\gamma}}{2} \frac{x_i^4}{|x|^4} \hat{\sigma}_i^2 - \sum_{i=1}^{n} \frac{x_i^4}{|x|^4} \check{\sigma}_i^2 \\
& \leq \ln\left(|x|\right) + \sum_{i=1}^{n} \frac{x_i^2}{|x|^2} (c_i + \frac{1}{2} \hat{\sigma}_i^2) - \sum_{i=1}^{n} \lambda_{ii} \frac{x_i^3}{|x|^2} + K_1 \\
& \leq \ln\left(|x|\right) - \frac{\min_i \lambda_{ii}}{\sqrt{n}} |x| + K_1 + K_2 \\
& \leq K_3 + K_1 + K_2 = K.
\end{aligned} \tag{2.26}$$

In the above, we used the fact that $\min_i \varepsilon_{ii} > 0$ and the function in variable t, $\ln\left(|t|\right) - \dfrac{t \min_i \lambda_{ii}}{\sqrt{n}}$ is bounded above on $(0, \infty)$. We also used the inequality $\sum_{i=1}^{n} x_i^3 \geq \dfrac{1}{\sqrt{n}} |x|^3$.

It then follows from (2.25) and (2.26) that

$$e^t \ln\left(|x(t)|\right) - \ln\left(|x(0)|\right) \leq K(e^t - 1) + \frac{\theta e^{\xi\gamma} \ln(\xi)}{\varepsilon}.$$

The desired result is obtained by repeating the argument in [71, Theorem 3.3].

(b) It is easily obtained from (a) that $\lim_{t\to\infty} \frac{\ln(t)}{t} = 0$. $\square$

This theorem says that the process $x(t)$ will growth at most polynomially. By virtue of this result, for any $\varepsilon > 0$, there is a positive random time T_ε such that for any $t \geq T_\varepsilon$, $|x(t)| \leq t^{1+\varepsilon}$ a.s.

2.4 Stochastic Permanence and Extinction

It is well known that in the study of stochastic population systems, stochastic permanence, which indicate that the species will survive forever, is one of the most important concepts. Many works have been devoted to stochastic permanence for different population models; see [32, 34, 35] among others. We first recall the definition of stochastic permanence [32, Definition 3.2].

Definition 2.5. The population system (2.13) is said to be stochastically permanent if for any $\varepsilon \in (0,1)$, there exist positive constants $H = H(\varepsilon)$ and $K = K(\varepsilon)$ such that

$$\liminf_{t\to\infty} P\{|x(t)| \geq H\} \geq 1-\varepsilon, \quad \liminf_{t\to\infty} P\{|x(t)| \leq K\} \geq 1-\varepsilon, \tag{2.27}$$

where $(x(t), \varphi(t))$ is the solution of the population system (2.13) with any initial condition $(x(0), \varphi(0)) \in \mathbb{R}^n_+ \times S_m$.

Theorem 2.6. *Assume that* (A) *and* (B) *are satisfied, and for* $i, j = 1, \ldots, n$, *there are constants* $\gamma_i > 0$, $\varepsilon_{ij} \geq 0$, $\varepsilon_{ii} > 0$, *such that*

$$u_i(t) + \sum_{k=1}^{m} \Big(r_i(k) - \sum_{j=1}^{n} a_{ij}(k) x_j(t) \Big) \varphi_k(t) \geq \gamma_i + \frac{1}{2} \big(\overline{\sigma}_i(\varphi(t)) \big)^2 - \sum_{j=1}^{n} \varepsilon_{ij} x_j(t), \quad t \geq 0, i = 1, \ldots, n. \tag{2.28}$$

Then the population system (2.13) *is stochastically permanent.*

Proof. Let $\hat{\sigma}^2 := \min_{1\le i\le n} \hat{\sigma}_i^2$. Since $\gamma := \min_i \gamma_i > 0$, then there are positive constants θ and κ such that

$$\gamma - 0.5\theta\hat{\sigma}^2 > 0, \quad \gamma - 0.5\theta\hat{\sigma}^2 - \frac{n\kappa}{\theta} > 0, \quad i = 1, \ldots, n. \tag{2.29}$$

Define

$$V_1(x) = \Big(\sum_{i=1}^n x_i\Big)^{-1}, \quad x \in \mathbb{R}_+^n.$$

Applying Itô's formula to $V_1(x(t))$ leads to

$$\begin{aligned} dV_1(x) = \Big\{ &- V_1^2(x)\sum_{i=1}^n x_i\Big[\sum_{k=1}^m \Big(r_i(k) - \sum_{j=1}^n a_{ij}(k)x_j\Big)\varphi_k(t) + u_i(t)\Big] \\ &+ V_1^3(x)\sum_{i=1}^n \left(\overline{\sigma}_i\left(\varphi(t)\right)\right)^2 x_i^2\Big\}dt - V_1^2(x)\sum_{i=1}^n x_i\overline{\sigma}_i\left(\varphi(t)\right)dw_i(t), \end{aligned}$$

where we suppressed t in $x(t)$ for simplicity.

Define

$$V_2(x) = \big(1 + V_1(x)\big)^\theta, \quad V_3(x) = e^{\kappa t}V_2(x), \quad x \in \mathbb{R}_+^n.$$

Then by Itô's formula,

$$dV_2(x) = \theta(1 + V_1(x))^{\theta-1}dV_1(x) + \frac{1}{2}\theta(\theta-1)(1 + V_1(x))^{\theta-2}(dV_1(x))^2.$$

That is,

$$\begin{aligned} dV_2(x) = \theta(1 + V_1(x))^{\theta-2}\Big\{ &- (1 + V_1(x))V_1^2(x)\sum_{i=1}^n x_i \\ &\times\Big[\sum_{k=1}^m \Big(r_i(k) - \sum_{j=1}^n a_{ij}(k)x_j\Big)\varphi_k(t) + u_i(t)\Big] \\ &+ (1 + V_1(x))V_1^3(x)\sum_{i=1}^n \left(\overline{\sigma}_i\left(\varphi(t)\right)\right)^2 x_i^2 + \frac{\theta-1}{2}V_1^4(x)\sum_{i=1}^n x_i^2\left(\overline{\sigma}_i\left(\varphi(t)\right)\right)^2\Big\}dt \\ &- \theta(1 + V_1(x))^{\theta-1}V_1^2(x)\sum_{i=1}^n x_i\overline{\sigma}_i\left(\varphi(t)\right)dw_i(t). \end{aligned}$$

It also follows from Itô's formula that

$$\begin{aligned} dV_3(x) &= \kappa e^{\kappa t} V_2(x) dt + e^{\kappa t} dV_2(x) \\ &= \theta e^{\kappa t} L\left(x(t), \varphi(t)\right) - \theta e^{\kappa t} (1+V_1(x))^{\theta-1} V_1^2(x) \sum_{i=1}^n x_i \overline{\sigma}_i\left(\varphi(t)\right) dw_i(t), \end{aligned} \tag{2.30}$$

where

$$\begin{aligned} L\left(x(t), \varphi(t)\right) &= (1+V_1(x))^{\theta-2} \Bigg\{ \frac{\kappa(1+V_1(x))^2}{\theta} \\ &\quad -(1+V_1(x)) V_1^2(x) \sum_{i=1}^n x_i \Bigg[\sum_{k=1}^m \Big(r_i(k) - \sum_{j=1}^n a_{ij}(k) x_j \Big) \varphi_k(t) + u_i(t) \Bigg] \\ &\quad +(1+V_1(x)) V_1^3(x) \sum_{i=1}^n x_i^2 \left(\overline{\sigma}_i\left(\varphi(t)\right)\right)^2 + \frac{\theta-1}{2} V_1^4(x) \sum_{i=1}^n x_i^2 \left(\overline{\sigma}_i\left(\varphi(t)\right)\right)^2 \Bigg\} \\ &= (1+V_1(x))^{\theta-2} \Bigg\{ \frac{\kappa(1+V_1(x))^2}{\theta} + V_1^3(x) \sum_{i=1}^n x_i^2 \left(\overline{\sigma}_i\left(\varphi(t)\right)\right)^2 \\ &\qquad + \frac{\theta+1}{2} V_1^4(x) \sum_{i=1}^n x_i^2 \left(\overline{\sigma}_i\left(\varphi(t)\right)\right)^2 \\ &\qquad -(1+V_1(x)) V_1^2(x) \sum_{i=1}^n x_i \Bigg[\sum_{k=1}^m \Big(r_i(k) - \sum_{j=1}^n a_{ij}(k) x_j \Big) \varphi_k(t) + u_i(t) \Bigg] \Bigg\}. \end{aligned} \tag{2.31}$$

To proceed, we show that $\sup_{t\geq 0} L\left(x(t), \varphi(t)\right) < \infty$. Observe that the following estimates hold.

$$\begin{aligned} &V_1^3(x) \sum_{i=1}^n x_i^2 \left(\overline{\sigma}_i(\varphi(t))\right)^2 \leq \hat{\sigma}^2 \frac{\sum_{i=1}^n x_i^2}{\big(\sum_{i=1}^n x_i\big)^2} V_1(x) \leq V_1(x) \hat{\sigma}^2, \\ &-V_1^4(x) \sum_{i=1}^n x_i^2 \leq -\frac{1}{n} V_1^4(x) \big(\sum_{i=1}^n x_i\big)^2 = -\frac{1}{n} V_1^2(x). \end{aligned} \tag{2.32}$$

We also have

$$\begin{aligned} &-(1+V_1(x))\ V_1^2(x) \sum_{i=1}^n x_i \Bigg[\sum_{k=1}^m \Big(r_i(k) - \sum_{j=1}^n a_{ij}(k) x_j \Big) \varphi_k(t) + u_i(t) \Bigg] \\ &\quad \leq -(1+V_1(x)) V_1^2(x) \sum_{i=1}^n x_i \Big[\gamma_i + \frac{1}{2} \big(\overline{\sigma}_i\left(\varphi(t)\right)\big)^2 - \sum_{j=1}^n \varepsilon_{ij} x_j(t) \Big] \\ &\quad \leq -V_1^3(x) \sum_{i=1}^n x_i \Big[\gamma_i + \frac{1}{2} \big(\overline{\sigma}_i\left(\varphi(t)\right)\big)^2 \Big] + (1+V_1(x)) \max_{1\leq i,j\leq n} \varepsilon_{ij} \\ &\quad \leq -V_1^4(x) \sum_{i=1}^n x_i^2 \Big[\gamma_i + \frac{1}{2} \big(\overline{\sigma}_i\left(\varphi(t)\right)\big)^2 \Big] + (1+V_1(x)) \max_{1\leq i,j\leq n} \varepsilon_{ij}. \end{aligned} \tag{2.33}$$

In views of (2.31), (2.32), and (2.33),

$$
\begin{aligned}
L\left(x(t),\varphi(t)\right) &\le (1+V_1(x))^{\theta-2}\Big\{V_1^2(x)\frac{\kappa}{\theta}+V_1(x)\Big(\frac{2\kappa}{\theta}+\hat{\sigma}^2+\max_{1\le i,j\le n}\varepsilon_{ij}\Big)\\
&\qquad+\max_{1\le i,j\le n}\varepsilon_{ij}+\frac{\kappa}{\theta}-V_1^4(x)\sum_{i=1}^n\Big(\gamma_i+\frac{1}{2}\big(\overline{\sigma}_i\left(\varphi(t)\right)\big)^2-\frac{\theta+1}{2}\big(\overline{\sigma}_i(\varphi(t))\big)^2\Big)x_i^2\Big\}\\
&\le (1+V_1(x))^{\theta-2}\Big\{V_1^2(x)\frac{\kappa}{\theta}+V_1(x)\Big(\frac{2\kappa}{\theta}+\hat{\sigma}^2+\max_{1\le i,j\le n}\varepsilon_{ij}\Big)\\
&\qquad+\max_{1\le i,j\le n}\varepsilon_{ij}+\frac{\kappa}{\theta}-\big(\gamma-0.5\theta\hat{\sigma}^2\big)V_1^4(x)\sum_{i=1}^n x_i^2\Big\}\\
&\le (1+V_1(x))^{\theta-2}\Big\{V_1(x)\Big(\frac{2\kappa}{\theta}+\hat{\sigma}^2+\max_{1\le i,j\le n}\varepsilon_{ij}\Big)\\
&\qquad+\max_{1\le i,j\le n}\varepsilon_{ij}+\frac{\kappa}{\theta}-\frac{V_1^2(x)}{n}\big(\gamma-0.5\theta\hat{\sigma}^2-\frac{n\kappa}{\theta}\big)\Big\}.
\end{aligned}
$$

It follows from (2.29) that

$$
M:=\sup_{t\ge 0}L\left(x(t),\varphi(t)\right)<\infty.
$$

Integrating and taking expectations on both sides of (2.30), we have

$$
E\big[V_3(x(t))\big]-V_3(x(0))\le M\int_0^t\theta e^{\kappa s}ds,
$$

i.e.,

$$
E\big[(1+V_1(x(t)))^\theta\big]\le e^{-\kappa t}\big(1+V_1(0)\big)^\theta+\frac{M\theta}{\kappa}.
$$

Note that for $x=(x_1,\ldots,x_n)'\in\mathbb{R}_+^n$, $\big(\sum_{i=1}^n x_i\big)^\theta\le n^\theta|x|^\theta$. Now for any given $\varepsilon\in(0,1)$, let $H>0$ be such that $\dfrac{H^\theta n^\theta M\theta}{\kappa}\le\varepsilon$. By Tchebychev's inequality, we obtain

$$
\begin{aligned}
P\big(|x(t)|<H\big)&=P\big(V_1^\theta(x(t))>\frac{1}{H^\theta n^\theta}\big)\le H^\theta n^\theta E\big[V_1(x(t))\big]^\theta\\
&\le H^\theta n^\theta E\big[(1+V_1(x(t)))^\theta\big]\\
&\le H^\theta n^\theta\Big[e^{-\kappa t}\big(1+V_1(0)\big)^\theta+\frac{M\theta}{\kappa}\Big].
\end{aligned}
$$

This implies that

$$\limsup_{t\to\infty} P\big(|x(t)| < H\big) \le \frac{H^\theta n^\theta M\theta}{\kappa} \le \varepsilon.$$

That is, $\liminf_{t\to\infty} P\big(|x(t)| \ge H\big) \ge 1-\varepsilon$. Hence the first inequality in (2.27) has been established. The second inequality can be obtain by using the boundedness of moments in Theorem 2.3 (b) and Tchebychev's inequality. The proof is thus completed. □

Remark 2.7. Assume that $u = 0$ and $\sigma_i(k) = \sigma_i$ for all $k = 1, \ldots, m$ and $i = 1, \ldots, n$, i.e., the matrix of intensities is independent of the environment. The assumption (2.28) now becomes $\min_{i,k} b_i(k) > 0$. In this case, the above theorem reveals the important fact that the unobservable environment mental noise cannot make the population extinct if the intrinsic growth rates of species are positive. This also presents a characterization of the white noise represented by a Stratonovich integral in our population system.

Recall that the population is said to reach the extinction if $\lim_{t\to\infty} |x(t)| = 0$ a.s., i.e., $\lim_{t\to\infty} \sum_{i=1}^n x_i(t) = 0$ a.s. We now provide a sufficient condition for extinction and estimates of the average in time of the underlying population.

Theorem 2.8. *Assume that* (A) *and* (B) *are satisfied. Then the following statements hold.*

(a) *For each* $i = 1, \ldots, n$, $\limsup_{t\to\infty} \frac{\ln x_i(t)}{t} \le \mu_i := \sup_{t\ge 0}\Big(u_i(t) + \overline{r}_i(\varphi(t)) - 0.5\,(\overline{\sigma}_i(\varphi(t)))^2\Big)$ a.s. *Hence if* $\mu_i < 0$ *for all* $i = 1, \ldots, m$, *the population will decay exponentially and reach the extinction.*

(b) *Suppose that for* $i = 1, \ldots, n_0 \le n, j = 1, \ldots, n$, *there are constants* $\gamma_i > 0$, $\varepsilon_{ij} \ge 0$, $\varepsilon_{ii} > 0$, *such that*

$$u_i(t) + \sum_{k=1}^m \Big(r_i(k) - \sum_{j=1}^n a_{ij}(k)x_j(t)\Big)\varphi_k(t) \ge \gamma_i + \frac{1}{2}\big(\overline{\sigma}_i\,(\varphi(t))\,\big)^2 - \sum_{j=1}^n \varepsilon_{ij}x_j(t), t \ge 0.$$

In addition, $\mu_i < 0$ *for* $i > n_0$*. Then*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \sum_{i=1}^{n_0} x_i(s)ds \geq \frac{\gamma_{1,n_0}}{\hat{\varepsilon}_{1,n_0}} \text{ a.s.},$$

where $\gamma_{1,n_0} := \min_{1\leq i\leq n_0} \gamma_i$, $\hat{\varepsilon}_{1,n_0} := \max_{1\leq i,j\leq n_0} \varepsilon_{ij}$.

(c) *Under the hypotheses of part* (b)*, suppose for all* $1 \leq i \leq n_0$ *and* $t \geq 0$,

$$u_i(t) + \sum_{k=1}^{m}\Big(r_i(k) - \sum_{j=1}^{n} a_{ij}(k)x_j(t)\Big)\varphi_k(t) - 0.5{n_0}^{-1}\min_{1\leq i\leq n_0}\hat{\sigma}_i^2 + \sum_{j=1}^{n}\varepsilon_{ij}x_j(t) \leq \beta_{1,n_0}.$$

Then

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\sum_{i=1}^{n_0}x_i(s)ds \leq \frac{n_0\beta_{1,n_0}}{\min_{1\leq i\leq n_0}\varepsilon_{ii}}, \text{ a.s..}$$

Proof. (a) By Itô's formula, for each $i = 1, \ldots, n$, we derive from (2.13) that

$$d[\ln x_i(t)] = \Big[\sum_{k=1}^{m}\Big(r_i(k) - \sum_{j=1}^{n} a_{ij}(k)x_j(t)\Big)\varphi_k(t) + u_i(t) - 0.5\left(\overline{\sigma}_i(\varphi(t))\right)^2\Big]dt$$
$$+\overline{\sigma}_i(\varphi(t))dw_i(t) \text{ a.s.}$$

Hence

$$\ln x_i(t) = \ln x_i(0) + \int_0^t\Big[\sum_{k=1}^{m}\Big(r_i(k) - \sum_{j=1}^{n}a_{ij}(k)x_j(s)\Big)\varphi_k(s) + u_i(s) - 0.5\left(\overline{\sigma}_i(\varphi(s))\right)^2\Big]ds$$
$$+\int_0^t\overline{\sigma}_i(\varphi(s))dw_i(s).$$

That is,

$$\begin{aligned}\ln x_i(t) &\leq \ln x_i(0) + \int_0^t\Big(\overline{r}_i(\varphi(s)) + u_i(s) - 0.5\left(\overline{\sigma}_i(\varphi(s))\right)^2\Big)ds + \int_0^t\overline{\sigma}_i(\varphi(s))dw_i(s)\\ &\leq \ln x_i(0) + t\mu_i + \int_0^t\overline{\sigma}_i(\varphi(s))dw_i(s).\end{aligned} \tag{2.34}$$

Dividing both sides by t and then letting $t \to \infty$ we obtain $\limsup_{t\to\infty}\frac{\ln x_i(t)}{t} \leq \mu_i$ a.s. The conclusion readily follows.

(b) By Itô's formula, we derive from (2.13) that

$$d\big[\ln\sum_{i=1}^{n_0}x_i(t)\big]\geq\left[\gamma_{1,n_0}-\sum_{j=1}^{n}\max_{1\leq i\leq n_0}\varepsilon_{ij}x_j(t)\right]dt+\frac{\sum\limits_{i=1}^{n_0}\overline{\sigma}_i(\varphi(t))x_i(t)dw_i(t)}{\sum\limits_{i=1}^{n_0}x_i(t)}. \tag{2.35}$$

Hence

$$\begin{aligned}&\ln\big[\sum_{i=1}^{n_0}x_i(t)\big]-\ln\big[\sum_{i=1}^{n_0}x_i(0)\big]\\&\geq t\gamma_{1,n_0}-\int\limits_0^t\sum_{j=1}^{n}\max_{1\leq i\leq n_0}\varepsilon_{ij}x_j(s)ds+\int_0^t\frac{\sum\limits_{i=1}^{n_0}\overline{\sigma}_i(\varphi(s))x_i(s)dw_i(s)}{\sum\limits_{i=1}^{n_0}x_i(s)}.\end{aligned}$$

That is,

$$\int\limits_0^t\sum_{j=1}^{n}\max_{1\leq i\leq n_0}\varepsilon_{ij}x_j(s)ds\geq t\gamma_{1,n_0}+\ln\big[\sum_{i=1}^{n_0}x_i(0)\big]-\ln\big[\sum_{i=1}^{n_0}x_i(t)\big]+\int_0^t\frac{\sum\limits_{i=1}^{n_0}\overline{\sigma}_i(\varphi(s))x_i(s)dw_i(s)}{\sum\limits_{i=1}^{n_0}x_i(s)}. \tag{2.36}$$

It follows from Theorem 2.4 that $\limsup\limits_{t\to\infty}\frac{1}{t}\ln\big[\sum\limits_{i=1}^{n_0}x_i(t)\big]\leq 0$. By the strong law of large numbers for local martingales [36],

$$\lim_{t\to\infty}\int_0^t\frac{\sum\limits_{i=1}^{n_0}\overline{\sigma}_i(\varphi(s))x_i(s)dw_i(s)}{\sum\limits_{i=1}^{n_0}x_i(s)}=0\text{ a.s.}$$

We can therefore divide both sides of (2.36) by t and then let $t\to\infty$ to obtain

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\sum_{j=1}^{n}\max_{1\leq i\leq n_0}\varepsilon_{ij}x_j(s)ds\geq\gamma_{1,n_0}. \tag{2.37}$$

Since $\mu_i<0$ for $i=n_0+1,\ldots,n$, $\lim\limits_{t\to\infty}\sum\limits_{i=n_0+1}^{n}x_i(t)=0$. Thus (2.37) yields the conclusion.

(c) We proceed as in (b). In view of (2.35), we obtain

$$d\big[\ln\sum_{i=1}^{n_0}x_i(t)\big]\leq\beta_{1,n_0}-\sum_{i=1}^{n_0}n_0^{-1}\min_{1\leq i\leq n_0}\varepsilon_{ii}x_i(t)+\frac{\sum\limits_{i=1}^{n_0}\overline{\sigma}_i(\varphi(t))x_i(t)dw_i(t)}{\sum\limits_{i=1}^{n_0}x_i(t)}.$$

Hence

$$\ln\big[\sum_{i=1}^{n_0} x_i(t)\big] - \ln\big[\sum_{i=1}^{n_0} x_i(0)\big]$$
$$\leq t\beta_{1,n_0} - \int_0^t \sum_{i=1}^{n_0} n_0^{-1} \min_{1\leq i\leq n_0} \varepsilon_{ii} x_i(s)ds + \int_0^t \frac{\sum_{i=1}^{n_0} \overline{\sigma}_i(\varphi(s))x_i(s)dw_i(s)}{\sum_{i=1}^{n_0} x_i(s)},$$

i.e.,

$$\int_0^t \sum_{i=1}^{n_0} n_0^{-1} \min_{1\leq i\leq n_0} \varepsilon_{ii} x_i(s)ds$$
$$\leq t\beta_{1,n_0} + \ln\big[\sum_{i=1}^{n_0} x_i(0)\big] - \ln\big[\sum_{i=1}^{n_0} x_i(t)\big] + \int_0^t \frac{\sum_{i=1}^{n_0} \overline{\sigma}_i(\varphi(s))x_i(s)dw_i(s)}{\sum_{i=1}^{n_0} x_i(s)}. \tag{2.38}$$

Using the same argument as in (b), to obtain (c), all we need is to show that

$$\lim_{t\to\infty} \frac{1}{t} \ln\big[\sum_{i=1}^{n_0} x_i(t)\big] = 0 \quad \text{a.s.}$$

This follows from Theorem 2.4 part (b) and the fact that for any $\omega \in \Omega$,

$$\lim_{t\to\infty} \frac{1}{t} \ln\big[\sum_{i=1}^{n_0} x_i(t,\omega)\big] < 0 \text{ implies } \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{i=1}^{n_0} x_i(s,\omega)ds = 0.$$

This completes the proof of the theorem. □

Remark 2.9. By virtue of Theorem 2.6, if condition (2.28) holds, the population system (2.13) is stochastically permanent. Now we can see from Theorem 2.8 that in such case, (2.13) will not reach extinction almost surely, i.e., $P\{|x(t)| \to 0, t \to \infty\} = 0$. In addition, we also have $P\{|x(t)| \to \infty, t \to \infty\} = 0$.

2.5 Feedback Controls

Our goal here is to design suitable and simple feedback controls so that the resulting population model (2.13) has the desired asymptotic properties such as permanence, extinction,

etc.

We first suppose that the species are certain insects and we wish to get rid of this population. Thus we design a feedback control u so that the species become extinct. Such controls can always be designed by the following theorem, which can be seen as a consequence of Theorem 2.8.

Theorem 2.10. *Assume that* (A) *is satisfied. If the feedback control u satisfies* (B) *and*

$$u_i(t) \leq -\kappa_i - \overline{r}_i(\varphi(t)) + 0.5\big(\overline{\sigma}_i(\varphi(t))\big)^2, \quad i = 1, \ldots, n, \tag{2.39}$$

for some constants $\kappa_i > 0$, then the controlled population system (2.13) *decays exponentially and reach the extinction. In particular, we can take*

$$u_i(t) = -\kappa_i - \overline{r}_i(\varphi(t)) + 0.5\big(\overline{\sigma}_i(\varphi(t))\big)^2, \quad i = 1, \ldots, n. \tag{2.40}$$

If we use the control (2.40), by virtue of Theorem 2.8 (a), $\limsup_{t\to\infty} \dfrac{\ln x_i(t)}{t} \leq -\kappa_i$. Moreover, it also follows from (2.34) that $E\Big[\ln \dfrac{x_i(t)}{x_i(0)}\Big] \leq -\kappa_i t$ for all $t \geq 0$. Hence it is worth to mention that constants κ_i can be chosen to yield a desired rate of extinction for $x(t)$.

Let us now consider the design of feedback controls to make the controlled system be stochastically permanent. Such controls can always be designed by the following theorem, which can be seen as a consequence of Theorem 2.6.

Theorem 2.11. *Assume that* (A) *is satisfied. If the feedback control u satisfies* (B) *and*

$$u_i(t) \geq \kappa_i + \frac{1}{2}\big(\overline{\sigma}_i\left(\varphi(t)\right)\big)^2 - \sum_{k=1}^{m}\Big(r_i(k) - \sum_{j=1}^{n} a_{ij}(k)x_j(t)\Big)\varphi_k(t) - \sum_{j=1}^{n}\varepsilon_{ij}x_j(t), i = 1, \ldots, n,$$

for some constants $\kappa_i > 0$, $\varepsilon_{ij} > 0$, $\varepsilon_{ii} > 0$, then the controlled population system (2.13) *is*

stochastically permanent. In particular, we can take

$$u_i(t) = \kappa_i + \frac{1}{2}\big(\overline{\sigma}_i\left(\varphi(t)\right)\big)^2 - \sum_{k=1}^{m}\Big(r_i(k) - \sum_{j=1}^{n} a_{ij}(k)x_j(t)\Big)\varphi_k(t) - \sum_{j=1}^{n}\varepsilon_{ij}x_j(t), i = 1, \ldots, n, \tag{2.41}$$

or

$$u_i(t) = \kappa_i + \frac{1}{2}\big(\overline{\sigma}_i\left(\varphi(t)\right)\big)^2 - \overline{r}\left(\varphi(t)\right), \quad i = 1, \ldots, n. \tag{2.42}$$

By virtue of Theorem 2.8, if we use the control (2.41) (resp., (2.42)), then constants κ_i and ε_{ij} (resp., κ_i) can be chosen depending on our desired asymptotic behavior of $\frac{1}{t}\int_0^t \sum_{i=1}^n x_i(s)ds$. In addition, we can maintain the persistence in the mean (see [35]) of n_0 species while making other species extinct. Note that the sub-ecosystem of species $1, \ldots, n_0$ is called persistent in the mean if there exist positive constants M_1 and M_2 such that

$$M_1 \leq \liminf_{t\to\infty} \frac{1}{t}\int_0^t \sum_{i=1}^{n_0} x_i(s)ds \leq \limsup_{t\to\infty} \frac{1}{t}\int_0^t \sum_{i=1}^{n_0} x_i(s)ds \leq M_2 \quad \text{a.s.}$$

The corresponding feedback control can be constructed by using Theorem 2.8.

Theorem 2.12. *Assume that* (A) *is satisfied. To maintain the persistence in the mean of the first n_0 species while making other species extinct, we can use the following feedback controls*

$$\begin{aligned} u_i(t) &= \kappa_i + \frac{1}{2}\big(\overline{\sigma}_i\left(\varphi(t)\right)\big)^2 - \sum_{k=1}^{m}\Big(r_i(k) - \sum_{j=1}^{n} a_{ij}(k)x_j(t)\Big)\varphi_k(t) - \sum_{j=1}^{n}\varepsilon_{ij}x_j(t), i = 1, \ldots, n_0, \\ u_i(t) &= -\kappa_i - \overline{r}_i(\varphi(t)) + 0.5\big(\overline{\sigma}_i(\varphi(t))\big)^2, \quad i = n_0 + 1, \ldots, n, \end{aligned} \tag{2.43}$$

for some constants $\kappa_i > 0$, $\varepsilon_{ij} \geq 0$, $\varepsilon_{ii} > 0$, or

$$\begin{aligned} u_i(t) &= \kappa_i - \overline{r}_i(\varphi(t)) + 0.5\big(\overline{\sigma}_i(\varphi(t))\big)^2, \quad i = 1, \ldots, n_0, \\ u_i(t) &= -\kappa_i - \overline{r}_i(\varphi(t)) + 0.5\big(\overline{\sigma}_i(\varphi(t))\big)^2, \quad i = n_0 + 1, \ldots, n. \end{aligned}$$

Again, by virtue of Theorem 2.8, constants κ_i and ε_{ij} are chosen depending on our desired

asymptotic behavior of $\frac{1}{t}\int_0^t \sum_{i=1}^{n_0} x_i(s)ds$ and desired rate of extinction of species $n_0+1,..,n$. When the population of species $n_0+1,\ldots,n$ is smaller than a very small number, it can be thought that they are extinct. Then the population system of the first n_0 species is stochastically permanent by virtue of Theorem 2.6. If the noise intensities $\Xi(k)$ is independent of k, our feedback control can remove the effect of the random environment from the system, and gives desired asymptotic properties. The following result follows from our preceding analysis and [19]; see Theorem 3.1 and Theorem 4.2.

