SAPIENZA UNIVERSITÀ DI ROMA

DOTTORATO DI RICERCA IN
MODELLI E METODI MATEMATICI PER LA TECNOLOGIA E LA SOCIETÀ
XXIV CICLO

Multi-scale model problems in lubrication theory and strain-gradient plasticity

Maria Chiricotto

Dipartimento di Scienze di Base e Applicate per l'Ingegneria
Sezione di Matematica

Acknowledgements.

I would like to thank and express my gratitude to all those who have supported and encouraged me in the realization of this thesis.

Lorenzo Giacomelli, for having introduced me to these beautiful subjects, for his many suggestions, for constant support during this research and for his inexhaustible patience.

Giuseppe Tomassetti, Micol Amar and Giuseppe Riey for all our mathematical discussions, for having given me the opportunity to work in a serene climate of friendship and collaboration.

I am also indebted with all my friends, MeMoMat-ti and not, for sharing my moments of weakness and my moments of enthusiasm, for their constant and affectionate presence.

A special thank to my family for having always believed in me.

Contents

Introduction

Over the recent years, an increasing interest in understanding the multi-scale nature of many physical phenomena has been registered. A wide range of phenomena we experience everyday, as well as applications in industrial processes, involve multiple length-scales ranging from macroscopic to molecular. For instance, applications of metals and liquids at micron, and even nano, scales are multiplying rapidly, and efforts are in progress in the materials and mechanics communities to measure and characterize their behavior. The difficulty in understanding the wide spectrum of rich phenomenologies that are observed, comes indeed from the fact that phenomena take place at different space and time scales. An insight into these problems requires a multidisciplinary approach, spanning from mechanical engineering to mathematical physics, from molecular dynamics to thermodynamics, each acting on different length-scales. The interaction and the constant feedback between these fields are leading to a more organic understanding of the dynamics that govern multi-scale phenomena. From the analytical viewpoint, an investigation of multi-scale problems requires the application of different techniques, including pde and ode methods (such as a priori estimates and compactness arguments), matched asymptotics expansion, and variational approaches (including relaxation and subdifferential techniques). All this apparatus has been applied in this dissertation to two different physical phenomena which we describe in the following sections. The first one concerns the dynamics of metals undergoing small plastic deformation in the framework of strain-gradient plasticity: we are interested in the effects of two different length-scales which have been introduced in recent models, with a particular attention to the feature that smaller specimens appear to have higher relative strength and hardness. The second one concerns the spreading of a droplet on a plain solid surface where both surface friction (at the liquid–solid interface) and contact-line friction (at the triple points where liquid, solid, and vapor meet) are accounted for. Common to both physical processes is the presence of at least two parameters, whose effects are of particular interest in this dissertation and which contribute in different ways to characterize the dynamics of the systems under consideration.

0.1 A multi-scale problem in strain-gradient plasticity

An increasing number of experimental evidences, including those from torsion in micron-dimensioned wires, nano/micro-indentation, and bending of micron-dimensioned thin-film (see e.g.[47, 83, 71, 84]), all show that, over a scale which extends from about a fraction to tens of microns, the strength of metallic components undergoing inhomogeneous plastic flow is inherently size-dependent: generally speaking, "smaller" specimens appear to be "stronger". Among the many evidences we mention a series of torsion experiments reported in [47] on copper wires of equal length and diameter ranging from $170\mu m$ down to $12\mu m$. The wire are twisted (with some rate) well into the plastic range, measuring the torque Q and the twist (i.e. the angle of rotation per unit length) Θ.

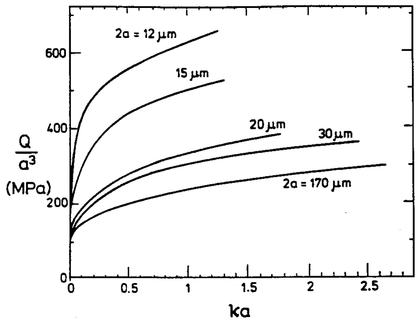

Figure 0.1: A series of experimental curves from [47]. Here Q is the torque needed to attain a twist k in a wire of radius a (here twist and radius are denoted by Θ and R, respectively).

The torsion data in Fig. 0.1 have been displayed in the form Q/R^3 *vs* ΘR where R the radius. The non-dimensional group ΘR may be interpreted as the magnitude of the shear strain at the surface of the wire. The group Q/R^3 gives a measure of the shear stress across the section of the wire in the same average sense. Had the wires been governed by a continuum theory with no constitutive length parameter, such as conventional plasticity is, all the curves of Fig. 0.1 would be the same and, by dimensional considerations, the torque Q needed to impart a twist Θ to a wire of radius R should obey

$$\frac{Q}{R^3} = f(R\Theta) \qquad (0.1.1)$$

where the function $f(\cdot)$ depends only on the material constituting wire. As noticed in [47], the experimental curves observed in Fig. 0.1 violate (0.1.1), and show that thinner wires have higher relative strength in the sense that a higher specific work input, Q/R^3, is needed to induce the same strain, $R\Theta$, in a thinner wire.

Though several observed plasticity phenomena display an effective size effect whereby the "smaller" is the size, the "stronger" is the response, classical plasticity theory cannot account for such experimental results, it being invariant with respect to spatial rescalings, i.e., neither any material intrinsic length-scale enters the constitutive law nor such size effects are predicted. This drawback has led to the development of theories that can capture such phenomena via dependencies on plastic-strain gradients, the *continuum gradient plasticity theories* [2, 3, 23, 40, 46, 47, 58, 60, 61].

In this dissertation we are concerned with small–strain theories, and we focus on a theory for isotropic materials introduced by Gurtin [60], elaborated in one space dimension by Gurtin, Anand, Gething, and Lele in [6], and developed by Gurtin, Anand, and Fried in [61] and [62]. Starting from the classical decomposition of the strain tensor into the sum of an *elastic strain* $\mathbf{E}^{\mathrm{e}}$, and a *plastic strain* $\mathbf{E}^{\mathrm{p}}$, size effects are incorporated through two distinct mechanisms:

- an *energetic mechanism*, by adding to the *elastic energy density*, $\psi_{\mathrm{e}}(\mathbf{E}^{\mathrm{e}})$, a *defect energy density*, $\psi_{\mathrm{d}}(\mathrm{curl}\mathbf{E}^{\mathrm{p}})$. In the framework of the Gurtin–Anand theory, $\mathrm{curl}\mathbf{E}^{\mathrm{p}}$ coincides with the Burgers tensor [62, §88.1-2] and provides a macroscopic description of "geometrically–necessary dislocations" (see also [29] for a discussion). This introduces an *energetic length-scale*, L which measures the contribution of the defect energy density to the system.

- a *dissipative mechanism*, by including a dependence on $\nabla\dot{\mathbf{E}}^{\mathrm{p}}$ in the *dissipation-rate density*. This introduces an *dissipative length-scale* ℓ (not necessarily microscopic) which measures the contribution of the dissipation-rate density to the system.

As will be shown in details in Chapter 1, Gurtin and Anand model is mainly grounded on the *microforce balance*

$$\mathbf{T}_0 = \mathbf{T}^{\mathrm{p}} - \mathrm{div}\mathbb{K}^{\mathrm{p}}, \tag{0.1.2}$$

equipped with termodinamically consistent constitutive relations for the *micro-stresses* $\mathbf{T}^{\mathrm{p}}$ and $\mathbb{K}^{\mathrm{p}}$, of the form

$$\mathbf{T}^{\mathrm{p}} = Y(E^{\mathrm{p}})g(d^{\mathrm{p}})\frac{\dot{\mathbf{E}}^{\mathrm{p}}}{d^{\mathrm{p}}}, \qquad \mathbb{K}^{\mathrm{p}} = \mathbb{K}^{\mathrm{p}}_{\mathrm{en}} + \mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}, \qquad \begin{cases} \mathbb{K}^{\mathrm{p}}_{\mathrm{en}} = \dfrac{\partial\psi_{\mathrm{d}}}{\partial\nabla\mathbf{E}^{\mathrm{p}}} \\ \mathbb{K}^{\mathrm{p}}_{\mathrm{diss}} = \ell^2 Y(E^{\mathrm{p}})g(d^{\mathrm{p}})\dfrac{\nabla\dot{\mathbf{E}}^{\mathrm{p}}}{d^{\mathrm{p}}} \end{cases} \tag{0.1.3}$$

where $Y(\cdot)$ is the flow resistance, $g(\cdot)$ is the rate-sensitivity and

$$d^{\mathrm{p}} := \sqrt{|\dot{\mathbf{E}}^{\mathrm{p}}|^2 + \ell^2|\nabla\dot{\mathbf{E}}^{\mathrm{p}}|^2}$$

is the *effective flow-rate*. The aim of Chapter 1 and Chapter 2, which are based on results obtained in [33] and [4], is to investigate, qualitatively and quantitatively, the role of *energetic length-scale* L and *dissipative length-scale* ℓ with respect to scale effects. To this aim, we will decouple the two length-scales:

- neglecting dissipative effects ($\ell = 0$) allows us to focus on the effects of the energetic length-scale L: in particular, we concentrate our attention on the development of boundary layers near $\partial\Omega$ and on an increase of the strain-hardening rate with L [6, §12];

- neglecting energetic effects ($L = 0$) allows us to focus on the strengthening effects of dissipative strain-rate gradients.

0.1.1 The case $\ell = 0$: Torsion problem

To focus on the role of the energetic length-scale L, in Chapter 1 we rule out dissipative size effects by setting $\ell = 0$. To reduce the complicated structure of the Gurtin-Anand model (0.1.2)-(0.1.3), we also assume constant flow resistance Y and rate-sensitivity g. Under these simplifying assumptions, the flow rule (0.1.3) is equivalent to the following differential inclusion:

$$\mathbf{T}^{\mathrm{p}} \in \partial\delta(\dot{\mathbf{E}}^{\mathrm{p}}) = \{\mathbf{A} \in \mathbb{R}^{3\times3}_{0,sym} : \ \delta(\widetilde{\mathbf{E}}^{\mathrm{p}}) - \delta(\dot{\mathbf{E}}^{\mathrm{p}}) \geq \mathbf{A} : (\widetilde{\mathbf{E}}^{\mathrm{p}} - \dot{\mathbf{E}}^{\mathrm{p}}) \ \ \forall\, \widetilde{\mathbf{E}}^{\mathrm{p}} \in \mathbb{R}^{3\times3}_{0,sym}\}. \tag{0.1.4}$$

By (0.1.4), after an explicit computation of $\mathbb{K}^{\mathrm{p}}$ the Gurtin–Anand flow rule (0.1.2) reads as:

$$\mathbf{T}_0 + \mu L^2 \left(\Delta\mathbf{E}^{\mathrm{p}} - \mathrm{sym}(\nabla \mathrm{div}\mathbf{E}^{\mathrm{p}}) + \frac{1}{3}(1+\eta)(\mathrm{div}\,\mathrm{div}\mathbf{E}^{\mathrm{p}})\mathbf{I} + \eta \,\mathrm{curlcurl}\mathbf{E}^{\mathrm{p}} \right) \in \partial\delta(\dot{\mathbf{E}}^{\mathrm{p}}) \tag{0.1.5}$$

where $-1 < \eta < 1$ is a dimensionless parameter. Note that when $L = 0$ the resulting law characterizes, according to the terminology of [62], the *Levy–Mises plastic response*. Looking closely to the energetic scale effects at the level of the one-dimensional problem is not appropriate, as the true role of the Burgers tensor (the curl of a vector field) can not be fully understood in such framework. Instead, it seems reasonable to investigate different symmetries which preserve the multi-dimensional nature of the problem. A first analysis suggests that, among these symmetries, the most interesting one is given by the *torsion problem* for a thin metallic wire, for which experimental evidences are also available (see above). We model a wire as an infinite right-cylinder Ω_R of radius R, subject to null tractions at the boundary and null initial conditions for the *twist* Θ and the *plastic–shear profile* $\gamma^{\mathrm{p}} = |\mathbf{E}^{\mathrm{p}}|$. The aim of Chapter 1 consists in quantifying the effects of the energetic lengthscale L on the torque Q that must be applied to induce a twist Θ with plastic–shear profile γ^{p} and which is given by the following expression:

$$Q = 2\pi\mu \int_0^R (\Theta\varrho - \gamma^{\mathrm{p}})\varrho^2 \mathrm{d}\varrho. \tag{0.1.6}$$

An important assumption we make is that the *twist* Θ is monotone: $\dot{\Theta} > 0$. This property is inspired both by the aforementioned experimental observation and by the fact that, because of the homogeneity of degree one of the dissipation rate density, the system (0.1.5) is unaffected by a monotone time re-parametrization, hence it allows also to replace the dependence of γ^{p} on time with a dependence on the twist. Furthermore, since the system does not contain intrinsic timescales, the ratio between the energetic length-scale L and the diameter $2R$ assumes a crucial role. In order to highlight the role of this parameter, we introduce a *normalized energetic lengthscale*:

$$\lambda := \sqrt{\frac{(1-\eta)}{2}}\frac{L}{R}. \tag{0.1.7}$$

In torsional symmetry the flow rule (0.1.5) reduces to a partial differential inclusion for the *normalized plastic-shear profile* γ and the *normalized twist* θ, that in terms of the normalized variables reads as:

$$\lambda^2\left(\frac{\partial^2\gamma}{\partial r^2}+\frac{1}{r}\frac{\partial\gamma}{\partial r}-\frac{1}{r^2}\gamma\right)-\gamma+\theta r\in\partial\left|\frac{\partial\gamma}{\partial\theta}\right| \quad \text{in } (0,1)\times(0,\infty), \tag{0.1.8a}$$

complemented by the initial-boundary conditions

$$\gamma(r,0)=0 \quad \text{and} \quad \frac{\partial\gamma}{\partial r}(1,\theta)+\frac{\gamma(1,\theta)}{2}=0 \quad \text{for } \theta\geq 0 \text{ and } r\in(0,1), \tag{0.1.8b}$$

the latter arising from requiring a null microscopic traction on the boundary. To construct and characterize solutions to (0.1.8), we will work with a relative *effective energy* functional

$$\mathscr{E}(\gamma,\theta) := \frac{1}{2}\int_0^1\left(\gamma^2+\lambda^2\left(\gamma'^2+\frac{\gamma'\gamma}{r}+\left(\frac{\gamma}{r}\right)^2\right)\right)r\,\mathrm{d}r-\theta\int_0^1\gamma r^2\mathrm{d}r$$

and a relative *dissipation* functional

$$\mathscr{D}(\dot\gamma) := \int_0^1|\dot\gamma|r\,\mathrm{d}r$$

in the natural space

$$H := \overline{C_c^\infty((0,1])}^{\|\cdot\|_r}, \quad \text{where} \quad \|g\|_r := \int_0^1\left(g'^2+\left(\frac{g}{r}\right)^2\right)r\mathrm{d}r.$$

Inspired by [72], writing (0.1.8) in its subdifferential formulation, we show the natural equivalence between this and an evolutionary variational inequality. This enable us to define the *energetic solution*, γ, of (0.1.8):

Definition 0.1. *Let* $\gamma\in W^{1,1}_{\mathrm{loc}}([0,+\infty);H)$. *We say that* γ *solves* (0.1.8) *if* $\gamma(0)=0$ *and*

$$\langle D_\gamma\mathscr{E}(\gamma,\theta),\tilde\gamma-\dot\gamma\rangle\geq\mathscr{D}(\dot\gamma)-\mathscr{D}(\tilde\gamma) \quad \textit{for all } \tilde\gamma\in H$$

for almost every $\theta>0$.

It follows from known results that this *energetic solution* exists and is unique [72]. The main result of Chapter 1 is an explicit characterization of this energetic solution, given in terms of solutions of suitable boundary-value problems. Three regimes are identified:

- an initial *elastic regime*, where $\theta \in [0, 1]$ and $\gamma = 0$;
- an intermediate *elasto-plastic regime*, where $\theta \in [1, \theta_\lambda)$, $\gamma = 0$ in $[0, c_\theta]$, and $\gamma := \gamma_\theta > 0$ in $(c_\theta, 1]$ where the pair $(c_\theta, \gamma_\theta)$ solves:

$$(\mathscr{P}_\theta) \begin{cases} \lambda^2 \left(\gamma'' + \dfrac{1}{r}\gamma' - \dfrac{\gamma}{r^2} \right) - \gamma = 1 - \theta r & \text{on } (c_\theta, 1) \\ \gamma(c_\theta) = \gamma'(c_\theta) = 0 \\ \gamma'(1) + \dfrac{\gamma(1)}{2} = 0. \end{cases} \tag{0.1.9}$$

  Here c_θ (representing the left-endpoint of the plastic region) is an additional unknown which is determined together with γ (at variance with the case $\lambda = 0$, when c_θ is given by $1/\theta$). When θ reaches the critical twist θ_λ (up to which (0.1.9) is well posed), the elasto-plastic boundary hits the origin $r = 0$, and the wire becomes fully plastified. Hence we have:

- an ultimate *plastic regime*, where $\theta > \theta_\lambda$, and $\gamma := \overline{\gamma}_\theta > 0$ in $(0, 1]$ where $\overline{\gamma}_\theta$ solves

$$(\mathscr{P}'_\theta) \begin{cases} \lambda^2 \left(\gamma'' + \dfrac{1}{r}\gamma' - \dfrac{\gamma}{r^2} \right) - \gamma = 1 - \theta r & \text{on } (0, 1) \\ \gamma(0) = 0 \\ \gamma'(1) + \dfrac{\gamma(1)}{2} = 0, \end{cases} \tag{0.1.10}$$

  which is well posed for all $\theta \in \mathbb{R}$.

Extending γ_θ to $(0, 1)$,

$$\gamma_\theta(r) := \ 0 \quad \text{if} \ \ r \in (0, c_\theta],$$

and patching γ_θ and $\overline{\gamma}_\theta$ together,

$$\gamma(r, \theta) := \begin{cases} 0 & \text{if } \ \theta \in [0, 1] \\ \gamma_\theta(r) & \text{if } \ \theta \in (1, \theta_\lambda) \\ \overline{\gamma}_\theta(r) & \text{if } \ \theta \geq \theta_\lambda, \end{cases} \tag{0.1.11}$$

we obtain the announced characterization of the energetic solution:

Theorem 0.1. *The function γ defined by* (0.1.11) *is the unique solution of* (0.1.8) *in the sense of Definition 0.1. Moreover,* $\gamma \in \mathrm{Lip}([0, +\infty); H)$.

The characterization of γ given by Theorem (0.1) allows us to work out a formal asymptotic expansion as $\lambda \to 0$ (for fixed θ) which confirms:

- the presence of two boundary layers of width $O(\lambda)$, near the external boundary of the wire and near the boundary of the plastified region;

- that the energetic scale is responsible for size-dependent strain-hardening, with the thinner wires being harder.

We also obtain a scaling law for the critical twist in terms of energetic scale λ:

$$\theta_\lambda \sim \frac{1}{\sqrt{6}\lambda} \quad \text{for} \ \ \lambda \ll 1. \tag{0.1.12}$$

0.1.2 The case $L = 0$: Traction problem

The effects of the dissipative length-scale ℓ may be singled out by focusing on the case $L = 0$ in the reduced *one-dimensional* model introduced by Gurtin, Anand, Lele and Gething in [6]. This theory alleviates most of the intricacies of the full model (0.1.3) and describes a body in the form of a strip of finite width I undergoing simple shear with a given shear stress τ. Under a simplified set of constitutive relations, the one-dimensional theory leads to a nonlocal flow rule in the form of a nonlinear partial differential equation for the *plastic strain* γ:

$$\begin{cases} \tau + L^2\partial_x^2\gamma = \frac{\partial_t\gamma}{d^{\mathrm{p}}} - \ell^2\partial_x\left(\frac{\partial_x\partial_t\gamma}{d^{\mathrm{p}}}\right) \\ \partial_x\tau = 0 \end{cases} \qquad (t,x) \in (0,\infty)\times I, \tag{0.1.13}$$

where

$$d^{\mathrm{p}} = \sqrt{|\partial_t\gamma|^2 + \ell^2|\partial_x\partial_t\gamma|^2}.$$

The flow rule (0.1.13) is to be considered together with initial-boundary conditions $\partial_t\gamma|_{\partial I} = 0$ and $\gamma(0,x) = \gamma_0(x)$, and with a traction condition given by imposing the constant (in space and time) traction $\tau = \tau_\ell$. These assumptions formally lead to the following *constrained boundary-value problem* for $u = \partial_t\gamma$:

$$\begin{cases} \tau_\ell = \frac{u}{\sqrt{u^2+\ell^2(u')^2}} - \ell^2\left(\frac{u'}{\sqrt{u^2+\ell^2(u')^2}}\right)' \\ u|_{\partial I} = 0, \quad ⨍_I u\,\mathrm{d}x = 1 \end{cases} \tag{0.1.14}$$

where the primes denote differentiation with respect to x. The presence of the normalized mean constraint is due to the scaling invariance of $(0.1.14)_1$ with respect to the transformation $u \to \alpha u$ $(\alpha \neq 0)$, which, in essence, expresses the rate–independence of $(0.1.13)_1$. In this framework, a sample may then be said to be *stronger* than a second one (made of the same material) if a higher stress τ_ℓ is needed to generate the same mean plastic flow. On the other hand, of course a material sample is *smaller* than a second one if the ratio $\ell/|I|$ is higher. Hence, "*smaller is stronger*" is equivalent to say that

(*A*) τ_ℓ is increasing with $\ell/|I|$.

This is exactly what the numerical simulations performed in [6] indicate. With a view toward establishing a variational description of (0.1.14), as formulated in [6] as a conjecture, we are led to consider the following variational problem: The *dissipational functional*

$$F(u) = \fint_I \sqrt{u^2 + \ell^2 (u')^2}\, \mathrm{d}x; \tag{0.1.15}$$

(B) has a minimum value, τ_ℓ, over all admissible fields u such that $u|_{\partial I} = 0$ and $\fint_I u = 1$;

(C) any minimizing fields u is a solution of $(0.1.14)_1$.

The natural space to analyze the problem is the space of functions with bounded variation. In some cases, it will be harmless to work in a bounded, open and connected set $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary rather than in an interval. By rescaling x, we may assume without loss of generality that $\ell = 1$. We thus define

$$BV_*(\Omega) = \left\{ u \in BV(\Omega) : \fint_\Omega u\, \mathrm{d}x = 1 \right\}.$$

Extending the (0.1.15) to $L^1(\Omega)$ and encoding the boundary conditions into the problem we define a functional $F^\circ : L^1(\Omega) \to [0, +\infty]$ as

$$F^\circ(u) = \begin{cases} \int_\Omega \sqrt{u^2 + |\nabla u|^2}\, \mathrm{d}x & \text{if } u \in W_0^{1,1}(\Omega) \\ +\infty & \text{if } u \in L^1(\Omega) \setminus W_0^{1,1}(\Omega). \end{cases}$$

We note that for a smooth u the integrand in (0.1.15) coincides with the norm of the $\mathbb{R}^{N+1}$-vector $(u, \nabla u)$. Hence we will show that the relaxation of F° coincides with the total variation of the R^{N+1}-valued measure (u, Du), denoted by $|(u, Du)|$ (see [5, Definition 1.4]):

Theorem 0.2. *Let F° be defined by* (2.1.5). *Then*

$$\overline{F^\circ}(u) = \begin{cases} \int_\Omega \sqrt{u^2 + |\nabla u|^2}\, \mathrm{d}x + |D^s u|(\Omega) + \int_{\partial\Omega} |u| \mathrm{d}\mathcal{H}^{N-1} & \textit{if } u \in BV(\Omega) \\ +\infty & \textit{if } u \in L^1(\Omega) \setminus BV(\Omega). \end{cases} \tag{0.1.16}$$

Furthermore, for all $u \in BV(\Omega)$ it holds:

$$\overline{F^\circ}(u) = |(u, Du)|(\Omega) + \int_{\partial\Omega} |u| \mathrm{d}\mathcal{H}^{N-1} \tag{0.1.17}$$

$$= \sup \left\{ \int_\Omega u\,(s - \operatorname{div} \mathbf{t})\, \mathrm{d}x + \int_{\partial\Omega} u \mathbf{t} \cdot \mathbf{n} \mathrm{d}\mathcal{H}^{N-1} : (s, \mathbf{t}) \in C^\infty(\overline{\Omega}),\ \|(s, \mathbf{t})\|_\infty \le 1 \right\}. \tag{0.1.18}$$

Here ∇u and $D^s u$ denote the absolutely continuous, resp. singular, part of Du with respect to the Lebesgue measure. Standard direct methods of the calculus of variations and the foregoing discussion enable us to answer positively to part (B). The 1-homogeneity of $\overline{F^\circ}$ enables us to identify a relation between the value of the minimum, the shear stress τ_ℓ, and the Lagrange multiplier of the constrained minimization problem, τ_Ω, as follows (see Proposition 2.1 and Remark 2.1 in Section 2.1.2):

Proposition 0.1. *Let* $\tau_\Omega := \frac{1}{|\Omega|} \min_{BV_*(\Omega)} \overline{F^\circ}$. *Then*

$$u_m \in \operatorname*{argmin}_{BV_*(\Omega)} \overline{F^\circ} \iff \begin{cases} u_m \in BV_*(\Omega) \\ \tau_\Omega \chi_\Omega \in \partial \overline{F^\circ}(u_m). \end{cases}$$

Here χ_Ω denotes the characteristic function of the set Ω and $\partial\overline{F^\circ}$ the subdifferential of $\overline{F^\circ}$ which we characterize at least in the sense of distributions, as it has been done for other problems with linear growth in the gradient [7, 8]. Identifying $\partial\overline{F^\circ}$ with the right-hand side of $(0.1.14)_1$, Proposition 0.1 shows that τ_Ω, seen as a Lagrange multiplier for the constrained minimization problem, is uniquely determined over all possible minimizers, a fact which corresponds to a weak, but dimension-independent, answer to (C) (see below for the one-dimensional case). We use the characterization of τ_Ω given in Proposition 0.1 to infer a monotonicity property of the shear stress with respect to the dissipative length-scale, and consequently to yield (A), as follows:

Theorem 0.3 ("Smaller is stronger")**.** *Let*

$$\lambda\Omega = \{x \in \mathbb{R}^N : \ x/\lambda \in \Omega\}.$$

The function $\lambda \mapsto \tau_{\lambda\Omega}$ *is decreasing (strictly if* $N = 1$*).*

Such property confirms that the strain-gradient theory under consideration is able to model the experimental evidence that smaller samples have higher relative strength.
In one space dimension, where the model is proposed, we are also able to give a complete answer to part (C) proving uniqueness, regularity, and qualitative properties of the minimizer in the space $SBV_*(I) = BV_*(I) \cap SBV(I)$ through:

Theorem 0.4. *The functional* $\overline{F^\circ}$ *has a unique minimizer* $u \in SBV_*(I)$*. The minimizer u is even, strictly decreasing in* $[0, \alpha)$*, smooth in* I*, and it solves the Euler-Lagrange equation* $(0.1.14)_1$ *(with* $\ell = 1$ *and* $\tau_\ell = \tau_I$ *defined by Proposition 2.1). Furthermore*

$$\lim_{x \to \alpha^-} u(x) > 0 \quad \textit{and} \quad \lim_{x \to \alpha^-} u'(x) = -\infty.$$

Besides non-generic domains (such as an N-sphere, where we expect results similar to those in Theorem 0.4 to hold), we believe that the multi-dimensional problem will not have smooth minimizers in general, as the mass constraint may produce solutions which jump down to zero in the interior. Hence, in general the corresponding Euler-Lagrange equation will not be satisfied by minimizers.

0.2 A multi-scale problem in lubrication theory

Wetting and spreading phenomena are of key importance in many processes, both natural and industrial. For example, in coating a liquid onto a solid or in the deposition of pesticides on plant leaves, it is essential that the liquid dynamically wets (or not) the solid surface. Though theories for the description of wetting phenomena have been extensively developed, the actual physics that govern them still remain unclear today. Part of this difficulty stems from the contact-line paradox arising in Navier-Stokes equations: the no-slip condition with a constant viscosity leads to a force singularity at advancing contact lines [67, 41]. To remove this paradox many models proposed the introduction of a "microscopic length-scale" [39, 75, 22]. In many theories, an effective slip condition at the liquid-solid interface is postulated to occur, for example the Navier slip condition $U = \mu B U_\zeta$ at the liquid-solid interface, $\zeta = 0$ (here U is the fluid's horizontal velocity in a two-dimensional framework and μ is the viscosity). The ratio $1/B$ is to be understood as a *friction coefficient* between the liquid and the wall. But as confirmed by recent investigations by Qian, Wang and Sheng [78] and by Ren and E [80], these slippage models fail to describe the dynamics *near the contact line region*. Among the variety of suggested models, we are concerned with an effective continuum model proposed by Ren and E [80] and by Ren, Hu and E [81] in which a further source of friction is encoded, coming from the deviation of the contact angle Θ from its static value Θ_S. In the simplest case of a linear friction law, this model turns into in the following conditions:

$$\begin{aligned} D\gamma(\cos\Theta - \cos\Theta_S) &= U_{CL} && \text{if } \Theta_S > 0 \quad \text{(partial wetting)},\\ D\gamma(\cos\Theta - 1) &= \max\{U_{CL}, 0\} && \text{if } \Theta_S = 0 \quad \text{(complete wetting)}. \end{aligned} \tag{0.2.1}$$

Here U_{CL} is the speed of the contact line, γ denotes the liquid-vapor surface tension, and $1/D$ is an effective friction coefficient *at the contact line*. Of interest to us is to discuss the dynamics of spreading with respect to the two parameters, b and d, which represents the normalized counterpart of B and D. This is done in Chapter 3 and Chapter 4 which are based on results obtained in [31, 32, 30] . First of all, we reduce the complexity of the Navier-Stokes system while retaining the effects of both capillary forces and frictional forces (viscous friction in the bulk, surface friction at the liquid-solid interface, and contact-line friction at the liquid-solid-vapor interface), considering this model in the lubrication regime (see e.g. [75, 53, 70]). In the lubrication approximation, the spreading of thin droplets may be modeled by a class of fourth order free boundary problems for the normalized height of the liquid film, $h(t, x)$, and the extent of the wetted region, $(-s(t), s(t))$ (for simplicity, we assume h to be symmetric with respect to $x = 0$):

$$\begin{cases} h_t + (hu)_x = 0, \ u = (h^2 + bh)h_{xxx} & \text{in } (0, s(t)) \\ h = 0, \ \frac{\mathrm{d}}{\mathrm{d}t}s(t) = \lim\limits_{x \to s(t)^-} u & \text{at } x = s(t) \\ h_x = h_{xxx} = 0 & \text{at } x = 0. \end{cases} \tag{0.2.2}$$

By formal asymptotic expansions of the traveling wave solutions to (0.2.2) (see Section 3.4) we know that fronts can only advance in the *complete wetting regime*, characterized by

$\theta_S = 0$: therefore the free boundary condition (0.2.1) reduces to

$$d\left(h_x^2 - \theta_S^2\right) = \frac{\mathrm{d}s}{\mathrm{d}t} \quad \text{at } x = s(t). \tag{0.2.3}$$

0.2.1 Asymptotic analysis

In the absence of contact-line friction, i.e. $1/d = 0$, the dynamics (0.2.2) are known to be influenced only logarithmically by the slippage model, at least at intermediate timescales (see [65] for $\theta_S > 0$ and [34] for the case of rough surfaces): more precisely,

$$\theta_m^3 \sim \theta_S^3 + 3s' \log\left(\frac{s\theta}{b}\right) \tag{0.2.4}$$

whiere θ_m is the *macroscopic contact angle*, defined as the slope of the unique even arc of parabola having the same mass and support at its zero. In the regime of complete wetting ($\theta_S = 0$), this leads to the following scaling law, which is often referred to as the logarithmic correction to Tanner's law [85] (see also [17] and [52]):

$$s \sim \left(\frac{t}{\log\left(\frac{1}{b^7 t}\right)}\right)^{1/7} \quad \text{for} \quad s_0^7 \log\left(\frac{1}{bs_0}\right) \ll t \ll b^{-7}. \tag{0.2.5}$$

Note that the appearance of an intermediate timescale is real: on one hand, it takes a certain time for the droplet to forget its initial shape; on the other hand, for large times $h \ll b$ on the whole support, hence the evolution is governed by slippage alone and s will scale like $t^{1/6}$. Again in complete wetting, analogous logarithmic corrections were obtained by de Gennes [39] for a related model in which the contact angle condition is replaced by the action of van der Waals forces.

In the presence of contact-line friction the situation is more complicated and more than one intermediate scaling law appears. This is due to the dependence of the scaling laws on whether θ_S is zero or not, and on the relation between the two normalized parameters b and d. To give a more precise quantitative description of these scaling laws, a matched asymptotic study is worked out in Chapter 3, relating the macroscopic contact angle to the speed of the contact line. It turns out that a crucial role is played by the parameter $k = dM/b^2$, which may be seen as a measure of the relative strength of surface friction versus contact-line friction (M is the mass of the droplet). Let us fix for simplicity $M = 1$ and discuss separately the case of complete and partial wetting.

If $\theta_S = 0$ the dynamics is governed by the following laws:

- for a stronger contact-line friction, $d \lesssim b^2$, the system bypasses the moderate timescale dominated by viscous friction and the droplet displays only an early timescale dominated by contact-line friction and a final timescale dominated by surface friction:

$$s \sim \begin{cases} (dt)^{1/5} & \text{if } \frac{s_0^5}{d} \ll t \ll \frac{b^5}{d^6} \quad (\text{and } s_0 \ll \frac{b}{d}) \\ (bt)^{1/6} & \text{if } t \gg \frac{b^5}{d^6}; \end{cases} \tag{0.2.6}$$

- for a stronger surface friction, $b^2 \ll d$ the droplet displays an early timescale dominated by contact-line friction, a moderate timescale dominated by viscous friction (which is logarithmically corrected by surface friction, as in the case of zero contact-line friction, see (0.2.5)), and a final timescale dominated by surface friction:

$$s \sim \begin{cases} (dt)^{1/5} & \text{if } \frac{s_0^5}{d} \ll t \ll \frac{1}{d^{7/2}\log^{5/2}\frac{d}{b^2}} \text{ (and } s_0^2 \ll \frac{1}{d\log\frac{d}{b^2}}) \\ \left(\frac{t}{\log\frac{1}{b^7 t}}\right)^{1/7} & \text{if } \frac{1}{d^{7/2}\log^{5/2}\frac{d}{b^2}} \ll t \ll b^{-7} \\ (bt)^{1/6} & \text{if } t \gg b^{-7}. \end{cases} \tag{0.2.7}$$

The lower bounds on the initial times, as already discussed, correspond to the time that the system needs to "forget" its initial shape and to relax to a quasi-static configuration.
If $\theta_S > 0$, the profile of a spreading droplet converges (exponentially) to the unique steady state with given mass and contact angle θ_S as $t \to +\infty$. We concentrate our attention to the case of a persistent macroscopic profile for all times:

$$\theta_S \gg b^2, \quad \text{i.e. } bs_\infty \ll 1.$$

For sufficiently large times, also in partial wetting the contact-line friction plays no role and the system evolves according with the Cox-Hocking relation (0.2.4). However there are still intermediate timescales which are influenced by contact-line friction:

(i) if $d \ll \theta_S$, then (0.2.4) is preceded by an early timescale dominated by contact-line friction;

(ii) if $\theta_S \ll d$, then (0.2.4) is preceded by an early timescale dominated by contact-line friction and a moderate timescale dominated by viscous friction.

These results highlight the role of the threshold parameter d/θ_S. In addition we are able to quantify the time in which (0.2.4) takes over: up to a logarithmic correction, it reads as:

$$(0.2.4) \iff t \gg \begin{cases} \frac{1}{d\theta_S^{5/2}} & \text{if } d \ll \theta_S \\ \frac{1}{\theta_S^{7/2}\log^{1/6}\left(\frac{\theta_S}{b^2}\right)} & \text{if } \theta_S \ll d. \end{cases}$$

The scaling laws in (0.2.6) and (0.2.7) may already be predicted by a simple heuristic argument (see §3.8). However, in this simple argument one has to assume *a-priori* that the microscopic contact angle θ is "relatively close" to θ_m. Now, especially in complete wetting where the slope might vary abruptly near the contact line, this strong assumption could be not valid and a discrepancy between the effective and microscopic contact angles may occur. To overcome this drawback, in §3.9 we work out a detailed matched asymptotic study of (0.2.2)-(0.2.3). From the pioneering works of Hocking [65, 66] and Cox [34], quite a few works has been devoted to matched asymptotic with speed-dependent contact angle conditions [56, 43, 63]. However, none of them includes (3.2.3), and the scaling assumptions used are not always sharp or easy to reconstruct. Hence, here we extend, modify and simplify the asymptotic in a way which includes (3.2.3) and keeps track of all the assumptions used (we actually argue for a much more general relation, potentially applicable to different boundary conditions, between speed and contact angle). We assume that the evolution within the liquid's bulk is "slow" and quasi-static, in the sense that

$$0 \le s^6 s' \ll 1 \quad \text{and} \quad bs \ll 1. \tag{0.2.8}$$

The second inequality in (0.2.8) ensures (via mass conservation) that $h(t,\cdot) \gg b$ on most of its support. Then the asymptotic yields

$$\theta_m^3 \sim \begin{cases} \theta^3 + 3s' \log\left(\frac{s\theta}{b}\right) & \text{if } b \ll s\theta \quad \text{and } s' \ll \theta^3 \\ 3s' \log\left(\frac{s(s')^{1/3}}{b}\right) & \text{if } b^3 \ll s^3 s' \quad \text{and } s' \gg \theta^3. \end{cases} \tag{0.2.9}$$

Of course, (0.2.9) recovers the earlier results in when $\theta \equiv \theta_S$ (see (0.2.4)). When instead $bs \gg 1$ but the evolution is "slow", an asymptotic relation between s and θ may be obtained:

$$\left(\frac{3}{2s^2}\right)^3 \sim \theta^3 \quad \text{if } bs \gg 1,\ s^5 s' \ll b, \text{ and } \theta > 0. \tag{0.2.10}$$

Ode arguments then enable us to pass from (0.2.9) and (0.2.10) to the early and moderate scaling laws in (0.2.6) and (0.2.7). In the particular case $1/d = 0$, (0.2.5) is also recovered. A different asymptotic which assumes a quasi-selfsimilar profile of the solution is adopted for the long-time scaling law. In this, as well as in earlier asymptotic studies, the local behavior near the contact line is described by an advancing traveling wave, that is, a solution of

$$\begin{cases} -U = (f^2 + bf) f_{\xi\xi\xi},\ f > 0 & \text{in } (0,+\infty), \\ f = 0,\ f_\xi = \theta & \text{at } \xi = 0 \end{cases} \tag{0.2.11}$$

whose profile is determined by "matching" it to the bulk region. The matching condition selects the solution to (0.2.11) which displays the "linear" (up to a log-correction) behavior at infinity. Though it is quite clear from the heuristics in Section 3.4 that such traveling wave exists and is unique, we were unable to find a proof in the literature. Therefore we provide it in Section 3.6. Actually, we will prove the following, slightly more general result:

Theorem 0.5. *For any $\theta \ge 0$ and any $U \in C([0,+\infty))$ non-negative, bounded, and such that* $\inf U > 0$ *if $\theta = 0$, there exists a unique solution $f \in C^1([0,+\infty)) \cap C^3((0,+\infty))$ of* (0.2.11) *such that $f_{\xi\xi}(\xi) \to 0$ as $\xi \to +\infty$.*

0.2.2 Existence of weak solutions

In Chapter 4, we perform a first analytical study for a generalized version of problem (0.2.2)-(0.2.3):

$$(P)\begin{cases} h_t + (m(h)h_{xxx})_x = 0,\ h > 0,\ h \text{ even} & \text{in } (0,t)\times(-s(t),s(t)) \\ h = 0,\ \dot{s}(t) = \lim\limits_{x\to s(t)} \dfrac{m(h)}{h} h_{xxx} & \text{at } (0,t)\times\{x = s(t)\} \\ \dot{s}(t) = d\left(h_x^2 - \theta_S^2\right) & \text{at } (0,t)\times\{x = s(t)\} \\ h(0,x) = h_0(x),\ h_0 \text{ even} & \text{in } (-s(t),s(t)), \end{cases} \tag{0.2.12}$$

where

$$m \in C^\infty((0,\infty)) \cap C([0,\infty)), \text{ with } m(h) \sim h^n\ (n > 0) \text{ as } h \to 0 \text{ and } m > 0 \text{ in } (0,\infty). \tag{0.2.13}$$

Thin-film equations with zero contact angle (i.e., replacing $(0.2.12)_3$ by $h_x = 0$) have been widely studied in the past two decades, and some results are also available for a constant, non-zero contact angle. We refer to Section 4.2 for a discussion. The main interest of our study lies in trying to capture a speed-dependent contact-angle condition in a weak formulation of (P). To this aim, the starting point is to translate the problem on the fixed domain $I = (-1,1)$:

$$\begin{cases} v_t - \dfrac{\dot{s}}{s} y v_y + \dfrac{1}{s^4}(m(v)v_{yyy})_y = 0,\ v > 0,\ v \text{ even} & \text{in } (0,t)\times I \\ v = 0,\ \dot{s}(t) = \lim\limits_{y\to 1} \dfrac{m(v)}{v}\dfrac{v_{yyy}}{s^3} & \text{at } (0,t)\times\{y = 1\} \\ \dot{s}(t) = d\left(\dfrac{v_y^2}{s^2} - \theta_S^2\right) & \text{at } (0,t)\times\{y = 1\} \\ v(0,y) = v_0(y),\ v_0 \text{ even} & \text{in } I. \end{cases} \tag{0.2.14}$$

Besides the specific form of the free boundary condition, we are interested in this fixed-domain formulation since it might have the potential to yield improvements in theory of thin-film equation. In this formulation, the surface energy functional is given by

$$E(v(t)) = \frac{1}{2}\int_I \left(\frac{v_y^2}{s} + s\theta_S^2\right) \mathrm{d}y.$$

As formally shown in §3.5, a sufficiently smooth solution to (0.2.14), is such that

$$E(v(t)) + \frac{d}{2}\int_0^t \left(\frac{v_y^2(t,1)}{s^2} - \theta_S^2\right)^2 + \iint_{\{v>0\}_t} \frac{1}{s^5} m(v) v_{yyy}^2 = E(v_0). \tag{0.2.15}$$

Our main result is the following:

Theorem 0.6. *Let m as in* (0.2.13)*. For any $v_0 \in H^1(I)$, even and non-negative, and any $s_0 > 0$ there exists a pair of functions (s, v), with $v \in C^{\frac{1}{2},\frac{1}{8}}([0,\infty)\times \overline{I}) \cap L^\infty_{\mathrm{loc}}([0,\infty); H^1(I))$, $v \geq 0$, and $s \in H^1((0,\infty))$, $s > 0$, which solves* (0.2.14) *with initial datum v_0 in the sense that, for all $T > 0$, it holds that:*

(i) $v_t \in L^2((0,T); (H^1(I))')$;

(ii) $v_{yyy} \in L^2_{\mathrm{loc}}(\{v > 0\})$ *and* $\sqrt{m(v)}v_{yyy} \in L^2(\{v > 0\})$;

(iii) *for all* $\varphi \in L^2((0,T); H^1(I))$

$$\int_0^T < v_t, \varphi > \,\mathrm{d}t = \int_0^T \int_I \frac{\dot{s}}{s} y v_y \varphi + \int_0^T \int_I \frac{1}{s^4} m(v) v_{yyy}\varphi_y; \tag{0.2.16}$$

(iv) $v(0,y) = v_0(y)$ *in* $H^1(I)$;

(v) $v(1) = 0$ *in* $L^2(0,T)$;

(vi) *v is even;*

(vii) *v dissipates $E(v)$ in the sense that*

$$E(v(t)) + \frac{1}{2d}\int_0^t \dot{s}^2 + \iint_{\{v>0\}_t} \frac{1}{s^5} m(v) v_{yyy}^2 \leq E(v_0). \tag{0.2.17}$$

The kinematic condition in (0.2.14) is captured in its weak form of mass conservation. The free boundary condition $(0.2.14)_3$, that is

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = d\left(\frac{v_y^2}{s^2} - \theta_S^2\right), \tag{0.2.18}$$

is encoded only very weakly, in the form of the energy inequality (0.2.17). More precisely, the extent in which (0.2.18) is recovered is the following: if the solution had sufficient additional regularity, such that on one hand (0.2.17) were satisfied as an equality, and on the other hand the formal computations leading to (0.2.15) were rigorous, then the solution would satisfy (0.2.18). A further weakness of Theorem 0.6 is that we are not able to prove that $v > 0$ a.e. in $(0,T)\times I$. In this respect, it is important to notice that even for the well-known case of a zero-contact angle condition, the standard entropy estimates in our fixed-domain framework would not yield a.e. positivity of the solution, since there the support of the test functions is fixed in the x-variable, that is, receding in the y-variable when s increases. This points to the necessity of a refinement of the standard entropy estimates (see §4.8 and (0.2.22) below), localized in such a way that the test function "follows" the free-boundary. We hope to come back to this topic in the future, and we leave it here as an open question.

A merit of our approach is the construction of approximating solutions (s, v) in which v *is positive and* (s, v) *satisfy the free boundary condition* (0.2.18). These approximating solutions are constructed as follows: we modify the mobility term

$$m_{\delta,\sigma}(\tau) = \delta + \frac{m(\tau)\tau^4}{\sigma m(\tau) + \tau^4 + \delta m(\tau)\tau^4}, \quad \tau \in \mathbb{R}, \tag{0.2.19}$$

for some $\delta > 0$ and $\sigma > 0$, and we raise the initial datum of an height $\varepsilon > 0$. Note that the approximation $m_{0,\sigma}$ corresponds to nowdays standard modification (see [16] and [10]), to obtain positive solutions given a positive initial datum. This leads to

$$\begin{cases} v_t - \frac{\dot{s}}{s} y v_y + \frac{1}{s^4}(m_{\delta,\sigma}(v) v_{yyy})_y = 0 & \text{in } (0,t) \times (0,1) \\ v_y = v_{yyy} = 0 & \text{at } (0,t) \times \{y = 0\} \\ v = \varepsilon, \ v_{yyy} = 0 & \text{at } (0,t) \times \{y = 1\} \\ \dot{s}(t) = d\left(\frac{v_y^2}{s^2} - \theta_S^2\right) & \text{at } (0,t) \times \{y = 1\} \\ v(0,y) = v_0(y) + \varepsilon & \text{in } (0,1). \end{cases} \tag{0.2.20}$$

The mentioned positive approximating solutions are obtained for $\delta = 0$ and $\varepsilon = \sigma$. In order to prove the existence of solutions (0.2.20), we consider the problem with *prescribed* free boundary $s(t)$:

$$\begin{cases} v_t - \frac{\dot{s}}{s} y v_y + \frac{1}{s^4}(m_{\delta,\sigma}(v) v_{yyy})_y = 0 & \text{in } (0,t) \times (0,1) \\ v_y = v_{yyy} = 0 & \text{at } (0,t) \times \{y = 0\} \\ v = \varepsilon, \ v_{yyy} = 0 & \text{at } (0,t) \times \{y = 1\} \\ v(0,y) = v_0(y) + \varepsilon & \text{in } (0,1), \end{cases} \tag{0.2.21}$$

where indeed the free-boundary condition (0.2.18) is removed. Since s is fixed (i.e., the contact-angle condition does not hold), the dissipative structure given by (0.2.15) is lost, so only local existence to (0.2.21) is avaible. To capture the contact-angle condition (0.2.18) and obtain local existence for the free-boundary problem, we apply a fixed point argument, which from the technical viewpoint, is the hardest part of the work and the crucial one. Once this condition is recovered, then also the dissipative structure given by (0.2.15) is, and some a-priori estimates, implying additional regularity and global existence, follow. To investigate sign property of solutions to (0.2.20), we adopt the technique proposed in [13]. It is based on the introduction of an auxiliary function G such that $G''(y) = \frac{1}{m_{\delta,\sigma}(y)}$ and which provide the following entropy-type estimate (uniform with respect to δ)

$$sup_{t \leq T} \int_0^1 G_{\sigma,\delta}(v(t)) + C^{-1} \iint_{Q_T} v_{yy}^2 \leq C(\varepsilon, T) \quad \text{for all } T < \infty. \tag{0.2.22}$$

This allows to pass to the limit as $\delta \to 0$ obtaining *positive* solutions to $(P_{\varepsilon,0,\sigma})$. Finally we pass to the limit as $\varepsilon = \sigma \to 0$ (in a nowadays standard fashion) and complete the proof of Theorem 0.6.

