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Vibrations of a Swept Box

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SUMMARY

The equations of motion of a uniform swept box with stringers and ribs are deduced. For the case of vibrations of a cantilever they are transformed into integral equations, an approximate method of solution of which is indicated.

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NOTATION

Oxyz	Oblique axes of reference (Fig. 1).
u, v, w	Oblique components of displacement.
p(x,t), q(x,t)	Oblique components of rotation of box sections about Ox, Oy respectively.
α	Angle between the axes Ox, Oy (complement of angle of sweep).
W(x,t)	Displacement of centre of box sections in Oz direction.
$\dot{u}$, $\dot{v}$, etc. $\ddot{u}$, $\ddot{v}$, etc.	Time derivatives of u, v, etc.
$\mu(x,y,z)$	Mass distribution of swept box.
T	Kinetic energy of box.
ℓ	Length of swept axis of box (Fig. 1).
OX, OY	Reference axes in Oxy plane, perpendicular to Oy and Ox axes respectively.
L_1, M_1	Oblique components of couple about OX, OY respectively.
i, j	Unit vectors in Ox, Oy directions.
i_1, j_1	Unit vectors in OX, OY directions.
U	Potential (strain) energy of box.
C_{ij}	Elastic constants given by equation (100) of Ref. 1.
Γ_{ij}	Constants defined by (2.10).
L = T-U	Lagrangian function of the box.
$I_y(x)$, $I_z(x)$	Moments of inertia of box sections.
m(x)	Mass per unit length of box.
$\eta(x)$	Position of centre of gravity of box sections.
t	Time
P(x), Q(x), $\Omega(x)$	Amplitudes of normal vibrations.
$\varkappa/2\pi$	Frequency of vibration.
k_y, k_z	Radii of gyration corresponding to I_y, I_z.
$\lambda^2 = 1/m\varkappa^2$	Frequency parameter.
$f_i(\xi,x)$	Kernel functions given by (4.10).

1. Introduction

The use of swept wings in high speed aircraft has naturally produced considerable interest in the dynamic behaviour of such structures. As far as complexity is concerned, the static and dynamic problems arising out of the sweeping of wings may well be compared with the problems encountered in high speed aerodynamics. Although the trend towards low thickness wings for high speed aircraft may lead to the treatment of wings as flat plates for the purposes of dynamic and aero-elastic investigations, it is of interest to consider the present problem as it should facilitate assessment of the effects of such a simplifying assumption on frequencies and modes of vibration, flutter characteristics, etc.

In this paper Hamilton's Principle is applied to deduce the equations of motion and relevant end conditions for a uniform swept box, reinforced by stringers and ribs in a manner typical of aircraft wings. The expression for the potential energy, forming part of the Lagrangian function, is obtained using generalised curvature-bending and twist-torque relations deduced in ref. 1. This reference and the present report use throughout oblique coordinates (Fig. 1) and the same notation, wherever possible.

The equations of motion, obtained thus, are integro-differential equations in terms of the vertical displacement and the twists about two oblique axes of the box sections (Fig. 1). Using the boundary conditions, they are transformed into integral equations in terms of certain derivatives of the above quantities. Finally, an approximate method of solution of these equations by reduction to a finite number of linear equations is indicated.

2. Deduction of the Lagrangian Function

First the kinetic energy of a swept box will be obtained, assuming in conjunction with ref. 1 that p, q and W are functions of x and t only. Referred to the oblique axes $Oxyz$ (Fig. 1), the displacements of a point $P(x,y,z)$ are

$$\begin{aligned} u &= z(p \cot \alpha + q \operatorname{cosec} \alpha) \\ v &= -z(p \operatorname{cosec} \alpha + q \cot \alpha) \\ w &= W + py \sin \alpha \end{aligned} \qquad \text{..........(2.1)}$$

and thus the components of velocity are

$$\begin{aligned} \dot{u} &= z(\dot{p} \cot \alpha + \dot{q} \operatorname{cosec} \alpha) \\ \dot{v} &= -z(\dot{p} \operatorname{cosec} \alpha + \dot{q} \cot \alpha) \\ \dot{w} &= \dot{W} + \dot{p}\, y \sin \alpha. \end{aligned} \qquad \text{..........(2.2)}$$

