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Some New Combinatorial Formulas For Cluster Monomials Of Type A Quivers

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SOME NEW COMBINATORIAL FORMULAS FOR CLUSTER MONOMIALS OF TYPE A QUIVERS

by

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DEDICATION

To my parents and my wife

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CHAPTER 1. INTRODUCTION

Cluster algebras were first introduced by S. Fomin and A. Zelevinsky in [6] to design an algebraic framework for understanding total positivity and canonical bases for quantum groups. A cluster algebra is a subring of a rational function field generated by a distinguished set of Laurent polynomials called cluster variables. The long-standing Positivity Conjecture, now proved in [13] and [8], asserts that the coefficients in any cluster variable are positive integers. From the combinatorial point of view, the Positivity Conjecture suggests that these coefficients should count some combinatorial objects. Lots of research focuses on building such combinatorial models. We give a brief summary of the pros and cons of four such models.

- **T-paths:** In [16], Schiffler obtained a formula for the cluster variables of a cluster algebra of finite type A (see §2.1 for the definition) in terms of T-paths. This formula has been modified and generalized to cluster algebras coming from surfaces [14, 15, 17, 18, 9].
- **Perfect matchings of a snake diagram:** A description that is similar to the T-path model but has a more graph-theoretic flavor [14]. Interesting combinatorics, for example the snake graph calculus [3, 4], arises in the study of this model. This formula is also restricted to cluster algebras coming from surfaces.
- **Compatible pairs in a Dyck path:** In [12], the cluster variables of rank 2 quivers, which do not necessarily come from surfaces, are described in terms of Dyck paths. A more general construction of the so-called compatible pairs is used in the study of greedy bases in [11], and another generalization called GCC is used in [1, 10].
- **Broken lines and Theta functions:** Discovered in [8], they are the most general

combinatorial models so far. But the model is so mysterious that even the finiteness of the number of broken lines is not immediate from the definition.

Our ultimate goal is to find a combinatorial model that is both general and effective in computation. Even though this goal appears out of reach for now, we feel that the model of maximal Dyck paths and compatible pairs has the potential to be generalized. This motivates the main goal of this thesis:

For a type A quiver, give a new formula for the cluster monomials using a combinatorial model similar to compatible pairs, and find the bijections to other known models.

We reach this goal by proving three equivalent formulas.

– In Theorem 5.1.1, we give a formula using a sequence of 0-1 sequences called GCS (globally compatible sequence), where each vertex of the quiver is assigned a 0-1 sequence satisfying a certain compatibility condition.

– In Theorem 5.2.2, we give a formula using globally compatible collections (GCCs) in Dyck paths. This formula has a similar flavor to the combinatorial formula for greedy bases in [11].

– In Chapter 3, we first use a combinatorial gadget called pipelines to decompose the **d**-vector of a cluster monomial into the ones of cluster variables, then give a formula for cluster variables using GCCs in Theorem 4.1.4.

The thesis is organized as follows. In Chapter 2 we recall the definition of cluster algebra and some facts about type A quivers. In §3.1 we define the **d**-vector of a cluster monomial and introduce its decomposition using pipelines. In the first section of Chapter 4, we define Dyck paths and then show the GCC formula for a cluster variable of complete extended linear quivers. In §4.2 we prove the GCC formula for cluster variables (Theorem 4.1.4) by establishing a bijection from GCCs to perfect matchings. In §4.3, we give an alternative proof of Theorem 4.1.4 using T-paths. Chapter 5 consists of three equivalent formulas for cluster monomials of type A quivers. The proof of the first two formulas is given in §5.4. Finally, we give some examples in Chapter 6.

CHAPTER 2. BACKGROUND ON CLUSTER ALGEBRAS AND TYPE A QUIVERS

In this chapter, we recall some definitions and fix notations about quivers and skew-symmetric cluster algebras (§2.1) and some special type A quivers (§2.2).

2.1 Quivers and skew-symmetric cluster algebras

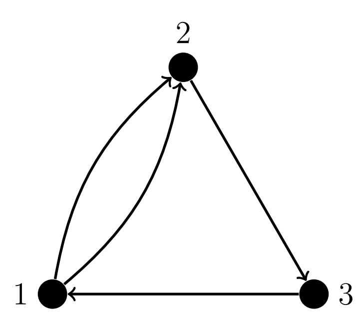
Recall that a finite oriented graph is a quadruple $Q = (Q_0, Q_1, h, t)$ formed by a finite set of vertices Q_0, a finite set of arrows Q_1 and two maps h and t from Q_1 to Q_0 which send an arrow α respectively to its head $h(\alpha)$ and its tail $t(\alpha)$. An arrow α whose head and tail coincide is a *loop*; a 2-*cycle* is a pair of distinct arrows β and γ such that $h(\beta) = t(\gamma)$ and $t(\beta) = h(\gamma)$. Similarly, it is clear how to define n-cycles for $n \geq 3$. A vertex is a *source* (respectively a *sink*) if it is not the head (resp. the tail) of any arrow.

In this thesis, a *quiver* is a finite oriented graph without loops or 2-cycles.

Given a quiver Q and a vertex $v \in Q_0$, the mutation $\mu_v(Q)$ is the new quiver Q' obtained as follows:

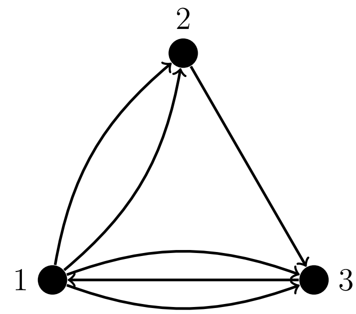
1. For every path of the form $u \to v \to w$, add a new arrow from u to w.
2. Reverse all arrows incident to v.
3. Remove all 2-cycles.

Example 2.1.1. Consider the following quiver.

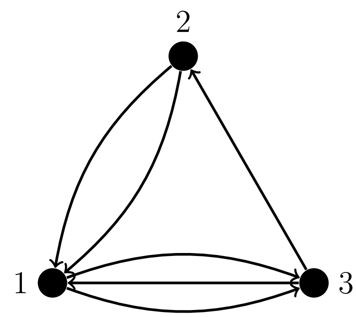


The sequence of steps to perform the mutation at the vertex 2 would be as follows:

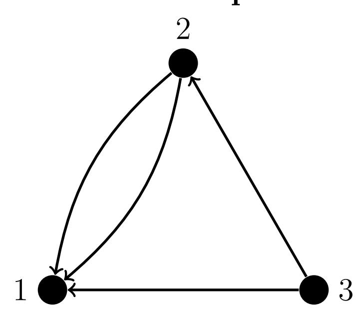
After Step 1



After Step 2



After Step 3



Next, we recall the definition of seeds and the mutation of seeds.

Let $Q = (Q_0, Q_1, h, t)$ be a quiver. Let $Q_0 = \{v_1, \ldots, v_n\}$ and let $F = \mathbb{Q}(x_1, \ldots, x_n)$ be the field of rational functions in $x_1, x_2, \ldots, x_n$ with rational coefficients. A *seed* is a pair $(Q, \mathbf{u})$ where $\mathbf{u} = \{u_1, u_2, \ldots, u_n\}$ is a set of elements of F which freely generate the field F.

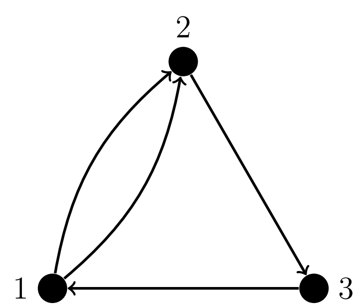
For any vertex $i \in Q_0$, we denote

$$\prod_{j \to i} u_j = \prod_{\alpha \in Q_1, h(\alpha)=i} u_{t(\alpha)}, \qquad \prod_{i \to j} u_j = \prod_{\alpha \in Q_1, t(\alpha)=i} u_{h(\alpha)}.$$

The mutation $\mu_i(Q, \mathbf{u})$ is the seed $(Q', \mathbf{u}')$ where $Q' = \mu_i(Q)$ and $\mathbf{u}'$ is obtained from $\mathbf{u}$ by replacing u_i by

$$u_i' = \frac{\prod_{j \to i} u_j + \prod_{i \to j} u_j}{u_i}.$$

Example 2.1.2. Consider the quiver in Example 2.1.1, where we identify the vertices with the variables $\{x_1, x_2, x_3\}$.



Then the quiver mutation at 2 maps $\{x_1, x_2, x_3\}$ to $\{x_1, x_2', x_3\}$, where

$$x_2' = \frac{\prod_{j \to 2} x_j + \prod_{2 \to j} x_j}{x_2} = \frac{x_1^2 + x_3}{x_2}.$$

If Q' is the mutated quiver in Example 2.1.1, then

$$\mu_2(Q, \{x_1, x_2, x_3\}) = (Q', \{x_1, \frac{x_1^2 + x_3}{x_2}, x_3\}).$$

Let $(Q, \{x_1 \dots, x_n\})$ be the initial seed. A *cluster* is a set $\mathbf{u}'$ which appears in a seed $(Q', \mathbf{u}')$ obtained from the initial seed by iterated mutations. An element in a cluster is called a *cluster variable*. A *cluster monomial* is a product of cluster variables in the same cluster. The (coefficient-free) *cluster algebra* $\mathcal{A}(Q)$ associated with Q is the subring of F

generated by all cluster variables.

Example 2.1.3. Consider the following seed $(\overset{1}{\bullet}\longrightarrow\overset{2}{\bullet},\ \{x_1, x_2\})$.

Mutating at 1,

$$(\overset{1}{\bullet}\longleftarrow\overset{2}{\bullet},\ \{\frac{x_2+1}{x_1}, x_2\}).$$

Mutating at 2,

$$(\overset{1}{\bullet}\longrightarrow\overset{2}{\bullet},\ \{\frac{x_2+1}{x_1}, \frac{x_1+x_2+1}{x_1x_2}\}).$$

For notational convenience, we rename the cluster variables $\frac{x_2+1}{x_1}$ and $\frac{x_1+x_2+1}{x_1x_2}$ as v_1 and v_2, respectively. Then, mutating at 1,

$$v_1' = \frac{\displaystyle\prod_{j\to 1} v_j + \prod_{1\to j} v_j}{v_1} = \frac{1+\dfrac{x_1+x_2+1}{x_1x_2}}{1+\dfrac{x_2+1}{x_1}} = \frac{x_1+1}{x_2}.$$

Hence the seed obtained is

$$(\overset{1}{\bullet}\longleftarrow\overset{2}{\bullet},\ \{\frac{x_1+1}{x_2}, \frac{x_1+x_2+1}{x_1x_2}\}).$$

Mutating at 2,

$$(\overset{1}{\bullet}\longrightarrow\overset{2}{\bullet},\ \{\frac{x_1+1}{x_2}, x_1\}).$$

Mutating at 1,

$$(\overset{1}{\bullet}\longleftarrow\overset{2}{\bullet},\ \{x_2, x_1\}).$$

Successive mutations will not produce new cluster variables due to symmetry. Hence, the set of all cluster variables is

$$\left\{x_1, x_2, \frac{x_1+1}{x_2}, \frac{x_2+1}{x_1}, \frac{x_1+x_2+1}{x_1x_2}\right\}$$

and the cluster algebra $\mathcal{A}(Q)$ is generated by the cluster variables.

Let us now extend the linear quiver to now include an additional vertex, such that

$$Q = 1 \to 2 \to 3$$

and the initial seed now becomes $(Q_0, \{x_1, x_2, x_3\})$. The set of cluster variables generated via the application of successive mutations would be

$$\left\{x_1, x_2, x_3, \frac{x_2+1}{x_1}, \frac{x_1+x_3}{x_2}, \frac{x_2+1}{x_2}, \frac{x_1+x_3(x_2+1)}{x_1x_2}, \frac{x_3+x_1(x_2+1)}{x_2x_3},\right.$$
$$\left., \frac{x_1(x_2+1)+x_3(x_2+1)}{x_1x_2x_3}\right\},$$

and the new cluster algebra is generated by these cluster variables.

Let $Q' = (Q'_0, Q'_1, h', t')$ be another quiver. We say that Q is a *subquiver* of Q' if

$$Q_0 \subseteq Q'_0 \text{ and } Q_1 \subseteq Q'_1$$

and $h(e) = h'(e)$ and $t(e) = t'(e)$ for any arrow $e \in Q_1$. We say that Q is a *full subquiver* of Q' if Q can be obtained from Q' by removing vertices $Q'_0 \setminus Q_0$ and their incident arrows.