Chapter 1

Torsion in strain gradient plasticity: energetic scale effects

1.1 Introduction

At the micron scale, metallic components undergoing non–uniform plastic flow are known to display size–dependent behavior: generally speaking, "smaller" specimens appear to be "stronger", with smaller specimens being, in general, stronger. Among the many evidences, of particular interest to us is a series of torsion experiments, reported in [47]. During these experiments, the wires are twisted (monotonically and with the same rate) well into the plastic range, and the relationship between torque Q and twist Θ (angle of rotation per unit length) is recorded. The inability of conventional plasticity in capturing size effects is medicated in strain-gradient plasticity theories through an explicit appearance of the plastic–strain gradient in the field equations [2, 3, 23, 40, 46, 47, 58, 60, 61]. This chapter is concerned with small–strain theories, and we focus on a theory for isotropic materials developed by Gurtin and Anand in [61]. For additional details, we refer to the recent monograph [62], where the theory is expounded.

1.1.1 Conventional plasticity

We begin by recalling the field equations from standard small–strain plasticity theory for isotropic materials, with specific reference to flow theories commonly used for metals. In small–strain plasticity, the unknowns are the *displacement* $\mathbf{u}(x,t) \in \mathbb{R}^3$ and the *plastic strain* $\mathbf{E}^{\mathrm{p}}(x,t) \in \mathbb{R}^{3\times3}_{0,sym}$. The *elastic strain* $\mathbf{E}^{\mathrm{e}}(x,t) \in \mathbb{R}^{3\times3}_{sym}$, defined by

$$\mathbf{E}^{\mathrm{e}} := \mathrm{sym}\nabla\mathbf{u} - \mathbf{E}^{\mathrm{p}} \tag{1.1.1a}$$

determines the stress $\mathbf{T}(x,t) \in \mathbb{R}^{3\times3}_{sym}$ through the constitutive equation

$$\mathbf{T} = \frac{\partial \psi_e}{\partial \mathbf{E}^e}, \quad \psi_e(\mathbf{E}^e) := \mu|\mathbf{E}^e_0|^2 + \tfrac{1}{2}\kappa|\mathrm{tr}\mathbf{E}^e|^2, \tag{1.1.1b}$$

where $\psi_e(\mathbf{E}^e)$ is the *elastic energy density* (see (1.2.18)).

and $\mu, \kappa > 0$. When body forces are null, the stress obeys the *force balance*:

$$\mathrm{div}\mathbf{T} = 0. \tag{1.1.1c}$$

Flow rules adopted in small–strain metal plasticity have typically the form

$$\mathbf{T}_0 = Y(e^p)g(|\dot{\mathbf{E}}^p|)\frac{\dot{\mathbf{E}}^p}{|\dot{\mathbf{E}}^p|}, \tag{1.1.2}$$

where $Y(\cdot) > 0$ is the *flow resistance*, $e^p(x,t) = \int_0^t |\dot{\mathbf{E}}^p(x,\mathrm{t})|\mathrm{dt}$ is the *accumulated plastic strain*, and $g(\cdot)$ is a (dimensionless) *rate–sensitivity function*. The simplest choice for the rate–sensitivity function is the power law $g(|\dot{\mathbf{E}}^p|) = \left(\frac{|\dot{\mathbf{E}}^p|}{d_0}\right)^m$, where $d_0 > 0$ is a *reference rate* and the parameter $m \geq 0$ is a measure of rate dependency: for $m = 0$, we have $g(\cdot) = 1$ and the flow rule (1.1.2) is not affected by a monotone time re-parametrization.

1.1.2 The Gurtin–Anand model

Ultimately, the the inability of (1.1.1)–(1.1.2) at capturing size effects is due to its invariance under the scaling $x \mapsto \alpha x$, $\mathbf{u} \mapsto \alpha\mathbf{u}$ ($\alpha > 0$). In the Gurtin–Anand theory [61, 62], size dependence is achieved by replacing (1.1.2) with a flow rule that explicitly accounts for the plastic–strain gradient in two ways:

1) an *energetic scale dependence*, and a corresponding *energetic lengthscale L*, are introduced by adding to the elastic energy density, a *defect energy density*

$$\psi_d(\nabla\mathbf{E}^p) = \tfrac{1}{2}\mu L^2\left((1-\eta)|\mathrm{curl}\mathbf{E}^p|^2 + \eta|\mathrm{curl}\mathbf{E}^p - (\mathrm{curl}\mathbf{E}^p)^T|^2\right) \tag{1.1.3}$$

(*cf.* [62, Eqs. (90.41)–(90.42)] with $\lambda_2 = \eta$), where $-1 < \eta < 1$ is a dimensionless parameter. In the framework of the Gurtin–Anand theory, $\mathrm{curl}\mathbf{E}^p$ coincides with the Burgers tensor [62, §88.1-2], which provides a macroscopic description of geometrically–necessary dislocations.

2) a *dissipative scale dependence*, and a corresponding *dissipative lengthscale ℓ*, are introduced by a dependence of the dissipation-rate density on spatial derivatives of the *plastic strain-rate*, $\dot{\mathbf{E}}^p$. The *dissipation-rate density* is given by:

$$\delta = Y(E^p)g(d^p)d^p, \quad \text{where} \quad d^p := \sqrt{|\dot{\mathbf{E}}^p|^2 + \ell^2|\nabla\dot{\mathbf{E}}^p|^2} \quad \text{and} \quad E^p(x,t) := \int_0^t d^p(x,s)\,\mathrm{d}s.$$

More specifically, Gurtin and Anand replace (1.1.2) with the *microforce balance*

$$\mathbf{T}_0 = \mathbf{T}^p - \mathrm{div}\mathbb{K}^p, \tag{1.1.4a}$$

and with the following constitutive equations for the *plastic stress* $\mathbf{T}^{\mathrm{p}}(x,t) \in \mathbb{R}^{3\times3}_{0,sym}$ and the *plastic microstress* $\mathbb{K}^{\mathrm{p}}(x,t) \in \mathbb{R}^{3\times3\times3}_{0,sym}$:

$$\mathbf{T}^{\mathrm{p}} = Y(E^{\mathrm{p}})g(d^{\mathrm{p}})\frac{\dot{\mathbf{E}}^{\mathrm{p}}}{d^{\mathrm{p}}}, \quad \mathbb{K}^{\mathrm{p}} = \mathbb{K}^{\mathrm{p}}_{\mathrm{en}} + \mathbb{K}^{\mathrm{p}}_{\mathrm{diss}}, \quad \begin{cases} \mathbb{K}^{\mathrm{p}}_{\mathrm{en}} = \dfrac{\partial \psi_{\mathrm{d}}}{\partial \nabla \mathbf{E}^{\mathrm{p}}} \\ \mathbb{K}^{\mathrm{p}}_{\mathrm{diss}} = \ell^2 Y(E^{\mathrm{p}})g(d^{\mathrm{p}})\dfrac{\nabla \dot{\mathbf{E}}^{\mathrm{p}}}{d^{\mathrm{p}}}. \end{cases} \tag{1.1.4b}$$

1.1.3 The goals

Quite a few efforts have been put into the mathematical analysis of this theory: besides [79], which deals with the (much more tractable) case in which hardening is present, in [54] the concept of "energetic solution" [72] is implemented for this model in the rate–independent case (which follows by formally substituting $g(\cdot) = 1$ in (1.1.4b)). However, we are not aware of analytical studies aiming to qualify and quantify the scale effects induced by ℓ and L. To our knowledge, only dimensional and numerical observation are available so far [60, 6], suggesting:

(a) the development of boundary layers near $\partial\Omega$, at least in case of no flux of the Burgers vector through $\partial\Omega$ [60, §10.2];

(b) an increase of the strain-hardening rate with L [6, §12];

(c) an increase of the strengthening with ℓ [6, §12].

The goal of this chapter is to obtain a more robust validation of the role of the energetic lengthscale L with respect to the observation in (a) and (b). To this aim: 1) we assume constant flow resistance and we rule out dissipative size effects by setting:

$$Y(\cdot) = \sqrt{2}k \tag{1.1.5a}$$

$$\ell = 0, \quad \text{i.e. } d^{\mathrm{p}} = |\dot{\mathbf{E}}^{\mathrm{p}}|, \tag{1.1.5b}$$

where $k > 0$ is the *yield strength under pure shear*; 2) we take the rate–independent limit $g(s) = 1$, so that $\delta(\dot{\mathbf{E}}^{\mathrm{p}}) = \sqrt{2}k|\dot{\mathbf{E}}^{\mathrm{p}}|$, and we replace the first of (1.1.4b) with

$$\begin{cases} \mathbf{T}^{\mathrm{p}} \in \mathbb{R}^{3\times3}_{0,sym} \text{ and } |\mathbf{T}^{\mathrm{p}}| \leq \sqrt{2}k & \text{if } \dot{\mathbf{E}}^{\mathrm{p}} = 0 \\ \mathbf{T}^{\mathrm{p}} = \sqrt{2}k\frac{\dot{\mathbf{E}}^{\mathrm{p}}}{|\dot{\mathbf{E}}^{\mathrm{p}}|} & \text{if } \dot{\mathbf{E}}^{\mathrm{p}} \neq 0. \end{cases} \tag{1.1.5c}$$

It is not hard to check that (1.1.5c) is equivalent to the following differential inclusion [64]:

$$\mathbf{T}^{\mathrm{p}} \in \partial\delta(\dot{\mathbf{E}}^{\mathrm{p}}) = \{\mathbf{A} \in \mathbb{R}^{3\times3}_{0,sym} : \ \delta(\widetilde{\dot{\mathbf{E}}}^{\mathrm{p}}) - \delta(\dot{\mathbf{E}}^{\mathrm{p}}) \geq \mathbf{A} : (\widetilde{\dot{\mathbf{E}}}^{\mathrm{p}} - \dot{\mathbf{E}}^{\mathrm{p}}) \ \ \forall\, \widetilde{\dot{\mathbf{E}}}^{\mathrm{p}} \in \mathbb{R}^{3\times3}_{0,sym}\}. \tag{1.1.6}$$

Using (1.1.5) and (1.2.19), after an explicit computation of $\mathbb{K}^{\rm p}$ (see §1.2) the flow rule (1.1.4) becomes

$$\mathbf{T}_0 + \mu L^2 \left(\Delta\mathbf{E}^{\rm p} - \mathrm{sym}(\nabla\mathrm{div}\mathbf{E}^{\rm p}) + \tfrac{1}{3}(1+\eta)(\mathrm{div}\,\mathrm{div}\mathbf{E}^{\rm p})\mathbf{I} + \eta\mathrm{curlcurl}\mathbf{E}^{\rm p}\right) \in \partial\delta(\dot{\mathbf{E}}^{\rm p}). \quad (1.1.7)$$

Note that when $L = 0$ the flow rule (1.1.7) reduces to (1.1.6) with $\mathbf{T}^{\rm p}$ replaced by $\mathbf{T}_0$; the resulting law characterizes, according to the terminology of [62], the *Levy–Mises plastic response*.

Though the effects in (a) and (b) seem to be observable, at least qualitatively, already at the level of the one-dimensional theory proposed in [6], we wish to explore them in a multidimensional setting where the role of the Burgers tensor (as the curl of a tensor field) should become more transparent on one hand and experimental results are available on the other hand. One such setting is, of course, that of the *torsion problem*, which has already been studied in the context of other strain–gradient plasticity theories [47, 58, 23], and which we introduce now.

1.1.4 The torsion problem

We model a thin metallic wire as an *infinite right-cylinder*

$$\Omega_R = \{x = (\varrho\cos\phi, \varrho\sin\phi, z) \in \mathbb{R}^3 : \varrho \in [0,R), \phi \in [0,2\pi)\} \quad (1.1.8)$$

subject to null tractions at the boundary $\partial\Omega_R$ (see $(1.2.17)_2$ below). Denoting by $(\mathbf{e}_{\langle 1\rangle}, \mathbf{e}_{\langle 2\rangle}, \mathbf{e}_{\langle 3\rangle})$ the local orthonormal frame (see (1.2.3)) associated to the cylindrical coordinates (ϱ, ϕ, z), we write down the following *Ansatz*:

$$\mathbf{u}(\varrho,\phi,z,t) = z\Theta(t)\varrho\mathbf{e}_{\langle 2\rangle}(\phi) \quad (1.1.9a)$$

$$\mathbf{E}^{\rm p}(\varrho,\phi,t) = \gamma^{\rm p}(\varrho,t)\mathrm{sym}(\mathbf{e}_{\langle 2\rangle}(\phi)\otimes\mathbf{e}_{\langle 3\rangle}), \quad (1.1.9b)$$

where both the *twist* Θ and the *plastic–shear profile* $\gamma^{\rm p}$ satisfy the null initial conditions:

$$\Theta(0) = 0, \quad \gamma^{\rm p}(\cdot,0) = 0. \quad (1.1.10)$$

The stress field that results from (1.1.9) and (1.1.1a–b) satisfies the balance equation (1.1.1c) and the null–traction condition for the standard forces. Moreover, as we shall see in §1.2.3, the torque that must be applied to induce a twist Θ with plastic–shear profile $\gamma^{\rm p}$ is given by the following expression:

$$Q = 2\pi\mu\int_0^R (\Theta\varrho - \gamma^{\rm p})\varrho^2 \mathrm{d}\varrho. \quad (1.1.11)$$

An important point to be made at first is that the system (1.1.7) is unaffected by a monotone time re-parametrization. This property is best exploited when the twist is *monotone*:

$$\dot{\Theta} > 0,$$

an additional working assumption that we make in this thesis. Indeed, this assumption enables us to replace the dependence of γ^{p} on time with a dependence on the twist by performing the substitution: $\gamma^{\mathrm{p}}(\varrho, t) \mapsto \gamma^{\mathrm{p}}(\varrho, \Theta)$. A second point is that, since the system has no intrinsic timescale, the only parameter that matters is the ratio between the energetic lengthscale L and the diameter $2R$. To highlight the role of this parameter, we introduce a normalized energetic length scale λ, proportional to L/R (see (1.4.1) below), and we work with the following normalized variables:

$$r := \frac{\varrho}{R}, \quad \theta := \frac{\Theta}{\Theta_{\mathrm{y}}}, \quad \gamma := \frac{\gamma^{\mathrm{p}}}{\gamma_{\mathrm{y}}}, \quad \text{where} \quad \gamma_{\mathrm{y}} := \frac{k}{\mu}, \quad \Theta_{\mathrm{y}} := \frac{\gamma_y}{R}. \tag{1.1.12}$$

The constants γ_{y} and Θ_{y} are the *yield shear* and the *yield twist*, respectively.

We show in §1.3 that, under the Ansatz (1.1.9), the flow rule (1.1.7) reduces to a partial differential inclusion in one dimension that, in terms of the normalized variables (1.1.12), reads:

$$\lambda^2 \left(\frac{\partial^2 \gamma}{\partial r^2} + \frac{1}{r}\frac{\partial \gamma}{\partial r} - \frac{1}{r^2}\gamma \right) - \gamma + \theta r \in \partial \left| \frac{\partial \gamma}{\partial \theta} \right| \quad \text{in } (0,1) \times (0, \infty), \tag{1.1.13a}$$

where

$$\partial\,|s| := \begin{cases} \{-1\} & \text{if } s < 0 \\ [-1, 1] & \text{if } s = 0 \\ \{1\} & \text{if } s > 0. \end{cases}$$

The assumption (1.1.10) and the null microscopic traction at the boundary (see (1.2.17)) yield

$$\gamma(r, 0) = 0 \quad \text{and} \quad \frac{\partial \gamma}{\partial r}(1, \theta) + \frac{\gamma(1, \theta)}{2} = 0 \quad \text{for } \theta \geq 0 \text{ and } r \in (0, 1). \tag{1.1.13b}$$

1.1.5 Solution of the torsion problem: $L = 0$

To get a first insight in the problem, it is convenient to consider the case $L = 0$. Then (1.1.13a) reduces to

$$\theta r - \gamma \in \partial \left| \frac{\partial \gamma}{\partial \theta} \right| \quad \text{in} \quad (0, R) \tag{1.1.14}$$

and there is no associated boundary condition. The unique solution of (1.1.14) with the initial condition $\gamma(r, 0) = 0$ is given by

$$\gamma(r, \theta) = (\theta r - 1)_+, \tag{1.1.15}$$

where $(s)_+ = \max\{s, 0\}$. From (1.1.15), two regimes may be identified:

1) an *elastic regime*, where $\theta \in [0, 1]$ and $\gamma = 0$;

2) an *elasto–plastic regime*, where $\theta \in (1, +\infty)$, $\gamma = 0$ in $[0, 1/\theta]$, and $\gamma > 0$ in $(1/\theta, 1]$.

Thus, an *elastic–plastic boundary* located at $r = 1/\theta$ separates the region where $\gamma = 0$, the so–called *elastic core*, from the rest of the body, where $\gamma > 0$. As θ increases, the elastic core shrinks down, but never disappears.

In terms of normalized variables, (1.1.11) is best written as $Q = Q_* q$, where $Q_* := \frac{2}{3}k\pi R^3$ and

$$q := 3\int_0^1 (\theta r - \gamma) r^2 \mathrm{d}r \qquad (1.1.16)$$

is the *normalized torque*. On substituting (1.1.15) into (1.1.16), we obtain, for $\theta \geq 1$,

$$q = 1 - \tfrac{1}{4}\theta^{-3} \quad \text{for all } \theta \geq 1. \qquad (1.1.17)$$

Notice that $q(\theta) \to 1$ as $\theta \to \infty$. Thus, Q_* is the *ultimate torque* that a wire can withstand according to the Levy–Mises theory.

1.1.6 Solution of the torsion problem: $L > 0$

When strain-gradient effects are accounted for, an expression for the torque as simple as (1.1.17) is not available. In order to get some insight, we need a detailed characterization of the solution of (1.1.13). In §1.4 we note that (1.1.13a) has a natural formulation in terms of an evolutionary variational inequality (see Definition 1.1), and hence it has a unique solution (see Proposition 1.1). Our main contribution is in §1.5, where we show that the unique solution γ of (1.1.13) may be characterized in terms of solutions of suitable boundary-value problems (see Theorem 1.1). As a by-product, our arguments provide an explicit construction of the solution; this construction allows us to identify three regimes:

1) an initial *elastic regime*, where $\theta \in [0, 1]$ and $\gamma = 0$;

2) an intermediate *elasto-plastic regime*, where $\theta \in [1, \theta_\lambda)$, $\gamma = 0$ in $[0, c_\theta]$, and $\gamma > 0$ in $(c_\theta, 1]$;

3) an ultimate *plastic regime*, where $\theta > \theta_\lambda$ and $\gamma > 0$ in $(0, 1]$.

A relevant feature is apparent from 3): the sample becomes fully plastified when θ attains a *critical twist* θ_λ, in contrast with the case $L = 0$, where plastic strain vanishes on $(0, 1/\theta)$.

1.1.7 Energetic scale effects

The characterization given by Theorem 1.1 allows for an easy computation of the plastic profile and the torque. Numerical results given in Figure 1.1 confirm both the presence of a boundary layer near $\partial\Omega$ and the higher relative strength of thinner wires. In addition, they

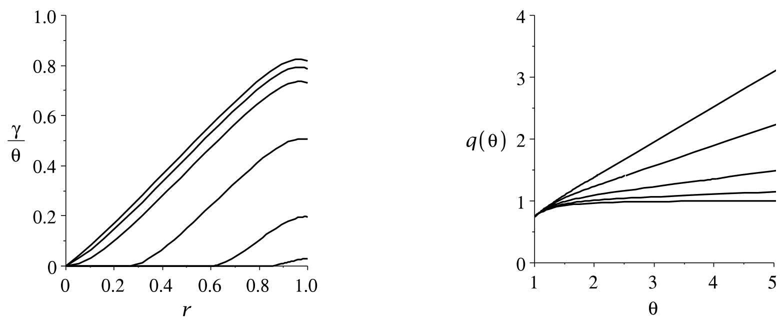

Figure 1.1: On the left, plots of $\frac{\gamma}{\theta}$ for $\lambda = 0.1$ and $\frac{\theta-1}{\theta_\lambda - 1} = \frac{1}{64}, \frac{1}{16}, \frac{1}{4}, 1, 2, 4$ (from bottom to top); on the right, plots of normalized torque vs. normalized twist for $\lambda = 0, 0.1, 0.2, 0.4, 0.8$ (from bottom to top).

show the presence of a boundary layer near c_θ, the left-endpoint of the plastic region. The characterization given by Theorem 1.1 also allows to quantify these effects in terms of the (normalized) energetic lengthscale λ. In §3.9, we develop a formal asymptotic expansion as $\lambda \to 0$ (for fixed θ). First we show that

$$c_\theta \sim \frac{1}{\theta} - \lambda \quad \text{for } \lambda \ll 1 \text{ and } \theta \in (1, \theta_\lambda), \tag{1.1.18}$$

$$\gamma \sim \theta r - 1 - \frac{1}{2}\lambda(3\theta - 1)e^{-\frac{1-r}{\lambda}} \quad \text{for } \lambda \ll 1,\ 1 - r \ll 1 \text{ and } \theta > 1. \tag{1.1.19}$$

Expansions (1.1.18) and (1.1.19) show the appearance of boundary layers of width $O(\lambda)$ near $r = c_\theta$ and $r = 1$, respectively. Using (1.1.18) and (1.1.19), we obtain a scaling law for the critical twist,

$$\theta_\lambda \sim \frac{1}{\sqrt{6}\lambda} \quad \text{for } \lambda \ll 1, \tag{1.1.20}$$

and we quantify the higher relative strength of thinner wires by finding the estimate

$$q(\theta) \sim \begin{cases} 1 - \frac{1}{4\theta^3} + \frac{3\lambda^2}{2\theta} + \frac{9}{2}\lambda^2(\theta - 1) & \text{if } 1 < \theta < \frac{1}{\sqrt{6}\lambda} \\ 1 + \frac{9}{2}\lambda^2(\theta - 1) & \text{if } \theta > \frac{1}{\sqrt{6}\lambda} \end{cases} \quad \text{for } \lambda \ll 1. \tag{1.1.21}$$

Comparing (1.1.21) with (1.1.17) and returning to the original variables, we see in particular that Q/Q_* is proportional to $(L/R)^2$. We remark that the theory under scrutiny does not predict any ultimate torque: we conjecture that a defect energy density with linear growth, as deduced in [49], may recover such feature.

1.1.8 Non-symmetric plastic distortion

The identification of curl$\mathbf{E}^\mathrm{p}$ as the macroscopic counterpart of the Burgers vector hinges on the assumption that, in the decomposition $\nabla\mathbf{u} = \mathbf{H}^\mathrm{e} + \mathbf{H}^\mathrm{p}$, the *plastic distortion* $\mathbf{H}^\mathrm{p}$ be symmetric. If this assumption is dropped then $\mathbf{H}^{\mathrm{p}} = \mathbf{E}^{\mathrm{p}} + \mathbf{W}^{\mathrm{p}}$, with $\mathbf{E}^{\mathrm{p}}$ symmetric and $\mathbf{W}^{\mathrm{p}}$ skew-symmetric. Thus, the additional kinematical unknown $\mathbf{W}^{\mathrm{p}}(x,t) \in \mathbb{R}^{3\times3}_{\mathrm{skw}}$, the so-called *plastic spin*, enters the theory [60]. As pointed out in [61], the ensuing flow rule is then much more complicated; not surprisingly, well-posedness has not been established for such model, unless one includes appropriate hardening terms [42], or restricts attention to particular symmetries [19]. In §1.7 we show that the trivial generalization of (1.1.9) with $\mathbf{W}^{\mathrm{p}} = 0$ provides a solution also to the flow rule proposed in [60], where $\psi_{\mathrm{d}} = \frac{1}{2}\mu L^2|\mathrm{curl}\mathbf{H}^{\mathrm{p}}|^2$ is postulated. This seems to indicate that, contrary to what intuition may suggest, plastic rotations do not affect the outcome of a torsion experiment.

1.2 Problem setup

1.2.1 Preliminaries

We adopt the following terminology and typographical convention: we use boldface small prints ($\mathbf{a}$, $\mathbf{b}$, *etc.*) to denote elements of $\mathbb{R}^3$, and we refer to them as "vectors"; we use boldface capitals ($\mathbf{A}$, $\mathbf{B}$, *etc.*) to denote elements of $\mathbb{R}^{3\times3}$, and we call them "tensors"; we use double struck capitals ($\mathbb{A}$, $\mathbb{B}$, *etc.*) to denote elements of $\mathbb{R}^{3\times3\times3}$, and we call them "second–order tensors". We denote the components of the vector $\mathbf{a}$, the tensor $\mathbf{A}$, and the second–order tensor $\mathbb{A}$ in the corresponding standard basis by $(\mathbf{a})_i$, $(\mathbf{A})_{ij}$, and $(\mathbb{A})_{ijk}$, respectively.

We use a single, a double, and a triple dot, to denote the scalar product between vectors, tensors, and second–order tensors, respectively, that is: $\mathbf{a}\cdot\mathbf{b} = (\mathbf{a})_i(\mathbf{b})_i$, $\mathbf{A}:\mathbf{B} = (\mathbf{A})_{ij}(\mathbf{B})_{ij}$, $\mathbb{A}\,\vdots\,\mathbb{B} = (\mathbb{A})_{ijk}(\mathbb{B})_{ijk}$. We maintain that $(\mathbf{A}\mathbf{a})_j = (\mathbf{A})_{ij}(\mathbf{a})_j$, and $(\mathbb{A}\mathbf{a})_{ij} = (\mathbb{A})_{ijk}(\mathbf{a})_k$. We denote by $\mathbb{R}^{3\times3}_{sym}$ and $\mathbb{R}^{3\times3}_0$ the sets of *symmetric*, resp. *traceless*, second–order tensors, and we let $\mathbb{R}^{3\times3}_{0,sym} = \mathbb{R}^{3\times3}_0 \cap \mathbb{R}^{3\times3}_{sym}$. Likewise we denote by $\mathbb{R}^{3\times3\times3}_{sym}$ and $\mathbb{R}^{3\times3\times3}_0$ the set of third–order tensors that are symmetric, resp. deviatoric, with respect to the first two indices, and we let $\mathbb{R}^{3\times3\times3}_{0,sym} = \mathbb{R}^{3\times3\times3}_0 \cap \mathbb{R}^{3\times3\times3}_{sym}$.

We denote by $\mathbf{a}\otimes\mathbf{b}$ the tensor defined componentwise by $(\mathbf{a}\otimes\mathbf{b})_{ij} = (\mathbf{a})_i(\mathbf{b})_j$. In a similar manner, we denote by $\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}$ the third–order tensor with components $(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c})_{ijk} = (\mathbf{a})_i(\mathbf{b})_j(\mathbf{c})_k$. In particular, we have

$$(\mathbf{a}\otimes\mathbf{b})\mathbf{c} = (\mathbf{b}\cdot\mathbf{c})\mathbf{a}. \tag{1.2.1}$$

We denote by $\mathrm{sym}\mathbf{A}$ and $\mathbf{A}_0$, respectively, the *symmetric part* and *deviatoric part* of any tensor $\mathbf{A}$, namely, $\mathrm{sym}\mathbf{A} = \frac{1}{2}\left(\mathbf{A}^T + \mathbf{A}\right)$, and $\mathbf{A}_0 = \mathbf{A} - \frac{1}{3}\mathrm{tr}(\mathbf{A})\mathbf{I}$, where $\mathbf{I}$ is the identity matrix; given that $(\mathbf{a}\otimes\mathbf{b})^T = \mathbf{b}\otimes\mathbf{a}$, and that $\mathrm{tr}(\mathbf{a}\otimes\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}$, we have

$$\mathrm{sym}(\mathbf{a}\otimes\mathbf{b}) = \frac{1}{2}\left(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a}\right) \quad \text{and} \quad (\mathbf{a}\otimes\mathbf{b})_0 = \mathbf{a}\otimes\mathbf{b} - \frac{1}{3}(\mathbf{a}\cdot\mathbf{b})\mathbf{I}. \tag{1.2.2}$$

For $x \in \mathbb{R}^3$, we denote by (ϱ, ϕ, z) its cylindrical coordinates,

$$x = (\varrho\cos\phi, \varrho\sin\phi, z), \quad (\varrho, \phi, z) \in (0, +\infty) \times [0, 2\pi) \times \mathbb{R}.$$

and we introduce the local *frame-field vectors*

$$\mathbf{e}_{\langle 1\rangle}(\phi) = (\cos\phi, \sin\phi, 0), \quad \mathbf{e}_{\langle 2\rangle}(\phi) = (-\sin\phi, \cos\phi, 0), \quad \mathbf{e}_{\langle 3\rangle} = (0, 0, 1). \tag{1.2.3}$$

It follows from (1.2.3) that

$$\mathbf{e}_{\langle i\rangle} \cdot \mathbf{e}_{\langle j\rangle} = \delta_{ij} \quad \text{and} \quad \frac{\partial \mathbf{e}_{\langle i\rangle}}{\partial \phi} \cdot \mathbf{e}_{\langle j\rangle} = \varepsilon_{ij3}, \tag{1.2.4}$$

where δ_{ij} and ε_{ijk} are, respectively, the Kroenecker and Levi–Civita symbols. We denote components in the frame–field (1.2.3) as follows:

$$(\mathbf{a})_{\langle i\rangle} = \mathbf{a} \cdot \mathbf{e}_{\langle i\rangle}, \quad (\mathbf{A})_{\langle ij\rangle} = \mathbf{A} : \mathbf{e}_{\langle i\rangle} \otimes \mathbf{e}_{\langle j\rangle}, \quad (\mathbb{A})_{\langle ijk\rangle} = \mathbf{A} \,\vdots\, \mathbf{e}_{\langle i\rangle} \otimes \mathbf{e}_{\langle j\rangle} \otimes \mathbf{e}_{\langle k\rangle}. \tag{1.2.5}$$

Although these components differ, in general, from those in the standard basis, the usual representation formulas in terms of components apply:

$$\mathbf{a} = (\mathbf{a})_{\langle i\rangle}\mathbf{e}_{\langle i\rangle}, \quad \mathbf{A} = (\mathbf{A})_{\langle ij\rangle}\mathbf{e}_{\langle i\rangle} \otimes \mathbf{e}_{\langle j\rangle}, \quad \mathbb{A} = (\mathbb{A})_{\langle ijk\rangle}\mathbf{e}_{\langle i\rangle} \otimes \mathbf{e}_{\langle j\rangle} \otimes \mathbf{e}_{\langle k\rangle}, \tag{1.2.6}$$

along with the usual component–wise multiplication rules:

$$(\mathbf{A}\mathbf{a})_{\langle i\rangle} = (\mathbf{A})_{\langle ij\rangle}(\mathbf{a})_{\langle j\rangle} \quad \text{and} \quad (\mathbb{A}\mathbf{a})_{\langle ij\rangle} = (\mathbb{A})_{\langle ijk\rangle}(\mathbf{a})_{\langle k\rangle}. \tag{1.2.7}$$

Given scalar functions f and g depending on $\mathbf{A}$, resp. $\mathbb{A}$, we use the notation $\frac{\partial f}{\partial \mathbf{A}}$ and $\frac{\partial g}{\partial \mathbb{A}}$ to denote the second–order, resp. third–order, tensors defined by

$$\left(\frac{\partial f}{\partial \mathbf{A}}\right)_{\langle ij\rangle} = \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{\langle ij\rangle}}, \quad \left(\frac{\partial g}{\partial \mathbb{A}}\right)_{\langle ijk\rangle} = \frac{\partial g(\mathbb{A})}{\partial \mathbb{A}_{\langle ijk\rangle}}. \tag{1.2.8}$$

Given a tensor field $\mathbf{A}$, we define its curl using local components:

$$(\mathrm{curl}\mathbf{A})_{\langle ij\rangle} = \varepsilon_{ikl}(\nabla\mathbf{A})_{\langle jlk\rangle}. \tag{1.2.9}$$

If $\mathbf{A}$ is symmetric, then the following identity holds [59, Eq. (13)]:

$$\mathrm{curlcurl}\mathbf{A} = -\Delta\mathbf{A} + 2\mathrm{sym}\nabla\mathrm{div}\mathbf{A} - \nabla\nabla\mathrm{tr}\mathbf{A} + (\Delta\mathrm{tr}\mathbf{A} - \mathrm{divdiv}\mathbf{A})\,\mathbf{I}. \tag{1.2.10}$$

We next summarize some useful rules of tensor calculus, to be used later on. Given a vector field $\mathbf{a}(\varrho, \theta, z)$, its gradient $\nabla\mathbf{a}$ can be represented as:

$$\nabla\mathbf{a} = \frac{\partial \mathbf{a}}{\partial \varrho} \otimes \mathbf{e}_{\langle 1\rangle} + \frac{1}{\varrho}\frac{\partial \mathbf{a}}{\partial \phi} \otimes \mathbf{e}_{\langle 2\rangle} + \frac{\partial \mathbf{a}}{\partial z} \otimes \mathbf{e}_{\langle 3\rangle}. \tag{1.2.11}$$

A similar formula holds for a tensor field $\mathbf{A}(\varrho, \theta, z)$. Moreover,

$$\mathrm{div}\mathbf{A} = \frac{\partial \mathbf{A}}{\partial \varrho}\mathbf{e}_{\langle 1\rangle} + \frac{1}{\varrho}\frac{\partial \mathbf{A}}{\partial \phi}\mathbf{e}_{\langle 2\rangle} + \frac{\partial \mathbf{A}}{\partial z}\mathbf{e}_{\langle 3\rangle}. \tag{1.2.12}$$

The implication

$$\mathbf{A}(\varrho, \phi) = \alpha(\varrho)\mathrm{sym}(\mathbf{e}_{\langle 2\rangle}(\phi) \otimes \mathbf{e}_{\langle 3\rangle}) \;\Rightarrow\; \mathrm{div}\mathbf{A} = \mathbf{0} \tag{1.2.13}$$

is easily verified using (1.2.1), $(1.2.2)_1$, (1.2.4), and (1.2.12).

1.2.2 Balance equations and traction conditions

Let Π denote an arbitrary subregion of body under scrutiny. Gurtin-Anand's discussion of the associated mechanics, which is nonstandard, is based on the belief that the power expended by each independent kinematical field be expressible in terms of an associated force system consistent with its own balance. Bearing in mind that the goal in the strain-gradient plasticity is to account for gradient of plastic strain-rate $\nabla\dot{\mathbf{E}}^{\mathrm{p}}$, we use the principle of virtual power to deduce the underlying balance laws. Consistent with the choice of descriptors $\dot{\mathbf{E}}^{\mathrm{e}}$ and $\dot{\mathbf{E}}^{\mathrm{p}}$, we therefore assume that the power is expended internally by

- an *elastic stress* $\mathbf{T}$ power-conjugate to $\dot{\mathbf{E}}^{\mathrm{e}}$,
- an plastic *microstress* $\mathbf{T}^{\mathrm{p}}$ power-conjugate to $\dot{\mathbf{E}}^{\mathrm{p}}$,
- a (third-order) *polar plastic microstress* $\mathbb{K}^{\mathrm{p}}$ power-conjugate to $\nabla\dot{\mathbf{E}}^{\mathrm{p}}$.

So the assumption, central to the Gurtin–Anand theory [61], is that the internal power expended within Π has the form

$$\mathcal{W}_{\mathrm{int}}(\Pi) = \int_{\Pi} \left\{ \mathbf{T} : \dot{\mathbf{E}}^{\mathrm{e}} + \mathbf{T}^{\mathrm{p}} : \dot{\mathbf{E}}^{\mathrm{p}} + \mathbb{K}^{\mathrm{p}} \vdots \nabla\dot{\mathbf{E}}^{\mathrm{p}} \right\} \mathrm{d}V.$$

Since $\dot{\mathbf{E}}^{\mathrm{p}}$ is symmetric and deviatoric, we may assume without loss of generality that $\mathbb{K}^{\mathrm{p}}$ is symmetric and deviatoric in its two first subscripts. The internal power is balanced by power expended externally by tractions that the exterior of the typical part Π exerts at the boundary $\partial\Pi$ and body forces acting within Π. As is standard, we consider, as power conjugates for the macroscopic velocity $\dot{\mathbf{u}}$, a *macroscopic surface traction* $\mathbf{t}_{\Pi}$ and an external *macroscopic body force* $\mathbf{b}$, presumed to account for inertia and each of whose working accompanies the macroscopic motion of the body. The internal power (1.2.2) contains terms $\nabla\dot{\mathbf{E}}^{\mathrm{p}}$, and – based on experience with other gradient theories – we assume that power is expended externally by a *microtraction* $\mathbf{K}_{\Pi}$, conjugate to the plastic strain $\dot{\mathbf{E}}^{\mathrm{p}}$, and whose working accompanies the flow of dislocations across the surfaces. Consistent with such assumption is the following form of the external power:

$$\mathcal{W}_{\mathrm{ext}}(\Pi) = \int_{\Pi} \mathbf{b} \cdot \dot{\mathbf{u}} \,\mathrm{d}V + \int_{\partial\Pi} \left\{ \mathbf{t}_{\Pi} \cdot \dot{\mathbf{u}} + \mathbf{K}_{\Pi} : \dot{\mathbf{E}}^{\mathrm{p}} \right\} \mathrm{d}S.$$

Again since $\dot{\mathbf{E}}^{\mathrm{p}}$ is symmetric-deviatoric, we assume that $\mathbf{K}_{\Pi}$ is symmetric-deviatoric. The principle of virtual powers applied to arbitrary body parts yields the standard–force and the micro–force balances:

$$\mathrm{div}\mathbf{T} + \mathbf{b} = 0, \quad \mathrm{div}\mathbb{K}^{\mathrm{p}} - \mathbf{T}^{\mathrm{p}} + \mathbf{T}_0 = 0, \tag{1.2.14}$$

along with relations between stresses and tractions:

$$\mathbf{t}_{\Pi} = \mathbf{T}\mathbf{n}_{\Pi}, \quad \mathbf{K}_{\Pi} = \mathbb{K}^{\mathrm{p}}\mathbf{n}_{\Pi} \quad \text{on } \partial\Pi \tag{1.2.15}$$

where $\mathbf{n}_\Pi$ denotes the outward unit normal to $\partial\Pi$. To arrive at an evolution problem for displacement and plastic strain, we shall supplement the balance statements (1.2.14) with constitutive equations for the stress descriptors $\mathbf{T}$, $\mathbf{T}^{\mathrm{p}}$, $\mathbb{K}^{\mathrm{p}}$, and with specifications for $\mathbf{b}$, $\mathbf{t}_\Omega$, and $\mathbf{K}_\Omega$.

1.2.3 The 3D problem

We now specialize the theory to the cylinder Ω_R described by (1.1.8). The *torque* sustained by the cylinder is, by definition,

$$Q := \mathbf{e}_{\langle 3\rangle} \cdot \int_\Sigma \varrho \mathbf{e}_{\langle 1\rangle} \times \mathbf{T}\, \mathbf{e}_{\langle 3\rangle} \mathrm{d}S, \tag{1.2.16}$$

where Σ is any cross section of Ω_R (for instance $\Sigma = \Omega_R \cap \{z = 0\}$ will do). We neglect inertia and other body forces, and we require the lateral side of the cylinder to be traction–free:

$$\begin{cases} \mathbf{T}\mathbf{n} = 0 \\ \mathbb{K}^{\mathrm{p}}\mathbf{n} = 0 \end{cases} \qquad \text{on } \partial\Omega_R, \tag{1.2.17}$$

$\mathbf{n}$ is the outward unit normal to $\partial\Omega_R$. We suppose that the cylinder is made of an isotropic material.

The most general quadratic expressions compatible with such symmetry are: for the elastic energy,

$$\psi_{\mathrm{e}}(\mathbf{E}^{\mathrm{e}}) = \mu|\mathbf{E}_0^{\mathrm{e}}|^2 + \frac{1}{2}\kappa|\mathrm{tr}\mathbf{E}^{\mathrm{e}}|^2, \tag{1.2.18}$$

where $\mathbf{E}_0^{\mathrm{e}}$ is the deviatoric part of $\mathbf{E}^{\mathrm{e}}$ and $\mu, \kappa > 0$; for the defect energy,

$$\psi_{\mathrm{d}}(\nabla\mathbf{E}^{\mathrm{p}}) = \frac{1}{2}\mu L^2 \Big((1-\eta)|\mathrm{curl}\mathbf{E}^{\mathrm{p}}|^2 + \eta|\mathrm{curl}\mathbf{E}^{\mathrm{p}} - (\mathrm{curl}\mathbf{E}^{\mathrm{p}})^T|^2\Big), \tag{1.2.19}$$

(*cf.* [62, Eqs. (90.41)–(90.42)], with $\lambda_2 = \eta$) where $L > 0$ is the energetic lengthscale and $-1 < \eta < 1$ is a dimensionless parameter. It follows from (1.1.1b), and (1.2.18), that

$$\mathbf{T} = 2\mu\mathbf{E}_0^{\mathrm{e}} + \kappa\mathrm{tr}(\mathbf{E}^{\mathrm{e}})\mathbf{I}. \tag{1.2.20}$$

The constitutive equations for the stress descriptors have already been given in (1.1.4b). In view of (1.1.5), they reduce to

$$\mathbf{T}^{\mathrm{p}} \in \partial\delta(\dot{\mathbf{E}}^{\mathrm{p}}) \quad \text{and} \quad \mathbb{K}^{\mathrm{p}} = \frac{\partial\psi_d(\nabla\mathbf{E}^{\mathrm{p}})}{\partial\nabla\mathbf{E}^{\mathrm{p}}}. \tag{1.2.21}$$

When worked out in components with the aid of (1.2.8) and (1.2.9), the constitutive equation $(1.2.21)_2$ turns into

$$\begin{aligned}(\mathbb{K}^{\mathrm{p}})_{\langle jqp\rangle} = \mu L^2 \Bigg[&(\nabla\mathbf{E}^{\mathrm{p}})_{\langle jqp\rangle} - \frac{1}{2}\Big((\nabla\mathbf{E}^{\mathrm{p}})_{\langle jpq\rangle} + (\nabla\mathbf{E}^{\mathrm{p}})_{\langle qpj\rangle}\Big) \\ &+ \frac{1}{3}(1+\eta)\delta_{jq}(\nabla\mathbf{E}^{\mathrm{p}})_{\langle rpr\rangle} - \frac{\eta}{2}(\varepsilon_{ipq}\varepsilon_{jrs} + \varepsilon_{ipj}\varepsilon_{qrs})(\nabla\mathbf{E})_{\langle isr\rangle}\Bigg]\end{aligned} \tag{1.2.22}$$

(*cf.* [62, Eq. (90.47)]), so that

$$\operatorname{div}\mathbb{K}^{\mathrm{p}} = \mu L^2\left(\Delta\mathbf{E}^{\mathrm{p}} - \operatorname{sym}(\nabla\operatorname{div}\mathbf{E}^{\mathrm{p}}) + \frac{1}{3}(1+\eta)(\operatorname{div}\operatorname{div}\mathbf{E}^{\mathrm{p}})\mathbf{I} + \eta\operatorname{curl}\operatorname{curl}\mathbf{E}^{\mathrm{p}}\right). \qquad (1.2.23)$$

(see also [62, Eq. (90.64)], with $\lambda = \eta$, which however contains a typo).

Substituting (1.2.23) and $(1.2.21)_1$ into (1.2.14) we obtain (1.1.7). The system governing the evolution of displacement $\mathbf{u}(x,t)$ and *plastic strain* $\mathbf{E}^{\mathrm{p}}(x,t)$ in the cylinder Ω_R is

$$\begin{cases} \operatorname{div}\mathbf{T} = \mathbf{0} & \text{in } \Omega_R \times (0,+\infty) \\ \operatorname{div}\mathbb{K}^{\mathrm{p}} + \mathbf{T}_0 \in \partial\delta(\dot{\mathbf{E}}^{\mathrm{p}}) & \text{in } \Omega_R \times (0,+\infty) \\ \mathbf{Tn} = 0 & \text{on } \partial\Omega_R \times (0,+\infty) \\ \mathbb{K}^{\mathrm{p}}\mathbf{n} = 0 & \text{on } \partial\Omega_R \times (0,+\infty) \\ \mathbf{E}^{\mathrm{p}}(\cdot,0) = 0 & \text{in } \Omega_R, \end{cases} \qquad (1.2.24)$$

where stress $\mathbf{T}(x,t) \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$ and polar microstress $\mathbb{K}^{\mathrm{p}}(x,t) \in \mathbb{R}^{3\times3\times3}_0$ are related to displacement gradient and plastic strain through (1.2.20) and (1.2.22).

1.3 The torsion problem

We now argue that the ansatz (1.1.9)-(1.1.10) yields a special class of solutions of the bulk system (1.2.24). As explained in the Introduction, we replace t with Θ as independent variable: we henceforth maintain that a superimposed dot denotes partial differentiation with respect to Θ. In place of (1.1.9), we then write

$$\begin{cases} \mathbf{u}(\varrho,\phi,z,\Theta) = z\Theta\varrho\mathbf{e}_{\langle 2\rangle}(\phi) \\ \mathbf{E}^{\mathrm{p}}(\varrho,\phi,\Theta) = \gamma^{\mathrm{p}}(\varrho,\Theta)\operatorname{sym}(\mathbf{e}_{\langle 2\rangle}(\phi)\otimes\mathbf{e}_{\langle 3\rangle}). \end{cases} \qquad (1.3.1)$$

Our first task is to verify that the stress $\mathbf{T}$ resulting from (1.3.1) satisfies $(1.2.24)_1$ and $(1.2.24)_3$. To begin with, we use (1.2.4) and (1.2.11) to obtain $\nabla\mathbf{u} = z\Theta\mathbf{e}_{\langle 2\rangle}\otimes\mathbf{e}_{\langle 1\rangle} - z\Theta\mathbf{e}_{\langle 1\rangle}\otimes\mathbf{e}_{\langle 2\rangle} + \varrho\Theta\mathbf{e}_{\langle 2\rangle}\otimes\mathbf{e}_{\langle 3\rangle}$, whence

$$\operatorname{sym}\nabla\mathbf{u} = \varrho\Theta\operatorname{sym}(\mathbf{e}_{\langle 2\rangle}\otimes\mathbf{e}_{\langle 3\rangle}). \qquad (1.3.2)$$

By combining (1.1.1a) with $(1.3.1)_2$ and (1.3.2), we get

$$\mathbf{E}^{\mathrm{e}} = (\varrho\Theta - \gamma^{\mathrm{p}})\operatorname{sym}(\mathbf{e}_{\langle 2\rangle}\otimes\mathbf{e}_{\langle 3\rangle}). \qquad (1.3.3)$$

On substituting (1.3.3) in (1.2.20), since $\operatorname{tr}\mathbf{E}^{\mathrm{e}} = 0$ we find

$$\mathbf{T} = 2\mu(\Theta\varrho - \gamma^{\mathrm{p}})\operatorname{sym}(\mathbf{e}_{\langle 2\rangle}\otimes\mathbf{e}_{\langle 3\rangle}). \qquad (1.3.4)$$

From (1.2.13) and (1.3.4) we conclude that $(1.2.24)_1$ is satisfied. Furthermore, since $\mathbf{n} = \mathbf{e}_{\langle 1\rangle}$ on $\partial\Omega_R$, and since $\mathrm{sym}(\mathbf{e}_{\langle 2\rangle} \otimes \mathbf{e}_{\langle 3\rangle})\mathbf{e}_{\langle 1\rangle} = 0$ by (1.2.1)–$(1.2.2)_1$, we conclude that $(1.2.24)_3$ holds true.