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The square of the modulus of the velocity vector is then

$$\dot{u}^2 + 2\dot{u}\dot{v}\cos\alpha + \dot{v}^2 + \dot{w}^2$$

$$= z^2(\dot{p}^2 + 2\dot{p}\dot{q}\cos\alpha + \dot{q}^2) + y^2\dot{p}^2\sin^2\alpha + 2y\dot{p}\dot{W}\sin\alpha + \dot{W}^2 \quad \ldots\ldots\ldots\ldots(2.3)$$

and hence the kinetic energy of a point mass $\mu(x,y,z)dx\,dy\,dz$

$$\frac{\mu(x,y,z)}{2}\left\{z^2(\dot{p}^2 + 2\dot{p}\dot{q}\cos\alpha + \dot{q}^2) + y^2\dot{p}^2\sin^2\alpha + 2y\dot{p}\dot{W}\sin\alpha + \dot{W}^2\right\}dx\,dy\,dz.$$

Integration of the last expression over a cross-section x = const. gives the kinetic energy of a cross-sectional element of thickness dx of the swept box.

$$dT = \tfrac{1}{2}\left\{I_y(x)(\dot{p}^2 + 2\dot{p}\dot{q}\cos\alpha + \dot{q}^2) + I_z(x)\dot{p}^2\sin^2\alpha + 2m(x)\eta(x)\dot{p}\dot{W}\sin\alpha + m(x)\dot{W}^2\right\}dx, \quad \ldots\ldots\ldots(2.4)$$

from which follows the total kinetic energy

$$T = \int_0^\ell dT. \quad \ldots\ldots\ldots\ldots(2.5)$$

In order to deduce an expression for the potential energy, consider a uniform swept box acted on by a uniform moment (i.e. neglecting shear),

$$L_1\,i_1 + M_1\,j_1.$$

The corresponding relative displacement for an element dx is

$$dpi + dqj$$

and hence the strain energy

$$dU = \tfrac{1}{2}(L_1 i_1 + M_1 j_1).(dpi + dqj) = \frac{\sin\alpha}{2}\left(L_1\frac{dp}{dx} + M_1\frac{dq}{dx}\right)dx \quad \ldots\ldots\ldots\ldots(2.6)$$

since obviously (Fig. 1)

$$i.i_1 = j.j_1 = \sin\alpha, \quad i.j_1 = i_1.j = 0.$$

Thus the total potential energy of a swept box is

$$U = \int_0^\ell dU. \quad \ldots\ldots\ldots\ldots(2.7)$$

By equation (99) of ref. 1,

$$\begin{pmatrix}\frac{dp}{dx}\\ \frac{dq}{dx}\end{pmatrix} = \begin{pmatrix}C_{11} & C_{12}\\ C_{12} & C_{22}\end{pmatrix}\begin{pmatrix}L_1\\ M_1\end{pmatrix} \qquad q = -\operatorname{cosec}\alpha\frac{dW}{dx} \quad \ldots\ldots\ldots\ldots(2.8)$$

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where the C_{ij} are known constants given by equation (100) of the same reference, and the last equation has been integrated taking into account the neglecting of shear deflections. Inversion of (2.8) gives

$$\begin{pmatrix} L_1 \\ M_1 \end{pmatrix} = \begin{pmatrix} \Gamma_{22} & -\Gamma_{12} \\ -\Gamma_{12} & \Gamma_{11} \end{pmatrix} \begin{pmatrix} \frac{dp}{dx} \\ \frac{dq}{dx} \end{pmatrix} \qquad \text{..........(2.9)}$$

where

$$\Gamma_{ij} = \frac{C_{ij}}{\Gamma}, \quad \Gamma = \begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix} \qquad \text{..........(2.10)}$$