2.2 Special classes of type A quivers

Here we define type A, linear, completely extended linear, and extended linear quivers. The relation among the four classes can be described as follows:

$$\begin{array}{ccccc} \{\text{type } A\} & \supset & \{\text{extended linear}\} & \supset & \{\text{completely extended linear}\} \\ & & & \rotatebox{-45}{$\supset$} & \{\text{linear}\} \end{array}$$

2.2.1 Type A quivers

By definition, type A quivers are those that are mutation equivalent to quivers of the form $\bullet \to \bullet \to \cdots \to \bullet$. In [2], A. Buan and D. Vatne showed that a type A quiver is a connected quiver such that

- all nontrivial simple cycles in the underlying graph have length 3, and the corresponding directed subgraphs are oriented (3-cycles);
- the vertex degrees of the underlying graph are at most 4; moreover, a degree-4 vertex belongs to two 3-cycles, a degree-3 vertex belongs to one 3-cycle.

2.2.2 Linear quivers

For two integers a and b, we denote $[a,b] = \{a, a+1, \ldots, b\}$ if $a \leq b$, and $[a,b] = \emptyset$ if $a > b$.

A *linear quiver* is a quiver with n vertices $\{v_1, v_2, \ldots, v_n\}$ (we also simply use i to denote the vertex v_i, if no confusion arises) and $n-1$ arrows in which any two consecutive vertices v_i and v_{i+1} ($i \in [1, n-1]$) are connected by a single arrow in either direction and there are no others arrows.

In order to have a convenient description for a linear quiver Q, we construct a sequence $\{\delta_i\}_{1 \leq i \leq n-1}$ such that $\delta_i = 0$ if there is an arrow going from the vertex v_i to the vertex v_{i+1}, and $\delta_i = 1$ otherwise. For example, for the quiver

$$1 \to 2 \to 3 \leftarrow 4 \to 5,$$

we have $(\delta_1, \delta_2, \delta_3, \delta_4) = (0, 0, 1, 0)$.

2.2.3 Completely extended linear quiver

We define a *completely extended linear quiver* Q' as one obtained from a linear quiver Q by attaching a 3-cycle to every edge, a 3-cycle to v_1, and a 3-cycle to v_n. (So Q' has $2n+3$ vertices.) By abuse of terminology, we also call the pair (Q, Q') a completely extended linear quiver, whenever we need to specify the linear quiver Q.

Let (Q, Q') be a completely extended linear quiver with $Q_0 = \{v_1, \dots, v_n\}$ and $Q'_0 = \{v_1, \dots, v_{2n+3}\}$. If $v_i \in Q'_0 \backslash Q_0$ is adjacent to both v_j and v_{j+1} then we also denote $v_{j,j+1} = v_i$. For the 3-cycle attached to v_1, the head (resp. tail) of the outgoing (resp. incoming) arrow is denoted $v_{1,0}$ (resp. $v_{1,1}$). We define $v_{n,0}$ and $v_{n,1}$ similarly.

Example 2.2.1. The quiver (Q, Q') in Figure 1 is a completely extended linear quiver, where Q is the linear part $1 \to 2 \to 3 \leftarrow 4$. In there, $v_{1,0} := 5$, $v_{1,1} := 6$, $v_{1,2} := 7$, $v_{2,3} := 8$, $v_{3,4} := 9$, $v_{4,0} := 10$ and $v_{4,1} := 11$.

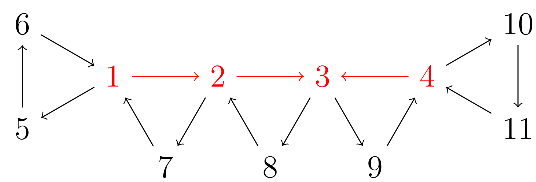


Figure 1: A completely extended linear quiver

For convenience, if $v_{j,k} = v_i$, then we denote the variable $x_{j,k} = x_i$.

2.2.4 Extended linear quivers

An *extended linear quiver* (Q, P) is obtained from a completely extended linear quiver (Q, Q') by removing some vertices (or none) in $Q'_0 \setminus Q_0$ and the arrows incident with them. Equivalently, we can characterize P as a quiver obtained from Q by adding some (or none) of the following:

- a 3-cycle or an edge hung on v_1, or
- a 3-cycle or an edge hung on v_n, or
- 3-cycles attached to some edges of Q.

There is an obvious way to obtain a completely extended linear quiver (Q, Q') from an extended linear quiver (Q, P) (up to relabeling vertices in $Q'_0 \setminus P_0$). An example is shown in

Figure 2.

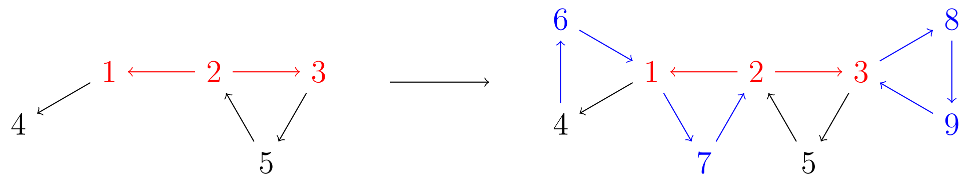


Figure 2: An extended linear quiver before and after being completed

CHAPTER 3. PARAMETRIZATION OF CLUSTER MONOMIALS BY d-VECTORS

3.1 Cluster monomials and d-vectors

It is well known that any cluster algebra associated to a type A quiver with n vertices can be constructed from a triangulation on the $(n+3)$-gon.

A *diagonal* on the $(n+3)$-gon is a line segment connecting two non-adjacent vertices. A connected curve on the polygon is called a *pseudo-diagonal* if it is isotopic to a diagonal (and its endpoints are the same as those of the diagonal) and if its interior is in the interior of the polygon. Two pseudo-diagonals are said to be *crossing* if they intersect in the interior of the polygon.

Let Q be a type A quiver with n vertices, and let $\{T_1, ..., T_n\}$ be the diagonals given by the corresponding triangulation on the $(n+3)$-gon and $\{T_{n+1}, ..., T_{2n+3}\}$ the boundary edges. It is also known from [7] that the cluster variables of $\mathcal{A}(Q)$ are in natural bijection with all the diagonals of the polygon.

Using this bijection, a cluster monomial of $\mathcal{A}(Q)$ can be identified with a finite set of pairwise non-crossing pseudo-diagonals, or equivalently with

$$\{(D_1, d_1), ..., (D_m, d_m)\},$$

where m is a positive integer, $D_1, ..., D_m$ are pairwise non-crossing pseudo-diagonals, and $d_1, ..., d_m$ are positive integers.

The following definition uses a natural intersection number, which was already considered in [5, 14, 17, 18, 15].

Definition 3.1.1. For two diagonals D, E on the $(n+3)$-gon, we define the intersection number $i(D,E)$ of D and E as follows:

$$i(D,E) := \begin{cases} 1, & \text{if } D \text{ and } E \text{ cross each other;} \\ -1, & \text{if } D \text{ and } E \text{ are the same;} \\ 0, & \text{otherwise.} \end{cases}$$

Then the **d**-vector of the cluster monomial $\{(D_1,d_1),...,(D_m,d_m)\}$ is defined by

$$(\sum_{j=1}^{m} d_j i(D_j,T_1),...,\sum_{j=1}^{m} d_j i(D_j,T_n)).$$

The **d**-vector $(a_1,...,a_n)$ of any cluster monomial satisfies the following property.

Lemma 3.1.2. (Property A) *For any 3-cycle $i \to j \to k \to i$ in Q such that a_i, a_j, a_k are positive and satisfy the triangle inequalities (i.e., the sum of any two numbers is strictly greater than the third), the sum $a_i + a_j + a_k$ is even.*

Proof. Let P_{ijk} be the set of crossing point of $D_1,\dots,D_m$ and T_i, T_j, T_k. Then $|P_{ijk}| = a_i + a_j + a_k$. For each $u \in [1,m]$, D_u only cross two diagonals, so contributes two points to P_{ijk}. That implies an even number of intersection points is made by all D_u, $u \in [1,m]$. Therefore, P_{ijk} has an even number of elements. $\square$

3.2 Construction of pipelines

Let $\mathcal{W}$ be the set of all integer vectors $(a_1,...,a_n)$ satisfying Property A. In this subsection we prove that the cluster monomials of $\mathcal{A}(Q)$ are in bijection with $\mathcal{W}$. We will define a map from $\mathcal{W}$ to the cluster monomials, which would then induce the immediate bijection. Let $[x]_+ = \max(x,0)$ for any real number x.

Let $\mathbf{a} = (a_1, ..., a_n) \in \mathcal{W}$. We define a function $\sigma : \mathbb{R}^3_{\geq 0} \to \mathbb{R}_{\geq 0}$ as follows:

$$\sigma(x,y,z) = \frac{[x+y-z]_+ - [x-y-z]_+ - [y-x-z]_+}{2} = \begin{cases} x, & \text{if } y > x+z; \\ y, & \text{if } x > y+z; \\ 0, & \text{if } z > x+y; \\ \dfrac{x+y-z}{2}, & \text{otherwise.} \end{cases}$$

For convenience, we denote

$$\sigma^{\mathbf{a}}_{ijk} = \sigma(a_i, a_j, a_k).$$

(If no confusion shall arise, we denote $\sigma_{ijk} = \sigma^{\mathbf{a}}_{ijk}$.)

We are ready to construct the so-called pipelines associated to $\mathbf{a}$.

Step 1: If $a_i > 0$ then draw a_i marking points on the diagonal T_i to separate it into $a_i + 1$ segments. If $a_i < 0$ then draw $-a_i$ pipes so that these pipes are pairwise non-crossing pseudo-diagonals isotopic to T_i and that they do not cross any other T_j $(j \neq i)$.

Step 2: If T_i and T_j are two sides of a triangle with the third side T_k, then for $1 \leq r \leq \sigma_{ijk}$, we join the two r-th marking points on T_i and T_j (ordering in the increasing distance from the common endpoint of T_i and T_j) by a pipe inside the triangle. Draw these pipes so that they are disjoint from each other and from the pipes constructed in Step 1.

Step 3: Suppose that T_i, T_j, T_k form a triangle. Then for each marking point on T_i that is not connected by a pipe to any marking point on T_j or T_k, we draw a pipe from the marking point to the common endpoint of T_j and T_k. Draw these pipes inside the triangle in such a way that they are non-crossing with each other and with the pipes constructed in Step 1,2.

A pipeline is a union of pipes connected consecutively through the marking points (but not through the vertices of the $(n+3)$-gon). Then the pipelines are pairwise non-crossing. Since the endpoints of each pipeline are non-adjacent vertices of the polygon, every pipeline is a pseudo-diagonal. Hence the union of these pipelines corresponds to a cluster monomial. Clearly the $\mathbf{d}$-vector of this cluster monomial is equal to $\mathbf{a}$.

Using the above construction of pipelines, it is straightforward to prove the following:

Proposition 3.2.1. *Let Q be a type A quiver. Then*

(1) $\mathbf{a} \in \mathbb{Z}^n$ is the $\boldsymbol{d}$-vector of some cluster monomial of $\mathcal{A}(Q)$ if and only if $\mathbf{a} \in \mathcal{W}$.

(2) Two distinct cluster monomials have different $\boldsymbol{d}$-vectors.

This allows us to denote the (unique) cluster monomial with $\mathbf{d}$-vector $(a_1, \ldots, a_n)$ by $x[a_1, \ldots, a_n]$ or $x[\mathbf{a}]$.

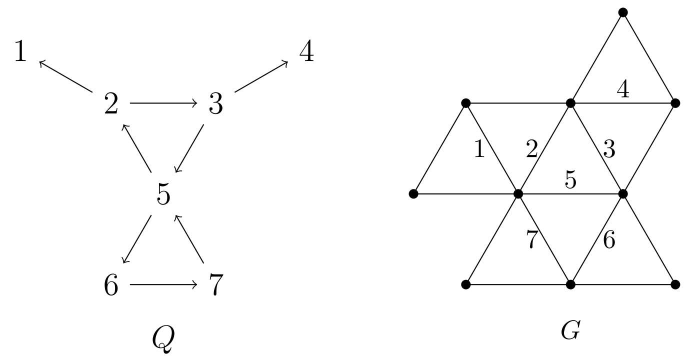
Each pipe Λ corresponds to a 0-1 sequence $\mathbf{b} = \mathbf{b}_\Lambda = (b_1, \ldots, b_n)$ such that

$$b_i = \begin{cases} 0, & \text{if the pipe } \Lambda \text{ is disjoint from } T_i; \\ 1, & \text{otherwise.} \end{cases} \tag{3.1}$$

Note that Λ corresponds to a linear full subquiver of Q. Let S be the multiset of all such sequences. Then

$$x[\mathbf{a}] = \prod_{\mathbf{b} \in S} x[\mathbf{b}]. \tag{3.2}$$

Example 3.2.2. The first two pictures in Figure 3 are a type A quiver with 7 vertices and its corresponding triangulation on the 10-gon. For clearer illustration, the 10-gon is drawn as a concave polygon. The bottom illustrates the construction of pipelines for $\mathbf{a} = (3, 3, 3, 2, 4, 3, 1)$.