Our next task it to show that, under (1.3.1), $(1.2.24)_2$ is translated into

$$\mu(\Theta\varrho - \gamma^{\mathrm{p}}) + \mu(1-\eta)\frac{L^2}{2}\left(\frac{\partial^2\gamma^{\mathrm{p}}}{\partial\varrho^2} + \frac{1}{\varrho}\frac{\partial\gamma^{\mathrm{p}}}{\partial\varrho^2} - \frac{\gamma^{\mathrm{p}}}{\varrho^2}\right) \in k\,\partial\left|\dot\gamma^{\mathrm{p}}\right| \quad \text{in} \quad (0,R), \tag{1.3.5}$$

and that the initial condition $(1.2.24)_5$ and the null–microtraction condition $(1.2.24)_4$ are translated into

$$\gamma^{\mathrm{p}}(\varrho,0) = 0 \quad \text{and} \quad \frac{\partial\gamma^{\mathrm{p}}}{\partial\varrho}(R,\Theta) + \frac{1}{2}\gamma^{\mathrm{p}}(R,\Theta) = 0. \tag{1.3.6}$$

First, we observe that, by (1.2.13), $\mathbf{E}^{\mathrm{p}}$ has null divergence:

$$\mathrm{div}\mathbf{E}^{\mathrm{p}} = \mathbf{0}. \tag{1.3.7}$$

Thus, since $\mathrm{tr}\mathbf{E}^{\mathrm{p}} = 0$, the identity (1.2.10) yields

$$\mathrm{curlcurl}\mathbf{E}^{\mathrm{p}} = -\Delta\mathbf{E}^{\mathrm{p}}.$$

Hence, on recalling (1.2.23), we see that $(1.2.24)_2$ reduces to

$$\mathbf{T}_0 + \mu(1-\eta)L^2\Delta\mathbf{E}^{\mathrm{p}} \in \partial\delta(\dot{\mathbf{E}}^{\mathrm{p}}). \tag{1.3.8}$$

Next, using the tensorial version of (1.2.11), and (1.2.4), we find from $(1.3.1)_2$ that

$$\nabla\mathbf{E}^{\mathrm{p}} = \frac{\partial\gamma^{\mathrm{p}}}{\partial\varrho}\mathrm{sym}(\mathbf{e}_{\langle 2\rangle} \otimes \mathbf{e}_{\langle 3\rangle}) \otimes \mathbf{e}_{\langle 1\rangle} - \varrho^{-1}\gamma^{\mathrm{p}}\mathrm{sym}(\mathbf{e}_{\langle 1\rangle} \otimes \mathbf{e}_{\langle 3\rangle}) \otimes \mathbf{e}_{\langle 2\rangle}. \tag{1.3.9}$$

Then, using the identity (1.2.12) with $\mathbf{A} = \nabla\mathbf{E}^{\mathrm{p}}$ we arrive at

$$\Delta\mathbf{E}^{\mathrm{p}} = \mathrm{div}\nabla\mathbf{E}^{\mathrm{p}} = \left(\frac{\partial^2\gamma^{\mathrm{p}}}{\partial\varrho^2} + \frac{1}{\varrho}\frac{\partial\gamma^{\mathrm{p}}}{\partial\varrho^2} - \frac{\gamma^{\mathrm{p}}}{\varrho^2}\right)\mathrm{sym}(\mathbf{e}_{\langle 2\rangle} \otimes \mathbf{e}_{\langle 3\rangle}). \tag{1.3.10}$$

From (1.3.4), taking into account (1.2.2) and $(1.2.4)_1$, we see that $\mathbf{T}_0 = \mathbf{T}$. Hence, plugging (1.3.4) and (1.3.10) into (1.3.8), we obtain that the inclusion $(1.2.24)_2$ is equivalent to:

$$2\mu\left[(\Theta\varrho - \gamma^{\mathrm{p}}) + (1-\eta)\frac{L^2}{2}\left(\frac{\partial^2\gamma^{\mathrm{p}}}{\partial\varrho^2} + \frac{1}{\varrho}\frac{\partial\gamma^{\mathrm{p}}}{\partial\varrho^2} - \frac{\gamma^{\mathrm{p}}}{\varrho^2}\right)\right]\mathrm{sym}\left(\mathbf{e}_{\langle 2\rangle} \otimes \mathbf{e}_{\langle 3\rangle}\right) \in \partial\delta(\dot\gamma^{\mathrm{p}}\mathrm{sym}\left(\mathbf{e}_{\langle 2\rangle} \otimes \mathbf{e}_{\langle 3\rangle}\right)), \tag{1.3.11}$$

granted the ansatz (1.3.1). Now, denoting by α and β any pair of scalars, and by $\mathbf{A} \neq \mathbf{0}$ a second–order tensor, we have

$$\alpha\mathbf{A} \in \partial\delta(\beta\mathbf{A}) \Leftrightarrow \alpha \in \frac{\sqrt{2}k}{|\mathbf{A}|}\partial|\beta|. \tag{1.3.12}$$

Since $|\mathrm{sym}\,(\mathbf{e}_{\langle 2\rangle}\otimes\mathbf{e}_{\langle 3\rangle})\,| = \frac{1}{\sqrt{2}}$, (1.3.12) implies that (1.3.5) is equivalent to (1.3.11), and hence to $(1.2.24)_2$. Finally, we consider the null micro–traction condition $(1.2.24)_4$. Since $\mathbf{n} = \mathbf{e}_{\langle 1\rangle}$, from $(1.2.4)_1$, (1.2.5) and (1.2.7) we have $(\mathbb{K}^{\mathrm{p}}\mathbf{n})_{\langle jq\rangle} = (\mathbb{K}^{\mathrm{p}})_{\langle jqp\rangle}(\mathbf{n})_{\langle p\rangle} = (\mathbb{K}^{\mathrm{p}})_{\langle jq1\rangle}$. Thus, by (1.2.6), $\mathbb{K}^{\mathrm{p}}\mathbf{n} = (\mathbb{K}^{\mathrm{p}}\mathbf{n})_{\langle jq\rangle}\mathbf{e}_{\langle j\rangle}\otimes\mathbf{e}_{\langle q\rangle} = (\mathbb{K}^{\mathrm{p}})_{\langle jq1\rangle}\mathbf{e}_{\langle j\rangle}\otimes\mathbf{e}_{\langle q\rangle}$. By working out (1.3.9) and (1.2.22), it turns out that all components $(\mathbb{K}^{\mathrm{p}})_{\langle jq1\rangle}$ vanish, except for

$$(\mathbb{K}^{\mathrm{p}})_{\langle 231\rangle} = (\mathbb{K}^{\mathrm{p}})_{\langle 321\rangle} = \frac{\mu L^2}{2}(1-\eta)\left(\frac{\partial\gamma^{\mathrm{p}}}{\partial\varrho} + \frac{1}{2}\gamma^{\mathrm{p}}\right).$$

Therefore, we conclude that

$$\mathbb{K}^{\mathrm{p}}\mathbf{n} = \mu L^2(1-\eta)\left(\frac{\partial\gamma^{\mathrm{p}}}{\partial\varrho} + \frac{1}{2}\gamma^{\mathrm{p}}\right)\mathrm{sym}(\mathbf{e}_{\langle 2\rangle}\otimes\mathbf{e}_{\langle 3\rangle}), \tag{1.3.13}$$

and hence $(1.2.24)_4$ yields the null-microtraction condition $(1.3.6)_2$. Finally, $(1.3.6)_1$ follows immediately from $(1.2.24)_5$.

By (1.2.18) and (1.3.3), and since $|\mathrm{sym}(\mathbf{e}_{\langle 2\rangle}\otimes\mathbf{e}_{\langle 3\rangle})|^2 = \frac{1}{2}$, the elastic–energy density is: $\psi_{\mathrm{e}} = \mu\mathbf{E}^{\mathrm{e}} : \mathbf{E}^{\mathrm{e}} = \frac{\mu}{2}(\varrho\Theta - \gamma^{\mathrm{p}})^2$. Moreover, using (1.3.9) and (1.2.9), we find that $(\mathrm{curl}\mathbf{E}^{\mathrm{p}})_{\langle ij\rangle} = 0$ if $i \neq j$ and

$$(\mathrm{curl}\mathbf{E}^{\mathrm{p}})_{\langle 11\rangle} = -\frac{1}{2}\frac{\gamma^{\mathrm{p}}}{\varrho}, \quad (\mathrm{curl}\mathbf{E}^{\mathrm{p}})_{\langle 22\rangle} = -\frac{1}{2}\frac{\partial\gamma^{\mathrm{p}}}{\partial\varrho}, \quad (\mathrm{curl}\mathbf{E}^{\mathrm{p}})_{\langle 33\rangle} = \frac{1}{2}\left(\frac{\partial\gamma^{\mathrm{p}}}{\partial\varrho} + \frac{\gamma^{\mathrm{p}}}{\varrho}\right).$$

Thus, by (1.2.19) the defect–energy density is:

$$\psi_{\mathrm{d}} = \mu\frac{L^2}{4}(1-\eta)\Big[\left(\frac{\partial\gamma^{\mathrm{p}}}{\partial\varrho}\right)^2 + \frac{\gamma}{\varrho}\frac{\partial\gamma^{\mathrm{p}}}{\partial\varrho} + \left(\frac{\gamma^{\mathrm{p}}}{\varrho}\right)^2\Big].$$

By integrating the free–energy density $\psi = \psi_{\mathrm{e}} + \psi_{\mathrm{d}}$ over any cross-section of Ω_R, we obtain the *free energy per unit length* along the cylinder axis:

$$\mathcal{F}(\gamma^{\mathrm{p}},\Theta) = 2\pi\mu\int_0^R \frac{1}{2}\left\{(\Theta\varrho - \gamma^{\mathrm{p}})^2 + (1-\eta)\frac{L^2}{2}\Big[\left(\frac{\partial\gamma^{\mathrm{p}}}{\partial\varrho}\right)^2 + \frac{\gamma}{\varrho}\frac{\partial\gamma^{\mathrm{p}}}{\partial\varrho} + \left(\frac{\gamma^{\mathrm{p}}}{\varrho}\right)^2\Big]\right\}\varrho\mathrm{d}\varrho.$$

By (1.3.1), $\delta(\dot{\mathbf{E}}^{\mathrm{p}}) = \sqrt{2}k|\mathrm{sym}(\mathbf{e}_{\langle 1\rangle}\otimes\mathbf{e}_{\langle 2\rangle})||\dot{\gamma}^{\mathrm{p}}| = k|\dot{\gamma}^{\mathrm{p}}|$. Again, integration over any cross section of Ω_R yields the *dissipation rate per unit length* along the axis:

$$\mathcal{D}(\dot{\gamma}^{\mathrm{p}}) = 2\pi k\int_0^R \left|\dot{\gamma}^{\mathrm{p}}\right|\varrho\mathrm{d}\varrho.$$

1.4 Formulation and solution to the torsion problem

1.4.1 Normalization

We pass to the normalized variables (1.1.12), we introduce the *normalized length scale*

$$\lambda := \sqrt{\frac{(1-\eta)}{2}}\frac{L}{R}, \tag{1.4.1}$$

and we define the linear operator

$$\mathcal{L}\gamma := \lambda^2 \left(\gamma'' + \frac{\gamma'}{r} - \frac{\gamma}{r^2} \right) - \gamma. \tag{1.4.2}$$

By virtue of (1.1.12), the partial differential inclusion (1.3.5) and the conditions (1.3.6) are equivalent to (1.1.13), which we rewrite for convenience:

$$\begin{cases} \mathcal{L}\gamma + \theta r \in \partial\,|\dot\gamma| & \text{in} \quad (0,1)\times(0,+\infty) \\ \gamma'(1,\theta) + \dfrac{\gamma(1,\theta)}{2} = 0 & \text{for} \;\; \theta > 0 \\ \gamma(r,0) = 0 & \text{for} \;\; r \in (0,1), \end{cases} \tag{1.4.3}$$

where now $\dot\gamma = \frac{\partial\gamma}{\partial\theta}$. In terms of the normalized variables the torque (1.1.11) can be written as in (1.1.16), and free energy and dissipation-rate are expressed resp. by $\mathcal{F}(\gamma^{\mathrm{p}},\Theta) = \mathcal{E}_*\mathscr{F}(\gamma,\theta)$ and $\mathcal{D}(\gamma^{\mathrm{p}},\Theta) = \mathcal{E}_*\mathscr{D}(\gamma,\theta)$, where $\mathcal{E}_* := 2\pi R^2 \frac{k^2}{\mu}$,

$$\mathscr{F}(\gamma,\theta) := \int_0^1 \frac{1}{2}\left((\theta r - \gamma)^2 + \lambda^2 \left(\gamma'^2 + \frac{\gamma'\gamma}{r} + \left(\frac{\gamma}{r}\right)^2 \right) \right) r\,\mathrm{d}r,$$

and

$$\mathscr{D}(\dot\gamma) := \int_0^1 |\dot\gamma| r \mathrm{d}r.$$

We'll find it more convenient to work with the *effective energy*

$$\begin{aligned} \mathscr{E}(\gamma,\theta) &:= \mathscr{F}(\gamma,\theta) - \frac{\theta^2}{8} \\ &= \frac{1}{2}\int_0^1 \left(\gamma^2 + \lambda^2 \left(\gamma'^2 + \frac{\gamma'\gamma}{r} + \left(\frac{\gamma}{r}\right)^2 \right) \right) r\,\mathrm{d}r - \theta \int_0^1 \gamma r^2 \mathrm{d}r. \end{aligned}$$

1.4.2 The evolutionary variational inequality

The structure of $\mathscr{E}$ suggests that the natural functional setting for (1.4.3) is the space

$$H := \overline{C_c^\infty((0,1])}^{\|\cdot\|_r}, \quad \text{where} \quad \|g\|_r := \int_0^1 \left(g'^2 + \left(\frac{g}{r}\right)^2 \right) r \mathrm{d}r.$$

Lemma 1.1.

$$\sup_{(0,1)} |\gamma|^2 \le \|\gamma\|_H^2 \quad \textit{and} \quad \lim_{r\to 0^+} \gamma(r) = 0 \quad \textit{for all} \;\; \gamma \in H. \tag{1.4.4}$$

Proof. Since $\|\gamma\|_H$ is finite, a sequence $r_n \to 0^+$ exists such that $\gamma^2(r_n) \to 0$ as $n \to +\infty$. For any $r > r_n$,

$$\left|\gamma^2(r) - \gamma^2(r_n)\right| = \left|2\int_{r_n}^r \gamma\gamma' \mathrm{d}r\right| \le \left(\int_0^r r(\gamma')^2 \mathrm{d}r\right)^{1/2} \left(\int_0^r \frac{\gamma^2}{r}\mathrm{d}r\right)^{1/2} \lesssim \|\gamma\|_H^2.$$

Passing to the limit as $n \to +\infty$ we obtain $(1.4.4)_1$. Passing to the limit as $n \to +\infty$ and as $r \to 0^+$, in this order, we obtain $(1.4.4)_2$. □

A simple computation shows that $D_\gamma \mathscr{E}(\gamma, \theta) \in H'$ is given by

$$\langle D_\gamma \mathscr{E}(\gamma, \theta), \tilde{\gamma} \rangle = a(\gamma, \tilde{\gamma}) - \langle \ell(\theta), \tilde{\gamma} \rangle, \tag{1.4.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between H' and H, $a : H \times H \to \mathbb{R}$ is the symmetric bilinear form defined by

$$a(\gamma, \tilde{\gamma}) := \int_0^1 \left(\gamma\tilde{\gamma} + \lambda^2 \left(\gamma'\tilde{\gamma}' + \frac{\gamma'\tilde{\gamma} + \tilde{\gamma}'\gamma}{2r} + \frac{\tilde{\gamma}\gamma}{r^2} \right) \right) r \, \mathrm{d}r, \tag{1.4.6}$$

and $\ell(\theta) : H \to \mathbb{R}$ is the linear form defined by

$$\langle \ell(\theta), \gamma \rangle := \theta \int_0^1 r^2 \gamma \, \mathrm{d}r. \tag{1.4.7}$$

On the other hand, a formal integration by parts shows that if γ is a smooth solution to (1.4.3), then

$$\langle D_\gamma \mathscr{E}(\gamma, \theta), \tilde{\gamma} \rangle = -\int_0^1 (\mathcal{L}\gamma + \theta r)\tilde{\gamma} \, r \mathrm{d}r.$$

This suggests to write (1.4.3) in its *subdifferential formulation*:

$$\partial \mathscr{D}(\dot{\gamma}) + D_\gamma \mathscr{E}(\gamma, \theta) \ni 0, \tag{1.4.8}$$

where $\partial \mathscr{D}$ is defined by

$$\xi \in \partial \mathscr{D}(\dot{\gamma}) \iff \langle -\xi, \widetilde{\gamma} - \dot{\gamma} \rangle \geq \mathscr{D}(\dot{\gamma}) - \mathscr{D}(\widetilde{\gamma}) \;\; \forall \widetilde{\gamma} \in H.$$

We can thus recognize in (1.4.5) and (1.4.8) the standard format of an *evolutionary variational inequality*:

Definition 1.1. *Let $\gamma \in W^{1,1}_{\mathrm{loc}}([0, +\infty); H)$. We say that γ solves* (1.4.3) *if $\gamma(0) = 0$ and*

$$a(\gamma, \tilde{\gamma} - \dot{\gamma}) - \langle \ell(\theta), \tilde{\gamma} - \dot{\gamma} \rangle \geq \mathscr{D}(\dot{\gamma}) - \mathscr{D}(\tilde{\gamma}) \quad \textit{for all } \; \tilde{\gamma} \in H \tag{1.4.9}$$

for almost every $\theta > 0$.

The proof of the next Lemma is standard; however, we reproduce it for completeness and later reference.

Lemma 1.2. *The bilinear form $a : H \times H \to \mathbb{R}$ defined in* (1.4.6) *is continuous and coercive. Furthermore,*

$$a(\gamma, \tilde{\gamma}) = \int_0^1 \left(\gamma\tilde{\gamma} + \lambda^2 \left(\gamma'\tilde{\gamma}' + \frac{\gamma\tilde{\gamma}}{r^2} \right) \right) r \, \mathrm{d}r + \frac{\lambda^2}{2} \gamma(1)\tilde{\gamma}(1) \quad \textit{for all } \gamma, \tilde{\gamma} \in H. \tag{1.4.10}$$

Proof. The reformulation (1.4.10) follows from noting that

$$\int_0^1 (\gamma'\tilde{\gamma} + \tilde{\gamma}'\gamma)\mathrm{d}r \overset{(1.4.4)_2}{=} \gamma(1)\tilde{\gamma}(1) \quad \text{for all } \gamma, \tilde{\gamma} \in H.$$

In view of (1.4.10), coercivity is immediate, and continuity follows on recalling $(1.4.4)_1$. □

According to [64, Theorem 7.3] the Lipschitz continuity of $\mathscr{D}$ with respect to the norm $\|\cdot\|_H$ and Lemma 1.2 give existence and uniqueness:

Proposition 1.1. *There exists a unique* $\gamma \in W^{1,1}_{\mathrm{loc}}([0,+\infty); H)$ *that solves* (1.4.3) *in the sense of Definition 1.1.*

1.5 Characterization of the solution

To characterize the solution to (1.4.3) we first try to get some hints from the explicit result available in the standard torsion problem, that is, when $\lambda = 0$. In terms of normalized variables (1.1.12), the solution for $\lambda = 0$ is given by (1.1.15), and has the following property: for each $\theta > 1$ there exists an *elasto-plastic radius* c_θ such that

$$\begin{cases} \dot{\gamma}(r,\theta) = 0 \ \text{ and } \ \gamma(r,\theta) = 0 & \text{if } r \in [0, c_\theta), \\ \dot{\gamma}(r,\theta) > 0 \ \text{ and } \ \gamma(r,\theta) > 0 & \text{if } r \in (c_\theta, 1]\,. \end{cases} \tag{1.5.1}$$

Moreover, $c_\theta = 1$ if $\theta \in [0,1]$. When looking for a solution of (1.4.3) for $\lambda > 0$, it is natural to search first among plastic profiles consistent with (1.5.1). For all fixed $\theta > 1$, a plastic profile consistent with (1.4.3) and (1.5.1) must satisfy

$$\lambda^2 \left(\gamma'' + \frac{1}{r}\gamma' - \frac{\gamma}{r^2} \right) - \gamma = 1 - \theta r \qquad \text{on } (c_\theta, 1), \tag{1.5.2}$$

along with the boundary condition $(1.4.3)_2$ and the left–end condition

$$\lim_{r \to c_\theta^+} \gamma(r,\theta) = 0, \tag{1.5.3}$$

the latter being implicit in the choice of H as ambient space if $c_\theta = 0$. There is however, an extra condition coming from (1.4.3) and from $\gamma(\cdot,\theta) \in H$, namely,

$$\lim_{r \to c_\theta^+} \gamma'(r,\theta) = 0 \quad \text{for } c_\theta > 0. \tag{1.5.4}$$

This condition is necessary for $r \mapsto \gamma'(\theta, r)$ to be continuous across c_θ: without such continuity, $r \mapsto \gamma''(\theta, r)$ would not be square-integrable across c_θ, whereas all the other terms in (1.4.3) are.

By putting together (1.5.2), (1.5.3), and (1.5.4), and the micro–free condition $(1.4.3)_2$ we obtain the following free boundary problem:

$$
(\mathscr{P}_\theta)\begin{cases}\lambda^2\left(\gamma'' + \dfrac{1}{r}\gamma' - \dfrac{\gamma}{r^2}\right) - \gamma = 1 - \theta r \quad \text{on } (c_\theta, 1)\\ \gamma(c_\theta) = \gamma'(c_\theta) = 0\\ \gamma'(1) + \dfrac{\gamma(1)}{2} = 0.\end{cases} \tag{1.5.5}
$$

Here c_θ is an additional unknown to be determined together with γ (at variance with the case $\lambda = 0$, when c_θ is given by $1/\theta$). It turns out that $(\mathscr{P}_\theta)$ is well posed up to a *critical twist* θ_λ:

Lemma 1.3. *Let $\lambda > 0$. There exists a* critical twist $\theta_\lambda > 1$ *such that Problem $(\mathscr{P}_\theta)$ has a unique solution*

$$
(c_\theta, \gamma_\theta) \in (0, 1) \times C^\infty([c_\theta, 1])
$$

for all $\theta \in (1, \theta_\lambda)$, and has no solution for $\theta > \theta_\lambda$. Furthermore:

(i) *c_θ is strictly decreasing and uniformly Lipschitz continuous with respect to θ, and*

$$
\lim_{\theta\to 1^+} c_\theta = 1, \quad \lim_{\theta\to\theta_\lambda^-} c_\theta = 0;
$$

(ii) *$c_\theta < 1/\theta$ for all $\theta \in (1, \theta_\lambda)$;*

(iii) *$\gamma_\theta > 0$ in $(c_\theta, 1]$ for all $\theta \in (1, \theta_\lambda)$;*

(iv) *if $1 < \theta_1 < \theta_2 < \theta_\lambda$, then $\gamma_{\theta_1} < \gamma_{\theta_2}$ in $[c_{\theta_1}, 1]$.*

It follows from part (i) in the above Lemma that, as θ attains the critical twist θ_λ, the elasto-plastic boundary hits the origin $r = 0$. Hence one expects that for $\theta \geq \theta_\lambda$ the plastic-shear profile solves

$$
(\mathscr{P}'_\theta)\begin{cases}\lambda^2\left(\gamma'' + \dfrac{1}{r}\gamma' - \dfrac{\gamma}{r^2}\right) - \gamma = 1 - \theta r \quad \text{on } (0, 1)\\ \gamma(0) = 0\\ \gamma'(1) + \dfrac{\gamma(1)}{2} = 0,\end{cases} \tag{1.5.6}
$$

which is well posed for all $\theta \in \mathbb{R}$:

Lemma 1.4. *Let $\lambda > 0$. For all $\theta \in \mathbb{R}$ there exists a unique solution $\overline{\gamma}_\theta \in H$ of $(\mathscr{P}'_\theta)$ in the sense that*

$$
a(\gamma, \tilde{\gamma}) = \int_0^1 (\theta r - 1)\tilde{\gamma} r \mathrm{d}r \quad \textit{for all } \tilde{\gamma} \in H,
$$

with $a(\cdot, \cdot)$ given by (1.4.6). *Furthermore:*

(i) $\overline{\gamma}_\theta \in C^\infty((0,1]) \cap C([0,1])$ *with* $\overline{\gamma}_\theta(0) = 0$*;*

(ii) *if* $\theta_1 < \theta_2$, *then* $\overline{\gamma}_{\theta_1} < \overline{\gamma}_{\theta_2}$ *in* $(0,1]$.

To construct a candidate solution for all $\theta \geq 0$, we extend γ_θ to $(0,1)$ by setting:

$$\gamma_\theta(r) := \ 0 \quad \text{if} \ \ r \in (0, c_\theta],$$

and we patch γ_θ and $\overline{\gamma}_\theta$ together by defining:

$$\gamma(r,\theta) := \begin{cases} 0 & \text{if} \ \ \theta \in [0,1] \\ \gamma_\theta(r) & \text{if} \ \ \theta \in (1, \theta_\lambda) \\ \overline{\gamma}_\theta(r) & \text{if} \ \ \theta \geq \theta_\lambda. \end{cases} \tag{1.5.7}$$

The resulting function turns out to be the right candidate:

Theorem 1.1. *The function* γ *defined by* (1.5.7) *is the unique solution of* (1.4.3) *in the sense of Definition 1.1. Moreover,* $\gamma \in \mathrm{Lip}([0,+\infty); H)$.

In the rest of this section we prove Lemma 1.3, Lemma 1.4, and Theorem 1.1.

Proof of Lemma 1.3. We introduce $\gamma_{(0)}$, $\gamma_{(1)}$ and $\gamma_{(2)}$ as the solutions of the following auxiliary problems:

$$\begin{cases} \mathcal{L}\gamma_{(0)}(r) = 1 \\ \gamma_{(0)}(1) = \gamma'_{(0)}(1) = 0, \end{cases} \quad \begin{cases} \mathcal{L}\gamma_{(1)}(r) = -r \\ \gamma_{(1)}(1) = \gamma'_{(1)}(1) = 0, \end{cases} \quad \begin{cases} \mathcal{L}\gamma_{(2)}(r) = 0 \\ \gamma_{(2)}(1) = 1, \ \gamma'_{(2)}(1) = -\frac{1}{2}. \end{cases} \tag{1.5.8}$$

It follows easily by comparison (see e.g. the proof of (iv) below) that $\gamma_{(0)}$, $-\gamma_{(1)}$ and $\gamma_{(2)}$ are positive, decreasing and convex in $(0,1)$.

If a pair (c,γ), with $c > 0$, is a solution of $(\mathscr{P}_\theta)$, then γ may be represented by

$$\gamma = \gamma_{(0)} + \theta\gamma_{(1)} + \alpha\gamma_{(2)} \tag{1.5.9}$$

for $\alpha \in \mathbb{R}$, and the boundary conditions $(1.5.5)_2$ imply:

$$\begin{cases} \theta\gamma_{(1)}(c) + \alpha\gamma_{(2)}(c) = -\gamma_{(0)}(c) \\ \theta\gamma'_{(1)}(c) + \alpha\gamma'_{(2)}(c) = -\gamma'_{(0)}(c). \end{cases} \tag{1.5.10}$$

Viceversa, if α and $c > 0$ are such that (1.5.10) holds, then (c,γ), with γ given by (1.5.9), is a solution of $(\mathscr{P}_\theta)$.

We now fix $c \in (0,1)$ and consider (1.5.10) as a linear system in (θ, α). Its determinant is given by

$$\delta(c) := \gamma_{(1)}(c)\gamma'_{(2)}(c) - \gamma_{(2)}(c)\gamma'_{(1)}(c).$$

Using (1.5.8), it is easily seen that δ satisfies

$$\begin{cases} \delta'(c) = -\dfrac{\delta(c)}{c} + \dfrac{c\gamma_{(2)}(c)}{\lambda^2} \quad \text{in } (0,1) \\ \delta(1) = 0 \end{cases} \tag{1.5.11}$$

which may be integrated explicitly:

$$\delta(c) = -\frac{1}{c\lambda^2}\int_c^1 r^2\gamma_{(2)}(r)\,\mathrm{d}r \quad \text{for all } c \in (0,1]. \tag{1.5.12}$$

Note that $\delta < 0$ in $(0,1)$ since $\gamma_{(2)} > 0$: therefore, for any $c \in (0,1)$, (1.5.10) has a unique solution,

$$\begin{aligned} \widehat{\theta}(c) &= \frac{-\gamma_{(0)}(c)\gamma'_{(2)}(c) + \gamma_{(2)}(c)\gamma'_{(0)}(c)}{\delta(c)} =: \frac{\nu(c)}{\delta(c)}, \\ \widehat{\alpha}(c) &= \frac{-\gamma_{(1)}(c)\gamma'_{(0)}(c) + \gamma_{(0)}(c)\gamma'_{(1)}(c)}{\delta(c)} =: \frac{\xi(c)}{\delta(c)}. \end{aligned} \tag{1.5.13}$$

In order to invert $\widehat{\theta}$, with the help of (1.5.8) we notice that the numerator ν of $\widehat{\theta}$ solves

$$\begin{cases} \nu'(c) = -\dfrac{\nu(c)}{c} + \dfrac{\gamma_{(2)}(c)}{\lambda^2} \text{ in } (0,1) \\ \nu(1) = 0, \end{cases} \tag{1.5.14}$$

which, as before, may be integrated explicitly:

$$\nu(c) = -\frac{1}{c\lambda^2}\int_c^1 r\gamma_{(2)}(r)\,\mathrm{d}r \quad \text{for all } c \in (0,1]. \tag{1.5.15}$$

Therefore

$$\begin{aligned} \widehat{\theta}'(c) &= \frac{\nu'(c)\delta(c) - \nu(c)\delta'(c)}{\delta^2(c)} \\ &\overset{(1.5.11),(1.5.14)}{=} \frac{\gamma_{(2)}(c)}{\lambda^2\delta^2(c)}(\delta(c) - c\nu(c)) \\ &\overset{(1.5.12),(1.5.15)}{=} -c\gamma_{(2)}(c)\frac{\int_c^1 r(r-c)\gamma_{(2)}(r)\mathrm{d}r}{\left(\int_c^1 r^2\gamma_{(2)}(r)\mathrm{d}r\right)^2} < 0 \quad \text{for all } c \in (0,1), \end{aligned} \tag{1.5.16}$$

which implies that $\widehat{\theta}$ is invertible.

We now notice that, letting $\hat{r} = r/\lambda$, the equation satisfied by $\gamma_{(2)}$ becomes the so–called *modified Bessel equation of order 1*:

$$\hat{r}^2\frac{d^2}{d\hat{r}^2}\gamma + \hat{r}\frac{d}{d\hat{r}}\gamma - (1+\hat{r}^2)\gamma = 0,$$

whose general solution is a linear combination of the *modified Bessel functions* $I_1(x)$ and $K_1(x)$; in particular, it is such that $\hat{r}\gamma_{(2)}(\hat{r}) \to C > 0$ as $\hat{r} \to 0^+$ [1, p. 374, §9.6.1 and p.375 9.6.7–8].

Then

$$\lim_{c\to 0^+} \widehat{\theta}'(c) = -C_2 < 0 \tag{1.5.17}$$

and, after simple computations using de l'Hôpital's rule,

$$\lim_{c\to 1^-} \widehat{\theta}'(c) = -\gamma_{(2)}(1) \lim_{c\to 1^-} \frac{-\int_c^1 r\gamma_{(2)}(r)\mathrm{d}r}{-2c^2\gamma_{(2)}(c)\left(\int_c^1 r^2\gamma_{(2)}(r)\mathrm{d}r\right)} = -\frac{1}{2}. \tag{1.5.18}$$

Since (by (1.5.16)) $\widehat{\theta}'$ is continuous in $(0, 1)$, (1.5.17) and (1.5.18) imply that

$$\widehat{\theta}'(c) \le -C_1 < 0 \quad \text{for all } c \in (0, 1). \tag{1.5.19}$$

In addition, recalling (3.2.6), (1.5.12), and (1.5.15),

$$\lim_{c\to 1^-} \widehat{\theta}(c) = \lim_{c\to 1^-} \frac{\int_c^1 s\gamma_{(2)}(s)\mathrm{d}s}{\int_c^1 s^2\gamma_{(2)}(s)\mathrm{d}s} = 1, \tag{1.5.20}$$

$$\lim_{c\to 0^+} \widehat{\theta}(c) = \lim_{c\to 0^+} \frac{\int_c^1 s\gamma_{(2)}(s)\mathrm{d}s}{\int_c^1 s^2\gamma_{(2)}(s)\mathrm{d}s} = \theta_\lambda < +\infty. \tag{1.5.21}$$

Combining (1.5.19), (1.5.20), and (1.5.21), we see that the function

$$\widehat{\theta}^{-1} : (1, \theta_\lambda) \ni \theta \longmapsto c = c_\theta \in [0, 1)$$

is strictly decreasing and uniformly Lipschitz continuous: it uniquely determines the solution of $(\mathscr{P}_\theta)$,

$$\gamma_\theta := \gamma_{(0)} + \theta\gamma_{(1)} + \widehat{\alpha}(c_\theta)\gamma_{(2)}. \tag{1.5.22}$$

Since $c_\theta > 0$, the regularity of γ_θ follows at once from that of $\gamma_{(0)}, \gamma_{(1)}$ and $\gamma_{(2)}$.

In order to prove (ii)-(iv) we make three observations. First, differentiating $(1.5.10)_1$ with respect to c and subtracting $(1.5.10)_2$ we obtain $\widehat{\theta}'(c)\gamma_{(1)}(c) + \widehat{\alpha}'(c)\gamma_{(2)}(c) = 0$, whence

$$\widehat{\alpha}'(c) = -\widehat{\theta}'(c)\frac{\gamma_{(1)}(c)}{\gamma_{(2)}(c)}, \quad \text{i.e.} \quad \frac{\mathrm{d}}{\mathrm{d}\theta}\widehat{\alpha}(c_\theta) = -\frac{\gamma_{(1)}(c)}{\gamma_{(2)}(c)} > 0. \tag{1.5.23}$$

Combining (1.5.23) with (1.5.22) (evaluated at $r = 1$) we obtain the following monotonicity property:

$$1 < \theta_1 < \theta_2 < \theta_\lambda \quad \Rightarrow \quad \gamma_{\theta_1}(1) < \gamma_{\theta_2}(1). \tag{1.5.24}$$

Second,

$$\gamma \text{ can not have a non-positive local minimum in } (1/\theta, 1). \tag{1.5.25}$$

Indeed, at a local minimum point $r_0 \in (1/\theta, 1)$ we would have

$$\gamma''(r_0) = \left(\frac{1}{\lambda^2} + \frac{1}{r_0^2}\right)\gamma(r_0) + 1 - \theta r_0 < 0,$$

which is impossible. Third,

$$\gamma(1) > 0. \tag{1.5.26}$$

Indeed, if $\gamma(1) < 0$ then $\gamma'(1) = -\gamma(1)/2 > 0$, whilst if $\gamma(1) = 0$ then $\gamma'(1) = 0$ and, as above, $\gamma''(1) < 0$ (since $\theta > 1$). Since $\gamma(c_\theta = 0)$, both would contradict (1.5.25).

We are now ready to prove (ii)-(iv).

(ii) We first show that $c_\theta < 1/\theta$. We recall that $\gamma'(c_\theta) = 0$ and we note that $\gamma''(c_\theta) = 1 - \theta c_\theta$. If by contradiction $c_\theta > 1/\theta$, then $\gamma''(c_\theta) < 0$, hence γ would be negative in a right-neighborhood of c_θ, in contradiction with (1.5.26) and (1.5.25). If, instead, $c_\theta = 1/\theta$, then $\gamma''(c_\theta) = 0$: differentiating the equation, this implies that $\gamma'''(c_\theta) = -\theta < 0$ and yields a contradiction as in the previous case.

(iii) We next show that $\gamma > 0$ in $(c_\theta, 1]$. If not, since $\gamma(c_\theta) = 0$ and $\gamma(1) > 0$, by (1.5.25) γ must have a non-positive minimum point $r_0 \in (c_\theta, 1/\theta]$. On the other hand, by (ii) $\gamma''(c_\theta) = 1 - \theta c_\theta > 0$, hence γ is positive in a right-neighborhood of c_θ. Since $\gamma(r_0) \le 0$, γ has a positive maximum in $r_1 \in (c_\theta, r_0) \subseteq (c_\theta, 1/\theta)$. But then $\gamma''(r_1) > 1 - \theta r_1 > 0$, a contradiction.

(iv) Finally, we show that (1.5.24) can be strengthened to:

$$1 < \theta_1 < \theta_2 < \theta_\lambda \quad \Rightarrow \quad \gamma_{\theta_1} < \gamma_{\theta_2} \text{ in } [c_{\theta_1}, 1].$$

Let $\overline{\theta} = \theta_2 - \theta_1 > 0$. The difference $\overline{\gamma} = \gamma_{\theta_2} - \gamma_{\theta_1}$ satisfies

$$\begin{cases} \mathcal{L}\overline{\gamma} + \overline{\theta} r = 0 \quad \text{in } (c_{\theta_1}, 1], \\ \overline{\gamma}'(1) + \frac{1}{2}\overline{\gamma}(1) = 0. \end{cases} \tag{1.5.27}$$

Since c_θ is strictly decreasing, (iii) implies that $\overline{\gamma}(c_{\theta_1}) > 0$. By (1.5.24) we also have $\overline{\gamma}(1) > 0$. If $\overline{\gamma}$ had a non-positive minimum at $r_0 \in (c_{\theta_1}, 1)$, by $(4.8.6)_1$ we would have

$$\overline{\gamma}''(r_0) = \left(\frac{1}{\lambda^2} + \frac{1}{r_0^2}\right)\overline{\gamma}(r_0) - \overline{\theta} r_0 < 0,$$

which is impossible. Hence $\overline{\gamma} > 0$ in $[c_{\theta_1}, 1]$ and the proof is complete.

Proof of Lemma 1.4. Let $f \in H'$ be defined by

$$\langle f, \tilde{\gamma} \rangle := \int_0^1 \tilde{\gamma}(\theta r - 1)\, r\mathrm{d}r, \quad \text{for all } \tilde{\gamma} \in H.$$

According to Lemma 1.2, and to the Lax-Milgram Theorem, there exists a unique function $\gamma = \overline{\gamma}_\theta \in H$ satisfying

$$a(\gamma, \tilde{\gamma}) = \langle f, \tilde{\gamma} \rangle \quad \text{for all } \tilde{\gamma} \in H,$$

that is to say,

$$\int_0^1 \tilde{\gamma}(\theta r - 1)\, r\mathrm{d}r \overset{(1.4.10)}{=} \int_0^1 \gamma\tilde{\gamma} r\mathrm{d}r + \lambda^2 \int_0^1 \left(\gamma'\tilde{\gamma}' + \frac{\tilde{\gamma}\gamma}{r^2} \right) r\mathrm{d}r + \frac{\gamma(1)\tilde{\gamma}(1)}{2} \tag{1.5.28}$$

for all $\tilde{\gamma} \in H$. Choosing first $\tilde{\gamma} \in C_c^\infty((0,1))$ in (1.5.28), we see that $\gamma \in H^2_{\mathrm{loc}}((0,1])$ and that

$$\lambda^2 \left(r\gamma'' + \gamma' - \frac{\gamma}{r} \right) - r\gamma = (1 - \theta r) r \quad \text{a.e. in } (0,1), \tag{1.5.29}$$

i.e. the equation in $(\mathscr{P}'_\theta)$ holds. Choosing then $\tilde{\gamma} \in C_c^\infty((0,1])$ in (1.5.28), integrating by parts and using (1.5.29), we see that

$$\gamma'(1)\tilde{\gamma}(1) = -\frac{1}{2}\gamma(1)\tilde{\gamma}(1),$$

hence the boundary condition in $(\mathscr{P}'_\theta)$ holds, too. It follows immediately from linear ODE theory that $\overline{\gamma}_\theta \in C^\infty((0,1])$; together with (1.4.4), (i) holds. Finally, (ii) follows by comparison arguments analogous to those used in the proof of Lemma 1.3, using that $\gamma(0) = 0$ for all θ.

Proof of Theorem 1.1. We extend the definition of c_θ with

$$c_\theta = \begin{cases} 1 & \text{if } \theta \in [0,1] \\ 0 & \text{if } \theta \geq \theta_\lambda. \end{cases} \tag{1.5.30}$$

We let $\gamma(\cdot) = \gamma(\cdot, \theta)$ and $c = c_\theta$ when no confusion arises. A few preliminary observations are in order. Let $a : H \times H \to \mathbb{R}$ be as in (1.4.6). We already know from Lemma 1.4 that

$$a(\gamma, \tilde{\gamma}) = \int_0^1 (\theta r - 1)\tilde{\gamma} r\mathrm{d}r \quad \text{for all } \tilde{\gamma} \in H \quad \text{and all } \theta \geq \theta_\lambda.$$

For $\theta \in (1, \theta_\lambda)$, we recall that $\gamma \in C^\infty([c,1])$ is such that $\gamma = 0$ in $(0,c)$ and

$$\begin{cases} \mathcal{L}\gamma = 1 - r\theta & \text{if } r \in (c,1) \\ \gamma(c) = \gamma'(c) = 0 \\ \gamma'(1) + \frac{\gamma(1)}{2} = 0 \end{cases} \tag{1.5.31}$$

Hence, for every $\tilde{\gamma} \in H$ we have

$$\begin{aligned}
\int_c^1 (r\theta - 1)\tilde{\gamma}\, r\, \mathrm{d}\, r \quad &\overset{(1.5.31a)}{=} \int_c^1 (-\mathcal{L}\gamma)\tilde{\gamma}\, r\, \mathrm{d}r \\
&\overset{(1.4.2)}{=} \int_c^1 \left[-\lambda^2 \left(\gamma''\tilde{\gamma}r + \gamma'\tilde{\gamma} - \frac{\gamma\tilde{\gamma}}{r}\right) + \gamma\tilde{\gamma}r\right] \mathrm{d}r \\
&\overset{(1.5.31b)}{=} \int_c^1 \left[\lambda^2 \left(\gamma'\tilde{\gamma}' + \frac{\gamma\tilde{\gamma}}{r^2}\right) + \gamma\tilde{\gamma}\right] r\, \mathrm{d}r - \lambda^2\gamma'(1)\tilde{\gamma}(1) \\
&\overset{(1.5.31c)}{=} \int_0^1 \left[\lambda^2\left(\gamma'\tilde{\gamma}' + \frac{\gamma\tilde{\gamma}}{r^2}\right) + \gamma\tilde{\gamma}\right] r\, \mathrm{d}r + \lambda^2 \frac{\gamma(1)\tilde{\gamma}(1)}{2} \\
&\overset{(1.4.10)}{=} a(\gamma, \tilde{\gamma}).
\end{aligned}$$

In view of (1.5.30), we conclude that

$$a(\gamma, \tilde{\gamma}) = \int_c^1 (\theta r - 1)\tilde{\gamma} r \mathrm{d}r \quad \text{for all } \tilde{\gamma} \in H \text{ and all } \theta \geq 0. \tag{1.5.32}$$

It follows from (ii) of Lemma 1.3 and (1.5.30) that

$$c_\theta < 1/\theta \quad \text{for all } \theta \in (0, +\infty). \tag{1.5.33}$$

We are now ready to complete the proof. First we show uniform Lipschitz continuity in H:

$$\|\gamma(\cdot, \theta_2) - \gamma(\cdot, \theta_1)\|_H \leq C|\theta_2 - \theta_1| \quad \text{for all } 0 \leq \theta_1 \leq \theta_2. \tag{1.5.34}$$

Let $0 \leq \theta_1 < \theta_2$, $\gamma_i(\cdot) := \gamma(\cdot, \theta_i)$, and $c_i := c_{\theta_i}$. By (i) in Lemma 1.3, $c_2 < c_1$. By Lemma 1.2,

$$\begin{aligned}
\|\gamma_2 - \gamma_1\|_H^2 \quad &\lesssim \quad a(\gamma_2 - \gamma_1, \gamma_2 - \gamma_1) \\
&\overset{(1.5.32)}{=} \int_0^1 (\theta_2 - \theta_1)(\gamma_2 - \gamma_1)r^2 \mathrm{d}r + \int_{c_2}^{c_1} (\theta_1 r - 1)\gamma_2 r \mathrm{d}r.
\end{aligned}$$

In (c_2, c_1), using (1.5.33) we have $\theta_1 r - 1 \leq \theta_1 c_1 - 1 < 0$. Therefore

$$\|\gamma_2 - \gamma_1\|_H^2 \lesssim (\theta_2 - \theta_1)\left(\int_0^1 \left(\frac{\gamma_2 - \gamma_1}{r}\right)^2 r\mathrm{d}r\right)^{1/2} \leq \|\gamma_2 - \gamma_1\|_H(\theta_2 - \theta_1)$$

which yields (1.5.34). Now (iv) of Lemma 1.3 and (ii) of Lemma 1.4 imply that

$$\dot{\gamma} \geq 0, \tag{1.5.35}$$

and the definition of γ implies that

$$\dot{\gamma}(\cdot, \theta) = 0 \quad \text{in } (0, c_\theta) \quad \text{if } c_\theta > 0. \tag{1.5.36}$$

It remains to show (1.4.9). By (1.5.34), $\dot\gamma \in H$ for a.e. $\theta \geq 0$. Thus, for all $\tilde\gamma \in H$ and a.e. $\theta \geq 0$ we have

$$\begin{aligned}
a(\gamma, \tilde\gamma - \dot\gamma) &\overset{(1.5.32)}{=} \int_c^1 (\theta r - 1)(\tilde\gamma - \dot\gamma) r \mathrm{d}r \\
&= \int_0^1 \theta r^2 (\tilde\gamma - \dot\gamma)\mathrm{d}r - \int_0^c \theta r^2 (\tilde\gamma - \dot\gamma)\mathrm{d}r - \int_c^1 (\tilde\gamma - \dot\gamma) r\mathrm{d}r \\
&\overset{(1.4.7),(1.5.36)}{=} \langle \ell(\theta), \tilde\gamma - \dot\gamma\rangle - \int_0^c \theta r^2 \tilde\gamma \mathrm{d}r - \int_c^1 \tilde\gamma r \mathrm{d}r + \int_c^1 \dot\gamma r\mathrm{d}r \\
&\overset{(1.5.35),(1.5.36)}{\geq} \langle \ell(\theta), \tilde\gamma - \dot\gamma\rangle - \int_0^c \theta r^2 |\tilde\gamma| \mathrm{d}r - \int_c^1 |\tilde\gamma| r \mathrm{d}r + \int_0^1 |\dot\gamma| r\mathrm{d}r \\
&\overset{(1.5.33)}{\geq} \langle \ell(\theta), \tilde\gamma - \dot\gamma\rangle - \int_0^1 |\tilde\gamma| r \mathrm{d}r + \int_0^1 |\dot\gamma| r\mathrm{d}r.
\end{aligned}$$

To prove uniqueness, let γ_1 and γ_2 be two solutions with the same initial condition. Then, for $\overline{\gamma} = \gamma_1 - \gamma_2$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{a(\overline{\gamma}, \overline{\gamma})}{2} &= a(\overline{\gamma}, \dot{\overline{\gamma}}) = a(\gamma_1, \dot\gamma_1 - \dot\gamma_2) - a(\gamma_2, \dot\gamma_1 - \dot\gamma_2) \\
&= -a(\gamma_1, \dot\gamma_2 - \dot\gamma_1) - a(\gamma_2, \dot\gamma_1 - \dot\gamma_2) \\
&\leq -\langle \ell(\theta), \dot\gamma_2 - \dot\gamma_1\rangle + \mathscr{D}(\dot\gamma_2) - \mathscr{D}(\dot\gamma_1) - \langle \ell(\theta), \dot\gamma_1 - \dot\gamma_2\rangle + \mathscr{D}(\dot\gamma_1) - \mathscr{D}(\dot\gamma_2) \\
&= 0
\end{aligned}$$

and the result follows from the coercivity of $a(\cdot, \cdot)$.