Substitution from (2.9) in (2.6) and then in (2.7) leads to the total potential energy

$$U = \tfrac{1}{2} \int_0^{\ell} \sin\alpha \left\{ \Gamma_{22} \left(\frac{dp}{dx}\right)^2 - 2\Gamma_{12} \frac{dp}{dx}\frac{dq}{dx} + \Gamma_{11} \left(\frac{dq}{dx}\right)^2 \right\} dx. \qquad \text{..........(2.11)}$$

Using (2.5) and (2.11) the Lagrangian Function

$$L = T - U \qquad \text{.........(2.12)}$$

can now be written down. However, before proceeding to the application of Hamilton's Principle to (2.12), it will be of interest to discuss the character of some of the quantities appearing in (2.4) in connection with the theoretical background of (2.8).

The equations (2.8) have been deduced in ref.1 assuming a uniform rectangular box with the swept axis Ox passing through the centres of the cross-sections x = const. Nevertheless the quantity $\eta(x)$, the position of the centre of gravity of these cross-sections has been retained in (2.4), in order to allow for the addition of masses which, while not affecting the elastic properties of the box, may change the value of η, and of course also of I_y and I_z. It is obvious that in the absence of additional masses

$$\eta(x) \equiv 0.$$

Finally it should be noted that the equation

$$q = -\operatorname{cosec}\alpha \frac{dW}{dx}$$

in (2.8) implies that only two of the three functions p, q, W are independent, a fact which has to be taken into account in the application of Hamilton's Principle.

3. Hamilton's Variational Equations of Motion

By Hamilton's Principle

$$\delta \int_{t_1}^{t_2} L \, dt = 0 \qquad \text{..........(3.1)}$$

provided that for $t = t_1$, $t = t_2$

$$\delta p = \delta W = 0 \qquad \text{..........(3.2a)}$$

where the latter can be seen by (2.8) to imply

$$\delta q = 0. \qquad \text{..........(3.2b)}$$

In addition there are the following clamped end conditions

$$p(0) = 0, \quad q(0) = 0, \quad W(0) = 0. \qquad \text{..........(3.3)}$$

Substituting in (3.1) for L from (2.12) and using (3.2)

$$\int_{t_1}^{t_2} dt \int_0^{\ell} \Big[I_y \{ \dot{p}\,\delta\dot{p} + (\dot{q}\,\delta\dot{p} + \dot{p}\,\delta\dot{q}) \cos\alpha + \dot{q}\,\delta\dot{q} \} + I_z \sin^2\alpha\, \dot{p}\,\delta\dot{p}$$

$$+ (\dot{W}\delta\dot{p} + \dot{p}\,\delta\dot{W}) m\eta \sin\alpha + m\dot{W}\delta\dot{W} - \sin\alpha \{ T_{22} p'\delta p' - T_{12}(q'\delta p' + p'\delta q')$$

$$+ T_{11} q'\delta q' \} \Big] dx$$

$$= -\int_{t_1}^{t_2} dt \Big[\int_0^{\ell} \Big[I_y \{ \ddot{p}\,\delta p + (\ddot{q}\,\delta p + \ddot{p}\,\delta q) \cos\alpha + \ddot{q}\,\delta q \}$$

$$+ I_z \sin^2\alpha\, \ddot{p}\,\delta p + (\ddot{W}\delta p + \ddot{p}\,\delta W) m\eta \sin\alpha + m\ddot{W}\delta W$$

$$- \sin\alpha \{ T_{22} p''\delta p - T_{12}(q''\delta p + p''\delta q) + T_{11} q''\delta q \} \Big] dx$$