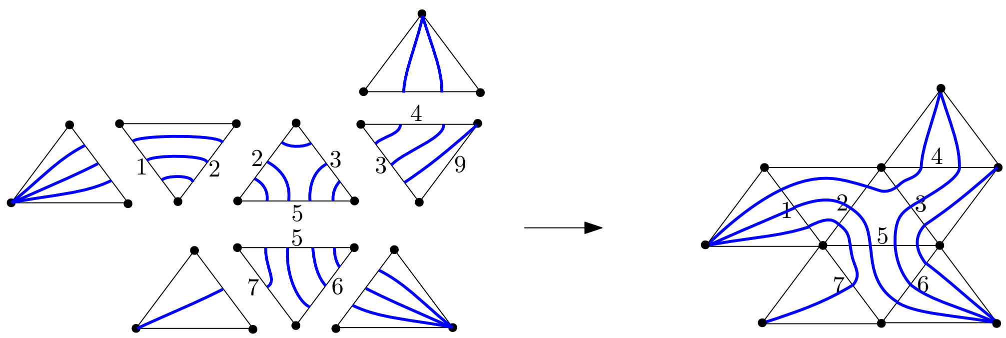


Figure 3: The construction of pipelines

There are 5 pipelines, passing through edges sets $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,5,7\}$, $\{3,4,5,6\}$, $\{3,5,6\}$, respectively. So the cluster monomial $x[\mathbf{a}]$ is decomposed as

$$x[3,3,3,2,4,3,1] = x[1,1,1,1,0,0,0] \cdot x[1,1,0,0,1,1,0] \cdot x[1,1,0,0,1,0,1] \cdot$$

$$\cdot\, x[0,0,1,1,1,1,0] \cdot x[0,0,1,0,1,1,0].$$

Lemma 3.2.3. *For $\mathbf{a} \in \mathbb{Z}^n$, assume that $[\mathbf{a}]_+$ satisfies Property A and $x[[\mathbf{a}]_+] = \prod x[\mathbf{b}]$ is a factorization into cluster variables in the same cluster. Then $x[\mathbf{b}]$ and x_i $(a_i < 0)$ are in the same cluster and that*

$$x[\mathbf{a}] = x[[\mathbf{a}]_+] \prod_{i=1}^{n} x_i^{[-a_i]_+} = \prod x[\mathbf{b}] \prod_{i} x_i^{[-a_i]_+}.$$

Proof. If $a_i < 0$, then there is no $\mathbf{b}$ satisfying $b_i > 0$. Thus no pseudo-diagonals corresponding to pipelines for $[\mathbf{a}]_+$ will cross the diagonal T_i. Therefore, after adding T_i we still get a set of non-crossing pseudo-diagonals, which means that $x[\mathbf{b}]$ and x_i ($a_i < 0$) are in the same cluster. It then follows that $x[\mathbf{a}] = x[[\mathbf{a}]_+]\prod_i x_i^{[-a_i]_+}$. $\square$

Remark 3.2.4. Assume that Q is a full subquiver of Q'. We compare cluster variables in various cluster algebras.

For simplicity denote the vertex sets $Q_0 = \{1, \ldots, n\}$ and $Q'_0 = \{1, \ldots, n'\}$. By definition, the cluster algebra with coefficients in $\mathbb{Z}[x_{n+1}^{\pm 1}, \ldots, x_{n'}^{\pm 1}]$, denoted $\mathcal{A}(Q, Q')$, is generated by only those cluster variables in $\mathcal{A}(Q')$ obtained from iteratively mutating the initial cluster variables $x_1, \ldots, x_n$ only at vertices in $\{1, \ldots, n\}$. (Vertices in $\{n+1, \ldots, n'\}$ are called frozen vertices. The coefficients are in $\mathbb{Z}[x_{n+1}^{\pm 1}, \ldots, x_{n'}^{\pm 1}]$.) Thus, there is a natural bijection sending a cluster variable in $\mathcal{A}(Q, Q')$ of the form $x[d_1, \ldots, d_n]$ to the cluster variable $x[d_1, \ldots, d_n, 0, \ldots, 0] \in \mathcal{A}(Q')$ of the same expression.

There is also a natural bijection sending a cluster variable $x[d_1, \ldots, d_n] \in \mathcal{A}(Q, Q')$ to the cluster variable $x[d_1, \ldots, d_n] \in \mathcal{A}(Q)$, given by setting x_i to 1 for $i \in [n+1, n']$. More generally, if Q is a full subquiver of P, and P is a full subquiver of Q', then there is a natural bijection sending a cluster variable $x[d_1, \ldots, d_n] \in \mathcal{A}(Q, Q')$ to $x[d_1, \ldots, d_n] \in \mathcal{A}(Q, P)$ given by setting x_i by 1 for $i \in Q'_0 \setminus P_0$.

If Q is a full subquiver of Q'', and Q' is the vertex-induced subquiver of Q'' whose vertex set consists of vertices in Q_0 and those adjacent to Q_0, then the cluster variables in $\mathcal{A}(Q, Q')$ have the same expressions as those in $\mathcal{A}(Q, Q'')$.

CHAPTER 4. CLUSTER VARIABLES

Let (Q, Q') be a completely extended linear quiver. By relabeling vertices of Q' if necessary, we assume $Q_0 = \{1, \ldots, n\}$. We shall give a formula of the cluster variable $x[\mathbf{a}]$ of $\mathcal{A}(Q')$, where $\mathbf{a} = (a_1, \ldots, a_{n'})$ such that $a_i = 1$ if $i \in [1, n]$ and $a_i = 0$ if $i \in [n+1, n']$.

First, we recall the following definition from [10].

4.1 Globally Compatible Collections

Let (a_1, a_2) be a pair of nonnegative integers. Let $c = \min(a_1, a_2)$. The *maximal Dyck path* of type $a_1 \times a_2$, denoted by $\mathcal{D} = \mathcal{D}^{a_1 \times a_2}$, is a lattice path from $(0, 0)$ to (a_1, a_2) that is as close as possible to the diagonal joining $(0, 0)$ and (a_1, a_2), but never goes above it. A *corner* is a subpath consisting of a horizontal edge followed by a vertical edge.

Definition 4.1.1. Let $\mathcal{D}_1$ (resp. $\mathcal{D}_2$) be the set of horizontal (resp. vertical) edges of a maximal Dyck path $\mathcal{D} = \mathcal{D}^{a_1 \times a_2}$. We label $\mathcal{D}$ with the *corner-first index* in the following sense:

(a) edges in $\mathcal{D}_1$ are indexed as $u_1, \ldots, u_{a_1}$ such that u_i is the horizontal edge of the i-th corner for $i \in [1, c]$ and u_{c+i} is the i-th of the remaining horizontal ones for $i \in [1, a_1 - c]$,

(b) edges in $\mathcal{D}_2$ are indexed as $v_1, \ldots, v_{a_2}$ such that v_i is the vertical edge of the i-th corner for $i \in [1, c]$ and v_{c+i} is the i-th of the remaining vertical ones for $i \in [1, a_2 - c]$.

(Here we count corners from bottom left to top right, count vertical edges from bottom to top, and count horizontal edges from left to right.)

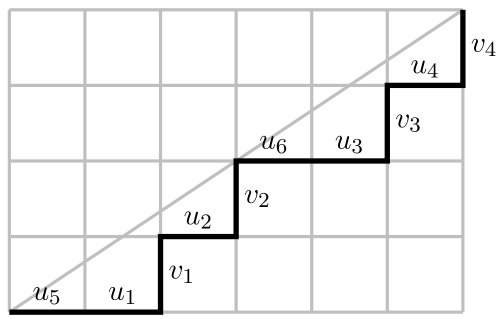


Figure 4: A maximal Dyck path

Definition 4.1.2. Let $S_1 \subseteq \mathcal{D}_1$, $S_2 \subseteq \mathcal{D}_2$, $s \in \mathbb{Z}_{\geq 0}$. We say that S_1 and S_2 are *s-compatible* if for every $1 \leq r \leq s$, either $u_r \notin S_1$ or $v_r \notin S_2$. In other words, neither of the first s corners are contained in the subpath $S_1 \cup S_2$.

Now for any arrow $(i+\delta_i) \to (i+1-\delta_i)$ of Q, we attach a Dyck path $\mathcal{D}^{(i)} = \mathcal{D}^{1\times 1}$, which consists of one horizontal edge and one vertical edge. (Recall that δ_i is defined in §2.2.2.)

Definition 4.1.3. Let $S_{i,r} \subseteq \mathcal{D}_r^{(i)}$ for $i \in [1, n-1]$, $r \in [1,2]$. We say that the collection $\{S_{i,r}\}$ is a *Globally Compatible Collection (abbreviated GCC)* if

$$S_{i,1} \text{ and } S_{i,2} \text{ are 1-compatible for } i \in [1, n-1],$$

and the following holds for $i \in [2, n-1]$,

(a) if $(\delta_{i-1}, \delta_i) = (0,0)$, then $|S_{i-1,2}| \neq |S_{i,1}|$;

(b) if $(\delta_{i-1}, \delta_i) = (1,1)$, then $|S_{i-1,1}| \neq |S_{i,2}|$;

(c) if $(\delta_{i-1}, \delta_i) = (0,1)$, then $|S_{i-1,2}| = |S_{i,2}|$;

(d) if $(\delta_{i-1}, \delta_i) = (1,0)$, then $|S_{i-1,1}| = |S_{i,1}|$.

Theorem 4.1.4. *The cluster variable with $\boldsymbol{d}$-vector $\mathbf{a}$ is*

$$x[\mathbf{a}] = \left(\prod_{i=1}^{n} x_i^{-1}\right) \sum \left(\prod_{i=0}^{n} y_i\right), \tag{4.1}$$

where the sum runs over all GCCs $\{S_{i,r}\}$, *and*

$$y_i := x_{i+\delta_i}^{|S_{i,2}|} x_{i+1-\delta_i}^{|S_{i,1}|} x_{i,i+1}^{1-|S_{i,1}|-|S_{i,2}|} = \begin{cases} x_i, & \text{if} \quad (|S_{i,1}|, |S_{i,2}|) = (\delta_i, 1-\delta_i) \\ x_{i+1}, & \text{if} \quad (|S_{i,1}|, |S_{i,2}|) = (1-\delta_i, \delta_i) \\ x_{i,i+1}, & \text{if} \quad (|S_{i,1}|, |S_{i,2}|) = (0,0) \end{cases}$$

for $i \in [1, n-1]$, *and*

$$y_0 := \begin{cases} x_{1,0}, & \text{if } |S_{1,1+\delta_1}| = 1-\delta_1, \\ x_{1,1}, & \text{otherwise,} \end{cases} \qquad y_n := \begin{cases} x_{n,0}, & \text{if } |S_{n-1,2-\delta_{n-1}}| = \delta_{n-1}, \\ x_{n,1}, & \text{otherwise.} \end{cases}$$

Remark 4.1.5. Note that the above theorem induces a formula for the cluster variables of any type A quiver. Indeed, let $\tilde{Q}$ be any type A quiver and non-initial $x[\mathbf{a}]$ be a cluster variable with $\mathbf{d}$-vector $\mathbf{a} = (a_1, \ldots, a_{\tilde{n}})$ (here $\tilde{n} = |\tilde{Q}_0|$). Then the subset of vertices $\{i \mid a_i = 1\}$ is equal to the set of vertices Q_0 of a linear full subquiver Q. By relabeling vertices if necessary, we assume $Q_0 = \{1, \ldots, n\}$ (thus $a_1 = \cdots = a_n = 1$ and $a_{n+1} = \cdots = a_{\tilde{n}} = 0$ and for convenience we denote $\mathbf{a} = 1^n 0^{\tilde{n}-n}$. By removing vertices in $\tilde{Q}_0$ but not in or adjacent to Q_0, we get an extended linear quiver (Q, P); from this extended linear quiver we can obtain a completely extended linear quiver Q'. Define $n' = |Q'_0|$, $m = |P_0|$. Thanks to Remark 3.2.4, a formula for cluster variable $x[1^n 0^{n'-n}] \in \mathcal{A}(Q, Q')$ induces a formula for $x[1^n 0^{m-n}] \in \mathcal{A}(Q, P)$ by setting x_i to 1 for $i \in Q'_0 \setminus P_0$, which is also a formula for $x[\mathbf{a}] = x[1^n 0^{\tilde{n}-n}] \in \mathcal{A}(\tilde{Q})$. (See Example 6.1.)

4.2 A bijection between perfect matchings and GCCs

In this section, we first recall the construction of snake diagram and the formula of cluster variables using perfect matching as in [14], then give a bijective proof of Theorem 4.1.4 via perfect matching.

Associated to a completely extended linear quiver (Q, Q'), we recursively construct the snake diagram by gluing n-tiles together as follows: we first put the 2^{nd}-tile to the right side of the 1^{st}-tile; suppose the i^{th}-tile is placed, we add the $(i+1)^{\text{th}}$-tile to the right side or on top of the i^{th}-tile such that the $(i-1)^{\text{th}}$, $(i)^{\text{th}}$ and $(i+1)^{\text{th}}$-tiles are in the same row or column if and only if $\delta_{i-1} \neq \delta_i$.