Theorem 2.13. *Assume that* (A) *is satisfied,* $\Xi(k)=\Xi$ *for all* k*, and* $x^*=(x_1^*,\ldots,x_n^*)'\in\mathbb{R}^n$ *is given. Then the following statements hold.*

(a) *Suppose* $x_i^*>0$ *for all* $i=1,\ldots,n$*. Let* $u(t)$ *be in Eq.* (2.41) *with*

$$\kappa_i-\sum_{i\neq j=1}^{n}\frac{\varepsilon_{ij}}{\varepsilon_{jj}}\kappa_j>0,\quad \sum_{j=1}^{n}\varepsilon_{ij}x_j^*=\kappa_i,\quad i=1,\ldots,n.$$

Then

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x_i(s)ds=x_i^*,\quad i=1,\ldots,n.$$

(b) *Suppose* $x_i^*>0$ *for* $i=1,\ldots,n_0<n$ *and* $x_i^*=0$ *for* $i=n_0+1,\ldots,n$*. Let* $u(t)$ *be in Eq.* (2.43) *with*

$$\kappa_i-\sum_{i\neq j=1}^{n_0}\frac{\varepsilon_{ij}}{\varepsilon_{jj}}\kappa_j>0,\quad \sum_{j=1}^{n_0}\varepsilon_{ij}x_j^*=\kappa_i,\quad i=1,\ldots,n_0.$$

Then

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x_i(s)ds=x_i^*,\quad i=1,\ldots,n_0,\quad \lim_{t\to\infty}x_i(t)=0,\quad i=n_0+1,\ldots,n.$$

Remark 2.14. Note that mathematically, it is possible to control a stochastic differential equation by adding a white noise [45]. It is also proved that for a population system in which

white noise is represented by a Itô's integral, a white nose with sufficient large intensity may make the underlying population extinct [32,34,35,42]. For this approach, we refer the reader to [45, 57]. Moreover, one can also consider another feedback control in the diffusion part, which can maintain the validity of Theorem 2.13 when the noise intensities $\Xi(k)$ depends on k. In this work, we only discuss a feedback control in drift part since it is more practical.

2.6 Numerical Examples

This section is devoted to a couple of examples. They are for demonstration purposes. We begin with the Wonham filter equations. After $\varphi(t)$ being found, we use it to obtain the feedback control $u(t)$. Although the filter provides precise results in the posterior probabilities, the system often has to be solved numerically because it is nonlinear and because observations are frequently collected in discrete moments.

To construct approximation algorithms, one may wish to discretize the stochastic differential equations (2.6) directly. However, such a procedure is numerically unstable due to the white noise perturbations [24, Section 13.3] and [43]. It may produce a non-probability vector (e.g., some components might be less than 0 or the sum of the components might be not 1). To overcome this difficulty, the authors in [43] suggested a method based on Clark transformations, whereas a logarithm transformation was used in [67] to build approximations. Note that we are mainly interested in sample path approximations of the filters. Using the approach suggested in [67] (see Section 8.4), we first transform the stochastic differential equations and then design a numerical procedure for the transformed system.

Let $v_j(t) := \ln \varphi_j(t)$ for $t \geq 0$ and $j = 1, \ldots, m$. It follows that $\varphi_j(t) = e^{v_j(t)}$. A straight-

forward application of Itô's formula to (2.6) leads to the following. For each $j = 1, \ldots, m$,

$$\begin{cases} dv_j(t) = \Big[q_{jj} + \sum_{k \neq j} q_{kj} \dfrac{\varphi_k(t)}{\varphi_j(t)} - \beta^{-2}(t)\big(f(j) - \overline{f}(\varphi(t))\big)\overline{f}(\varphi(t)) \\ \qquad\qquad -\dfrac{1}{2}\beta^{-2}(t)\big(f(j) - \overline{f}(\varphi(t))\big)^2\Big]dt + \beta^{-2}(t)\big(f(j) - \overline{f}(\varphi(t))\big)dy(t), \\ v_j(0) \ = \ln \varphi_j^0(0). \end{cases} \tag{2.44}$$

Then we use Euler-Maruyama type approximations of (2.5) (see [67, p. 186]), (2.44), and (2.10) to mimic the dynamics of our population system. Note that in the above, by using the transformation $\varphi_j(t) = e^{v_j(t)}$, we have assumed implicitly that $\{\varphi_j\}$ is bounded below from zero. The relaxation of this condition can be found in [67]. Now we will use the above results for numerical examples. To demonstrate the validity of our model, we denote by $\hat{x}(t)$ the actual population process defined by (2.4) and compare $x(t)$ with $\hat{x}(t)$. By path mean square error, we mean $\frac{1}{N}\sum_{j=1}^{N} |x^j - \hat{x}^j|^2$, where N is the number of iterations, x^j and $\hat{x}^j$ are j th iterations of sample path approximations of $x(t)$ and $\hat{x}(t)$, respectively. Our numerical experiments show that our method is effective (see Figures. 1-5).

Example 2.15. We first consider a single species ecosystem in random environment (also called a logistic model with regime switching). Let $x(t)$ denote the population size of a certain species at time t. Suppose that the Markov chain $\alpha(\cdot) \in \{1, 2\}$ that models random environment. The generator of the continuous-time Markov chain is given by $Q = \begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix}$, and $b(1) = 3, a(1) = 4, \sigma(1) = 2, b(2) = -2, a(2) = 1, \sigma(2) = 1$. In this case, the corresponding population system (2.3) is stochastically permanent (see [34, Theorem 6.1]). Now we suppose that the Markov chain can only be observed through $dy(t) = f(\alpha(t))dt + 2dB(t)$, where $f(1) = -1$ and $f(2) = 1$. Then the population size $x(t)$ and Wonham's filter $\varphi(t)$ satisfy the

following equations

$$\begin{cases} dx(t) &= x(t)\big[5-4x(t)+u(t)\big]\varphi_1(t)dt + x(t)\big[-1.5-x(t)+u(t)\big]\varphi_2(t)dt \\ &\quad +x(t)\big[2\varphi_1(t)+\varphi_2(t)\big]dw(t), \\ d\varphi_1(t) &= \Big[-2\varphi_1(t)+3\varphi_2(t)-\big(1-\overline{f}(\varphi(t))\big)\overline{f}(\varphi(t))\varphi_1(t)\Big]dt \\ &\quad +\big(1-\overline{f}(\varphi(t))\big)\varphi_1(t)dy(t), \\ d\varphi_2(t) &= \Big[2\varphi_1(t)-3\varphi_2(t)-\big(2-\overline{f}(\varphi(t))\big)\overline{f}(\varphi(t))\varphi_2(t)\Big]dt \\ &\quad +\big(2-\overline{f}(\varphi(t))\big)\varphi_2(t)dy(t), \end{cases} \tag{2.45}$$

where $\overline{f}(\varphi(t)) = -\varphi_1(t)+\varphi_2(t)$ and $u(t)$ is a feedback control.

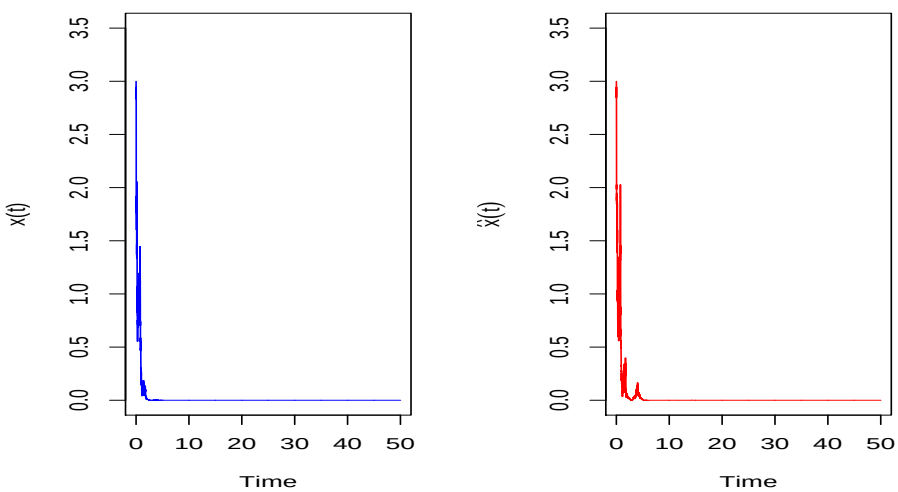

Figure 1: Sample paths of $x(t)$ and $\hat{x}(t)$, with $u(t)$ in (2.46)

Suppose that the species is a certain insect of the ecosystem for which we would like to get rid of. By Theorem 2.10, we add a feedback control

$$u(t) = -\kappa - \Big[5\varphi_1(t)+2.5\varphi_2(t)\Big] + 0.5\Big[2\varphi_1(t)+\varphi_2(t)\Big]^2, \tag{2.46}$$

where $\kappa > 0$ is chosen depending on the rate of extinction. Taking $\kappa = 1$, we perform a

computer simulation of 10,000 iterations of a sample path of $x(t)$ with step size $\triangle = 0.005$ and initial conditions $x(0) = 3, \varphi_1(0) = 0.1, \varphi_2(0) = 0.9$, $u(t)$ in (2.46). The corresponding sample paths of $\hat{x}(t)$ are shown in Figure 1. For these sample paths of $x(t)$ and $\hat{x}(t)$, the path mean square error is only 6×10^{-4}. It shows that the feedback control work very well.

Example 2.16. Consider a Lotka-Volterra model of two species competitive ecosystem with a hidden Markov chain with

$$b(1) = \begin{pmatrix} -0.5 \\ -3 \end{pmatrix}, b(2) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, a(1) = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, a(2) = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}, \sigma(1) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \sigma(2) = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

and $\alpha(\cdot)$ is a continuous-time Markov chain generated by $Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. In this case, a

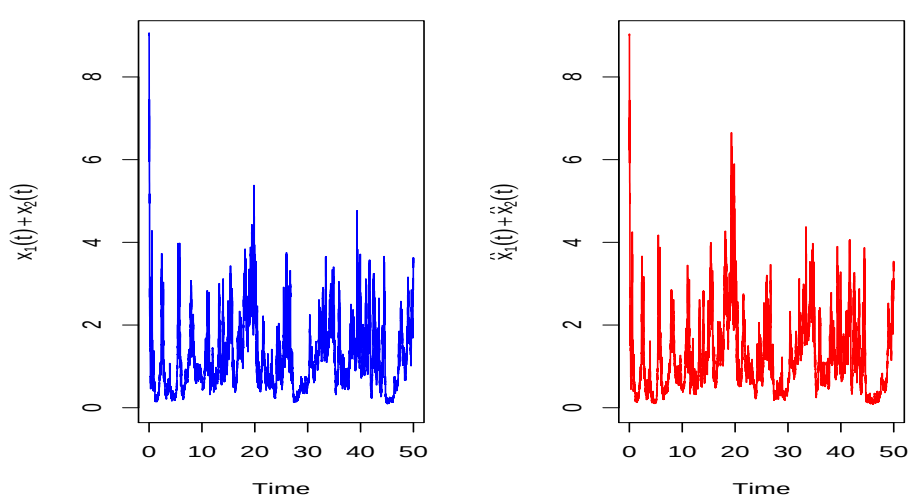

Figure 2: Sample paths of $x_1(t) + x_2(t)$ and $\hat{x}_1(t) + \hat{x}_2(t)$, with $u(t)$ in (2.48)

similar argument as in Theorem 2.8 (see also [32]) tell us that the corresponding population system (2.3) reaches the extinction. Now we suppose that the Markov chain can only be

observed through $dy(t) = \big(\alpha(t)\big)^2 dt + (t^2+1)dB(t)$. We would like to find a feedback control such that the controlled population size $(x_1(t), x_2(t))$ is stochastically permanent. The population size $(x_1(t), x_2(t))$ satisfy

$$\begin{cases} dx_1(t) = x_1(t)\big[1.5 - 2x_1(t) - 3x_2(t) + u_1(t)\big]\varphi_1(t)dt \\ \qquad\qquad +x_1(t)\big[-1 - x_1(t) - x_2(t) + u_1(t)\big]\varphi_2(t)dt + 2x_1(t)\varphi_1(t)dw_1(t), \\ dx_2(t) = x_2(t)\big[-1 - x_1(t) - 2x_2(t) + u_2(t)\big]\varphi_1(t)dt \\ \qquad\qquad +x_2(t)\big[1 - 3x_2(t) + u_2(t)\big]\varphi_2(t)dt + 2x_2(t)dw_2(t), \end{cases} \tag{2.47}$$

and $(\varphi_1(t), \varphi_2(t))$ satisfies Wonham's equation. By Theorem 2.11, we can use the following

$$\begin{aligned} u_1(t) &= \kappa_1 - \big[1.5\varphi_1(t) - \varphi_2(t)\big] + 2\varphi_1^2(t), \\ u_2(t) &= \kappa_2 + 2 - \big[-\varphi_1(t) + \varphi_2(t)\big], \end{aligned} \tag{2.48}$$

where κ_1, κ_2 are positive constants. We can choose κ_1, κ_2 to give a desired asymptotic behavior of $\frac{1}{t}\int_0^t \big(x_1(s) + x_2(s)\big)ds$. Taking $\kappa_1 = \kappa_2 = 2$, we perform a computer simulation of 10,000 iterations of sample paths of $x_i(t)$ and $\hat{x}(t)$ with step size $\triangle = 0.005$ and initial condition $x_1(0) = 4$, $x_2(0) = 5, \varphi_1(0) = 0.9, \varphi_2(0) = 0.1$, $(u_1(t), u_2(t))$ in (2.49). Since in this case we are interested in $x_1(t) + x_2(t)$ and $\hat{x}_1(t) + \hat{x}_2(t)$, then we plot sample paths of these processes in Figure 2. The path mean square errors are only 0.07866089 and 0.06132534, for species 1 and 2, respectively. The histograms of the 10,000 iterations of $x_1(t) + x_2(t)$ and $\hat{x}_1(t) + \hat{x}_2(t)$ is shown in Figure 3.

Repeating the simulation $N = 1,000$ times, the corresponding frequency distributions of $x_1(50) + x_2(50)$ and $\hat{x}_1(50) + \hat{x}_2(50)$ are displayed in Figure 4. Approximately, we have $E|x_1(50) + x_2(50) - \hat{x}_1(50) - \hat{x}_2(50)|^2 \simeq 0.1399255$. It can be seen that the solutions of our model are very close to the actual evolution of population process on both qualitative and

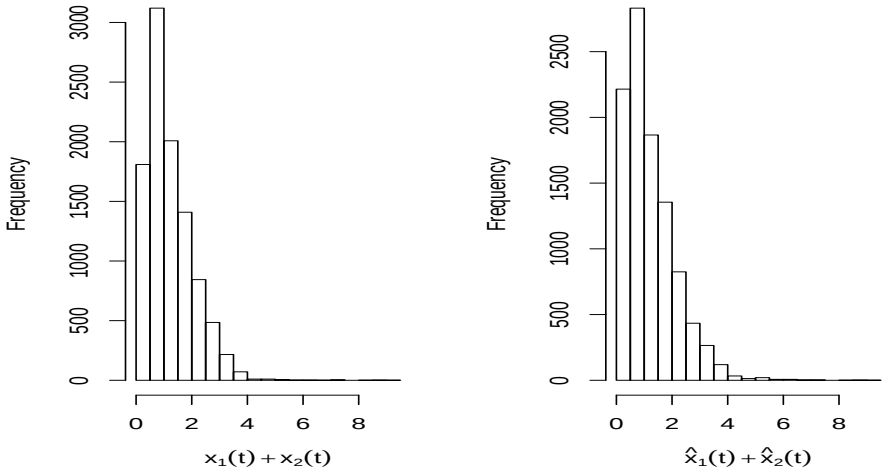

Figure 3: Histograms of paths of $x_1(t) + x_2(t)$ and $\hat{x}_1(t) + \hat{x}_2(t)$, with $u(t)$ in (2.48)

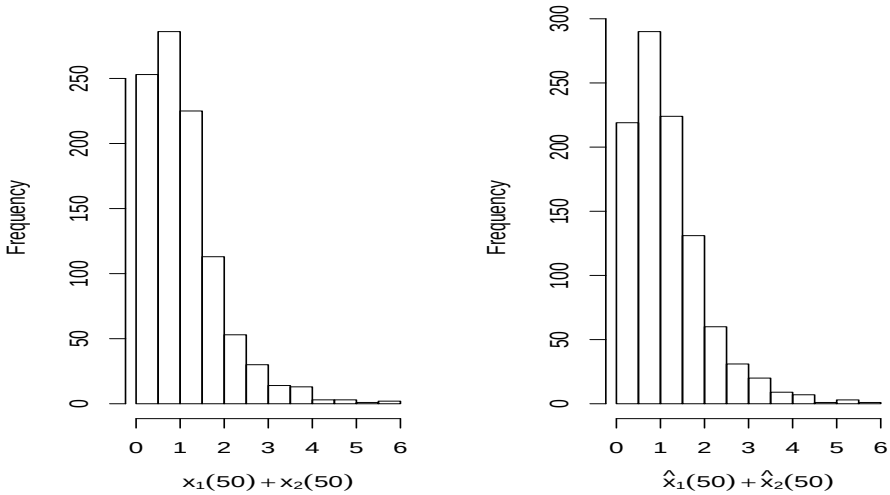

Figure 4: Histograms of $x_1(50) + x_2(50)$ and $\hat{x}_1(50) + \hat{x}_2(50)$, with $u(t)$ in (2.48)

quantitative aspects.

Suppose we wish to maintain the second species and make the first species extinct. By

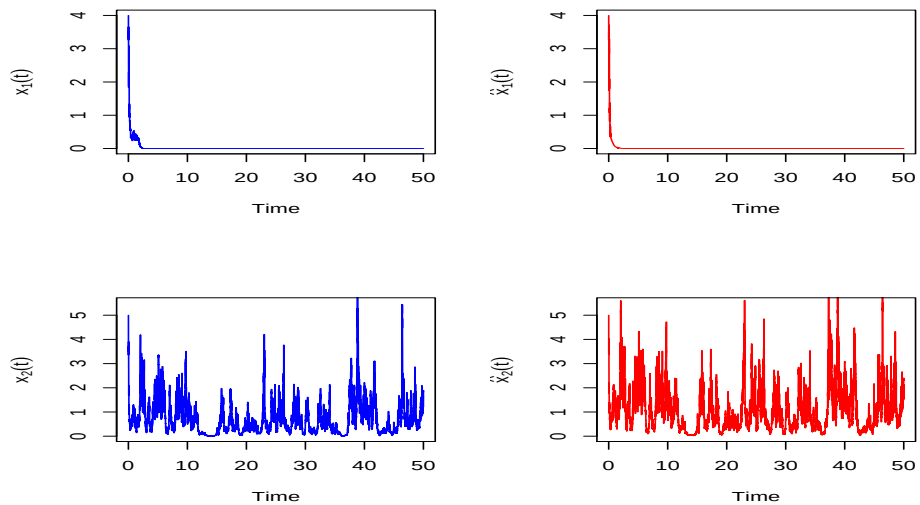

Figure 5: Sample paths of $x_i(t)$ and $\hat{x}_i(t)$, $i = 1, 2$, with $u(t)$ in (2.49)

Theorem 2.12, we can use the following

$$u_1(t) = 0, \quad u_2(t) = \kappa + 2 - \big[-\varphi_1(t) + \varphi_2(t) \big], \tag{2.49}$$

where κ is a positive constant. We can choose κ to give a desired asymptotic behavior of $\frac{1}{t}\int_0^t x_2(s)ds$. Taking $\kappa = 3$, to visualize the effect of our feedback control, we plot sample paths of the controlled population process $x_i(t)$ and $\hat{x}_i(t)$ in Figure 5. The path mean square errors are only 0.002089295 and 0.2592157, for species 1 and 2, respectively. Not only do the above observations and calculations support the theoretical results but also show the efficiency of our feedback controls.

2.7 Further Remarks

This chapter is devoted to the study of stochastic competitive Lotka-Volterra models in random environments with an unobservable Markov chain. Under the framework of the Wonham filtering, we first converted the underlying system to a fully observable system. Next

we established the regularity, positivity, stochastic boundedness, permanence and extinction. The design of feedback controls for stochastic permanence and extinction as well as other desired asymptotic properties has been obtained. These results pave a way for practical consideration for control problems of ecosystems under partial observation.

CHAPTER 3 OPTIMAL HARVESTING STRATEGIES FOR STOCHASTIC COMPETITIVE LOTKA-VOLTERRA ECOSYSTEMS

3.1 Introduction

This chapter develops optimal harvesting strategies for stochastic Lotka-Volterra models of ecosystems that are represented by stochastic differential equations with regime switching modeled by a continuous-time Markov chain. As noted by many researchers, one of the most important problems in modern natural resources management is the establishment of ecologically, environmentally, and economically reasonable wildlife management and harvesting policies; see [54] and references therein. It is noted that simple-minded myopic unconstrained harvesting strategies and over-harvesting could lead to detrimental effect causing local extinctions or depletion of numerous species; see the examples documented in [30]. Thus the study on the optimal harvesting strategies has significant impact on the environment, ecology, economy, and the society.

Building on the corresponding models without controls in [70,71], the problem we consider belongs to a class of singular stochastic control problems motivated by the establishment of reasonable wildlife management and harvesting policies. There has been resurgent interests in determining the optimal harvesting strategies in the presence of stochastic fluctuations recently. Radner and Shepp [51] considered certain optimal corporate strategies. Alvarez and Shepp [1] and Alvarez [3] studied optimal harvesting plans for the stochastic Verhulst-Pearl logistic model and a similar model in the presence of a state-dependent yield structure. Similar problems for another logistic population model were investigated in [39] by Lunggu

and Oksendal. The papers [40] and [4] were one of the first in the analysis of the harvesting problem for interacting populations. All of the aforementioned works dealt with species living in a static environment. Recently, Song, Stockbridge, and Zhu [53], and Song and Zhu [54] treated such class of singular control problems in random environments modeled by a Markov chain, where the first one deals with a single species and the second one deals with multiple species in the class of constrained harvesting options. Some results on numerical methods for the above singular control formulations can be found in Jin, Yin, and Zhu [20].

In virtually all ecosystems, many species interact with each other and compete for resources, food, habitat, or territory. Therefore, it is more practical and natural to consider multiple interacting species. One of such most important population models is competitive Lotka-Volterra ecosystems; see for example [6,32,58,70,71]. In particular, the hybrid stochastic Lotka-Volterra ecosystems capture both stochastic fluctuations in intrinsic growth rates as well as the abrupt changes in a random environment. However, to the best of our knowledge, there have not been published results for the optimal harvesting problems. Our objective is to fill in this gap. In fact, the known results on interacting population systems are of limited scope (see [4, 40]). The difficulty arises from the complexity in the model of our interest, in which the methods in [1–3] are no longer applicable. In this chapter, we establish properties and characterizations of the value functions and develop optimal harvesting policies in some special cases. It is worth to remark that the optimal harvesting problems under consideration are not simple generalizations of the corresponding models in a static environment (see Theorem 3.7) and also not a trivial combination of logistic population systems (see Theorem 3.7 as well as Example 3.9). Moreover, let us add that singular stochastic control has many applications in various areas, for them we refer the reader to [17, 50, 53] and many references

therein for such examples; see also [68] and [44] for comprehensive treatments of switching diffusion process and applications.

In contrast to the existing results, our new contributions in this chapter are as follows.

(i) In lieu of a single species, we treat multi-species. Our model is a Lotka-Volterra ecosystem with regime switching.

(ii) By constructing upper bounds for the value functions, not only do we prove the finiteness of the harvesting value but also derive further properties such as the continuity of the value function, and the impact of large noise on extinction. When $n = 1$, we characterize the value function as a viscosity solution of a coupled system of quasi-variational inequalities. In particular, Theorem 3.5 and Corollary 3.11 are nontrivial extensions of [53, Theorem 4.9] and [1, Proposition 1], respectively.

(iii) We construct explicit chattering harvesting strategies and the corresponding lower bounds for the value functions by using the idea of harvesting only one species at a time. We further show that this is a reasonable candidate for the best lower bound that one can expect. Moreover, in some cases, the lower bounds provide a good approximation of the value function. In particular, Theorem 3.7 and Corollary 3.10 are nontrivial extensions of [53, Theorem 2.4] and [3, Lemma 3], respectively.

The rest of this chapter is organized as follows. Section 3.2 begins with the problem formulation. Section 3.3 is devoted to properties and upper bounds for the value function. Section 3.4 considers chattering harvesting policies and we use them to establish a lower bound for the value function. Finally, the paper is concluded with some further remarks in Section 3.5. To facilitate the reading, all proofs are placed in Section 3.6.

3.2 Formulation

We work with a complete filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ with the filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions (i.e., it is right continuous, increasing, and $\mathcal{F}_0$ contains all the null sets). For $i = 1, 2, \ldots, n$, let $\xi_i(t)$ be the population size of the ith species in the ecosystem at time t, and denote by $\xi(t) = (\xi_1(t), \ldots, \xi_n(t))' \in \mathbb{R}^n$ (where z' denotes the transpose of z for $z \in \mathbb{R}^{l_1 \times l_2}$ with $l_1, l_2 \geq 1$). Consider a competitive ecosystem of n species given by

$$d\xi(t) = \operatorname{diag}(\xi(t)) \Big\{ \big[b(\alpha(t)) - A(\alpha(t))\xi(t) \big] dt + \Xi(\alpha(t)) dw(t) \Big\}, \tag{3.1}$$

and a constant initial condition $\xi(0) = x$. In the model, $w(\cdot) = (w_1(\cdot), \ldots, w_n(\cdot))'$ is an n-dimensional standard Brownian motion, and $b(\alpha) = (b_1(\alpha), \ldots, b_n(\alpha))'$, $A(\alpha) = (a_{ij}(\alpha))$, $\Xi(\alpha) = \operatorname{diag}(\sigma_1(\alpha), \ldots, \sigma_n(\alpha))$ with $\alpha \in \mathcal{M} = \{1, \ldots, m\}$ represent the different intrinsic growth rates, the community matrices, and noise intensities in different external environments, respectively; $\alpha(t)$ is a finite state Markov chain.

The population model (3.1) was proposed and studied in details in [70, 71]. Necessary and sufficient conditions for permanence and extinction were proved in [34]. In a recent work [58], we designed feedback controls for permanence and extinction when the Markov chain is unobservable. In this work, we consider that the ecosystem is subject to harvesting. Our formulation follows that of [54] closely. Denote

$$F(x, \alpha) = \operatorname{diag}(x) \big[b(\alpha) - A(\alpha)x \big], \quad G(x, \alpha) = \operatorname{diag}(x)\, \Xi(\alpha).$$

For later use, we introduce the generator of the process $(\xi(t), \alpha(t))$. For a function $h(\cdot, \cdot) : [0, \infty)^n \times \mathcal{M} \mapsto \mathbb{R}$ such that $h(\cdot, \alpha)$ is twice continuously differentiable function for each

$\alpha \in \mathcal{M}$, we define

$$\mathcal{L}h(x,\alpha) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}(x,\alpha)x_i\Big(b_i(\alpha) - \sum_{j=1}^{n} a_{ij}(\alpha)x_j\Big) + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} x_i^2 \sigma_i^2(\alpha) + \sum_{j\in\mathcal{M}} q_{\alpha j}\big[h(x,j) - h(x,\alpha)\big].$$

where $\nabla h(\cdot,\alpha)$ and $\nabla^2 h(\cdot,\alpha)$ denote the gradient and Hessian matrix of $h(\cdot,\alpha)$, respectively.

Let $Z(t) = (Z_1(t),\ldots,Z_n(t))' \in \mathbb{R}^n$, where $Z_i(t)$ denote the total number harvested (to be defined shortly) from the species i up to time t. Then $X(t) = (X_1(t),\ldots,X_n(t))' \in \mathbb{R}^n$, the population size of the harvested population, satisfies

$$X(t) = x + \int_0^t F(X(s),\alpha(s))ds + \int_0^t G(X(s),\alpha(s))dw(s) - Z(t), \tag{3.2}$$

with initial conditions

$$X(0-) = x \in \mathbb{R}_+^n, \quad \alpha(0) = \alpha \in \mathcal{M}. \tag{3.3}$$

Let $f_i(\cdot,\cdot) : [0,\infty)^n \times \mathcal{M} \mapsto (0,\infty)$ represent the instantaneous marginal yields accrued from exerting the harvesting strategy Z_i for the species i, also known as the price of species i. Let $\tau = \inf\{t \geq 0 : X_i(t) = 0, \text{ for all } i = 1,\ldots,n\}$ be the extinction time of the ecosystem. Let $r > 0$ be the discounting factor and $E_{x,\alpha}$ denote the expectation with respect to the probability law when the process $(X(t),\alpha(t))$ starts with initial condition (x,α). For an appropriate control process $Z(\cdot)$, the expected total discounted reward is defined by

$$\begin{aligned} J(x,\alpha,Z) :&= E_{x,\alpha}\int_0^\tau e^{-rs} f(X(s-),\alpha(s-))\cdot dZ(s) \\ &= E\int_0^\tau e^{-rs} f(X^x(s-),\alpha^\alpha(s-))\cdot dZ(s), \end{aligned} \tag{3.4}$$

Harvesting strategy. An n-dimensional admissible harvesting strategy is a stochastic pro-

cess $Z(t)$ satisfying the following conditions:

(a) $Z_i(t)$ is nonnegative for any $t \geq 0$ and nondecreasing with respect to t,

(b) $Z_i(t)$ is cadlag and adapted to $\mathcal{F}_t = \sigma\{w(s), \alpha(s), 0 \leq s \leq t\}$, and

(c) $J(x, \alpha, Z) < \infty$, for any $(x, \alpha) \in \mathbb{R}^n_+ \times \mathcal{M}$, where $J(\cdot)$ is the functional defined above.

(d) $X_i(t) \geq 0$, for any $t \leq \tau$, where $\tau = \inf\{t \geq 0 : X_i(t) = 0, \text{ for all } i = 1, \ldots, n\}$ is the extinction time of the population system.