1.6 Formal asymptotic for $\lambda \ll 1$

For a fixed θ, we let $\gamma(\cdot) := \gamma(\cdot, \theta)$ denote the solution characterized in Theorem 1.1, and c_θ as in Lemma 1.3.

1.6.1 The bulk

We expand γ and (for $\theta < \theta_\lambda$) c_θ in powers of $\lambda \ll 1$:

$$\gamma = \gamma_0 + \lambda\gamma_1 + \dots, \quad c_\theta = c_0 + \lambda c_1 + \dots.$$

At leading order, we see from (1.5.5) and (1.5.6) that

$$\gamma_0(r) = (\theta r - 1)_+ \quad \text{and} \quad c_0 = 1/\theta.$$

Due to the incompatibility of γ_0 with the boundary conditions at $r = 1/\theta$ and at $r = 1$, a boundary layer will form near each of the two points. In this section we address these local behaviors, and we use the former to determine the leading order value of the torque for $\lambda \ll 1$.

1.6.2 The boundary layer near the free boundary

For $\theta < \theta_\lambda$, we zoom into the free boundary with the help of the change of variables

$$\gamma(r) = \lambda g(x), \quad x := \frac{r - c_\theta}{\lambda} \quad \text{(i.e. } r = c_\theta + \lambda x \text{ and } \tfrac{d}{dx} = \lambda \tfrac{d}{dr}\text{)},$$

which leaves the slope invariant: $\gamma'(r) = g_x(x)$. Therefore we will use

$$\theta = \gamma_0'(r_0) = \lim_{x\to+\infty} g_x(x) \quad \text{for all } r_0 \in (1/\theta, 1) \tag{1.6.1}$$

in order to match g with the bulk solution, γ_0. Neglecting the condition at $r = 1$, (1.5.5) reads as

$$\begin{cases} g_{xx} + \lambda(c_\theta + \lambda x)^{-1} g_x - \lambda^2(c_\theta + \lambda x)^{-2} g - g = \frac{1}{\lambda}(1 - \theta(c_\theta + \lambda x)) \\ g(0) = g_x(0) = 0. \end{cases} \tag{1.6.2}$$

We expand g and c in powers of λ: $g = g_0 + \lambda g_1 + \ldots$, $c = 1/\theta + \lambda c_1 + \ldots$. At leading order in λ we have

$$\begin{cases} (g_0)_{xx} - g_0 = \frac{1}{\lambda}(1 - \theta(c_0 + \lambda c_1 + \lambda x)) = -\theta(c_1 + x) \\ g_0(0) = (g_0)_x(0) = 0. \end{cases} \tag{1.6.3}$$

The general solution of the ODE $(1.6.3)_1$ is $g_0 = \theta(c_1 + x) + ae^x + be^{-x}$; the initial conditions $(1.6.3)_2$ yield:

$$g_0 = \theta(c_1 + x) - \tfrac{1}{2}\theta(1 + c_1)e^x + \tfrac{1}{2}\theta(1 - c_1)e^{-x}.$$

Using the matching condition (1.6.1) yields $c_1 = -1$. Therefore

$$c_\theta = \frac{1}{\theta} - \lambda + O(\lambda^2) \quad \text{and} \quad g = \theta(x - 1 + e^{-x}) + O(\lambda) \quad \text{for } \lambda \ll 1 \text{ and } \theta < \theta_\lambda, \tag{1.6.4}$$

the former coinciding with (1.1.18).

As we will see, in order to quantify the dependence of the torque on λ we need to work out the next order correction to c_θ. It follows from (1.6.2) and (1.6.4) that

$$\begin{cases} (g_1)_{xx} + \theta(g_0)_x - g_1 = -\theta c_2 \\ g_1(0) = (g_1)_x(0) = 0. \end{cases} \tag{1.6.5}$$

The general solution of the ODE $(1.6.5)_1$ is $g_1 = ae^x + be^{-x} + \theta^2 + \theta c_2 - \theta^2 xe^{-x}/2$; the initial conditions $(1.6.5)_2$ yield

$$g_1 = -\left(\frac{\theta^2}{4} + \frac{\theta c_2}{2}\right)e^x - \left(\frac{3\theta^2}{4} + \frac{\theta c_2}{2}\right)e^{-x} + \theta^2 + \theta c_2 - \frac{\theta^2}{2}xe^{-x}.$$

The matching condition for the slope requires $(g_1)_x$ be bounded as $x \to +\infty$. Hence $c_2 = -\theta/2$ and

$$c_\theta \sim \frac{1}{\theta} - \lambda - \frac{\theta}{2}\lambda^2 \quad \text{for } \lambda \ll 1 \text{ and } \theta < \theta_\lambda. \tag{1.6.6}$$

1.6.3 The boundary layer near $r = 1$

Here we motivate the expansion (1.1.19). We zoom into $r = 1$ with the help of the change of variables

$$\gamma(r) = \theta - 1 + \lambda g(x), \quad x = \frac{1-r}{\lambda} \quad (\text{i.e. } r = 1 - \lambda x \text{ and } \tfrac{d}{dx} = -\lambda \tfrac{d}{dr}),$$

which again leaves the slope invariant (up to the sign). Therefore we'll use as matching condition:

$$\lim_{x \to +\infty} g_x(x) = -\theta.$$

Neglecting the boundary conditions at $r = c_\theta$, it follows from (1.5.5) and (1.5.6) that

$$\begin{cases} g_{xx} - \lambda(1-\lambda x)^{-1} g_x - \lambda^2 (1-\lambda x)^{-2} g - g = \theta x + \lambda(1-\lambda x)^{-2}(\theta - 1) \\ g_x(0) = \frac{1}{2}(\theta - 1 + \lambda g(0)). \end{cases}$$

We expand g in powers of λ: $g = g_0 + \lambda g_1 + \ldots$. At leading order in λ we have

$$\begin{cases} (g_0)_{xx} - g_0 = \theta x \\ g_0'(0) = \frac{1}{2}(\theta - 1). \end{cases}$$

The general solution and the matching condition yield $g_0 = -\theta x + \frac{1}{2}(1 - 3\theta)e^{-x}$. In terms of the original variables,

$$\gamma \sim \theta - 1 - \lambda\theta x + \frac{1}{2}\lambda(1-3\theta)e^{-x} = \theta - 1 - \theta(1-r) + \frac{1}{2}\lambda(1-3\theta)e^{-\frac{1-r}{\lambda}}$$

and (1.1.19) follows. In particular,

$$\gamma(1) \sim \theta - 1 - \frac{1}{2}\lambda(3\theta - 1) \quad \text{for } \lambda \ll 1. \tag{1.6.7}$$

1.6.4 The asymptotic for the torque

For $1 < \theta < \theta_\lambda$, we have

$$\begin{aligned} q(\theta) &\overset{(1.1.16)}{=} 3\int_0^1 (\theta r - \gamma) r^2 \mathrm{d}r \\ &\overset{(1.4.3)}{=} 3\int_0^{c_\theta} \theta r^3 \mathrm{d}r + 3\int_{c_\theta}^1 \left(r^2 - \lambda^2 (r^2\gamma'' + r\gamma' - \gamma)\right) \mathrm{d}r \\ &= \frac{3\theta}{4} c_\theta^4 + 1 - c_\theta^3 - 3\lambda^2 \left([r^2\gamma']_{c_\theta}^1 - \int_{c_\theta}^1 (r\gamma' + \gamma) \mathrm{d}r \right) \\ &\overset{(1.4.3)}{=} 1 - \frac{c_\theta^3}{4}(4 - 3c_\theta\theta) + \lambda^2 \frac{9}{2}\gamma(1). \end{aligned}$$

For $\theta > \theta_\lambda$, we have instead

$$\begin{aligned} q(\theta) &\overset{(1.1.16)}{=} 3\int_0^1 (\theta r - \gamma) r^2 \mathrm{d}r \\ &\overset{(1.4.3)}{=} 3\int_0^1 \Big(r^2 - \lambda^2(r^2\gamma'' + r\gamma' - \gamma)\Big)\mathrm{d}r \\ &\overset{(1.4.3)}{=} 1 + \lambda^2 \frac{9}{2}\gamma(1). \end{aligned}$$

Plugging (1.6.6) and (1.6.7) (in its leading order form, $\gamma(1) = (\theta - 1)(1 + o(1))$) yields, after straightforward computations,

$$q(\theta) \sim \begin{cases} 1 - \frac{1}{4\theta^3} + \frac{3\lambda^2}{2\theta} + \frac{9}{2}\lambda^2(\theta - 1) & \text{if } \theta \in (1, \theta_\lambda) \\ 1 + \frac{9\lambda^2}{2}(\theta - 1) & \text{if } \theta > \theta_\lambda \end{cases} \qquad \text{for } \lambda \ll 1.$$

Note that the $O(\lambda)$-term in the expansion for $\theta < \theta_\lambda$ vanishes, which points for the aforementioned necessity of a second-order expansion of c_θ. Since $\gamma \in \mathrm{Lip}([0, \infty); H)$, q need be continuous across $\theta = \theta_\lambda$: therefore

$$-\frac{1}{4\theta_\lambda^3} + \frac{3\lambda^2}{2\theta_\lambda} \ll 1 \iff \theta_\lambda \sim \frac{1}{\sqrt{6}\lambda},$$

which yields (1.1.20) and (1.1.21).

1.7 Plastic spin

In the small–strain theory proposed by Gurtin in [60], the plastic distortion $\mathbf{H}^{\mathrm{p}} := \nabla\mathbf{u} - \mathbf{H}^{\mathrm{e}}$ is not symmetric:

$$\mathbf{H}^{\mathrm{p}} = \mathbf{E}^{\mathrm{p}} + \mathbf{W}^{\mathrm{p}}, \quad \mathbf{E}^{\mathrm{p}} \text{ symmetric}, \quad \mathbf{W}^{\mathrm{p}} \text{ skew-symmetric}. \tag{1.7.1}$$

The defect energy and effective flow rate considered in [60] are:

$$\psi_{\mathrm{d}} = \tfrac{1}{2}\mu L^2 |\mathrm{curl}\mathbf{H}^{\mathrm{p}}|^2, \quad \text{resp.} \quad d^{\mathrm{p}} = \sqrt{|\dot{\mathbf{E}}^{\mathrm{p}}|^2 + \chi|\dot{\mathbf{W}}^{\mathrm{p}}|^2 + \ell^2|\nabla\dot{\mathbf{E}}^{\mathrm{p}}|^2}, \tag{1.7.2}$$

where $\chi > 0$ is a constitutive parameter that measures the importance of dissipation associated to plastic rotations (see also [9] for a discussion in the case of simple shear). Note that $(1.7.2)_1$ generalizes (1.2.19) in the particular case $\eta = 0$. Within our working assumptions (1.1.5), Gurtin's theory leads to the following flow rule:

$$\mathbf{T}_0 + \mu L^2\left(\Delta\mathbf{H}^{\mathrm{p}} - \nabla\mathrm{div}\mathbf{H}^{\mathrm{p}} + \frac{1}{3}(\mathrm{div}\,\mathrm{div}\mathbf{H}^{\mathrm{p}})\mathbf{I}\right) \in \partial\delta_\chi(\dot{\mathbf{H}}^{\mathrm{p}}), \tag{1.7.3}$$

where now the dissipation is $\delta_\chi(\dot{\mathbf{H}}^p) = \sqrt{2}k\sqrt{|\dot{\mathbf{E}}^p|^2 + \chi|\dot{\mathbf{W}}^p|^2}$ and

$$\partial\delta_\chi(\dot{\mathbf{H}}^p) := \{\mathbf{A}_0 \in \mathbb{R}^{3\times3}_0 : \ \delta_\chi(\widetilde{\mathbf{H}}^p) - \delta_\chi(\dot{\mathbf{H}}^p) \geq \mathbf{A}_0 : (\widetilde{\mathbf{H}}^p - \dot{\mathbf{H}}^p) \ \ \forall\, \widetilde{\mathbf{H}}^p \in \mathbb{R}^{3\times3}_0\}.$$

As announced in the Introduction, a solution of (1.7.3) is readily constructed by taking the solution of (1.1.7) for $\eta = 0$ and by setting $\mathbf{W}^p = 0$. Indeed, if $\mathbf{E}^p$ is given by the ansatz (1.1.9), with γ^p solving (1.3.5)–(1.3.6) with $\eta = 0$ and $\mathbf{W}^p = 0$, then

$$\partial\delta_\chi(\dot{\mathbf{H}}^p) = \partial\delta_\chi(\dot{\mathbf{E}}^p) := \{\mathbf{A}_0 \in \mathbb{R}^{3\times3}_0 : \ \delta_\chi(\widetilde{\mathbf{H}}^p) - \delta_\chi(\dot{\mathbf{E}}^p) \geq \mathbf{A}_0 : (\widetilde{\mathbf{H}}^p - \dot{\mathbf{E}}^p) \ \ \forall\, \widetilde{\mathbf{H}}^p \in \mathbb{R}^{3\times3}_0\}.$$

We decompose $\widetilde{\mathbf{H}}^p$ as in (1.7.1) and we use that $\delta_\chi(\dot{\mathbf{E}}^p) = \delta(\dot{\mathbf{E}}^p)$, that $\delta(\widetilde{\mathbf{E}}^p) = \delta(\widetilde{\mathbf{H}}^p) \leq \delta_\chi(\widetilde{\mathbf{H}}^p)$ and that $\mathbf{A}_0 : \widetilde{\mathbf{H}}^p = \mathbf{A}_0 : \widetilde{\mathbf{E}}^p$ if $\mathbf{A}_0 \in \mathbb{R}^{3\times3}_{0,sym}$: then

$$\begin{aligned}
\partial\delta_\chi(\dot{\mathbf{H}}^p) &\supset \{\mathbf{A}_0 \in \mathbb{R}^{3\times3}_{0,sym} : \ \delta(\widetilde{\mathbf{E}}^p) - \delta(\dot{\mathbf{E}}^p) \geq \mathbf{A}_0 : (\widetilde{\mathbf{E}}^p - \dot{\mathbf{E}}^p) \ \ \forall\, \widetilde{\mathbf{H}}^p \in \mathbb{R}^{3\times3}_0\} \\
&= \{\mathbf{A}_0 \in \mathbb{R}^{3\times3}_{0,sym} : \ \delta(\widetilde{\mathbf{E}}^p) - \delta(\dot{\mathbf{E}}^p) \geq \mathbf{A}_0 : (\widetilde{\mathbf{E}}^p - \dot{\mathbf{E}}^p) \ \ \forall\, \widetilde{\mathbf{E}}^p \in \mathbb{R}^{3\times3}_{0,sym}\} \\
&\overset{(1.1.6)}{=} \partial\delta(\dot{\mathbf{E}}^p) \overset{(1.3.8)}{\ni} \mathbf{T}_0 + \mu L^2 \Delta\mathbf{E}^p \overset{(1.3.7)}{\ni} \mathbf{T}_0 + \mu L^2 \left(\Delta\mathbf{E}^p - \nabla\mathrm{div}\mathbf{E}^p + \tfrac{1}{3}(\mathrm{div}\,\mathrm{div}\mathbf{E}^p)\mathbf{I}\right) \\
&= \mathbf{T}_0 + \mu L^2 \left(\Delta\mathbf{H}^p - \nabla\mathrm{div}\mathbf{H}^p + \tfrac{1}{3}(\mathrm{div}\,\mathrm{div}\mathbf{H}^p)\mathbf{I}\right) \quad \text{(since } \mathbf{W}^p = 0).
\end{aligned}$$

Chapter 2

Mass constrained minimization of a one-homogeneous functional arising in strain-gradient plasticity

2.1 Introduction

2.1.1 The model

This chapter stems from the following conjecture, formulated in [6]:

Conjecture. *Let $I = (-\alpha, \alpha)$. The functional*

$$F(u) = \fint_I \sqrt{u^2 + \ell^2 (u')^2}\, \mathrm{d}x \tag{2.1.1}$$

(*a*) *has a minimum, τ_ℓ, over all u such that*

$$u|_{\partial I} = 0, \qquad \fint_I u \,\mathrm{d}x = 1\ ; \tag{2.1.2}$$

(*b*) *any minimum u is a solution of*

$$\tau_\ell = \frac{u}{\sqrt{u^2 + \ell^2 (u')^2}} - \ell^2 \left(\frac{u'}{\sqrt{u^2 + \ell^2 (u')^2}} \right)' . \tag{2.1.3}$$

The conjecture originates from a strain-gradient theory of plasticity introduced by Gurtin in [60] and developed by Anand and Gurtin in [61] (see also [62]).

In order to investigate the role of the dissipative length-scale ℓ, it is convenient to look at a reduced one-dimensional model, introduced in [6], which alleviates most of the intricacies of the full model in [60, 61] but yet may allow to extract its essence: it describes the plastic strain γ in a strip of finite width $|I|$ undergoing simple shear with shear stress τ. In the case of null internal-variable hardening (which is of interest here) and after a suitable rescaling, this model leads to the following evolution equation for (τ, γ):

$$\begin{cases} \tau + L^2 \partial_x^2 \gamma = \frac{\partial_t \gamma}{d} - \ell^2 \partial_x \left(\frac{\partial_x \partial_t \gamma}{d} \right) \\ \partial_x \tau = 0 \end{cases} \qquad (t, x) \in (0, \infty) \times I, \tag{2.1.4}$$

where

$$d = \sqrt{|\partial_t \gamma|^2 + \ell^2 |\partial_x \partial_t \gamma|^2}$$

which holds provided the strip is "fully plastified", i.e. $\partial_t \gamma > 0$ in $\mathring{I}$. The evolution is complemented by initial-boundary conditions of the form

$$\partial_t \gamma|_{\partial I} = 0, \quad \gamma(0, x) = \gamma_0(x),$$

and by either "displacement" or "traction" condition. The latter, which is of interest here, amounts to prescribe the stress τ at the boundary of I (and hence everywhere since, in view of $(2.1.4)_2$, τ is spatially constant).

Setting $L = 0$ allows to isolate and analyze the dependence of the flow on the dissipative length-scale ℓ. Assuming $m = 0$, imposing a constant (in space *and* time) traction τ_ℓ, and letting $u = \partial_t \gamma$, the evolution (2.1.4) reduces to (2.1.3) (with primes denoting differentiation with respect to x).

Using the scale invariance $u \mapsto au$ $(a \neq 0)$, we may normalize the mean of the plastic flow to one. Such normalization leads to the problem considered in the conjecture and is natural in order to capture scale effects. Indeed, we can then say that a sample is *stronger* than a second one (made of the same material) if a higher stress τ_ℓ is needed to generate the same mean plastic flow. On the other hand, of course a sample is *smaller* than a second one if the ratio $\ell/|I|$ is higher. Hence, *smaller is stronger* is equivalent to say that

(c) τ_ℓ is increasing with $\ell/|I|$.

This is exactly what the numerical simulations performed in [6] indicate.

2.1.2 Main results

The goal of this part is to provide a rigorous validation of (a), (b), and (c). By the rescaling $x \mapsto x/\ell$, we may assume without loss of generality that $\ell = 1$. In some cases, it will be harmless to work in a bounded, open and connected set $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary rather than in an interval $I \subset \mathbb{R}$. Given $u : \Omega \to \mathbb{R}$, $\overline{u}$ denotes its extension by zero:

$$\overline{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

First of all, we extend the functional F given by (2.1.1) to $L^1(\Omega)$ and we encode the boundary conditions $(2.1.2)_1$: let $F^\circ : L^1(\Omega) \to [0, +\infty]$ be defined as

$$F^\circ(u) = \begin{cases} \int_\Omega \sqrt{u^2 + |\nabla u|^2}\, \mathrm{d}x & \text{if } u \in W_0^{1,1}(\Omega) \\ +\infty & \text{if } u \in L^1(\Omega) \setminus W_0^{1,1}(\Omega) \end{cases} \tag{2.1.5}$$

We recall that the relaxation $\overline{G}$ of a functional $G : L^1(\Omega) \to [0, +\infty]$ is defined by

$$\overline{G}(u) = \inf \left\{ \liminf_{n\to+\infty} G(u_n) : \; u_n \in L^1(\Omega), \; u_n \to u \text{ in } L^1(\Omega) \right\} \tag{2.1.6}$$

and that $\overline{G}$ is lower semi-continuous with respect to the $L^1(\Omega)$-topology. The relaxation of F° is characterized as follows: [1]

Theorem 2.1. *Let F° be defined by* (2.1.5). *Then*

$$\overline{F^\circ}(u) = \begin{cases} \int_\Omega \sqrt{u^2 + |\nabla u|^2}\, \mathrm{d}x + |D^s u|(\Omega) + \int_{\partial\Omega} |u| \mathrm{d}\mathcal{H}^{N-1} & \textit{if } u \in BV(\Omega) \\ +\infty & \textit{if } u \in L^1(\Omega) \setminus BV(\Omega) \,. \end{cases} \tag{2.1.7}$$

Furthermore, for all $u \in BV(\Omega)$ it holds:

$$\overline{F^\circ}(u) = |(\overline{u}, D\overline{u})|(\mathbb{R}^N) \;=\; |(u, Du)|(\Omega) + \int_{\partial\Omega} |u| \mathrm{d}\mathcal{H}^{N-1} \tag{2.1.8}$$

$$= \sup \left\{ \int_\Omega u\,(s - \operatorname{div} \mathbf{t})\, \mathrm{d}x + \int_{\partial\Omega} u\mathbf{t}\cdot\mathbf{n} \mathrm{d}\mathcal{H}^{N-1} : (s, \mathbf{t}) \in C^\infty(\overline{\Omega}),\; \|(s, \mathbf{t})\|_\infty \le 1 \right\}. \tag{2.1.9}$$

Let

$$BV_*(\Omega) = \left\{ u \in BV(\Omega) : \; \fint_\Omega u\, \mathrm{d}x = 1 \right\}.$$

The positive answer to part (a) of the conjecture follows from Theorem 2.1 and standard direct methods (for related results see, for instance, [5, 37] and the references quoted therein):

Corollary 2.1 (Existence of minimizers)**.** *There exists at least one minimizer of $\overline{F^\circ}$ among all $u \in BV_*(\Omega)$.*

In order to introduce the results concerning parts (b) and (c) of the conjecture, it is convenient to have the notion of sub-differential at hand. To this aim, we let

$$p = \max\{2, N\}, \qquad q = \frac{p}{p-1},$$

[1] Here $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure (with a slight abuse of notation, we hereafter identify u with its trace on $\partial\Omega$), $\mathbf{n}$ is the outward unit normal to $\partial\Omega$, ∇u and $D^s u$ denote the absolutely continuous part and the singular part of Du with respect to the Lebesgue measure, respectively, $|\mu|$ denotes the total variation of a measure μ (see [5, Definition 1.4]), $(s, \mathbf{t})$ denotes the $\mathbb{R}^{N+1}$-vector $(s, t_1, \dots, t_N)$, and $(\overline{u}, D\overline{u})$ denotes the $\mathbb{R}^{N+1}$-valued measure $(\overline{u}\mathcal{L}^N, D_1\overline{u}, \dots, D_N\overline{u})$ (with another slight abuse of notation, we hereafter identify a $\mathcal{L}^N$-integrable function $u \in L^1(\Omega, \mathbb{R}^N)$ with the measure $u\mathcal{L}^N$).

and we hereafter consider the functional $\phi : L^q(\Omega) \to [0,+\infty]$ defined by

$$\phi(u) = \begin{cases} |(\overline{u}, D\overline{u})|(\mathbb{R}^N) & \text{if } u \in BV(\Omega) \\ +\infty & \text{if } u \in L^q(\Omega) \setminus BV(\Omega)\,. \end{cases} \tag{2.1.10}$$

The subdifferential of ϕ at u_m, denoted by $\partial\phi(u_m) \subset L^p(\Omega)$, is defined by:

$$u^* \in \partial\phi(u_m) \iff \int_\Omega u^*(u-u_m)\,\mathrm{d}x + \phi(u_m) \le \phi(u) \quad \text{for all } u \in L^q(\Omega). \tag{2.1.11}$$

Remark 2.1. *Classical embedding theorems imply that $BV(\Omega) \subset L^q(\Omega) \subset L^1(\Omega)$, so that the functional ϕ coincides with the restriction to $L^q(\Omega)$ of the relaxation $\overline{F^\circ}$. Hence ϕ and $\overline{F^\circ}$ have the same minimum and the same minimizers in $BV_*(\Omega)$, and the identifications* (2.1.8) *and* (2.1.9) *continue to hold.*

We start the discussion on (b) and (c) with a characterization of the minimum value, which crucially relies on the 1-homogeneity of ϕ:

Proposition 2.1. *Let $\tau_\Omega := \frac{1}{|\Omega|} \min_{BV_*(\Omega)} \phi$. Then*

$$u_m \in \operatorname*{argmin}_{BV_*(\Omega)} \phi \iff \begin{cases} u_m \in BV_*(\Omega) \\ \tau_\Omega \chi_\Omega \in \partial\phi(u_m). \end{cases}$$

Here χ_Ω denotes the characteristic function of the set Ω. Proposition 2.1 already provides a weak answer to (b): τ_Ω, seen as a Lagrange multiplier for the constrained minimization problem, is uniquely determined over all possible minimizers, a fact which would yield (b) if one could identify $\partial\phi$ with the right-hand side of (2.1.3). We slightly postpone this discussion, and we first notice that the characterization of τ_Ω given in Proposition 2.1 already allows to justify (c) through a scaling argument:

Theorem 2.2 ("Smaller is stronger"). *Let*

$$\lambda\Omega = \{x \in \mathbb{R}^N : \ x/\lambda \in \Omega\}.$$

The function $\lambda \mapsto \tau_{\lambda\Omega}$ is decreasing (strictly if $N = 1$).

Let us now return to part (b) of the conjecture. In one space dimension, where the conjecture is formulated, we are able to give a complete answer to part (b) in the space

$$SBV_*(I) = BV_*(I) \cap SBV(I)\,,$$

where $u \in SBV(I)$ if and only if $u \in BV(I)$ and the singular part of its variation is given only by the jump part. This means that

$$\mathrm{d}^s u = \sum_{i\in\mathbf{N}} [u(x_i^+) - u(x_i^-)]\delta_{x_i},$$

where x_i are the jump points of u, δ_x is the Dirac mass concentrated on x, and $u(x_i^\pm)$ are the left, resp. right, limits at x_i. We prove the following:

Theorem 2.3. *The functional ϕ given by* (2.1.10) *has a unique minimizer $u \in SBV_*(I)$. The minimizer u is even, strictly decreasing in* $[0, \alpha)$*, smooth in I, and it solves the Euler-Lagrange equation* (2.1.3) *(with $\ell = 1$ and $\tau_\ell = \tau_I$ defined by Proposition 2.1). Furthermore*

$$\lim_{x \to \alpha^-} u(x) > 0 \quad \textit{and} \quad \lim_{x \to \alpha^-} u'(x) = -\infty.$$

Note in particular that u jumps at ∂I, in the sense that it does not attain the boundary value zero at ∂I: this observation confirms the numerical simulations performed in [6].

Besides non-generic domains (such as an N-sphere, where we expect results similar to those in Theorem 2.3 to hold), we believe that the multi-dimensional problem will not have smooth minimizers in general, as the mass constraint may produce solutions which jump down to zero in the interior. Hence, in general the corresponding Euler-Lagrange equation will not be satisfied by minimizers. However, yet it is possible to characterize the subdifferential $\partial\phi$ at least in the sense of distributions, as it has been done for other problems with linear growth in the gradient [7, 8]. To this aim, we let

$$X(\Omega) = \left\{ \mathbf{z} \in (L^\infty(\Omega))^N : \operatorname{div} \mathbf{z} \in L^p(\Omega) \right\}$$

and we recall that for any $u \in BV(\Omega)$ and any $\mathbf{z} \in X(\Omega)$ the functional $(\mathbf{z}, Du) : C_c^\infty(\Omega) \to \mathbb{R}$ defined by

$$\langle (\mathbf{z}, Du), \varphi \rangle = - \int_\Omega u\varphi \operatorname{div} \mathbf{z} \, \mathrm{d}x - \int_\Omega u\mathbf{z} \cdot \nabla\varphi \, \mathrm{d}x$$

is a Radon measure which is absolutely continuous with respect to $|Du|$ (see [7, §C.2]). Furthermore, the trace $[\mathbf{z}, \mathbf{n}] \in L^\infty(\partial\Omega)$ of the normal component of $\mathbf{z} \in X(\Omega)$ is well defined (see §2.7). We may now state the characterization of $\partial\phi$:

Theorem 2.4 (Characterization of $\partial\phi$)**.** *Let $u \in BV(\Omega)$ and $v \in L^p(\Omega)$. Then $v \in \partial\phi(u)$ if and only if there exists $(s, \mathbf{z}) \in L^\infty(\Omega) \times X(\Omega)$ such that:*

(*i*) $\|(s, \mathbf{z})\|_\infty \leq 1$;

(*ii*) $v = s - \operatorname{div} \mathbf{z}$ *in* $L^p(\Omega)$;

(*iii*) $\phi(u) = \displaystyle\int_\Omega su \, \mathrm{d}x + \int_\Omega \mathrm{d}(\mathbf{z}, Du) + \int_{\partial\Omega} u[\mathbf{z}, \mathbf{n}] \, \mathrm{d}\mathcal{H}^{N-1} = \int_\Omega uv \, \mathrm{d}x.$

A recent discussion on the existence, the (non-)uniqueness, and the Euler-Lagrange equation of minimizers of the total variation with mass and side constraints may be found in [68]. The thesis is organized as follows. In Section 2.2 we give the proofs of Theorem 2.1 and Corollary 2.1. In §2.3 we prove Proposition 2.1 and Theorem 2.2. In §2.4 we look at the uniqueness part of Theorem 2.3. Namely, we show that any minimizer is positive in I and can not have jump points in I: these two properties, combined with the (not strict) convexity of ϕ, suffice to give uniqueness of the minimizer in SBV. In §2.5 we prove the regularity part of Theorem 2.3. The crucial observation is that, though solutions to the Euler-Lagrange equation need not be concave, their square root does: this gives an a-priori Lipschitz bound in the interior for solutions of suitable approximating problems, and thus a smooth minimizer in the limit. In §2.6 we use ode methods to characterize the solutions of (2.1.3): as a consequence, we show that u jumps and u' blows up at the boundary. Finally, in §2.7 we prove Theorem 2.4.

2.2 Relaxation results and existence of a minimizer

In this section we prove Theorem 2.1 and Corollary 2.1. We begin with the counterpart of Theorem 2.1 when boundary conditions are neglected.

Lemma 2.1. *Let $A \subset \mathbb{R}^N$ be an open, bounded set, and let $F_A : L^1(A) \to [0, +\infty]$ be defined as*

$$F_A(u) = \begin{cases} \int_A \sqrt{u^2 + |\nabla u|^2}\,\mathrm{d}x & \text{if } u \in W^{1,1}(A) \\ +\infty & \text{if } u \in L^1(A) \setminus W^{1,1}(A). \end{cases} \tag{2.2.1}$$

Then its relaxation $\overline{F_A}$ (see (2.1.6)) is characterized by

$$\overline{F_A}(u) = \begin{cases} \int_A \sqrt{u^2 + |\nabla u|^2}\,\mathrm{d}x + |D^s u|(A) & \text{if } u \in BV(A) \\ +\infty & \text{if } u \in L^1(A) \setminus BV(A)\,. \end{cases} \tag{2.2.2}$$

Proof. The result is an immediate consequence of Theorem 3.2 in [36]. □

In order to prove Theorem 2.1,we need to notice that F satisfies the so-called "fundamental estimate" (see [37, Def. 18.2]).

Lemma 2.2. *Let $\mathcal{A}$ be the family of open subsets of Ω. For every $\varepsilon > 0$ and for every $A', A'', B \in \mathcal{A}$, with $A' \Subset A''$, there exists a constant $M > 0$ with the following property: for every $u, v \in L^p(\Omega)$ there exists a cut-off[2] φ between A' and A'' such that:*

$$F_{A'\cup B}(\varphi u + (1-\varphi)v) \leq (1+\varepsilon)[F_{A''}(u) + F_B(v)] + \varepsilon(\|u\|_{L^1(S)} + \|v\|_{L^1(S)} + 1) + M\|u - v\|_{L^1(S)}, \tag{2.2.3}$$

where $S = (A'' \setminus A') \cap B$.

Proof. The lemma is an immediate consequence of [37, Theorem 19.1] with $g(\xi) = |\xi|$, $c_1 = c_2 = c_3 = c_4 = 1$, and $a \equiv 0$. □

Now we can extend the result in Lemma 2.1 to the case of homogeneous Dirichlet boundary conditions.

[2] a cut-off φ between A' and A'' is a function $\varphi \in C_0^\infty(A'')$such that $0 \leq \varphi \leq 1$ in A'' and $\varphi = 1$ in a neighborhood of $\overline{A'}$.

Proof of Theorem 2.1. Let

$$\Phi(u) := \begin{cases} \int_\Omega \sqrt{u^2+|\nabla u|^2}\,\mathrm{d}x + |D^s u|(\Omega) + \int_{\partial\Omega} |u|\mathrm{d}\mathcal{H}^{N-1} & \text{if } u \in BV(\Omega) \\ +\infty & \text{if } u \in L^1(\Omega)\setminus BV(\Omega)\,. \end{cases} \tag{2.2.4}$$

First of all we prove the equivalences in (2.1.8) and (2.1.9), i.e. that

$$\Phi(u) = |(\overline{u}, D\overline{u})|(\mathbb{R}^N) \;=\; |(u, Du)|(\Omega) + \int_{\partial\Omega} |u|\mathrm{d}\mathcal{H}^{N-1} \tag{2.2.5}$$

$$= \;\sup\left\{\int_\Omega u\,(s - \operatorname{div}\mathbf{t})\,\mathrm{d}x + \int_{\partial\Omega} u\mathbf{t}\cdot\mathbf{n}\mathrm{d}\mathcal{H}^{N-1} : (s,\mathbf{t}) \in C^\infty(\Omega),\; \|(s,\mathbf{t})\|_\infty \le 1\right\} \tag{2.2.6}$$

for all $u \in BV(\Omega)$. By the Radon-Nikodým Theorem [5, Theorem 1.28], $(\overline{u}, D\overline{u})$ may be uniquely decomposed into the sum $(\overline{u}, \nabla\overline{u}) + (0, D^s\overline{u})$, which are absolutely continuous, resp. singular, with respect to $\mathcal{L}^{N+1}$. Since the two measures are mutually singular, we obtain

$$|(\overline{u}, D\overline{u})|(\mathbb{R}^N) = \int_\Omega \sqrt{u^2+|\nabla u|^2}\mathrm{d}x + |D^s\overline{u}|(\mathbb{R}^N). \tag{2.2.7}$$

Furthermore (see [5, Corollary 3.89])

$$|D^s\overline{u}|(\mathbb{R}^N) = |D^s u|(\Omega) + \left|u\mathbf{n}\mathcal{H}^{N-1}\right|(\partial\Omega) = |D^s u|(\Omega) + \int_{\partial\Omega} |u|\mathrm{d}\mathcal{H}^{N-1}$$

and the first equality in (2.2.5) follows. The proof of the second one is even simpler and we omit it. For the latter, we just need to recall the characterization of the total variation of a Radon measure [5, Proposition 1.47],

$$|(\overline{u}, D\overline{u})|(\mathbb{R}^N) = \sup\left\{\int_{\mathbb{R}^N} (\overline{u}s\mathrm{d}x + \mathbf{t}\cdot\mathrm{d}D\overline{u}) :\; (s,\mathbf{t}) \in C_c^\infty(\mathbb{R}^N),\; \|(s,\mathbf{t})\|_\infty \le 1\right\},$$

and, since $\overline{u}$ is supported in $\overline{\Omega}$, the integration by parts' formula for BV functions [5, (3.85)]:

$$\int_{\mathbb{R}^N} \mathbf{t}\cdot\mathrm{d}D\overline{u} = \int_\Omega \mathbf{t}\cdot\mathrm{d}Du = \int_{\partial\Omega} u\mathbf{t}\cdot\mathbf{n}\mathrm{d}\mathcal{H}^{N-1} - \int_\Omega u\operatorname{div}\mathbf{t}\,\mathrm{d}x.$$

We now show that $\Phi = \overline{F^\circ}$. Firstly we prove that $\Phi \le \overline{F^\circ}$. For this, it suffices to show that Φ is lower semi-continuous. Indeed, since $\Phi(u) \le F^\circ(u)$ for all $u \in L^1(\Omega)$, we then have that

$$\Phi(u) \le \liminf_{n\to+\infty} \Phi(u_n) \le \liminf_{n\to+\infty} F^\circ(u_n) \quad \text{for all } u_n \to u \in L^1(\Omega).$$

Let A be an open ball such that $\overline{\Omega} \subset A$. We have

$$\Phi(u) \overset{(2.2.4)}{=} \begin{cases} \int_A \sqrt{|\overline{u}|^2+|\nabla\overline{u}|^2}\mathrm{d}x + |D^s\overline{u}|(A) & \text{if } u \in BV(\Omega) \\ +\infty & \text{if } u \in L^1(\Omega)\setminus BV(\Omega). \end{cases}$$

It follows from Lemma 2.1 that $\Phi(u) \equiv \overline{F_A}(\overline{u})$ for all $u \in L^1(\Omega)$, and since $\overline{F_A}$ is lower semi-continuous, also Φ is.

We now prove the opposite inequality, $\overline{F^\circ}(u) \leq \Phi(u)$. If $u \notin BV(\Omega)$ the inequality is trivial. Else, let $\{\tilde{w}_n\} \subset W^{1,1}(\Omega)$ be an optimal sequence for the relaxation of F_Ω, i.e.

$$\tilde{w}_n \to u \text{ in } L^1(\Omega) \quad \text{and} \quad \overline{F_\Omega}(u) = \lim_{n\to+\infty} F_\Omega(\tilde{w}_n). \tag{2.2.8}$$

Let

$$\Omega_n = \{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > 1/n\}$$

(with n sufficiently large so that Ω_n is not empty), let A_n be an open set with Lipschitz boundary such that $\overline{\Omega}_n \subset A_n$ and $\overline{A}_n \subset \Omega_{2n}$, and let $B_n = \Omega \setminus \overline{A}_n$. Lemma 2.1 in [25] (with $A = B_n$, $w = u$ and $\theta = u$) guarantees that a sequence $\{w_{n,k}\} \subset W^{1,1}(B_n)$ exists such that

$$\lim_{k\to+\infty} w_{n,k} = u \text{ in } L^1(B_n), \ \ w_{n,k} = u \text{ on } \partial B_n, \ \text{ and } \limsup_{k\to+\infty} \int_{B_n} |\nabla w_{n,k}| \mathrm{d}x \leq |Du|(B_n). \tag{2.2.9}$$

We apply Lemma 2.2 with $\varepsilon = 1/n$, $A'' = \Omega$, $A' = \Omega_{2n}$, and $B = B_n$. Note that $A' \cup B = \Omega$ and that $S = (A'' \setminus A') \cap B = \Omega \setminus \Omega_{2n} =: S_n$. Then for all n sufficiently large there exists $M_n > 0$ such that for any $k \in \mathbf{N}$ there exists a cut-off $\varphi_{n,k}$ between Ω_{2n} and Ω such that

$$\begin{aligned} F_\Omega(\varphi_{n,k}\tilde{w}_k + (1-\varphi_{n,k})w_{n,k}) \ &\leq \ (1+\frac{1}{n})(F_\Omega(\tilde{w}_k) + F_{B_n}(w_{n,k})) \\ &\quad + \frac{1}{n}\left(\|\tilde{w}_k\|_{L^1(S_n)} + \|w_{n,k}\|_{L^1(S_n)} + 1\right) \\ &\quad + M_n\|\tilde{w}_k - w_{n,k}\|_{L^1(S_n)}. \end{aligned} \tag{2.2.10}$$

Set $z_{n,k} = \varphi_{n,k}\tilde{w}_k + (1-\varphi_{n,k})w_{n,k}$. By definition, $z_{n,k}|_{\partial\Omega} = w_{n,k}|_{\partial\Omega} = u$; in addition,

$$\begin{aligned} \int_\Omega |z_{n,k} - u| \mathrm{d}x \ &= \ \int_{\Omega_{2n}} |\tilde{w}_k - u| \mathrm{d}x + \int_{\Omega\setminus\Omega_{2n}} |\varphi_{n,k}(\tilde{w}_k - u) + (1-\varphi_{n,k})(w_{n,k} - u)| \mathrm{d}x \\ &\leq \ \int_{\Omega_{2n}} |\tilde{w}_k - u| \mathrm{d}x + \int_{B_n} (|\tilde{w}_k - u| + |w_{n,k} - u|) \mathrm{d}x, \end{aligned}$$

hence $z_{n,k} \to u$ in $L^1(\Omega)$ as $k \to +\infty$. Therefore, passing to the limit as $k \to +\infty$ in (2.2.10) we obtain

$$\begin{aligned} \overline{F_\Omega}(u) \ &\overset{(2.1.6)}{\leq} \ \liminf_{k\to+\infty} F_\Omega(z_{n,k}) \leq \limsup_{k\to+\infty} F_\Omega(z_{n,k}) \\ &\overset{(2.2.8),(2.2.9)}{\leq} \ (1+\frac{1}{n})\left(\overline{F_\Omega}(u) + \limsup_{k\to+\infty} F_{B_n}(w_{n,k})\right) + \frac{1}{n}\left(1 + 2\|u\|_{L^1(S_n)}\right), \end{aligned}$$

and since

$$\limsup_{k\to+\infty} F_{B_n}(w_{n,k}) \leq \limsup_{k\to+\infty} \int_{B_n} (|w_{n,k}| + |\nabla w_{n,k}|) \,\mathrm{d}x \overset{(2.2.9)}{\leq} \int_{B_n} |u| \mathrm{d}x + |Du|(B_n),$$

we have that for all n there exists $k(n)$ such that

$$\overline{F_\Omega}(u) - \frac{1}{n} \leq F_\Omega(z_{n,k}) \leq (1 + \frac{1}{n})\left(\overline{F_\Omega}(u) + \int_{B_n} |u| \mathrm{d}x + |Du|(B_n)\right) + \frac{1}{n}\left(2 + 2\|u\|_{L^1(S_n)}\right)$$

for all $k \geq k(n)$. Defining the diagonal sequence $z_n = z_{n,k(n)}$ and using the monotonicity of positive measures (see e.g. [5, Remark 1.3]), we conclude that z_n is also an optimal sequence for the relaxation of F, which in addition attains the boundary value:

$$z_n \to u \text{ in } L^1(\Omega), \quad z_n = u \text{ on } \partial\Omega, \quad \text{and} \quad \overline{F_\Omega}(u) = \lim_{n\to+\infty} F_\Omega(z_n). \tag{2.2.11}$$

On the other hand, again Lemma 2.1 in [25] (this time with $w = 0$ in Ω and $\theta = u$ on $\partial\Omega$) guarantees that $v_n \in W^{1,1}(\Omega)$ exists such that

$$v_n = u \text{ on } \partial\Omega, \quad v_n \to 0 \text{ in } L^1(\Omega), \quad \text{and} \quad \limsup_{n\to+\infty} \int_\Omega |\nabla v_n| \mathrm{d}x \leq \int_{\partial\Omega} |u| \mathrm{d}\mathcal{H}^{N-1}. \tag{2.2.12}$$

Finally, let $u_n = z_n - v_n$. By construction $u_n \in W_0^{1,1}(\Omega)$ and $u_n \to u$ in $L^1(\Omega)$. Hence, using also triangle inequality,

$$\overline{F^\circ}(u) \overset{(2.1.6)}{\leq} \liminf_{n\to+\infty} F^\circ(u_n) \overset{(2.1.6),(2.2.1)}{=} \liminf_{n\to+\infty} F_\Omega(u_n) \leq \liminf_{n\to+\infty} \left(F_\Omega(z_n) + F_\Omega(v_n)\right). \tag{2.2.13}$$

We note that

$$\limsup_{n\to+\infty} F_\Omega(v_n) \leq \limsup_{n\to+\infty} \int_\Omega (|v_n| + |\nabla v_n|) \mathrm{d}x \overset{(2.2.12)}{\leq} \int_{\partial\Omega} |u| \mathrm{d}\mathcal{H}^{N-1}.$$

Hence, passing to the limit as $n \to +\infty$ in (2.2.13) and using (2.2.11) we conclude that

$$\overline{F^\circ}(u) \leq \overline{F_\Omega}(u) + \int_{\partial\Omega} |u| \mathrm{d}\mathcal{H}^{N-1} = \Phi(u),$$

and the proof is complete. □

The existence of a minimizer can now be obtained by standard direct methods: we present its proof for completeness.

Proof of Corollary 2.1. Assume that $\{u_n\} \subseteq BV_*(\Omega)$ is a minimizing sequence for $\overline{F^\circ}$. By the growth condition and the Rellich's Theorem there exists a subsequence, still denoted by $\{u_n\}$, such that $u_n \rightharpoonup u \in BV(\Omega)$ and $u_n \to u$ in $L^1(\Omega)$. In particular,

$$1 = \lim_{n\to+\infty} \fint_\Omega u_n \, \mathrm{d}x = \fint_\Omega u \, \mathrm{d}x,$$

so that $u \in BV_*(\Omega)$. Corollary 2.1 now follows from the lower semi-continuity of $\overline{F^\circ}$. □

2.3 Characterization and monotonicity of the minimum

In this section we prove Proposition 2.1 and Theorem 2.2.