Next, we label the edges as follows.

- The common edge of the i^{th}-tile and the $(i+1)^{\text{th}}$-tile is labeled $T_{i,i+1}$.
- Denote by $Pl^{(i)}$ the parallelogram bounded by the main diagonals of the i^{th}-tile and the $(i+1)^{\text{th}}$-tile and two boundary edges. Any edge forming an angle of 135° with the main diagonal of the i^{th}-tile will be labeled $T_i^{(j)}$ (where j indicates the parallelogram to which the edge belongs).

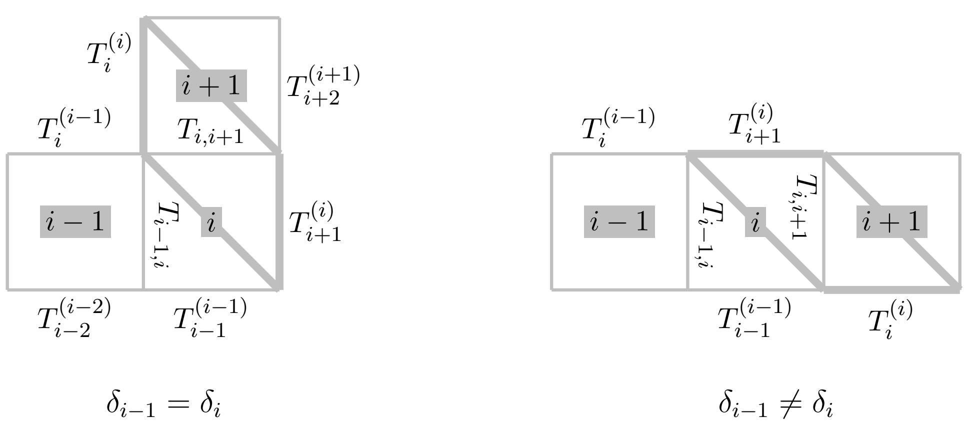


Figure 5: Labels of edges in $(i-1)^{\text{th}}$, i^{th} and $(i+1)^{\text{th}}$-tiles in two cases

For convenience, we let $Pl^{(0)}$ be the right triangle with legs $T_{1,0}$ and $T_{1,1}$, and let $Pl^{(n)}$ the right triangle with legs $T_{n,0}$ and $T_{n,1}$. The edges of the first and the last tiles are labeled as in Figure 6.

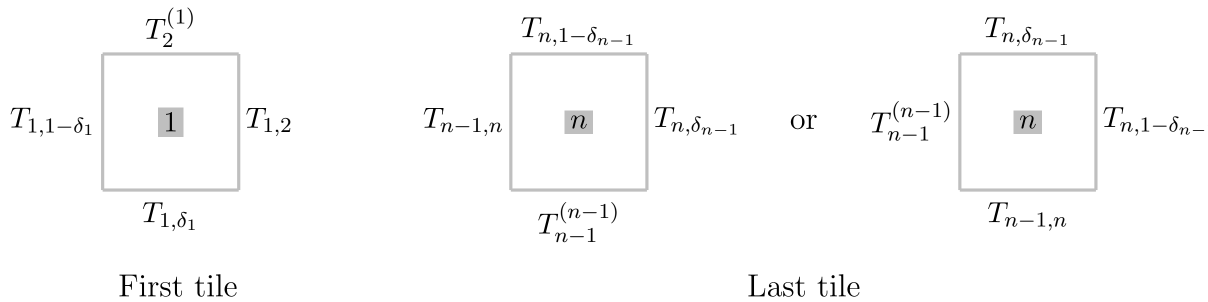


Figure 6: Labels of edges in the first and last tiles

Example 4.2.1. Associated to the completely extended quiver in Example 2.2.1, we have the following snake diagram.

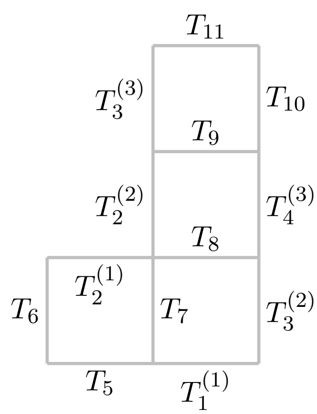


Figure 7: A snake diagram

A *perfect matching* of the snake diagram is a set of edges such that each vertex is incident to exactly one edge in the set.

For the fixed completely extended linear quiver (Q, Q'), let $\mathcal{M}$ be the set of all perfect matchings in the associated snake diagram, and $\mathcal{G}$ be the set of all GCCs. We shall construct a bijective map $\psi_1 : \mathcal{M} \to \mathcal{G}$ and its inverse ψ_2. First we prove a simple lemma.

Lemma 4.2.2. *For any perfect matching γ and $i \in [0, n]$, there is exactly one edge of γ that lies in $Pl^{(i)}$.*

Proof. The statement is obviously true for $i = 0$ and $i = n$. Suppose the statement is false for some $i \in [1, n-1]$. Since γ is a perfect matching, we are in one of the following two cases.

Case 1: both $T_i^{(i)}, T_{i+1}^{(i)}$ are in γ. If we remove $Pl^{(i)}$ (4 vertices and 3 edges), then the rest of the graph has two components which have odd number of vertices and have perfect matchings. This is a contradiction.

Case 2: none of the three edges $T_i^{(i)}, T_{i+1}^{(i)}, T_{i,i+1}$ lies in γ. Then we remove the three edges (but do not remove the vertices) and apply the same argument as in Case 1. $\square$

Remark 4.2.3. Thanks to the above lemma, we can write a perfect matching γ as $\{\gamma_1, \dots, \gamma_n\}$ where $\gamma_i \in Pl^{(i)}$.

Definition 4.2.4. (i) We define a map $\psi_{\mathcal{M},\mathcal{G}} : \mathcal{M} \to \mathcal{G}$ by sending $\gamma \in \mathcal{M}$ to $\{S_{i,r}\} \in \mathcal{G}$ such that for $i \in [1, n-1]$,

$$(|S_{i,1}|, |S_{i,2}|) = \begin{cases} (\delta_i, 1-\delta_i), & \text{if } T_i^{(i)} \in \gamma, \\ (1-\delta_i, \delta_i), & \text{if } T_{i+1}^{(i)} \in \gamma, \\ (0,0), & \text{if } T_{i,i+1} \in \gamma. \end{cases}$$

(By Lemma 4.2.2, exactly one of the three cases occurs.)

(ii) We define a map $\psi_{\mathcal{G},\mathcal{M}} : \mathcal{G} \to \mathcal{M}$ by sending $\{S_{i,r}\} \in \mathcal{G}$ to the set of edges $\gamma = \{\gamma_0, \gamma_1, \dots, \gamma_n\}$ such that

$$\gamma_i = \begin{cases} T_i^{(i)}, & \text{if } \ (|S_{i,1}|, |S_{i,2}|) = (\delta_i, 1-\delta_i) \\ T_{i+1}^{(i)}, & \text{if } \ (|S_{i,1}|, |S_{i,2}|) = (1-\delta_i, \delta_i) \\ T_{i,i+1}, & \text{if } \ (|S_{i,1}|, |S_{i,2}|) = (0,0) \end{cases}$$

for $i \in [1, n-1]$, and

$$\gamma_0 = \begin{cases} T_{1,0} & \text{if } |S_{1,1+\delta_1}| = 1-\delta_1, \\ T_{1,1} & \text{otherwise,} \end{cases} \qquad \gamma_n = \begin{cases} T_{n,0} & \text{if } |S_{n-1,2-\delta_{n-1}}| = \delta_{n-1}, \\ T_{n,1} & \text{otherwise.} \end{cases}$$

We assign a weight $w(u)$ for each edge u of the snake diagram as follows for all i, j:

$$w(T_j^{(i)}) = x_j, \quad w(T_{j,j+1}) = x_{j,j+1} \tag{4.2}$$

For a perfect matching $\gamma = (\gamma_0, \ldots, \gamma_n)$, define its weight $w(\gamma) = \prod_{i=0}^n w(\gamma_i)$. In [14] it is proved that the cluster variable with $\mathbf{d}$-vector $\mathbf{a}$ is

$$x[\mathbf{a}] = \left(\prod_{i=1}^n x_i^{-1}\right) \sum_\gamma w(\gamma) \tag{4.3}$$

(compare with 4.1.)

Theorem 4.2.5. *The maps $\psi_{\mathcal{M},\mathcal{G}}$ and $\psi_{\mathcal{G},\mathcal{M}}$ are well-defined and are inverses of each other. Moreover, $w(\gamma_i) = y_i$, thus $\psi_{\mathcal{M},\mathcal{G}}$ induces a bijective proof of Theorem 4.1.4 using* (4.3)*.*

Proof. (i) We show that $\psi_{\mathcal{M},\mathcal{G}}$ is well-defined, that is, $\psi_{\mathcal{M},\mathcal{G}}(\gamma) = \{S_{i,r}\}$ satisfies the condition (4.1.3). It's clear from the construction that for every $i \in [1, n-1]$, $(|S_{i,1}|, |S_{i,2}|) \neq (1,1)$. Next, we prove (a) and (c) of (4.1.3), since (b) and (d) can be proved similarly.

For (a), we suppose $(\delta_{i-1}, \delta_i) = (0,0)$ and need to show that $T_{i-1}^{(i-1)} \in \gamma \Leftrightarrow T_{i+1}^{(i)} \notin \gamma$. This is true because the two edges $T_{i-1}^{(i-1)}$ and $T_{i+1}^{(i)}$ are incident to the same deg-2 vertex, thus exactly one of them is in γ. (See the left diagram in Figure 5.)

For (c), we suppose $(\delta_{i-1}, \delta_i) = (0,1)$ and need to show that $T_{i-1}^{(i-1)} \in \gamma \Leftrightarrow T_{i+1}^{(i)} \in \gamma$. These two edges are opposite edges of a tile which is the middle of three tiles in a row or a column. Deleting these two edges will separate the snake diagram into two graphs with even number of vertices each. Thus the two edges must be both in γ or not in γ. (See the right diagram in Figure 5.)

(ii) We show that $\psi_{\mathcal{G},\mathcal{M}}$ is well-defined, that is, $\psi_{\mathcal{G},\mathcal{M}}(\{S_{i,r}\}) = \gamma$ is a perfect matching. Since the snake diagram has $2n+2$ vertices and γ has $n+1$ edges, it suffices to show that all edges in γ are disjoint. We assume the contrary that γ_c shares a vertex with γ_d for some $0 \le c < d \le n$. Since $\gamma_c \in Pl^{(c)}$ and $\gamma_d \in Pl^{(d)}$, $Pl^{(c)}$ and $Pl^{(d)}$ much be consecutive, thus $d = c+1$.

We first assume $1 \le c \le n-2$. We shall discuss two cases $(\delta_c, \delta_{c+1}) = (0,0)$ and $(0,1)$, and omit the other two cases $(1,0)$ and $(1,1)$ since the proof is similar.

Case $(\delta_c, \delta_{c+1}) = (0,0)$: since $\{S_{i,r}\}$ is a GCC, we must have $(|S_{c,1}|, |S_{c,2}|, |S_{c+1,1}|, |S_{c+1,2}|) = (0,1,0,1)$, $(0,1,0,0)$, $(1,0,1,0$, or $(0,0,1,0)$. Correspondingly,

$$(\gamma_c, \gamma_{c+1}) = (T_c^{(c)}, T_{c+1}^{(c+1)}), (T_c^{(c)}, T_{c+1,c+2}), (T_{c+1}^{(c)}, T_{c+2}^{(c+1)}), \text{ or } (T_{c,c+1}, T_{c+2}^{(c+1)}).$$

It is obvious from Figure 8 that γ_c and γ_{c+1} are disjoint, a contradiction as expected.

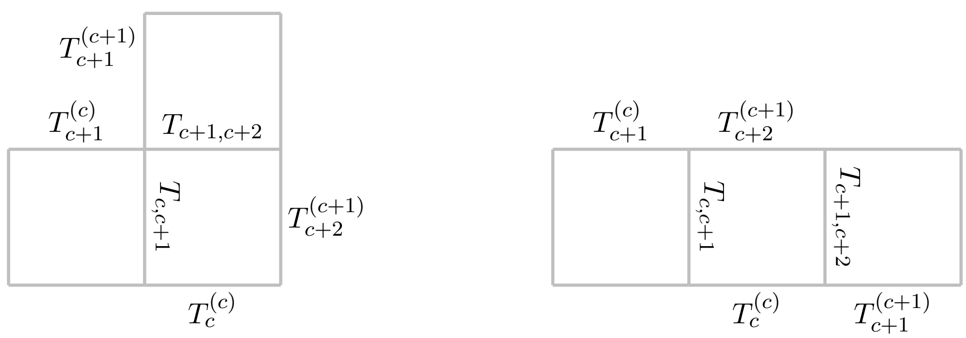


Figure 8: Left: $(\delta_c, \delta_{c+1}) = (0,0)$ and Right: $(\delta_c, \delta_{c+1}) = (0,1)$

Case $(\delta_c, \delta_{c+1}) = (0,1)$: similar as the above case,

$$(\gamma_c, \gamma_{c+1}) = (T_c^{(c)}, T_{c+2}^{(c+1)}), (T_{c+1}^{(c)}, T_{c+1}^{(c+1)}), (T_{c+1}^{(c)}, T_{c+1,c+2}), (T_{c,c+1}, T_{c+1}^{(c+1)}) \text{ or } (T_{c,c+1}, T_{c+1,c+2}).$$

We get the expected contradiction by observing Figure 8.