Thus, (3.4) represents the total discounted reward from harvesting. Note that τ might be infinite. Let $\mathcal{A}_{x,\alpha}$ denote the collection of all admissible harvesting strategies with initial conditions given by (3.3). Then the optimal harvesting problem is to maximize the expected total discounted reward from harvesting and find an optimal harvesting strategy $Z^* \in \mathcal{A}_{x,\alpha}$ such that

$$V(x, \alpha) := J(x, \alpha, Z^*) = \sup_{Z \in \mathcal{A}_{x,\alpha}} J(x, \alpha, Z). \tag{3.5}$$

For each time t, note that $X(t-)$ is the state before harvesting starts at time t, while $X(t)$ is the state immediately after. Hence $X(0)$ may not be equal to $X(0-)$ due to an instantaneous harvest $Z(0)$ at time 0. Throughout the paper we use the convention that $Z(0-) = 0$. If Z consists of an immediate harvest at time t, then this jump size is denoted by $\Delta Z(t) := Z(t) - Z(t-)$, and $Z^c(t) := Z(t) - \sum_{0 \leq s \leq t} \Delta Z(s)$ denotes the continuous part of Z. Also note that $\Delta X(t) := X(t) - X(t-) = -\Delta Z(t)$ for any $t \geq 0$. Denote the solution to (3.2) with initial condition specified by (3.3) by $(X^x(t), \alpha^\alpha(t))$ if necessary. For $x, y \in \mathbb{R}^n$, with $x = (x_1, \ldots, x_n)'$ and $y = (y_1, \ldots, y_n)'$, we write $x \leq y$ if $x_j \leq y_j$ for each $j = 1, \ldots, n$. We also define $x \cdot y := \sum_{j=1}^{n} x_j y_j$.

For convenience, let us combine frequently referred hypotheses in the following.

(A) (i) $a_{ij}(\alpha) \geq 0$ and $a_{ii}(\alpha) > 0$, for any $i, j = 1, \ldots, n, \alpha \in \mathcal{M}$,

(ii) For each i and α, $f_i(\cdot, \alpha)$ is continuous. Moreover, $f_i(x, \alpha) \geq f_i(y, \alpha)$ for each $\alpha \in \mathcal{M}$ whenever $x \leq y$.

Condition (A)(i) means that the ecosystem under consideration is of competitive type. Condition (A)(ii) is motivated by the fact that "the law of decreasing demand guarantees that the profitability of a harvested individual increases as its density decreases" (see [2]). It is argued that the smaller the population becomes, the higher the harvesting costs are. However, as long as the harvesting costs increase at a smaller rate than that of the revenues, then (A)(ii) still holds.

3.3 Properties and Upper Bounds of Value Functions

This section is devoted to several properties of the value function. We first establish a verification theorem whose proof utilizes the generalized Itô formula, the monotonicity of f, and the regularity of $(X^x(t), \alpha^\alpha(t))$. Then we obtain some upper bounds for the value function. Further properties when $n = 1$ are also provided. We present a number of results below. The proofs are relegated to Section 3.6.

Theorem 3.1. *Assume* (A). *Suppose that there exists a function* $W : \mathbb{R}^n_+ \times \mathcal{M} \mapsto \mathbb{R}_+$ *such that* $W(\cdot, \alpha)$ *is twice continuously differentiable for each* $\alpha \in \mathcal{M}$ *and that* $W(\cdot)$ *solves the following coupled system of quasi-variational inequalities*

$$\sup_{(x,\alpha)} \left\{ (\mathcal{L} - r)W(x, \alpha), \max_i \left[f_i(x, \alpha) - \frac{\partial W}{\partial x_i}(x, \alpha) \right] \right\} \leq 0, \tag{3.6}$$

where $(\mathcal{L} - r)W(x, \alpha) = \mathcal{L}W(x, \alpha) - rW(x, \alpha)$. *The following assertions hold.*

(a) *We have*

$$V(x,\alpha) \leq W(x,\alpha) \quad \textit{for all } (x,\alpha) \in \mathbb{R}^n_+ \times \mathcal{M}.$$

(b) *Define the non-intervention region* $\mathcal{C} = \cap_{i=1}^n \mathcal{C}_i$, *with*

$$\mathcal{C}_i := \{(x,\alpha) \in \mathbb{R}^n_+ \times \mathcal{M} : f_i(x,\alpha) - \frac{\partial W}{\partial x_i}(x,\alpha) < 0\}.$$

Suppose that $(\mathcal{L} - r)W(x,\alpha) = 0$, *for all* $(x,\alpha) \in \mathcal{C}$, *and that there exists a harvesting strategy* $\widetilde{Z} \in \mathcal{A}_{x,\alpha}$ *and a corresponding process* $\widetilde{X}$ *such that the following statements hold:*

(i) $(\widetilde{X}(t), \alpha(t)) \in \mathcal{C}$ *for Lebesgue almost all* $0 \leq t \leq \tau$.

(ii) $\int_0^t \Big[\nabla W(\widetilde{X}(s), \alpha(s)) - f(\widetilde{X}(s)), \alpha(s)\Big] \cdot d\widetilde{Z}^c(s) = 0$ *for any* $t \leq \tau$.

(iii) $\lim_{N\to\infty} E_{x,\alpha}\Big[e^{-rT_N} W(\widetilde{X}(T_N), \alpha(T_N))\Big] = 0$, *where for each* $N = 1, 2, \ldots,$

$$\beta_N := \inf\{t \geq 0 : |X(t)| \geq N\}, \quad T_N := N \wedge \beta_N \wedge \tau. \tag{3.7}$$

(iv) *If* $\widetilde{X}(s) \neq \widetilde{X}(s-)$, *then*

$$W(\widetilde{X}(s), \alpha(s)) - W(\widetilde{X}(s-), \alpha(s-)) = -f(\widetilde{X}(s-)), \alpha(s-)) \cdot \Delta\widetilde{Z}(s).$$

Then $V(x,\alpha) = W(x,\alpha)$ *for all* $(x,\alpha) \in \mathbb{R}^n_+ \times \mathcal{M}$, *and* $\widetilde{Z}$ *is an optimal harvesting strategy.*

By virtue of Theorem 3.1 (a), any twice continuously differentiable solution to (3.6) is an upper bound for the value function. Furthermore, the additional conditions in Theorem 3.1(b) help us to find an optimal harvesting strategy. In practice, it is, however, usually very difficult to find an explicit solution. We now give some explicit upper bounds for the value

function.

To proceed, for each $i = 1, \ldots, n$, $\mathbf{e}_i$ denotes the standard unit vector in the ith direction. It is worth to mention that in the following result, we will show that the value function is continuous at 0 without using the approach in [53]; see Lemma 4.3 and Proposition 4.7.

Theorem 3.2. *Assume that* (A) *holds and for each* $i = 1, \ldots, n$ *and each* $\alpha \in \mathcal{M}$, $f_i(\cdot, \alpha)$ *is continuously differentiable. Then the following assertions hold.*

(a) *There exist positive integers* N *and* K *such that*

$$V(x, \alpha) \leq \sum_{i=1}^{n} \int_0^{x_i} f_i(\varphi\, \mathbf{e}_i, \alpha) d\varphi + KN \sum_{i=1}^{n} x_i^{1/N} \quad \textit{for all} \quad (x, \alpha) \in \mathbb{R}_+^n \times \mathcal{M}. \tag{3.8}$$

Therefore, $\lim_{x \to 0} V(x, \alpha) = 0$ *for all* $\alpha \in \mathcal{M}$.

(b) *Suppose that for each* $i = 1, \ldots, n$ *and each* $\alpha \in \mathcal{M}$,

$$\begin{aligned} M_i(\alpha) := \sup_{\varphi > 0} \Bigg[& \varphi f_i(\varphi\, \mathbf{e}_i, \alpha) \Big(b_i(\alpha) - a_{ii}(\alpha) \varphi \Big) \\ & + \sum_{k=1}^{m} q_{\alpha k} \int_0^{\varphi} f_i(u\, \mathbf{e}_i, k) du - r \int_0^{\varphi} f_i(u\, \mathbf{e}_i, \alpha) du \Bigg] < \infty. \end{aligned} \tag{3.9}$$

Then there exists a positive number M *such that*

$$V(x, \alpha) \leq \sum_{i=1}^{n} \int_0^{x_i} f_i(\varphi\, \mathbf{e}_i, \alpha) d\varphi + \frac{M}{r} \quad \textit{for all} \quad (x, \alpha) \in \mathbb{R}_+^n \times \mathcal{M}.$$

Remark 3.3. Note that (3.9) holds for a wide class of price functions $f_i(\cdot)$. For instance,

(i) if $f_{i_0}(\cdot, \alpha)$ is independent of α, then (3.9) holds for $i = i_0$ and all $\alpha \in \mathcal{M}$,

(ii) if $\lim_{\varphi \to \infty} \varphi f_i(L_i \varphi, \alpha) = \infty$, then (3.9) is satisfied . Moreover, if there are positive numbers $C_1, C_2, \gamma_1 \geq 0$, $\gamma_2 \in (\gamma_1, \gamma_1 + 1)$ such that $C_2 \varphi^{-\gamma_2} < f_{i_0}(L_{i_0} \varphi, \alpha) < C_1 \varphi^{-\gamma_1}$ for all $\alpha \in \mathcal{M}$ and for all sufficiently large φ, then (3.9) holds for $i = i_0$ and all $\alpha \in \mathcal{M}$.

Using the same arguments as in the preceding theorem, we obtain the following result for the case all species have the same price.

Theorem 3.4. *Assume that* (A) *holds. Moreover, for each* $\alpha \in \mathcal{M}$, $f(\cdot, \alpha) : [0, \infty) \mapsto (0, \infty)$ *is non-increasing, continuously differentiable, and*

$$f_i(x, \alpha) = f\Big(\sum_{j=1}^{n} x_j, \alpha\Big), \quad (x, \alpha) \in \mathbb{R}_+^n \times \mathcal{M}.$$

Then the following assertions hold.

(a) *There exist positive integers* N *and* K *such that*

$$V(x, \alpha) \leq \int_0^{\sum_{i=1}^n x_i} f(\varphi, \alpha) d\varphi + KN \sum_{i=1}^{n} x_i^{1/N} \quad \textit{for all} \quad (x, \alpha) \in \mathbb{R}_+^n \times \mathcal{M}.$$

(b) *Suppose that for each* $i = 1, \ldots, n$ *and each* $\alpha \in \mathcal{M}$,

$$\begin{aligned} M_i(\alpha) := \sup_{x \in \mathbb{R}_+^n} \Bigg[& x_i f(\sum_{j=1}^{n} x_j, \alpha)\Big(b_i(\alpha) - a_{ii}(\alpha) x_i\Big) \\ & + \sum_{k=1}^{m} q_{\alpha k} \int_0^{\sum_{j=1}^n x_j} f(\varphi, k) d\varphi - r \int_0^{\sum_{j=1}^n x_j} f(\varphi, \alpha) d\varphi \Bigg] < \infty. \end{aligned} \tag{3.10}$$

Then for each $(x, \alpha) \in \mathbb{R}_+^n \times \mathcal{M}$, *we have*

$$V(x, \alpha) \leq \int_0^{\sum_{i=1}^n x_i} f(\varphi, \alpha) d\varphi + \frac{M}{r},$$

for some positive constant M.

To proceed, we derive the next result.

Theorem 3.5. *Assuming that* (A) *holds, we have the following results.*

(a) *For each* $\alpha \in \mathcal{M}$ *and any* $0 < y \leq x$, *we have*

$$V(x, \alpha) \geq f(x, \alpha) \cdot (x - y) + V(y, \alpha). \tag{3.11}$$

(b) *If* $n = 1$, $V(\cdot, \alpha)$ *is continuous on* $[0, \infty)$ *for each* $\alpha \in \mathcal{M}$. *Moreover,* $V(\cdot)$ *is a viscosity*

solution of the system of quasi-variational inequalities

$$\max\left\{(\mathcal{L}-r)W(x,\alpha), f(x,\alpha)-\frac{dW}{dx}(x,\alpha)\right\}=0, \quad (x,\alpha)\in\mathbb{R}_+\times\mathcal{M}, \tag{3.12}$$

that is,

(i) *for any* $(x^0,\alpha^0)\in\mathbb{R}_+\times\mathcal{M}$ *and* $W(\cdot,\alpha)\in C^2(\mathbb{R}_+)$ *satisfying* $(V-W)(x,\alpha)\geq (V-W)(x^0,\alpha^0)=0$, *for all* x *in a neighborhood of* x^0 *and all* $\alpha\in\mathcal{M}$, *we have*

$$\max\left\{(\mathcal{L}-r)W(x^0,\alpha^0), f(x_0,\alpha_0)-\frac{dW}{dx}(x_0,\alpha_0)\Big]\right\}\leq 0.$$

(ii) *for any* $(x^0,\alpha^0)\in\mathbb{R}_+\times\mathcal{M}$ *and* $W(\cdot,\alpha)\in C^2(\mathbb{R}_+)$ *satisfying* $(V-W)(x,\alpha)\leq (V-W)(x^0,\alpha^0)=0$, *for all* x *in a neighborhood of* x^0 *and all* $\alpha\in\mathcal{M}$, *we have*

$$\max\left\{(\mathcal{L}-r)W(x^0,\alpha^0), f(x_0,\alpha_0)-\frac{dW}{dx}(x_0,\alpha_0)\right\}\geq 0.$$

Example 3.6. Consider a stochastic logistic population model with harvesting,

$$dX(t)=X(t)\Big[b(\alpha(t))-a(\alpha(t)X(t)\Big]+\sigma(\alpha(t))X(t)dw(t)-dZ(t). \tag{3.13}$$

This model without harvesting was studied in details in [34]. The optimal harvesting problem when $\mathcal{M}=\{1\}$ and a constant price function was also explicitly solved in [1]. Unfortunately, due to the presence of random environment modulated by a Markov chain, we are unable to find an explicit optimal harvesting strategy and the corresponding value function. However, by virtue of Theorem 3.5, the value function $V(\cdot)$ is a viscosity solution of the coupled system of quasi-variational inequalities (3.12) with the boundary condition $V(0,\alpha)=0, \alpha\in\mathcal{M}$. Moreover, [20] provides us with ways to approximate the optimal harvesting strategy and the corresponding value function using numerical methods.

3.4 Chattering Harvesting Strategies

The term "chattering harvesting strategy" was introduced in [53, Theorem 2.4] and previously exploited in [2, Corollary 1]. A chattering strategy is an admissible harvesting policy that instantaneously harvests a sufficiently small amount many times in a sufficiently small interval of time until the population system becomes extinct.

For optimal harvesting problems in one dimension [1, 53], when the discounted factor is sufficiently large, driving the process instantaneously to extinction is the optimal harvesting. Another interesting result is that the chattering harvesting strategy might give an approximation of an optimal harvesting, [53, Theorem 2.4]. In light of these observations, we now study the chattering harvesting policy for the stochastic competitive Lotka-Volterra models. Compared to the case of a single species in [53], our ecosystem is of multi-species and coefficients in the model are not linear growth, then the analysis is more delicate. Since our ecosystem has more than one species, a question naturally arise: should we make all species extinct at a time or in some specific order? The authors in [40] stated a conjecture that it is almost surely never optimal to harvest from more than one population at a time. Our chattering harvesting strategy will be designed by using this idea. To proceed, let S_n be the set of all permutations of $\{1, \ldots, n\}$, and

$$H(x, \alpha) = \sup_{(i_1,\ldots,i_n)\in S_n} H_{(i_1,\ldots,i_n)}(x, \alpha), \quad (x, \alpha) \in \mathbb{R}^n_+ \times \mathcal{M},$$

where

$$\begin{aligned} H_{(1,2,\ldots,n)}(x, \alpha) :&= \int_0^{x_1} f_1(\varphi, x_2, \ldots, x_n, \alpha)d\varphi + \int_0^{x_2} f_2(0, \varphi, x_3, \ldots, x_n, \alpha)d\varphi \\ &\quad + \cdots + \int_0^{x_n} f_n(0, \ldots, 0, \varphi, \alpha)d\varphi, \quad (x, \alpha) \in \mathbb{R}^n_+ \times \mathcal{M}, \end{aligned}$$

and for any permutation $(i_1, \ldots, i_n)$ of $\{1, \ldots, n\}$, $H_{(i_1,\ldots,i_n)}(x, \alpha)$ is analogously defined.

Theorem 3.7. *Suppose that* (A) *holds and for each* $\alpha \in \mathcal{M}$, $f(\cdot, \alpha)$ *is uniformly Lipschitz continuous. Then for any* $(x, \alpha) \in \mathbb{R}^n_+ \times \mathcal{M}$ *and* $\varepsilon > 0$, *there exists a admissible harvesting strategy* $Z^\varepsilon \in \mathcal{A}_{x,\alpha}$ *under which*

$$J(x, \alpha, Z^\varepsilon) \geq H(x, \alpha) - \varepsilon.$$

The harvesting strategy Z^ε *is a policy that instantaneously harvests a sufficiently small amount many times of species in some specific order in a sufficiently small interval of time until the ecosystem becomes extinct. As a result,*

$$V(x, \alpha) \geq H(x, \alpha), \quad \textit{for all} \quad (x, \alpha) \in \mathbb{R}^n_+ \times \mathcal{M}.$$

The chattering harvesting strategies $\{Z^\varepsilon\}$ that we obtained above give us a lower bound for the value function. For the system of n species, in a very small time interval, we harvest only one species until its extinction. In the next time interval, we harvest another species until its extinction. We then repeat this process until the extinction of the whole population system. Continuing in this way, we harvest only one population at a time. One can try to harvest a small amount of all species at a time and make them all extinct at the same time, but the stated lower bound cannot be attained.

Remark 3.8. Using the same argument as in [53, Remark 3.2], we can show that if $f(x, \alpha)$ is strictly decreasing in x for each $\alpha \in \mathcal{M}$, then assumptions in Theorem 3.1(b) can never be satisfied, i.e., no optimal harvesting strategy Z^* (defined in (3.5)) can be constructed by using Theorem 3.1. Moreover, there might be no optimal harvesting policy Z^* at all. Indeed, consider the logistic population model (3.13). Assume (A) and for each $\alpha \in \mathcal{M}$, $f(\cdot, \alpha)$ is uniformly Lipschitz, continuously differentiable and strictly decreasing. Moreover,

suppose that r is sufficiently large so that $(\mathcal{L}-r)H(x,\alpha)<0$ for all $(x,\alpha)\in\mathbb{R}_+\times\mathcal{M}$ with $H(x,\alpha)=\int_0^x f(\varphi,\alpha)d\varphi$. By virtue of Theorem 3.1(a) and Theorem 3.7,

$$V(x,\alpha)=H(x,\alpha),\quad (x,\alpha)\in\mathbb{R}_+\times\mathcal{M}.$$

Now for any admissible strategy $Z\in\mathcal{A}_{x,\alpha}$ and a corresponding harvested process X, a similar argument as in [53, Proposition 2.3] (or [2, Lemma 2]) leads to

$$J(x,\alpha,Z)\le H(x,\alpha)+E_{x,\alpha}\int_0^\tau e^{-rs}(\mathcal{L}-r)H\big(X(s),\alpha(s)\big)ds. \tag{3.14}$$

If $P(\tau>0)>0$, then $E_{x,\alpha}\int_0^\tau e^{-rs}(\mathcal{L}-r)H\big(X(s),\alpha(s)\big)ds<0$. It follows from (3.14) that $J(x,\alpha,Z)<H(x,\alpha)$. Otherwise, Z is the policy that drives the system instantaneously to extinction w.p.1, i.e.,

$$J(x,\alpha,Z)=xf(x,\alpha)<\int_0^x f(\varphi,\alpha)d\varphi=H(x,\alpha)=V(x,\alpha),$$

due to the fact that $f(\cdot,\alpha)$ is strictly decreasing for each α. Hence there is no optimal harvesting strategy at all. However, for sufficiently small ε, chattering harvesting strategies $\{Z^\varepsilon\}$ are ε-optimal or near-optimal harvesting ones.

Example 3.9. Suppose $f_1(x_1,x_2,\alpha)=\dfrac{1}{1+x_1},\quad f_2(x_1,x_2,\alpha)=\dfrac{1}{1+x_1+x_2}$. Let us consider the harvesting problem for a competitive Lotka-Volterra ecosystems of 2 species

$$\begin{aligned}
dX_1(t)&=X_1(t)\Big(b_1(\alpha(t))-a_{11}(\alpha(t))X_1(t)-a_{12}(\alpha(t))X_2(t)\Big)dt\\
&\qquad+\sigma_1(\alpha(t))X_1(t)dw_1(t)-dZ_1(t)\\
dX_2(t)&=X_2(t)\Big(b_2(\alpha(t))-a_{21}(\alpha(t))X_1(t)-a_{22}(\alpha(t))X_2(t)\Big)dt\\
&\qquad+\sigma_2(\alpha(t))X_2(t)dw_2(t)-dZ_2(t),
\end{aligned}\tag{3.15}$$

with initial conditions $x = (x_1, x_2)' \in \mathbb{R}^2_+$, $\alpha \in \mathcal{M}$. We have

$$H_{(1,2)}(x,\alpha) = \int_0^{x_1} f_1(\varphi, x_2, \alpha)d\varphi + \int_0^{x_2} f_2(0,\varphi,\alpha)d\varphi = \ln|1+x_1| + \ln|1+x_2|,$$

and $H_{(2,1)}(x,\alpha) = \ln|1+x_1+x_2|$. Hence $V(x,\alpha) \geq H(x,\alpha) = \ln|1+x_1| + \ln|1+x_2|$. Now we suppose that $b_1(\alpha) \leq r$ and $b_2(\alpha) \leq r$ for all $\alpha \in \mathcal{M}$. Then detail computations give us that $(\mathcal{L} - r)H(x,\alpha) \leq 0$ for all $(x,\alpha) \in \mathbb{R}^2_+ \times \mathcal{M}$. By virtue of Theorem 3.1, $V(x,\alpha) \leq H(x,\alpha)$. Therefore, $V(x,\alpha) = H(x,\alpha)$. The chattering harvesting strategies can provide an approximation of the optimal harvesting. Indeed, since $V(x,\alpha) = H_{(1,2)}(x,\alpha)$, for given $\varepsilon > 0$, by chattering harvesting the first species, and then the second one as in Theorem 3.7, in a sufficiently small time interval, we have a ε-optimal harvesting strategy.

We now consider the case

$$a_{12}(\alpha) = a_{21}(\alpha) = 0, \quad f_1(x,\alpha) = f_1(x_1,\alpha), \quad f_2(x,\alpha) = f_2(x_2,\alpha), \quad (x,\alpha) \in \mathbb{R}^2_+ \times \mathcal{M},$$

that is, there is no interaction between species in the ecosystem, then (3.15) is just a trivial combination of logistic population systems. By virtue of Theorem 3.7,

$$V(x,\alpha) \geq \int_0^{x_1} f_1(\varphi, 0, \alpha)d\varphi + \int_0^{x_2} f_2(0,\varphi,\alpha)d\varphi, \quad (x,\alpha) \in \mathbb{R}^2_+ \times \mathcal{M}. \tag{3.16}$$

In practice, different species interact with each other and compete for resources, food, habitat, or territory. Therefore, it is natural to ask whether (3.16) still holds. In other words, is it true that

$$\int_0^{x_1} f_1(\varphi, 0, \alpha)d\varphi + \int_0^{x_2} f_2(0,\varphi,\alpha)d\varphi,$$

a lower bound for the value function? We claim that this statement might be false. Indeed, suppose that two species have the same price, that is, there are nonincreasing functions

$f(\cdot,\alpha) : [0,\infty) \mapsto (0,\infty)$ such that $f_1(x_1,x_2,\alpha) = f_2(x_1,x_2,\alpha) = f(x_1+x_2,\alpha)$. Detail computations give us

$$H(x,\alpha) = \int_0^{x_1} f(\varphi + x_2,\alpha)d\varphi + \int_0^{x_2} f(\varphi,\alpha)d\varphi = \int_0^{x_1+x_2} f(\varphi,\alpha)d\varphi,$$

and

$$\int_0^{x_1} f_1(\varphi,0,\alpha)d\varphi + \int_0^{x_2} f_2(0,\varphi,\alpha)d\varphi = \int_0^{x_1} f(\varphi,\alpha)d\varphi + \int_0^{x_2} f(\varphi,\alpha)d\varphi.$$

Suppose that $b_1(\alpha) \le r$, $b_2(\alpha) \le r$ for all $\alpha \in \mathcal{M}$. Again, by Theorem 3.1 (a) and Theorem 3.7, we can show that $V(x,\alpha) = H(x,\alpha)$. In many cases, for instance, taking

$$f(\varphi,\alpha) = \frac{1}{1+\varphi} \quad (x,\alpha) \in \mathbb{R}_+^2 \times \mathcal{M},$$

we have

$$H(x,\alpha) = V(x,\alpha) < \int_0^{x_1} f_1(\varphi,0,\alpha)d\varphi + \int_0^{x_2} f_2(0,\varphi,\alpha)d\varphi, \quad \text{for all} \quad (\varphi,\alpha) \in \mathbb{R}_+ \times \mathcal{M}.$$

Thus $\int_0^{x_1} f_1(\varphi,0,\alpha)d\varphi + \int_0^{x_2} f_2(0,\varphi,\alpha)d\varphi$ cannot be a lower bound for the value function.

As a consequence of Theorem 3.1(a), in an extreme case the discounting is so severe that the population evolves almost surely towards extinction independently of its initial state. In this case, it is optimal to harvest the entire population instantaneously. As pointed out in [3], if this condition holds, other ecological and preservation issues may enter to preclude this tactic. However, this may not be the case for the chattering harvesting strategies because we can use this strategy and stop it before the possible extinction. Then the ecosystem evolutes and approaches its stationary state. Hence it is interesting to study the efficiency of the chattering harvesting strategies in some special cases. To proceed, we assume the following condition.

(B) Either

(i) $f_i(\cdot, \alpha)$ is continuously differentiable with bounded first order partial derivatives and (3.9) holds for each (i, α), and

$$H(x, \alpha) = \sum_{i=1}^{n} \int_0^{x_i} f_i(\varphi\, \mathbf{e}_i, \alpha) d\varphi, \quad (x, \alpha) \in \mathbb{R}_+^n \times \mathcal{M},$$

(ii) or for each (i, α), there is a continuously differentiable function with bounded derivative $f(\cdot, \alpha) : [0, \infty) \mapsto (0, \infty)$ satisfying (3.10) and

$$f_i(x, \alpha) = f(\sum_{j=1}^{n} x_j, \alpha), \quad i = 1, \ldots, n, (x, \alpha) \in \mathbb{R}_+^n \times \mathcal{M}.$$

Corollary 3.10. *Assume that* (A) *and* (B) *hold. Then there is a positive constant M such that*

$$H(x, \alpha) \leq V(x, \alpha) \leq H(x, \alpha) + M, \quad \textit{for all} \quad (x, \alpha) \in \mathbb{R}_+^n \times \mathcal{M}. \tag{3.17}$$

Moreover, if there exists (i, α) such that

$$\lim_{x_i \to \infty} \int_0^{x_i} f_i(\varphi\, \mathbf{e}_i, \alpha) d\varphi = \infty, \tag{3.18}$$

then

$$\lim_{x_i \to \infty} \frac{H(x, \alpha)}{V(x, \alpha)} = 1.$$

Intuitively, $H(x, \alpha)$ can be seen as the current harvesting potential since for any $\varepsilon > 0$, the value $H(x, \alpha) - \varepsilon$ can be obtained by using a suitable chattering harvesting strategy. By virtue of Corollary 3.10, the value function is the sum of the current harvesting potential $H(x, \alpha)$ and the maximum present expected value of the accumulate net convenience yields accrued from postponing the harvesting decision and keeping the population alive after a small time

interval. Moreover, under condition (3.18), chattering harvesting strategies become optimal when some initial population x_i is sufficiently large.

When $n = 1$, $\mathcal{M} = 1$ and $f(x) \equiv 1$, the author in [1] studied how the value of harvesting reacts when stochastic fluctuations are so severe, that is, the noise intensities are very large. It is proved that in that case, the value of harvesting approaches the value which is attained by instantaneously depleting the population. This is reasonable since for sufficiently large noise intensities, the population tends to extinct [32, 34]. It indicates that the harvesting activity should be done in a very short time interval. Due to the complexity in the model of our interest, the approach in [1] is no longer applicable, so we proceed using the upper bounds of the value function.

Corollary 3.11. *Assume that* (A) *and* (B) *hold. Let* $\kappa := \inf_{(i,\alpha)} \sigma_i^2(\alpha)$. *Then*

$$\lim_{\kappa\to\infty} V(x,\alpha) = H(x,\alpha),$$

uniformly in (x,α) *such that* $|x| \le M$ *and* $\alpha \in \mathcal{M}$, *for any positive constant* M.

Example 3.12. Let us consider the optimal harvesting problem for a competitive Lotka-Volterra ecosystems of 3 species, with initial conditions $x = (x_1, x_2, x_3)' \in \mathbb{R}^3_+$, $\alpha \in \mathcal{M} = \{1, 2\}$, and

$$\begin{aligned}
&f_1(x,1) = 1, \quad f_1(x,2) = 1 + \frac{1}{x_1+1},\\
&f_2(x,1) = \frac{1}{\sqrt{x_1+x_2+1}}, \quad f_2(x,2) = 1 + \frac{e^{-x_1}}{\sqrt{2+x_2+x_3}},\\
&f_3(x,1) = f_3(x,2) = \frac{1}{x_1^2+1} + \frac{1}{x_3+1}.
\end{aligned}$$

Since $f_1(x,1) = f_1(x_1,1)$, $f_2(x,1) = f_2(x_1,x_2,1)$, and $f_3(x,1) = f_3(x_1,x_3,1)$, then the sequence of chattering harvesting strategies which approximates $H(x,1)$ make species extinct

in the order of corresponding numbers in $(1,2,3)$, i.e.,

$$H(x,1) = H_{(1,2,3)}(x,1) = \int_0^{x_1} d\varphi + \int_0^{x_2} \frac{1}{\sqrt{\varphi+1}} d\varphi + \int_0^{x_3} (1 + \frac{1}{\varphi+1}) d\varphi$$
$$= x_1 + 2(\sqrt{x_2+1} - 1) + x_3 + \ln(x_3+1).$$

Similarly, the sequence of chattering harvesting strategies which approximates $H(x,2)$ make species extinct in the order of corresponding numbers in $(1,3,2)$. Therefore, we obtain that

$$H(x,2) = H_{(1,3,2)}(x,2) = \int_0^{x_1} \Big(1 + \frac{1}{\varphi+1}\Big) d\varphi + \int_0^{x_3} \Big(1 + \frac{1}{\varphi+1}\Big) d\varphi + \int_0^{x_2} \Big(1 + \frac{1}{\sqrt{2+\varphi}}\Big) d\varphi$$
$$= x_1 + \ln|1+x_1| + x_3 + \ln|1+x_3| + x_2 + 2(\sqrt{x_2+2} - \sqrt{2}).$$

By virtue of Corollary 3.10, there exists a positive constant M such that

$$H(x,\alpha) \le V(x,\alpha) \le H(x,\alpha) + M \quad \text{for all} \quad (x,\alpha) \in \mathbb{R}^3_+ \times \mathcal{M}.$$

Moreover, we have $\lim_{|x|\to\infty} \frac{H(x,\alpha)}{V(x,\alpha)} = 1$. This tells us that if initial population of some species i is very large, then the sequence of chattering harvesting strategies provides a good approximation for the optimal harvesting. Finally, by virtue of Corollary 3.11, $\lim_{\kappa\to\infty} V(x,\alpha) = H(x,\alpha)$, uniformly in (x,α) such that $|x| \le M$ and $\alpha \in \mathcal{M}$, for any positive constant M, where $\kappa := \inf_{(i,\alpha)} \sigma_i^2(\alpha)$. In other words, the intertemporal profits accrued by waiting and postponing the harvesting decision are arbitrarily small for sufficiently large noise intensities.