Proof of Proposition 2.1. We recall that u_m is a minimizer of ϕ in $BV_*(\Omega)$ and

$$\tau_\Omega := \frac{\phi(u_m)}{|\Omega|}. \tag{2.3.1}$$

By (2.1.11), the constant function $x \mapsto \tau_\Omega \chi_\Omega(x)$ belongs to $\partial\phi(u_m)$ if and only if

$$\tau_\Omega \int_\Omega (u - u_m)\,\mathrm{d}x + \phi(u_m) \leq \phi(u) \quad \text{for all } \; u \in L^p(\Omega). \tag{2.3.2}$$

If $u_m \in BV_*(\Omega)$ and $\tau_\Omega \chi_\Omega(x) \in \partial\phi(u_m)$, then by (2.3.2) $\phi(u_m) \leq \phi(u)$ for all $u \in BV_*(\Omega)$, hence u_m is a minimizer. Let us look at the converse. If $u \notin BV(\Omega)$ then $\phi(u) = +\infty$ and (2.3.2) is obviously true. If $u \in BV(\Omega)$ and $\fint_\Omega u \leq 0$, then

$$\tau_\Omega \int_\Omega (u - u_m)\,\mathrm{d}x + \phi(u_m) \overset{(2.3.1)}{=} \phi(u_m) \fint_\Omega (u - u_m)\,\mathrm{d}x + \phi(u_m) \leq 0 \leq \phi(u),$$

hence (2.3.2). Else, since ϕ is positively 1-homogeneous and u_m is a minimizer in $BV_*(\Omega)$ we have

$$\begin{aligned}\phi(u) &= \left(\fint_\Omega u\,\mathrm{d}x\right)\phi\left(\frac{u}{\fint_\Omega u\,\mathrm{d}x}\right) \geq \left(\fint_\Omega u\,\mathrm{d}x\right)\phi(u_m) \\ &= \phi(u_m) + \left(\fint_\Omega u\,\mathrm{d}x - 1\right)\phi(u_m) \overset{(2.3.1)}{=} \phi(u_m) + \tau_\Omega \int_\Omega (u - u_m)\,\mathrm{d}x,\end{aligned}$$

hence (2.3.2) holds for all $u \in L^p(\Omega)$. □

Proof of Theorem 2.2. The proof relies on a scaling argument. Assume $\lambda_1 < \lambda_2$, let u_i be a minimizer of ϕ in $BV_*(\lambda_i\Omega)$, and let

$$\tau_i := \tau_{\lambda_i\Omega} = \frac{1}{\lambda_i^N|\Omega|}\phi(u_i) = \frac{1}{\lambda_i^N|\Omega|}\left(|(u_i, Du_i)|(\lambda_i\Omega) + \int_{\partial(\lambda_i\Omega)} |u_i|\mathrm{d}\mathcal{H}^{N-1}\right).$$

Let

$$u(x) = u_1\left(\frac{\lambda_1}{\lambda_2}x\right) \in BV_*(\lambda_2\Omega).$$

Then $Du = \frac{\lambda_1}{\lambda_2}Du_1$ and

$$|(u, Du)|(B) = \left|\left(\left(\tfrac{\lambda_2}{\lambda_1}\right)^N u_1, \left(\tfrac{\lambda_2}{\lambda_1}\right)^{N-1} Du_1\right)\right|\left(\frac{\lambda_1}{\lambda_2}B\right) \tag{2.3.3}$$

for any Borel set $B \subset \Omega$. Therefore

$$\begin{aligned}
\tau_2 &= \frac{1}{\lambda_2^N|\Omega|}\left(|(u_2, Du_2)|(\lambda_2\Omega) + \int_{\partial(\lambda_2\Omega)} |u_2(x)| \mathrm{d}\mathcal{H}^{N-1}(x)\right) \\
&\leq \frac{1}{\lambda_2^N|\Omega|}\left(|(u, Du)|(\lambda_2\Omega) + \int_{\partial(\lambda_2\Omega)} |u(x)| \mathrm{d}\mathcal{H}^{N-1}(x)\right) \qquad \text{(since } u_2 \text{ is a minimizer)} \\
&\overset{(2.3.3)}{=} \frac{1}{\lambda_2^N|\Omega|}\left(\left|\left(\left(\tfrac{\lambda_2}{\lambda_1}\right)^N u_1, \left(\tfrac{\lambda_2}{\lambda_1}\right)^{N-1} Du_1\right)\right|(\lambda_1\Omega) + \int_{\partial(\lambda_2\Omega)} \left|u_1\left(\tfrac{\lambda_1}{\lambda_2}x\right)\right| \mathrm{d}\mathcal{H}^{N-1}(x)\right) \\
&= \frac{1}{\lambda_1^N|\Omega|}\left(\left|\left(u_1, \tfrac{\lambda_1}{\lambda_2}Du_1\right)\right|(\lambda_1\Omega) + \tfrac{\lambda_1}{\lambda_2}\int_{\partial(\lambda_1\Omega)} |u_1(\hat{x})| \mathrm{d}\mathcal{H}^{N-1}(\hat{x})\right) \\
&\leq \frac{1}{\lambda_1^N|\Omega|}\left(|(u_1, Du_1)|(\lambda_1\Omega) + \int_{\partial(\lambda_1\Omega)} |u_1(\hat{x})| \mathrm{d}\mathcal{H}^{N-1}(\hat{x})\right) \qquad \text{(since } \lambda_1 < \lambda_2\text{)} \\
&= \tau_1,
\end{aligned}$$

and the latter inequality is strict if minimizers are not constant, a fact which is true if $N = 1$ (see Theorem 2.7 in §2.6). □

2.4 Uniqueness of minimizers in SBV_*

In this section look at the one-dimensional case, $\Omega = I = (-\alpha, \alpha)$. We prove:

Theorem 2.5. *The functional ϕ given by* (2.1.10) *has at most one minimizer $u \in SBV_*(I)$.*

The argument for Theorem 2.5 is based on two lemmas. Firstly we show that, along minimizers, ϕ does not degenerate in I, in the sense that:

Lemma 2.3. *Any minimizer $u \in BV_*(I)$ of ϕ is positive in I.*

This property allows to evaluate the variation of ϕ along competitors of a minimizer in I. A suitable choice of such competitors yields:

Lemma 2.4. *No minimizer $u \in BV_*(I)$ of ϕ jumps in the interior of I.*

The proofs of Lemmas 2.3 and 2.4 will be given at the end of this section. We now prove Theorem 2.5.

Proof of Theorem 2.5. Let u_1 and u_2 be two minimizers in $SBV_*(I)$ and set

$$u = u_1 - u_2, \quad u_t = tu_1 + (1-t)u_2, \quad t \in (0, 1).$$

Since ϕ is convex and u_i are minimizers, we have

$$\phi(u_t) = \phi(u_1) = \phi(u_2) \quad \text{for all } t \in (0, 1), \tag{2.4.1}$$

i.e. u_t is a minimizer for every $t \in (0,1)$. Note that

$$u_t' = tu_1' + (1-t)u_2', \quad \mathrm{d}^s u_t = t\mathrm{d}^s u_1 + (1-t)\mathrm{d}^s u_2,$$

and the same holds for u. Then set

$$f(s,p) = \sqrt{s^2+p^2}.$$

In view of Lemma 2.4 and since $u_i \in SBV_*(I)$, we have

$$\frac{\phi(u_{t+h})-\phi(u_t)}{h} = \int_I \frac{f(u_{t+h},u_{t+h}')-f(u_t,u_t')}{h}\,\mathrm{d}x + u_1(\alpha^-) - u_2(\alpha^-) + u_1(-\alpha^+) - u_2(-\alpha^+).$$

By Lemma 2.3 u_t, being a minimizer, is strictly positive in I for every $t \in (0,1)$, hence

$$\frac{f(u_{t+h},u_{t+h}')-f(u_t,u_t')}{h} \to \partial_s f(u_t,u_t')u + \partial_p f(u_t,u_t')u' \quad \text{a.e. in } I \text{ as } \ h \to 0.$$

In addition, taking into account that

$$|f(s,p)-f(s_0,p_0)| \leq |s-s_0| + |p-p_0|,$$

it follows

$$\left|\frac{f(u_{t+h},u_{t+h}')-f(u_t,u_t')}{h}\right| \leq |u| + |u'| \in L^1(I).$$

Therefore we may use Lebesgue's dominated convergence theorem to conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(u_t) = \int_I \Big(\partial_s f(u_t,u_t')u + \partial_p f(u_t,u_t')u'\Big)\,\mathrm{d}x + u_1(\alpha^-) - u_2(\alpha^-) + u_1(-\alpha^+) - u_2(-\alpha^+).$$

By the same argument, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\phi(u_t) = \int_I \Big(\partial_s^2 f(u_t,u_t')u^2 + 2\partial_s\partial_p f(u_t,u_t')uu' + \partial_p^2 f(u_t,u_t')(u')^2\Big)\,\mathrm{d}x. \tag{2.4.2}$$

A simple computation of the integrand in (2.4.2) shows that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\phi(u_t) = \int_I \frac{\big(u_1u_2' - u_2u_1'\big)^2}{((u_t)^2+(u_t')^2)^{3/2}}\,\mathrm{d}x\,.$$

In view of (2.4.1), this implies that

$$u_1u_2' = u_2u_1' \quad \text{a.e. in } I.$$

Since u_i are absolutely continuous and positive in I, we obtain

$$\log(u_1) = \log(u_2) + C, \quad \text{i.e.} \quad u_1 = Cu_2.$$

Recalling the constraint on the mass which must be satisfied by minimizers, it follows that $C = 1$. Hence $u_1 = u_2$ and the thesis is achieved. □

We conclude the section with the proofs of Lemmas 2.3 and 2.4.

Proof of Lemma 2.3. We proceed in three steps.

(I) Every minimizer is non-negative. If not, we would have

$$M = \fint_I u_+ \,\mathrm{d}x > \fint_I u \,\mathrm{d}x = 1. \tag{2.4.3}$$

But then, letting $\tilde{u} = u_+/M \in BV_*(I)$, we obtain

$$\begin{aligned}\phi(\tilde{u}) &= \frac{1}{M}\left(\int_I \sqrt{u_+^2 + (u_+')^2}\,\mathrm{d}x + |\mathrm{d}^s u_+|(I) + |u_+((-\alpha)^+)| + |u_+(\alpha^-)|\right) \\ &\le \frac{1}{M}\phi(u) \overset{(2.4.3)}{<} \phi(u),\end{aligned}$$

in contradiction with u being a minimizer.

(II) No minimizer is zero in an open set $J \subseteq I$. Assume it is, and let $J = (x_0, x_1) \subseteq I$ be a maximal interval such that $u = 0$ a.e. in J. Since $\fint u\,\mathrm{d}x = 1$, we have $J \subset I$. Hence, up to exchanging x with $-x$ we may assume without loss of generality that

$$-\alpha < x_0 < x_1 \le \alpha, \quad u = 0 \text{ a.e. in } (x_0, x_1), \quad u \not\equiv 0 \text{ in } (-\alpha, x_0). \tag{2.4.4}$$

We construct a re-scaled function of the form

$$\tilde{u}(x) = \begin{cases} Au\left(-\alpha + \frac{x_0+\alpha}{x_1+\alpha}(x+\alpha)\right) & \text{if } -\alpha \le x < x_1 \\ u(x) & \text{if } x_1 \le x \le \alpha. \end{cases}$$

We choose A such that mass is conserved: since

$$\begin{aligned}\int_{-\alpha}^{\alpha} \tilde{u}\mathrm{d}x &= A\int_{-\alpha}^{x_1} u\left(-\alpha + \frac{x_0+\alpha}{x_1+\alpha}(x+\alpha)\right)\mathrm{d}x + \int_{x_1}^{\alpha} u(x)\,\mathrm{d}x \\ &= A\frac{x_1+\alpha}{x_0+\alpha}\int_{-\alpha}^{x_0} u(\hat{x})\mathrm{d}\hat{x} + \int_{x_1}^{\alpha} u(x)\,\mathrm{d}x,\end{aligned}$$

we let

$$A = \frac{x_0+\alpha}{x_1+\alpha} < 1\,. \tag{2.4.5}$$

Then, using (2.4.4), for the absolutely continuous part of the functional we have

$$\begin{aligned}\int_{-\alpha}^{\alpha} \sqrt{\tilde{u}^2 + \tilde{u}'^2}\,\mathrm{d}x &= \tfrac{x_0+\alpha}{x_1+\alpha}\int_{-\alpha}^{x_0}\left(\tfrac{x_1+\alpha}{x_0+\alpha}\right)\sqrt{u^2 + \left(\tfrac{x_0+\alpha}{x_1+\alpha}\right)^2 u'^2}\,\mathrm{d}\hat{x} + \int_{x_1}^{\alpha}\sqrt{u^2+(u')^2}\,\mathrm{d}x \\ &= \int_{-\alpha}^{x_0}\sqrt{u^2 + \left(\tfrac{x_0+\alpha}{x_1+\alpha}\right)^2 u'^2}\,\mathrm{d}\hat{x} + \int_{x_1}^{\alpha}\sqrt{u^2+(u')^2}\,\mathrm{d}x \\ &\overset{(2.4.5)}{\le} \int_{-\alpha}^{\alpha}\sqrt{u^2+(u')^2}\,\mathrm{d}x. \end{aligned} \tag{2.4.6}$$

The same argument holds for the singular part,

$$||\mathrm{d}^s\tilde{u}||(-\alpha,\alpha) \;=\; \tfrac{x_0+\alpha}{x_1+\alpha}||\mathrm{d}^s u||(-\alpha,x_0] + ||\mathrm{d}^s u||[x_1,\alpha) \leq ||\mathrm{d}^s u||(-\alpha,\alpha), \tag{2.4.7}$$

and the boundary part

$$|\tilde{u}((-\alpha)^+)| + |\tilde{u}(\alpha^-)| = \tfrac{x_0+\alpha}{x_1+\alpha}|u((-\alpha)^+)| + |u(\alpha^-)| \leq |u((-\alpha)^+)| + |u(\alpha^-)|. \tag{2.4.8}$$

Summing (2.4.6)-(2.4.8) we see that $\phi(\tilde{u}) \leq \phi(u)$, thus $\phi(\tilde{u}) = \phi(u)$ since u is a minimizer. On the other hand,

$$\phi(\tilde{u}) = \phi(u) \quad\iff\quad \begin{cases} u' = 0 \;\text{ a.e. in }\; (-\alpha, x_0) \\ |d^s u|(-\alpha, x_0) = 0 \\ |u((-\alpha)^+)| = 0 \end{cases}$$

which implies that $u \equiv 0$ in $(-\alpha, x_0)$. This contradicts (2.4.4) and completes the proof of (II).

(III) Conclusion. We argue by contradiction. Up to exchanging x with $-x$, we may assume without loss of generality that there exists $x_0 \in I$ such that $u(x_0^-) = 0$. For $\varepsilon > 0$ to be chosen later, let

$$a = \inf\{\xi : ||u||_{L^\infty(\xi,x_0)} < \varepsilon\}.$$

In view of (II) and since $u(x_0^-) = 0$, choosing ε sufficiently small we have

$$-\alpha < a \quad\text{and}\quad x_0 - a < \tfrac{1}{2}.$$

We now distinguish two cases. If $u(x_0^+) = 0$, we let

$$b = \sup\{\xi : ||u||_{L^\infty(x_0,\xi)} < \varepsilon\}.$$

As before,

$$b < \alpha \quad\text{and}\quad b - x_0 < 1/2$$

for ε sufficiently small. If instead $u(x_0^+) > 0$, we choose ε so small that $\varepsilon < u(x_0^+)$ and we let $b = x_0$. In conclusion, we have

$$-\alpha < a < x_0 \leq b < \alpha, \quad b - a < 1, \tag{2.4.9}$$

and

$$u(a^-) \geq \varepsilon \geq u(a^+), \quad u(b^-) \leq \varepsilon \leq u(b^+). \tag{2.4.10}$$

Let $K = (a, b)$. We define the function

$$\tilde{u}(x) = \begin{cases} Au(x) & x \in I \setminus K \\ A\varepsilon & x \in K \end{cases}$$

where we choose

$$A = \frac{\int_I u \,\mathrm{d}x}{\int_{I\setminus K} u \,\mathrm{d}x + |K|\varepsilon}$$

so that $\int_I \tilde{u}\mathrm{d}x = \int_I u\mathrm{d}x$. By definition of a and b, $\int_K u \,\mathrm{d}x < |K|\varepsilon$, hence $A < 1$. Therefore

$$\begin{aligned} \phi(u) - \phi(\tilde{u}) \quad > \quad & \int_K \sqrt{u^2 + (u')^2}\,\mathrm{d}x + |\mathrm{d}^s u|(a,b) + |u(a^-) - u(a^+)| + |u(b^-) - u(b^+)| \\ & - A\left(|K|\varepsilon + |u(a^-) - \varepsilon| + |\varepsilon - u(b^+)|\right). \end{aligned} \tag{2.4.11}$$

We recall that $u \geq 0$ (by step (I)) and that $u(x_0^-) = 0$ (by assumption). If $u(x_0^+) = 0$ and $b > x_0$, we have

$$\begin{aligned} \int_a^b \sqrt{u^2 + u'^2}\,\mathrm{d}x + |\mathrm{d}^s u|(a,b) &\geq |\mathrm{d}u|(a,b) \\ &\geq |u(a^+) - u(x_0^-)| + |u(x_0^+) - u(x_0^-)| + |u(x_0^+) - u(b^-)| = u(a^+) + u(b^-). \end{aligned}$$

If instead $u(x_0^+) > 0$ and $b = x_0$,

$$\begin{aligned} \int_a^b \sqrt{u^2 + u'^2}\,\mathrm{d}x + |\mathrm{d}^s u|(a,b) &= \int_a^{x_0} \sqrt{u^2 + u'^2}\,\mathrm{d}x + |\mathrm{d}^s u|(a,x_0) \\ &\geq |\mathrm{d}u|(a,x_0) \geq |u(a^+) - u(x_0^-)| = u(a^+) = u(a^+) + u(b^-) \end{aligned}$$

(since $u(b^-) = u(x_0^-) = 0$). Hence, in both cases, (2.4.11) may be rewritten as

$$\begin{aligned} \phi(u) - \phi(\tilde{u}) \quad > \quad & \underbrace{-A|K|\varepsilon + u(a^+) + |u(a^-) - u(a^+)| - A|u(a^-) - \varepsilon|}_{=:M_a} \\ & + \underbrace{u(b^-) + |u(b^-) - u(b^+)| - A|\varepsilon - u(b^+)|}_{=:M_b}. \end{aligned}$$

In view of (2.4.9) and (2.4.10), we have

$$\begin{aligned} M_a &= -A|K|\varepsilon + u(a^+) + u(a^-) - u(a^+) - A(u(a^-) - \varepsilon) \\ &= (1 - A)u(a^-) + A\varepsilon(1 - |K|) > 0 \end{aligned}$$

and

$$M_b = u(b^-) + u(b^+) - u(b^-) - A(u(b^+) - \varepsilon) = (1 - A)u(b^+) + A\varepsilon > 0.$$

Hence $\phi(u) > \phi(\tilde{u})$, which is a contradiction. □

Proof of Lemma 2.4. We argue by contradiction. Up to exchanging x with $-x$, we may assume that there exists $x_0 \in I$ such that $u(x_0^-) > u(x_0^+)$. Then, for $\varepsilon \geq 0$ let

$$u_\varepsilon(x) = \begin{cases} A_\varepsilon u(x) & \text{in } (-\alpha, x_0) \\ A_\varepsilon(u(x) + \varepsilon) & \text{in } (x_0, \alpha), \end{cases}$$

where we choose

$$A_\varepsilon = \frac{\int_I u\,\mathrm{d}x}{\int_I u\,\mathrm{d}x + \varepsilon(\alpha - x_0)},$$

so that the mass in preserved. We have

$$\phi(u_\varepsilon) = A_\varepsilon\Big(u(-\alpha^+) + \int_{-\alpha}^{x_0} \sqrt{u^2 + (u')^2}\,\mathrm{d}x + |\mathrm{d}^s u|(-\alpha, x_0) + u(x_0^-) - (u(x_0^+) + \varepsilon) + \\ + \int_{x_0}^{\alpha} \sqrt{(u+\varepsilon)^2 + (u')^2}\,\mathrm{d}x + |\mathrm{d}^s u|(x_0, \alpha) + (u(\alpha^-) + \varepsilon)\Big).$$

By Lemma 2.3, $u > 0$ in I. Hence, arguing as in the proof of Theorem 2.5, we have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\phi(u_\varepsilon)|_{\varepsilon=0} = \left(\frac{\mathrm{d}A_\varepsilon}{\mathrm{d}\varepsilon}|_{\varepsilon=0}\right)\phi(u) + A_0 \int_{x_0}^{\alpha} \frac{u}{\sqrt{u^2 + (u')^2}}\,\mathrm{d}x.$$

Since

$$A_0 = 1, \qquad \frac{\mathrm{d}A_\varepsilon}{\mathrm{d}\varepsilon}|_{\varepsilon=0} = -\frac{\alpha - x_0}{\int_I u\,\mathrm{d}x},$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\phi(u_\varepsilon)|_{\varepsilon=0} = -\frac{\alpha - x_0}{\int_I u\,\mathrm{d}x}\phi(u) + \int_{x_0}^{\alpha} \frac{u}{\sqrt{u^2 + (u')^2}}\,\mathrm{d}x \le (\alpha - x_0)\left(1 - \frac{\phi(u)}{\int_I u\,\mathrm{d}x}\right).$$

Since u is not constant (it has a jump in the interior), $\phi(u) > \int_I u\,\mathrm{d}x$: hence the latter factor is negative, in contradiction with u being a minimizer. □

2.5 Existence of a smooth minimizer

The goal of this section is to prove that there exists a minimizer for the one-dimensional problem which is smooth in the bulk. As in the previous section, we let $\Omega = I = (-\alpha, \alpha)$, $\alpha > 0$.

Theorem 2.6 (Existence of a smooth minimizer)**.** *There exists a minimizer u of ϕ in $BV_*(I)$ which is smooth in I and solves*

$$\tau_I = \frac{u}{\sqrt{u^2 + (u')^2}} - \left(\frac{u'}{\sqrt{u^2 + (u')^2}}\right)' \quad \text{in } I. \tag{2.5.1}$$

Furthermore u is even and non-increasing in $[0, \alpha)$.

Our approach is based on a-priori concavity estimates, in the spirit of the arguments developed in [24]. However, we shall not prove that the minimizer is concave, but rather that its square root $v = \sqrt{u}$ is. To this aim, by letting

$$\phi_\varepsilon(u) = \int_I f_\varepsilon(u, u')\,\mathrm{d}x, \quad \text{where} \quad f_\varepsilon(s, p) = \sqrt{s^2 + p^2} + \varepsilon^2 p^2,$$

we relax the minimum problem into one which is well posed in the space

$$\mathcal{H}_*(I) = \left\{ u \in H^1_0(I) : \fint_I u \, \mathrm{d}x = 1 \right\}.$$

Lemma 2.5. *For any $\varepsilon > 0$, there exists a minimizer u_ε of ϕ_ε in $\mathcal{H}_*(I)$. Furthermore, u_ε may be chosen to be even, non-increasing in $[0, \alpha)$ and positive in I.*

Since $f_\varepsilon \in C^2((0, +\infty) \times \mathbb{R})$ and u_ε is positive in the bulk, u_ε is a smooth solution of the Euler-Lagrange equation, with uniform bounds in BV; in addition, the Lagrange multiplier μ_ε is larger than one:

Lemma 2.6. *The minimizer u_ε given in Lemma 2.5 is smooth in I. Furthermore, there exists $\mu_\varepsilon \geq 1$ such that*

$$-\partial_p^2 f_\varepsilon(u_\varepsilon, u'_\varepsilon)u''_\varepsilon - \partial_s\partial_p f_\varepsilon(u_\varepsilon, u'_\varepsilon)u'_\varepsilon + \partial_s f_\varepsilon(u_\varepsilon, u'_\varepsilon) = \mu_\varepsilon \quad \textit{in } \ I, \tag{2.5.2}$$

and a positive constant C exists such that

$$\int_I \sqrt{u_\varepsilon^2 + (u'_\varepsilon)^2} \, \mathrm{d}x \quad \leq \quad C. \tag{2.5.3}$$

The core of the argument is the concavity of $\sqrt{u_\varepsilon}$.

Lemma 2.7. *Let u_ε be as in Lemma 2.5. Then $\sqrt{u_\varepsilon}$ is concave in I.*

In turn, concavity yields a uniform sup-bound on u'_ε and a uniform lower bound on u_ε.

Lemma 2.8. *Let u_ε be as in Lemma 2.5. Then there exists a positive constant C such that*

$$\sup_{|x|<\alpha-\delta} |u'_\varepsilon(x)| \leq \frac{C}{\delta} \quad \textit{for all } \ \delta \in (0, \alpha).$$

Furthermore

$$u_\varepsilon(x) \geq \frac{1}{4\alpha^2} (\alpha - |x|)^2 \quad \textit{for all } \ x \in I.$$

The proofs of lemmas 2.5-2.8 will be given at the end of this section. We now prove Theorem 2.6.

Proof of Theorem 2.6. For every $\varepsilon > 0$ let $u_\varepsilon \in \mathcal{H}_*(I)$ be the minimizer of ϕ_ε given in Lemma 2.5. Using the bounds in lemmas 2.6 and 2.8, up to a subsequence we have

$$u_\varepsilon \rightharpoonup u \quad \text{in } \ W^{1,\infty}_{loc}(I) \cap BV(I).$$

By the lower semi-continuity of ϕ, for all $\tilde{u} \in \mathcal{H}_*(I)$ we have

$$\phi(u) \leq \liminf_{\varepsilon\to 0} \phi(u_\varepsilon) \leq \liminf_{\varepsilon\to 0} \phi_\varepsilon(u_\varepsilon) \leq \liminf_{\varepsilon\to 0} \phi_\varepsilon(\tilde{u}) = \phi(\tilde{u}).$$

By density, the inequality holds for all $\tilde{u} \in BV_*(I)$; hence u is a minimizer of ϕ. Because of the properties of u_ε, u is even, non-increasing in $[0, \alpha)$, and positive in I.

By Proposition 2.1, $\tau_I \in \partial\phi(u)$. On the other hand, since u is positive in I and Lipschitz continuous on compact subsets of I, for any $\eta \in C_c^\infty(I)$ the function $h \mapsto \phi(u + h\eta)$ is differentiable at $h = 0$: therefore

$$\int_I \left(\frac{u}{\sqrt{u^2 + u'^2}} \eta + \frac{u'}{\sqrt{u^2 + u'^2}} \eta' \right) \mathrm{d}x = \tau_I \int_I \eta \, \mathrm{d}x,$$

which means that

$$\left(\frac{u'}{\sqrt{u^2 + u'^2}} \right)' = \frac{u}{\sqrt{u^2 + u'^2}} - \tau_I \in L^\infty(I). \tag{2.5.4}$$

Since u is even, (2.5.4) implies that

$$\frac{u'(x)}{\sqrt{u^2(x) + u'^2(x)}} = -\tau_I x + \int_0^x \frac{u}{\sqrt{u^2 + u'^2}} \, \mathrm{d}y =: \ell(x) \in W^{1,\infty}(I) \tag{2.5.5}$$

Let now $K \Subset I$. We have

$$|\ell| = \left| \frac{u'}{\sqrt{u^2 + (u')^2}} \right| \leq 1 - \delta \quad \Longleftrightarrow \quad (u')^2(1 - (1 - \delta)^2) \leq (1 - \delta)^2 u^2.$$

Hence, choosing δ so small that

$$\|u'\|^2_{L^\infty(K)}(1 - (1 - \delta)^2) \leq (1 - \delta)^2 \left(\inf_K u \right)^2,$$

we have

$$|\ell(x)| < 1 - \delta \quad \text{for all} \ \ x \in K. \tag{2.5.6}$$

In K we may therefore invert (2.5.5) with respect to $u'(x)$, and by the arbitrariness of K we obtain that

$$u'(x) = \frac{\ell(x) u(x)}{\sqrt{1 - \ell^2}} \quad \text{a.e. in} \ \ I. \tag{2.5.7}$$

It follows from (2.5.5), (2.5.6), and (2.5.7) that u'' is well defined and belongs to $L^\infty(K)$. Hence $u \in W^{2,\infty}_{loc}(I)$ and u solves the Euler-Lagrange equation point-wise. A standard bootstrap argument then implies that u is smooth in I. □

We conclude this section with the proof of lemmas 2.5-2.8.

Proof of Lemma 2.5. We divide the proof into various steps.

(i). The proof of existence is standard, but we reproduce it for the sake of completeness. Let us fix $\varepsilon > 0$. Note that f_ε is continuous in $\mathbb{R} \times \mathbb{R}$ and convex with respect to p, since

$$\partial_p^2 f_\varepsilon = \frac{s^2}{(s^2 + p^2)^{3/2}} + 2\varepsilon^2.$$

Therefore (see [35, Theorem 3.4]) ϕ_ε is (sequentially) weakly lower semi-continuous and coercive on $\mathcal{H}_*(I)$. Let now $\{u_n\}$ be a minimizing sequence. Then, up to a subsequence, $u_n \rightharpoonup u_\varepsilon$ in $H^1_0(I)$ and $u_\varepsilon \to u$ in $L^1(I)$. Hence $u_\varepsilon \in \mathcal{H}_*(I)$ and, by the lower semi-continuity of ϕ_ε, u_ε is a minimizer.

(ii). The proof that u_ε is non-negative is completely analogous to that of Lemma 2.3, therefore we omit the details.

(iii). Let us extend u_ε by zero outside of I, and let $u^\star_\varepsilon$ be the Schwarz symmetrization of u_ε (see [27, §3.3]). By definition $u^\star_\varepsilon$ is even, non-increasing in $[0,\alpha)$, and zero outside of I. We claim that $u^\star_\varepsilon$ is also a minimizer of ϕ_ε. Let $u_n \in C^\infty_c(I)$ such that $u_n \to u_\varepsilon$ in $H^1_0(I)$. Since $\|(u^\star_n)'\|_{L^2(I)} \le \|u'_n\|_{L^2(I)}$ (see [27, Thm. 4.3]), there exists a subsequence such that $u^\star_n \rightharpoonup w$ in $H^1_0(I)$ and $u^\star_n \to w$ in $L^2(I)$. By the non-expansivity of the symmetrization (see [27, Cor. 3.1]), $\|u^\star_n - u^\star_\varepsilon\|_{L^2(I)} \le \|u_n - u_\varepsilon\|_{L^2(I)}$: hence $w = u^\star_\varepsilon$. Since f_ε is convex and non-decreasing with respect to p for all $(s,p) \in [0,+\infty)^2$, $\phi_\varepsilon(u^\star_n) \le \phi_\varepsilon(u_n)$ (see [27, Thm. 4.3]). Therefore, the weak lower semi-continuity of ϕ_ε implies that

$$\phi_\varepsilon(u^\star_\varepsilon) \le \liminf_{n\to+\infty} \phi_\varepsilon(u^\star_n) \le \lim_{n\to+\infty} \phi_\varepsilon(u_n) = \phi_\varepsilon(u_\varepsilon).$$

Finally, $⨍_I u^\star_\varepsilon \,\mathrm{d}x = ⨍_I u_\varepsilon \,\mathrm{d}x$ (see [27, Thm. 3.1]). Hence $u^\star_\varepsilon$ is also a minimizer.

(iv). It remains to prove that $u^\star_\varepsilon$ is positive in I. To this aim, we could argue as in the proof of Lemma 2.3, but in view of (iii) we may provide a simpler argument. Assume by contradiction that there exists $\delta > 0$ such that $u^\star_\varepsilon > 0$ in $[0, \alpha-\delta)$ and $u^\star_\varepsilon = 0$ in $(\alpha-\delta, \alpha)$. Let

$$v(x) = A u^\star_\varepsilon(Ax), \quad A = \frac{\alpha-\delta}{\alpha} < 1.$$

Then v and $u^\star_\varepsilon$ have the same mass and, since $A < 1$,

$$\begin{aligned}\phi_\varepsilon(v) &= A\int_0^\alpha \sqrt{(u^\star_\varepsilon(Ax))^2 + A^2[(u^\star_\varepsilon)'(Ax)]^2}\,\mathrm{d}x + \varepsilon^2 A^4 \int_0^\alpha [(u^\star_\varepsilon)'(Ax)]^2\,\mathrm{d}x \\ &= \int_0^{\alpha-\delta} \sqrt{(u^\star_\varepsilon)^2 + A^2[(u^\star_\varepsilon)']^2}\,\mathrm{d}x + \varepsilon^2 A^3 \int_0^{\alpha-\delta} [(u^\star_\varepsilon)']^2\,\mathrm{d}x \\ &< \phi_\varepsilon(u^\star_\varepsilon),\end{aligned}$$

in contradiction with $u^\star_\varepsilon$ being a minimizer. □

Proof of Lemma 2.6. Let $\eta \in C^\infty_c(I)$ such that $\int_I \eta \,\mathrm{d}x = 0$. Since u_ε is positive and continuous in I, $u_\varepsilon + h\eta \ge c > 0$ in $\mathrm{supp}(\eta) \Subset I$ for h sufficiently small. In addition, $⨍_I (u_\varepsilon + h\eta)\,\mathrm{d}x = 1$ for all h. In $[c,+\infty)\times\mathbb{R}$, the functions f_ε, $\partial_s f_\varepsilon$ and $\partial_p f_\varepsilon$ are smooth and grow at most linearly with respect to p: therefore $h \mapsto \phi_\varepsilon(u_\varepsilon + h\eta)$ is differentiable at $h = 0$, and since u_ε is a minimizer we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}h}\phi_\varepsilon(u_\varepsilon + h\eta)\Big|_{h=0} = \int_I \Big(\partial_s f_\varepsilon(u_\varepsilon, u'_\varepsilon)\eta + \partial_p f_\varepsilon(u_\varepsilon, u'_\varepsilon)\eta'\Big)\,\mathrm{d}x.$$

By the arbitrariness of η, this shows that there exists $\mu_\varepsilon \in \mathbb{R}$ such that

$$\begin{aligned}\left(\partial_p f_\varepsilon(u_\varepsilon, u_\varepsilon')\right)' &= \partial_s f_\varepsilon(u_\varepsilon, u_\varepsilon') - \mu_\varepsilon \\ &= \frac{u_\varepsilon}{\sqrt{u_\varepsilon^2 + (u_\varepsilon')^2}} - \mu_\varepsilon := \varphi_\varepsilon \in L^\infty(I).\end{aligned} \tag{2.5.8}$$

In turn, taking into account the symmetry of u_ε, this means that

$$\left(\frac{1}{\sqrt{u_\varepsilon^2(x) + (u_\varepsilon'(x))^2}} + 2\varepsilon^2\right) u_\varepsilon'(x) = \int_0^x \varphi_\varepsilon(y)\, \mathrm{d}y \quad \text{a.e. in } (0, \alpha).$$

In particular

$$|u_\varepsilon'(x)| \le \frac{1}{2\varepsilon^2} \int_I |\varphi_\varepsilon|(y)\, \mathrm{d}y \quad \text{for all } x \in I,$$

therefore $u_\varepsilon \in W^{1,\infty}(I)$. We may now argue as in the proof of Theorem 2.6 to conclude that $u_\varepsilon \in C^\infty(I)$ and u_ε solves the Euler-Lagrange equation (2.5.2).

Since $u_\varepsilon \in W^{1,\infty}(I)$ with zero boundary conditions, we may now multiply (2.5.8) by u_ε and integrate over I. After one integration by parts, and recalling that $⨍ u_\varepsilon \,\mathrm{d}x = 1$, we obtain

$$\begin{aligned}\mu_\varepsilon = ⨍_I \mu_\varepsilon u_\varepsilon \,\mathrm{d}x &= ⨍_I \left(\partial_p f_\varepsilon(u_\varepsilon, u_\varepsilon') \cdot u_\varepsilon' + \partial_s f_\varepsilon(u_\varepsilon, u_\varepsilon') \cdot u_\varepsilon\right) \mathrm{d}x \\ &= \frac{1}{|I|}\left(\phi_\varepsilon(u_\varepsilon) + \varepsilon^2 \int_I (u_\varepsilon')^2 \,\mathrm{d}x\right) \ge \frac{1}{|I|}\phi_\varepsilon(u_\varepsilon) \ge ⨍_I u_\varepsilon \mathrm{d}x = 1.\end{aligned}$$

Finally, (2.5.3) is immediate: taken any $\tilde{u} \in \mathcal{H}_*(I)$, we have

$$\int_I \sqrt{u_\varepsilon^2 + (u_\varepsilon')^2}\,\mathrm{d}x + \varepsilon^2 \int_I u_\varepsilon'^2 \,\mathrm{d}x = \phi_\varepsilon(u_\varepsilon) \le \phi_\varepsilon(\tilde{u}) \le \phi_1(\tilde{u}) =: C.$$

□

Proof of Lemma 2.7. Let $v_\varepsilon = \sqrt{u_\varepsilon}$ and

$$g_\varepsilon(s, p) = s\sqrt{s^2 + 4p^2} + 4\varepsilon^2 s^2 p^2, \quad (s, p) \in [0, +\infty) \times \mathbb{R}.$$

Simple computations starting from (2.5.2) or, more simply, observing that

$$v_\varepsilon \text{ minimizes } \int_I g_\varepsilon(v_\varepsilon, v_\varepsilon')\,\mathrm{d}x \text{ among all } v^2 \in H_0^1(I) \text{ s.t. } ⨍_I v^2 \,\mathrm{d}x = 1,$$

yield

$$-\partial_p^2 g_\varepsilon(v_\varepsilon, v_\varepsilon')v_\varepsilon'' - \partial_s\partial_p g_\varepsilon(v_\varepsilon, v_\varepsilon')v_\varepsilon' + \partial_s g_\varepsilon(v_\varepsilon, v_\varepsilon') = 2\mu_\varepsilon v_\varepsilon \quad \text{in } I. \tag{2.5.9}$$

For $s \geq 0$, we have

$$\begin{aligned}
\partial_s g_\varepsilon &= \frac{1}{(s^2+4p^2)^{1/2}}(2s^2+4p^2) + 8\varepsilon^2 s p^2, \\
\partial_p^2 g_\varepsilon &= \frac{4s^3}{(s^2+4p^2)^{3/2}} + 8\varepsilon^2 s^2 \geq 0 \\
\partial_s \partial_p g_\varepsilon &= \frac{16p^3}{(s^2+4p^2)^{3/2}} + 16\varepsilon^2 s p.
\end{aligned}$$

Hence (2.5.9) may be rewritten as

$$v_\varepsilon'' = G_\varepsilon(v_\varepsilon, v_\varepsilon'),$$

where

$$\begin{aligned}
\partial_p^2 g_\varepsilon(s,p) \cdot G_\varepsilon(s,p) &= (-2\mu_\varepsilon s + \partial_s g_\varepsilon - p \cdot \partial_s \partial_p g_\varepsilon) \\
&= 2s\left(-\mu_\varepsilon - 4\varepsilon^2 p^2 + \frac{s^3+6sp^2}{(s^2+4p^2)^{3/2}}\right) \\
&\overset{\mu_\varepsilon \geq 1}{\leq} 2s\left(-1 + \frac{s^3+6sp^2}{(s^2+4p^2)^{3/2}}\right).
\end{aligned}$$

A simple computation shows that

$$s^3 + 6sp^2 \leq (s^2+4p^2)^{3/2} \;\Leftrightarrow\; 0 \leq 12s^2p^4 + 64p^6.$$

Therefore $G_\varepsilon(s,p) \leq 0$ for all $(s,p) \in [0,+\infty) \times \mathbb{R}$, which means that v_ε in concave in I. □

Proof of Lemma 2.8. We first prove the Lipschitz bounds. Since $v_\varepsilon = \sqrt{u_\varepsilon}$ is concave, the differential quotients are decreasing:

$$\frac{v_\varepsilon(x_1) - v_\varepsilon(x)}{x_1 - x} \geq v_\varepsilon'(x) \geq \frac{v_\varepsilon(x_2) - v_\varepsilon(x)}{x_2 - x} \quad \text{for all } x_1 < x < x_2.$$

In particular, choosing $x_1 = -\alpha$ and $x_2 = \alpha$, we have

$$\frac{u'}{2\sqrt{u}} = |v_\varepsilon'(x)| \leq v_\varepsilon(x) \max\left\{\frac{1}{\alpha - x}, \frac{1}{\alpha + x}\right\} \leq \frac{2}{\delta} \sup_{|x|<\alpha-\delta} v_\varepsilon(x) \quad \text{for all } |x| < \alpha - \delta$$

for every $\delta \in (0,\alpha)$. In terms of $u_\varepsilon = v_\varepsilon^2$, the last inequality reads as

$$|u_\varepsilon'(x)| \leq \frac{4}{\delta} \sup_{|x|<\alpha-\delta} u_\varepsilon(x) \quad \text{for all } |x| < \alpha - \delta,$$

and the Lipschitz bound follows from Lemma 2.6 (since u_ε are uniformly bounded in L^∞).

We now prove the lower bound. Since v_ε is concave and $v_\varepsilon = 0$ on ∂I, v_ε assumes its maximum in a point $x_\varepsilon \in I$. Because of the constraint,

$$1 = \fint v_\varepsilon^2 \, \mathrm{d}x \leq v_\varepsilon^2(x_\varepsilon). \tag{2.5.10}$$

Again because of concavity,

$$\begin{aligned} v_\varepsilon(x) &\geq \min\left\{v_\varepsilon(x_\varepsilon)\frac{\alpha - x}{\alpha - x_\varepsilon}, v_\varepsilon(x_\varepsilon)\frac{\alpha + x}{\alpha + x_\varepsilon}\right\} \overset{(2.5.10)}{\geq} \min\left\{\frac{\alpha - x}{\alpha - x_\varepsilon}, \frac{\alpha + x}{\alpha + x_\varepsilon}\right\} \\ &\geq \inf_{|\eta|<\alpha} \min\left\{\frac{\alpha - x}{\alpha - \eta}, \frac{\alpha + x}{\alpha + \eta}\right\} = \frac{1}{2\alpha}(\alpha - |x|) \end{aligned}$$

which proves the lower bound. □

2.6 The smooth minimizer jumps at the boundary

In this section we complete the proof of Theorem 2.3 by showing that:

Theorem 2.7. *The minimizer u given in Theorem 2.6 is strictly decreasing and such that*

$$\lim_{x\to\alpha^-} u(x) > 0 \quad \textit{and} \quad \lim_{x\to\alpha^-} u'(x) = -\infty.$$

In order to prove Theorem 2.7, we first characterize the solutions to the Euler-Lagrange equation.

Lemma 2.9. *For all $\tau > 1$ and all $A > 0$ there exists a unique maximal solution $u \in C^2([0, x_\tau))$ of*

$$\begin{cases} \tau = \dfrac{u}{\sqrt{u^2 + (u')^2}} - \left(\dfrac{u'}{\sqrt{u^2 + (u')^2}}\right)' \qquad \textit{in } I \\ u(0) = A, \; u'(0) = 0. \end{cases} \tag{2.6.1}$$

Furthermore $x_\tau < \infty$, u is strictly decreasing in $[0, x_\tau)$, and

$$\lim_{x\to x_\tau^-} u(x) = \frac{\tau - 1}{\tau} > 0, \quad \lim_{x\to x_\tau^-} u'(x) = -\infty. \tag{2.6.2}$$

Proof. Since (2.5.1) is invariant under $u \mapsto u/A$, we may assume without loss of generality that $A = 1$. We rewrite the first equation in (2.6.1) as

$$u'' = 2\frac{u'^2}{u} + u - \tau\frac{1}{u^2}\left(u^2 + u'^2\right)^{3/2},$$

so that existence and uniqueness of a classical local solution for the Cauchy problem (2.6.1) is a standard result. Moreover, since $u''(0) = 1 - \tau < 0$, we have $u' < 0$ in a right-neighbourhood of $x = 0$. As long as $u' < 0$, we may use u as the independent variable: letting

$$v(u) = u'^2(x(u)) + u^2, \tag{2.6.3}$$

we have

$$\begin{aligned} v'(u) &= 2u'(x(u))u''(x(u))x'(u) + 2u = 2\left(u''(x(u)) + u\right) \\ &= 2\left(2\frac{u'^2}{u} + u - \tau\frac{1}{u^2}\left(u^2 + u'^2\right)^{3/2} + u\right) = \frac{2}{u}\left(2v - \frac{\tau}{u}v^{3/2}\right) \end{aligned}$$

and $v(1) = 1$. We let $w(u) = u^\gamma v$, with γ to be chosen later, and we compute

$$\begin{aligned} w'(u) &= \gamma u^{\gamma-1}v + u^\gamma v' = \gamma u^{\gamma-1}v + u^\gamma\frac{2}{u}\left(2v - \frac{\tau}{u}v^{3/2}\right) = \\ &= \frac{1}{u}\left(\gamma w + 4w - 2\tau u^{\gamma-1}v^{3/2}\right) = \frac{1}{u}\left(\gamma w + 4w - 2\tau u^{-1-\gamma/2}w^{3/2}\right). \end{aligned}$$

We choose $\gamma = -2$, so that $w(u)$ is a solution of a first-order separable ode:

$$\begin{cases} uw'(u) = 2(w - \tau w^{3/2}) \\ w(1) = 1. \end{cases} \tag{2.6.4}$$

An integration gives

$$w(u) = \frac{u^2}{(1 + \tau(u-1))^2},$$

which in terms of $v(u)$ reads as

$$v(u) = \frac{u^4}{(1 + \tau(u-1))^2}.$$

Recalling (2.6.3), this gives

$$\begin{aligned} u' &= -\sqrt{\frac{u^4}{(1 + \tau(u-1))^2} - u^2} \\ &= -\frac{u\sqrt{(1-\tau)((\tau+1)u^2 - 2u\tau - (1-\tau))}}{|1 + \tau(u-1)|} \quad \text{as long as } u'(x) < 0. \end{aligned}$$

One easily checks that

$$(1-\tau)(u^2(\tau+1) - 2u\tau + (\tau-1)) \geq 0 \quad \text{for all } u \in \left[\frac{\tau-1}{\tau+1}, 1\right]$$

whereas

$$1 + \tau(u-1) = 0 \iff u = \frac{\tau-1}{\tau} \in \left(\frac{\tau-1}{\tau+1}, 1\right).$$

Therefore $u'(x)$ never changes sign and blows up for a positive value of u: in other words, there exists $x_\tau \in (0, +\infty]$ such that

$$\lim_{x\to x_\tau^-} u(x) = \frac{\tau-1}{\tau} > 0 \quad \text{and} \quad \lim_{x\to x_\tau^-} u'(x) = -\infty,$$

and of course these two conditions imply that $x_\tau < +\infty$. □

We are now ready to prove Theorem 2.7.