The cases of $c = 0$ and $c = n-1$ are proved by a similar argument.

(iii) The fact that $\psi_{\mathcal{M},\mathcal{G}}$ and $\psi_{\mathcal{G},\mathcal{M}}$ are inverses of each other, and $w(\gamma_i) = y_i$, follows easily from their definitions. $\square$

Example 4.2.6. We put a perfect matching $\gamma = \{T_1, T_3, T_6, T_8, T_{10}\}$ on the snake diagram in Example 4.2.1. Applying the map $\psi_{\mathcal{M},\mathcal{G}}$ to γ, we get $\psi_{\mathcal{M},\mathcal{G}}(\gamma) = ((0,1),(0,0),(1,0))$.

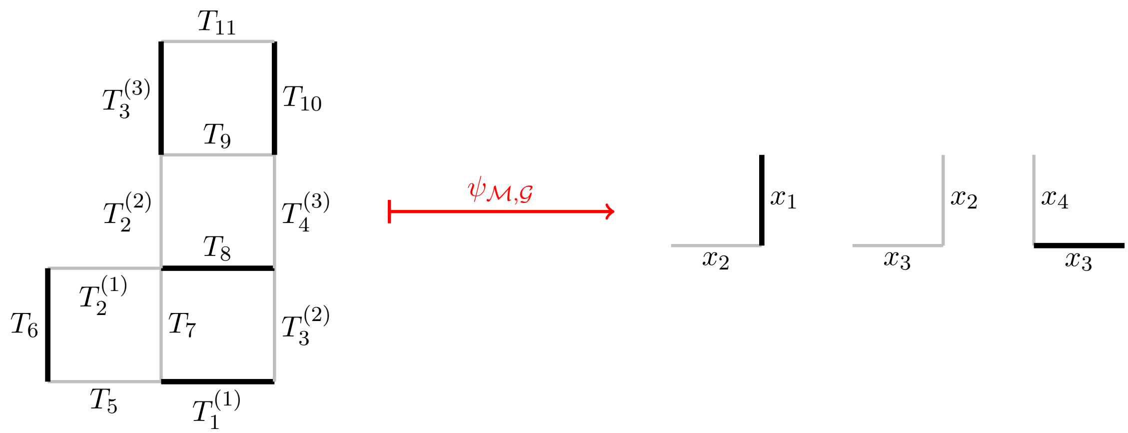


Figure 9: An example of the map $\psi_{\mathcal{M},\mathcal{G}}$

4.3 A bijection between T-paths and GCCs

In this section, we give an alternative proof of Theorem 4.1.4 via a model called T-paths. We first recall the construction of T-paths and the formula of cluster variables using T-paths as in [16], then give the bijective proof of the theorem.

The quiver being considered here is still a completely extended linear quiver (Q, Q'). Let P be a convex polygon with $n+3$ vertices. A diagonal of P is a line segment connecting two non-adjacent vertices. Two diagonals are said to be crossing if they intersect in the interior of P. A triangulation T of P is a maximal set of non-crossing diagonals together with the boundary edges of P. Any triangulation has n diagonals and $n+3$ boundary edges.

Our initial triangulation of P will consist of the set $T = \{T_1, \ldots, T_n\} \cup \{T_{1,0}, T_{1,1}, T_{n,0}, T_{n,1}\} \cup$

$\{T_{i,i+1} : i \in [1, n-1]\}$, where the first set is the set of diagonals and the last two sets constitute the set of boundary edges.

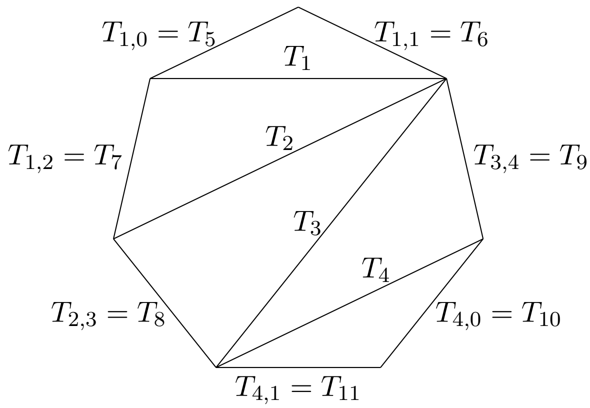


Figure 10: The initial triangulation of the quiver (Q, Q') in Example 2.2.1.

The process of constructing the initial triangulation starts with placing the diagonal T_1. We obtain $T_{1,0}$ by rotating T_1 in the counterclockwise direction. The edge $T_{1,1}$ is obtained by rotating T_1 in the clockwise direction. Let $\mathfrak{v}$ be the common vertex of $T_{1,0}$ and $T_{1,1}$.

Suppose that the diagonal T_i ($i \in [1, n-1]$) is drawn. The diagonal T_{i+1} is obtained by rotating T_i in the counterclockwise direction if $\delta_i = 0$, in the clockwise direction if $\delta_i = 1$. The boundary edge between T_i and T_{i+1} is labeled $T_{i,i+1}$.

When $i = n$ then $T_{n,1}$ is the boundary edge clockwise from T_n and $T_{n,0}$ is the boundary edge counterclockwise from T_n. Denote the common vertex of $T_{n,0}$ and $T_{n,1}$ by $\mathfrak{w}$.

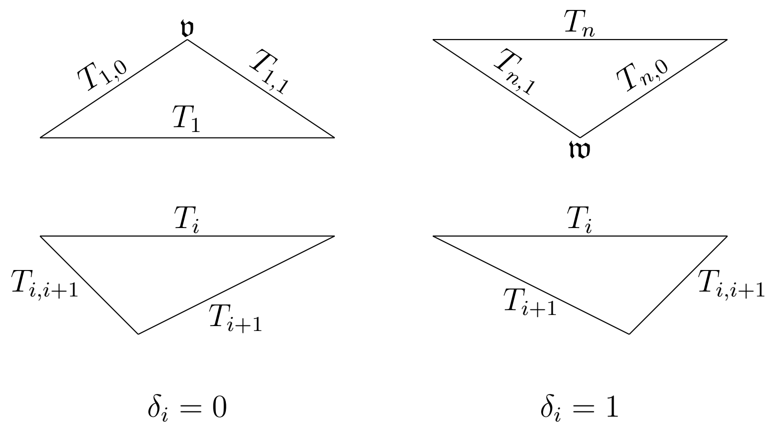


Figure 11: Labels of boundary edges and diagonals of the polygon P

We can view both the snake diagram and the triangulation T as graphs. Then there is a natural graph homomorphism p between them that sends an edge of the snake diagram to an edge of the triangulation as follows:

$$p(T_j^{(i)}) = T_j, \quad p(T_{j,j+1}) = T_{j,j+1} \tag{4.4}$$

In [7], Fomin and Zelevinsky showed that the cluster variables of $\mathcal{A}(Q)$ are in bijection with the diagonals of the polygon P where the initial set of cluster variables $\{x_1, \ldots, x_n\}$ corresponds to $\{T_1, \ldots, T_n\}$.

Let $M_{\mathfrak{v},\mathfrak{w}}$ be the diagonal connecting $\mathfrak{v}$ and $\mathfrak{w}$, thus crossing the diagonals $T_1, \ldots, T_n$. For $i \in [1, n]$, let p_i be the intersection of $M_{\mathfrak{v},\mathfrak{w}}$ and T_i.

Definition 4.3.1. [16] A T-path α from $\mathfrak{v}$ to $\mathfrak{w}$ is the sequence

$$\alpha = w_0 \xrightarrow{T_{i_1}} w_1 \xrightarrow{T_{i_2}} \cdots \xrightarrow{T_{i_{l(\alpha)}}} w_{l(\alpha)}$$

such that

(1) $\mathfrak{v} = w_0, w_1, \ldots, w_{l(\alpha)} = \mathfrak{w}$ are vertices of P.

(2) $i_k \in \{0, 1, \ldots, 2n+2\}$ such that T_{i_k} connects the vertices w_{k-1} and w_k for each $k =$

$1, 2, \ldots, l(\alpha)$.

(3) $i_j \neq i_k$ if $j \neq k$.

(4) $l(\alpha)$ is odd.

(5) T_{i_k} crosses $M_{\mathfrak{v},\mathfrak{w}}$ if k is even.

(6) If $j < k$ and both T_{i_j} and T_{i_k} cross M then p_{i_j} is closer to $\mathfrak{v}$ than p_{i_k} is to $\mathfrak{v}$.

Let $\mathcal{P}$ be the set of all T-path from $\mathfrak{v}$ to $\mathfrak{w}$. For any $\alpha \in \mathcal{P}$, let

$$x(\alpha) = \prod_{k \text{ odd}} x_{i_k} \prod_{k \text{ even}} x_{i_k}^{-1}. \tag{4.5}$$

Let $\mathbf{a} = (a_1, \ldots, a_{n'}) \in \{0,1\}^{n'}$ such that $a_i = 1$ if and only if $i \in Q$. The following formula of the cluster variable $x[\mathbf{a}]$ is proved in [16]:

$$x[\mathbf{a}] = \sum_{\alpha \in \mathcal{P}} x(\alpha) \tag{4.6}$$

Definition 4.3.2. We define a map $\psi_{\mathcal{G},\mathcal{P}} : \mathcal{G} \to \mathcal{P}$ by sending $\{S_{i,r}\} \in \mathcal{G}$ to the T-path α obtained by first constructing a path $\alpha'_1\alpha'_2 \ldots \alpha'_{2n+1}$ from $\mathfrak{v}$ to $\mathfrak{w}$ where $\alpha'_{2i} = T_i$ for $i \in [1, n]$,

$$\alpha'_{2i+1} = \begin{cases} T_i, & \text{if } \quad (|S_{i,1}|, |S_{i,2}|) = (\delta_i, 1-\delta_i) \\ T_{i+1}, & \text{if } \quad (|S_{i,1}|, |S_{i,2}|) = (1-\delta_i, \delta_i) \\ T_{i,i+1}, & \text{if } \quad (|S_{i,1}|, |S_{i,2}|) = (0,0) \end{cases}$$

for $i \in [1, n-1]$, and

$$\alpha'_1 = \begin{cases} T_{1,0} & \text{if } |S_{1,1+\delta_1}| = 1 - \delta_1, \\ T_{1,1} & \text{otherwise,} \end{cases} \qquad \alpha'_{2n+1} = \begin{cases} T_{n,0} & \text{if } |S_{n-1,2-\delta_{n-1}}| = \delta_{n-1}, \\ T_{n,1} & \text{otherwise,} \end{cases}$$

then define α to be the path obtained from α' by canceling duplicate pairs.

We shown in the theorem below, $\psi_{\mathcal{G},\mathcal{P}}$ is a bijection. Then we define $\psi_{\mathcal{P},\mathcal{G}} = \psi_{\mathcal{G},\mathcal{P}}^{-1} : \mathcal{P} \to \mathcal{G}$.

Theorem 4.3.3. *The maps $\psi_{\mathcal{G},\mathcal{P}}$ is a well-defined bijection. Moreover, for $\alpha = \psi_{\mathcal{G},\mathcal{P}}(\{S_{i,r}\})$,*

$$\prod_{i=1}^{n} x_i^{-1} \prod_{i=0}^{n} y_i = x(\alpha),$$

thus $\psi_{\mathcal{G},\mathcal{P}}$ induces a bijective proof of Theorem 4.1.4 using (4.6).