3.5 Further Remarks

In this chapter, we have developed optimal harvesting strategies for regime-switching Lotka-Volterra systems. One of the interesting aspects of our results is the chattering strategies developed. Although the idea was exploited in [40], it has not been fully developed prior to our work. This work is devoted to Lotka-Volterra ecosystems. Nevertheless, the methods developed can be adopted to other optimal controls involving harvesting.

3.6 Proofs of Technical Results

Proof of Theorem 3.1. The proof is similar to that of [53, Theorem 2.1]; see also [40]. Since we use part (a) frequently in this chapter, for convenience, we provide a detail proof for this part. Fix some $(x, \alpha) \in \mathbb{R}^n_+ \times \mathcal{M}$ and $Z \in \mathcal{A}_{x,\alpha}$, and let X denote the corresponding harvested process. Choose N sufficiently large so that $|x| < N$. By virtue of [70],

$$\beta_N \to \infty, \quad \text{a.s. as} \quad N \to \infty, \tag{3.19}$$

where β_N and T_N were defined in (3.7). Then Dynkin's formula leads to

$$\begin{aligned} &E_{x,\alpha}\left[e^{-rT_N}W(X(T_N),\alpha(T_N))\right] - W(x,\alpha) \\ &\quad = E_{x,\alpha}\int_0^{T_N} e^{-rs}(\mathcal{L}-r)W(X(s),\alpha(s))ds - E_{x,\alpha}\int_0^{T_N} e^{-rs}\nabla W(X(s),\alpha(s))\cdot dZ^c(s) \\ &\qquad + E_{x,\alpha}\sum_{0\le s\le T_N} e^{-rs}\Big[W(X(s),\alpha(s-)) - W(X(s-),\alpha(s-))\Big]. \end{aligned}$$

It follows from (3.6) that

$$\begin{aligned} &E_{x,\alpha}\left[e^{-rT_N}W(X(T_N),\alpha(T_N))\right] - W(x,\alpha) \\ &\quad \le -E_{x,\alpha}\int_0^{T_N} e^{-rs}\nabla W(X(s),\alpha(s))\cdot dZ^c(s) + E_{x,\alpha}\sum_{0\le s\le T_N} e^{-rs}\Delta W(X(s),\alpha(s-)), \end{aligned} \tag{3.20}$$

where $\Delta W(X(s),\alpha(s-)) = W(X(s),\alpha(s-)) - W(X(s-),\alpha(s-))$. By virtue of the mean value theorem, we obtain

$$\Delta W(X(s),\alpha(s-)) = \nabla W(X_Z(s),\alpha(s-))\cdot\Delta X(s) = -\nabla W(X_Z(s),\alpha(s-))\cdot\Delta Z(s),$$

for some point $X_Z(s)$ on the line connecting the points $X(s)$ and $X(s-)$. Using (3.6) again, also noting that $f(\cdot,\alpha)$ is nonincreasing for each $\alpha \in \mathcal{M}$ and $\Delta Z(s) \ge 0$, we have

$$\Delta W(X(s),\alpha(s-)) \le -f(X(s),\alpha(s-))\cdot\Delta Z(s). \tag{3.21}$$

Since $W(\cdot)$ is nonnegative, it follows from (3.20) and (3.21) that

$$\begin{aligned} W(x,\alpha) &\geq \left[E_{x,\alpha}\int_0^{T_N} e^{-rs} f(X(s),\alpha(s))\cdot dZ^c(s) + E_{x,\alpha}\sum_{0\leq s\leq T_N} e^{-rs} f(X(s-),\alpha(s-))\cdot \Delta Z(s)\right] \\ &= E_{x,\alpha}\int_0^{T_N} e^{-rs} f(X(s-),\alpha(s-))\cdot dZ(s). \end{aligned}$$

Letting $N\to\infty$, it follows from (3.19) and the bounded convergence theorem that

$$W(x,\alpha) \geq E_{x,\alpha}\int_0^{\tau} e^{-rs} f(X(s)-,\alpha(s-))\cdot dZ(s).$$

Taking supremum over all $Z\in\mathcal{A}_{x,\alpha}$, we obtain $W(x,\alpha)\geq V(x,\alpha)$. $\square$

Proof of Theorem 3.2.

(a) Let $W(x,\alpha)=\sum_{i=1}^n \int_0^{x_i} f_i(\varphi\,\mathbf{e}_i,\alpha)d\varphi + KN\sum_{i=1}^n x_i^{1/N}$, $\quad (x,\alpha)\in\mathbb{R}_+^n\times\mathcal{M}$. Then

$$\frac{\partial W}{\partial x_i}(x,\alpha) = f_i(x_i\,\mathbf{e}_i,\alpha)+\frac{K}{x_i^{1-1/N}},\quad \frac{\partial^2 W}{\partial x_i^2}(x,\alpha)=\frac{\partial f_i}{\partial x_i}(x_i\,\mathbf{e}_i,\alpha)-\frac{K(N-1)}{Nx_i^{2-1/N}},\quad (x,\alpha)\in\mathbb{R}_+^n\times\mathcal{M},$$

where K and N are to be specified. For each $i=1,\ldots,n$, $f_i(x,\alpha)-\dfrac{\partial W}{\partial x_i}(x,\alpha)<0$, for all $(x,\alpha)\in\mathbb{R}_+^n\times\mathcal{M}$. By virtue of Theorem 3.1(a), it suffices to show that $(\mathcal{L}-r)W(x,\alpha)\leq 0$ for all $(x,\alpha)\in\mathbb{R}_+^n\times\mathcal{M}$. Indeed,

$$\begin{aligned} (\mathcal{L}-r)W(x,\alpha) &= \sum_{i=1}^n f_i(x_i\,\mathbf{e}_i,\alpha)x_i\left(b_i(\alpha)-\sum_{j=1}^n a_{ij}(\alpha)x_j\right)+\sum_{i=1}^n\sum_{k=1}^m q_{\alpha k}\int_0^{x_i} f_i(\varphi\,\mathbf{e}_i,\alpha)d\varphi \\ &\quad +\sum_{i=1}^n Kx_i^{1/N}\left(b_i(\alpha)-\sum_{j=1}^n a_{ij}(\alpha)x_j\right)-\sum_{i=1}^n\frac{K(N-1)}{2N}\sigma_i^2(\alpha)x_i^{1/N} \\ &\quad +\frac{1}{2}\sum_{i=1}^n\frac{\partial f_i}{\partial x_i}(x_i\,\mathbf{e}_i,\alpha)\sigma_i^2(\alpha)x_i^2 - r\sum_{i=1}^n\int_0^{x_i} f_i(\varphi\,\mathbf{e}_i,\alpha)d\varphi - rKN\sum_{i=1}^n x_i^{1/N} \\ &\leq \sum_{i=1}^n x_i\left(f_i(x_i\,\mathbf{e}_i,\alpha)b_i(\alpha)+\sum_{k\neq\alpha} q_{\alpha k}f_i(0,\alpha)\right) \\ &\quad -\sum_{i=1}^n K\Big(rN-b_i(\alpha)\Big)x_i^{1/N}-\sum_{i=1}^n Ka_{ii}(\alpha)x_i^{1+1/N} \\ &\leq \sum_{i=1}^n\left(C_1x_i - C_2x_i^{1+1/N}-\frac{rKN}{2}x_i^{1/N}\right) \end{aligned} \tag{3.22}$$

with C_1, C_2 are positive constants being independent of i, α and $N > \max_{i,\alpha} \dfrac{2b_i(\alpha)}{r}$. Note that in the above, we used the fact that $f_i(\cdot, \alpha)$ is nonincreasing for each (i, α). It can be shown that for sufficiently large K, $(\mathcal{L} - r)W(x, \alpha) \leq 0$ for all $(x, \alpha) \in \mathbb{R}^n_+ \times \mathcal{M}$. The conclusion follows.

(b) Let

$$W(x, \alpha) = \sum_{i=1}^{n} \int_0^{x_i} f_i(\varphi\, \mathbf{e}_i, \alpha) d\varphi + \frac{M}{r}, \quad (x, \alpha) \in \mathbb{R}^n_+ \times \mathcal{M}.$$

Then

$$\frac{\partial W}{\partial x_i}(x, \alpha) = f_i(x_i\, \mathbf{e}_i, \alpha), \quad \frac{\partial^2 W}{\partial x_i^2}(x, \alpha) = \frac{\partial f_i}{\partial x_i}(x_i\, \mathbf{e}_i, \alpha), \quad (x, \alpha) \in \mathbb{R}^n_+ \times \mathcal{M},$$

It suffices to show that $(\mathcal{L} - r)W(x, \alpha) \leq 0$ for all $(x, \alpha) \in \mathbb{R}^n_+ \times \mathcal{M}$. Indeed,

$$\begin{aligned}
(\mathcal{L} - r)W(x, \alpha) &= \sum_{i=1}^{n} f_i(x_i\, \mathbf{e}_i, \alpha) x_i \Big(b_i(\alpha) - \sum_{j=1}^{n} a_{ij}(\alpha) x_j \Big) + \sum_{i=1}^{n} \sum_{k=1}^{m} q_{\alpha k} \int_0^{x_i} f_i(\, \mathbf{e}_i \varphi, k) d\varphi \\
&\quad + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x_i\, \mathbf{e}_i, \alpha) \sigma_i^2(\alpha) x_i^2 - r \sum_{i=1}^{n} \int_0^{x_i} f_i(\varphi\, \mathbf{e}_i, \alpha) d\varphi - M \\
&\leq \sum_{i=1}^{n} \Bigg[x_i f_i(\, \mathbf{e}_i x_i, \alpha) \Big(b_i(\alpha) - a_{ii}(\alpha) x_i \Big) \\
&\qquad + \sum_{k=1}^{m} q_{\alpha k} \int_0^{x_i} f_i(\varphi\, \mathbf{e}_i, k) d\varphi - r \int_0^{x_i} f_i(\varphi\, \mathbf{e}_i, \alpha) d\varphi \Bigg] - M.
\end{aligned}$$

The conclusion follows from (3.9). □

Proof of Theorem 3.5.

(a) The proof is similar to [53, Lemma 4.1]. The details are thus omitted.

(b) Using the same argument as in Lemma 4.2 [53], for each $\alpha \in \mathcal{M}$ and any $0 < y \leq x$, we obtain

$$V(x, \alpha) \leq V(y, \alpha) + \max_{j \in \mathcal{M}} V(x - y, j). \tag{3.23}$$

Hence for $y < x < z$, using (3.11) and (3.23) yield

$$V(x,\alpha) \le V(y,\alpha) + \max_{j\in\mathcal{M}} V(x-y,j) \le V(x,\alpha) + \max_{j\in\mathcal{M}} V(x-y,j),$$

and

$$V(x,\alpha) - \max_{j\in\mathcal{M}} V(z-x,j) \le V(z,\alpha) - \max_{j\in\mathcal{M}} V(z-x,j) \le V(x,\alpha).$$

By virtue of Theorem 3.2(a),

$$\lim_{z\downarrow x} V(z-x,\alpha) = \lim_{y\uparrow x} V(x-y,\alpha) = 0,$$

for each $\alpha \in \mathcal{M}$. Taking the limit when $y \uparrow x$ and $z \downarrow x$, we arrive at

$$V(x,\alpha) \le \lim_{z\downarrow x} V(z,\alpha) \le V(x,\alpha) \le \lim_{y\uparrow x} V(y,\alpha) \le V(x,\alpha).$$

Hence $V(\cdot,\alpha)$ is continuous on $[0,\infty)$ for each $\alpha \in \mathcal{M}$. Now the proof of Theorem 4.9 [53] still holds in our case. The conclusion follows. $\square$

Proof of Theorem 3.7. Fix some $(x,\alpha) \in \mathbb{R}^n_+ \times \mathcal{M}$ and $\varepsilon > 0$. Throughout the proof, we use K_0 to denote a generic positive constant depending only on x, m, and constant coefficients of (3.1). The exact value of K_0 may be different in different appearances.

Without loss of generality, let the permutation $(1,2,\ldots,n)$ of $\{1,\ldots,n\}$. We will design a chattering harvesting strategy Z^ε such that

$$J(x,\alpha,Z^\varepsilon) \ge H_{(1,2,\ldots,n)}(x,\alpha) - \varepsilon.$$

The order of each number in $(1,2,\ldots,n)$ is the order of extinction of the corresponding species in the system. We first describe Z^ε and then for simplicity, we only give a detail proof for the case $n = 2$. For $n > 2$, we use the analogously argument.

Let k and γ be positive integers. Define

$$\rho = k^{-\gamma}, \quad \delta_i = x_i/k^i, \quad x_i^j = x_i - j\delta_i, \quad i = 1, \ldots, n, \quad j = 1, \ldots, k^i,$$

and

$$t_j = j\rho/k^n, \quad j = 0, 1, \ldots, k^n - 1.$$

We construct a harvesting strategy $Z^\varepsilon = (Z_1^\varepsilon, \ldots, Z_n^\varepsilon)'$, where Z_i^ε increases only on the set $\{t_j : j = 0, \ldots, k^i - 1\}$. The corresponding harvested process is denoted by $X = (X_1, \ldots, X_n)'$. Note that $X(t_0-) = x = (x_1^0, \ldots, x_n^0)'$. Define $\Delta Z(t_0) = Z(t_0) = (\delta_1, \ldots, \delta_n)'$. Then $X(t_0) = (x_1^1, \ldots, x_n^1)'$, and it therefore implies

$$X(t_1-) = X(t_0) + \int_{t_0}^{t_1} F(X(s), \alpha(s))ds + \int_{t_0}^{t_1} G(X(s), \alpha(s))dw(s). \tag{3.24}$$

For each real number a, we denote $a^+ := \max\{a, 0\}$. At time $t = t_1$, define

$$\Delta Z_i(t_1) = (X_i(t_1-) - x_i^2)^+,$$

so that $X_i(t_1) \leq x_i^2$ and allow the process X to diffuse until time $t = t_2$. For $j = 1, \ldots, k-1$, we define

$$\Delta Z_i(t_j) = \left(X_i(t_j-) - x_i^{j+1}\right)^+.$$

Then

$$X(t_{j+1}-) = X(t_j) + \int_{t_j}^{t_{j+1}} F(X(s), \alpha(s))ds + \int_{t_j}^{t_{j+1}} G(X(s), \alpha(s))dw(s), \tag{3.25}$$

where $X(t_j) = X(t_j-) - \Delta Z(t_j)$. Hence $\Delta Z_1(t_{k-1}) = X_1(t_{k-1}-)$ and species 1 is extinct at time t_{k-1}. Next we harvest remaining species. In general, define

$$\Delta Z_i(t_j) := \left(X_i(t_j-) - x_i^{j+1}\right)^+, \quad i = 2, .., n, j = k^{i-1}, \ldots, k^i - 1,$$

and

$$X_i(t_{j+1}-) = X_i(t_j) + \int_{t_j}^{t_{j+1}} F_i(X(s), \alpha(s))ds + \int_{t_j}^{t_{j+1}} G_i(X(s), \alpha(s))dw(s).$$

Hence $\Delta Z_i(t_{k^i-1}) = X_i(t_{k^i-1}-)$, species i is extinct at time t_{k^i-1}, and the population system is extinct at time t_{k^n-1}. In the above, we adopt the convention that $X_i(t) = 0$ if $t \geq t_{k^i-1}$. We will show that for sufficiently large γ and k, $J(x, \alpha, Z^\varepsilon) \geq H_{(1,2,\ldots,n)}(x, \alpha) - \varepsilon$. For simplicity, let $n = 2$.

The expected total discounted reward from the first species corresponding to the harvesting strategy Z^ε is

$$J_1(x, \alpha, Z^\varepsilon) = E_{x,\alpha} \sum_{j=0}^{k-1} e^{-rt_j} f_1(X(t_j-), \alpha(t_j-))\Delta Z_1(t_j).$$

Define

$$R_1(x, \alpha) := \sum_{j=0}^{k-1} f_1(x_1^j, x_2^j, \alpha)\delta_1.$$

We want to estimate $|J_1(x, \alpha, Z^\varepsilon) - R_1(x, \alpha)|$. In fact, we have

$$\begin{aligned}
|J_1(x, \alpha, Z^\varepsilon) - R_1(x, \alpha)| &\leq \sum_{j=0}^{k-1} E_{x,\alpha}|e^{-rt_j} f_1(X(t_j-), \alpha(t_j-))\Delta Z_1(t_j)) - f_1(x_1^j, x_2^j, \alpha)\delta_1| \\
&\leq \sum_{j=1}^{k-1} \Big[E_{x,\alpha}\Big| \big[f_1(X(t_j-), \alpha(t_j-)) - f_1(x_1^j, x_2^j, \alpha)\big]\delta_1 \Big| \\
&\qquad + E_{x,\alpha}\Big| f_1(X(t_j-), \alpha(t_j-))\big[\Delta Z_1(t_j) - \delta_1\big] \Big| \\
&\qquad + E_{x,\alpha}\Big| \big[e^{-rt_j} - 1\big] f_1(X(t_j-), \alpha(t_j-))\Delta Z_1(t_j) \Big| \Big] \\
&:= \sum_{j=1}^{k-1} (A_j^1 + B_j^1 + C_j^1).
\end{aligned}$$

In what follows, we analyze the terms A_j^1, B_j^1, C_j^1 separately. First we note that for any $j = 0, \ldots, k-1$, $|X_i(t_j)| \leq x_i$. Using the similar argument as in Theorem 3.1 [70], it can be shown that for any $p > 0$, $E|X(t)|^p \leq K_0 \quad \text{for all} \quad t \in [t_j, t_{j+1})$. Observe that for each

$\alpha \in \mathcal{M}$, $F_i(\cdot, \alpha)$ and $G_i(\cdot, \alpha)$ are polynomials with order 2 and 1, respectively. It follows that

$$\begin{aligned} E\Big| \textstyle\int_{t_j}^{t_{j+1}} F_i(X(s), \alpha(s))ds + \int_{t_j}^{t_{j+1}} G_i(X(s), \alpha(s))dw(s)\Big|^2 &\leq K_0(t_{j+1} - t_j) \\ &= K_0 t_1, \end{aligned} \tag{3.26}$$

and as a consequence,

$$E|\Delta Z_i(t_j)| = E|(X_i(t_j-) - x_i^{j+1})^+| \leq K_0, \tag{3.27}$$

where recall that K_0 is a generic positive constant depending only on x, m, and constant coefficients of (3.1). It follows from (3.24) and the Chebyshev inequality that

$$\begin{aligned} P\{\Delta Z_i(t_1) = 0\} &= P\{X_i(t_1-) \leq x_i^2\} \\ &= P\Big\{x_i^1 + \int_{t_0}^{t_1} F_i(X(s), \alpha(s))ds + \int_{t_0}^{t_1} G_i(X(s), \alpha(s))dw(s) \leq x_i^2\Big\} \\ &= P\Big\{\int_{t_0}^{t_1} F_i(X(s), \alpha(s))ds + \int_{t_0}^{t_1} G_i(X(s), \alpha(s))dw(s) \leq -\delta_i\Big\} \\ &\leq P\Big\{\Big|\int_{t_0}^{t_1} F_1(X(s), \alpha(s))ds + \int_{t_0}^{t_1} G_1(X(s), \alpha(s))dw(s)\Big| \geq \delta_i\Big\} \\ &\leq \frac{K_0 t_1}{\delta_i^2}. \end{aligned} \tag{3.28}$$

Note that $X_i(t_1) = x_i^2$ if $\Delta Z_i(t_1) > 0$. Hence we have

$$\begin{aligned} P\big\{X_i(t_1) \neq x_i^2\big\} &\leq P\big\{\Delta Z_i(t_1) = 0\big\} \\ &\leq \frac{K_0 t_1}{\delta_i^2}. \end{aligned} \tag{3.29}$$

Using the same argument as that of (3.28) and (3.29), we obtain

$$\begin{aligned} P\{\Delta Z_i(t_2) = 0\} &= P\{\Delta Z_i(t_2) = 0, X_i(t_1) = x_i^2\} + P\{\Delta Z_i(t_2) = 0, X_i(t_1) \neq x_i^2\} \\ &\leq \frac{K_0 t_1}{\delta_i^2} + \frac{K_0 t_1}{\delta_i^2} = \frac{K_0 t_2}{\delta_i^2}, \end{aligned}$$

and

$$P\big\{X_i(t_2) \neq x_i^3\big\} \leq P\big\{\Delta Z_i(t_2) = 0\big\} \leq \frac{K_0 t_2}{\delta_i^2}.$$

Continuing in this manner, it follows that

$$P\{\Delta Z_i(t_j) = 0\} \leq \frac{K_0 t_j}{\delta_i^2}, \quad j = 1, 2, \ldots, k-1, \tag{3.30}$$

and

$$P\big\{X_i(t_j) \neq x_i^{j+1}\big\} \leq \frac{K_0 t_j}{\delta_i^2}, \quad j = 1, 2, \ldots, k-1. \tag{3.31}$$

Using the conditions that $f_1(\cdot, \alpha)$ is Lipschitz continuous for each $\alpha \in \mathcal{M}$ with Lipschitz constant $L > 0$ and uniformly bounded, we obtain

$$\begin{aligned}
A_j^1 &= E\Big|f_1(X(t_j-), \alpha(t_j-)) - f_1(x_1^j, x_2^j, \alpha)\Big|\delta_1 \\
&\leq E\Big|f_1(X(t_j-), \alpha) - f_1(x_1^j, x_2^j, \alpha)\Big|\delta_1 + E\Big|f_1(X(t_j-), \alpha(t_j-)) - f_1(X(t_j-), \alpha)\Big|\delta_1 \\
&\leq L\Big(E|X_1(t_j-) - x_1^j| + E|X_2(t_j-) - x_2^j|\Big)\delta_1 + K_0 P\{\alpha(t_j-) \neq \alpha\}\delta_1 \\
&\leq L\Big(E|X_1(t_j-) - x_1^j| + E|X_2(t_j-) - x_2^j|\Big)\delta_1 + K_0 t_j \delta_1,
\end{aligned} \tag{3.32}$$

where in the last inequality we used the property of the Markov chain $\alpha(\cdot)$. Using (3.25), (3.26), and (3.31), we obtain

$$\begin{aligned}
E|X_i(t_j-) - x_i^j| &\leq E|X_i(t_{j-1}) - x_i^j| \\
&\qquad + E\left|\int_{t_{j-1}}^{t_j} F_i(X(s), \alpha(s))ds + \int_{t_{j-1}}^{t_j} G_i(X(s), \alpha(s))ds\right| \\
&\leq E^{1/2}|X_i(t_{j-1}) - x_i^j|^2 E^{1/2}\Big[I_{X_i(t_{j-1}) \neq x_i^j}\Big] + K_0\sqrt{t_1} \\
&\leq \frac{K_0 x_i^j \sqrt{t_{j-1}}}{\delta_i} + K_0\sqrt{t_1} \leq \frac{K_0\sqrt{t_j}}{\delta_i} + K_0\sqrt{t_1}.
\end{aligned} \tag{3.33}$$

Since $\delta_1 \in (0, 1)$, then $t_j < \sqrt{t_j}$ and $\sqrt{t_1} < \sqrt{t_j}$. With these observations, using (3.32) and (3.33), we arrive at

$$\begin{aligned}
A_j^1 &\leq L\Big(\frac{K_0\sqrt{t_j}}{\delta_1} + \frac{K_0\sqrt{t_j}}{\delta_2} + K_0\sqrt{t_1}\Big)\delta_1 + K_0 t_j \delta_1 \\
&\leq K_0\Big(\sqrt{t_j} + \frac{\delta_1\sqrt{t_j}}{\delta_2}\Big),
\end{aligned} \tag{3.34}$$

Next we estimate B_j^1. Since $f_1(\cdot,\cdot)$ is uniformly bounded, it follows that

$$\begin{aligned}
B_j^1 &\le K_0 E|\Delta Z_1(t_j) - \delta_1| \\
&= K_0 E|\big(\Delta Z_1(t_j) - \delta_1\big) I_{\{\Delta Z_1(t_j)=0\}}| + K_0 E|\big(\Delta Z_1(t_j) - \delta_1\big) I_{\{\Delta Z_1(t_j)\neq 0\}} I_{\{X_1(t_{j-1})=x_1^j\}}| \\
&\qquad + K_0 E|\big(\Delta Z_1(t_j) - \delta_1\big) I_{\{\Delta Z_1(t_j)\neq 0\}} I_{\{X_1(t_{j-1})\neq x_1^j\}}| \\
&:= B_{j1} + B_{j2} + B_{j3}.
\end{aligned} \tag{3.35}$$

By virtue of (3.30),

$$B_{j1} \le \delta_1 \frac{K_0 t_j}{\delta_1^2} = \frac{K_0 t_j}{\delta_1}. \tag{3.36}$$

It follows from (3.26) that

$$\begin{aligned}
B_{j2} &= K_0 E|\big(\Delta Z_1(t_j) - \delta_1\big) I_{\{\Delta Z_1(t_j)\neq 0\}} I_{\{X_1(t_{j-1})=x_1^j\}}| \\
&= K_0 E|\big(X(t_j-) - x_1^{j+1} - \delta_1\big) I_{\{\Delta Z_1(t_j)\neq 0\}} I_{\{X_1(t_{j-1})=x_1^j\}}| \\
&= K_0 E|\big(X(t_j-) - x_1^{j}\big) I_{\{\Delta Z_1(t_j)\neq 0\}} I_{\{X_1(t_{j-1})=x_1^j\}}| \\
&= K_0 E|\Big[\int_{t_{j-1}}^{t_j} F_1(X(s),\alpha(s))ds + \int_{t_{j-1}}^{t_j} G_1(X(s),\alpha(s))dw(s)\Big] I_{\{\Delta Z_1(t_j)\neq 0\}} I_{\{X_1(t_{j-1})=x_1^j\}}| \\
&\le K_0\sqrt{t_1}.
\end{aligned} \tag{3.37}$$

For the term B_{j3}, using the Cauchy-Schwarz inequality, (3.27), and (3.31), we obtain

$$\begin{aligned}
B_{j3} &\le K_0 E^{1/2}|\big(\Delta Z_1(t_j) - \delta_1\big) I_{\{\Delta Z_1(t_j)\neq 0\}}|^2 E^{1/2}|I_{\{X_1(t_{j-1})\neq x_1^j\}}|^2 \\
&\le K_0 \frac{\sqrt{t_{j-1}}}{\delta_1} \\
&\le K_0 \frac{\sqrt{t_j}}{\delta_1}.
\end{aligned} \tag{3.38}$$

From (3.35), (3.36), (3.37), and (3.38), we have

$$\begin{aligned}
B_j^1 &\le \frac{K_0 t_j}{\delta_1} + K_0\sqrt{t_1} + \frac{K_0\sqrt{t_j}}{\delta_1} \\
&\le K_0 \frac{\sqrt{t_j}}{\delta_1},
\end{aligned} \tag{3.39}$$

since $\sqrt{t_1} < \sqrt{t_j}$ and $t_j < \sqrt{t_j}$. For the term C_j, we again use the uniform boundedness of $f_1(\cdot,\cdot)$ and (3.27) to obtain

$$\begin{aligned} C_j^1 &= E\Big|\big[e^{-rt_j} - 1\big] f_1(X(t_j-), \alpha(t_j-))\Delta Z_1(t_j)\Big| \\ &\le K_0(1 - e^{-rt_j}) \\ &\le K_0 t_j. \end{aligned} \tag{3.40}$$

It follows from (3.34), (3.39), and (3.40) that

$$\begin{aligned} |J_1(x,\alpha,Z^\varepsilon) - R_1(x,\alpha)| &\le \sum_{j=1}^{k-1}\left(K_0\Big(\sqrt{t_j} + \frac{\delta_1\sqrt{t_j}}{\delta_2}\Big) + K_0\frac{\sqrt{t_j}}{\delta_1} + K_0 t_j\right) \\ &\le K_0\sqrt{t_1}\Big[\frac{\delta_1+1}{\delta_1} + \frac{\delta_1}{\delta_2}\Big]\sum_{j=1}^{k-1}\sqrt{j} \\ &\le K_0 k^{-0.5\gamma-1} k k^2 = K_0 k^{-0.5\gamma+2}, \end{aligned} \tag{3.41}$$

where we used

$$t_1 = k^{-\gamma-2}, \quad \sum_{j=1}^{k-1}\sqrt{j} \le \sum_{j=1}^{k-1}(j+1) \le K_0 k^2, \quad \frac{\delta_1+1}{\delta_1} + \frac{\delta_1}{\delta_2} \le \frac{K_0}{\delta_1} + \frac{\delta_1}{\delta_2} \le K_0 k.$$

The expected total discounted reward from the second species corresponding to the harvesting strategy Z^ε is

$$\begin{aligned} J_2(x,\alpha,Z^\varepsilon) = E_{x,\alpha}\Bigg(&\sum_{j=0}^{k-1} e^{-rt_j} f_2(X(t_j-),\alpha(t_j-))\Delta Z_2(t_j) \\ &+ \sum_{j=k}^{k^2-1} e^{-rt_j} f_2(0, X_2(t_j-),\alpha(t_j-))\Delta Z_2(t_j)\Bigg). \end{aligned} \tag{3.42}$$