Proof of Theorem 2.7. The minimizer u given in Theorem 2.6 satisfies (2.6.1) in $(-\alpha,\alpha)$, and is even and positive. Hence u coincides in $I = (-\alpha,\alpha)$ with the solution obtained in Lemma 2.9 with $A = u(0)$. Therefore, in order to prove Theorem 2.7 it suffices to show that $x_\tau = \alpha$, where $J = (-x_\tau, x_\tau) \supseteq I$ is the maximal interval in which the solution of (2.6.1) is defined. We argue by contradiction and we assume that $\alpha < x_\tau$. Then we may define

$$\tilde{u}(x) = Au(Bx) \text{ for all } x \in I, \quad \text{where } B = \frac{x_\tau}{\alpha} \text{ and } \frac{A}{B} = \frac{\int_I u\,\mathrm{d}x}{\int_J u\,\mathrm{d}x} < 1. \tag{2.6.5}$$

Of course, A is chosen so that mass is conserved:

$$\int_I \tilde{u}\,\mathrm{d}x = A\int_I u(Bx)\,\mathrm{d}x = \frac{A}{B}\int_J u(\hat{x})\,d\hat{x} = \int_I u\,\mathrm{d}x.$$

The function $\tilde{u}$ satisfies the following equation:

$$\tau = \frac{\tilde{u}}{\sqrt{\tilde{u}^2 + \frac{1}{B^2}(\tilde{u}')^2}} - \frac{1}{B^2}\left(\frac{\tilde{u}'}{\sqrt{\tilde{u}^2 + \frac{1}{B^2}(\tilde{u}')^2}}\right)' \quad \text{in } I. \tag{2.6.6}$$

Let $I_\varepsilon := [-\alpha+\varepsilon, \alpha-\varepsilon]$. We multiply (2.6.6) by $\tilde{u}$ and integrate over I_ε. By an integration by parts we obtain

$$\begin{aligned}
\tau\int_{I_\varepsilon} \tilde{u}\,\mathrm{d}x &= \int_{I_\varepsilon} \frac{\tilde{u}^2}{\sqrt{\tilde{u}^2 + \frac{1}{B^2}(\tilde{u}')^2}}\mathrm{d}x - \frac{1}{B^2}\frac{\tilde{u}\tilde{u}'}{\sqrt{\tilde{u}^2 + \frac{1}{B^2}(\tilde{u}')^2}}\Bigg|_{\partial I_\varepsilon} + \frac{1}{B^2}\int_{I_\varepsilon}\frac{(\tilde{u}')^2}{\sqrt{(\tilde{u}')^2 + \frac{1}{B^2}(\tilde{u}')^2}}\mathrm{d}x \\
&= \int_{I_\varepsilon}\sqrt{\tilde{u}^2 + \frac{1}{B^2}(\tilde{u}')^2}\,\mathrm{d}x - \frac{1}{B^2}\frac{\tilde{u}\tilde{u}'}{\sqrt{\tilde{u}^2 + \frac{1}{B^2}(\tilde{u}')^2}}\Bigg|_{\partial I_\varepsilon}.
\end{aligned}$$

Since the first derivative blows up at the boundary, passing to the limit as $\varepsilon \to 0$ we get

$$\tau\int_I \tilde{u}\,\mathrm{d}x = \int_I \sqrt{\tilde{u}^2 + \frac{1}{B^2}(\tilde{u}')^2}\,\mathrm{d}x + \frac{1}{B}\left(\tilde{u}\,(-\alpha) + \tilde{u}\,(\alpha)\right),$$

which in terms of u reads as follows:

$$\tau\frac{A}{B}\int_J u\mathrm{d}\hat{x} = \frac{A}{B}\left(u(-x_\tau) + u(x_\tau) + \int_J \sqrt{u^2 + (u')^2}\mathrm{d}\hat{x}\right) = \frac{A}{B}\phi_J(u).$$

Therefore, recalling Proposition 2.1,

$$\tau = \frac{\phi_J(u)}{\int_J u} = \frac{\phi_I(u)}{\int_I u}. \tag{2.6.7}$$

We now show that, since u is decreasing, the function

$$t > 0 \mapsto F(t) = \frac{\int_0^t \sqrt{u^2 + (u')^2}\,\mathrm{d}x + u(t)}{\int_0^t u\,\mathrm{d}x}$$

is strictly decreasing, which contradicts (2.6.7) and thus proves the theorem. Indeed,

$$F'(t) = \frac{\left(\sqrt{u^2 + (u')^2}(t) + u'(t)\right)\left(\int_0^t u\,\mathrm{d}x\right) - \left(\int_0^t \sqrt{u^2 + (u')^2}\,\mathrm{d}x + u(t)\right)u(t)}{\left(\int_0^t u\,\mathrm{d}x\right)^2} < 0$$

if and only if

$$\frac{(\sqrt{u^2 + (u')^2}(t) + u'(t))}{\int_0^t \sqrt{u^2 + (u')^2}\,\mathrm{d}x + u(t)} < \frac{u}{\int_0^t u\,\mathrm{d}x},$$

which is true in view of the following chain of inequalities:

$$\frac{(\sqrt{u^2 + (u')^2}(t) + u'(t))}{\int_0^t \sqrt{u^2 + (u')^2}\,\mathrm{d}x + u(t)} < \frac{(\sqrt{u^2 + (u')^2}(t) + u'(t))}{\int_0^t u\,\mathrm{d}x} < \frac{(u + |u'| + u')}{\int_0^t u\,\mathrm{d}x} = \frac{u}{\int_0^t u\,\mathrm{d}x}$$

where in the last equality we have used the monotonicity of u in $[0, x_\tau)$. □

2.7 The subdifferential of ϕ

In this section we prove Theorem 2.4. To this aim, we recall (see [7, §C.2]) that for any $u \in BV(\Omega)$ and any $\mathbf{z} \in X(\Omega)$ the functional $(\mathbf{z}, Du) : C_c^\infty(\Omega) \to \mathbb{R}$ defined by

$$\langle (\mathbf{z}, Du), \varphi \rangle = -\int_\Omega u\varphi \operatorname{div} \mathbf{z}\,\mathrm{d}x - \int_\Omega u\mathbf{z} \cdot \nabla\varphi\,\mathrm{d}x \tag{2.7.1}$$

is a Radon measure which is absolutely continuous with respect to $|Du|$. Furthermore, there exists a linear operator $[\cdot, \mathbf{n}] : X(\Omega) \to L^\infty(\partial\Omega)$, such that

$$\|[\mathbf{z}, \mathbf{n}]\|_\infty \le \|\mathbf{z}\|_\infty, \tag{2.7.2}$$

which represents the trace on $\partial\Omega$ of the normal component of $\mathbf{z}$ in the sense that

$$\int_{\mathbb{R}^N} w \operatorname{div} \mathbf{z}\,\mathrm{d}x + \int_{\mathbb{R}^N} \mathrm{d}(\mathbf{z}, Dw) = \int_{\partial\Omega} u[\mathbf{z}, \mathbf{n}]\mathrm{d}\mathcal{H}^{N-1} \quad \text{for all } \mathbf{z} \in X(\Omega),\ w \in BV(\Omega). \tag{2.7.3}$$

We will need the following estimate:

Lemma 2.10. *For all $u \in BV(\Omega)$, all $s \in L^\infty(\Omega)$, and all $\mathbf{z} \in X(\Omega)$, it holds:*

$$\left| \int_\Omega (su\,\mathrm{d}x + \mathrm{d}(\mathbf{z}, Du)) \right| \le \|(s, \mathbf{z})\|_\infty |(u, Du)|(\Omega). \tag{2.7.4}$$

Proof. Let $\{u_n\} \subset W^{1,1}(\Omega)$ be an optimal sequence for Lemma 2.1, that is,

$$\int_\Omega \sqrt{u_n^2 + |\nabla u_n|^2} \mathrm{d}x \to \int_\Omega \sqrt{u^2 + |\nabla u|^2} \mathrm{d}x + |D^s u|(\Omega) \overset{(2.1.7),(2.1.8)}{=} |(u, Du)|(\Omega) \qquad (2.7.5)$$

as $n \to +\infty$. Up to a subsequence, we also have that $u_n \to u$ weakly in $BV(\Omega)$ and in $L^{\frac{N}{N-1}}(\Omega)$: hence

$$\langle (\mathbf{z}, Du), \varphi \rangle \overset{(2.7.1)}{=} \lim_{n\to+\infty} \left(- \int_\Omega u_n \varphi \operatorname{div} \mathbf{z} \, \mathrm{d}x - \int_\Omega u_n \mathbf{z} \cdot \nabla\varphi \, \mathrm{d}x \right) = \lim_{n\to+\infty} \int_\Omega \varphi \mathbf{z} \cdot \nabla u_n \mathrm{d}x$$

for all $\varphi \in C_c^\infty(\Omega)$. Therefore

$$\left| \int_\Omega \varphi \, (su \, \mathrm{d}x + \mathrm{d}(\mathbf{z}, Du)) \right| = \lim_{n\to\infty} \left| \int_\Omega \varphi (s u_n + \mathbf{z} \cdot \nabla u_n) \mathrm{d}x \right| \overset{(2.7.5)}{\leq} \|\varphi\|_\infty \|(s, \mathbf{z})\|_\infty |(u, Du)|(\Omega)$$

and the conclusion follows from the arbitrariness of φ. □

We also recall that given a normed space E and a functional $\psi : E \to [0, +\infty]$, the *polar transformation* of ψ is defined by

$$\widetilde{\psi} : E^* \to [0, +\infty], \quad \widetilde{\psi}(u) = \sup \left\{ \frac{\langle v, u \rangle}{\psi(v)} : \ v \in E \right\}, \qquad (2.7.6)$$

where E^* denotes the dual of E, with pairing $\langle \cdot, \cdot \rangle$, and where we use the convention that $0/0 = 0$ and $0/\infty = 0$. For any $v \in L^p(\Omega)$, let

$$U(v) = \{(s, \mathbf{z}) \in L^\infty(\Omega) \cap X(\Omega) \ : v = s - \operatorname{div} \mathbf{z} \ \text{ a.e. in } \ \Omega\} \qquad (2.7.7)$$

and

$$\psi : L^p(\Omega) \to [0, +\infty], \quad \psi(v) = \inf \{\|(s, \mathbf{z})\|_\infty : \ (s, \mathbf{z}) \in U(v)\} \qquad (2.7.8)$$

with the usual understanding that $\psi(v) = +\infty$ if $U(v) = \emptyset$. The proof of Theorem 2.4 is based on the following result.

Proposition 2.2. *Let ψ be defined by* (2.7.8). *Then $\psi = \widetilde{\phi}$.*

In the next lemma we summarize some properties of the polar transformation which we need. The proofs may be found in [7, Lemma 1.5, Prop. 1.6 and Theorem 1.8].

Lemma 2.11. *Let E be a normed space, and E^* be its dual.*

(*i*) *if $\psi_1, \psi_2 : E \to [0, +\infty]$ are such that $\psi_1 \leq \psi_2$, then $\widetilde{\psi}_1 \geq \widetilde{\psi}_2$.*

If ψ is convex, lower semi-continuous and positive homogeneous of degree 1, *then:*

(*ii*) $\widetilde{\widetilde{\psi}}|_E = \psi$;

(*iii*) *$v \in \partial\psi(u)$ if and only if $\widetilde{\psi}(v) \leq 1$ and $\langle v, u \rangle = \psi(u)$.*

We are now ready to prove Proposition 2.2.

Proof of Proposition 2.2. The polar transformations of ψ and ϕ are given respectively by

$$\widetilde{\psi} : L^q(\Omega) \to [0, +\infty], \qquad \widetilde{\psi}(u) = \sup\left\{ \frac{\int_\Omega uv \, \mathrm{d}x}{\psi(v)} : \ v \in L^p(\Omega) \right\},$$

$$\widetilde{\phi} : L^p(\Omega) \to [0, +\infty], \qquad \widetilde{\phi}(v) = \sup\left\{ \frac{\int_\Omega uv \, \mathrm{d}x}{\phi(u)} : \ u \in L^q(\Omega) \right\}.$$

We first argue that $\psi \geq \widetilde{\phi}$. Let $v \in L^p(\Omega)$. If $\psi(v) = +\infty$ the claim is obvious, hence we assume that $\psi(v) < \infty$. For any $(s, \mathbf{z}) \in U(v)$, we have

$$\begin{aligned}
\widetilde{\phi}(v) \quad &= \quad \sup_{u \in L^q(\Omega)} \frac{\int_\Omega uv \, \mathrm{d}x}{\phi(u)} \overset{(2.7.7)}{=} \sup_{u \in L^q(\Omega)} \frac{\int_\Omega u(s - \operatorname{div} \mathbf{z}) \, \mathrm{d}x}{\phi(u)} \\
&= \quad \sup_{u \in BV(\Omega)} \frac{\int_\Omega u(s - \operatorname{div} \mathbf{z}) \, \mathrm{d}x}{\phi(u)} \qquad \text{(since otherwise } \phi(u) = +\infty) \\
&\overset{(2.7.3)}{=} \quad \sup_{u \in BV(\Omega)} \frac{\int_\Omega us \, \mathrm{d}x + \int_\Omega \mathrm{d}(\mathbf{z}, Du) - \int_{\partial\Omega} u[\mathbf{z}, \mathbf{n}] \, \mathrm{d}\mathcal{H}^{N-1}}{\phi(u)} \\
&\overset{(2.7.4),(2.7.2)}{\leq} \quad \sup_{u \in BV(\Omega)} \frac{\|(s, \mathbf{z})\|_\infty |(u, Du)|(\Omega) + \|\mathbf{z}\|_\infty \int_{\partial\Omega} |u| \mathrm{d}\mathcal{H}^{N-1}}{\phi(u)} \\
&\leq \quad \|(s, \mathbf{z})\|_\infty \sup_{u \in BV(\Omega)} \frac{|(u, Du)|(\Omega) + \int_{\partial\Omega} |u| \mathrm{d}\mathcal{H}^{N-1}}{\phi(u)} \overset{(2.1.8),(2.1.10)}{=} \|(s, \mathbf{z})\|_\infty.
\end{aligned}$$

The inequality now follows taking the infimum over all $(s, \mathbf{z}) \in U(v)$:

$$\widetilde{\phi}(v) \leq \inf_{(s,\mathbf{z}) \in U(v)} \|(s, \mathbf{z})\|_\infty = \psi(v).$$

To prove the opposite inequality, we note that ψ is convex, lower semi-continuous and positive homogeneous of degree 1. Therefore, by Lemma 2.11 (i) and (ii), $\psi \leq \widetilde{\phi}$ if and only if $\phi \leq \widetilde{\psi}$. Let us define

$$D = \left\{ (s, \mathbf{z}) \in C^\infty\left(\overline{\Omega}; \mathbb{R}^{N+1}\right) : \ \|(s, \mathbf{z})\|_\infty \leq 1 \right\}.$$

Then

$$\begin{aligned}
\widetilde{\psi}(u) \quad &= \quad \sup_{v \in L^p(\Omega)} \frac{\int_\Omega uv \, \mathrm{d}x}{\psi(v)} \geq \sup_{(s,\mathbf{z}) \in D} \frac{\int_\Omega u(s - \operatorname{div} \mathbf{z}) \, \mathrm{d}x}{\psi(s - \operatorname{div} \mathbf{z})} \\
&\geq \quad \sup_{(s,\mathbf{z}) \in D} \frac{\int_\Omega u(s - \operatorname{div} \mathbf{z}) \, \mathrm{d}x}{\|(s, \mathbf{z})\|_\infty} \qquad \text{(by definition of } \psi) \\
&\geq \quad \sup_{(s,\mathbf{z}) \in D} \int_\Omega u(s - \operatorname{div} \mathbf{z}) \, \mathrm{d}x \qquad \text{(by definition of } D).
\end{aligned}$$

If $u \notin BV(\Omega)$, then (see [5, Prop. 3.6])

$$\sup\left\{\int_\Omega u \operatorname{div} \mathbf{z}\, \mathrm{d}x : \mathbf{z} \in C_0^\infty\left(\Omega; \mathbb{R}^N\right),\ \|\mathbf{z}\|_\infty \leq 1\right\} = +\infty$$

and therefore (choosing $s = 0$) $\widetilde{\psi}(u) = +\infty$. Otherwise, integrating by parts (here, since $\mathbf{z}$ is smooth, the classical theory of BV functions suffices, see e.g. [5, (3.85)]) we get:

$$\widetilde{\psi}(u) \geq \sup_{(s,\mathbf{z})\in D}\left(\int_\Omega us\, \mathrm{d}x + \int_\Omega \mathbf{z}\cdot \mathrm{d}Du + \int_{\partial\Omega} u\mathbf{z}\cdot\mathbf{n}\mathrm{d}\mathcal{H}^{N-1}\right) \overset{(2.1.9)}{=} \phi(u)\,.$$

□

The characterization of $\partial\phi$ given by Theorem 2.4 now follows from part (*iii*) of Lemma 2.11.

Proof of Theorem 2.4. Since ϕ is convex, lower semi-continuous and positive homogeneous of degree 1, part (*iii*) of Lemma 2.11 implies that

$$v \in \partial\phi(u) \iff \begin{cases} \widetilde{\phi}(v) \leq 1 \\ \int_\Omega uv\, \mathrm{d}x = \phi(u). \end{cases}$$

By Proposition 2.2, $\psi = \widetilde{\phi}$ where ψ is defined in (2.7.8). Therefore: $\forall v \in L^p(\Omega)$

$$\widetilde{\phi}(v) \leq 1 \iff \psi(v) \leq 1 \iff \exists (s,\mathbf{z}) \in U(v) : \|(s,\mathbf{z})\|_\infty \leq 1.$$

In addition, by (2.7.3) we obtain

$$\phi(u) = \int_\Omega uv\, \mathrm{d}x = \int_\Omega su\, \mathrm{d}x + \int_\Omega \mathrm{d}(\mathbf{z}, Du) + \int_{\partial\Omega} u[\mathbf{z},\mathbf{n}]\, \mathrm{d}\mathcal{H}^{N-1}$$

and the proof is complete. □

2.8 Appendix

In this appendix we sketch the derivation of the partial differential equation (2.1.4) from the full model in (0.1.2)-(0.1.3). Let (x, y, z) denote rectangular cartesian coordinates. We restrict attention to the plane-strain shearing of a body which occupies a strip of finite length I in the y-direction, but is unbounded in the x- and z-directions. The plane-strain shearing condition means

$$\mathbf{Tn} = \tau\mathbf{e}_1 \quad \text{on } \partial(\mathbb{R}\times I\times\mathbb{R})$$

with τ constant in space and time. We make the ansatz that the displacement vector has the form $\mathbf{u} = (u(y,t), 0, 0)$. Accordingly, the ansatz for $\mathbf{E}^{\mathrm{e}}$ and $\mathbf{E}^{\mathrm{p}}$ are

$$\mathbf{E}^{\mathrm{e}} = \begin{pmatrix} 0 & \gamma^{\mathrm{e}}(y,t) & 0 \\ \gamma^{\mathrm{e}}(y,t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{E}^{\mathrm{p}} = \begin{pmatrix} 0 & \gamma^{\mathrm{p}}(y,t) & 0 \\ \gamma^{\mathrm{p}}(y,t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the decomposition (1.1.1a) reduces to

$$\gamma^{\mathrm{e}} := \frac{\partial u}{\partial y} - \gamma^{\mathrm{p}}$$

with γ^{e} and γ^{p} the *elastic and plastic strains* respectively. The explicit derivation of (2.1.4) is based on the assumptions that the regime is fully plastified, i.e. the plastic strain satisfies $\dot{\gamma}^{\mathrm{p}} > 0$ in $\mathring{I}$, and that $\eta = 0$. Under these assumptions (1.1.6) reads as

$$\mathbf{T}_0 + \mu L^2 \left(\Delta\mathbf{E}^{\mathrm{p}} - \mathrm{sym}(\nabla \mathrm{div}\mathbf{E}^{\mathrm{p}}) + \tfrac{1}{3}(\mathrm{div}\,\mathrm{div}\mathbf{E}^{\mathrm{p}})\mathbf{I}\right) = Y(E^{\mathrm{p}})\frac{\dot{\mathbf{E}}^{\mathrm{p}}}{d^{\mathrm{p}}} - \ell^2 \mathrm{div}\left(\mathrm{Y}(\mathrm{E}^{\mathrm{p}})\frac{\nabla\dot{\mathbf{E}}^{\mathrm{p}}}{\mathrm{d}^{\mathrm{p}}}\right) \quad (2.8.1)$$

where $d^{\mathrm{p}} = \sqrt{|\dot{\mathbf{E}}^{\mathrm{p}}|^2 + \ell^2|\nabla\dot{\mathbf{E}}^{\mathrm{p}}|^2}$. Since $\gamma^{\mathrm{p}} = \gamma^{\mathrm{p}}(y,t)$, it follows that $\mathrm{div}\mathbf{E}^{\mathrm{p}} = 0$, hence (2.8.1) reduces to

$$\mathbf{T}_0 + \mu L^2 \Delta\mathbf{E}^{\mathrm{p}} = Y(E^{\mathrm{p}})\frac{\dot{\mathbf{E}}^{\mathrm{p}}}{d^{\mathrm{p}}} - \ell^2 \mathrm{div}\left[\mathrm{Y}(\mathrm{E}^{\mathrm{p}})\frac{\nabla\dot{\mathbf{E}}^{\mathrm{p}}}{\mathrm{d}^{\mathrm{p}}}\right]. \quad (2.8.2)$$

One easily sees that $\mathbf{T}_0 = \mathbf{T}$, hence $\mathrm{div}\mathbf{T} = 0$. Together with the boundary conditions, this implies that

$$\mathbf{T}_0 = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Under the further assumption of a constant flow resistance $Y(\cdot) = \sqrt{2}k$, after simple computations we conclude that (2.8.2) translates into

$$\tau + \mu L^2 \partial^2_{yy}\gamma^{\mathrm{p}} = \sqrt{2}k\left[\frac{\dot{\gamma}^{\mathrm{p}}}{d^{\mathrm{p}}} - \ell^2 \partial_y \left(\frac{\partial_y \dot{\gamma}^{\mathrm{p}}}{d^{\mathrm{p}}}\right)\right] \quad (2.8.3)$$

which, after a suitable rescaling, provides (2.1.4).

Chapter 3

Droplets spreading under contact-line friction: asymptotic analysis

3.1 Introduction

3.1.1 The model

Understanding the dynamics of wetting phenomena of droplets on solid substrates is still an ongoing challenge. The difficulty comes from the classical theory of fluids. Indeed, in the Navier-Stokes equations, the constant viscosity coupled with a no-slip boundary condition at the liquid-solid interface results in a nonphysical singularity at moving contact lines, i.e. an infinite rate of energy dissipation [67, 41]. Many models have been proposed in order to remove this singularity (see e.g. [39, 75, 22]). All of them introduce at least one "microscopic" lengthscale in the problem. The most common approach is to introduce effective slip conditions at the liquid-solid interface: the simplest slippage model, the so-called Navier slip, reads as

$$U = \mu B U_\zeta \quad \text{at the liquid-solid interface, } \zeta = 0. \tag{3.1.1}$$

Here we adopt a two-dimensional framework, $(\xi, \zeta) \in \mathbb{R} \times \mathbb{R}_+$ with the solid substrate at $\zeta = 0$, U denotes the horizontal component of the velocity field within the liquid phase, μ denotes the liquid's viscosity and $\mu B \geq 0$ is the so-called slip length. The ratio $1/B$ is to be understood as a friction coefficient between the liquid and the solid.

Away from the contact line where the liquid, the solid and the surrounding vapor meet, slippage models for single-phase flows have survived an extensive crosscheck by MD simulations (see e.g. [86] and the discussion in [78, 80]). However, recent investigations by Qian, Wang and Sheng [78] and by Ren and E [80] have confirmed that, *near the contact* *line region*, slippage models such as (3.1.1) cease to provide a valid description of the dynamics: there, the main driving force which is responsible for the slip is the unbalanced Young's stress. Of particular interest in this note is the contribution by Ren and E [80] and by Ren, Hu and E [81]. There, by a combination of molecular dynamics and continuum thermodynamics, an effective continuum model is derived, in which the unbalanced Young's stress results from the deviation of the contact angle Θ from its static value Θ_S. Such deviation drives the motion of the contact line in a way which, in the simplest case of a linear friction law, reads as follows:

$$
\begin{aligned}
D\gamma(\cos\Theta - \cos\Theta_S) &= U_{CL} && \text{if } \Theta_S > 0 \quad \text{(partial wetting)},\\
D\gamma(\cos\Theta - 1) &= \max\{U_{CL}, 0\} && \text{if } \Theta_S = 0 \quad \text{(complete wetting)}.
\end{aligned}
\tag{3.1.2}
$$

Here U_{CL} is the speed of the contact line, γ denotes the liquid-vapor surface tension, and $1/D$ is an effective friction coefficient *at the contact line*. Note that the dynamic contact angle is strictly larger than the static one if the wet region expands, smaller (or equal, in complete wetting) if it contracts.

All together, (3.1.1) and (3.1.2) introduce two parameters in the problem, B and D, which account for the effective friction at the liquid-solid and liquid-solid-vapor interfaces, respectively. The general goal of this chapter is to discuss the effect of these parameters on the evolution of a droplet, assumed for simplicity to be symmetric, which spreads over an horizontal substrate. To this aim, it is convenient to argue in the regime of lubrication approximation, which we introduce now.

3.1.2 Lubrication approximation and its dissipative structure

Lubrication approximation (see e.g. [75]) is a tool to reduce the complexity of the Navier-Stokes system while retaining the effects of both capillary forces and frictional forces (viscous friction in the bulk, surface friction at the liquid-solid interface, and contact-line friction at the liquid-solid-vapor interface). Lubrication approximation is based on a separation of the (macroscopic) lengthscales, which (in the presence of a contact line) has been rigorously justified in two model cases [53, 70]. Namely, the typical vertical lengthscale Z is assumed to be much smaller than the typical horizontal lengthscale X, and the typical timescale is chosen so to retain the effects of both surface tension and viscosity:

$$\varepsilon = \frac{Z}{X} \ll 1, \quad T = \frac{3\mu}{\gamma}\frac{X^4}{Z^3}.$$

Introducing new independent variables according to the above scaling,

$$(t, x, z) := \left(\frac{\tau}{T}, \frac{\xi}{X}, \frac{\zeta}{Z}\right),$$

and performing a careful asymptotic expansion in ε (see Section 3.3), one obtains a limiting evolution which consists in a fourth order free boundary problem for the normalized height of the liquid film, $h(t,x)$, and the extent of the wetted region, $(s_-(t), s_+(t))$:

$$\begin{cases} h_t + (hu)_x = 0, \ u = (h^2+bh)h_{xxx}, \ h > 0 & \text{in } (s_-(t), s_+(t)) \\ h = 0, \ \frac{\mathrm{d}}{\mathrm{d}t} s_\pm(t) = \lim\limits_{x \to s_\pm(t)^\mp} u & \text{at } x = s_\pm(t) \end{cases} \tag{3.1.3}$$

and the free boundary condition (3.1.2) translates into

$$d\left(h_x^2 - \theta_S^2\right) = \begin{cases} \pm \frac{\mathrm{d}s_\pm}{\mathrm{d}t} & \text{if } \theta_S > 0 \\ \max\left\{\pm \frac{\mathrm{d}s_\pm}{\mathrm{d}t}, 0\right\} & \text{if } \theta_S = 0 \end{cases} \quad \text{at } x = s_\pm(t). \tag{3.1.4}$$

Here u represents the normalized mean horizontal velocity of the liquid phase, $\theta_S = \varepsilon^{-1}\Theta_S$ is the normalized static contact angle, and

$$b = \frac{3\mu B}{Z}, \quad d = \frac{3D\mu X}{2Z}.$$

Now, it follows from a simple asymptotic expansion near the contact lines (see Section 3.4) that the equation in (3.1.3) does not possess receding traveling waves with zero contact angle (see [21, 28] for the general structure of traveling waves for thin-film equations): in other words, for instance, $\frac{\mathrm{d}s_+}{\mathrm{d}t} \geq 0$ whenever $h_x = 0$ at $x = s_+(t)$. Therefore (3.1.4) simplifies to

$$d\left(h_x^2 - \theta_S^2\right) = \pm \frac{\mathrm{d}s_\pm}{\mathrm{d}t} \quad \text{at } x = s_\pm(t). \tag{3.1.5}$$

The free boundary problem (3.1.3)-(3.1.5) preserves the dissipative structure of the original system. The energy

$$E(h(t)) = \int_{s_-(t)}^{s_+(t)} \frac{1}{2}(h_x^2 + \theta_S^2)\mathrm{d}x \tag{3.1.6}$$

corresponds, to leading order in lubrication approximation, to the surface energy of the droplet, and accounts (via θ_S and the Young's law) for all the three surface tension coefficients (liquid/solid, liquid/vapor and solid/vapor) which enter into the system (see e.g. [20]). As formally shown in Section 3.5, a sufficiently smooth solution to (3.1.3)-(3.1.5) is such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{s_-(t)}^{s_+(t)} \frac{1}{2}(h_x^2 + \theta_S^2)\mathrm{d}x = -\frac{1}{2d}\left(\left(\frac{\mathrm{d}s_-}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}s_+}{\mathrm{d}t}\right)^2\right) - \int_{s_-(t)}^{s_+(t)} \frac{u^2}{h+b}\mathrm{d}x. \tag{3.1.7}$$

or, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{s_-(t)}^{s_+(t)} \frac{1}{2}(h_x^2+\theta_S^2)\mathrm{d}x = -\frac{d}{2}\left[(h_x^2(t,s_-(t)) - \theta_S^2)^2 + (h_x^2(t,s_+(t)) - \theta_S^2)^2\right] - \int_{s_-(t)}^{s_+(t)} m(h)h_{xxx}^2\mathrm{d}x. \tag{3.1.8}$$

The two terms at the right-hand side of (3.1.7) encode the two different means of free energy dissipation: the latter, which is standard in this field, represents viscous friction both in the liquid's bulk and at the liquid/solid interface; the former instead represents friction *at the contact line* and is specific to the free boundary condition proposed in [80]. As expected, it vanishes when the effective friction coefficient $1/d$ does.

3.1.3 Scaling laws without contact-line friction

Assume now that the droplet is symmetric, i.e. $s_- = -s_+ = s$, and has unit mass, i.e.

$$M = 1$$

(the case of a general M can be easily recovered by scaling, see §3.7). In classical models, (3.1.5) is replaced by its frictionless counterpart, $1/d = 0$:

$$h_x \equiv -\theta_S \quad \text{at} \;\; x = s(t),$$

which amounts to assume an instantaneous enforcement of equilibrium at the contact line. In this case, the droplet's dynamics are known to be influenced only logarithmically by the slippage model, at least at intermediate timescales. This fact has been first observed by Hocking for $\theta_S > 0$ (see also Cox [34] for the case of rough surfaces) by matched asymptotic methods. More precisely, in [65] a relation is obtained between the contact-line velocity and the *macroscopic contact angle*, θ_m, defined there as the slope of the unique even arc of parabola having the same mass and support at its zero:

$$p(s,x) = \frac{3}{4s^3}(s^2 - x^2)_+, \quad \theta_m = |\partial_x p(s,s)| = \frac{3}{2s^2}. \tag{3.1.9}$$

In the present two-dimensional case, it reads as follows:

$$\theta_m^3 \sim \theta_S^3 + 3s' \log\left(\frac{s\theta}{b}\right). \tag{3.1.10}$$

In the case $\theta_S = 0$, the same logarithmic correction was obtained by Hocking in [66] and leads to the following scaling law for the speed of the contact line, which is often referred to as the logarithmic correction to Tanner's law [85]:

$$s \sim \left(\frac{t}{\log\left(\frac{1}{b^7 t}\right)}\right)^{1/7}. \tag{3.1.11}$$

The scaling law (3.1.11) was then inferred in [17] by a different formal argument which used quasi-selfsimilar solutions, and rigorously derived in [52] for the boundary of the "macroscopic support", $(-a(t), a(t)) = \{h(t,\cdot) > b\}$, i.e. replacing $s(t)$ by $a(t)$ in (3.1.9) and (3.1.11). In the latter two contributions, the time window of validity of (3.1.11) is also obtained:

$$s_0^7 \log\left(\frac{1}{bs_0}\right) \ll t \ll b^{-7}. \qquad (3.1.12)$$

Note that the appearance of an intermediate timescale is real: on one hand, it takes a certain time for the droplet to forget its initial shape; on the other hand, for large times $h \ll b$ on the whole support, hence the evolution is governed by slippage alone and s will scale like $t^{1/6}$. Again in complete wetting, analogous logarithmic corrections were obtained by de Gennes [39] for a related model in which the contact angle condition is replaced by the action of van der Waals forces.

3.1.4 Scaling laws with contact-line friction

In the presence of contact-line friction the situation is more complicated, since the scaling laws will depend not only on whether θ_S is zero or not, but also on the relation between the two parameters b and d. In particular, due to presence of two parameters, more than one intermediate scaling law should be expected in general. Indeed, in [81], formal considerations based on the dissipation relation (3.1.7) have been worked out in the complete wetting regime, $\Theta_S = 0$. Three timescales are identified:

- an early stage, dominated by contact-line friction, where $s(t) \sim t^{1/5}$;
- a moderate stage, dominated by viscous friction, where $s(t) \sim t^{1/7}$;
- a final stage, dominated by surface friction, where $s(t) \sim t^{1/6}$.

Such behavior has been validated by numerical simulations of (3.1.3)-(3.1.5). The goal of this contribution is to give a more precise and more quantitative description of these scaling laws, in the spirit of (3.1.10), (3.1.11) and (3.1.12), covering also the case of partial wetting (see §3.2.2). As a by-product, we will obtain a matched asymptotic expansion of solutions to (3.1.3)-(3.1.5) for a wide class of free boundary conditions relating the speed and the contact angle.

3.2 Results and outline

3.2.1 Traveling waves

In Section 3.4 we heuristically classify the traveling-wave solutions to (3.1.3)-(3.1.5), i.e. the solutions to

$$\begin{cases} -U = (f^2 + bf) f_{\xi\xi\xi}, \ f > 0 & \text{in } (0, +\infty), \\ f = 0, \ f_\xi = \theta & \text{at } \xi = 0. \end{cases} \qquad (3.2.1)$$

In particular, we argue that (3.1.3) is expected to have a unique advancing front which displays a "linear" (up to a log-correction) behavior at infinity. This is an important prerequisite, since in the case of a spreading droplet, the local behavior near the contact line is that of an advancing traveling wave, whose profile is determined by "matching" it to the bulk region. This procedure has been followed in the past by many authors [65, 66, 34, 56, 43, 63] in order to obtain qualitative information on the macroscopic dynamics. In all of these papers, the matching condition indeed selects the solution to (3.2.1) which displays the "linear" behavior at infinity. Though it is quite clear from the heuristics in Section 3.4 that such traveling wave exists and is unique, we were unable to find a proof in the literature. Therefore we will provide it in Section 3.6 (see also [31]). In fact, it is harmless to consider a velocity field U which, instead of being constant, varies smoothly between two limiting positive values. Thus, we will prove the following, slightly more general result:

Theorem 3.1. *For any $\theta \geq 0$ and any $U \in C([0,+\infty))$ non-negative, bounded, and such that* $\inf U > 0$ *if $\theta = 0$, there exists a unique solution $f \in C^1([0,+\infty)) \cap C^3((0,+\infty))$ of* (3.2.1) *such that $f_{\xi\xi}(\xi) \to 0$ as $\xi \to +\infty$.*

Its proof follows the general approach of [45], where a similar equation was considered in a bounded domain: the proof of existence is based on the construction of a solution operator via the Green's function, whereas uniqueness relies on estimates of the solution's behavior near the domain's boundaries. However the details differ quite a bit from those in [45], due to the unboundedness of the domain and the different boundary conditions (both at zero and at infinity).

3.2.2 Scaling laws

From Section 3.7 on, we restrict our analysis to the case of a symmetric droplet: hence we look at

$$\begin{cases} h_t + (hu)_x = 0, \ u = (h^2 + bh)h_{xxx} & \text{in } (0, s(t)) \\ h = 0, \ \frac{\mathrm{d}}{\mathrm{d}t}s(t) = \lim\limits_{x \to s(t)^-} u & \text{at } x = s(t) \\ h_x = h_{xxx} = 0 & \text{at } x = 0 \end{cases} \tag{3.2.2}$$

with the contact-line condition

$$d\left(h_x^2 - \theta_S^2\right) = \frac{\mathrm{d}s}{\mathrm{d}t} \quad \text{at } x = s(t). \tag{3.2.3}$$

In §3.7 we perform a renormalization of (3.2.2)-(3.2.3) which highlights the crucial role of the parameter

$$k = \frac{dM}{b^2},$$

which may be seen as a measure of the relative strength of surface friction versus contact-line friction. In summarizing the further results of this chapter (see also [32]), we assume once again that

$$M = 1$$

and we disregard universal constants.

Scaling laws in complete wetting

If $\theta_S = 0$, we will argue that:

(A) for a stronger contact-line friction, $d \lesssim b^2$, the droplet displays an early timescale dominated by contact-line friction and a final timescale dominated by surface friction:

$$s \sim \begin{cases} (dt)^{1/5} & \text{if } \frac{s_0^5}{d} \ll t \ll \frac{b^5}{d^6} \quad (\text{and } s_0 \ll \frac{b}{d}) \\ (bt)^{1/6} & \text{if } t \gg \frac{b^5}{d^6}; \end{cases} \tag{3.2.4}$$

(B) for a stronger surface friction, $b^2 \ll d$, the droplet displays an early timescale dominated by contact-line friction, a moderate timescale dominated by viscous friction, and a final timescale dominated by surface friction:

$$s \sim \begin{cases} (dt)^{1/5} & \text{if } \frac{s_0^5}{d} \ll t \ll \frac{1}{d^{7/2}\log^{5/2}\frac{d}{b^2}} \ (\text{and } s_0^2 \ll \frac{1}{d\log\frac{d}{b^2}}) \\ \left(\frac{t}{\log\frac{1}{b^7 t}}\right)^{1/7} & \text{if } \frac{1}{d^{7/2}\log^{5/2}\frac{d}{b^2}} \ll t \ll b^{-7} \\ (bt)^{1/6} & \text{if } t \gg b^{-7}. \end{cases} \tag{3.2.5}$$

The scaling laws in (B) quantify those predicted in [81]. A main difference may be noted:

- for a stronger contact-line friction, case (A), the system bypasses the moderate timescale dominated by viscous friction.

One also notices that:

- for a stronger surface friction, case (B), the moderate regime is logarithmically corrected by surface friction, as in the case of zero contact-line friction (see (3.1.11)-(3.1.12));

- all timescales, besides the final one, depend both on surface and on contact-line friction.

As already pointed out in §3.1.3, the lower bounds on the initial times are real: they correspond to the time that the system needs to “forget” its initial shape and to relax to a quasi-static configuration.

A heuristic argument and its limitation

In §3.8 we present, in the case of complete wetting, a simple heuristic argument based on the dissipation relation (3.1.7) and already used in this framework (see the discussion in §3.8). It turns out that this argument is already capable to predict (3.2.4) and (3.2.5). However, it relies on quite a heavy hypothesis: the quasi-static equilibrium configuration of h (see (3.1.9)) must be postulated *up to the contact-line*. This corresponds to assuming *a-priori* that the microscopic contact angle θ is "relatively close" to θ_m. Such fact may not be true, especially in complete wetting, since the slope might vary abruptly near the contact line. It should instead be demonstrated: indeed, the discrepancy between effective and microscopic contact angles is probably the main object of interest in this matter, especially in this case where a speed-dependent contact angle condition is postulated.

Matched asymptotic analysis

In order to overcome such a strong limitation, in §3.9 we work out a matched asymptotic study of (3.2.2)-(3.2.3). After the works of Hocking [65, 66] and of Cox [34], matched asymptotic with speed-dependent contact angle conditions have been extensively performed in the past [56, 43, 63]. However, none of them includes (3.2.3), and the scaling assumptions used are not always sharp or easy to reconstruct. Hence, here we extend, modify and simplify the asymptotic in a way which includes (3.2.3) and keeps track of all the assumptions used. Up to the extent we need for (3.2.3), we may argue for a rather general relation between speed and contact angle,

$$|h_x(t, s(t))| = \theta = \theta(s'(t), \theta_S), \quad \theta \gtreqless \theta_S \ \text{ for } \ s' \gtreqless 0, \tag{3.2.6}$$

which makes the results potentially applicable to different boundary conditions and therefore, we believe, of independent interest. The asymptotic is based on the assumptions that the evolution is "slow" and quasi-static, and yields the following: if

$$0 \le s^6 s' \ll 1 \quad \text{and} \quad bs \ll 1, \tag{3.2.7}$$

then

$$\theta_m^3 \sim \begin{cases} \theta^3 + 3s' \log\left(\frac{s\theta}{b}\right) & \text{if } \ b \ll s\theta \quad \text{ and } \ s' \ll \theta^3 \\ 3s' \log\left(\frac{s(s')^{1/3}}{b}\right) & \text{if } \ b^3 \ll s^3 s' \quad \text{and } \ s' \gg \theta^3, \end{cases} \tag{3.2.8}$$

where θ_m is defined as in (3.1.9). The first assumption in (3.2.7) says that the droplet spreads and spreads slowly: in particular, it rules out of the analysis an initial timescale during which the evolution is governed by the droplet's initial shape. The second one ensures (via mass conservation) that $h(t, \cdot) \gg b$ on most of its support, which motivates calling θ_m a *macroscopic contact angle*. Of course, (3.2.8) recovers the earlier results in [65, 66] when

$\theta \equiv \theta_S$. In §3.9 we also obtain an asymptotic relation between s and θ, valid when $h \ll b$ but the evolution is "slow" and quasi-static:

$$\left(\frac{3}{2s^2}\right)^3 \sim \theta^3 \quad \text{if } bs \gg 1,\ s^5 s' \ll b, \text{ and } \theta > 0. \tag{3.2.9}$$

In §3.10 and §3.11 we consider the specific contact-angle condition (3.2.3) in the regime of complete wetting, and we use ode arguments to pass from (3.2.8) and (3.2.9) to the early and moderate scaling laws in (3.2.4) and (3.2.5). In the particular case $1/d = 0$, (3.1.11)-(3.1.12) are also recovered. The scaling laws for long time are obtained in §3.12 by a different asymptotic which assumes a quasi-selfsimilar profile of the solution. As a consequence, one may conclude that θ_m and θ are indeed "relatively close" to each other, which a-posteriori justifies the heuristic argument described in §3.2.2.

Scaling laws in partial wetting

In the case of partial wetting, $\theta_S > 0$, the profile of a spreading droplet converges (exponentially, see §3.13) to the unique steady state with given mass and contact angle θ_S as $t \to +\infty$: assuming $M = 1$,

$$h \to \frac{3}{4s_\infty^3}(s_\infty^2 - x^2)_+ \quad \text{and} \quad s \uparrow s_\infty = \sqrt{\frac{3}{2\theta_S}} \quad \text{as } t \to +\infty.$$

We focus on the most interesting case of

$$\theta_S \gg b^2, \quad \text{i.e. } bs_\infty \ll 1,$$

which guarantees the persistence for all times of a macroscopic profile. In §3.13 we argue that, for sufficiently large times, the system evolves according with the Cox-Hocking relation (3.1.10) between the effective and the microscopic contact angle. Hence, also in partial wetting the contact-line friction plays no role for large times. However, it turns out that there are still intermediate timescales which are influenced by contact-line friction. We illustrate the results in words for $M = 1$, neglecting a (logarithmically short) transition timescale (the reader is referred to §3.13 for the precise statements):

(i) if $d \ll \theta_S$, then (3.1.10) is preceded by an early timescale dominated by contact-line friction;

(ii) if $\theta_S \ll d$, then (3.1.10) is preceded by an early timescale dominated by contact-line friction and a moderate timescale dominated by viscous friction.

These results identify the ratio d/θ_S as threshold parameter in the partial wetting regime. In addition, the upper bounds on the timescales permit to quantify the time in which (3.1.10) takes over: again up to a logarithmic correction, the analysis in §3.13 shows that

$$(3.1.10) \quad \Longleftrightarrow \quad t \gg \begin{cases} \frac{1}{d\theta_S^{5/2}} & \text{if } d \ll \theta_S \\ \frac{1}{\theta_S^{7/2}\log^{1/6}\left(\frac{\theta_S}{b^2}\right)} & \text{if } \theta_S \ll d. \end{cases}$$

3.3 Lubrication approximation

Consider a Newtonian liquid placed over a flat solid surface and surrounded by vapor (assumed to have zero viscosity). Let μ and γ denote the viscosity of the liquid and the liquid-vapor surface tension, respectively. We consider a one dimensional geometry, $(\xi, \zeta) \in \mathbb{R} \times (0, \infty)$, with the solid substrate coinciding with $\{\zeta = 0\}$. The region occupied by the liquid at time τ is denoted by $L(\tau)$, and $L = \cup_{\tau>0} L(\tau)$. The so-called "lubrication approximation" of the Navier-Stokes equations is based on a separation of the (macroscopic) length scales: the typical vertical length scale Z is much smaller than the typical horizontal length scale X, and the typical time-scale is chosen so to retain the effects of both surface tension and viscosity:

$$\varepsilon = \frac{Z}{X} \ll 1, \quad T = \frac{3\mu}{\gamma}\frac{X^4}{Z^3}.$$

Introducing new independent variables according to the above scaling,

$$(t, x, z) := \left(\frac{\tau}{T}, \frac{\xi}{X}, \frac{\zeta}{Z}\right),$$

and performing a careful asymptotic expansion in ε (see e.g. [75, 81], the limiting evolution is described by the normalized thickness of the liquid film,

$$h(t, x) := \frac{1}{Z}\mathcal{L}^1\left(\{\zeta > 0 : \ (t, x, \zeta) \in L\}\right),$$

and the normalized average horizontal velocity u,

$$u(t, x) := \frac{T}{X}\int_{\{\zeta>0: \ (t,x,\zeta)\in L\}} U(t, x, \zeta)\mathrm{d}\zeta.$$

Namely, one obtains

$$h_t + (hu)_x = 0, \ u = (h^2 + bh)h_{xxx} \quad \text{in } \{h > 0\}, \tag{3.3.1}$$

where $b = 3\mu B/Z$ and

$$\{h(t) > 0\} := \{x : \ h(t, x) > 0\}, \quad \{h > 0\} := \bigcup_{t>0}\{h(t) > 0\}.$$

We now translate (3.1.2) in the lubrication regime. Let $\{h(t) > 0\} = (s_-(t), s_+(t))$. By symmetry reasons, it suffices to consider the left contact line, $x = s_-(t)$: let therefore

$$\theta := h_x(t, s_-(t)).$$

At $x = s_-(t)$ we have

$$\Theta = \tan(\varepsilon\theta) \sim \varepsilon\theta \quad \text{for } \varepsilon \ll 1.$$

Accordingly, let $\Theta_S = \varepsilon\theta_S$. Because of the scaling, $U = \frac{X}{T}u = \frac{\gamma}{3\mu}\varepsilon^3 u$. Therefore (3.1.2) reads as

$$\frac{\gamma}{3\mu}\varepsilon^3 u = D\gamma\left(\cos(\varepsilon\theta) - \cos(\varepsilon\theta_S)\right) \sim \frac{\varepsilon^2}{2}D\gamma(\theta_S^2 - \theta^2) \quad \text{at } (t, s_-(t)).$$

Letting $d = \frac{3D\mu X}{2Z}$, we obtain

$$u \sim d(\theta_S^2 - \theta^2) \quad \text{at } (t, s_-(t)).$$

By symmetry, we conclude that the lubrication approximation of (3.1.2) is

$$u = \pm d\left(h_x^2 - \theta_S^2\right) \quad \text{at } x = s_\pm(t). \tag{3.3.2}$$

Collecting (3.3.1) and (3.3.2) and including the kinematic condition $s'(t) = u$ at $x = s_\pm(t)$, we obtain the free boundary problem (3.2.2).

3.4 Traveling waves

A traveling wave solution to (3.1.3) is of the form

$$h(t, x) = f(\xi), \quad \xi = x + Ut$$

where $U \in \mathbb{R}$ is the wave speed and, of course, $s_-(t) = -Ut$ and $s_+(t) \equiv +\infty$. Hence $U > 0$ ($U < 0$) correspond to an advancing (resp. receding) front. Substituting into (3.1.3) and integrating once, we obtain that f solves

$$\begin{cases} -U = (f^2 + bf)f_{\xi\xi\xi}, \ f > 0 & \text{in } (0, +\infty), \\ f = 0, \ f_\xi = \theta & \text{at } \xi = 0, \end{cases} \tag{3.4.1}$$

with θ to be determined using (3.1.4), which now reads as

$$d\left(\theta^2 - \theta_S^2\right) = \begin{cases} U & \text{if } \theta_S > 0 \\ \max\{U, 0\} & \text{if } \theta_S = 0. \end{cases} \tag{3.4.2}$$

The admissible behaviors of the solutions to (3.4.1) near $\xi = 0$ may be easily ascertained by formal expansions (see [21] and the detailed analysis in [28] for the case $U < 0$). Near the contact line,

$$f(\xi) \sim \begin{cases} \sqrt{\frac{8U}{3b}}\xi^{3/2} & \text{if } \theta = 0 \text{ and } U \geq 0 \\ \theta\xi - \frac{U}{2b\theta}\xi^2\log\xi & \text{if } \theta > 0 \end{cases} \quad \text{as } \xi \to 0. \tag{3.4.3}$$

In particular, as is well-known, traveling waves with $\theta = 0$ only exist if $U \geq 0$. Therefore (3.4.2) simplifies to

$$d\left(\theta^2 - \theta_S^2\right) = U$$

for all θ_S. Rewriting it, we determine θ:

$$\theta := \sqrt{\frac{U}{d} + \theta_S^2}. \tag{3.4.4}$$

It follows immediately from (3.4.4) that $U \geq -d\theta_S^2$, i.e. a front can not recede too fast. In addition, (3.4.4) implies that fronts can only advance in the *complete wetting regime*, characterized by $\theta_S = 0$. On the other hand, (3.4.4) with $\theta = 0$ implies that $U \leq 0$. Hence the former behavior in (3.4.3) is excluded (besides the trivial case $U = 0$), and we conclude that

$$f(\xi) \sim \theta\xi - \frac{U}{2b\theta}\xi^2 \log\xi \quad \text{as } \xi \to 0 \quad \text{for any } U \geq -d\theta_S^2. \tag{3.4.5}$$

The local behavior given by (3.4.5) will be used in Section 3.5 in order to motivate the aforementioned dissipative structure of (3.1.3).