Proof. In order to prove Theorem 4.3.3, we shall show that all maps below are bijective, and that their composition is $\psi_{\mathcal{G},\mathcal{P}}$:

$$\mathcal{G} \xrightarrow{\psi_{\mathcal{G},\mathcal{M}}} \mathcal{M} \xrightarrow{\mathcal{L}} \{\text{complete } T\text{-paths from } \mathfrak{v} \text{ to } \mathfrak{w}\} \xrightarrow{\pi} \mathcal{P}.$$

(i) We first define $\mathcal{L}$, which is exactly the folding map in [14, §4.3]. As defined in [14, 18], a complete T-path α from $\mathfrak{v}$ to $\mathfrak{w}$ is similar to a T-path from $\mathfrak{v}$ to $\mathfrak{w}$ defined in Definition 4.3.1, in the sense that we require 1), 2), and

5') the $2j$-th edge $T_{i_{2j}} = T_j$ (i.e., $i_{2j} = j$),

6') $T_{i_1} \le T_{i_2} \le \cdots$,

where we use the order

$$T_{1,0} < T_{1,1} < T_1 < T_{1,2} < T_2 < T_{2,3} < \cdots < T_n < T_{n,0} < T_{n,1}. \tag{4.7}$$

Note that we do not require edges to be distinct in α. It is easy to see that a complete T-path has length $2n+1$. For simplicity, we denote α using its edge sequence. For $\gamma \in \mathcal{M}$, we define (recall that p is defined in (4.4)):

$$\mathcal{L}(\gamma) = L_1 L_2 \cdots L_{2n+1}, \quad \text{where } L_{2j} = T_j \text{ for } j \in [1,n],\ \ L_{2j+1} = p(\gamma_j) \text{ for } j \in [0,n]. \tag{4.8}$$

Note that the starting point of each L_i is determined by $L_1 \cdots L_{i-1}$. The union of a perfect matching γ with diagonals of all tiles form a path α'_γ in the snake diagram. If we consider the quotient map from the snake diagram to the triangulation of P, by identifying the diagonal edge i with $T^{(i)}$ and identifying diagonal edge $i+1$ with $T^{(i)}_{i+1}$, then the image of α'_γ is the complete T-path $\mathcal{L}(\gamma)$.

(ii) We show that $\mathcal{L}$ has a well-defined inverse map $\mathcal{L}^{-1}$ (which is the unfolding map in [14, §4.5]), thus $\mathcal{L}$ is bijective. Indeed, $\mathcal{L}^{-1}$ sends a complete T-path $\theta = L_1 \cdots L_{2n+1}$ to $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, where γ_j is the unique edge in $Pl^{(j)} \cap p^{-1}(L_{2j+1})$, that is, $\gamma_1 = L_1$, $\gamma_n = L_{2n+1}$, and $\gamma_j = T^{(j)}_j$ (resp. $T^{(j)}_{j+1}, T_{j,j+1}$) if L_{2j+1} is T_j (resp. $T_{j+1}, T_{j,j+1}$) for $j \in [1, n-1]$.

Next we show that γ is indeed a perfect matching, it suffices to prove that the edges in γ are disjoint, because it has the correct number $(= n+1)$ of edges.

For $j, j' \in [0, n-2]$ with $j < j'$, γ_j and $\gamma_{j'}$ are disjoint if $j' > j+1$ because $Pl^{(j)}$ and $Pl^{(j')}$ are disjoint. So we assume $j' = j+1$. We shall only discuss the case $\delta_j = \delta_{j+1} = 0$ since other cases can be proved similarly.

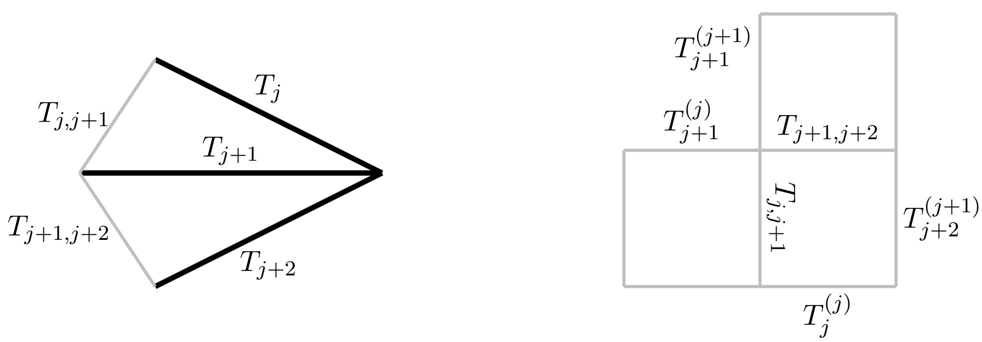


Figure 12: Parts of the polygon and the snake diagram corresponding to the subquiver $j \to j+1 \to j+2$

The subpath $L_{2j+1}L_{2j+2}L_{2j+3}L_{2j+4}$ of $\mathcal{L}(\alpha)$ is one of the following:

$$T_jT_{j+1}T_{j+1}T_{j+2},\ T_jT_{j+1}T_{j+1,j+2}T_{j+2},\ T_{j,j+1}T_{j+1}T_{j+2}T_{j+2},\ T_{j+1}T_{j+1}T_{j+2}T_{j+2}.$$

By looking at Figure 12, we see that γ_j and γ_{j+1} are disjoint in each case.

(iii) We show that π is bijective by giving its inverse π^{-1}. Suppose that $\alpha = T_{i_1}T_{i_2}\cdots T_{i_{l(\alpha)}}$ is a T-path from $\mathfrak{v}$ to $\mathfrak{w}$. If $n = 1$, then α is already a complete T-path, so we define $\pi^{-1}(\alpha) = \alpha$. Now assume $n > 1$. The sequence $\pi^{-1}(\alpha) = L = L_1L_2\cdots L_{2n+1}$ is obtained as a result of the following algorithm.

1. Initialize $L := \alpha$.
2. Let j run from 1 to n: if $L_{2j} \neq T_j$, then insert T_jT_j to L so that the resulting L is nondecreasing with the order given in (4.7).
3. Define $\pi^{-1}(\alpha) := L$.

We claim that L is a complete T-path. Conditions 1) 2) 6') are obviously satisfied, and 5') can be proved by induction.

Combining (i)(ii)(iii) and Theorem 4.2.5, we have proved that $\psi_{\mathcal{G},\mathcal{P}}$ is bijective.

Finally, we show that $\prod_{i=1}^n x_i^{-1}\prod_{i=0}^n y_i = x(\alpha)$. By the construction of $\pi^{-1}(\alpha)$ in (iii), $x(\alpha)$ remains unchanged if we replace α by the complete T-path $\pi^{-1}(\alpha) = T_{i_1}T_{i_2}\cdots T_{i_{2n+1}}$; this is because each time we insert the pair T_jT_j, the extra contribution to the product (4.5) is $x_jx_j^{-1} = 1$. So it suffices to show

$$\prod_{i=1}^{n} x_i^{-1}\prod_{i=0}^{n} y_i = \prod_{k \text{ even}} x_{i_k}^{-1}\prod_{k \text{ odd}} x_{i_k}.$$

By 5'), $\prod_{k \text{ even}} x_{i_k}^{-1} = \prod_{i=1}^n x_i^{-1}$, so it suffices to show $\prod_{i=0}^n y_i = \prod_{k \text{ odd}} x_{i_k}$, or to show that $y_j = x_{i_{2j+1}}$ for $j \in [0,n]$. Indeed, $T_{i_{2j+1}} = L_{2j+1} = p(\gamma_j)$ by (4.8), thus $x_{i_{2j+1}} = w(\gamma_j)$ by the definition of the weight w in (4.2). Moreover, Theorem 4.2.5 asserts that $w(\gamma_j) = y_j$. Thus $y_j = x_{i_{2j+1}}$. □

Example 4.3.4. With the T-path $\alpha = T_5T_1T_9T_4T_{11}$, the complete T-path is $\pi^{-1}(\alpha) = T_5T_1T_2T_2T_3T_3T_9T_4T_{11}$. Then $\psi_{\mathcal{P},\mathcal{G}}(\alpha) = ((1,0),(1,0),(0,0))$ as you can see in Figure 13.

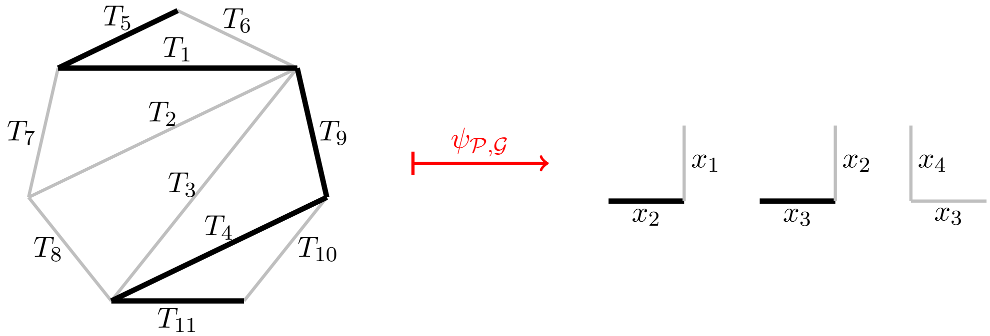


Figure 13: An example of the map $\psi_{\mathcal{P},\mathcal{G}}$

CHAPTER 5. CLUSTER MONOMIALS

In this chapter, we give three equivalent formulas for computing the cluster monomial $x[\mathbf{a}]$ for $\mathbf{a} \in \mathcal{W}$.

First we reduce to a special case. Lemma 3.2.3, we can replace $\mathbf{a}$ by $[\mathbf{a}]_+$, thus can assume $a_i \geq 0$. Moreover, for any edge $i \to j$ of Q that is not in a 3-cycle, we can add a vertex k and two arrows $j \to k$ and $k \to i$ (the vertex k is a frozen vertex). Indeed, assume the modified quiver is Q'. Then by Remark 3.2.4, once we have a formula for cluster monomials for Q', we can set $x_i = 1$ for all $i \in Q'_0 \setminus Q_0$ and obtain a formula for cluster monomials of $\mathcal{A}(Q)$. In the rest of the thesis, we assume that $\mathbf{a} = [\mathbf{a}]_+$ and

$$Q \text{ is of type } A \text{ with more than one vertex, and every edge of } Q \text{ is in a 3-cycle.} \tag{5.1}$$

For every 3-cycle $i \to j \to k \to i$, σ_{ijk} is a nonnegative integer by Proposition 3.2.1, and it is not hard to verify that $\sigma_{kij} + \sigma_{ijk} \leq a_i$.

5.1 A formula using 0-1 sequences

Fix a deg-2 vertex i_0 of Q (which exists because of (5.1)). For $i \in Q_0$, denote by $d(i)$ the distance (i.e., the length of the shortest directed path) from i_0 to i. Let $\mathbf{s}_i = (s_{i,1}, s_{i,2}, \dots, s_{i,a_i}) \in \{0,1\}^{a_i}$ be a 0-1 sequence, and define

$$|\mathbf{s}_i| = \sum_{r=1}^{a_i} s_{i,r} \ , \qquad |\bar{\mathbf{s}}_i| = \sum_{r=1}^{a_i} (1 - s_{i,r}) = a_i - |\mathbf{s}_i|.$$

We say that a sequence of 0-1 sequences $\mathbf{s} := (\mathbf{s}_1, \dots, \mathbf{s}_n)$ is a *globally compatible sequence (abbreviated GCS)* if the following holds for any 3-cycle $i \to j \to k \to i$:

- If $d(i) < d(j) < d(k)$, then $(s_{i,t}, s_{j,t}) \neq (1,0)$ for $1 \leq t \leq \sigma_{ijk}$;
- If $d(j) < d(k) < d(i)$, then $(s_{i,a_i+1-t}, s_{j,a_j+1-t}) \neq (1,0)$ for $1 \leq t \leq \sigma_{ijk}$;

- If $d(k) < d(i) < d(j)$, then $(s_{i,a_i+1-t}, s_{j,t}) \neq (1,0)$ for $1 \leq t \leq \sigma_{ijk}$.

Theorem 5.1.1. *For any* $\mathbf{d}$*-vector* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$ *(i.e.,* $\mathbf{a} \in \mathcal{W} \cap \mathbb{Z}^n_{\geq 0}$*), we have the following formula for the corresponding cluster monomial:*

$$x[\mathbf{a}] = \left(\prod_{l=1}^{n} x_l^{-a_l}\right) \sum_{\mathbf{s}} \left(\prod_i x_i^{e_i}\right), \quad \textit{where } e_i = \sum_{i\to j} |\,\bar{\mathbf{s}}_j\,| + \sum_{k\to i} |\mathbf{s}_k| - \sum_{i\to j\to k\to i} \sigma_{jki}, \tag{5.2}$$

here $\mathbf{s}$ *runs through all GCSs. (Note that because of the rotational symmetry, each 3-cycle contributes three terms* σ_{jki}*,* σ_{kij} *and* σ_{ijk} *to the last sum.)*

This formula specializes to the formula given in [10] for a linear quiver.

5.2 A formula using Dyck paths

In §4.1, we defined global compatibility in the case linear quivers. Now a more general definition of global compatibility in the case type A quivers shall be given.