Define

$$R_2(x,\alpha) := \sum_{j=k}^{k^2-1} f_2(0, x_2^j, \alpha)\delta_2.$$

We will estimate

$$\Big|E_{x,\alpha}\sum_{j=k}^{k^2-1} e^{-rt_j} f_2(0, X_2(t_j-),\alpha(t_j-))\Delta Z_2(t_j) - R_2(x,\alpha)\Big|$$

as the way we did for $|J_1(x,\alpha,Z^\varepsilon)-R_1(x,\alpha)|$. We have

$$\begin{aligned}
&\Big|E_{x,\alpha}\sum_{j=k}^{k^2-1} e^{-rt_j}\ f_2(0,X_2(t_j-),\alpha(t_j-))\Delta Z_2(t_j)-R_2(x,\alpha)\Big| \\
&\qquad\le \sum_{j=k}^{k^2-1} E_{x,\alpha}|e^{-rt_j}f_2(0,X_2(t_j-),\alpha(t_j-))\Delta Z_2(t_j))-f_2(0,x_2^j,\alpha)\delta_2| \\
&\qquad\le \sum_{j=k}^{k^2-1}\Big[E_{x,\alpha}\Big|\big[f_2(0,X_2(t_j-),\alpha(t_j-))-f_2(0,x_2^j,\alpha)\big]\delta_2\Big| \\
&\qquad\quad +E_{x,\alpha}\Big|f_2(0,X_2(t_j-),\alpha(t_j-))\big[\Delta Z_2(t_j)-\delta_2\big]\Big| \\
&\qquad\quad +E_{x,\alpha}\Big|\big[e^{-rt_j}-1\big]f_2(0,X_2(t_j-),\alpha(t_j-))\Delta Z_2(t_j)\Big|\Big] \\
&\qquad := \sum_{j=k}^{k^2-1}(A_j^2+B_j^2+C_j^2).
\end{aligned} \tag{3.43}$$

Note that (3.26), (3.27), (3.30), and (3.31) still hold for $i=2$ and all $j=k,\dots,k^2-1$. Hence using the same arguments for A_j^1, B_j^1, C_j^1, we obtain

$$\begin{aligned}
&A_j^2\le K_0\sqrt{t_j}, \\
&B_j^2\le K_0\frac{\sqrt{t_j}}{\delta_2}, \\
&C_j^2\le K_0 t_j,\quad j=k,\dots,k^2-1.
\end{aligned} \tag{3.44}$$

Now we observe that $t_1=k^{-\gamma-2}$, $\sum_{j=k}^{k^2-1}\sqrt{j}\le\sum_{j=k}^{k^2-1}(j+1)\le K_0k^4$, $\frac{1}{\delta_2}\le K_0k^2$. It follows from (3.43) and (3.44) that

$$\begin{aligned}
&\Big|\sum_{j=k}^{k^2-1} e^{-rt_j}\ f_2(0,X_2(t_j-),\alpha(t_j-))\Delta Z_2(t_j)-R_2(x,\alpha)\Big| \\
&\qquad\le K_0\sum_{j=k}^{k^2-1}\frac{\sqrt{t_j}}{\delta_2} \\
&\qquad\le K_0\sqrt{t_1}\frac{1}{\delta_2}\sum_{j=k}^{k^2-1}\sqrt{j} \\
&\qquad\le K_0k^{-0.5\gamma-1}k^2k^4 \\
&\qquad= K_0k^{-0.5\gamma+5},
\end{aligned} \tag{3.45}$$

By virtue of (3.41) and (3.45), for $\gamma = 12$ and sufficiently large k, we obtain

$$\begin{aligned} &|J_1(x,\alpha,Z^\varepsilon) - R_1(x,\alpha)| \le \frac{\varepsilon}{4}, \\ &\Big|\sum_{j=k}^{k^2-1} e^{-rt_j} f_2(0, X_2(t_j-), \alpha(t_j-))\Delta Z_2(t_j) - R_2(x,\alpha)\Big| \le \frac{\varepsilon}{4}. \end{aligned} \tag{3.46}$$

Moreover, since $f_1(\cdot,\alpha)$ is nonincreasing and continuous, then for sufficiently large k,

$$\begin{aligned} R_1(x,\alpha) &= \sum_{j=0}^{k-1} f_1(x_1^j, x_2^j, \alpha)\delta_1 \ge \sum_{j=0}^{k-1} f_1(x_1^j, x_2, \alpha)\delta_1 \\ &\ge \int_0^{x_1} f_1(\varphi, x_2, \alpha)d\varphi - \frac{\varepsilon}{4}. \end{aligned} \tag{3.47}$$

Similarly, for sufficiently large k,

$$\begin{aligned} R_2(x,\alpha) &= \sum_{j=0}^{k^2-1} f_2(0, x_2^j, \alpha)\delta_2 - \sum_{j=0}^{k-1} f_2(0, x_2^j, \alpha)\delta_2 \\ &\ge \sum_{j=0}^{k^2-1} f_2(0, x_2^j, \alpha)\delta_2 - \frac{x_2}{k} f_2(0,0,\alpha) \\ &\ge \int_0^{x_2} f_2(0,\varphi,\alpha)d\varphi - \frac{\varepsilon}{4}. \end{aligned} \tag{3.48}$$

It follows from (3.42), (3.46), (3.47), and (3.48) that

$$\begin{aligned} J(x,\alpha,Z^\varepsilon) &\ge J_1(x,\alpha,Z^\varepsilon) + E_{x,\alpha}\sum_{j=k}^{k^2-1} e^{-rt_j} f_2(0, X_2(t_j-), \alpha(t_j-))\Delta Z_2(t_j) \\ &\ge R_1(x,\alpha) - \frac{\varepsilon}{4} + R_2(x,\alpha) - \frac{\varepsilon}{4} \\ &\ge \int_0^{x_1} f_1(\varphi, x_2, \alpha)d\varphi + \int_0^{x_2} f_2(0,\varphi,\alpha)d\varphi - \varepsilon. \end{aligned}$$

Since ε is arbitrary,

$$V(x,\alpha) \ge H_{(1,2)}(x,\alpha) = \int_0^{x_1} f_1(\varphi, x_2, \alpha)d\varphi + \int_0^{x_2} f_2(0,\varphi,\alpha)d\varphi.$$

Moreover, since we can interchange the order of species in the harvesting policy, we also have

$$V(x,\alpha) \ge H_{(2,1)}(x,\alpha) = \int_0^{x_2} f_2(x_1, \varphi, \alpha)d\varphi + \int_0^{x_1} f_1(\varphi,0,\alpha)d\varphi.$$

The conclusion follows. □

Proof of Corollary 3.10. By virtue of Theorem 3.7, $V(x,\alpha) \geq H(x,\alpha)$ for each $(x,\alpha) \in \mathbb{R}^n_+ \times \mathcal{M}$. If (B)(i) holds, from Theorem 3.2, we have $V(x,\alpha) \leq H(x,\alpha) + M$ for a sufficiently large positive number M. Otherwise, (B)(ii) is satisfied. In this case,

$$\begin{aligned} H(x,\alpha) = H_{(1,2,\ldots,n)}(x,\alpha) &= \sum_{i=1}^{n-1} \int_0^{x_i} f(\varphi + x_{i+1} + \cdots + x_n, \alpha) d\varphi + \int_0^{x_n} f(\varphi,\alpha) \\ &= \int_0^{\sum_{i=1}^n x_i} f(\varphi,\alpha) d\varphi. \end{aligned}$$

Hence it follows from Theorem 3.4 that $V(x,\alpha) \leq H(x,\alpha) + M$ for some positive number M. Therefore (3.17) always holds. Suppose that (3.18) is satisfied. Then $\lim\limits_{x_i \to \infty} H(x,\alpha) = \lim\limits_{x_i \to \infty} \int_0^{x_i} f_i(\varphi\, \mathbf{e}_i, \alpha) d\varphi = \infty$. The conclusion follows. $\square$

Proof of Corollary 3.11. For any $M > 0$ and $\varepsilon > 0$, let N be sufficiently large such that

$$\frac{1}{N} \sum_{i=1}^n x_i^{1/N} < \varepsilon, \quad \text{whenever} \quad |x| \leq M, \alpha \in \mathcal{M}.$$

Let $W(x,\alpha) = H(x,\alpha) + \frac{1}{N} \sum\limits_{i=1}^n x_i^{1/N}, \quad (x,\alpha) \in \mathbb{R}^n_+ \times \mathcal{M}$. Suppose that (B)(i) holds. The other case is analogously proved. Proceeding as in Theorem 3.2, it follows from (3.22) that

$$\begin{aligned} (\mathcal{L} - r)W(x,\alpha) &\leq \sum_{i=1}^n x_i \Big(f_i(x_i\, \mathbf{e}_i, \alpha) b_i(\alpha) + \sum_{k \neq \alpha} q_{\alpha k} f_i(0,\alpha) \Big) \\ &\quad - \sum_{i=1}^n \frac{rN - b_i(\alpha)}{N^2} x_i^{1/N} - \sum_{i=1}^n \frac{a_{ii}(\alpha)}{N^2} x_i^{1+1/N} - \sum_{i=1}^n \frac{N-1}{2N^3} \kappa x_i^{1/N} \\ &\leq \sum_{i=1}^n \Big(C_1 x_i - \frac{C_2}{N^2} x_i^{1+1/N} - \frac{\kappa}{3N^2} x_i^{1/N} \Big) \end{aligned}$$

with C_1, C_2 being positive constants independent of i, α and $N > \sup\limits_{i,\alpha} \dfrac{b_i(\alpha)}{r}$. It can be shown that for sufficiently large κ, $(\mathcal{L} - r)W(x,\alpha) \leq 0$ for all $(x,\alpha) \in \mathbb{R}^n_+ \times \mathcal{M}$. Hence

$$H(x,\alpha) \leq V(x,\alpha) \leq W(x,\alpha) \leq H(x,\alpha) + \varepsilon \quad \text{whenever} \quad |x| \leq M, \alpha \in \mathcal{M}.$$

The conclusion follows from the fact that $\varepsilon > 0$ is arbitrarily small. $\square$

CHAPTER 4 NUMERICAL METHODS FOR OPTIMAL HARVESTING STRATEGIES IN RANDOM ENVIRONMENTS UNDER PARTIAL OBSERVATIONS

4.1 Introduction

This chapter focuses on optimal harvesting problems for ecosystems formulated by stochastic differential equations with regime switching represented by a continuous-time Markov chain. The problem belongs to the class of singular stochastic control problems motivated by the establishment of ecologically, environmentally, and economically reasonable wildlife management and harvesting policies. Recently, there has been a resurgent interest in determining the optimal harvesting strategies in the presence of stochastic fluctuations. Radner and Shepp [51] derived the optimal strategy of a model for corporate strategy. Alvarez and Shepp [1] studied the optimal harvesting plan for the stochastic Verhulst-Pearl logistic model. All the aforementioned works dealt with species living in an environment with a fixed configuration. Recently, Song, Stockbridge, and Zhu [53] and [59] considered singular control problems in random environments modeled by a Markov chain. Note that the paper [53] dealt with a single species and [59] treated multiple species with interactions.

Suppose that there is a single species $X(t)$ whose growth is subject to the usual fluctuations as well as the abrupt changes of a random environment. Harvesting strategies are introduced to derive financial benefit as well as to control the growth of the population. Let $Z(t)$ denote the total amount harvested from the species up to time t. The goal is to find a harvesting strategy $Z(t)$ that maximizes the expected total discounted reward from harvesting, up to the time when the population falls to a given threshold (e.g., extinction),

which has the following economic interpretation. Let $X(t)$ be the value at time t of asset/security/investment and $Z(t)$ represent the total amount paid in dividends up to time t. Then $\mathbb{R}_+ = (0, \infty)$ can be regarded as the solvency set, and (4.13) becomes the problem of finding the optimal stream of dividends from the collection of assets until the time of bankruptcy; see [5, 11, 20].

Harvesting may occur instantaneously, so results in a singular stochastic control problem in the sense that the optimal harvesting strategy $Z(t)$ may not be absolutely continuous with respect to the Lebesgue measure of the time variable. For instance, if the discounted reward and noise intensity are sufficiently large, driving the population to extinction instantly or chattering harvesting strategies might be optimal or near-optimal; see [1, 53, 59]. Similarly, for insurance problems, insurance companies may distribute dividends on discrete-time intervals resulting in unbounded payment rate. In other words, in contrast to regular stochastic control problems, in which the displacement of the state due to control is differentiable in time, the harvesting problem considered in this work allows the displacement to be discontinuous. To find the value function and the harvesting strategy, one usually solves a so-called Hamilton-Jacobi-Bellman (HJB) equation. However, for singular control problems with regime switching, the HJB equation is in fact a coupled system of nonlinear quasi-variational inequalities. A closed-form solution is virtually impossible to obtain. The Markov chain approximation methodology developed by Kushner and Dupuis [28] becomes a viable alternative. As pointed out in [28], a probabilistic approach using the Markov chain approximation method for controlled diffusions has the following advantages. First, the Markov chain approximation method allows one to use physical insights derived from the dynamics of the controlled diffusion in obtaining a suitable approximation scheme. Second,

the Markov chain approximation method does not require much regularity of the controlled processes (solutions of the controlled stochastic differential equations) nor does it rely on the uniqueness properties of the associated HJB equations. This is particularly appealing when the not much information concerning the regularity of the associated PDEs is known. Though it is important to develop methods for numerical approximations for singular control problems, the results are still scare. For singular controlled diffusions without regime switching, Budhiraja and Ross [9] and Kushner and Martins [27] are two of the representative works that carry out a convergence analysis using weak convergence and relaxed control formulation for singular control problems in the setting of Itô diffusions. Recently, some works have been devoted to numerical methods for singular controls with regime switching. Jin, Yin, and Zhu [20] developed numerical algorithms for finding optimal dividend pay-out and reinsurance policies under a generalized singular control formulation. A numerical algorithm for optimal dividend payment and investment strategies of regime-switching jump diffusion models with capital injections was then introduced in Jin and Yin [21].

In our work, we focus on the harvesting problem for a partially observed system with a hidden Markov chain. So far, the work on numerical solutions has mostly concentrated on the case the Markov chain being observable. In reality, the environment (Markov chain) can often be only observed with noise. That is, at any given instance, the exact state of residency of the Markov chain is not known. Thus, we cannot see $\alpha(t)$ directly but only have noise-corrupted observation in the form of $\alpha(t)$ plus noise. An effective way to handle control problems of such partially observed systems is to converted them to completely observed ones, which can be done by using a Wonham filter (see, for example, [63]). In the literature, the Wonham filters have been used widely to investigate control problems with partial observations; see [58, 64]

for applications in engineering, finance, and ecology.

Compared to the aforementioned works on numerical methods for singular control problems, in the current work, we take a step towards more useful and realistic model where the Markov chain is unobservable. Although main ideas developed are crucial to the analysis of the current work, there are key differences in the model that make our analysis more delicate. Using a Wonham filter, we convert the partially observed system into a fully observed controlled diffusion. We then design approximation procedures for the optimal strategies and the value function. We need to use a couple of step sizes $h = (h_1, h_2)$. The parameter $h_1 > 0$ is a discretization parameter for state variables, and $h_2 > 0$ is the step size for time variable. In the actual computing, the computations are involved due to the presence of the Wonham filter.

In contrast to the existing results, our new contributions in this chapter are as follows.

(i) We use Wonham's filter to formulate the harvesting problem in random environments when the Markov chain is only observable in white Gaussian noise.

(ii) We convert the partially observed system to a fully observed system by replacing the unknown Markovian states by their posterior probability estimates.

(iii) We develop numerical approximation schemes based on the Markov chain approximation method. Although Markov chain approximation techniques have been used extensively in various control problems, the work on combination of such method for a singular control problem with partial observation seems to be scarce to the best of our knowledge.

The rest of this chapter is organized as follows. Section 4.2 begins with the problem formula-

tion. Section 4.3 presents the numerical algorithm based on the Markov chain approximation method. In Section 4.4, we establish the convergence of the algorithm. Finally, the chapter is concluded with a numerical example for illustration; some further remarks are also provided.

4.2 Formulation

For $i = 1, \ldots, r$, let $X^i(t)$ be the population size of the ith species in the ecosystem at time t and denote $X(t) = \big(X^1(t), \ldots, X^r(t)\big)' \in \mathbb{R}^r$ (with z' denoting the transpose of $z \in \mathbb{R}^{r_1 \times r_2}$ with $r_1, r_2 \geq 1$). Suppose that species $X^i(t)$ live in random environments. In addition to the random fluctuations of the population, we also assume that the growth of the species is subject to abrupt changes within a finite number of configurations of the environment. For simplicity, we assume that the switching among different environments is memoryless and that the waiting time for the next switch is exponentially distributed. In fact, this phenomenon is frequently observed in nature; see [52, 68]. Thus we can model the random environments and other random factors in the ecological system by a continuous-time Markov chain $\alpha(t)$ taking values in $\mathcal{M} = \{1, 2, \ldots, m\}$ with the generator given by $Q = (q^{ij}) \in \mathbb{R}^{m \times m}$. Assume throughout this chapter that both the Markov chain $\alpha(t)$ and the r-dimensional standard Wiener process $w(\cdot) = \big(w^1(\cdot), \ldots, w^r(\cdot)\big)'$ are defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}(t), P)$, where $\{\mathcal{F}(t)\}$ is a filtration satisfying the usual conditions (i.e., it is right continuous, increasing, and $\mathcal{F}(0)$ contains all the null sets).

In an effort to capture the salient feature that continuous dynamics and discrete events coexist in the ecosystem, we model the evolution in the absence of harvesting by the stochastic differential equation

$$dX(t) = b(X(t), \alpha(t))dt + \sigma(X(t), \alpha(t))dw(t), \quad X(0) = x_0 \in \mathbb{R}^r_+, \alpha(0) = \alpha_0 \in \mathcal{M}, \quad (4.1)$$

where $b(\cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^r$, $\sigma(\cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^{r\times r}$ are suitable functions. Furthermore, we assume that the Brownian motion $w(\cdot)$ and the Markov chain $\alpha(\cdot)$ are independent, a commonly used assumption in the literature.

We attempt to answer the question: Can we solve optimal harvesting problems if the Markov chain is hidden and we can only treat a partially observed system? In particular, we cannot see $\alpha(t)$ directly but only have noise-corrupted observation in the form of $\alpha(t)$ plus noise. That is, we can observe the following process

$$dy(t) = g(\alpha(t))dt + \sigma_0 dB(t), \quad y(0) = 0, \tag{4.2}$$

where σ_0 is a positive constant, $g : \mathcal{M} \mapsto \mathbb{R}$ is a one-to-one function, $B(t)$ is a one-dimensional standard Brownian motion being independent of $w(t)$ and $\alpha(t)$.

To proceed, we denote by $1\!\!1_E$ the indicator function of the event E, and use the following notation throughout this chapter.

$$\begin{aligned} p^j(t) &:= 1\!\!1_{\{\alpha(t)=j\}}, \quad j = 1, \ldots, m, \\ \varphi^j(t) &:= P\left(\alpha(t) = j | y(s), 0 \le s \le t\right), \quad j = 1, \ldots, m. \end{aligned} \tag{4.3}$$

Since $\varphi^j(t)$ is the probability vector conditioned on the observation $\sigma\{y(s), 0 \le s \le t\}$, $\varphi^j(t) \ge 0$ and $\sum_{j=1}^{m} \varphi^j(t) = 1$. Based on this property, it is sufficient to work with $\varphi(t) := \left(\varphi^1(t), \ldots, \varphi^{m-1}(t)\right)'$. Such approach helps us to reduce one dimension in the actual computation. The actual state space for $\varphi(t)$ is

$$S_{m-1} := \{\varphi := (\varphi^1, \ldots, \varphi^{m-1})' \in \mathbb{R}^{m-1} : \varphi^j \ge 0, \sum_{j=1}^{m-1} \varphi^j \le 1\}. \tag{4.4}$$

For given functions $b(\cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^r$, $\sigma(\cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^{r\times r}$, and $g : \mathcal{M} \to \mathbb{R}$, we define

$$\bar{b}(x, \varphi) = \sum_{j=1}^{m} \varphi^j b(x, j), \quad \bar{\sigma}(x, \varphi) = \sum_{j=1}^{m} \varphi^j \sigma(x, j), \quad \bar{g}(\varphi) = \sum_{j=1}^{m} \varphi^j g(j),$$

for each $(x,\varphi)\in\mathbb{R}^r\times S_{m-1}$, where S_{m-1} was defined in (4.4) and $\varphi^m=1-\sum\limits_{i=1}^{m-1}\varphi^i$.

We first recall some results of Wonham's filter. As was mentioned, the Markov chain $\alpha(t)$ is observed through (4.2). It was proved in [63] that the posterior probability $\varphi^j(\cdot)$ satisfies

$$\begin{cases} d\varphi^j(t)=\Big[\sum\limits_{k=1}^m q^{ki}\varphi^k(t)-\sigma_0^{-2}\big(g(j)-\overline{g}(\varphi(t))\big)\overline{g}(\varphi(t))\varphi^j(t)\Big]dt \\ \qquad\qquad +\sigma_0^{-2}\big(g(j)-\overline{g}(\varphi(t))\big)\varphi^j(t)dy(t),\quad j=1,\dots,m, \\ \varphi^j(0)=\varphi_0^j,\quad j=1,\dots,m, \end{cases} \tag{4.5}$$

where $(\varphi_0^1,\dots,\varphi_0^m)'$ is the initial distribution of $\alpha(t)$. Introduce a one dimensional innovation process

$$d\overline{w}(t)=\sigma_0^{-1}\big(dy(t)-\overline{g}(\varphi(t))dt\big),\quad \overline{w}(0)=0.$$

Then the first $m-1$ equations in (4.5) can be rewritten as

$$d\varphi^j(t)=\sum_{k=1}^m q^{kj}\varphi^k(t)dt+\sigma_0^{-1}\varphi^j(t)\big(g(j)-\overline{g}(\varphi(t))\big)d\overline{w}(t),\quad j=1,\dots,m-1. \tag{4.6}$$

With the use of (4.3), (4.1) can be written as

$$X(t)=x_0+\int_0^t\sum_{j=1}^m p^j(s)b(X(s),j)ds+\int_0^t\sum_{j=1}^m p^j(s)\sigma(X(s),j)dw(s), \tag{4.7}$$

Replacing the hidden state $p^j(t)$ by its estimate $\varphi^j(t)$ in (4.7), we arrive at

$$X(t)=x_0+\int_0^t\sum_{j=1}^m \varphi^j(s)b(X(s),j)ds+\int_0^t\sum_{j=1}^m \varphi^j(s)\sigma(X(s),j)dw(s),$$

i.e.,

$$X(t)=x_0+\int_0^t\overline{b}(X(s),\varphi(t))ds+\int_0^t\overline{\sigma}(X(s),\varphi(t))dw(s), \tag{4.8}$$

Let $Z(t)=(Z^1(t),\dots,Z^r(t))'\in\mathbb{R}^r$, where $Z^i(t)$ denote the total number harvested (to be defined shortly) from the species i up to time t. Then $\xi(t)=(\xi^1(t),\dots,\xi^r(t))'\in\mathbb{R}^r$, the

population size of the harvested population, satisfies

$$\xi(t) = x_0 + \int_0^t \overline{b}(\xi(s), \varphi(t))ds + \int_0^t \overline{\sigma}(\xi(s), \varphi(t))dw(s) - Z(t), \tag{4.9}$$

with initial data

$$\xi(0-) = x_0 \in \mathbb{R}^r_+, \quad \varphi(0) = \varphi_0 \in S_{m-1}. \tag{4.10}$$

At each time t, $\xi(t-)$ is the state before harvest starting at time t, while $\xi(t)$ is the state immediately after. Hence $\xi(0)$ may not be equal to $\xi(0-)$ due to an instantaneous harvest $Z(0)$ at time 0. Throughout the work we use the convention that $Z(0-) = 0$. If Z consists of an immediate harvest at time t, then this jump size is denoted by $\Delta Z(t) := Z(t) - Z(t-)$, and $Z^c(t) := Z(t) - \sum_{0 \le s \le t} \Delta Z(s)$ denotes the continuous part of Z. Since $Z(\cdot)$ is not required to be absolutely continuous with respect to the Lebesgue measure of the time variable, it is referred to as singular control. Also note that $\Delta\xi(t) := \xi(t) - \xi(t-) = -\Delta Z(t)$ for any $t \ge 0$. Hence (4.6) and (4.9) form a controlled process $(\xi(t), \varphi(t)) \in \mathbb{R}^r \times S_{m-1}$ with complete observation and the initial condition (x_0, φ_0).

An admissible harvesting strategy is a stochastic process $Z(t)$ satisfying the following conditions:

(a) $Z(t)$ is right continuous, nonnegative, and nondecreasing with respect to t,

(b) $Z(t)$ is adapted to $\sigma\{w(s), \varphi(s) : 0 \le s \le t\}$, augmented by the P-null sets,

(c) $J(x_0, \varphi_0, Z) < \infty$, for any $(x_0, \varphi_0) \in \mathbb{R}^r_+ \times S_{m-1}$, where $J(\cdot)$ is the functional defined below.

(d) $\xi(t) \ge 0$, for any $t \le \tau$, where $\tau = \inf\{s \ge 0 : \xi_i(s) = 0, \text{ for all } i = 1, \ldots, r\}$ is the extinction time of the system.

Note that τ might be infinite. If $\tau = \infty$ a.s., then the corresponding harvesting strategy belongs to the class of singular control with state constraints [54]. Let $\mathcal{A}_{x_0,\varphi_0}$ denote the collection of all admissible harvesting strategies with initial conditions given by (4.10).

Let $f^i(\cdot) : \mathcal{M} \mapsto \mathbb{R}_+ := (0, \infty)$ represent the instantaneous marginal yields accrued from exerting the harvesting strategy Z^i for species i, also known as the price of species i. Define $\overline{f}^i(\varphi) = \sum_{j=1}^m \varphi^j f^i(j)$ for each $\varphi \in S_{m-1}$, with $\varphi^m = 1 - \sum_{j=1}^{m-1} \varphi^j$. Then $\mathbb{R}_+$ is regarded as the survival set of each species and we impose $\xi^i(t) = 0$ for $t \geq \inf\{s \geq 0 : \xi^i(s) = 0\}$. For a fixed harvesting process $Z \in \mathcal{A}_{x_0,\varphi_0}$, the expected total discounted reward from harvesting is defined by

$$J(x_0, \varphi_0, Z) := \sum_{i=1}^r E_{x_0,\varphi_0} \int_0^\tau e^{-as} \overline{f}^i(\varphi(s)) dZ^i(s), \tag{4.11}$$

where $a > 0$ is the discounting factor and E_{x_0,φ_0} denotes the expectation with respect to the probability law when the process $(\xi(t), \varphi(t))$ starts with initial condition (x_0, φ_0). The goal is to maximize the expected total discounted reward from harvesting and find an optimal harvesting strategy Z^* such that

$$J(x_0, \varphi_0, Z^*) = V(x_0, \varphi_0) := \sup_{Z \in \mathcal{A}_{x_0,\varphi_0}} J(x_0, \varphi_0, Z). \tag{4.12}$$

To proceed, we introduce the generator of the combined process $(X(t), \varphi(t))$. For any twice continuously differentiable function $W(\cdot, \cdot) : \mathbb{R}^r \times S_{m-1} \mapsto \mathbb{R}$, we define

$$\begin{aligned}\mathcal{L}W(x, \varphi) = &\sum_{i=1}^r W_{x_i}(x, \varphi) \overline{b}^i(x, \varphi) + \frac{1}{2} \sum_{i,j=1}^r \overline{a}^{ij}(x, \varphi) W_{x^i x^j}(x, \varphi) + \sum_{j=1}^{m-1} W_{\varphi^j}(x, \varphi) \sum_{k=1}^m q^{kj} \varphi^k \\ &+ \frac{1}{2\sigma_0^2} \sum_{j=1}^{m-1} \left[\sum_{k=1}^{m-1} \varphi^j \varphi^k \Big(g(j) - \overline{g}(\varphi) \Big) \Big(g(k) - \overline{g}(\varphi) \Big) \right] W_{\varphi^j \varphi^k}(x, \varphi),\end{aligned} \tag{4.13}$$

where

$$\overline{b}(x,\varphi) = \big(\overline{b}^1(x,\varphi),\ldots,\overline{b}^r(x,\varphi)\big)' \in \mathbb{R}^r, \quad \overline{a}(x,\varphi) = (\overline{a}^{ij}(x,\varphi)) = \overline{\sigma}(x,\varphi)\overline{\sigma}'(x,\varphi) \in \mathbb{R}^{r\times r}.$$

Our standing assumptions are as follows.

(A1) $\overline{b}(\cdot,\cdot)$ and $\overline{\sigma}(\cdot,\cdot)$ satisfy the usual local Lipschitz condition and there exists a positive constant C_1 such that

$$x'\overline{b}(x,\varphi) + |\overline{\sigma}(x,\varphi)|^2 \le C_1(|x|^2+1) \quad \text{for all} \quad (x,\varphi) \in \mathbb{R}^r \times S_{m-1}. \tag{4.14}$$

(A2) There exists a positive constant C_2 such that $\overline{a}(x,\varphi) - C_2 I_r$ is positive definite for each $(x,\varphi) \in \mathbb{R}^r \times S_{m-1}$, and

$$\overline{a}^{ii}(x,\varphi) - \sum_{j:j\neq i} |\overline{a}^{ij}(x,\varphi)| \ge 0 \quad \text{for all } (x,\varphi) \in \mathbb{R}^r \times S_{m-1}, i = 1,\ldots,r, \tag{4.15}$$

where I_r is the $r \times r$ identity matrix.

Under (A1), for any initial condition $(x_0,\varphi_0) \in \mathbb{R}^n_+ \times S_{m-1}$, the system (4.6)-(4.8) has the unique global solution $(x(t),\varphi(t)) \in \mathbb{R}^r \times S_{m-1}$ for all $t \ge 0$ (see [68]). Assumption (A2) is imposed for convenience. There are several ways of relaxing the condition (4.15) for which we refer to [27, p.110]. Meanwhile, the first condition in (A2) is a non-degeneracy requirement for the diffusion part. If it does not hold, one can use a trick in [27, p.288-289] which requires more complex notation and the use of another Brownian motion.

Before proceeding further, recall that if the value functions are sufficiently smooth, they are solutions of the following system of HJB equations (see [53])

$$\begin{aligned} &\max\Big(\big(\mathcal{L}-r\big)W(x,\varphi), \overline{f}^i(\varphi) - W_{x^i}(x,\varphi), i = 1,\ldots,r\Big) = 0, \quad (x,\varphi) \in \mathbb{R}^r_+ \times S_{m-1}, \\ &W(x,\varphi) = 0, \quad (x,\varphi) = \{0\} \times S_{m-1}. \end{aligned} \tag{4.16}$$

Using the same argument as in [53, Theorem 2.1], one can also establish a verification theorem that leads to an optimal harvesting strategy. However, due to the presence of partial observation, it is very difficult to find value functions and optimal harvesting policies explicitly except for some special cases. Our task to follow is to construct a numerical procedure for solving the optimal control problem.