For large ξ there is a one-parameter family of quadratic behaviors,

$$f(\xi) \sim A\xi^2 + \frac{U}{6A^2\xi}, \quad A \in \mathbb{R},$$

plus a single "linear" (logarithmically corrected) one if $U > 0$:

$$f(\xi) \sim (3U)^{1/3}\xi(\log\xi)^{1/3} \quad \text{as } \xi \to +\infty \quad \text{if } U > 0. \tag{3.4.6}$$

These heuristics suggest that for any $U > -d\theta_S^2$ there is a one-parameter family of traveling-wave solutions, a uniqueness criterion being a suitable condition at $+\infty$. In Section 3.6 we will make this assertion rigorous by proving Theorem 3.1. Before that, let us use (3.4.5) in order to formally infer the dissipation relation (3.1.7).

3.5 The dissipative structure

We now formally show that, for sufficiently smooth solutions, the dissipation relation (3.1.7) holds. For the ease of the presentation, we argue in the case of a symmetric droplet, the extension to the general case being harmless. We thus consider (3.2.2)-(3.2.3). Let $E(h)$ be the symmetric version of (3.1.6). We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E(h(t)) &= \frac{s'(t)}{2}(h_x^2(t,s(t)) + \theta_S^2) + \int_0^{s(t)} h_x h_{xt}\,\mathrm{d}x \\ &= \frac{s'(t)}{2}(h_x^2(t,s(t)) + \theta_S^2) + [h_x h_t]_0^{s(t)} - \int_0^{s(t)} h_t h_{xx}\,\mathrm{d}x. \end{aligned} \tag{3.5.1}$$

Since $h(t, s(t)) = 0$ for all t, we have

$$h_t(t, s(t)) = -s'(t)h_x(t, s(t)).$$

Therefore, using the boundary conditions in (3.2.2)-(3.2.3), the two boundary terms in (3.5.1) combine into

$$\begin{aligned}
\frac{s'(t)}{2}(h_x^2(t, s(t)) + \theta_S^2) + [h_x h_t]_0^{s(t)} &= \frac{s'(t)}{2}(h_x^2(t, s(t)) + \theta_S^2) - s'(t)h_x^2(t, s(t)) \\
&= \frac{s'(t)}{2}(\theta_S^2 - h_x^2(t, s(t))) \\
&= -\frac{1}{2d}(s'(t))^2. \qquad (3.5.2)
\end{aligned}$$

For the integral term in (3.5.1), after one integration by parts we obtain

$$\begin{aligned}
-\int_0^{s(t)} h_t h_{xx} \mathrm{d}x &= \int_0^{s(t)} h_{xx}(hu)_x \mathrm{d}x \\
&= [h_{xx}hu]_0^{s(t)} - \int_0^{s(t)} (h^3 + bh^2)h_{xxx}^2 \mathrm{d}x. \qquad (3.5.3)
\end{aligned}$$

The boundary term in (3.5.3) is zero at zero. At $s(t)$, we assume that h has the same local expansion of a traveling wave (see (3.4.5)): then, with $\theta = |h_x(t, s(t))|$ and $\xi = s(t) - x$,

$$\lim_{x\to s(t)^-} h(t, x)h_{xx}(t, x)u(t, x) = \lim_{\xi\to 0^+} -\left(\theta\xi \cdot \frac{s'(t)}{b\theta} \log\xi \cdot s'(t)\right) = 0. \qquad (3.5.4)$$

Combining (3.5.2)-(3.5.4) into (3.5.1) we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{s(t)} \frac{1}{2}(h_x^2 + \theta_S^2)\mathrm{d}x = -\frac{(s'(t))^2}{2d} - \int_0^{s(t)} (h^3 + bh^2)h_{xxx}^2 \mathrm{d}x,$$

and the symmetric versions of (3.1.7) and (3.1.8) follow observing that

$$(h^3 + bh^2)h_{xxx}^2 = h(h^2 + bh)h_{xxx}^2 = h\frac{u^2}{h^2 + bh} = \frac{u^2}{h + b}$$

and using boundary conditions.

3.6 Proof of Theorem 3.1

Scaling both the unknown function and the independent variable as

$$v(r) = b^{-1} f(y), \quad r = b^{-1}y,$$

(3.2.1) may be rewritten as follows:

$$\begin{cases} v''' = -\dfrac{U}{v^2+v}, \ v > 0 & \text{in } (0,+\infty) \\ v = 0, \ v' = \theta & \text{at } r = 0 \\ v'' \to 0 & \text{as } r \to +\infty, \end{cases} \tag{3.6.1}$$

where throughout this section $'$ denotes the derivative with respect to r. Hence, we will equivalently show that for any $\theta \geq 0$ and any non-negative $U \in C([0,+\infty))$ such that $U \leq U_1$, and $U \geq U_0 > 0$ if $\theta = 0$, there exists a unique solution $v \in C^1([0,\infty)) \cap C^3((0,\infty))$ of (3.6.1). We split the proof into various steps.

3.6.1 Approximating problems

For any $\varepsilon > 0$, let us consider the following approximating problem:

$$(P_\varepsilon) \qquad \begin{cases} v_\varepsilon''' = -\dfrac{U}{v_\varepsilon^2 + v_\varepsilon} & \text{in } r \in (0, 1/\varepsilon) \\ v_\varepsilon = \varepsilon, \ v_\varepsilon' = \theta & \text{at } r = 0 \\ v_\varepsilon'' = 0 & \text{at } r = 1/\varepsilon. \end{cases}$$

We associate to (P_ε) the following linear problem:

$$(PL_\varepsilon) \qquad \begin{cases} v_\varepsilon''' = f & \text{in } (0, 1/\varepsilon) \\ v_\varepsilon = \varepsilon, \ v_\varepsilon' = \theta & \text{at } r = 0 \\ v_\varepsilon'' = 0 & \text{at } r = 1/\varepsilon. \end{cases}$$

We also introduce the Green's function associated to the homogeneous part of (PL_ε):

$$\begin{cases} G_{\varepsilon rrr} = \delta(r-t) & \text{on } (0,1/\varepsilon) \times (0,1/\varepsilon) \\ G_\varepsilon(0,t) = G_{\varepsilon r}(0,t) = 0 & \\ G_{\varepsilon rr}(1/\varepsilon, t) = 0. & \end{cases}$$

Simple computations show that G_ε is in fact independent of ε, and is given by

$$G_\varepsilon(r,t) = G(r,t) = \begin{cases} G_+(r,t) = \frac{t^2}{2} - rt & \text{if } r \geq t \\ G_-(r,t) = -\frac{r^2}{2} & \text{if } r \leq t. \end{cases} \tag{3.6.2}$$

It is standard to check that for any $f \in C([0,+\infty))$ the function

$$r \mapsto \varepsilon + \theta r + \int_0^{1/\varepsilon} G(r,t) f(t) \mathrm{d}t$$

is a $C^3((0,+\infty))$-solution of (PL_ε).

To prove the existence of a solution to (P_ε), we apply Schauder's fixed point theorem. Let S be the closed, bounded and convex subset of the real Banach space $X = C([0,1/\varepsilon])$ defined by

$$S = \{g \in C([0,1/\varepsilon]) :\ \varepsilon \le g \le M_\varepsilon\}, \tag{3.6.3}$$

where $M_\varepsilon > 0$ is a constant to be chosen below. On S we define the (nonlinear) operator F by setting

$$F : S \ni g \mapsto v, \quad \text{where} \quad v(r) := \varepsilon + \theta r - \int_0^{1/\varepsilon} G(r,t)\frac{U(t)}{g^2(t)+g(t)}\mathrm{d}t.$$

Note that $v \in C([0,1/\varepsilon])$ and $G_{rr} \le 0$. Since $g \ge \varepsilon > 0$, we then have

$$v''(r) \ge 0, \quad v'(r) \ge \theta, \quad \text{and} \quad v(r) \ge \varepsilon + \theta r \quad \text{for all } r \in [0,1/\varepsilon]. \tag{3.6.4}$$

In addition

$$v''(r) = -\int_0^{1/\varepsilon} G_{rr}(r,t)\frac{U(t)}{g^2(t)+g(t)}\mathrm{d}t \overset{(3.6.2),(3.6.3)}{\le} \int_0^{1/\varepsilon} \frac{U_1}{\varepsilon}\mathrm{d}t = \frac{U_1}{\varepsilon^2}. \tag{3.6.5}$$

Hence

$$v \le \varepsilon + \frac{\theta}{\varepsilon} + \frac{U_1}{2\varepsilon^4} =: M_\varepsilon,$$

so that $F(S) \subset S$. Together with $v(0) = \varepsilon$ and $v'(0) = \theta$, (3.6.4) and (3.6.5) imply that $F(S)$ is a bounded subset of $C^2([0,1/\varepsilon])$: in particular, by the Ascoli-Arzelà Theorem, $F(S)$ is relatively compact in $C^0([0,1/\varepsilon])$, and the existence of a fixed point v_ε follows from Schauder's fixed point theorem:

$$v_\varepsilon(r) = \varepsilon + \theta r - \int_0^{1/\varepsilon} G(r,t)\frac{U(t)}{v_\varepsilon^2(t)+v_\varepsilon(t)}\mathrm{d}t, \tag{3.6.6}$$

and from (3.6.4) we also have

$$v_\varepsilon''(r) \ge 0, \quad v_\varepsilon'(r) \ge \theta, \quad \text{and} \quad v_\varepsilon(r) \ge \varepsilon + \theta r \quad \text{for all } r \in [0,1/\varepsilon]. \tag{3.6.7}$$

3.6.2 Existence for $\theta > 0$

We now pass to the limit as $\varepsilon \downarrow 0$ in the approximating problem (P_ε). First we consider the case $\theta > 0$. It follows from (3.6.7) that

$$v_\varepsilon^2(t) + v_\varepsilon(t) \ge (\varepsilon+\theta t)^2 + (\varepsilon + \theta t) = (\varepsilon+\theta t)(1+\varepsilon+\theta t) \ge \theta t.$$

Therefore

$$\begin{aligned}
v_\varepsilon'(r) &\overset{(3.6.2)}{=} \theta + \int_0^r t\frac{U(t)}{v_\varepsilon^2(t)+v_\varepsilon(t)}\mathrm{d}t + r\int_r^{1/\varepsilon}\frac{U(t)}{v_\varepsilon^2(t)+v_\varepsilon(t)}\mathrm{d}t \\
&\le \theta + U_1\left(\int_0^r t\frac{1}{\theta t}\mathrm{d}t + r\int_r^{1/\varepsilon}\frac{\mathrm{d}t}{(\varepsilon+\theta t)(1+\varepsilon+\theta t)}\right) \\
&= \theta + U_1\frac{r}{\theta}\left(1+\log\left(\frac{\varepsilon+\theta/\varepsilon}{1+\varepsilon+\theta/\varepsilon}\right)-\log\left(\frac{\varepsilon+\theta r}{1+\varepsilon+\theta r}\right)\right) \\
&\le \theta + U_1\frac{r}{\theta}\left(1+\log\left(1+\frac{1}{\theta r}\right)\right). \qquad (3.6.8)
\end{aligned}$$

Similarly,

$$v_\varepsilon''(r) = \int_r^{1/\varepsilon}\frac{U(t)\mathrm{d}t}{v_\varepsilon^2(t)+v_\varepsilon(t)} \le U_1\int_r^{1/\varepsilon}\frac{\mathrm{d}t}{v_\varepsilon^2} \le U_1\int_r^{1/\varepsilon}\frac{\mathrm{d}t}{\theta^2 t^2} \le \frac{U_1}{\theta^2 r}. \qquad (3.6.9)$$

Together with $v_\varepsilon(0) = \varepsilon$, the estimates (3.6.8) and (3.6.9) imply that

$$\|v_\varepsilon\|_{C^1([0,R])} + \|v_\varepsilon\|_{C^2([R^{-1},R])} \le K_R \quad \text{for all } R > 0.$$

Then, by the Ascoli-Arzelà theorem, a subsequence (which we do not relabel) exists such that

$$v_\epsilon \to v \quad \text{in} \quad C_{loc}([0,+\infty)) \cap C^2_{loc}((0,\infty)).$$

In particular, $v(0) = 0$. By (3.6.7), $v > 0$ in $(0,\infty)$: hence, passing to the limit in the equation of (P_ε) we see that v satisfies the differential equation in (3.6.1). Finally, (3.6.7), (3.6.8) and (3.6.9) imply that

$$\theta \le v'(r) \le \theta + U_1\frac{r}{\theta}\left(1+\log\left(1+\frac{1}{\theta r}\right)\right) \quad \text{and} \quad 0 \le v''(r) \le \frac{U_1}{\theta^2 r}, \qquad (3.6.10)$$

hence the boundary conditions are satisfied. This proves the existence of a solution to (3.6.1) if $\theta > 0$.

3.6.3 Existence for $\theta = 0$

In the case $\theta = 0$, we begin noting that

$$\begin{aligned}
v_\varepsilon(r) &\overset{(3.6.2),(3.6.6)}{=} \varepsilon + \int_0^r \frac{\frac{t}{2}(2r-t)U(t)}{v_\varepsilon^2(t)+v_\varepsilon(t)}\mathrm{d}t + \frac{r^2}{2}\int_r^{1/\varepsilon}\frac{U(t)\mathrm{d}t}{v_\varepsilon^2(t)+v_\varepsilon(t)} \\
&\ge \frac{U_0}{2}\int_0^r\frac{t(2r-t)}{v_\varepsilon^2(t)+v_\varepsilon(t)}\mathrm{d}t \overset{(3.6.7)}{\ge} \frac{U_0}{3}\frac{r^3}{v_\varepsilon^2(r)+v_\varepsilon(r)}.
\end{aligned}$$

Hence

$$v_\varepsilon^3(r) + v_\epsilon^2(r) \ge C^{-1}r^3, \qquad (3.6.11)$$

where here and in the rest of this proof $C \geq 1$ denotes a generic positive constant, independent of ε and r. The bound in (3.6.11) implies that

$$v_\varepsilon \geq C^{-1} \min\left\{r, r^{3/2}\right\}, \tag{3.6.12}$$

which in turn yields

$$\begin{aligned} v_\varepsilon^2 + v_\varepsilon &\geq C^{-1} \min\left\{r^2 + r, r^3 + r^{3/2}\right\} = C^{-1} \begin{cases} r^{3/2} & \text{if } r \leq 1 \\ r^2 & \text{if } r \geq 1 \end{cases} \\ &= C^{-1} \max\left\{r^{3/2}, r^2\right\}. \end{aligned}$$

Therefore

$$\begin{aligned} v_\varepsilon'(r) &= \int_0^r t \frac{U(t)}{v_\varepsilon^2(t) + v_\varepsilon(t)} \mathrm{d}t + r \int_r^{1/\varepsilon} \frac{U(t)}{v_\varepsilon^2(t) + v_\varepsilon(t)} \mathrm{d}t \\ &\leq C\left(\int_0^r \frac{t}{t^{3/2}} \mathrm{d}t + r \int_r^{1/\varepsilon} \frac{1}{t^{3/2}} \mathrm{d}t\right) \leq C r^{1/2} \end{aligned} \tag{3.6.13}$$

and, similarly,

$$v_\varepsilon''(r) = \int_r^{1/\varepsilon} \frac{U(t)\mathrm{d}t}{v_\varepsilon^2(t) + v_\varepsilon(t)} \leq C \int_r^{1/\varepsilon} \frac{\mathrm{d}t}{t^2} \leq \frac{C}{r}. \tag{3.6.14}$$

The argument is now identical to *II*, with (3.6.8) and (3.6.9) replaced by (3.6.13) and (3.6.14), respectively.

3.6.4 Uniqueness

Let v_1 and v_2 be two solutions of (3.6.1) and let $w = v_1 - v_2$. Then w satisfies

$$\begin{cases} w''' = U\left(\frac{1}{v_2^2+v_2} - \frac{1}{v_1^2+v_1}\right) & \text{in } (0, +\infty) \\ w(0) = w'(0) = 0 \\ w''(r) \to 0 \quad \text{as } r \to +\infty. \end{cases}$$

Since the function $v \mapsto \frac{1}{v^2+v}$ is decreasing and U is non-negative,

$$w w''' = U(v_1 - v_2)\left(\frac{1}{v_2^2 + v_2} - \frac{1}{v_1^2 + v_1}\right) \geq 0. \tag{3.6.15}$$

Let us define the auxiliary function

$$h(r) := w w'' - \frac{w'^2}{2}.$$

Note that $h'(r) = ww''' \geq 0$, i.e. h is increasing. We claim that

$$h(0) = 0. \tag{3.6.16}$$

If (3.6.16) holds, then the monotonicity of h implies that that $h(r) \geq 0$ for $r > 0$. Thus

$$ww'' \geq \frac{w'^2}{2} \geq 0 \quad r > 0. \tag{3.6.17}$$

As a consequence of (3.6.15) and (3.6.17),

$$0 \leq 2w''w''' = ((w'')^2)'.$$

On the other hand, $w''(r) \to 0$ as $r \to +\infty$, which implies that $w'' \equiv 0$: since $w(0) = w'(0) = 0$, we conclude that $w \equiv 0$.

It remains to show (3.6.16). In view of (3.6.10) and (3.6.12),

$$v_i \geq \begin{cases} \theta r & \text{if } \theta > 0 \\ C^{-1} r^{3/2} & \text{if } \theta = 0 \end{cases} \qquad \text{for } r \leq 1.$$

Hence

$$0 \leq -v_i'''(r) = \frac{U}{v_i^2 + v_i} \leq U_1 \begin{cases} \frac{1}{\theta r} & \text{if } \theta > 0 \\ C r^{-3/2} & \text{if } \theta = 0 \end{cases} \qquad \text{for } r \leq 1.$$

Consequently, we have that

$$0 \leq v_i''(r) \leq C_i \begin{cases} -\log r & \text{if } \theta > 0 \\ r^{-1/2} & \text{if } \theta = 0 \end{cases} \qquad \text{for } r \leq 1/2 \tag{3.6.18}$$

(C_i depends on i through, say, $v_i''(1/2)$) and after two other integrations

$$0 \leq v_i(r) - \theta r \leq C_i \begin{cases} -r^2 \log r & \text{if } \theta > 0 \\ C r^{3/2} & \text{if } \theta = 0 \end{cases} \qquad \text{for } r \leq 1/2. \tag{3.6.19}$$

Therefore, for $r \leq 1/2$ we have

$$|ww''| \overset{(3.6.18),(3.6.19)}{\leq} C \left\{ \begin{matrix} r^2 \log^2 r & \text{if } \theta > 0 \\ r & \text{if } \theta = 0 \end{matrix} \right\} \to 0 \qquad \text{as } r \to 0,$$

and (3.6.16) follows since $w'(0) = 0$.

3.7 Renormalization

In the rest of the Chapter we perform the qualitative analysis of (3.2.2)-(3.2.3). It is convenient to scale all quantities in such a way that the mass is 1 and the equation is parameter-free:

$$x = \frac{M}{b}\hat{x} \quad \text{and} \quad s = \frac{M}{b}\hat{s}, \quad h = b\hat{h}, \quad t = \frac{M^4}{b^7}\hat{t}.$$

In particular, the nonlinearity $m(h) = h^3 + bh^2$ turns into $m(\hat{h}) = \hat{h}^3 + \hat{h}^2$: the transition between the two regimes of m, $h \sim b$, in the new variables occurs at $\hat{h} \sim 1$. The free boundary condition (3.2.3) reads as

$$\frac{dM}{b^2}\left(\hat{h}_{\hat{x}}^2 - \frac{M^2}{b^4}\theta_S^2\right) = \frac{\mathrm{d}\hat{s}}{\mathrm{d}\hat{t}} \quad \text{at } \hat{x} = \hat{s}(\hat{t}).$$

Hence, introducing the parameters

$$\alpha_S = \frac{M}{b^2}\theta_S, \quad k = \frac{dM}{b^2}$$

and removing all hats, (3.2.2)-(3.2.3) read as

$$\begin{cases} h_t + (hu)_x = 0, \ u = (h^2 + h)h_{xxx}, \ h > 0 & \text{in } (0, s(t)) \\ h_x = h_{xxx} = 0 & \text{at } x = 0 \\ h = 0, \ s'(t) = \lim\limits_{x \to s(t)^-} u(t, x) = k\left(h_x^2 - \alpha_S^2\right) & \text{at } x = s(t) \end{cases} \tag{3.7.1}$$

and the dissipation relation (3.1.7) transforms into

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{s(t)} \frac{1}{2}(h_x^2 + \alpha_S^2)\mathrm{d}x = -\frac{1}{2k}(s'(t))^2 - \int_0^{s(t)} \frac{u^2}{h+1}\mathrm{d}x. \tag{3.7.2}$$

3.8 A heuristic argument in complete wetting

As we mentioned earlier, in the case of complete wetting the scaling law (3.1.11) was first observed by Hocking [66] and then rigorously derived in [52] for the boundary $a(t)$ of the "macroscopic support", $(-a(t), a(t)) = \{h(t, \cdot) > b\}$. While Hocking uses careful matched asymptotic expansions, the heuristic behind the rigorous results in [52] is much simpler: it relies on the energy dissipation mechanism encoded by (3.7.2) and it is inspired by that used by de Gennes in [39]; more recently, Glasner [55] has given a detailed interpretation to these heuristic in terms of gradient flows. The essential simplification consists in assuming that most of the energy is contained and dissipated in the macroscopic support (though near its boundary). This allows to avoid all the subtleties of "matching" with a microscopic region near the contact line. However, (3.7.2) contains a term which acts *at the contact line*.

Hence, in revisiting the heuristic in the present case, one is forced to argue in the whole support $(-s(t), s(t))$ rather than just on the macroscopic one.

The crucial assumption is that the evolution is quasi-static, in the sense that the droplet's profile is, at leading order, in equilibrium given mass and support:

$$h \sim \frac{1}{s^3}(s^2 - x^2) \tag{3.8.1}$$

(here and after we disregard universal constants). Then, by a simple computation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^s h_x^2 \mathrm{d}x \sim -\frac{s'}{s^4}.$$

In order to compute the rate of dissipation in $(0, s(t))$, we pick the simplest possible form of the velocity field u such that $u = 0$ at $x = 0$ and $u = s'(t)$ at $x = s(t)$:

$$u \sim \frac{xs'}{s}.$$

Then

$$\int_0^s \frac{u^2}{h+1}\mathrm{d}x \sim \frac{s'^2}{s^2}\left(\int_{\{h\geq 1\}} \frac{x^2}{h}\mathrm{d}x + \int_{\{h\leq 1\}} x^2 \mathrm{d}x\right).$$

In view of (3.8.1), the first integral on the right-hand side is zero if $s \gg 1$. Simple computations using (3.8.1) then yield

$$\int_0^s \frac{u^2}{h+1}\mathrm{d}x \sim \begin{cases} s^2(s')^2 \log\frac{1}{s} & \text{if } s \ll 1 \\ s(s')^2 & \text{if } s \gg 1. \end{cases} \tag{3.8.2}$$

Plugging (3.8.1) and (3.8.2) into (3.7.2) we obtain

$$-\frac{s'}{s^4} \sim \begin{cases} -\frac{(s')^2}{k} - s^2(s')^2 \log\frac{1}{s} & \text{if } s \ll 1 \\ -\frac{(s')^2}{k} - s(s')^2 & \text{if } s \gg 1, \end{cases}$$

that is,

$$\frac{1}{s'} \sim \begin{cases} \frac{s^4}{k} + s^6 \log\frac{1}{s} & \text{if } s \ll 1 \\ \frac{s^4}{k} + s^5 & \text{if } s \gg 1. \end{cases} \tag{3.8.3}$$

We note that

$$\frac{s^4}{k} \gg s^6 \log\frac{1}{s} \iff \frac{1}{k} \gg s^2 \log\frac{1}{s^2} \tag{3.8.4}$$

$$\frac{s^4}{k} \gg s^5 \iff \frac{1}{k} \gg s. \tag{3.8.5}$$

Hence we must distinguish two cases.

(1). If $k \ll 1$, then (3.8.4) is always satisfied for $s \ll 1$, and (3.8.3) reads as

$$\frac{1}{s'} \sim \begin{cases} \frac{s^4}{k} & \text{if } s \ll \frac{1}{k} \\ s^5 & \text{if } s \gg \frac{1}{k}. \end{cases} \tag{3.8.6}$$

We assume that $s_0 \ll 1/k$, so that both the regimes in (3.8.6) are seen. Then, solving (3.8.6) renders

$$s \sim \begin{cases} s_0 + (kt)^{1/5} \sim (kt)^{1/5} & \text{if } \frac{s_0^5}{k} \ll t \ll \frac{1}{k^6} \\ t^{1/6} & \text{if } t \gg \frac{1}{k^6}. \end{cases} \tag{3.8.7}$$

(2). If $k \gg 1$, then (3.8.5) is never satisfied for $s \gg 1$, whereas for $s \ll 1$ (3.8.4) may be inverted as follows:

$$(3.8.4) \iff \frac{1}{k \log k} \gg s^2.$$

Therefore (3.8.3) reads as

$$\frac{1}{s'} \sim \begin{cases} \frac{s^4}{k} & \text{if } s^2 \ll \frac{1}{k \log k} \\ s^6 \log \frac{1}{s} & \text{if } \frac{1}{k \log k} \ll s^2 \ll 1 \\ s^5 & \text{if } s \gg 1. \end{cases}$$

Assuming that $s_0^2 \ll \frac{1}{k \log k}$ and solving this ode (see §3.11.2 for details) yields

$$s \sim \begin{cases} s_0 + (kt)^{1/5} \sim (kt)^{1/5} & \text{if } \frac{s_0^5}{k} \ll t \ll \frac{1}{k^{7/2} \log^{5/2} k} \\ \left(\frac{t}{\log \frac{1}{t}}\right)^{1/7} & \text{if } \frac{1}{k^{7/2} \log^{5/2} k} \ll t \ll 1 \\ t^{1/6} & \text{if } t \gg 1. \end{cases} \tag{3.8.8}$$

Returning to the original variables, (3.8.7) and (3.8.8) coincide with (3.2.4), resp. (3.2.5).

It must be pointed out that (3.8.1) implicitly postulates that the microscopic contact angle θ is "close" to θ_m. To convince the reader we note that, had we used the equivalent formulation of (3.7.2),

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^{s(t)} \frac{1}{2} h_x^2 \mathrm{d}x \overset{(3.7.2),(3.7.1)}{=} -\frac{k}{2} h_x^4|_{x=s(t)} - \int_0^{s(t)} \frac{u^2}{h+1} \mathrm{d}x,$$

with the contact-angle given by

$$h_x|_{x=s(t)} \overset{(3.8.1)}{\sim} -\frac{1}{s^2}, \tag{3.8.9}$$

we would have obtained exactly the same result. But the postulate (3.8.9) is not legitimate a priori and should instead be demonstrated: the slope might vary abruptly near the contact line, and such discrepancy is indeed the main issue to be clarified within this theory. Therefore, in the next section we work out a formal asymptotic study which avoids such postulate.

3.9 Matched asymptotic and the macroscopic contact angle

We work under the more general boundary condition (3.2.6), which under the normalization performed in (3.7) reads as

$$|h_x(t, s(t))| = \alpha(s'(t), \alpha_S), \quad \alpha(s', \alpha_S) \gtreqless \alpha_S \text{ for } s' \gtreqless 0. \tag{3.9.1}$$

Note that the contact-angle condition in (3.7.1) is included in (3.9.1) by letting

$$\alpha(s', \alpha_S) = \sqrt{\frac{s'}{k} + \alpha_S^2}. \tag{3.9.2}$$

The asymptotic is based on two main assumptions:

(I) the evolution within the liquid's bulk is "quasi-static";

(II) the evolution within the liquid's bulk is "slow".

The former is of a qualitative nature. In order to make it more precise, it is convenient to introduce a variable transformation which differs from those used in earlier studies and yields sharp scaling assumptions. It fixes the free boundary and preserves mass:

$$h(t, x) = \frac{1}{s(t)} H(t, y), \quad y = \frac{x}{s(t)} \in (0, 1).$$

Then

$$s^6 s'(yH)_y - s^7 H_t = ((H^3 + sH^2)H_{yyy})_y \quad \text{in } (0, \infty) \times (0, 1). \tag{3.9.3}$$

A quasi-static evolution of the liquid's bulk means that, except maybe for a region where $H \ll 1$, H depends on time only through the modulations given by s and s'. Hence (3.9.3) reads as

$$(s^6 s')(yH)_y \sim ((H^3 + sH^2)H_{yyy})_y,$$

which may be integrated once with respect to y (from $y = 0$), obtaining

$$(s^6 s')yH \sim (H^3 + sH^2)H_{yyy} \quad \text{in } (0, 1). \tag{3.9.4}$$

We now think of H and its derivatives to be $O(1)$; then (3.9.4) shows a scaling-wise natural way to quantify the notion of a "slow" evolution within the liquid's bulk:

$$s^6 s' \ll 1 \text{ if } s \ll 1 \tag{3.9.5}$$

and

$$s^5 s' \ll 1 \text{ if } s \gg 1. \tag{3.9.6}$$

Note that four conditions are to be imposed for H, whereas (3.9.4) is of third order: we'll use

$$H_y|_{y=0} = 0, \quad H|_{y=1} = 0, \quad \int_0^{s(t)} H(s(t), x)\mathrm{d}x = \frac{1}{2} \tag{3.9.7}$$

to determine H, and

$$H_y|_{y=1} = -\alpha s^2 \tag{3.9.8}$$

to determine a relation between s and s'. Provided (3.9.5) holds, we obtain the following asymptotic:

$$\left(\frac{3}{2s^2}\right)^3 \sim \begin{cases} \alpha^3 + 3s' \log(s\alpha) & \text{if } 1 \ll s\alpha \text{ and } s' \ll \alpha^3 \\ 3s' \log(s(s')^{1/3}) & \text{if } 1 \ll s^3 s' \text{ and } s' \gg \alpha^3. \end{cases} \tag{3.9.9}$$

If instead (3.9.6) holds, then

$$\left(\frac{3}{2s^2}\right)^3 \sim \alpha^3 \quad \text{if } \alpha > 0. \tag{3.9.10}$$

Returning to the original variables we obtain (3.2.8) and (3.2.9). In the rest of the section we provide the details for both. The first one is by far less obvious.

3.9.1 Slow evolution with a macroscopic profile: the outer expansion

We first consider the case $s \ll 1$, which in view of mass conservation implies that $H \gg 1$ in the liquid's bulk, i.e., a macroscopic profile exists. Since $s \ll 1$, (3.9.4) and (3.9.8) simplify to

$$(s^6 s')yH \sim H^3 H_{yyy} \quad \text{in } (0,1) \tag{3.9.11}$$

and

$$H_y|_{y=1} = 0, \tag{3.9.12}$$

respectively. In view of (3.9.5), we expand H in powers of $s^6 s'$:

$$H = H_0(y) + (s^6 s')H_1(y) + \text{ l.o.t..}$$

At zeroth order, (3.9.11) and (3.9.7) read as

$$\begin{cases} (H_0)_{yyy} = 0 \quad \text{in } (0,1). \\ (H_0)_y|_{y=0} = 0, \ H_0|_{y=1} = 0, \ \int_0^1 H_0(y)\mathrm{d}y = \frac{1}{2}. \end{cases} \tag{3.9.13}$$

A simple calculation shows that the solution of (3.9.13) is

$$H_0(y) = \frac{3}{4}(1 - y^2).$$

Since the contact-angle condition (3.9.12) can not be satisfied, we proceed to first order. For H_1, we obtain

$$\begin{cases} (H_1)_{yyy} = \frac{y}{H_0^2} = \frac{16y}{9(1-y^2)^2} & \text{in } (0,1) \\ (H_1)_y|_{y=0},\ H_1|_{y=1} = 0,\ \int_0^1 H_1(y)\mathrm{d}y = 0. \end{cases}$$

Three integrations yield, after lengthy but straightforward computations,

$$H_1(y) = \frac{8-9B}{18}(1-y^2) + \frac{4}{9}\left((1+y)\log(1+y) + (1-y)\log(1-y) - 2\log 2\right),$$

where $B = (H_1)_{yy}(0)$ has to be determined via the mass constraint. After an additional calculus exercise, one sees that $B = -4/9$: therefore

$$\begin{aligned} H_y &\sim (H_0 + s^6 s' H_1)_y = -\frac{3}{2}y + s^6 s'\left(-\frac{4}{3}y + \frac{4}{9}\log\left(\frac{1+y}{1-y}\right)\right) \\ &\sim -\frac{3}{2} + \frac{4}{9}s^6 s' \log\left(\frac{1}{1-y}\right) \quad \text{as } y \to 1. \end{aligned} \tag{3.9.14}$$

Since H_y has a logarithmic singularity as $y \to 1$, yet we can not impose the contact-angle condition (3.9.12). This points to the necessity of an inner expansion which permits to cancel the singularity by a suitable matching. Before proceeding we observe that, in terms of the original variables, (3.9.14) reads as

$$h_x \sim -\frac{3}{2s^2} + \frac{4}{9}s^4 s' \log\left(\frac{s}{s-x}\right) \quad \text{for } \frac{s}{s-x} \gg 1. \tag{3.9.15}$$

3.9.2 Slow evolution with a macroscopic profile: the inner expansion

Near the free boundary we follow [65, 66] and use the scaling of a traveling wave,

$$h(t,x) = f(\xi), \quad \xi = s(t) - x.$$

We impose the touchdown condition, $f(0) = 0$, the contact angle condition, $f_\xi = \alpha$ at $\xi = 0$, and the kinematic condition, $f = fu = 0$ at $\xi = 0$. Then, after one integration, we see that for each $t > 0$

$$\begin{cases} f_{\xi\xi\xi} = -\dfrac{s'}{f^2 + f} & \text{for } \xi > 0 \\ f = 0, & \text{at } \xi = 0 \\ f_\xi = \alpha & \text{at } \xi = 0. \end{cases} \tag{3.9.16}$$

In order to achieve a matching with the solution in the outer region, f_ξ must be no more than logarithmically large at infinity. This singles out the unique solution of (3.9.16) such that $f_{\xi\xi} \to 0$ as $\xi \to +\infty$, as given by Theorem 3.1 in §3.2.1. A simple asymptotic expansion of (3.9.16) shows that this solution is such that

$$f(\xi) \sim (3s')^{1/3}\xi(\log\xi)^{1/3} \quad \text{as } \xi \to +\infty \quad \text{if } s' > 0. \tag{3.9.17}$$

In order to infer the asymptotic form of f_ξ up to order 0 in ξ, we distinguish two regimes.

(1). $\beta = \frac{s'}{\alpha^3} \ll 1$. In this case we rescale (3.9.16) according to $\hat{\xi} = \alpha\xi$, so that

$$f_{\hat{\xi}\hat{\xi}\hat{\xi}} = -\frac{\beta}{f^2 + f}, \quad f_{\hat{\xi}}(0) = 1,$$

and we linearize around $\beta = 0$: $f = f_0 + \beta f_1 + ...$ At leading order in β we have

$$f_0 = \hat{\xi}. \tag{3.9.18}$$

At first order in β we have

$$(f_1)_{\hat{\xi}\hat{\xi}\hat{\xi}} = -\frac{1}{\hat{\xi}^2 + \hat{\xi}} \quad \text{for } \xi > 0, \quad f_1(0) = (f_1)_{\hat{\xi}}(0) = 0.$$

After two integrations (using the boundary conditions), we obtain

$$\begin{aligned} (f_1)_{\hat{\xi}} &= (1+\hat{\xi})\log(1+\hat{\xi}) - \hat{\xi}\log\hat{\xi} = (1+\hat{\xi})\left(\log\hat{\xi} + \log\left(1 + \frac{1}{\hat{\xi}}\right)\right) - \hat{\xi}\log\hat{\xi} \\ &\sim 1 + \log\hat{\xi} \quad \text{as } \hat{\xi} \to +\infty. \end{aligned} \tag{3.9.19}$$

Recombining (3.9.18) and (3.9.19), we see that

$$f_{\hat{\xi}} \sim (f_0 + \beta f_1)_\xi \sim 1 + \beta\left(1 + \log\hat{\xi}\right) \quad \text{for } \hat{\xi} \gg 1.$$

Recalling that $\beta \ll 1$, in terms of the outer variable the previous expression reads as follows:

$$-h_x \sim \alpha + \frac{s'}{\alpha^2}\log(\alpha(s-x)) \quad \text{for } \alpha(s-x) \gg 1. \tag{3.9.20}$$

(2). $\beta = \frac{s'}{\alpha^3} \gg 1$. In this case we scale (3.9.16) according to $\hat{\xi} = (s')^{1/3}\xi$, so that

$$f_{\hat{\xi}\hat{\xi}\hat{\xi}} = -\frac{1}{f^2 + f}, \quad f_{\hat{\xi}}(0) = \frac{1}{\beta^{1/3}}.$$

At leading order in $\beta^{-1/3}$ we obtain that

$$\begin{cases} f_{\hat{\xi}\hat{\xi}\hat{\xi}} = -\frac{1}{f^2+f} \quad \text{in } (0, +\infty) \\ f(0) = f_{\hat{\xi}}(0) = 0, \ \lim\limits_{\hat{\xi}\to+\infty} f_{\hat{\xi}\hat{\xi}}(\hat{\xi}) = 0. \end{cases} \tag{3.9.21}$$

Theorem 3.1 guarantees that (3.9.21) has a unique solution, and the asymptotic in (3.9.17) yields

$$f_{\hat{\xi}} \sim \log^{1/3}(\hat{\xi}^3) \quad \text{as } \hat{\xi} \to +\infty.$$

In terms of the outer variables, this means that

$$-h_x \sim \left(s'\log\left(s'(s-x)^3\right)\right)^{1/3} \quad \text{for } s'(s-x)^3 \gg 1. \tag{3.9.22}$$

3.9.3 Slow evolution with a macroscopic profile: the matching

In the outer region, where $h \gg 1$, the velocity field $u = (h^2 + h)h_{xxx} \sim h^2 h_{xxx}$ has the same scaling of h_x^3. Therefore, in order to get a relation between the velocity and the macroscopic contact angle, it is natural to cube the expressions obtained for h_x. For the outer profile, at order one in $s^6 s'$ we find from (3.9.15) that

$$h_x^3 \sim -\left(\frac{3}{2s^2}\right)^3 + 3s' \log\left(\frac{s}{s-x}\right) \quad \text{for } (s-x) \ll s. \tag{3.9.23}$$

For the inner profile, (3.9.20) (at order one in s'/α^3) and (3.9.22) yield

$$h_x^3 \sim \begin{cases} -\alpha^3 - 3s' \log(\alpha(s-x)) \text{ for } (s-x) \gg \frac{1}{\alpha} & \text{if } s' \ll \alpha^3 \\ -3s' \log\left((s')^{1/3}(s-x)\right) \text{ for } (s-x) \gg \frac{1}{(s')^{1/3}} & \text{if } s' \gg \alpha^3. \end{cases} \tag{3.9.24}$$

Having carefully tracked the scaling assumptions both in the outer and in the inner region allows to simplify the matching with respect to [65, 66]. Indeed, we just have to notice that the range of validity of the expansions (3.9.23) and (3.9.24) overlap if $1 \ll s\alpha$ when $s' \ll \alpha^3$, and if $s^3 s' \gg 1$ when $s' \gg \alpha^3$. In these cases we may equate them, and after a cancelation of the $\log(s-x)$ terms we obtain (3.9.9).

3.9.4 Slow evolution without macroscopic profile

Since $s \gg 1$, $H^3 + sH^2 \sim sH^2$, so that (3.9.4) takes the form

$$(s^5 s')y \sim HH_{yyy}.$$

Because of (3.9.6), we expand H in powers of $s^5 s'$: $H = H_0 + (s^5 s')H_1$+l.o.t.. At zeroth order, as in §3.9.1 we recover

$$H_0(y) = \frac{3}{4}(1 - y^2).$$

This solution meets the boundary condition $(H_0)_y = -\alpha s^2$ provided $\alpha > 0$, and in terms of the original variables we obtain (3.9.10).

3.10 Intermediate scaling law in complete wetting without contact-line friction

As a first example, which we shall anyway need later on, we recover the well-known logarithmic correction to Tanner's law stated in (3.1.11)-(3.1.12) in the case that $\alpha \equiv 0$. We will neglect universal constants.

Since $\alpha \equiv 0$, only the second regime in (3.9.9) is relevant. Hence, if

$$s \ll 1 \tag{3.10.1}$$

and if

$$s^6 s' \ll 1, \; s^3 s' \gg 1, \tag{3.10.2}$$

then

$$\frac{1}{s^6} \sim s' \log\left(s^3 s'\right). \tag{3.10.3}$$

We now analyze (3.10.1)-(3.10.3) in the (s, s') plane. First of all, we make (3.10.3) explicit (in what follows we shall often use this type of argument; we provide its details here once for all):

$$\begin{aligned} \frac{1}{s^6} \sim s' \log\left(s^3 s'\right) \quad &\iff \quad \frac{1}{s^3} \sim s^3 s' \log\left(s^3 s'\right) \overset{(3.10.2)}{\gg} 1 \\ &\iff \quad \frac{1}{s^3 \log\left(\frac{1}{s^3}\right)} \sim s^3 s' \\ &\iff \quad s' \sim \frac{1}{s^6 \log\left(\frac{1}{s}\right)}. \end{aligned} \tag{3.10.4}$$

Then we observe that

$$\begin{aligned} s^6 s' \ll 1 \quad &\overset{(3.10.4)}{\iff} \quad \frac{1}{\log\left(\frac{1}{s}\right)} \ll 1 \quad \iff \quad s \ll 1, \\ 1 \ll s^3 s' \quad &\overset{(3.10.4)}{\iff} \quad s^3 \log\left(\frac{1}{s}\right) \ll 1 \quad \iff \quad s \ll 1. \end{aligned}$$

Hence (3.10.1)-(3.10.3) are equivalent to (3.10.1) and (3.10.4). If (3.10.1) is initially true, i.e. $s_0 := s(0) \ll 1$, we may integrate (3.10.4): since

$$\left(s^7 \log\left(\frac{1}{s}\right)\right)' \overset{(3.10.1)}{\sim} s^6 \log\left(\frac{1}{s}\right) s',$$

we obtain

$$s^7 \log\left(\frac{1}{s}\right) \sim t \quad \text{provided} \quad s_0^7 \log\left(\frac{1}{s_0}\right) \ll t. \tag{3.10.5}$$

We now check for how long (3.10.1) remains true:

$$s \ll 1 \quad \iff \quad s^7 \ll 1 \quad \iff \quad \frac{t}{\log\left(\frac{1}{t}\right)} \ll 1 \quad \iff \quad t \ll 1,$$

and in this case (3.10.5) may be inverted as before, yielding

$$s^7 \sim \frac{t}{\log\left(\frac{1}{t}\right)} \quad \text{provided} \quad s_0^7 \log\left(\frac{1}{s_0}\right) \ll t \ll 1 \;\; \text{and} \;\; s_0 \ll 1. \tag{3.10.6}$$

Note that the time window is not empty since $s_0 \ll 1$. Returning to the original variables we recover (3.1.11)-(3.1.12). Large timescales will be analyzed in §3.12.

3.11 Intermediate scaling laws in complete wetting with contact-line friction

We now focus on the specific boundary condition proposed in [80] in the case of complete wetting, $\alpha_S = 0$. In view of (3.9.2), we then have

$$\alpha = \sqrt{s'/k}. \tag{3.11.1}$$

We will neglect universal constants, and argue that:

(I) if $k \lesssim 1$ and $s_0 \ll \frac{1}{k}$, then

$$s(t) \sim (kt)^{1/5} \quad \text{if } \frac{s_0^5}{k} \ll t \ll \frac{1}{k^6}; \tag{3.11.2}$$

(II) if $k \gg 1$ and $s_0^2 \ll \frac{1}{k \log k}$, then

$$s(t) \sim \begin{cases} (kt)^{1/5} & \text{if } \frac{s_0^5}{k} \ll t \ll \frac{1}{k^{7/2} \log^{5/2} k} \\ \left(\frac{t}{\log(\frac{1}{t})}\right)^{1/7} & \text{if } \frac{1}{k^{7/2} \log^{5/2} k} \ll t \ll 1. \end{cases} \tag{3.11.3}$$

Note that the time windows in (3.11.2) and $(3.11.3)_1$ are not empty in view of the assumptions on s_0. Returning to the original variables and letting $M = 1$ we obtain the early and moderate timescales in (3.2.4) and (3.2.5). Large timescales will be analyzed in the next section.

The rest of the section is devoted to showing that (3.9.9) and (3.9.10) imply (3.11.2) and (3.11.3). In §3.11.1 we show that, under (3.11.1), (3.9.9) and (3.9.10) are equivalent to

$$s' \sim \frac{k}{s^4} \quad \text{if } s \ll \frac{1}{k} \qquad \text{for } k \lesssim 1, \tag{3.11.4}$$

$$s' \sim \begin{cases} \frac{k}{s^4} & \text{if } s^2 \ll \frac{1}{k \log k} \\ \frac{1}{s^6 \log(\frac{1}{s})} & \text{if } \frac{1}{k \log k} \ll s^2 \ll 1. \end{cases} \qquad \text{for } k \gg 1. \tag{3.11.5}$$

In §3.11.2 we easily infer (3.11.2) and (3.11.3) from (3.11.4) and (3.11.5).

3.11.1 The ode's for s

Plugging (3.11.1) into (3.9.9), we obtain that if

$$s \ll 1 \tag{3.11.6}$$

and

$$s^6 s' \ll 1, \tag{3.11.7}$$

then

$$\frac{1}{s^6} \sim \begin{cases} \left(\frac{s'}{k}\right)^{3/2} + \frac{3}{2} s' \log\left(\frac{s^2 s'}{k}\right) & \text{if } k \ll s^2 s' \quad \text{and } k^3 \ll s' \\ s' \log\left(s^3 s'\right) & \text{if } 1 \ll s^3 s' \quad \text{and } s' \ll k^3. \end{cases} \tag{3.11.8}$$

The relation in $(3.11.8)_1$ may be split into two regimes:

$$\frac{1}{s^6} \quad \sim \quad \begin{cases} \left(\frac{s'}{k}\right)^{3/2} & \text{if } \left(\frac{s'}{k^3}\right)^{1/2} \gg \log\left(\frac{s^2 s'}{k}\right) \\ s' \log\left(\frac{s^2 s'}{k}\right) & \text{if } \left(\frac{s'}{k^3}\right)^{1/2} \ll \log\left(\frac{s^2 s'}{k}\right). \end{cases}$$

Therefore (3.11.8) is equivalent to

$$\frac{1}{s^6} \sim \begin{cases} \left(\frac{s'}{k}\right)^{3/2} & \text{if } k \ll s^2 s' \quad \text{and } \left(\frac{s'}{k^3}\right)^{1/2} \gg \log\left(\frac{s^2 s'}{k}\right) \\ s' \log\left(\frac{s^2 s'}{k}\right) & \text{if } k \ll s^2 s' \quad \text{and } 1 \ll \left(\frac{s'}{k^3}\right)^{1/2} \ll \log\left(\frac{s^2 s'}{k}\right) \\ s' \log\left(s^3 s'\right) & \text{if } 1 \ll s^3 s' \quad \text{and } s' \ll k^3. \end{cases} \tag{3.11.9}$$

Plugging (3.11.1) into (3.9.10), we obtain

$$\frac{1}{s^6} \sim \left(\frac{s'}{k}\right)^{3/2} \quad \text{if } \alpha > 0,\ s^5 s' \ll 1, \text{ and } s \gg 1. \tag{3.11.10}$$

We now analyze each regime in (3.11.9) and (3.11.10).