$$+ \sin\alpha \Big| T_{22} p'\delta p - T_{12}(q'\delta p + p'\delta q) + T_{11} q'\delta q \Big|_0^{\ell} \Big]$$

$$= 0. \qquad \text{..........(3.4)}$$

Choosing δp, δq as independent variations, one finds immediately that for $0 \leq x \leq \ell$,

$$T_{22}p'' - T_{12}q'' - \ddot{W} m\eta - I_z \sin\alpha\, \ddot{p} - I_y(\ddot{p} \operatorname{cosec}\alpha + \ddot{q} \cot\alpha) = 0$$

$$\text{..........(3.5)}$$

and

$$T_{22}\, p'(\ell) - T_{12}\, q'(\ell) = 0. \qquad \text{..........(3.6)}$$

Since the coefficient of δp must disappear.

When dealing with the remaining terms of (3.4), δW will be transformed into δq using (2.8). By the help of the last condition of (3.3), the following transformation of the terms involving δW is possible:

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$$\int_0^{\ell}(m\ddot{W} + \sin\alpha\, m\eta\,\ddot{p})\delta W\, d\xi = \int_0^{\ell} F(\xi,t)\delta W(\xi,t)d\xi$$

$$= \int_0^{\ell} F(\xi,t)\, d\xi \int_0^{\xi}\delta W' dx$$

$$= \int_0^{\ell}\delta W'\, dx \int_x^{\ell} F(\xi,t)\, d\xi$$

$$= \int_0^{\ell}(-\sin\alpha\,\delta q)dx \int_x^{\ell} F(\xi,t)d\xi . \qquad \ldots\ldots\ldots\ldots (3.7)$$

Hence, substituting from (3.7) in (3.4),

$$\Gamma_{12}p'' - \Gamma_{11}q'' + I_y(\cot\alpha\,\ddot{p} + \operatorname{cosec}\alpha\,\ddot{q}) - \int_x^{\ell}\left\{m\ddot{W} + \sin\alpha\, m\eta\ddot{p}\right\}d\xi = 0 \qquad \ldots\ldots\ldots\ldots (3.8)$$

$$\Gamma_{12}\, p'(\ell) - \Gamma_{11}\, q'(\ell) = 0. \qquad \ldots\ldots\ldots\ldots (3.9)$$

The boundary conditions (3.5) and (3.9) have been chosen to satisfy Hamilton's Principle at the free end of the cantilever. The equations (3.5) and (3.8) are the desired equations of motion of a swept box; they are supplemented by the relation holding between q and W, given in (2.8), so that there are actually three equations. Before giving an interpretation of the various terms appearing in these equations, they will be transformed into integral equations in the next section.

4. Transformation of the Equations of Motion

Integration of (3.5) and (3.8) with respect to x from ξ to ℓ using (3.6) and (3.9) gives

$$\Gamma_{22}p'(\zeta,t) - \Gamma_{12}q'(\zeta,t) = -\int_{\zeta}^{\ell}\Big\{m(x)\eta(x)\ddot{W}(x,t) + I_z(x)\sin\alpha\,\ddot{p}(x,t) + I_y(x)\Big(\ddot{p}(x,t)\operatorname{cosec}\alpha + \ddot{q}(x,t)\cot\alpha\Big)\Big\}dx \qquad \ldots\ldots\ldots\ldots (4.1)$$

$$-\Gamma_{12}p'(\zeta,t) + \Gamma_{11}q'(\zeta,t) = -\int_{\zeta}^{\ell}\Big\{I_y(x)\Big(\cot\alpha\,\ddot{p}(x,t) + \operatorname{cosec}\alpha\,\ddot{q}\Big) - \int_x^{\ell}\Big(m(\xi)\ddot{W}(\xi,t) + \sin\alpha\, m(\xi)\eta(\xi)\ddot{p}(\xi,t)\Big)d\xi\Big\}dx. \qquad \ldots\ldots\ldots\ldots (4.2)$$