4.3 Numerical Algorithm

The basic idea behind the numerical method is to find a controlled Markov chain in discrete time to approximate the controlled diffusions. The method is similar to [9,20,27,28]. However, some important modifications are required due to the presence of a combination of singular control and partial observation.

4.3.1 Approximation Algorithm of the Wonham Filter

In this section, we deal with the numerical algorithms for the two components system. We begin with the Wonham filter equations. To construct approximation algorithms, one may wish to discretize the stochastic differential equations (4.5) directly. However, such a procedure is numerically unstable due to the white noise perturbations. It may produce a non-probability vector (e.g., some components might be less than 0 or the sum of the components might not equal to 1). To overcome this difficulty, we use the approach suggested in [67, Section 8.4], in which a logarithm transformation is used to transform the stochastic differential equations and then a numerical procedure for the transformed system is constructed.

Define

$$v^j(t) := \ln \varphi^j(t) \quad \text{for} \quad t \geq 0, \quad j = 1, \ldots, m.$$

It follows that $\varphi^j(t) = e^{v^j(t)}$. An application of Itô's formula to (4.6) leads to that for each $j = 1, \ldots, m$,

$$\begin{cases} dv^j(t) = \Big[\sum_{k=1}^{m} q^{kj} \dfrac{\varphi^k(t)}{\varphi^j(t)} - \dfrac{1}{2\sigma_0^2}\big(g(j) - \overline{g}\big(\varphi(t)\big)\big)^2\Big]dt + \dfrac{1}{2\sigma_0}\big(g(j) - \overline{g}\big(\varphi(t)\big)\big)d\overline{w}(t), \\ v^j(0) \ = \ln(\varphi_0^j). \end{cases} \tag{4.17}$$

We use the constant step size $h_2 > 0$ for time variable. There are a couple of ways to construct discrete-time approximation of Wonham's filters. One possibility is the approach in [67] (see Section 8.4). Here employing the approximation algorithm constructed in [64], we can discretize (4.17) as follows.

$$\begin{cases} v_0^{h_2,j} \ = \ln(\varphi_0^j), \qquad \varphi_0^{h_2,j} = \varphi_0^j, \\ \overline{g}_n^{h_2} \ = \sum_{k=1}^{m} g(k)\varphi_n^{h_2,k}, \\ r_n^{h_2,j} \ = \sum_{k=1}^{m} q^{kj} \dfrac{\varphi_n^{h_2,k}}{\varphi_n^{h_2,j}} - \dfrac{1}{2\sigma_0^2}\big(g(j) - \overline{g}_n^{h_2}\big)^2, \\ v_{n+1}^{h_2,j} \ = v_n^{h_2,j} + h_2 r_n^{h_2,j} + \dfrac{1}{\sigma_0}\big(g(j) - \overline{g}_n^{h_2}\big)\sqrt{h_2}\zeta_n, \\ \varphi_{n+1}^{h_2,j} = \exp\big(v_{n+1}^{h_2,j}\big) / \sum_{k=1}^{m} \exp\big(v_{n+1}^{h_2,k}\big), \end{cases} \tag{4.18}$$

where $\{\zeta_n\}$ is a sequence of independent and identically distributed random variables.

Let $\{\varphi_n^{h_2}\}$ be the sequence of discretized posterior probabilities in which $\varphi_n^{h_2} = (\varphi_n^{h_2,1}, \ldots, \varphi_n^{h_2,m-1})'$. We use the last equation in (4.18) to reinforce that $\varphi_n^{h_2} \in S_{m-1}$ for each n. Such property is convenient for convergence verifications of the problem under consideration. Note that $\varphi_n^{h_2,j}$ appeared as the denominator in (4.18) and we have focused on the case that $\varphi_n^{h_2,j}$ stays away from 0. A modification can be made to take into consideration the case of $\varphi_n^{h_2,j} = 0$. This is done as follows. In lieu of (4.18), let M be a fixed but otherwise

arbitrarily large real number. Construct the approximation according to:

$$\begin{cases} v_0^{h_2,j} &= \ln(\varphi_0^j), \qquad \varphi_0^{h_2,j} = \varphi_0^j, \\ \overline{g}_n^{h_2} &= \sum_{k=1}^m g(k)\varphi_n^{h_2,k}, \\ r_n^{h_2,j} &= \Big[\sum_{k=1}^m q^{kj}\frac{\varphi_n^{h_2,k}}{\varphi_n^{h_2,j}} - \frac{1}{2\sigma_0^2}\big(g(j)-\overline{g}_n^{h_2}\big)^2\Big]\mathbb{1}_{\{\varphi_n^{h_2,k}\geq e^{-M}\}} - M\mathbb{1}_{\{\varphi_n^{h_2,k}<e^{-M}\}}, \\ v_{n+1}^{h_2,j} &= v_n^{h_2,j} + h_2 r_n^{h_2,j} + \frac{1}{\sigma_0}\big(g(j)-\overline{g}_n^{h_2}\big)\sqrt{h_2}\zeta_n, \\ \varphi_{n+1}^{h_2,j} &= \exp\big(v_{n+1}^{h_2,j}\big)/\sum_{k=1}^m \exp\big(v_{n+1}^{h_2,k}\big), \end{cases} \tag{4.19}$$

The conditions needed for the convergence analysis of the Wonham filter approximation are as follows.

(A3) $\{\zeta_n\}$ is a sequence of independent and identically distributed random variables satisfying

$$E\zeta_n = 0, \quad E\zeta_n^2 = 1, \quad E|\zeta_n|^{2+\gamma} < \infty, \text{ for some } \gamma > 0.$$

4.3.2 Approximation Algorithm of the Harvested Process

In what follows, we construct a discrete-time finite state Markov chain to approximate the harvested process $\xi(t)$. Let $h_1 > 0$ be a discretization parameter for state variable, and recall that $h_2 > 0$ is the step size for time variable that we used above. Denote $h = (h_1, h_2)$. In the later presentation, for simplicity, we also use φ^h for φ^{h_2}. By writing $h \to 0$, we mean $h = (h_1, h_2) \to (0, 0)$.

Let $U \in (0, \infty)$ be an upper bound introduced for numerical purpose only. Moreover, assume without loss of generality that the boundary point U is an integer multiple of h_1. Define

$$L_{h_1} := \{x = (k^1 h_1, \dots, k^r h_1)' \in \mathbb{R}^r : k^i = 0, \pm 1, \pm 2, \dots\} \cap [0, U + h_1]^r.$$

Let $\{\xi_n^h : n = 0, 1, \dots\}$ be a discrete-time controlled Markov chain with state space L_{h_1} such that the controlled Markov chain well approximates the local behavior of the controlled diffusion (4.9). We define the difference

$$\Delta\xi_n^h = \xi_{n+1}^h - \xi_n^h.$$

At any discrete-time step n, we can either exercise a harvesting action, a reflection action, or not to apply a control. Each of these is described precisely in what follows. If we do not apply a control, then the increment $\Delta\xi_n^h$ is to "behave like" an increment of $\int \overline{b}dt + \int \overline{\sigma}dw$ over a small time interval. We call this a "diffusion step". We can write

$$\Delta\xi_n^h = \Delta\xi_n^h I_{\{\text{diffusion step at } n\}} + \Delta\xi_n^h I_{\{\text{harvesting step at } n\}} + \Delta\xi_n^h I_{\{\text{reflection step at } n\}}. \tag{4.20}$$

The chain and the control will be chosen so that there is exactly one term on the right-hand side of (4.20) is positive. Let $\pi^h = (\pi_0^h, \pi_1^h, \dots)$ denote the sequence of control actions at time 0, 1, ... We take $\pi_n^h = -i$, 0, or i, if we exercise a reflection on species i, not to apply a control, or harvesting action on species i at time n, respectively. Let $\{\psi_n^h\}$ be the sequence of discretized posterior probabilities associated with $\{\xi_n^h\}$, to be defined shortly. Let $E_{x,\psi,n}^{h,\pi}$, $Cov_{x,\psi,n}^{h,\pi}$ denote the conditional expectation and covariance given by

$$\{\xi_k^h, \psi_k^h, \pi_k^h, k \leq n, \xi_n^h = x, \psi_n^h = \psi, \pi_n^h = \pi\},$$

respectively. By stating that $\{\xi_n^h\}$ is a controlled discrete-time Markov chain on a finite state space L_{h_1} with transition probabilities from state x to another state y, denoted by $p^h((x,y)|\pi,\psi)$, we mean that the transition probabilities are functions of control action π and posterior probability ψ. The sequence $\{\xi_n^h\}$ is said to be locally consistent with respect

to (4.9) if it satisfies

$$
\begin{aligned}
&E^{h,0}_{x,\psi,n}\Delta\xi^h_n = \overline{b}(x,\psi)h_2 + o(h_2), \quad Cov^{h,0}_{x,\psi,n}\Delta\xi^h_n = \overline{a}(x,\psi)h_2 + o(h_2),\\
&\sup_{n,\ \omega}|\Delta\xi^h_n| \to 0, \quad h \to 0.
\end{aligned}
\tag{4.21}
$$

We define $\xi^h_0 = x_0$ and $\psi^h_0 = \varphi_0$, where φ_0 is the initial value of the Wonham filter. For each $n \geq 0$, if $\pi^h_n = i$, we assume that the harvesting amount for species i at time n is h_1. Hence the harvesting amount for the chain at time n is $\Delta z^h_n = h_1 \sum_{i=1}^{r} \mathbf{e}_i \mathbb{1}_{\{\pi^h_n = i\}}$. If the i th component of ξ^h_n equals $U + h_1$ for some i, a reflection step on species $i_0 = \inf_i\{\xi^{h,i}_n = U + h_1\}$ is exerted definitely. i.e., $\pi^h_n = -i_0$. Moreover, we require that reflection takes the i_0 th component of the chain from $U + h_1$ to U. We denote by Δg^h_n the random vector that is the reflection amount for the chain at time n, then $\Delta g^h_n = h_1 \sum_{i=1}^{r} \mathbf{e}_i \mathbb{1}_{\{\pi^h_n = -i\}}$. To reflect the fact that reflection and harvesting terms change the population process instantaneously, we define $\psi^h_{n+1} = \psi^h_n$ if $\pi^h_n = i$ or $\pi^h_n = -i$. If $\pi^h_n = 0$ and $\psi^h_n = \varphi^h_k$ for some integer k, we define $\psi^h_{n+1} = \varphi^h_{k+1}$. Hence we have defined the sequences $\{\psi^h_n\}$ and $\{\xi^h_n\}$ recursively.

As described above, the control at each step, is specified by the choice of an action: diffusion, harvesting, or reflection. Denote $\mathcal{F}^h_n = \sigma\{\xi^h_k, \psi^h_k, \pi^h_k, k \leq n\}$. The sequence π^h is said to be admissible if π^h_n is $\sigma\{\xi^h_0, \ldots, \xi^h_n, \psi^h_0, \ldots, \psi^h_n, \pi^h_0, \ldots, \pi^h_{n-1}\} -$ adapted, and for any $x \in L_{h_1}$, we have

$$
\begin{aligned}
&P\{\xi^h_{n+1} = x|\mathcal{F}^h_n\} = P\{\xi^h_{n+1} = x|\xi^h_n, \psi^h_n, \pi^h_n\} = p^h(\xi^h_n, x|\psi^h_n, \pi^h_n),\\
&P\Big\{\pi^h_n = -\inf_i\{\xi^{h,i}_n = U + h_1\}\Big|\xi^{h,i}_n = U + h_1 \text{ for some } i, \mathcal{F}^h_n\Big\} = 1,\\
&P\{\pi^h_n = -i|\xi^{h,i}_n \leq U, \mathcal{F}^h_n\} = 0,
\end{aligned}
$$

where $\xi^{h,i}_n$ denote the i th component of the vector ξ^h_n. The class of all admissible control sequences for initial state (x_0, φ_0) will be denoted by $\mathcal{A}^h_{x_0,\varphi_0}$.

For each triple $(x, \psi, i) \in L_{h_1} \times S_{m-1} \times \{0, i, -i : i = 1, \ldots, r\}$, we first define a family of the interpolation intervals $\Delta t^h(x, \psi, i)$. For diffusion steps, if the state of the chain is x and the state of the discretized Wonham's filter is ψ, $\Delta t^h(x, \psi, i = 0)$ will be taken to be h_2; whereas for harvesting steps and reflection steps, $\Delta t^h(x, \psi, i)$ will be taken to be 0. This reflects the fact that for the controlled diffusion process, reflection and harvesting terms can change the state instantaneously. Therefore, we define

$$\begin{aligned} &\Delta t^h(x, \psi, i) = h_2 I_{\{i=0\}}, \\ &t_0^h = 0, \\ &\Delta t_k^h = \Delta t^h(\xi_k^h, \psi_k^h, \pi_k^h), \\ &t_n^h = \sum_{k=0}^{n-1} \Delta t_k^h. \end{aligned} \tag{4.22}$$

Let

$$\eta_h := \inf\{n : \xi_n^h = 0 \in L_{h_1}\}, \quad \tau_h := t_{\eta_h}^h.$$

For $(x_0, \varphi_0) \in L_{h_1} \times S_{m-1}$ and π^h is admissible, the cost function for the controlled Markov chain is defined as

$$J^h(x_0, \varphi_0, z^h) = \sum_{i=1}^{r} E \sum_{k=1}^{\eta^h - 1} e^{-a t_k^h} \overline{f}^i(\psi_k^h) \Delta z_k^{h,i}, \tag{4.23}$$

where $\Delta z_k^h = \left(\Delta z_k^{h,1}, \ldots, \Delta z_k^{h,r}\right)'$. The value function of the controlled Markov chain is

$$V^h(x_0, \varphi_0) = \sup_{z^h} J^h(x_0, \varphi_0, z^h). \tag{4.24}$$

4.3.3 Transition Probabilities and Local Consistency

Let $\mathbf{e_i} \in \mathbb{R}^r$ be the standard unit vector in the i th direction, $i = 1, \ldots, r$. Now we define the approximation to the first and the second derivatives of $V(\cdot, \cdot)$ by finite difference method

using stepzise $h_1 > 0$ for the state variable as

$$
\begin{aligned}
&V(x,\varphi) \to V^h(x,\varphi),\\
&V_{x^i}(x,\varphi) \to \frac{V^h(x+h_1\mathbf{e_i},\varphi) - V^h(x,\varphi)}{h_1}, \quad \text{if } \overline{b}^i(x,\varphi) \geq 0,\\
&V_{x^i}(x,\varphi) \to \frac{V^h(x,\varphi) - V^h(x-h_1\mathbf{e_i},\varphi)}{h_1}, \quad \text{if } \overline{b}^i(x,\varphi) < 0,\\
&V_{x^ix^i}(x,\varphi) \to \frac{V^h(x+h_1\mathbf{e_i},\varphi) - 2V^h(x,\varphi) + V^h(x-h_1\mathbf{e_i},\varphi)}{h_1^2},\\
&V_{x^ix^j}(x,\varphi) \to \frac{2V^h(x,\varphi) + V^h(x+h_1\mathbf{e_i}+h_1\mathbf{e_j},\varphi) + V^h(x-h_1\mathbf{e_i}-h_1\mathbf{e_j},\varphi)}{2h_1^2}\\
&\quad -\frac{V^h(x+h_1\mathbf{e_i},\varphi) + V^h(x-h_1\mathbf{e_i},\varphi) + V^h(x+h_1\mathbf{e_j},\varphi) + V^h(x-h_1\mathbf{e_j},\varphi)}{2h_1^2}, \quad \text{if } \overline{a}^{ij}(x,\varphi) \geq 0,\\
&V_{x^ix^j}(x,\varphi) \to -\frac{2V^h(x,\varphi) + V^h(x+h_1\mathbf{e_i}-h_1\mathbf{e_j},\varphi) + V^h(x-h_1\mathbf{e_i}+h_1\mathbf{e_j},\varphi)}{2h_1^2}\\
&\quad +\frac{V^h(x+h_1\mathbf{e_i},\varphi) + V^h(x-h_1\mathbf{e_i},\varphi) + V^h(x+h_1\mathbf{e_j},\varphi) + V^h(x-h_1\mathbf{e_j},\varphi)}{2h_1^2}, \quad \text{if } \overline{a}^{ij}(x,\varphi) < 0,
\end{aligned}
\tag{4.25}
$$

For the first and the second derivatives with respect to the posterior probability, we use similar approximations. We proceed to figure out the transition probabilities. We define for a real number a that $a^+ = \max\{a, 0\}$, $a^- = -\min\{0, a\}$. Then $a = a^+$ if $a \geq 0$ and $a = -a^-$ if $a < 0$. Moreover, $|a| = a^+ + a^-$ and $a = a^+ - a^-$. To find transition probabilities of the controlled Markov chain, we plug all the necessary expressions into the first part of system (4.16), then use the symmetry of the $\overline{a}(x,\varphi)$ matrix, combine like terms and divide by the coefficient of $V^h(x,\varphi)$. The transition probabilities are coefficients of the resulting equation. For $x \in L_{h_1}$ and $\psi \in S_{m-1}$, we define the transition probabilities at diffusion steps in the following way,

$$
\begin{aligned}
p^h(x, x+h_1\mathbf{e_i}|\psi,\pi=0) &= \frac{\Big(\overline{a}^{ii}(x,\psi)/2 - \sum\limits_{j:j\neq i} |\overline{a}^{ij}(x,\psi)|/2 + \overline{b}^{i+}(x,\psi)h_1\Big)h_2}{h_1^2},\\
p^h(x, x-h_1\mathbf{e_i}|\psi,\pi=0) &= \frac{\Big(\overline{a}^{ii}(x,\psi)/2 - \sum\limits_{j:j\neq i} |\overline{a}^{ij}(x,\psi)|/2 + \overline{b}^{i-}(x,\psi)h_1\Big)h_2}{h_1^2},
\end{aligned}
\tag{4.26}
$$

$$p^h(x, x + h_1\mathbf{e_i} + h_1\mathbf{e_j}|\psi, \pi = 0) = p^h(x, x - h_1\mathbf{e_i} - h_1\mathbf{e_j}|\psi, \pi = 0) = \frac{\overline{a}^{ij+}(x,\psi)h_2}{2h_1^2},$$
$$p^h(x, x + h_1\mathbf{e_i} - h_1\mathbf{e_j}|\psi, \pi = 0) = p^h(x, x - h_1\mathbf{e_i} + h_1\mathbf{e_j}|\psi, \pi = 0) = \frac{\overline{a}^{ij^-}(x,\psi)h_2}{2h_1^2},$$
$$p^h(x, x|\psi, \pi = 0) = 1 - \sum p^h(x, x \pm h_1\mathbf{e_i}|\psi, \pi = 0) - \sum p^h(x, x + h_1\mathbf{e_i} \pm h_1\mathbf{e_j}|\psi, \pi = 0). \tag{4.27}$$

Assumption (A2) guarantees that $p^h(x, x + h_1\mathbf{e_i}|\psi, \pi = 0) \geq 0$ and $p^h(x, x - h_1\mathbf{e_i}|\psi, \pi = 0) \geq 0$ for all $(x, \psi) \in L_{h_1} \times S_{m-1}$. Moreover, by choosing proper h_1 and h_2 (for instance, $h_2 = o(h_1^2)$), we can reasonably assume that $p^h(x, x|\psi, \pi = 0) \geq 0$, that is, the transition probabilities in (4.26) and (4.27) are well-defined. At reflection steps and harvesting steps, we define

$$p^h(x, x - h_1\mathbf{e_i})|\psi, \pi = \pm i) = 1. \tag{4.28}$$

The definition of the transition function at 0 is not important since in the analysis of the control problem, the chain will be stopped the first time it hits 0. For the sake of specificity, we set $\overline{a}^{ij}(x,\psi) = \overline{a}^{ji}(x,\psi) = \overline{b}^i(x,\psi) = 0$ for all $j = 1, \ldots, r$ if $x^i = 0$ and $p^h(0, 0|\psi, \pi = 0) = 1$.

Let $\xi_n^{h,i}$ denote the i th component of the vector ξ_n^h. Using the above transition probabilities, we have

$$\begin{aligned} E_{x,\psi,n}^{h,0}\Delta\xi_n^{h,i} &= E_{x,\psi,n}^{h,0}(\xi_{n+1}^{h,i} - \xi_n^{h,i}) \\ &= \overline{b}^i(x,\psi)h_2, \end{aligned} \tag{4.29}$$

for $i \neq j$,

$$\begin{aligned} E_{x,\psi,n}^{h,0}\big(\Delta\xi_n^{h,i}\Delta\xi_n^{h,j}\big) &= \frac{2\overline{a}^{ij^+}(x,\psi)h_2}{2h_1^2}h_1^2 - \frac{2\overline{a}^{ij^-}(x,\psi)h_2}{2h_1^2}h_1^2 \\ &= \overline{a}^{ij}(x,\psi)h_2, \end{aligned} \tag{4.30}$$

and $E_{x,\psi,n}^{h,0}\big(\Delta\xi_n^{h,i}\big)^2 = \overline{a}^{ii}(x,\psi)h_2 + o(h_2)$, when $h \to 0$. Note that $\overline{b}(\cdot,\cdot)$ is bounded on $L_{h_1} \times S_{m-1}$. The local consistence of the controlled Markov chain $\{\xi_n^h\}$ with transition probabilities

defined in (4.26) and (4.27) follows.

4.4 Convergence

4.4.1 Continuous-Time Interpolation and Time Rescaling

One of the main goals of the study is to show that the value function of the controlled Markov chain defined in (4.24) converges, as $h = (h_1, h_2) \to (0, 0)$, to the value function of the limit control problem. This convergence result allows for the computation of near optimal policies for the control problem by numerical method. We next introduce the continuous-time interpolation and time rescaling techniques that will be used in the proof of our main convergence result.

The continuous-time interpolations of various processes will be constructed to be piecewise constant on the time interval $[t_n^h, t_{n+1}^h), n \geq 0$. For use in this construction, we define $n^h(t) = \max\{n : t_n^h \leq t\}, t \geq 0$. We first define discrete time processes associated with the controlled Markov chain as follows. Let $z_0^h = g_0^h = B_0^h = M_0^h = 0$ and define for $n \geq 1$,

$$\begin{aligned} z_n^h &= \sum_{k=0}^{n-1} \Delta z_k^h, \quad g_n^h = \sum_{k=0}^{n-1} \Delta g_k^h, \\ B_n^h &= \sum_{k=0}^{n-1} 1\!\!1_{\{\pi_k^h = 0\}} E_k^h \Delta \xi_k^h, \quad M_n^h = \sum_{k=0}^{n-1} (\Delta \xi_k^h - E_k^h \Delta \xi_k^h) 1\!\!1_{\{\pi_k^h = 0\}}. \end{aligned} \tag{4.31}$$

The piecewise constant interpolations, denoted by $(\xi^h(\cdot), \psi^h(\cdot), z^h(\cdot), g^h(\cdot), B^h(\cdot), M^h(\cdot))$ are naturally defined as

$$\begin{aligned} &\xi^h(t) = \xi_{n^h(t)}^h, \quad \psi^h(t) = \psi_{n^h(t)}^h, \\ &z^h(t) = z_{n^h(t)}^h, \quad g^h(t) = g_{n^h(t)}^h, \\ &B^h(t) = B_{n^h(t)}^h, \quad M^h(t) = M_{n^h(t)}^h \quad t \geq 0. \end{aligned} \tag{4.32}$$

Define $\mathcal{F}^h(t) = \sigma\{\xi^h(s), \psi^h(s), g^h(s), z^h(s) : s \leq t\} = \mathcal{F}_{n^h(t)}^h$. Using the representation of

diffusion steps, harvesting steps, reflection steps in (4.20), we obtain

$$\xi_n^h = x_0 + \sum_{k=0}^{n-1} \Delta\xi_k^h \mathbb{1}_{\{\pi_k^h \le -1\}} + \sum_{k=0}^{n-1} \Delta\xi_k^h \mathbb{1}_{\{\pi_k^h \ge 1\}} + \sum_{k=0}^{n-1} \Delta\xi_k^h \mathbb{1}_{\{\pi_k^h = 0\}} \tag{4.33}$$

Using the interpolations defined above, we have

$$\xi^h(t) = x_0 + B^h(t) + M^h(t) - z^h(t) - g^h(t). \tag{4.34}$$

Recall that $\Delta t_k^h = h_2$ if $\pi_k^h = 0$ and $\Delta t_k^h = 0$ if $\pi_k^h \ge 1$ or $\pi_k^h \le -1$. It follows that

$$\begin{aligned} B^h(t) &= \sum_{k=0}^{n^h(t)-1} \overline{b}(\xi_k^h, \psi_k^h)\Delta t_k^h \\ &= \int_0^t \overline{b}(\xi^h(s), \psi^h(s))ds - \int_{t_{n^h(t)}^h}^t \overline{b}(\xi^h(s), \psi^h(s))ds \\ &= \int_0^t \overline{b}(\xi^h(s), \psi^h(s))ds + \varepsilon_1^h(t), \end{aligned} \tag{4.35}$$

with $\{\varepsilon_1^h(\cdot)\}$ is an $\mathcal{F}^h(t)$-adapted process satisfying

$$\lim_{h\to 0} \sup_{t\in[0,T_0]} E|\varepsilon_1^h(t)| = 0, \qquad \text{for any } 0 < T_0 < \infty.$$

We now attempt to represent $M^h(\cdot)$ in a form similar to the diffusion term in (4.9). Factor

$$\overline{a}(x,\varphi) = \overline{\sigma}(x,\varphi)\overline{\sigma}'(x,\varphi) = P(x,\varphi)D^2(x,\varphi)P'(x,\varphi),$$

where $P(\cdot)$ is an orthogonal matrix, $D(\cdot) = \text{diag}\{d^1(\cdot), \ldots, d^r(\cdot)\}$. By assumption (A2), $\inf_{(x,\varphi)} d^i(x,\varphi) > 0$ for all $i = 1, \ldots, r$. Define $D_0(\cdot) = \text{diag}\{1/d^1(\cdot), \ldots, 1/d^r(\cdot)\}$. Define

$$\begin{aligned} w^h(t) &= \int_0^t D_0(\xi^h(s), \psi^h(s))P'(\xi^h(s), \psi^h(s))dM^h(s) \\ &= \sum_{k=0}^{n^h(t)-1} D_0(\xi_k^h, \psi_k^h)P'(\xi_k^h, \psi_k^h)(\Delta\xi_k^h - E_k^h\Delta\xi_k^h)\mathbb{1}_{\{\pi_k^h=0\}}. \end{aligned} \tag{4.36}$$

Then we can write

$$M^h(t) = \int_0^t \overline{\sigma}(\xi^h(s), \psi^h(s))dw^h(s) + \varepsilon_2^h(t), \tag{4.37}$$

with $\{\varepsilon_2^h(\cdot)\}$ is an $\mathcal{F}^h(t)$-adapted process satisfying

$$\lim_{h\to 0}\sup_{t\in[0,T_0]} E|\varepsilon_2^h(t)| = 0, \qquad \text{for any } 0 < T_0 < \infty.$$

Using (4.35) and (4.37), we can write (4.34) as

$$\xi^h(t) = x_0 + \int_0^t \overline{b}(\xi^h(s),\psi^h(s))ds + \int_0^t \overline{\sigma}(\xi^h(s),\psi^h(s))dw^h(s) - z^h(t) - g^h(t) + \varepsilon^h(t), \quad (4.38)$$

with $\varepsilon^h(\cdot)$ is an $\mathcal{F}^h(t)$-adapted process satisfying

$$\lim_{h\to 0}\sup_{t\in[0,T_0]} E|\varepsilon^h(t)| = 0, \qquad \text{for any } 0 < T_0 < \infty.$$

The modified dynamics of (4.9) corresponding to (4.38) is given by

$$\xi(t) = x_0 + \int_0^t \overline{b}(\xi(s),\varphi(t))ds + \int_0^t \overline{\sigma}(\xi(s),\varphi(t))dw(s) - z(t) - g(t), \quad (4.39)$$

with the presence of the reflection component $g(\cdot)$ and the harvesting component $z(\cdot)$.

The cost function in (4.23) can also be rewritten as

$$J^h(x_0,\varphi_0,z^h(\cdot)) = \sum_{i=1}^r E\int_0^{\tau_h} e^{-as}\overline{f}^i(\psi^h(s))dz^{h,i}(s), \quad (4.40)$$

where $z^h(\cdot) = (z^{h,1}(\cdot),\ldots,z^{h,r}(\cdot))'$. In fact, since we have defined $p^h(0,0|\psi,\pi=1) = 1$, we can rewrite (4.40) with $\tau_h = \infty$.

Time rescaling. Next we will introduce the time rescaling that will be used in our work. Our ultimate goal is to show that V^h converges to V in a large enough interval $[0,U]$ as $h = (h_1,h_2) \to (0,0)$. A common approach for proving the convergence of V^h to V is to begin by showing that the collection $\{\xi^h(\cdot),\psi^h(\cdot),w^h(\cdot),g^h(\cdot),z^h(\cdot)\}$ is tight, and then characterize the subsequential weak limits suitably. However, for singular control problems, showing the tightness of the above family becomes problematic since in general, the family $\{g^h(\cdot),z^h(\cdot)\}$ may fail to be tight. To overcome this difficulty, the analysis must be done in

a "stretched-out" time scale, analogously to the approach previously used by Kushner [27], Budhiraja and Ross [9] on singular control problems.

First the rescaled time increments $\{\Delta\widehat{t}_n^h : n = 0, 1, \dots\}$ are defined as follows

$$\begin{aligned} &\Delta\widehat{t}_n^h = h_2 \mathbb{1}_{\{\pi_n^h = 0\}} + h_1 \mathbb{1}_{\{\pi_n^h \le -1\}} + h_1 \mathbb{1}_{\{\pi_n^h \ge 1\}}, \\ &\widehat{t}_0 = 0, \qquad \widehat{t}_n = \sum_{k=0}^{n-1} \Delta\widehat{t}_k^h, \quad n \ge 1. \end{aligned} \tag{4.41}$$

The time scale is stretched out by h_1 at the reflection and harvesting steps.

Definition 4.1. The rescaled time process $\widehat{T}^h(\cdot)$ is the unique continuous nondecreasing process satisfying the following:

(a) $\widehat{T}^h(0) = 0$;

(b) the derivative of $\widehat{T}^h(\cdot)$ is 1 on $(\widehat{t}_n^h, \widehat{t}_{n+1}^h)$ if $\pi_n^h = 0$, i.e., n is a diffusion step;

(c) the derivative of $\widehat{T}^h(\cdot)$ is 0 on $(\widehat{t}_n^h, \widehat{t}_{n+1}^h)$ if $\pi_n^h \neq 0$, i.e., n is a reflection step or a harvesting step.