Definition 5.2.1. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$ be a $\mathbf{d}$-vector. For each pair $i \to j$ in Q, let $\mathcal{D}^{(i,j)}$ be a maximal Dyck path $\mathcal{D}^{a_i \times a_j}$. We label $\mathcal{D}^{(i,j)}$ with the corner-first index (described in Definition 4.1.1), whose horizontal edges are denoted $u_1^{(i,j)}, \ldots, u_{a_i}^{(i,j)}$ and vertical edges are denoted by $v_1^{(i,j)}, \ldots, v_{a_j}^{(i,j)}$. We say that the collection

$$\left\{ S_\ell^{(i,j)} \subseteq \mathcal{D}_\ell^{(i,j)} \;\middle|\; i \to j \text{ is an arrow, } \ell \in \{1,2\} \right\}$$

is a GCC if and only if for any $k \to i \to j$ in Q:

- if $j \to k$ is also an arrow in Q, then $S_1^{(i,j)}$ and $S_2^{(i,j)}$ are σ_{ijk}-compatible, and

$$v_r^{(k,i)} \in S_2^{(k,i)} \Longleftrightarrow u_{a_i+1-r}^{(i,j)} \notin S_1^{(i,j)}, \text{ for all } r \in [1, a_i];$$

- otherwise,

$$v_r^{(k,i)} \in S_2^{(k,i)} \Longleftrightarrow u_r^{(i,j)} \notin S_1^{(i,j)}, \text{ for all } r \in [1, a_i].$$

Theorem 5.2.2. *Assume $n > 1$. For any **d**-vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$, we have the following formula for the corresponding cluster monomial:*

$$x[\mathbf{a}] = \left(\prod_{l=1}^{n} x_l^{-a_l}\right) \sum \left(\prod_{i \to j} x_i^{\left|S_2^{(i,j)}\right|} x_j^{\left|S_1^{(i,j)}\right|}\right) \cdot \prod_{i \to j \to k \to i} x_i^{-\sigma_{jki}}, \tag{5.3}$$

where the sum runs over all GCCs. (Note that because of the rotational symmetry, each 3-cycle contributes three terms to the last product.)

5.3 A method using decomposition of denominator vectors

The third method of computing the cluster monomial with given **d**-vector $\mathbf{a}$ is to first decompose $\mathbf{a}$ into a sum of 0-1 sequences $\mathbf{b}$'s using pipelines. By Remark 4.1.5, for each **d**-vector $\mathbf{b}$, we can find the cluster variable $x[\mathbf{b}]$ of $\mathcal{A}(Q)$. Then $x[\mathbf{a}]$ is computed as

$$x[\mathbf{a}] = \prod x[\mathbf{b}].$$

5.4 Proof of Main Theorems

In order to compute cluster variables, we have explained in §5.3 that it suffices to have the formula for a completely extended linear quiver, namely Theorem 4.1.4. This theorem follows from Theorem 4.2.5. In this section, we show how to derive Theorem 5.1.1 and Theorem 5.2.2 from Theorem 4.1.4.

5.4.1 Proof of Theorem 5.2.2

Let $x'[\mathbf{a}]$ be the right hand side of the formula in Theorem 5.2.2. We shall show that (i) the GCCs for the **d**-vector $\mathbf{a}$ are in one-to-one correspondence with the collections of GCCs for the **d**-vectors $\mathbf{b}$'s described in (3.1); (ii) $x'[\mathbf{a}] = \prod_{\mathbf{b}} x'[\mathbf{b}]$; and (iii) $x'[\mathbf{b}] = x[\mathbf{b}]$ using Theorem 4.1.4. It then follows that $x'[\mathbf{a}] = x[\mathbf{a}]$.

(i) Let $\{S_\ell^{(i,j)}\}$ be any GCC for the **d**-vector $\mathbf{a}$. For each pipeline Λ, let $\mathbf{b} = \mathbf{b}_\Lambda$, we construct a GCC $\{S_\ell^{(i,j),\Lambda}\}$ for the $\mathbf{d}$-vector $\mathbf{b}$ by requiring the following for each arrow $i \to j$:

- if Λ intersects the edge i at the r-th marking point, then $|S_1^{(i,j),\Lambda}| = 1$ if and only if $u_r^{(i,j)} \in S_1^{(i,j)}$,
- if Λ intersects the edge j at the r-th marking point, then $|S_2^{(i,j),\Lambda}| = 1$ if and only if $v_r^{(i,j)} \in S_2^{(i,j)}$,

(in both case the marking points are ordered in the increasing distance from the common endpoint of i and j).

To verify that $\{S_\ell^{(i,j),\Lambda}\}$ is a GCC for the $\mathbf{d}$-vector $\mathbf{b}$, we need to check the conditions in Definition 5.2.1. The only nontrivial condition to check is that for a 3-cycle $k \to i \to j \to k$, $S_1^{(i,j),\Lambda}$ and $S_2^{(i,j),\Lambda}$ are $\sigma_{ijk}^{\mathbf{b}}$-compatible. That reduces to showing that $(|S_1^{(i,j),\Lambda}|, |S_1^{(i,j),\Lambda}|) \neq (1,1)$ in the case $(b_i, b_j, b_k) = (1,1,0)$. In this case, Λ intersects edges i and j at the r-th marking points for some $r \leq \sigma_{ijk}^{\mathbf{a}}$. Then either $u_r^{(i,j)} \notin S_1^{(i,j)}$ or $v_r^{(i,j)} \notin S_2^{(i,j)}$. In the former case, $|S_1^{(i,j),\Lambda}| = 0$; in the latter case, $|S_2^{(i,j),\Lambda}| = 0$. Therefore $(|S_1^{(i,j),\Lambda}|, |S_2^{(i,j),\Lambda}|) \neq (1,1)$.

It is easy to see that a unique GCC $\{S_\ell^{(i,j)}\}$ is determined if we take any collection of GCCs $\{S_\ell^{(i,j),\Lambda}\}$ for all pipelines Λ. So we have the desired one-to-one correspondence.

(ii) We show that $\prod_{\mathbf{b}} x'[\mathbf{b}] = x'[\mathbf{a}]$. Since $\sum \mathbf{b} = \mathbf{a}$, it suffices to show that, for each GCC $\{S_\ell^{(i,j)}\}$, letting $\{S_\ell^{(i,j),\Lambda}\}$ be defined as in (i), the following holds (recall that $\mathbf{b} = \mathbf{b}_\Lambda$ depends on Λ):

$$\prod_{\Lambda} \left(\prod_{i \to j} x_i^{\left|S_2^{(i,j),\Lambda}\right|} x_j^{\left|S_1^{(i,j),\Lambda}\right|} \cdot \prod_{i \to j \to k \to i} x_i^{-\sigma_{jki}^{\mathbf{b}}} \right) = \prod_{i \to j} x_i^{\left|S_2^{(i,j)}\right|} x_j^{\left|S_1^{(i,j)}\right|} \cdot \prod_{i \to j \to k \to i} x_i^{-\sigma_{jki}^{\mathbf{a}}}.$$

Since the left hand side is equal to

$$\prod_{i\to j} x_i^{\sum_\Lambda \left|S_2^{(i,j),\Lambda}\right|} x_j^{\sum_\Lambda \left|S_1^{(i,j),\Lambda}\right|} \cdot \prod_{i\to j\to k\to i} x_i^{-\sum_\Lambda \sigma^{\mathbf{b}}_{jki}},$$

it suffices to show that $\sum_\Lambda |S_2^{(i,j),\Lambda}| = |S_2^{(i,j)}|$, $\sum_\Lambda |S_1^{(i,j),\Lambda}| = |S_1^{(i,j)}|$, and $\sum_\Lambda \sigma^{\mathbf{b}}_{jki} = \sigma^{\mathbf{a}}_{jki}$. The first two are clear. To show the last equality: first note that if Λ is disjoint from the edge j, then $b_j = 0$ and thus $\sigma^{\mathbf{b}}_{jki} = 0$. So we only need to consider those Λ that intersect j. Let Λ_r $(1 \le r \le a_j)$ be the pipeline that intersects j at the r-th marking point (ordered in the increasing distance to the common endpoint of j and k). If $r \le \sigma^{\mathbf{a}}_{jki}$, then $(b_j, b_k, b_i) = (1,1,0)$, thus $\sigma^{\mathbf{b}}_{ijk} = 1$; otherwise, either $b_j = 0$ or $b_k = 0$, thus $\sigma^{\mathbf{b}}_{ijk} = 0$. Therefore $\sum_\Lambda \sigma^{\mathbf{b}}_{jki} = \sigma^{\mathbf{a}}_{jki}$.

(iii) We show that $x'[\mathbf{b}] = x[\mathbf{b}]$. By Remark 3.2.4, it suffices to show that, in the setting of Theorem 4.1.4, the right hand side of (4.1) is equal to $x'[\mathbf{a}]$. It breaks down to show that, for $i' \in [0,n]$, the following equality holds for the i'-th 3-cycle $i \to j \to k \to i$ in Q' (for $i' \in [1, n-1]$, the i'-th 3-cycle is the one that contains vertices v_i, v_{i+1} and $v_{i,i+1}$; the 0-th 3-cycle is $v_1 \to v_{1,0} \to v_{1,1} \to v_1$; the n-th 3-cycle is $v_n \to v_{n,0} \to v_{n,1} \to v_n$):

$$\left(x_i^{\left|S_2^{(i,j)}\right|} x_j^{\left|S_1^{(i,j)}\right|}\right)\left(x_j^{\left|S_2^{(j,k)}\right|} x_k^{\left|S_1^{(j,k)}\right|}\right)\left(x_k^{\left|S_2^{(k,i)}\right|} x_i^{\left|S_1^{(k,i)}\right|}\right) \cdot x_i^{-\sigma_{jki}} x_j^{-\sigma_{kij}} x_k^{-\sigma_{ijk}} = y_{i'} \tag{5.4}$$

We shall only prove the case when $i' \in [1, n-1]$ and $\delta_{i'} = 0$, because other cases can be proved in a similar way. In this case, the i'-th 3-cycle is $v_i \to v_{i+1} \to v_{i,i+1}$ (where $i = i'$), and the left hand side of (5.4) is equal to

$$\left(x_i^{\left|S_{i,2}\right|} x_j^{\left|S_{i,1}\right|}\right)\left(x_j^0 x_k^{1-\left|S_{i,2}\right|}\right)\left(x_k^{1-\left|S_{i,1}\right|} x_i^0\right) \cdot x_k^{-1} = x_i^{\left|S_{i,2}\right|} x_j^{\left|S_{i,1}\right|} x_k^{1-\left|S_{i,1}\right|-\left|S_{i,2}\right|} = y_{i'}.$$

So (5.4) holds.

5.4.2 Proof that Theorem 5.2.2 implies Theorem 5.1.1

We give a bijection between GCSs and GCCs. Let $\mathbf{s}$ be a GCS. Consider a 3-cycle $i \to j \to k \to i$, labeled in the way that $d(k) < d(i) < d(j)$. Then we define

$$S_1^{(k,i)} = \{u_r \in \mathcal{D}_1^{(k,i)} \big| s_{k,r} = 1\}, \qquad S_2^{(k,i)} = \{v_r \in \mathcal{D}_2^{(k,i)} \big| s_{i,r} = 0\}$$

$$S_1^{(i,j)} = \{u_r \in \mathcal{D}_1^{(i,j)} \big| s_{i,a_i+1-r} = 1\}, \quad S_2^{(i,j)} = \{v_r \in \mathcal{D}_2^{(i,j)} \big| s_{j,r} = 0\}$$

$$S_1^{(j,k)} = \{u_r \in \mathcal{D}_1^{(j,k)} \big| s_{j,a_j+1-r} = 1\}, \quad S_2^{(j,k)} = \{v_r \in \mathcal{D}_2^{(j,k)} \big| s_{k,a_k+1-r} = 0\}$$

It is then easy to check that the conditions of GCSs and GCCs, as well as the two theorems, are equivalent under this bijection.

CHAPTER 6. EXAMPLES

In this chapter, we give two examples to illustrate the computation of cluster variables and cluster monomials using methods introduced in previous sections.

6.1 All cluster variables of a type A quiver

We compute some cluster variables of $\mathcal{A}(\mathcal{Q})$, where $\mathcal{Q}$ is the following type A quiver.

$$\begin{tikzcd}
1 \arrow[dr] & & 6 \arrow[r] \arrow[dd] & 7 \\
& 2 \arrow[ur] \arrow[dl] & & \\
5 \arrow[uu] & & 3 \arrow[ul] \arrow[r] & 4
\end{tikzcd}$$

As observed in Remark 4.1.5, the set of non-initial cluster variables is in one-to-one correspondence with the set of $\mathbf{d}$-vectors $\mathbf{a} = (a_1, \ldots, a_n)$, where $a_i \in \{0, 1\}$ and $\{i \,|\, a_i = 1\}$ is the vertex set of a linear full subquiver of $\mathcal{Q}$. So we can compute all cluster variables using Theorem 4.1.4.

For example, we consider the $\mathbf{d}$-vector $\mathbf{a} = (1, 1, 1, 0, 0, 0, 0)$. Then the subset of vertices $\{i|a_i = 1\}$ is equal to the set of vertices Q_0 of the full linear subquiver $Q = 1 \to 2 \leftarrow 3$. It is a subquiver of an extended linear subquiver P and after completing P, we get a completely extended linear quiver Q' as shown in Figure 14.