Comparing the equations (4.1) and (4.2) with (2.9) leads immediately to the conclusion that the right hand sides of (4.1) and (4.2) are respectively equal to L_1 and M_1 of (2.9). A further reference to (2.1)

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suggests that the terms under the integrals are rates of change of moment of momentum and of momentum of a cross sectional element of thickness dx.

$$\bar{L} = \iint \mu (y\ddot{w} - z\ddot{v})\, dydz$$

$$\bar{M} = \iint \mu\, z\ddot{u}\, dydz$$

$$\bar{Z} = \iint \mu\, \ddot{w}\, dydz$$

which on consideration of the equilibrium of such an element of the box will be found to be related to L_1 and M_1 in the manner suggested by (4.1) and (4.2).

The equations (4.1) and (4.2) are integro-differential equations which are easily transformed into integral equations using (3.3). However, before doing so, in order to simplify the further treatment, normal vibrations and absence of additional masses (see end of section 2) will be assumed, i.e.

$$p(x,t) = P(x)\sin \varkappa t, \quad q(x,t) = Q(x)\sin \varkappa t, \quad W = \Omega(x)\sin \varkappa t, \qquad \ldots\ldots\ldots\ldots (4.3)$$

$$\eta(x) \equiv 0 \qquad \ldots\ldots\ldots\ldots (4.4)$$

substituting (4.3) and (4.4) in (4.1) and (4.2), and inverting the order of integration in (4.2),

$$\lambda^2 \Gamma_{22} P'(\zeta) - \lambda^2 \Gamma_{12} Q'(\zeta) = \int_\zeta^\ell \left\{ k_z^2 \sin\alpha\, P(x) + k_y^2 \left(P(x)\operatorname{cosec}\alpha + Q(x)\cot\alpha \right) \right\} dx \qquad \ldots\ldots\ldots\ldots (4.5)$$

$$-\lambda^2 \Gamma_{12} P'(\zeta) + \lambda^2 \Gamma_{11} Q'(\zeta) = \int_\zeta^\ell \left\{ k_y^2 \left(P(x)\cot\alpha + Q(x)\operatorname{cosec}\alpha - (x-\xi)\Omega(x) \right) \right\} dx \qquad \ldots\ldots\ldots\ldots (4.6)$$

where

$$I_y = m\, k_y^2, \quad I_z = m\, k_z^2, \quad \lambda^2 = \frac{1}{m \varkappa^2} \quad \ldots\ldots (4.7)$$

by (3.3)

$$P(0) = \Omega(0) = Q(0) = \Omega'(0) = 0,$$

hence

$$P(x) = \int_0^x P'(\xi)\, d\xi,$$

$$Q(x) = \int_0^x Q'(\xi)\, d\xi$$

$$\Omega(x) = \int_0^x d\xi \int_0^\xi \Omega''(\eta)\, d\eta = \int_0^x (x-\eta)\Omega''(\eta)\, d\eta = -\operatorname{cosec}\alpha \int_0^x (x-\xi) Q'(\xi)\, d\xi,$$

where in the last step use has been made of (2.8).

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Introducing these last expressions in (4.5) and (4.6), and inverting the orders of integration

$$\lambda^2\Gamma_{22}P'(x) - \lambda^2\Gamma_{12}Q'(x) = \int_0^{\ell}\left\{P'(\xi)f_1(\xi,x) + Q'(\xi)f_2(\xi,x)\right\}d\xi \qquad \ldots\ldots\ldots\ldots (4.8)$$

$$-\lambda^2\Gamma_{12}P'(x) + \lambda^2\Gamma_{11}Q'(x) = \int_0^{\ell}\left\{P'(\xi)f_3(\xi,x) + Q'(\xi)f_4(\xi,x)\right\}d\xi \qquad \ldots\ldots\ldots\ldots (4.9)$$

with $0 \leq x \leq \ell$.