Thus $\widehat{T}^h(\cdot)$ does not increase at these t at which a harvesting step or a reflection step occurs. It follows from the above definition that

$$\widehat{T}^h(\widehat{t}_n) = t_n^h \quad \text{and} \quad \widehat{T}^h(\widehat{t}_{n+1}) - \widehat{T}^h(\widehat{t}_n) = \Delta t_n^h.$$

Moreover, for $t \ge 0$ and $\delta > 0$, $0 \le \widehat{T}^h(t+\delta) - \widehat{T}^h(t) \le \delta$. Define the rescaled and interpolated process $\widehat{\xi}^h(t) = \xi^h(\widehat{T}^h(t))$ and likewise define $\widehat{\psi}^h(\cdot)$, $\widehat{z}^h(\cdot)$, $\widehat{g}^h(\cdot)$, $\widehat{B}^h(\cdot)$, $\widehat{M}^h(\cdot)$, $\widehat{w}^h(\cdot)$, and the filtration $\widehat{\mathcal{F}}^h(\cdot)$ similarly. It follows from (4.34) that

$$\widehat{\xi}^h(t) = x_0 + \widehat{B}^h(t) + \widehat{M}^h(t) - \widehat{z}^h(t) - \widehat{g}^h(t). \tag{4.42}$$

Using the same argument that produced (4.38) we obtain

$$\widehat{\xi}^h(t) = x_0 + \int_0^t \overline{b}(\widehat{\xi}^h(s), \widehat{\psi}^h(s))d\widehat{T}^h(s) + \int_0^t \overline{\sigma}(\widehat{\xi}^h(s), \widehat{\psi}^h(s))d\widehat{w}^h(s) - \widehat{z}^h(t) - \widehat{g}^h(t) + \widehat{\varepsilon}^h(t), \tag{4.43}$$

with $\widehat{\varepsilon}^h(\cdot)$ is an $\widehat{\mathcal{F}}^h(\cdot)$-adapted process satisfying

$$\lim_{h\to 0} \sup_{t\in[0,T_0]} E|\widehat{\varepsilon}^h(t)| = 0, \qquad \text{for any } 0 < T_0 < \infty. \tag{4.44}$$

Denote

$$\widehat{H}^h(\cdot) = \Big(\widehat{\xi}^h(\cdot), \widehat{\psi}^h(\cdot), \widehat{w}^h(\cdot), \widehat{z}^h(\cdot), \widehat{g}^h(\cdot), \widehat{T}^h(\cdot)\Big), \quad h = (h_1, h_2).$$

To proceed, we give the definition of existence and uniqueness of weak solution and state some more assumptions.

Definition 4.2. By a weak solution of (4.6)-(4.39) we mean that there exists a probability space $(\Omega, \mathcal{F}, P)$, a filtration $\mathcal{F}(t)$, and process $\big(\xi(\cdot), \psi(\cdot), z(\cdot), g(\cdot), w(\cdot), \overline{w}(\cdot)\big)$ such that $w(\cdot)$ and $\overline{w}(\cdot)$ are independent $\mathcal{F}(t)$-Wiener processes, $z(\cdot)$ and $g(\cdot)$ are $\mathcal{F}(t)$-adapted, and (4.6)-(4.39) are satisfied. For an initial condition (x_0, φ_0), by the weak sense uniqueness, we mean that irrespective of probability space, the probability law of solution $\big(\xi(\cdot), \psi(\cdot), z(\cdot), g(\cdot), w(\cdot), \overline{w}(\cdot)\big)$ to (4.6)-(4.39) is determined by the probability law of $\big(\psi(\cdot), z(\cdot), w(\cdot), \overline{w}(\cdot)\big)$.

(A4) For each initial condition, there exists a solution to (4.6)-(4.39) and this solution is unique in the weak sense.

4.4.2 Proof of Convergence

In this subsection, we use the weak convergence methods to obtain the convergence of the algorithms. We refer the readers to [8, 16] for standard references and [26, 28] for a brief

account of concepts and results in the theory of weak convergence that we will use in the sequel. Let $D[0,\infty)$ denote the space of functions that are right continuous and have left-hand limits endowed with the Skorohod topology. All the weak analysis will be on this space or its k-fold products $D^k[0,\infty)$ for appropriate k.

Theorem 4.3. *Assume* (A1)-(A4). *Let the approximating chain* $\{\xi_n^h\}$ *be constructed with transition probabilities defined in* (4.26)-(4.28),

$$H^h(\cdot) = \Big(\xi^h(\cdot), \psi^h(\cdot), w^h(\cdot), z^h(\cdot), g^h(\cdot), T^h(\cdot)\Big)$$

be the continuous-time interpolation defined in (4.31)-(4.32), (4.36), *Definition 4.1, and* $\widehat{H}^h(\cdot)$ *be the corresponding rescaled processes. Then* $\widehat{H}^h(\cdot)$ *is tight. As a result,* $\widehat{H}^h(\cdot)$ *has a weakly convergent subsequence with the limit denoted by*

$$\widehat{H}(\cdot) = \Big(\widehat{\xi}(\cdot), \widehat{\psi}(\cdot), \widehat{w}(\cdot), \widehat{z}(\cdot), \widehat{g}(\cdot), \widehat{T}(\cdot)\Big),$$

having continuous paths w.p.1.

Proof. It follows from the definition of $\{\psi_n^h\}$ and interpolation intervals constructed in (4.22) that if n is a harvesting step or a reflection step, $\Delta t_n^h = t_{n+1}^h - t_n^h = 0$ and $\psi_{n+1}^h = \psi_n^h$. Otherwise, n is a diffusion step, $\Delta t_n^h = t_{n+1}^h - t_n^h = h_2$ and $\psi_{n+1}^h = \varphi_{k+1}^h$ if $\psi_n^h = \varphi_k^h$. By virtue of the continuous time interpolation $\psi^h(\cdot)$ in (4.32), we have

$$\psi^h(t) = \varphi_k^h, \quad \text{for} \quad t \in [kh_2, kh_2 + h_2), \quad k = 0, 1, \ldots,$$

with the sequence $\{\varphi_n^h\}$ constructed in (4.18). With this observation, the tightness of $\psi^h(\cdot)$ can be obtained as in [67, Theorem 8.15]. For other components, we use the same estimations as in [27, Theorem 5.3] using the tightness criteria in [25, p. 47]. Let $T_0 < \infty$ be a positive

constant and τ_0 be a stopping time which is not bigger than T_0. Then for any $\delta > 0$,

$$E^h_{\tau_0}\left|w^h(\tau_0+\delta) - w^h(\tau_0)\right|^2 = O(\delta) + \varepsilon^h(\delta), \tag{4.45}$$

where terms $E|\varepsilon^h(\delta)| \to 0$ uniformly in τ_0 as $h \to 0$. Taking $\limsup_{h\to 0}$ followed by $\lim_{\delta\to 0}$ yield the tightness of $\{w^h(\cdot)\}$. The tightness of $\{\widehat{\psi}^h(\cdot)\}$ and $\{\widehat{w}^h(\cdot)\}$ are obtained due to the stretching out of the timescale.

Following the definition of "stretched out" timescale,

$$\begin{aligned} E^h_{\tau_0}|\widehat{z}^h(\tau_0+\delta) - \widehat{z}^h(\tau_0)|^2 &\le r h_1^2 E^h_{\tau_0}(\text{number of harvesting steps in} \\ &\qquad\qquad \text{interpolated interval } [\tau_0, \tau_0+\delta))^2 \\ &\le r h_1^2 \max\{1, \delta^2/h_1^2\} \\ &\le r(h_1^2 + \delta^2). \end{aligned} \tag{4.46}$$

Similarly,

$$E^h_{\tau_0}|\widehat{g}^h(\tau_0+\delta) - \widehat{g}^h(\tau_0)|^2 \le r(h_1^2+\delta^2). \tag{4.47}$$

Thus $\{\widehat{z}^h(\cdot), \widehat{g}^h(\cdot)\}$ is tight. The tightness of $\{\widehat{T}^h(\cdot)\}$ follows from the fact that

$$0 \le \widehat{T}^h(\tau_0+\delta) - \widehat{T}^h(\tau_0) \le \delta.$$

Next we prove the tightness of $\{\widehat{\xi}^h(\cdot)\}$. It follows from (4.42), (4.46), and (4.47) that

$$\begin{aligned} E^h_{\tau_0}|\widehat{\xi}^h(\tau_0+\delta) - \widehat{\xi}^h(\tau_0)|^2 &\le 4E^h_{\tau_0}|\widehat{B}^h(\tau_0+\delta) - \widehat{B}^h(\tau_0)|^2 + 4E^h_{\tau_0}|\widehat{M}^h(\tau_0+\delta) - \widehat{M}^h(\tau_0)|^2 \\ &\quad + 4E^h_{\tau_0}|\widehat{z}^h(\tau_0+\delta) - \widehat{z}^h(\tau_0)|^2 + 4E^h_{\tau_0}|\widehat{g}^h(\tau_0+\delta) - \widehat{g}^h(\tau_0)|^2 \\ &\le K\delta^2 + K\delta + 8r(h_1^2+\delta^2), \end{aligned}$$

where K is a positive constant depending only on upper bounds of $\overline{b}(\cdot,\cdot)$ and $\overline{\sigma}(\cdot,\cdot)$ on $L_{h_1} \times S_{m-1} \subset [0, U+1]^r \times S_{m-1}$. This show the tightness of $\{\widehat{\xi}^h(\cdot)\}$. Hence $\widehat{H}^h(\cdot)$ is tight. By virtue of Prohorov's Theorem, $\widehat{H}^h(\cdot)$ has a weakly convergent subsequence with the limit

$\widehat{H}(\cdot)$.

By the definition of $\widehat{T}^h(\cdot)$, it is Lipschitz continuous with Lipschitz coefficient 1. By virtue of the Skorohod representation, such property also holds for $\widehat{T}(\cdot)$. Since sizes of jumps of $\widehat{\xi}^h(\cdot)$, $\widehat{w}^h(\cdot)$, $\widehat{z}^h(\cdot)$, and $\widehat{g}^h(\cdot)$ go to 0 as $h \to 0$, then their limits have continuous paths w.p.1 (see [26, p. 1007]). Finally, consider the tight sequence $\big(\psi^h(\cdot), \widehat{\psi}^h(\cdot), \widehat{T}^h(\cdot)\big)$ with the weak limit $\big(\widetilde{\psi}(\cdot), \widehat{\psi}(\cdot), \widehat{T}(\cdot)\big)$. Using the same argument as in (see [67, Section 8.4]),we obtain that $\widetilde{\psi}(\cdot)$ solves the Wonham filter equation, then it has continuous paths w.p.1. It then follows from $\widehat{\psi}^h(\cdot) = \psi^h(\widehat{T}^h(\cdot))$ that $\widehat{\psi}(\cdot) = \widetilde{\psi}(\widehat{T}(\cdot))$. Therefore, $\widehat{\psi}(\cdot)$ has also continuous paths w.p.1. This completes the proof. $\square$

In what follows, for notational simplicity, we still denote the convergent subsequence of $\widehat{H}^h(\cdot)$ by $\widehat{H}^h(\cdot)$. By Skorohod's representation, with a slight abuse of notation, we can always assume that the convergence is also pathwise w.p.1 in the topology of the path space and is uniform on bounded time interval. We proceed to characterize the limit process.

Theorem 4.4. *Under conditions of Theorem 4.3, let $\widehat{\mathcal{F}}(t)$ be the σ-algebra generated by*

$$\{\widehat{\xi}(s), \widehat{\psi}(s), \widehat{w}(s), \widehat{z}(s), \widehat{g}(s), \widehat{T}(s) : s \le t\}.$$

Then the following assertions hold.

(a) *$\widehat{w}(t)$ is an $\widehat{\mathcal{F}}(t)$- martingale with quadratic variation process $\widehat{T}(t)I_r$.*

(b) *$\widehat{z}(\cdot)$, $\widehat{g}(\cdot)$, and $\widehat{T}(\cdot)$ are nondecreasing and nonnegative.*

(c) *The limit processes satisfy*

$$\widehat{\xi}(t) = x_0 + \int_0^t \overline{b}(\widehat{\xi}(s), \widehat{\psi}(s))d\widehat{T}(s) + \int_0^t \overline{\sigma}(\widehat{\xi}(s), \widehat{\psi}(s))d\widehat{w}(s) - \widehat{z}(t) - \widehat{g}(t). \tag{4.48}$$

Proof. (a) Let $\widehat{E}_t^h$ denote the expectation conditioned on $\widehat{\mathcal{F}}^h(t) = \mathcal{F}^h(\widehat{T}^h(t))$. Recall that $w^h(\cdot)$ is an $\mathcal{F}^h(\cdot)$- martingale and by the definition of $\widehat{w}^h(\cdot)$, for any $\delta > 0$,

$$\begin{aligned} &\widehat{E}_t^h\big(\widehat{w}^h(t+\delta) - \widehat{w}^h(t)\big) = 0, \\ &\widehat{E}_t^h\big(\widehat{w}^h(t+\delta)\widehat{w}^h(t+\delta)' - \widehat{w}^h(t)\widehat{w}^h(t)'\big) = \big(\widehat{T}^h(t+\delta) - \widehat{T}^h(t)\big)I_r + \widehat{\varepsilon}^h(\delta), \end{aligned} \tag{4.49}$$

where $E|\widehat{\varepsilon}^h(\delta)| \to 0$ as $h \to 0$. To characterize $\widehat{w}(\cdot)$, let q be an arbitrary integer, $t > 0$, $\delta > 0$ and $\{t_k : k \le q\}$ be such that $t_k \le t < t+\delta$ for each k. Let $\Psi(\cdot)$ be a real-valued and continuous function of its arguments with compact support. Then in view of (4.49), we have

$$E\Psi(\widehat{H}^h(t_k), k \le q)\Big[\widehat{w}^h(t+\delta) - \widehat{w}^h(t)\Big] = 0, \tag{4.50}$$

and

$$E\Psi(\widehat{H}^h(t_k), k \le q)\Big[\big(\widehat{w}^h(t+\delta)\widehat{w}^h(t+\delta)' - \widehat{w}^h(t)\widehat{w}^h(t)' - \big(\widehat{T}^h(t+\delta) - \widehat{T}^h(t)\big)I_r - \widehat{\varepsilon}^h(\delta)\Big] = 0. \tag{4.51}$$

By using the Skorohod representation and the dominated convergence theorem, letting $h \to 0$ in (4.50), we obtain

$$E\Psi(\widehat{H}(t_k), k \le q)\Big[\widehat{w}(t+\delta) - \widehat{w}(t)\Big] = 0. \tag{4.52}$$

Since $\widehat{w}(\cdot)$ has continuous paths w.p.1, (4.52) implies that $\widehat{w}(\cdot)$ is a continuous $\widehat{\mathcal{F}}(\cdot)$-martingale. Moreover, (4.51) gives us that

$$E\Psi(\widehat{H}(t_k), k \le q)\Big[\widehat{w}(t+\delta)\widehat{w}(t+\delta)' - \widehat{w}(t)\widehat{w}(t)' - \big(\widehat{T}(t+\delta) - \widehat{T}(t)\big)I_r\Big] = 0. \tag{4.53}$$

Then part (a) follows.

(b) The monotonicity and non-negativity of $\widehat{z}(\cdot)$, $\widehat{g}(\cdot)$, and $\widehat{T}(\cdot)$ follow immediately from that of $\widehat{z}^h(\cdot)$, $\widehat{g}^h(\cdot)$, and $\widehat{T}^h(\cdot)$, respectively.

(c) The proof of this part is motivated by that of [28, Theorem 10.4.1]. By virtue of Skorohod representation, the uniform convergence of $\big(\widehat{\xi}^h(\cdot), \widehat{\psi}^h(\cdot), \widehat{T}^h(\cdot)\big)$ to $\big(\widehat{\xi}(\cdot), \widehat{\psi}(\cdot), \widehat{T}(\cdot)\big)$ on bounded time interval, we obtain

$$\int_0^t \overline{b}(\widehat{\xi}^h(s), \widehat{\psi}^h(s))d\widehat{T}^h(s) - \int_0^t \overline{b}(\widehat{\xi}(s), \widehat{\psi}(s))d\widehat{T}(s) \to 0 \quad \text{as} \quad h \to 0, \tag{4.54}$$

uniformly in t on any bounded time interval w.p.1. For each positive constant δ and a process $\widehat{\nu}(\cdot)$, define the piecewise constant process $\widehat{\nu}^\delta(\cdot)$ by $\widehat{\nu}^\delta(t) = \widehat{\nu}(k\delta)$ for $t \in [k\delta, k\delta + \delta), k = 0, 1, ...$ Then, by the tightness of $(\widehat{\xi}^h(\cdot), \widehat{\psi}^h(\cdot))$, (4.43) can be rewritten as

$$\widehat{\xi}^h(t) = x_0 + \int_0^t \overline{b}(\widehat{\xi}^h(s), \widehat{\psi}^h(s))d\widehat{T}^h(s) + \int_0^t \overline{\sigma}(\widehat{\xi}^{h,\delta}(s), \widehat{\psi}^{h,\delta}(s))d\widehat{w}^h(s) - \widehat{z}^h(t) - \widehat{g}^h(t) + \widehat{\varepsilon}^{h,\delta}(t), \tag{4.55}$$

where $\lim_{\delta\to 0} \limsup_{h\to 0} E|\widehat{\varepsilon}^{h,\delta}(t)| = 0$. Owing to the fact that $\widehat{\xi}^{h,\delta}$ and $\widehat{\psi}^{h,\delta}$ take constant values on the intervals $[n\delta, n\delta + \delta)$, we have

$$\int_0^t \overline{\sigma}(\widehat{\xi}^{h,\delta}(s), \widehat{\psi}^{h,\delta}(t))d\widehat{w}^h(s) \to \int_0^t \overline{\sigma}(\widehat{\xi}^\delta(s), \widehat{\psi}^\delta(t))d\widehat{w}(s) \quad \text{as } h \to 0, \tag{4.56}$$

which are well defined w.p.1 since they can be written as finite sums. Combining (4.54)-(4.56), we have

$$\widehat{\xi}(t) = x_0 + \int_0^t \overline{b}(\widehat{\xi}(s), \widehat{\psi}(s))d\widehat{T}(s) + \int_0^t \overline{\sigma}(\widehat{\xi}^\delta(s), \widehat{\psi}^\delta(t))d\widehat{w}(s) - \widehat{z}(t) - \widehat{g}(t) + \widehat{\varepsilon}^\delta(t), \tag{4.57}$$

where $\lim_{\delta\to 0} E|\widehat{\varepsilon}^\delta(t)| = 0$. Taking limit in the above equation as $\delta \to 0$ yields the result. □

Theorem 4.5. *Under conditions of Theorem 4.3, for $t < \infty$, define the reverse $R(t) = \inf\{s : \widehat{T}(s) > t\}$. For any process $\widehat{\nu}(\cdot)$, define the rescaled process $\nu(\cdot)$ by $\nu(t) = \widehat{\nu}(R(t))$. Let $\mathcal{F}(t)$ be the σ-algebra generated by $\{\xi(s), \psi(s), w(s), z(s), g(s), R(s) : s \le t\}$. The following assertions hold:*

(a) *$R(\cdot)$ is right continuous, nondecreasing, and $R(t) \to \infty$ as $t \to \infty$ w.p.1.*

(b) *$z(\cdot)$ and $g(\cdot)$ are right-continuous, nondecreasing, nonnegative, and $\mathcal{F}(t)$-adapted processes.*

(c) *$w(\cdot)$ is a standard $\mathcal{F}(t)$-Wiener process, $\psi(\cdot)$ satisfies the system of Wonham filter equations* (4.5)*, and*

$$\xi(t) = x_0 + \int_0^t \overline{b}(\xi(s), \psi(s))ds + \int_0^t \overline{\sigma}(\xi(s), \psi(t))dw(s) - z(t) - g(t). \tag{4.58}$$

Proof. (a) We will argue via contradiction that $\widehat{T}(t) \to \infty$ as $t \to \infty$ w.p.1. Suppose $P[\sup_{t\geq 0} \widehat{T}(t) < \infty] > 0$. Then there exist positive constants ε and T_0 such that

$$P[\sup_{t\geq 0} \widehat{T}(t) < T_0 - 1] > \varepsilon. \tag{4.59}$$

We first observe that

$$t + r|z^h(t) + g^h(t)| \geq \sum_{k=0}^{n^h(t)-1} \Big(h_2 \mathbb{1}_{\{\pi^h=0\}} + h_1 \mathbb{1}_{\{\pi^h\geq 1\}} + h_1 \mathbb{1}_{\{\pi^h\leq -1\}} \Big).$$

Since $\widehat{T}^h(\cdot)$ is nondecreasing and $\widehat{T}^h(\widehat{t}^h_n) = t^h_n$,

$$\begin{aligned} \widehat{T}^h\big(t + r|z^h(t) + g^h(t)|\big) &\geq \widehat{T}^h\Big(\sum_{k=0}^{n^h(t)-1} \big(h_2 \mathbb{1}_{\{\pi^h=0\}} + h_1 \mathbb{1}_{\{\pi^h\geq 1\}} + h_1 \mathbb{1}_{\{\pi^h\leq -1\}} \big) \Big) \\ &= \widehat{T}^h(\widehat{t}^h_{n^h(t)}) = t^h_{n^h(t)} \geq t - 1. \end{aligned} \tag{4.60}$$

The last inequality above is a consequence of the inequalities $t^h_{n^h(t)} \leq t < t^h_{n^h(t)+1} = t^h_{n^h(t)} + h_2 < t^h_{n^h(t)} + 1$.

It follows from (4.34) that for each fixed $t \geq 0$, $\sup_h E\Big(|z^h(t) + g^h(t)|\Big) < \infty$. Hence for a sufficiently large K,

$$P\{r|z^h(T_0) + g^h(T_0)| \geq 2K\} \leq \frac{rE\big|z^h(T_0) + g^h(T_0)\big|}{2K} < \frac{\varepsilon}{2}. \tag{4.61}$$

In views of (4.60) and (4.61), we obtain

$$\begin{aligned} P\big[\widehat{T}^h(T_0+2K) < T_0 - 1\big] &\le P\big[\widehat{T}^h\big(T_0 + r|z^h(T_0) + g^h(T_0)|\big) < T_0 - 1, r|z^h(T_0) + g^h(T_0)| < 2K\big] \\ &\quad + P\big[r|z^h(T_0) + g^h(T_0)| \ge 2K\big] \\ &< \frac{\varepsilon}{2} \qquad \text{for all small } h = (h_1, h_2). \end{aligned} \tag{4.62}$$

Since $\widehat{T}^h$ converges weakly to $\widehat{T}$, it follows from (4.62) that $\liminf_{h\to 0} P\big[\widehat{T}^h(T_0+2K) < T_0-1\big] \le \varepsilon/2$. This contradicts (4.59) (see [8, Theorem 1.2.1]). Hence $\widehat{T}(t) \to \infty$ as $t \to \infty$ w.p.1. Thus $R(t) < \infty$ for all t and $R(t) \to \infty$ as $t \to \infty$. Since $\widehat{T}(\cdot)$ is nondecreasing and continuous, $R(\cdot)$ is nondecreasing and right-continuous.

(b) follows the fact that $\widehat{z}(\cdot)$ and $\widehat{g}(\cdot)$ are continuous, nondecreasing, nonnegative, and $R(\cdot)$ is right-continuous.

(c) We first note that although $R(\cdot)$ might fail to be continuous, $w(\cdot) = \widehat{w}(R(\cdot))$ has continuous paths w.p.1. Indeed, consider the tight sequence $\big(w^h(\cdot), \widehat{w}^h(\cdot), \widehat{T}^h(\cdot)\big)$ with the weak limit $\big(\widetilde{w}(\cdot), \widehat{w}(\cdot), \widehat{T}(\cdot)\big)$. Since $\widehat{w}^h(\cdot) = w^h(\widehat{T}^h(\cdot))$, we must have that $\widehat{w}(\cdot) = \widetilde{w}(\widehat{T}(\cdot))$. It follows from the definition of $R(\cdot)$ that for each $t \ge 0$, we have $\widehat{T}(R(t)) = t$. Hence $w(t) = \widehat{w}(R(t)) = \widetilde{w}\big(\widehat{T}(R(t))\big) = \widetilde{w}(t)$. Since magnitude of jumps of $w^h(\cdot)$ go to 0 as $h \to 0$, $\widetilde{w}(\cdot)$ also has continuous paths w.p.1. This shows that $w(\cdot) = \widehat{w}(R(\cdot))$ has continuous paths w.p.1. By the same argument for the tight sequence $\big(\psi^h(\cdot), \widehat{\psi}^h(\cdot), \widehat{T}^h(\cdot)\big)$, we obtain that $\psi(\cdot)$ also has continuous paths w.p.1. Moreover, it satisfies the system of Wonham filter equation (4.5) (see [67, Section 8.4]).

Before characterizing $w(\cdot)$, we note that for $t \ge 0$, $\{R(s) \le t\} = \{\widehat{T}(t) \ge s\} \in \widehat{\mathcal{F}}(t)$ since $\widehat{T}(t)$ is $\widehat{\mathcal{F}}(t)$-measurable. Thus $R(s)$ is an $\widehat{\mathcal{F}}(t)$-stopping time for each $s \ge 0$. Since $\widehat{w}(t)$ is

an $\widehat{\mathcal{F}}(t)$-martingale with quadratic variation process $\widehat{T}(t)I_r$,

$$\begin{aligned} &E\big[\widehat{w}(R(t)\wedge n)|\widehat{\mathcal{F}}(R(s))\big] = \widehat{w}(R(s)\wedge n), \quad n = 1, 2, \ldots, \\ &E\widehat{w}(R(t)\wedge n)\widehat{w}(R(t)\wedge n)' = E\widehat{T}(R(t)\wedge n)I_r, \end{aligned} \tag{4.63}$$

and $\widehat{T}(R(t)\wedge n) \leq \widehat{T}(R(t)) = t$. Hence for each fixed $t \geq 0$, the family $\{\widehat{w}(R(t)\wedge n), n \geq 1\}$ is uniformly integrable. By that uniform integrability, we obtain from (4.63) that

$$E\big[\widehat{w}(R(t))|\widehat{\mathcal{F}}(R(s))\big] = \widehat{w}(R(s)),$$

that is $E\big[w(t)|\mathcal{F}(s)\big] = w(s)$. This proves that $w(\cdot)$ is a continuous $\mathcal{F}(\cdot)$ -martingale. We next consider its quadratic variation. By the Burkholder-Davis-Gundy inequality, there exists a positive constant C independent of $n = 1, 2, \ldots$ such that

$$\begin{aligned} E|\widehat{w}(R(t)\wedge n)|^2 &\leq CE\Big[\Big(\sup_{0\leq s\leq R(t)} |\widehat{w}(R(s)\wedge n)|^2\Big)\Big] \\ &\leq CE|\widehat{T}(R(t)\wedge n)| \leq Ct. \end{aligned}$$

Thus the families $\{\widehat{w}(R(t)\wedge n), n \geq 1\}$ and $\{\widehat{T}(R(t)\wedge n), n \geq 1\}$ are uniformly integrable for each fixed $t \geq 0$. Combining with the fact that $\widehat{w}(\cdot)$, $\widehat{T}(\cdot)$ have continuous paths, for nonnegative constants $s \leq t$, we have

$$\begin{aligned} \widehat{w}(R(s)\wedge n)\widehat{w}(R(s)\wedge n)' - \widehat{T}(R(s)\wedge n)I_r &= E\big[\widehat{w}(R(t)\wedge n)\widehat{w}(R(t)\wedge n)' - \widehat{T}(R(t)\wedge n)I_r|\widehat{\mathcal{F}}(R(s))\big] \\ &\to E\big[\widehat{w}(R(t))\widehat{w}(R(t))' - \widehat{T}(R(s))I_r|\widehat{\mathcal{F}}(R(s))\big] \\ &= E\big[w(t)w(t)' - tI_r|\mathcal{F}(s)\big]. \end{aligned} \tag{4.64}$$

Note that the first equation in (4.64) follows from the martingale property of $\widehat{w}(\cdot)\widehat{w}(\cdot)' - \widehat{T}(\cdot)I_r$ with respect to $\widehat{\mathcal{F}}(t)$. Letting $n \to \infty$ in (4.64), we arrive at

$$E\big[w(t)w(t)' - tI_r|\mathcal{F}(s)\big] = w(s)w(s)' - sI_r.$$

Therefore, $w(\cdot)$ is an $\mathcal{F}(t)$ - Wiener process. A rescaling of (4.48) yields

$$\xi(t) = x_0 + \int_0^t \overline{b}(\xi(s), \psi(s))ds + \int_0^t \overline{\sigma}(\xi(s), \psi(t))dw(s) - z(t) - g(t).$$

The proof is complete. □

Theorem 4.6. *Under conditions of Theorem 4.3, let $V^h(x_0, \varphi_0)$ and $V(x_0, \varphi_0)$ be value functions defined in* (4.24) *and* (4.12)*, respectively. Then $V^h(x_0, \varphi_0) \to V(x_0, \varphi_0)$ as $h \to 0$.*

Proof. We first show that as $h \to 0$,

$$\begin{aligned} J^h(x_0, \varphi_0, z^h) &\to \sum_{i=1}^r E \int_0^\tau e^{-at} \overline{f}^i(\psi(t))dz^i(t) \\ &= J(x_0, \varphi_0, z(\cdot)), \end{aligned} \tag{4.65}$$

where $\tau = \inf\{s : \xi(s) = 0\}$. Indeed, for a harvesting strategy $z^h = \{z_n^h\}$, we have

$$\begin{aligned} J^h(x_0, \varphi_0, z^h) &= \sum_{i=1}^r E \sum_{k=0}^{\eta_h - 1} e^{-at_k^h} \overline{f}^i(\psi_k^h)\Delta z_k^{h,i} \\ &= \sum_{i=1}^r E \int_0^{\widehat{t}_{\eta_h}^h} e^{-a\widehat{T}^h(t)} \overline{f}^i(\widehat{\psi}^h(t))d\widehat{z}^{h,i}(t). \end{aligned} \tag{4.66}$$

By a small modification of the proof in Theorem 4.5 (a), we have $\widehat{T}^h(t) \to \infty$ as $t \to \infty$ w.p.1. It also follows from the representation (4.34) and estimates on $B^h(\cdot)$ and $M^h(\cdot)$ that $\{z^h(n+1) - z^h(n) : n, h\}$ is uniformly integrable. Thus, by the definition of $\widehat{T}^h(\cdot)$,

$$\begin{aligned} \sum_{i=1}^r E \textstyle\int_{T_0}^\infty e^{-a\widehat{T}^h(t)} \overline{f}^i(\widehat{\psi}^h(t))d\widehat{z}^{h,i}(t) &\le \sum_{i=1}^r E \int_{\min\{t:\widehat{T}^h(t) \ge T_0\}}^\infty K e^{-as} dz^{h,i}(s) \\ &\le \sum_{i=1}^r E \int_{T_0}^\infty K e^{-as} dz^{h,i}(s) \to 0, \end{aligned}$$

uniformly in h as $T_0 \to \infty$. In the above argument, we have employed the fact that we can replace $\widehat{t}_{\eta_h}^h$ in (4.66) by infinity, and $\widehat{T}^h(T_0) \le T_0$. Then by the weak convergence, the Skohorod representation (therefore, the uniform convergence of $\left(\widehat{z}^h(\cdot), \widehat{\psi}^h(\cdot), \widehat{T}^h(\cdot)\right)$ to

$\big(\widehat{z}(\cdot), \widehat{\psi}(\cdot), \widehat{T}(\cdot)\big)$ on bounded time interval), and uniform integrability, for any $T_0 > 0$,

$$\sum_{i=1}^{r} E \int_0^{T_0} e^{-a\widehat{T}^h(t)} \overline{f}^i(\widehat{\psi}^h(t)) d\widehat{z}^{h,i}(t) \to \sum_{i=1}^{r} E \int_0^{T_0} e^{-a\widehat{T}(t)} \overline{f}^i(\widehat{\psi}(t)) d\widehat{z}^i(t).$$

Therefore, we obtain

$$\sum_{i=1}^{r} E \int_0^{\infty} e^{-a\widehat{T}^h(t)} \overline{f}^i(\widehat{\psi}^h(t)) d\widehat{z}^{h,i}(t) \to \sum_{i=1}^{r} E \int_0^{\infty} e^{-a\widehat{T}(t)} \overline{f}^i(\widehat{\psi}(t)) d\widehat{z}^i(t),$$

or equivalently,

$$\sum_{i=1}^{r} E \int_0^{\widehat{t}^h_{\eta_h}} e^{-a\widehat{T}^h(t)} \overline{f}^i(\widehat{\psi}^h(t)) d\widehat{z}^{h,i}(t) \to \sum_{i=1}^{r} E \int_0^{\widehat{\tau}} e^{-a\widehat{T}(t)} \overline{f}^i(\widehat{\psi}(t)) d\widehat{z}^i(t),$$

where $\widehat{\tau} = \inf\{s : \widehat{\xi}(s) = 0\}$. On inversion of the timescale, the above expression can be written as

$$\sum_{i=1}^{r} E \int_0^{\tau} e^{-at} \overline{f}^i(\psi(t)) dz^i(t).$$

Thus, $J^h(x_0, \varphi_0, z^h) \to J(x_0, \varphi_0, z(\cdot))$ as $h \to 0$.