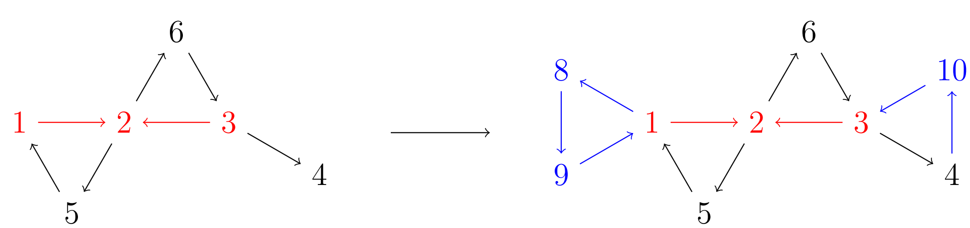
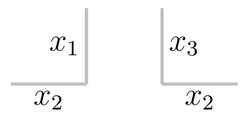


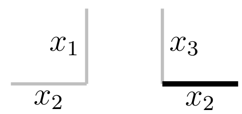
Figure 14: P and its completed version Q'

In P, we have $v_{1,2} = 5$, $v_{2,3} = 6$, $v_{3,0} = 4$. Two 3-cycles are added and the new vertices are $v_{1,0} = 8$, $v_{1,1} = 9$ and $v_{3,1} = 10$.

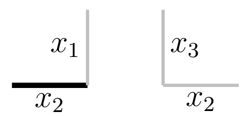
All GCCs are described as follows.

$\beta_1 =$ 

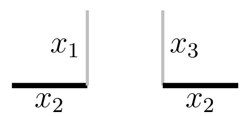
, $x(\beta_1) = (x_5x_6x_9x_{10})/(x_1x_2x_3)$

$\beta_2 =$ 

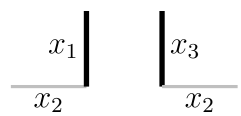
, $x(\beta_2) = (x_2x_4x_5x_9)/(x_1x_2x_3)$

$\beta_3 =$ 

, $x(\beta_3) = (x_2x_6x_8x_{10})/(x_1x_2x_3)$

$\beta_4 =$ 

, $x(\beta_4) = (x_2^2x_4x_8)/(x_1x_2x_3)$

$\beta_5 =$ 

, $x(\beta_5) = (x_1x_3x_9x_{10})/(x_1x_2x_3)$

The cluster variable of $\mathcal{A}(Q')$ with **d**-vector $(1,1,1,0,0,0,0,0,0,0)$ is

$$\sum_{i=1}^{5} x(\beta_i) = \frac{x_5x_6x_9x_{10} + x_2x_4x_5x_9 + x_2x_6x_8x_{10} + x_2^2x_4x_8 + x_1x_3x_9x_{10}}{x_1x_2x_3}.$$

Setting $x_8 = x_9 = x_{10} = 1$, we get the following cluster variable of $\mathcal{A}(\mathcal{Q})$:

$$x[1,1,1,0,0,0,0] = \frac{x_5x_6 + x_2x_4x_5 + x_2x_6 + x_2^2x_4 + x_1x_3}{x_1x_2x_3}.$$

The table below shows some **d**-vectors and their corresponding cluster variables of $\mathcal{A}(\mathcal{Q})$.

(1,0,0,0,0,0,0)	$\frac{x_2 + x_5}{x_1}$	(1,1,0,0,0,0,0)	$\frac{x_1x_3 + x_2x_6 + x_5x_6}{x_1x_2}$
(0,1,0,0,0,0,0)	$\frac{x_1x_3 + x_5x_6}{x_2}$	(0,1,1,0,0,0,0)	$\frac{x_1x_3 + x_2x_4x_5 + x_5x_6}{x_2x_3}$
(0,0,1,0,0,0,0)	$\frac{x_2x_4 + x_6}{x_3}$	(0,0,1,1,0,0,0)	$\frac{x_2x_4 + x_3x_6 + x_6}{x_3x_4}$
(0,0,0,1,0,0,0)	$\frac{1 + x_3}{x_4}$	(1,0,0,0,1,0,0)	$\frac{x_1 + x_2 + x_5}{x_1x_5}$
		(0,0,1,0,0,1,0)	$\frac{x_2x_4 + x_6 + x_3x_4x_7}{x_3x_6}$

$$(1,1,1,0,0,0,0) \qquad \frac{x_5x_6 + x_2x_4x_5 + x_2x_6 + x_2^2x_4 + x_1x_3}{x_1x_2x_3}$$

$$(0,1,1,0,1,0,0) \qquad \frac{x_1x_3 + x_2x_3 + x_2x_4x_5 + x_5x_6}{x_2x_3x_5}$$

$$(0,1,1,1,0,0,0) \qquad \frac{x_1x_3 + x_1x_3^2 + x_2x_4x_5 + x_5x_6 + x_3x_5x_6}{x_2x_3x_4}$$

$$(0,0,1,1,0,1,0) \qquad \frac{x_2x_4 + x_6 + x_3x_6 + x_3x_4x_7}{x_3x_4x_6}$$

$$(1,1,1,1,0,0,0) \qquad \frac{x_1x_3 + x_1x_3^2 + x_2^2x_4 + x_2x_4x_5 + x_2x_6 + x_2x_3x_6 + x_5x_6 + x_3x_5x_6}{x_1x_2x_3x_4}$$

6.2 A cluster monomial of a type A quiver

We compute $x[2,2,2]$, the cluster monomial with **d**-vector $(2,2,2)$ of $\mathcal{A}(Q)$, where

$$Q \quad = \quad \begin{tikzcd} 1 \arrow[r] & 2 \arrow[dl] \\ 3 \arrow[u] & \end{tikzcd}$$

– Using formula (5.2): choose $i_0 = 1$. Then $d(1) = 0$, $d(2) = 1$, $d(3) = 2$, $\sigma_{123} = \sigma_{231} = \sigma_{312} = 1$. A GCS $\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)$ satisfies

$$(s_{1,1}, s_{2,1}), (s_{3,2}, s_{1,2}), \text{ and } (s_{2,2}, s_{3,1}) \neq (1,0).$$

For instance, such a GCS can be

$$\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3) = \left(\begin{matrix} 1 \\ 1 \end{matrix}, \begin{matrix} 0 \\ 1 \end{matrix}, \begin{matrix} 1 \\ 1 \end{matrix} \right).$$

Doing some computations, we have

$$|\mathbf{s}_1| = 2, \quad |\bar{\mathbf{s}}_1| = 0,$$

$$|\mathbf{s}_2| = 1, \quad |\bar{\mathbf{s}}_2| = 1,$$

$$|\mathbf{s}_3| = 2, \quad |\bar{\mathbf{s}}_3| = 0.$$

That implies

$$e_1 = |\bar{\mathbf{s}}_2| + |\mathbf{s}_3| - \sigma_{231} = 2,$$

$$e_2 = |\bar{\mathbf{s}}_3| + |\mathbf{s}_1| - \sigma_{312} = 1,$$

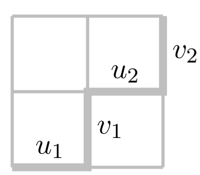
$$e_3 = |\bar{\mathbf{s}}_1| + |\mathbf{s}_2| - \sigma_{123} = 0.$$

Hence

$$\prod_i x_i^{e_i} = x_1^2 x_2.$$

Computing all possible GCSs gives $x[2,2,2] = x_1^{-2}x_2^{-2}x_3^{-2}(x_1+x_2+x_3)^3$. (It's not hard to see that $x[2,2,2]$ has 27 terms; indeed, since each pair has 3 choices $(0,0),(1,1),(0,1)$, the total number of GCSs is $3\times 3\times 3 = 27$.)

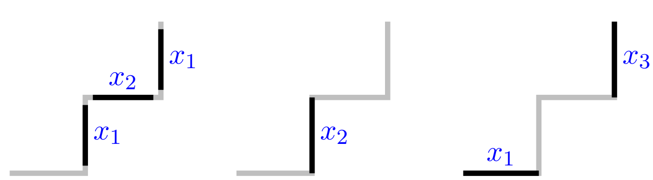
– Using formula (5.3): Corresponding to each arrow $1\to 2$, $2\to 3$, or $3\to 1$, we have a Dyck path of size 2×2.



A GCC is chosen by following the rules:

- we do not choose both u_1 and v_1 in each Dyck path,
- we choose v_r in the i-th Dyck path if and only if we do not choose u_{3-r} in the $(i+1)$-th Dyck path for $r = 1, 2$ (by convention, the 4th Dyck path is the 1st one).

For example, we have a GCC.



Now we compute the corresponding product of this GCC. We have

$$\prod_{i\to j} x_i^{\left|S_2^{(i,j)}\right|} x_j^{\left|S_1^{(i,j)}\right|} = x_1^{|S_2^{(1,2)}|} x_2^{|S_1^{(1,2)}|} \cdot x_2^{|S_2^{(2,3)}|} x_3^{|S_1^{(2,3)}|} \cdot x_3^{|S_2^{(3,1)}|} x_1^{|S_1^{(3,1)}|}$$

$$= x_1^2 x_2 \cdot x_2 \cdot x_1 x_3$$

$$= x_1^3 x_2^2 x_3$$

and

$$\prod_{i\to j\to k\to i} x_i^{-\sigma_{jki}} = x_1^{-\sigma_{231}} x_2^{-\sigma_{312}} x_3^{-\sigma_{123}} = x_1^{-1} x_2^{-1} x_3^{-1}.$$

Hence

$$\left(\prod_{i\to j} x_i^{\left|S_2^{(i,j)}\right|}\right) \cdot \prod_{i\to j\to k\to i} x_i^{-\sigma_{jki}} = x_1^3 x_2^2 x_3 \cdot x_1^{-1} x_2^{-1} x_3^{-1} = x_1^2 x_2.$$

Computing on all possible GCCs gives

$$x[2,2,2] = x_1^{-2} x_2^{-2} x_3^{-2} (x_1 + x_2 + x_3)^3.$$

– Using formula (4.1): first observe that there are 3 pipelines as shown in Figure 15.

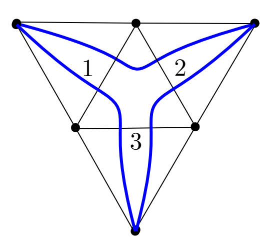


Figure 15: Pipelines

According to §5.3, $x[2,2,2]$ is decomposed as the product of $x[1,1,0]$, $x[0,1,1]$ and $x[1,0,1]$. Now we compute $x[1,1,0]$ using Theorem 4.1.4. The pair $(|S_{1,1}|, |S_{1,2}|)$ can be $(1,0)$, $(0,1)$ or $(0,0)$. Correspondingly, we have $(y_1, y_0, y_2) = (x_1, x_{1,0}, x_{2,0})$, $(x_2, x_{1,1}, x_{2,1})$ or

$(x_{1,2}, x_{1,1}, x_{2,0})$. Then

$$\begin{aligned} x[1,1,0] &= (x_1^{-1}x_2^{-1}) \sum y_0 y_1 y_2 \\ &= (x_1^{-1}x_2^{-1})(x_1 x_{1,0} x_{2,0} + x_2 x_{1,1} x_{2,1} + x_{1,2} x_{1,1} x_{2,0}) \\ &= (x_1^{-1}x_2^{-1})(x_1 + x_2 + x_3) \end{aligned}$$

where the last equality is obtained by setting $x_{1,2} = x_3$ and $x_{1,0} = x_{1,1} = x_{2,0} = x_{2,1} = 1$.

Similarly,

$$x[0,1,1] = (x_2^{-1}x_3^{-1})(x_1 + x_2 + x_3),$$

$$x[1,0,1] = (x_1^{-1}x_3^{-1})(x_1 + x_2 + x_3).$$

Thus $x[2,2,2] = x_1^{-2}x_2^{-2}x_3^{-2}(x_1 + x_2 + x_3)^3$.

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ABSTRACT

SOME NEW COMBINATORIAL FORMULAS FOR CLUSTER MONOMIALS OF TYPE A QUIVERS

by

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August 2016

Co-Advisor: Dr. Kyungyong Lee

Co-Advisor: Dr. Frank Okoh

Major: Mathematics

Degree: Doctor of Philosophy

Lots of research focuses on the combinatorics behind various bases of cluster algebras. This dissertation studies the natural basis of a type A cluster algebra, which consists of all cluster monomials. We introduce a new kind of combinatorial formulas for the cluster monomials in terms of the so-called globally compatible collections. We give bijective proofs of these formulas by comparing with the well-known combinatorial models of the T-paths and of the perfect matchings in a snake diagram.

AUTOBIOGRAPHICAL STATEMENT

Ba Uy Nguyen was born in Ho Chi Minh City in 1981. He grew up here and began attending the University of Science, Ho Chi Minh City in the fall of 1999. He earned a B.S. in Mathematics and Computer Science in 2003 and a M.S. in Mathematics in 2007, both from the University of Science. In 2010, he came to Wayne State University in Detroit, Michigan for his Ph.D. in Mathematics. At Wayne State, he has been working on two projects in the field of Cluster Algebras. He graduated in August 2016. During the time at Wayne State University, he was a Graduate Teaching Assistant.