$$\left.\begin{aligned} \frac{f_1(\xi,x)}{k_z^2\sin\alpha + k_y^2\operatorname{cosec}\alpha} &= \frac{f_2(\xi,x)}{k_y^2\cot\alpha} = \frac{f_3(\xi,x)}{k_y^2\cot\alpha} = (\ell - \overline{\xi,x}) \\ f_4(\xi,x) &= \int_{\overline{\xi,x}}^{\ell}\left\{k_y^2\operatorname{cosec}\alpha + \operatorname{cosec}\alpha\,(\eta-x)(\eta-\xi)\right\}d\eta \end{aligned}\right\} \ldots (4.10)$$

where $\overline{\xi,x} = \max(\xi,x)$.

The equations (4.8) and (4.9) represent a system of simultaneous homogeneous Fredholm equations of the second kind. An approximate method of solution of these equations will be indicated in the next section.

5. A Method of Solution of the Integral Equations

The method of solution, to be suggested here, has been applied to similar problems in references 2, 3. As it has been presented there in great detail it will be sufficient to concentrate attention here on a modification of this method which, it is hoped, will reduce the amount of computation involved as well as improve the accuracy of the results obtained. However, before going any further, it should be noted that the method of solution given in references 2, 3 is one of many which could probably equally well be applied.

The principle of the method of solution in the above mentioned references is to replace the integrals in the integral equations by finite sums. For this purpose the range of values of the independent variable, in the present case $0 \leq x \leq \ell$ is subdivided into n equal intervals. The integrals over each of these subdivisions are then replaced by the product of the values of the integrand at their mid-points and the length of the interval ℓ/n. By giving the independent variable of the converted integral equation successively the values corresponding to the mid-points of the subdivisions, one obtains a set of n simultaneous linear equations in terms of the approximate values of the solutions of the integral equation at the mid-points.

It follows clearly from the last paragraph that the choice of the mid-points of subdivisions in the reduction of the integrals is arbitrary, and that the degree of accuracy of approximation to the solutions of the integral equation will depend entirely on the suitability of this choice. As a result of unpublished work, it was found that in the case of vibrations of a uniform cantilever the use of the mid-points as "reference points" necessitated the introduction of a comparatively large number of degrees of freedom, i.e. subdivisions, to ensure satisfactory accuracy for a few of the lower frequencies. Hence an attempt was made to develop a method by which more appropriate "reference points" could be found. In this way it was hoped to reduce the number of simultaneous equations and at the same time to improve the accuracy of the solutions.

This method, which proved to be very satisfactory in the problem mentioned above, assumes the knowledge of at least an approximation to one solution of the integral equation. In the case of the vibrating cantilever, a polynomial was used as an approximation to the fundamental mode. Using this approximate solution, the following equations determining values x_{ii} of reference points in the i^{th} subdivisions may be written down:

$$\int_{\frac{\ell}{n}(i-1)}^{\frac{\ell}{n}i} f(\xi,x_{ii})g(\xi)d\xi = \frac{\ell}{n} f(x_{ii},x_{ii})g(x_{ii}), \quad i = 1,2,\ldots,n$$

where f is the kernel and g the approximate solution of the integral equation. Once the x_{ii} have been found, their suitability can be checked by substitution in the equations

$$\int_{\frac{\ell}{n}(j-1)}^{\frac{\ell}{n}j} f(\xi,x_{ii})\, g(\xi)d\xi = \frac{\ell}{n} f(x_{ij},x_{ii})\, g(x_{ij})$$

assuming in the first place $x_{ij} = x_{ii}$. In the case of the cantilever it was found that the x_{ij} varied very little for $j = \text{const.}$, $i = 1,2,\ldots,n$.

The use of the modified method of solution in the problem treated earlier in this paper is obvious. It is intended to apply it to the case of a swept box, already built, and to compare the theoretical and experimental results.

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FIG. 1.