Next, we prove that

$$\limsup_h V^h(x_0, \varphi_0) \le V(x_0, \varphi_0). \tag{4.67}$$

For any small positive constant ε, let $\widetilde{z}^h$ be an ε-optimal harvesting strategy for the chain $\{\xi^h_n\}$, i.e.,

$$V^h(x_0, \varphi_0) = \sup_{z^h} J^h(x_0, \varphi_0, z^h) \le J^h(x_0, \varphi_0, \widetilde{z}^h) + \varepsilon.$$

Choose a subsequence $\{\widetilde{h}\}$ of $\{h\}$ such that

$$\limsup_{h\to 0} V^h(x_0, \varphi_0) = \lim_{\widetilde{h}\to 0} V^{\widetilde{h}}(x_0, \varphi_0) \le \limsup_{\widetilde{h}\to 0} J^{\widetilde{h}}(x_0, \varphi_0, \widetilde{z}^{\widetilde{h}}) + \varepsilon. \tag{4.68}$$

Without loss of generality (passing to an additional subsequence if needed), we may assume

that

$$\widehat{H}^{\widetilde{h}}(\cdot) = \Big(\widehat{\xi}^{\widetilde{h}}(\cdot), \widehat{\varphi}^{\widetilde{h}}(\cdot), \widehat{w}^{\widetilde{h}}(\cdot), \widehat{z}^{\widetilde{h}}(\cdot), \widehat{g}^{\widetilde{h}}(\cdot), \widehat{T}^{\widetilde{h}}(\cdot)\Big)$$

converges weakly to

$$\widehat{H}(\cdot) = \Big(\widehat{\xi}(\cdot), \widehat{\varphi}(\cdot), \widehat{w}(\cdot), \widehat{z}(\cdot), \widehat{g}(\cdot), \widehat{T}(\cdot)\Big),$$

and $z(\cdot) = \widehat{z}(R(\cdot))$. It follows from our claim in the beginning of the proof that

$$\lim_{\widetilde{h}\to 0} J^{\widetilde{h}}(x_0, \varphi_0, \widetilde{z}^{\widetilde{h}}) = J(x_0, \varphi_0, z(\cdot)) \leq V(x_0, \varphi_0), \tag{4.69}$$

where $J(x_0, \varphi_0, z(\cdot)) \leq V(x_0, \varphi_0)$ since $V(x_0, \varphi_0)$ is the maximizing cost function. Since ε is arbitrarily small, (4.67) follows from (4.68) and (4.69).

To prove the reverse inequality $\liminf_h V^h(x_0, \varphi_0) \geq V(x_0, \varphi_0)$, for any small positive constant ε, we choose a particular ε-optimal harvesting strategy for (4.6) and (4.39) such that the approximation can be applied to the chain $\{\xi_n^h\}$ and the associated cost compared with $V^h(x_0, \varphi_0)$. By an adaption of the method used for singular control problems [27, 41], for given $\varepsilon > 0$, there is a ε-optimal harvesting strategy $z(\cdot)$ and Wiener process $w(\cdot)$ for (4.6)-(4.39) with the following properties: (a) There are $T_\varepsilon < \infty$, $\rho > 0$, and $\delta > 0$ such that $z(\cdot)$ are constants on the intervals $[n\rho, n\rho + \rho)$, only one of the components can jump at a time, and the jumps take values in the discrete set $\{k\delta : k = 1, 2, \dots\}$; also $z(\cdot)$ is bounded and is constant on $[T_\varepsilon, \infty)$; (b) there is a $\theta > 0$ such that

$$\begin{aligned} &P\big(\Delta z_i(n\rho) = k\delta | z(m\rho), m < n, w(s), \psi(s), s \leq n\rho\big) \\ &\qquad = q_{nki}\big(k\rho, z(m\rho), m < n, w(p\theta), \psi(p\theta), p\theta \leq n\rho\big), \quad i = 1, ..., n, \end{aligned}$$

where $q_{nki}(\cdot)$ can be supposed to be continuous in the w variables. Next, we adapt this harvesting strategy to the chain $\{\xi_n^h\}$ by a harvesting strategy $z^h = \{z_n^h\}$ using the same

method as in [27, p. 1459]. As a preparation, we first note the following. Suppose that we wish to apply a harvesting action of "impulsive" magnitude Δz to the chain at some interpolated time t_0. Define $n_h = \min\{k : t_k^h \geq t_0\}$, with t_k^h was defined in (4.22). Then starting at step n_h, apply $[\Delta z/h_1]$ successive harvesting steps. Let $z^h(\cdot)$ denote the piecewise interpolation of the harvesting strategy just defined. Since the values of the times t_n^h do not increase during the sequence of successive harvesting steps just described, $z^h(\cdot)$ is just a step function with jump $h_1[\Delta z/h_1]$ at time $t_{n_h}^h$. Moreover, $z^h(\cdot)$ is constant on time intervals $[n\rho, n\rho + \rho)$. Therefore, when using such controls, there is no need to rescale time in the convergence proof and $\{z^h(\cdot)\}$ is tight in the Skorohod topology.

With the observations in the last paragraph, we are ready to define the "adapted" form of $z(\cdot)$ to use on $\{\xi_n^h\}$. Let $z^h(\cdot)$ denote the interpolated form of the "adaption". We will define $z^h(\cdot)$ such that it has the same number of impulsive changes as does $z(\cdot)$. Each of the impulses is to be realized for the chain via the method used in the above observation. By the weak convergence argument analogous to that of preceding theorems, but without the time rescaling, we obtain the weak convergence $\big(\xi^h(\cdot), \psi^h(\cdot), w^h(\cdot), z^h(\cdot), g^h(\cdot)\big) \to \big(\widetilde{\xi}(\cdot), \widetilde{\psi}(\cdot), \widetilde{w}(\cdot), \widetilde{z}(\cdot), \widetilde{g}(\cdot)\big)$, and the limit solves (4.6)-(4.39), where $\big(\widetilde{\psi}(\cdot), \widetilde{w}(\cdot), \widetilde{z}(\cdot)\big)$ has the distribution of $\big(\psi(\cdot), w(\cdot), z(\cdot)\big)$. By the weak sense uniqueness assumption (A4), $\big(\widetilde{\xi}(\cdot), \widetilde{\psi}(\cdot)\big)$ is the unique solution to (4.6)-(4.39) with the ε-optimal strategy $\widetilde{z}(\cdot)$. It follows that $J(x_0, \varphi_0, \widetilde{z}(\cdot)) \geq V(x_0, \varphi_0) - \varepsilon$. By the optimality of $V^h(x_0, \varphi_0)$ and the above weak convergence,

$$V^h(x_0, \varphi_0) \geq J^h(x_0, \varphi_0, z^h) \to J(x_0, \varphi_0, \widetilde{z}(\cdot)).$$

Thus,

$$\liminf_{h\to 0} V^h(x_0,\varphi_0) \geq V(x_0,\varphi_0) - \varepsilon.$$

Since ε is arbitrarily small, the conclusion follows. $\square$

4.5 A Numerical Example

We consider a single species ecosystem in random environment subjected to the harvesting as follows

$$d\xi(t) = \xi(t)\Big(b(\alpha(t)) - c(\alpha(t))\xi(t)\Big)dt + \sigma(\alpha(t))dw(t) - dZ(t). \tag{4.70}$$

Suppose that the Markov chain $\alpha(\cdot) \in \{1,2\}$ that models random environment. The generator of the continuous-time Markov chain is given by $Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, and

$$b(1) = 2, c(1) = 2, \sigma(1) = 1, b(2) = 7, c(2) = 1, \sigma(2) = 0.5.$$

We suppose that the Markov chain can only be observed through $dy(t) = g(\alpha(t))dt + dB(t)$, where $g(1) = -1$, $g(2) = 1$. Introduce the innovation process

$$d\overline{w}(t) = dy(t) - g(1)\varphi^1(t)dt - g(2)\varphi^2(t)dt, \quad \overline{w}(0) = 0.$$

Using the Wonham filter, we can convert the incompletely observed system to the following system with complete observation

$$\begin{cases} d\xi(t) &= \xi(t)\big[b(1) - c(1)\xi(t)\big]\varphi^1(t)dt + \xi(t)\big[b(2) - c(2)\xi(t)\big]\varphi^2(t)dt \\ & \qquad + \big[\sigma(1)\varphi^1(t) + \sigma(2)\varphi^2(t)\big]dw(t) - dZ(t), \\ d\varphi^1(t) &= \big[q^{11}\varphi^1(t) + q^{21}\varphi^2(t)\big]dt + \big[g(1) - g(1)\varphi^1(t) - g(2)\varphi^2(t)\big]\varphi^1(t)d\overline{w}(t), \\ \varphi^2(t) &= 1 - \varphi^1(t), \end{cases} \tag{4.71}$$

Let $a = 0.05$, $f(1) = f(2) = 1$. In this case, $\varphi(t) = \varphi^1(t)$ and $\overline{f}(\varphi(t)) = 1$. Then for an admissible strategy $Z(\cdot)$,

$$J(x, \varphi, Z) = E \int_0^\tau e^{-as} dZ(s).$$

Based on the algorithm constructed in Section 4.3, we carry out the computation by value iterations. Let Z_0 be the harvesting policy that drive the system to extinct immediately. Then $J(x, \varphi, Z_0) = x$ for all (x, φ). Letting Z_0 be the initial harvesting strategy, we set the initial values

$$V_0^h(x, \varphi) = x, \quad x = 0, h_1, 2h_1, \ldots, U = 10, \quad \varphi = 0, h_1, 2h_1, \ldots, 1.$$

We outline how to find an improved values of $V(\cdot, \cdot)$ as follows. At each level $x = h_1, 2h_1, \ldots, U$, denote by $\pi(x, n)$ the action we choose, where $\pi(x, n) = 2$ if we choose a harvesting action and $\pi(x, n) = 1$ if we choose not to do such action. Thus, initially we choose $\pi(x, 0) = 2$ for all x and we seek for better policies which give larger values. If we take a harvesting action, an improved value of $V(x, \varphi)$ is calculated by using

$$V_{n+1}^{h,2}(x, \varphi) = V_n^h(x - h_1, \varphi) + \overline{f}(\varphi) h_1.$$

Otherwise, we choose not to harvest. Plugging all the necessary finite difference expressions into the first part of system (4.16), we obtain

$$\begin{aligned} V^h(x, \varphi) = \frac{1}{1 + ah_2} \Bigg[& V^h(x + h_1, \varphi) p^h(x, x + h_1 | \varphi, \pi) + V^h(x - h_1, \varphi) p^h(x, x - h_1 | \varphi, \pi) \\ & + V^h(x, \varphi + h_1) \frac{\left(q^{11}\varphi + q^{21} - q^{21}\varphi\right)^+ h_2 h_1 + \frac{1}{2}\varphi^2(-2 + 2\varphi)^2 h_2}{h_1^2} \\ & + V^h(x, \varphi - h_1) \frac{\left(q^{11}\varphi + q^{21} - q^{21}\varphi\right)^- h_2 h_1 + \frac{1}{2}\varphi^2(-2 + 2\varphi)^2 h_2}{h_1^2} \\ & + V^h(x, \varphi) \left(p^h(x, x | \varphi, \pi) - \frac{|q^{11}\varphi + q^{21} - q^{21}\varphi| h_2 h_1 + \varphi^2(-2 + 2\varphi)^2 h_2}{h_1^2} \right) \Bigg], \end{aligned}$$

where $\frac{1}{1+ah_2} \approx e^{-ah_2}$ plays the role of discounting part. Hence an improved value $V_{n+1}^{h,1}(x,\varphi)$ is calculated by using

$$\begin{aligned} V_{n+1}^{h,1}(x,\varphi) = \frac{1}{1+ah_2}\Bigg[& V_n^h(x+h_1,\varphi)p^h(x,x+h_1|\varphi,\pi) + V_n^h(x-h_1,\varphi)p^h(x,x-h_1|\varphi,\pi) \\ & + V_n^h(x,\varphi+h_1)\frac{\left(q^{11}\varphi+q^{21}-q^{21}\varphi\right)^+ h_2h_1 + \frac{1}{2}\varphi^2(-2+2\varphi)^2h_2}{h_1} \\ & + V_n^h(x,\varphi-h_1)\frac{\left(q^{11}\varphi+q^{21}-q^{21}\varphi\right)^- h_2h_1 + \frac{1}{2}\varphi^2(-2+2\varphi)^2h_2}{h_1^2} \\ & + V_n^h(x,\varphi)\left(p^h(x,x|\varphi,\pi) - \frac{|q^{11}\varphi+q^{21}-q^{21}\varphi|h_2h_1 + \varphi^2(-2+2\varphi)^2h_2}{h_1^2}\right)\Bigg], \end{aligned}$$

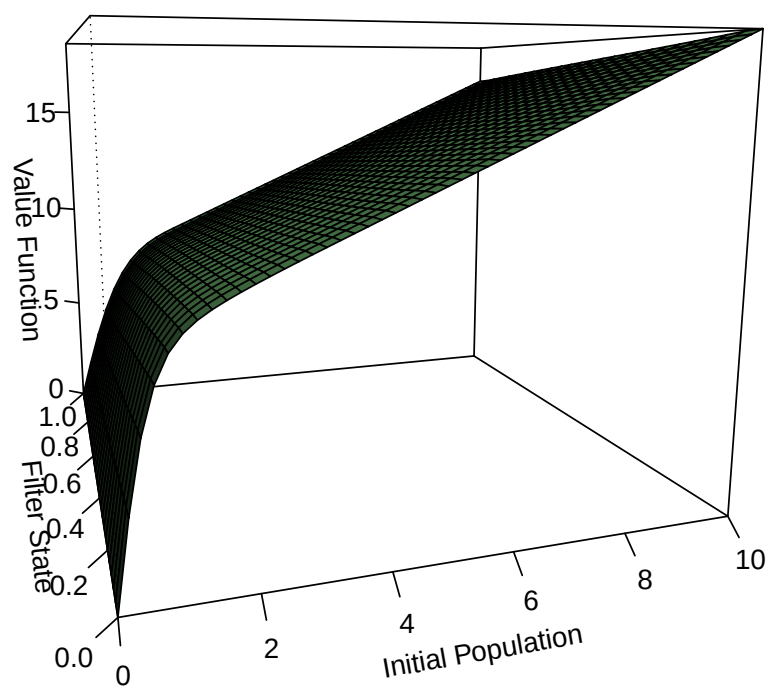

Figure 6: Optimal value function versus initial population and initial filter state

Therefore, we can find the optimal action and the corresponding improved $V_{n+1}^h(x,\varphi)$ as

follows.

$$\pi(x,n) := \arg max\{i = 1,2 : V_{n+1}^{h,i}(x,\varphi)\},$$

and

$$V_{n+1}^{h}(x,\varphi) := V_{n+1}^{h,\pi(x,n)}(x,\varphi).$$

The iterations stop as soon as the increment $V_{n+1}^{h}(\cdot) - V_{n}^{h}(\cdot)$ is below some tolerance.

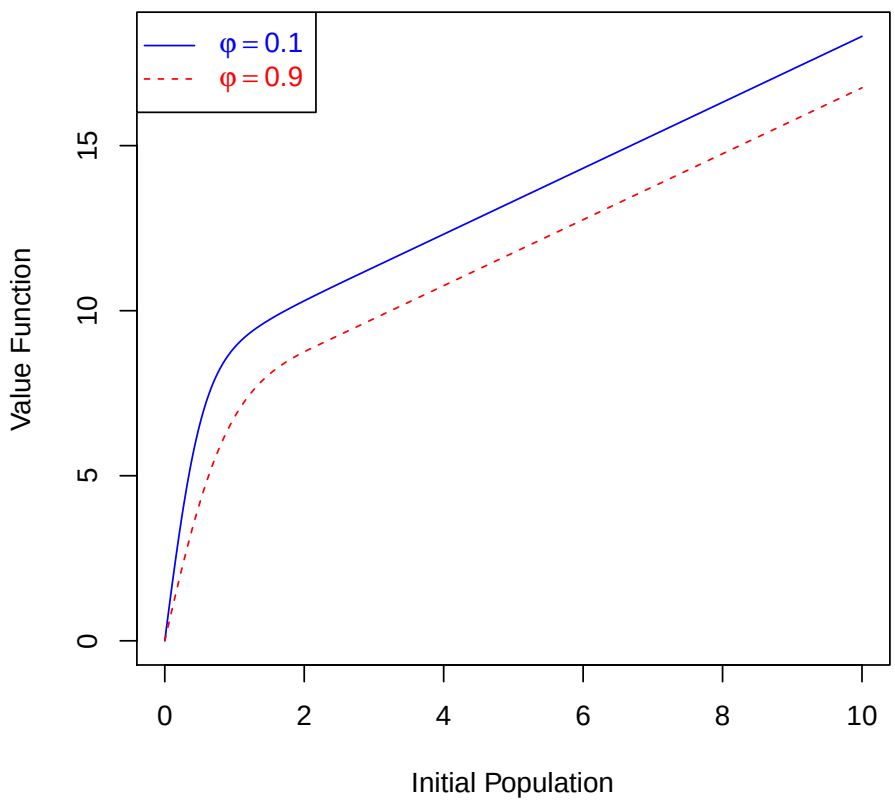

Figure 7: Optimal value function versus initial population with $\varphi = 0.1$ and $\varphi = 0.9$

Figure 6 shows the value function $V(x,\varphi)$ as a function of initial population x and initial filter state φ. To highlight the effect of initial filter states, in Figure 7, we plot graphs of

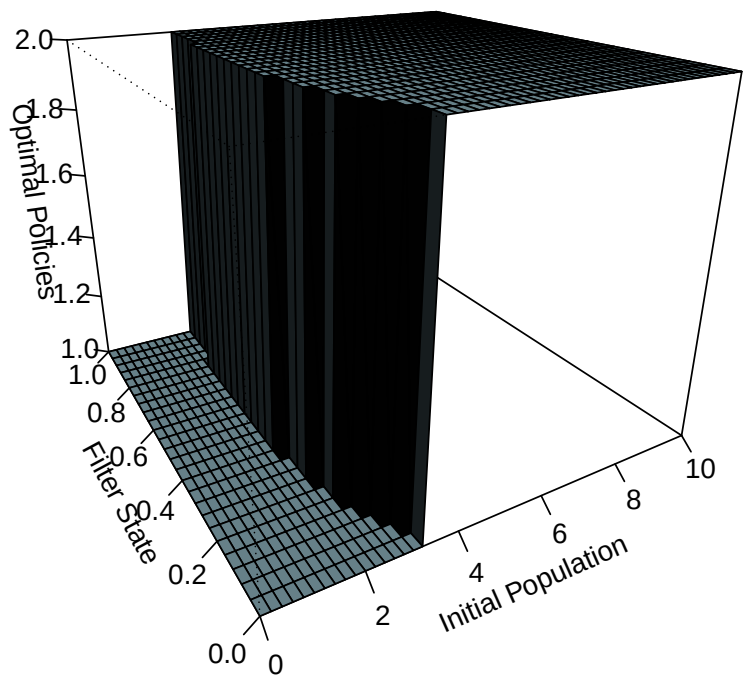

Figure 8: Optimal policies versus initial population and initial filter state

$V(\cdot, 0.1)$ and $V(\cdot, 0.9)$. Note that $J(\cdot, \cdot, Z_0)$ is also referred as current harvesting potential, and $V(\cdot, \cdot) - J(\cdot, \cdot, Z_0)$ can be seen as the maximum present expected value of the accumulate net convenience yields accrued from postponing the harvesting decision and keeping the population alive after a small time interval; see [1] and [59]. Figure 8 and Figure 9 provide optimal policies, with "1" denoting not to harvest and "2" denoting harvesting actions. It can be seen from Figure 6 and Figure 7 that the value function is concave and the optimal harvesting policy is a barrier strategy. This notice agrees with observations and results in [20] and [11]. To be more specific, if the population is higher than some barrier

level, a harvesting action is chosen, and the value function increases with unity slope (since we take $f(1) = f(2) = 1$). Moreover, the barrier levels are different in different Wonham filter states.

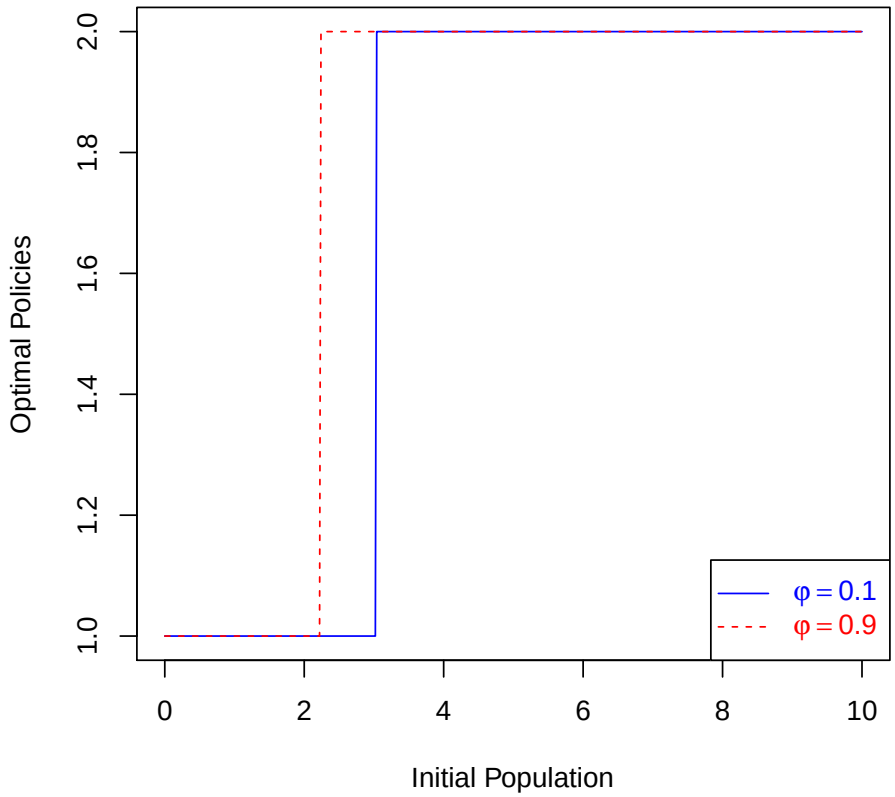

Figure 9: Optimal policies versus initial population with $\varphi = 0.1$ and $\varphi = 0.9$

4.6 Further Remarks

This chapter focused on numerical methods for solution of optimal harvesting strategies in random environments. Note that different from applications in finance, in which the $X(t)$ is observed, here we have another observation $y(t)$ process. Such a model is natural for ecological systems. They also appear in many communication systems, networked systems, as well as cyber-physical systems. The novelties of our approach include that (1) we depicted

the random environment as a hidden Markov chain; (2) we treated the resulting singular control problem under partial observations; (3) we used Wonham filter as a bridge and built numerical approximation methods based on Markov chain approximation techniques to solve the optimal control problem under partial observations.

The convergence of the algorithms was proved. A numerical example was used to demonstrate the performance of our algorithm. The problem considered here can be modified to treat dividend optimization in insurance risk management, and networked control systems. Not only can the approach be applied the harvest problem under consideration, but also it opens up the domain for treating more general singular control problems under partial observations.

CHAPTER 5 CONCLUDING REMARKS AND FUTURE DIRECTIONS

In this dissertation, we have concentrated on asymptotic properties and controls for stochastic population systems with Markovian switching. First, we study ecological properties of Lotka-Volterra models with partial observations. Then, stochastic permanence and extinction using feedback controls are investigated. Next, we keep working on Lotka-Volterra systems under a different objective in which we aim at constructing optimal harvesting strategies. Finally, we focus on the optimal harvesting problem for a general switching diffusion with partial observations by Markov chain approximation method.

Although the dissertation is mainly devoted to ecosystems, the methods and techniques developed can be used in certain related systems with a hidden Markov chain or involving optimal controls of harvesting. Thus the results and the simulation study will be of interests not only to people working in ecological systems, but also for researchers in other disciplines as well.

Several directions may be worthwhile for further study and investigation. One can study the design of feedback controls of an ecosystem modulated by a regime-switching jump diffusion system in which the hidden Markov chain is observed in white noise. Such models are more realistic since many sudden-environmental shocks, e.g., earthquakes, hurricanes, epidemics, etc. can be taken into account. Using switching diffusions with delays for modeling population dynamics has drawn much attention recently; see [33]. A related problem of interest is to develop optimal harvesting strategies for such models. The Markov chain approximation method developed for numerical methods of controlled stochastic systems with

delays in [29] appears promising. Although for ecological applications, one usually examines continuous-time systems, it is also interesting to study similar problems in discrete time for other applications with a hidden Markov chain.

Regarding to the harvesting problem, we have assumed that the Markov chain takes value in a finite set $\mathcal{M}$. Frequently, one deals with the situation that the random environment has many discrete states (i.e., the state space of $\mathcal{M}$ is large). To reduce the computation complexity, we may consider the cases that the Markov chain is nearly decomposable. To be more specific, the generator may be written as $Q^\varepsilon = (\widetilde{Q}/\varepsilon) + \widehat{Q}$, where $\widetilde{Q}/\varepsilon$ models the fast varying dynamics and $\widehat{Q}$ depicts the slowly varying motions. Such a structure enables us to reduce the computational complexity by using time-scale separation techniques. The second possibility is that the Markov chain is time varying with a generator $Q(t)$. Assume that the rate of change of the generator $Q(t)$ varies slowly in time that the Markov chain can achieve its equilibrium before there is any significant change in the rate. Treating queueing models, a singularly perturbed model was proposed in [46]. Such an idea can be adopted to our current problem to treat time-varying systems with similar feature. These problems deserve in-depth study and investigation.

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ABSTRACT

NONLINEAR STOCHASTIC SYSTEMS AND CONTROLS: LOTKA-VOLTERRA TYPE MODELS, PERMANENCE AND EXTINCTION, OPTIMAL HARVESTING STRATEGIES, AND NUMERICAL METHODS FOR SYSTEMS UNDER PARTIAL OBSERVATIONS

by

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Major: Mathematics (Applied)

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This dissertation focuses on a class of stochastic models formulated using stochastic differential equations with regime switching represented by a continuous-time Markov chain, which also known as hybrid switching diffusion processes. Our motivations for studying such processes in this dissertation stem from emerging and existing applications in biological systems, ecosystems, financial engineering, modeling, analysis, and control and optimization of stochastic systems under the influence of random environments, with complete observations or partial observations.

The first part is concerned with Lotka-Volterra models with white noise and regime switching represented by a continuous-time Markov chain. Different from the existing literature, the Markov chain is hidden and can only be observed in a Gaussian white noise in our

work. We use a Wonham filter to estimate the Markov chain from the observable evolution of the given process, and convert the original system to a completely observable one. We then establish the regularity, positivity, stochastic boundedness, and sample path continuity of the solution. Moreover, stochastic permanence and extinction using feedback controls are investigated.

The second part develops optimal harvest strategies for Lotka-Volterra systems so as to establish economically, ecologically, and environmentally reasonable strategies for populations subject to the risk of extinction. The underlying systems are controlled regime-switching diffusions that belong to the class of singular control problems. We construct upper bounds for the value functions, prove the finiteness of the harvesting value, and derive properties of the value functions. Then we construct explicit chattering harvesting strategies and the corresponding lower bounds for the value functions by using the idea of harvesting only one species at a time. We further show that this is a reasonable candidate for the best lower bound that one can expect.

In the last part, we study optimal harvesting problems for a general systems in the case that the Markov chain is hidden and can only be observed in a Gaussian white noise. The Wonham filter is employed to convert the original problem to a completely observable one. Then we treat the resulting optimal control problem. Because the problem is virtually impossible to solve in closed form, our main effort is devoted to developing numerical approximation algorithms. To approximate the value function and optimal strategies, Markov chain approximation methods are used to construct a discrete-time controlled Markov chain. Convergence of the algorithm is proved by weak convergence method and suitable scaling.

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