- Within $(3.11.9)_1$, we have

$$\frac{1}{s^6} \sim \left(\frac{s'}{k}\right)^{3/2} \quad \Longleftrightarrow \quad s' \sim \frac{k}{s^4}. \tag{3.11.11}$$

Hence

$$\begin{aligned} k \ll s^2 s' \quad &\overset{(3.11.11)}{\Longleftrightarrow} \quad k \ll \frac{k}{s^2} \quad \Longleftrightarrow \quad (3.11.6) \\ \left(\frac{s'}{k^3}\right)^{1/2} \gg \log\left(\frac{s^2 s'}{k}\right) \quad &\overset{(3.11.11)}{\Longleftrightarrow} \quad \frac{1}{ks^2} \gg \log\left(\frac{1}{s^2}\right) \\ &\Longleftrightarrow \quad s^2 \log\left(\frac{1}{s}\right) \ll \frac{1}{k}, \end{aligned} \tag{3.11.12}$$

and (3.11.7) is absorbed by (3.11.6) and (3.11.12):

$$s^6 s' \ll 1 \quad \overset{(3.11.11)}{\Longleftrightarrow} \quad s^2 \ll \frac{1}{k} \quad \overset{(3.11.6)}{\Longleftarrow} \quad (3.11.12).$$

We now distinguish two cases. If $k \lesssim 1$, (3.11.6) guarantees that (3.11.12) holds, and (3.11.4) follows for $s \ll 1$ (the window $1 \ll s \ll \frac{1}{k}$ in (3.11.4) will follow from (3.11.10)). If $k \gg 1$, we may rewrite the constraint in (3.11.12) as

$$s^2 \log\left(\frac{1}{s}\right) \ll \frac{1}{k} \iff s^2 \ll \frac{1}{k \log k} \ll 1. \tag{3.11.13}$$

Hence (3.11.13) enforces (3.11.6) and $(3.11.5)_1$ follows.

• Within $(3.11.9)_2$ we have

$$\frac{1}{s^6} \sim s' \log\left(\frac{s^2 s'}{k}\right) \iff \frac{1}{ks^4} \sim \frac{s^2 s'}{k} \log\left(\frac{s^2 s'}{k}\right).$$

Then

$$ks^4 \ll 1 \iff s^2 s' \gg k \tag{3.11.14}$$

and in this case

$$\begin{aligned} \frac{1}{s^6} \sim s' \log\left(\frac{s^2 s'}{k}\right) \quad &\iff \quad \frac{s^2 s'}{k} \sim \frac{1}{ks^4 \log\left(\frac{1}{ks^4}\right)} \\ &\iff \quad s' \sim \frac{1}{s^6 \log\left(\frac{1}{ks^4}\right)}. \end{aligned} \tag{3.11.15}$$

In particular,

$$\begin{aligned} \left(\frac{s'}{k^3}\right)^{1/2} \ll \log\left(\frac{s^2 s'}{k}\right) \quad &\overset{(3.11.15)}{\iff} \quad \frac{1}{k^3 s^6 \log\left(\frac{1}{ks^4}\right)} \ll \log^2\left(\frac{1}{ks^4 \log\left(\frac{1}{ks^4}\right)}\right) \\ &\overset{(3.11.14)}{\iff} \quad \frac{1}{k^3 s^6} \ll \log^3\left(\frac{1}{ks^4}\right) \\ &\overset{(3.11.14)}{\iff} \quad \frac{1}{k} \ll s^2 \log\frac{1}{s^2}. \end{aligned} \tag{3.11.16}$$

Together with (3.11.6), (3.11.16) implies that $(3.11.9)_2$ is seen only if $1 \ll k$. In this case, the constraints in $(3.11.9)_2$ may be written as follows:

$$1 \ll \left(\frac{s'}{k^3}\right)^{1/2} \ll \log\left(\frac{s^2 s'}{k}\right) \overset{(3.11.16),(3.11.15)}{\iff} \frac{1}{s^2}\log^{1/3}\frac{1}{s} \ll \frac{1}{ks^4} \ll \frac{1}{s^2}\log\frac{1}{s} \tag{3.11.17}$$

$$\overset{(3.11.6)}{\iff} \quad \frac{1}{k^3 \log^3 k} \ll s^6 \ll \frac{1}{k^3 \log k}. \tag{3.11.18}$$

By (3.11.17) we deduce that $\log\left(\frac{1}{ks^4}\right) \sim \log\left(\frac{1}{s}\right)$. Therefore (3.11.15) reads as

$$s' \sim \frac{1}{s^6 \log\left(\frac{1}{s}\right)} \tag{3.11.19}$$

and holds provided (3.11.6), (3.11.7), (3.11.14) and (3.11.18) are satisfied. Noting that (3.11.6) is implied by (3.11.18) (since $k \gg 1$) and that

$$\begin{aligned} (3.11.7) \quad &\iff \quad s^6 s' \ll 1 \overset{(3.11.19)}{\iff} (3.11.6) \\ (3.11.14) \quad &\iff \quad ks^4 \ll 1 \iff k^{3/2} s^6 \ll 1 \impliedby (3.11.18), \end{aligned}$$

we conclude that

$$s' \sim \frac{1}{s^6 \log\left(\frac{1}{s}\right)} \quad \text{if } \frac{1}{k^3 \log^3 k} \ll s^6 \ll \frac{1}{k^3 \log k} \quad \text{and } k \gg 1. \tag{3.11.20}$$

• For $(3.11.9)_3$ we argue exactly as in §3.10: we obtain that (3.11.19) holds provided $s' \ll k^3$ and (3.11.6) are satisfied. Now

$$s' \ll k^3 \iff \frac{1}{k^3} \ll s^6 \log\left(\frac{1}{s}\right).$$

Because of (3.11.6), also $(3.11.9)_3$ is seen only if $1 \ll k$, and in this case

$$s' \ll k^3 \iff s^6 \gg \frac{1}{k^3 \log k}. \tag{3.11.21}$$

Combining (3.11.6), (3.11.20), and (3.11.21) we obtain $(3.11.5)_2$.

• Within (3.11.10), we have

$$s' \sim \frac{k}{s^4} \quad \text{if } \alpha > 0,\ s^5 s' \ll 1 \text{ and } s \gg 1.$$

Since

$$s^5 s' \sim ks \ll 1 \iff s \ll \frac{1}{k},$$

the regime in (3.11.10) is not empty only if $k \ll 1$, and (3.11.4) follows for $1 \ll s \ll \frac{1}{k}$.

3.11.2 The timescales

We now infer from (3.11.4) and (3.11.5) the scaling laws for s given by (3.11.2) and (3.11.3).

(I) If $k \lesssim 1$ and $s_0 \ll 1/k$, it follows from (3.11.4) that

$$s^5 \sim s_0^5 + 5kt \sim kt \quad \text{provided } t \gg \frac{s_0^5}{k},$$

and

$$s \ll \frac{1}{k} \iff t \ll \frac{1}{k^6},$$

whence (3.11.2).

(II) If $k \gg 1$, we assume that $s_0^2 \ll \frac{1}{k \log k}$, so that both regimes in (3.11.5) are seen. According to $(3.11.5)_1$, we have

$$s^5 \sim s_0^5 + 5kt \sim kt \quad \text{provided } t \gg \frac{s_0^5}{k}, \tag{3.11.22}$$

which holds as long as

$$s^2 \ll \frac{1}{k \log k} \overset{(3.11.22)}{\iff} t \ll \left(\frac{1}{k^7 \log^5 k}\right)^{1/2} =: t_1. \tag{3.11.23}$$

As $t \sim t_1$, the free boundary enters the second regime in (3.11.5), which has already been analyzed in §3.10: it follows from (3.10.6) that

$$s(t) \sim \left(\frac{t}{\log\left(\frac{1}{t}\right)} \right)^{1/7} \quad \text{if} \quad \max\left\{ t_1, s_1^7 \log\left(\frac{1}{s_1}\right) \right\} \ll t \ll 1 \quad \text{and} \quad s_1 \ll 1, \tag{3.11.24}$$

with initial condition $s_1 := s(t_1) = (kt_1)^{1/5}$. Note that $s_1 \ll 1$ since $k \gg 1$ and t_1 is given by (3.11.23). Since

$$s_1^7 \log\left(\frac{1}{s_1}\right) \overset{(3.11.5)}{\sim} \frac{s_1^5}{k} \overset{(3.11.22)}{\sim} t_1,$$

the lower bounds on t in (3.11.24) coincide. Therefore we conclude that

$$s(t) \sim \left(\frac{t}{\log\left(\frac{1}{t}\right)} \right)^{1/7} \quad \text{if} \quad t_1 \ll t \ll 1. \tag{3.11.25}$$

Gathering (3.11.22), (3.11.23) and (3.11.25) we obtain (3.11.3).

3.12 Long time scaling laws in complete wetting

The asymptotic of this section is based on two main assumptions:

(I) the timescale is “large”;

(II) the evolution is “quasi-selfsimilar”.

We will argue that

$$s(t) \sim t^{1/6} \quad \text{if} \quad t \gg \max\left\{ 1, \frac{1}{k^6} \right\}. \tag{3.12.1}$$

Comparing (3.12.1) with (3.11.2) and (3.11.3), we see that the whole remaining range of timescales is covered by (3.12.1). In terms of the original variables, we obtain the final timescale in (3.2.4) and (3.2.5).

We now motivate (3.12.1). In complete wetting, $h \to 0$ as $t \to +\infty$: hence $h \ll 1$ everywhere for sufficiently large times, and conservation of mass implies that

$$s \gg 1, \tag{3.12.2}$$

which partially encodes (I). Since $h^3 + h^2 \sim h^2$ everywhere, we may replace the equation in (3.7.1) with

$$h_t + (h^2 h_{xxx})_x = 0. \tag{3.12.3}$$

Following (I), we introduce the selfsimilar variable transformation of (3.12.3) which preserves mass:

$$h = t^{-1/6} f(t,y), \quad y = xt^{-1/6} \in (0, a(t)), \quad \text{where } a(t) = t^{-1/6} s(t).$$

Then

$$\begin{cases} \frac{1}{6}(yf)_y - t f_t = (f^2 f_{yyy})_y, \quad f > 0 \quad \text{in } (0,a) \\ f_y|_{y=0} = f_{yyy}|_{y=0} = 0, \ f|_{y=a} = 0, \ \int_0^a f \mathrm{d}y = 1/2 \end{cases} \tag{3.12.4}$$

while the boundary condition reads as

$$f_y^2|_{y=a} = \frac{1}{kt^{1/6}} \left(\lim_{y \to a(t)} f f_{yyy} \right). \tag{3.12.5}$$

Since (3.12.5) is not time independent, an exact selfsimilar profile does not exist. However, if

$$kt^{1/6} \gg 1 \tag{3.12.6}$$

(which completes (I)), the contact-angle condition is only a perturbation of $f_y|_{y=a} = 0$. Hence we assume that f is quasi-selfsimilar in the sense that it has an expansion of the form

$$f(t,y) = f_0(y) + (k^6 t)^{-1} f_1(y) + \ldots,$$

which encodes (II). Then, at leading order, (3.12.4) reads as

$$\begin{cases} \frac{1}{6} y f_0 = f_0^2 f_{0yyy}, \quad f > 0 \quad \text{in } (0,a) \\ f_{0y}|_{y=0} = f_{0yyy}|_{y=0} = 0, \ f_0|_{y=a} = 0, \ f_{0y}|_{y=a} = 0, \ \int_0^a f_0(y) \mathrm{d}y = 1/2. \end{cases} \tag{3.12.7}$$

As is well-known [14], (3.12.7) has a unique solution (f_0, a). Therefore, recalling (3.12.2) and (3.12.6), we obtain (3.12.1).

3.13 Partial wetting with contact line friction

In the case of partial wetting, $\alpha_S > 0$, the profile of a spreading droplet converges to the unique steady state with mass 1 and contact angles α_S as $t \to +\infty$:

$$h \to \frac{3}{4s_\infty^3}(s_\infty^2 - x^2)_+, \ s \uparrow s_\infty = \sqrt{\frac{3}{2\alpha_S}}, \ s' \to 0 \quad \text{as } t \to +\infty. \tag{3.13.1}$$

We focus on the most interesting case of

$$\alpha_S \gg 1 \overset{(3.13.1)}{\Longleftrightarrow} s \le s_\infty \ll 1, \tag{3.13.2}$$

which guarantees the persistence for all times of a macroscopic profile. Because of $s \ll 1$, (3.9.10) may be ignored and we only have to look at (3.9.9), which we rewrite for the reader's convenience:

$$\left(\frac{3}{2s^2}\right)^3 \sim \begin{cases} \alpha^3 + 3s' \log(s\alpha) & \text{if } 1 \ll s\alpha \text{ and } s' \ll \alpha^3 \\ 3s' \log(s(s')^{1/3}) & \text{if } 1 \ll s^3 s' \text{ and } s' \gg \alpha^3. \end{cases} \tag{3.13.3}$$

In view of (3.13.1), for sufficiently large times (3.13.3) reduces to

$$\left(\frac{3}{2s^2}\right)^3 \sim \alpha_S^3 + 3s' \log(s\alpha_S), \tag{3.13.4}$$

which is equivalent to the well-known Cox-Hocking relation between the effective and the microscopic contact angle. In terms of the original variables, it coincides with (3.1.10).

The relation (3.13.4) yields an exponential convergence of s to s_∞. Indeed, let

$$s = \sqrt{\frac{3}{2\alpha_S}}\hat{s}, \quad t = \sqrt{\frac{27}{2\alpha_S^7}} \log \alpha_S \hat{t}.$$

In view of (3.13.1) and (3.13.2), $\log(\alpha_S s) \sim \log(\sqrt{\alpha_S})$ as $t \to +\infty$. Hence (3.13.4) reads as

$$\frac{d\hat{s}}{d\hat{t}} \sim \frac{1 - \hat{s}^6}{\hat{s}^6}.$$

An integration shows that $1 - \hat{s}(\hat{t}) \sim e^{-6\hat{t}}$ as $\hat{t} \to +\infty$, i.e.

$$\sqrt{\frac{3}{2\alpha_S}} - s(t) \sim e^{-Dt} \quad \text{as } t \to +\infty, \qquad D := \sqrt{\frac{8\alpha_S^7}{3\log^2 \alpha_S}}.$$

In order to infer the timescale of validity of (3.13.4), we have to give a closer look to (3.13.3) in order to identify the intermediate scaling laws which precede (3.13.4). We will argue that:

(i) if $|k \log k| \lesssim \alpha_S$, then

$$s(t) \sim (kt)^{1/5} \quad \text{for } \frac{s_0^5}{k} \ll t \ll \frac{1}{k\alpha_S^{5/2}};$$

(ii) if $k \lesssim \alpha_S \ll k \log k$, then

$$s(t) \sim \begin{cases} (kt)^{1/5} & \text{if } \frac{s_0^5}{k} \ll t \ll \frac{1}{k^{7/2}\log^{5/2} k} \\ \left(\frac{t}{\log\left(\frac{1}{t}\right)}\right)^{1/7} & \text{if } \frac{1}{k^{7/2}\log^{5/2} k} \ll t \ll \frac{1}{k^{7/6}\alpha_S^{7/3}\log^{1/6}(\alpha_S^2 k)} =: t_2. \end{cases}$$

(iii) if $\alpha_S \ll k$, then

$$s(t) \sim \begin{cases} (kt)^{1/5} & \text{if } \frac{s_0^5}{k} \ll t \ll \frac{1}{k^{7/2}\log^{5/2} k} \\ \left(\frac{t}{\log\left(\frac{1}{t}\right)}\right)^{1/7} & \text{if } \frac{1}{k^{7/2}\log^{5/2} k} \ll t \ll \frac{1}{\alpha_S^{7/2}\log^{1/6}\alpha_S}. \end{cases}$$

Preliminarily we observe that

$$\alpha = \sqrt{\frac{s'}{k} + \alpha_S^2} \sim \begin{cases} \left(\frac{s'}{k}\right)^{1/2} & \text{if } s' \gg k\alpha_S^2 \\ \alpha_S & \text{if } s' \ll k\alpha_S^2. \end{cases} \tag{3.13.5}$$

Because of (3.13.5), (3.13.3) coincides with the case of complete wetting as long as $s' \gg k\alpha_S^2$. Therefore (3.11.2) and (3.11.3) hold under the additional constraints that $s \ll 1$ and $s' \gg k\alpha_S^2$: imposing them, a few simple computations yield (i), (ii), and (iii) up to $t = t_2$. When $s' \ll k\alpha_S^2$, then $\alpha \sim \alpha_S$ and $(3.13.3)_1$ coincides with (3.13.4). Instead, $(3.13.3)_2$ yields

$$s(t) \sim \left(\frac{t}{\log\left(\frac{1}{t}\right)}\right)^{1/7} \quad \text{if } \frac{1}{k^{7/2}\log^{5/2} k} \ll t \ll 1$$

with the additional constraints that $\alpha_S^3 \ll s' \ll k\alpha_S^2$ and that $s^3 s' \gg 1$. Hence this regime is seen only if $\alpha_S \ll k$: in this case, a few more computations imposing the bounds on the speed yield (iii).

Chapter 4

Droplets spreading under contact-line friction: existence of weak solutions

4.1 Introduction and main results

In this Chapter we consider the thin-film equation with the free boundary condition proposed in [80] and discussed in Chapter 3:

$$\dot{s}(t) = d\left((h_x|_{x=s(t)})^2 - \theta_S^2\right) \tag{4.1.1}$$

where the superposed dot denotes the material time derivative. For simplicity, we consider the case of a symmetric droplet in $(-s(t), s(t))$. Furthermore, we replace the mobility $m(h) = h^3 + bh^2$ discussed in Chapter 3 by a more general mobility:

$$(P)\begin{cases} h_t + (m(h)h_{xxx})_x = 0, \; h > 0, \; h \text{ even} & \text{in } (0,t)\times(-s(t), s(t)) \\ h = 0, \; \dot{s}(t) = \lim\limits_{x\to s(t)^-} \dfrac{m(h)}{h} h_{xxx} & \text{at } (0,t)\times\{x = s(t)\} \\ \dot{s}(t) = d\left(h_x^2 - \theta_S^2\right) & \text{at } (0,t)\times\{x = s(t)\} \\ h(0,x) = h_0(x), \; h_0 \text{ even} & \text{in } (-s(t), s(t)), \end{cases} \tag{4.1.2}$$

where

$$m \in C^\infty((0,\infty)), \quad \text{with } m(h) \sim h^n, \; n > 0 \text{ as } h \to 0. \tag{4.1.3}$$

The parameter $n > 0$ is related to the slip condition imposed at the liquid-solid interface: in particular the equation with $n = 2$ corresponds to Navier slip, $n = 3$ means no slip, while $n \in (0, 3)$ models various relaxed slip conditions. The case with $n = 1$ may also be seen as the lubrication approximation of the two-dimensional Hele-Shaw flow in half-space [53].

The energy is given (see (3.1.6)) by

$$E(h) = \int_{-s(t)}^{s(t)} \frac{1}{2}(h_x^2 + \theta_S^2)\, \mathrm{d}x. \tag{4.1.4}$$

Arguing exactly as in Section 3.5, one sees that solutions of (P) formally satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-s(t)}^{s(t)} \frac{1}{2}(h_x^2 + \theta_S^2)\, \mathrm{d}x = -\frac{\dot{s}^2(t)}{2d} - \int_{-s(t)}^{s(t)} m(h) h_{xxx}^2\, \mathrm{d}x. \tag{4.1.5}$$

We translate the problem on the fixed domain $I = (-1, 1)$ by using the simple change of variable

$$y = \frac{x}{s(t)} \ \in I \tag{4.1.6}$$

and by defining the new function

$$v(t, y) = h(t, ys(t)), \tag{4.1.7}$$

so that

$$v_t = h_t + h_x y\dot{s} \quad \text{and } v_y = h_x s.$$

Then the free boundary condition is replaced by

$$\dot{s}(t) = d\left(\frac{(v_y|_{y=1})^2}{s^2} - \theta_S^2\right), \tag{4.1.8}$$

and the system (4.1.2) reads as

$$(P_v)\begin{cases} v_t - \dfrac{\dot{s}}{s} y v_y + \dfrac{1}{s^4}(m(v)v_{yyy})_y = 0,\ v > 0,\ v \text{ even} & \text{in } (0,t)\times I \\ v = 0,\ \dot{s}(t) = \lim\limits_{y\to 1} \dfrac{m(v)}{v}\dfrac{v_{yyy}}{s^3} & \text{at } (0,t)\times\{y=1\} \\ \dot{s}(t) = d\left(\dfrac{v_y^2}{s^2} - \theta_S^2\right) & \text{at } (0,t)\times\{y=1\} \\ v(0,y) = v_0(y),\ v_0 \text{ even} & \text{in } I. \end{cases} \tag{4.1.9}$$

The surface energy functional (4.1.4) in the new variables is replaced by

$$E(v) = \frac{1}{2}\int_I \left(\frac{v_y^2}{s} + s\theta_S^2\right) \mathrm{d}y, \tag{4.1.10}$$

and the energy balance (4.1.5) reads now as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_I \left(\frac{v_y^2}{s} + s\theta_S^2\right) \mathrm{d}y = -\frac{\dot{s}^2}{2d} - \frac{1}{s^5}\int_I m(v) v_{yyy}^2\, \mathrm{d}y. \tag{4.1.11}$$

We let

$$\{v > 0\}_T := \{(t, y) \in \mathrm{dom}(v) :\ t \le T,\ v(t, y) > 0\}$$

and we denote by $< \cdot, \cdot >$ the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$. Our goal is to prove the existence of non-negative weak solutions to (P_v) in the following sense:

Theorem 4.1. *Let m be as in* (4.1.3). *For any $v_0 \in H^1(I)$, even and non-negative, and any $s_0 > 0$ there exists a pair of functions (s, v) with $v \in C^{\frac{1}{2},\frac{1}{8}}([0,\infty) \times \overline{I}) \cap L^\infty_{\mathrm{loc}}([0,\infty); H^1(I))$, $v \geq 0$, and $s \in H^1((0,\infty))$, $s > 0$, which solves (P_v) with initial datum v_0 in the sense that, for all $T > 0$, it holds that:*

(i) $v_t \in L^2((0,T); (H^1(I))')$;

(ii) $v_{yyy} \in L^2_{\mathrm{loc}}(\{v > 0\})$ *and* $\sqrt{m(v)}v_{yyy} \in L^2(\{v > 0\})$;

(iii) *for all* $\varphi \in L^2((0,T); H^1(I))$

$$\int_0^T < v_t, \varphi > \,\mathrm{d}t = \int_0^T \int_I \frac{\dot{s}}{s} y v_y \varphi + \int_0^T \int_I \frac{1}{s^4} m(v) v_{yyy} \varphi_y; \tag{4.1.12}$$

(iv) $v(0, y) = v_0(y)$ *in* $H^1(I)$;

(v) $v(t, 1) = 0$ *in* $L^2(0,T)$;

(vi) *v is even;*

(vii) *v dissipates E in the sense that*

$$E(v(t)) + \frac{1}{2d} \int_0^t \dot{s}^2 + \iint_{\{v>0\}_t} \frac{1}{s^5} m(v) v_{yyy}^2 \leq E(v_0). \tag{4.1.13}$$

The kinematic condition, $\dot{s}(t) = \lim_{y\to 1} \frac{m(v)}{v} \frac{v_{yyy}}{s^3}$, is captured in its weak form of mass conservation, which may be obtained from testing (4.1.12) by s:

$$s(t) \int_I v(y,t)\,\mathrm{d}y = s_0 \int_I v_0(y)\,\mathrm{d}y. \tag{4.1.14}$$

The free boundary condition (4.1.8) is encoded only very weakly, in the form of the energy inequality (4.1.13). By "very weakly" we mean the following: if the solution had sufficient additional regularity, such that on one hand (4.1.13) were satisfied as an equality, and on the other hand the formal computations in Section 3.5 were rigorous (cf. (3.1.8)), so that

$$E(v(t)) + \int_0^t \frac{\dot{s}}{2} \left(\frac{v_y^2(1)}{s^2} - \theta_S^2 \right) + \iint_{\{v>0\}_t} \frac{1}{s^5} m(v) v_{yyy}^2 = E(v_0), \tag{4.1.15}$$

then the (4.1.8) would be implied. A further weakness of Theorem 4.1 is that we are not able to prove that $v > 0$ a.e. in $(0,T) \times I$. In this respect, it is important to notice that even for the well-known case of a zero-contact angle condition, the standard entropy estimates (see §4.2) in our fixed-domain framework would not yield a.e. positivity of the solution, since there the support of the test functions is fixed in the x-variable, that is, receding in the y-variable when s increases. This points to the necessity of a refinement of the standard entropy estimates, localized in such a way that the test function "follows" the free-boundary. We hope to come back to this topic in the future, and as such we leave it as an open question.

A merit of our approach is the construction of approximating solutions (s, v) in which v *is positive and* (s, v) *satisfy the free boundary condition* (4.1.8). More precisely, they are (suitably symmetrized) strong solutions (see §4.3) of the following problem:

$$(P_\sigma)\begin{cases} v_t - \frac{\dot{s}}{s} y v_y + \frac{1}{s^4}(m_\sigma(v) v_{yyy})_y = 0, \ v > 0 & (t, y) \in (0, \infty) \times (0, 1) \\ v_y = v_{yyy} = 0 & (t, y) \in (0, \infty) \times \{y = 0\} \\ v = \sigma, \ m_\sigma(v) v_{yyy} = 0, & (t, y) \in (0, \infty) \times \{y = 1\} \\ \dot{s}(t) = d\left(\frac{v_y^2}{s^2} - \theta_S^2\right) & (t, y) \in (0, \infty) \times \{y = 1\} \\ v(0, y) = v_0(y) + \sigma, & \text{in } (0, 1). \end{cases} \tag{4.1.16}$$

Here $\sigma > 0$ and m_σ is (a simple modification of) the standard regularization for thin-film equations: following [16] and [10], we let

$$m_\sigma(\tau) = \frac{\tau^4 m(\tau)}{\sigma m(\tau) + \tau^4}.$$

We believe that this approximation is a good candidate for a consistent scheme that captures the main features of the limiting problem. An even more consistent candidate would emerge from replacing the boundary condition $(m_\sigma(v) v_{yyy})|_{y=1} = 0$ (a zero-flux condition) by the stronger kinematic condition $s^3 \dot{s}(t) = \left(\frac{m_\sigma(v)}{v} v_{yyy}\right)|_{y=1}$: indeed, since solutions of (P_σ) are positive, and therefore smooth, a control on the trace of third derivative is conceivable. However, at present we have to leave it as a further open question.

Besides the specific free-boundary condition, this chapter stands as a first investigation of different formulations for thin-film equations, which lie in between the weak and the classical ones. We believe that this kind of formulations has the potential to yield improvements in the theory, e.g. conditions for the uniqueness of global weak solutions, and therefore deserves to be explored. It should be noted in this respect that this approach to the problem raises some new technical issues: these are described in Section 4.3, where both the proof of Theorem 4.1 and the plan of this chapter are outlined. Before that, however, let us give a brief overview on thin-film equations.

4.2 An overview on thin-film equations

Thin-film equations are fourth-order degenerate diffusion equations of the form

$$h_t + (m(h) h_{xxx})_x = 0 \tag{4.2.1}$$

where $m(h) = h^n$ for $n > 0$ (for simplicity, we adopt a one-dimensional framework). The diffusion coefficient m is positive for $h > 0$, but vanishes at zero. By n we denote its growth exponent near zero. Equation (4.2.1) can be seen as the prototype of a family of parabolic equations of higher order which arises in several applications to material sciences and fluid dynamics, and in which $h(t, x)$ is required to be non-negative. For instance, in the Cahn-Hilliard model of phase separation for binary mixtures, h plays the role of the concentration of one component (see [44]). As we have seen in Chapter 3 in lubrication theory, h denotes the height of a viscous droplet spreading on a solid surface in which inertia is negligible and the dynamics are governed by viscosity and capillarity forces. Instead, as discussed in Section 4.1, the exponent n is related to the slip condition imposed at the liquid-solid surface.

The second-order counter-part of degenerate diffusion equations is the well-known *porous medium equation* (see e.g. [77, 87]):

$$h_t - \triangle\Phi(h) = 0, \tag{4.2.2}$$

where $\Phi'(h) > 0$ for $h > 0$ and $\phi(h) \sim h^m$ as $h \to 0$. Here $m > 1$ makes the equation degenerate. Comparing (4.2.1) to (4.2.2) some similarities emerge: for instance, both equations are parabolic and in divergence form, with a nonlinear diffusion coefficient which provides instantaneous smoothing of the solutions in regions where h is positive. However, strong differences emerge, too. The most crucial one is the lack of comparison or maximum principle, which in general does not hold for higher-order equations: for instance, classical solutions to the linear non-degenerate parabolic equation $h_t + h_{xxxx} = 0$ may in general change sign even in the case of strictly positive initial data [13].

In spite of the lack of comparison principle, the degeneracy of the operator as $h \to 0$ allows to establish a special form of “minimum principle”: the existence of non-negative solutions starting from a non-negative initial datum. This was first proved in 1990 by Bernis and Friedman [13]. In this pioneering paper, in one space dimension they were able to show the existence of nonnegative and Hölder continuous weak solutions for all values $n \geq 1$, provided that the initial data were nonnegative, and positivity of solutions for $n \geq 4$. We point out once again that this kind of a weak maximum principle is due to the nonlinear and degenerate structure of (4.2.1), and is not common to fourth-order parabolic equations. The positivity of solutions was later extended to $n \geq 7/2$ by Beretta, Bertsch, and Dal Passo [10] and by Bertozzi and Pugh [16], where a rich structure of qualitative and regularity properties of solutions to (4.2.1) are also shown, depending on the growth exponent n. The approach in these papers relies on two essential estimates. The first one is the well-known energy estimate

$$\frac{1}{2}\int_\Omega h_x^2(t)\,\mathrm{d}x + \iint_{\{h>0\}_t} m(h)|h_{xxx}|^2\,\mathrm{d}x\,\mathrm{d}t \leq \frac{1}{2}\int_\Omega h_x^2(0)\,\mathrm{d}x. \tag{4.2.3}$$

The second key a-priori estimate is a class of integral inequalities, so called "entropy estimates", which play an important role also for proving results on finite speed propagation of support (see e.g. [10, 16, 18, 12, 11]). The simplest form of entropy estimate can be formally obtained by testing the equation with a function $G'(y)$ that satisfies

$$G''(y) = \frac{1}{m(y)}.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega G(h)\,\mathrm{d}x = -\int_\Omega h_{xx}^2\,\mathrm{d}x. \tag{4.2.4}$$

More generally, choosing G such that

$$G''(h) = \frac{h^{\alpha+n-1}}{m(h)},$$

one arrives at the entropy estimates of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{h^{\alpha+1}}{\alpha(\alpha+1)}\,\mathrm{d}x \lesssim -\int_\Omega |(h^{\frac{\alpha+n+1}{2}})_{xx}|^2\,\mathrm{d}x - \int_\Omega |(h^{\frac{\alpha+n+1}{4}})_x|^4\mathrm{d}x \tag{4.2.5}$$

for $\alpha \in \left(\frac{1}{2}-n, 2-n\right)$. In particular, as shown in [10, 16], it follows from (4.2.5) that an initially positive solution remains positive for all times if $n \geq \frac{7}{2}$ (i.e. $\alpha+1 \leq -2$). This feature may then be used to build up an approximating procedure and construct non-negative "entropy" solutions to (4.2.1) for $0 < n < 3$, as limits of solutions of approximating problems with very carefully modified initial data and mobilities, such that $m(h) \sim h^4$ as $h \to 0$.

Let us point out that the growth exponent $n = 3$ appears to be a borderline value with respect to the qualitative behavior of solutions to (4.2.1). For instance, in [14] it is proved that compactly supported source type solutions (i.e. solutions that start as a Dirac mass at the origin and spread out in a self-similar way while preserving the mass) do not exist for $n \geq 3$. Technically, this is reflected by the entropy estimates: for $n \geq 3$ there is no $\alpha > -1$ such that the entropy estimates hold, hence $\int h_0^{\alpha+1}$ is unbounded for compactly supported initial data.

The entropy inequality (4.2.5) guarantees that entropy solutions have sufficient regularity to ensure the zero contact angle condition for almost every t. Hence the solutions constructed in [13, 10, 16] may be seen as weak solutions of the following free-boundary problem:

$$\begin{cases} h_t + (m(h)h_{xxx})_x = 0,\ h > 0, & \text{in } (0,t)\times(s_-(t), s_+(t)) \\ h = 0,\ \dot{s}_\pm(t) = \lim\limits_{x\to s_\pm(t)^\mp} \dfrac{m(h)}{h} h_{xxx} & \text{at } (0,t)\times\{x = s_\pm(t)\} \\ h_x = 0 & \text{at } (0,t)\times\{x = s_\pm(t)\} \\ h(0,x) = h_0(x) & \text{in } (s_-(t), s_+(t)). \end{cases} \tag{4.2.6}$$

The theory of entropy solutions described so far was later extended to higher space dimensions in [38, 18, 57], where new difficulties arise: for instance, not strong enough being the norms controlled by energy and entropy estimate, Hölder continuity (or even boundedness) is lost and the identification of the limit becomes harder. The existence of weak solutions with non-zero contact angle is instead much less investigated: in the case $n = 1$, it was obtained in one space dimension by Otto [76] for a prescribed, positive contact angle; results in this direction for a generic n were obtained in [20]. More recently, a study of (4.2.6) as a classical free-boundary problem has been initiated: global existence of classical solutions with initial data close to the equilibrium solution $(x)_+^2$ (with $s_+ = +\infty$) were obtained in [51] (see also [50]). In [69], analogous results have been obtained the case of $n = 2$ with a prescribed, non-zero contact angle, for initial data close to the traveling-wave solution. So far, we have not been able to extend the latter result to the case of the free-boundary condition (4.1.1): the reason is that, while $h_x(t, s_+(t)) = -1$ is linear and (scaling-wise) of low order, condition (4.1.1), rewritten in form of

$$h_x(t, s_+(t)) = -\sqrt{\frac{1}{d}\left(\lim_{x\to s_+(t)} h^{n-1}h_{xxx}\right) + \theta_S^2}$$

is nonlinear and (scaling-wise) of highest order (it depends on the trace of the third derivative for a fourth-order problem).

Though the analytical development for entropy solutions is now sufficiently well established, many questions remain unanswered. Among the most mathematically intriguing problems there is of course the (non-)uniqueness of entropy solutions for $0 < n < 3$. We refer to [10] for an example of non-uniqueness. Another outstanding question is to identify a threshold condition on the exponent n such that initially positive solutions can/cannot develop finite-time singularities of the form $h(t, x) \to 0$ as $t \uparrow t^* < \infty$, a phenomenon which was observed numerically and by matched asymptotics in [15] for sufficiently small values of n. Among the open problems are also a more robust notion of weak solutions with non zero contact angle, regularity properties (such as continuity and even boundedness) in higher space dimension, and the development of a full theory of classical solutions for the formulation (4.2.6).

4.3 Plan of the proof of the main result

The proof of Theorem 4.1 is based on a multi-step approximating procedure. As we said, a solution to (P_v) will be obtained as limit of solutions to (P_σ). In turn, a solution to (P_σ) will be obtained as limit, as $\delta \to 0$, of problems in which we replace the diffusivity m_σ, which is itself degenerate as $v \to 0$ and unbounded as $v \to \infty$, by an approximating family of non-degenerate and bounded diffusivities $m_{\delta,\sigma}$:

$$m_{\delta,\sigma}(\tau) = \delta + \frac{|\tau|^{n+4}}{\sigma|\tau|^n + \tau^4 + \delta|\tau|^{n+4}}, \quad \tau \in \mathbb{R}, \tag{4.3.1}$$

for some $\delta > 0$ and $\sigma > 0$. We also need to raise the initial datum v_0 of an height $\varepsilon > 0$. Letting

$$\Omega = (0,1), \quad Q_t = (0,t) \times \Omega,$$

we consider the following problems:

$$(P_{\varepsilon,\delta,\sigma}) \begin{cases} v_t - \frac{\dot{s}}{s} y v_y + \frac{1}{s^4}(m_{\delta,\sigma}(v) v_{yyy})_y = 0, & \text{in } Q_t \\ v_y = v_{yyy} = 0 & \text{at } (0,t) \times \{y = 0\} \\ v = \varepsilon, \ v_{yyy} = 0 & \text{at } (0,t) \times \{y = 1\} \\ \dot{s}(t) = d\left(\frac{v_y^2}{s^2} - \theta_S^2\right) & \text{at } (0,t) \times \{y = 1\} \\ v(0,y) = v_0(y) + \varepsilon, & \text{in } \Omega. \end{cases} \tag{4.3.2}$$

Letting

$$H^1_\varepsilon(\Omega) = \{v \in H^1(\Omega) \quad \text{s.t.} \ \ v(1) = \varepsilon\}$$

a solution of $(P_{\varepsilon,\delta,\sigma})$ is defined as follows

Definition 4.1. *Let $T > 0$, $\varepsilon > 0$, $\delta \geq 0$, $\sigma > 0$. Let $v_0 \in H^1(\Omega)$ be non-negative and $s_0 > 0$. A pair of functions (s, v), with $v \in L^\infty([0,T]; H^1_\varepsilon(\Omega))$ and $s \in H^1(0,T)$, is called a solution of $(P_{\varepsilon,\delta,\sigma})$ in $(0,T)$ with initial datum v_0 if*

(i) $v_t \in L^2([0,T]; (H^1(\Omega))')$;

(ii) $v \in L^2((0,T), H^3(\Omega))$;

(iii) *for all* $\varphi \in C^\infty([0,T) \times \overline{\Omega})$

$$\int_0^T < v_t, \varphi > \ \mathrm{d}t = \iint_{Q_T} \frac{\dot{s}}{s} y v_y \varphi - \iint_{Q_T} \frac{1}{s^4} m_{\delta,\sigma}(v) v_{yyy} \varphi_y; \tag{4.3.3}$$

(iv) $v(1) = \varepsilon$ *in* $L^2(0,T)$;

(v) $v_y(0) = 0$ *in* $L^2(0,T)$;

(vi) $v(0,y) = v_{0\varepsilon}(y)$ *in* $H^1(\Omega)$;

(vii) $s(t) > 0$ *in* $[0,T]$ *and* $\dot{s}(t) = d\left(\frac{v_y^2(t,1)}{s^2} + \theta_S^2\right)$ *in* $L^2(0,T)$.

In order to obtain global existence for $(P_{\varepsilon,\delta,\sigma})$, we first *prescribe* the free boundary $s(t)$ and consider the following problems:

$$(P_{\varepsilon,\delta,\sigma,s})\begin{cases} v_t - \frac{\dot{s}}{s} y v_y + \frac{1}{s^4}(m_{\delta,\sigma}(v) v_{yyy})_y = 0 & \text{in } Q_t \\ v_y = v_{yyy} = 0 & \text{at } (0,t)\times\{y=0\} \\ v = \varepsilon,\ v_{yyy} = 0 & \text{at } (0,t)\times\{y=1\} \\ v(0,y) = v_0(y) + \varepsilon & \text{in } \Omega, \end{cases} \tag{4.3.4}$$

where indeed the free-boundary condition (4.1.8) is removed. In Section 4.5 we prove local existence of solutions for $(P_{\varepsilon,\delta,\sigma,s})$ (see Proposition 4.1). The reason for these solutions to be only local is that, once s is fixed (i.e., the contact-angle condition does not hold), the dissipative structure is lost (compare (4.1.15)). In Section 4.6 we apply a contraction argument to obtain a local existence result for $(P_{\varepsilon,\delta,\sigma})$ (see Proposition 4.3). This is, from the technical viewpoint, both the hardest part of the work and the crucial one in order to capture the contact-angle condition. Once this condition is recovered, then also the dissipative structure is, and local existence can be upgraded to global existence (see Proposition 4.4 in Section 4.7). In Section 4.8 we prove an entropy-type estimate for solutions to $(P_{\varepsilon,\delta,\sigma})$ which is uniform with respect to δ (see Lemma 4.5): this allows to pass to the limit as $\delta \to 0$ obtaining *positive* solutions to $(P_{\varepsilon,0,\sigma})$ (see Proposition 4.5). Finally, in Section 4.9 we pass to the limit as $\varepsilon = \sigma \to 0$ (in a nowadays standard fashion) and complete the proof of Theorem 4.1.

4.4 Preliminaries

We frequently use the following interpolation inequalities due to Gagliardo-Nirenberg (see [48], [73], and [74]). We consider the one dimensional case, and we let ∂^j denote the j-th order derivative.

Theorem 4.2 (Gagliardo-Nirenberg inequalities)**.** *Let* $0 < q < p$, $1 \le r \le \infty$, $m \in \mathbf{N}$, $j \in [1, m-1]$, *such that* $\frac{1}{r} < m + \frac{1}{p}$, *and let* $I \subset \mathbb{R}$ *be an interval. Positive constants* C_1, C_2 *exist such that the following inequality holds for all* $u \in L^q(I)$ *such that* $\partial^m u \in L^r(I)$*:*

$$\int_I |\partial^j u|^p \,\mathrm{d}x \le C_1 \left(\int_I |\partial^m u|^r \,\mathrm{d}x \right)^{\frac{\alpha p}{r}} \left(\int_I |u|^q \,\mathrm{d}x \right)^{\frac{(1-\alpha)p}{q}} + C_2 \left(\int_I |u|^q \,\mathrm{d}x \right)^{\frac{p}{q}} \tag{4.4.1}$$

where α *is given by*

$$\frac{1}{p} = j + \alpha\left(\frac{1}{r} - m\right) + (1-\alpha)\frac{1}{q}. \tag{4.4.2}$$

Furthermore, $C_2 = 0$ *if* I *is unbounded or if* $u = 0$ *somewhere in* $\overline{I}$*.*

The particular cases we are interested in are the following ones:

(i) If $j = 0$, $p = \infty$, $m = 2$, $r = q = 2$, and u vanishes somewhere in I, the corresponding inequality reads as:

$$\sup |u| \lesssim \|u\|_2^{3/4} \|\partial^2 u\|_2^{1/4}. \tag{4.4.3}$$

If $\partial u(0) = 0$, replacing u by ∂u we get

$$\sup |\partial u|^2 \lesssim \|\partial u\|_2^{3/4} \|\partial^3 u\|_2^{1/4}. \tag{4.4.4}$$

(ii) If $j = 1$, $p = r = q = 2$, $m = 2$ the corresponding inequality follows

$$\|\partial u\|_2 \leq C_1 \|u\|_2^{1/2} \|\partial^2 u\|_2^{1/2} + C_2 \|u\|_2. \tag{4.4.5}$$

Replacing u by ∂u we get

$$\|\partial^2 u\|_2 \leq C_1 \|\partial u\|_2^{1/2} \|\partial^3 u\|_2^{1/2} + C_2 \|\partial u\|_2. \tag{4.4.6}$$

We recall here the following interpolation Theorem by Simon:

Theorem 4.3. *([82], Corollary 8.4) Let $X \subset B \subset Y$ with compact imbedding $X \hookrightarrow B$ (X, B and Y are Banach spaces). Let F be bounded in $L^p(0, T; X)$ where $1 \leq p < \infty$, and F_t be bounded in $L^1(0, T; Y)$. Then F is relatively compact in $L^p(0, T; B)$. Let F be bounded in $L^\infty(0, T; X)$, and F_t be bounded in $L^r(0, T; Y)$ where $r > 1$. Then F is relatively compact in $C(0, T; B)$.*

4.5 Local existence of solutions for approximating problems with a prescribed free boundary

The aim of this section is to show local existence of weak solutions to $(P_{\varepsilon,\delta,\sigma,s})$. We will use the following assumptions on s:

$$\int_0^\infty \dot{s}^2 \leq k^2 \quad \text{and} \quad 0 < s_m \leq s(t) \quad \forall\, t \tag{4.5.1}$$

for some positive k and s_m.

Proposition 4.1. *Let $m_{\delta,\sigma} \in C^1(\mathbb{R}, [\delta, \delta^{-1}])$ and s satisfying (4.5.1). Suppose $T < T_{\delta,k}$ (see Lemma 4.2) and $v_0 \in H^1(\Omega)$. Then there exists a weak solution $v \in L^\infty((0, T); H^1_\varepsilon(\Omega)) \cap L^2((0, T); H^3(\Omega))$ to $(P_{\varepsilon,\delta,\sigma,s})$ in $(0, T)$ with initial datum v_0 in the following sense:*

$$\iint_0^T < v_t, \varphi >= \iint_{Q_T} \frac{\dot{s}}{s} y v_y \varphi + \iint_{Q_T} \frac{1}{s^4} m_{\delta,\sigma}(v) v_{yyy} \varphi_y \tag{4.5.2}$$

for all $\varphi \in L^2((0, T); H^1(\Omega))$. Furthermore $v_t \in L^2((0, T), (H^1(\Omega))')$ and *(i)* $v(0) = v_0$ *in* $H^1(\Omega)$*;*

(ii) $\partial v(t,0) = 0$ *in* $L^2(0,T)$*;*

(iii) $v(t,1) = \varepsilon$ *in* $L^2(0,T)$.

The strategy for Proposition 4.1 is based on a density argument with respect to s: Indeed, after having proved the existence for smooth $s(t)$, we will extend this result for the general hypothesis (4.5.1) on s. So the starting point will be to prove (by a Galerkin type method) the following existence result for $s \in C^1(0,T)$.

Proposition 4.2. *Let* $T > 0$*. Let* $v_0 \in H^1_\varepsilon(\Omega)$ *and let* $m_{\delta,\sigma} \in C^1(\mathbf{R}, [\delta, \delta^{-1}])$ *for some* $\delta > 0$*. We suppose*

$$s \in C^1[0,T] \cap H^1_0(0,T), \; s(0) = s_0 \quad \text{and} \quad 0 < s_m \leq s \tag{4.5.3}$$

for some positive constants s_m, s_0*. Then there exists a weak solution* v *of to* $(P_{\varepsilon,\delta,\sigma,s})$ *in* $(0,T)$ *with initial datum* v_0 *in the sense of Proposition 4.1.*

In the course of the proof of Proposition 4.2 we will use the following interpolation inequality:

Lemma 4.1.

$$\begin{aligned} \left| \int_0^t a(t) \int_\Omega y f_y g_{yy} \,\mathrm{d}y \,\mathrm{d}t \right| \leq \;& t^{1/2} \left(\int_0^t a^2 \right)^{1/2} \left(\sup_t \int_\Omega f_y^2 \sup_t \int_\Omega g_y^2 \right)^{1/2} \\ +\;& t^{1/4} \left(\int_0^t a^2 \right)^{1/2} \left(\sup_t \int_\Omega f_y^2 \right)^{1/2} \left(\left(\sup_t \int_\Omega g_y^2 \right) \left(\iint_{Q_t} g_{yyy}^2 \right) \right)^{1/4} \end{aligned} \tag{4.5.4}$$

Proof. Inequality (4.5.4) is obtained by using Hölder inequality:

$$\begin{aligned} \left| \int_0^t a(t) \int_\Omega y f_y g_{yy} \,\mathrm{d}y \,\mathrm{d}t \right| &\leq \left(\int_0^t a^2 \right)^{1/2} \left(\int_0^t \Big(\int_\Omega y f_y g_{yy} \Big)^2 \right)^{1/2} \\ &\leq \left(\int_0^t a^2 \right)^{1/2} \left(\int_0^t \Big(\int_\Omega f_y^2 \Big) \Big(\int_\Omega g_{yy}^2 \Big) \right)^{1/2} \\ &\overset{(4.4.6)}{\leq} \left(\int_0^t a^2 \right)^{1/2} \left[\int_0^t \left(\int_\Omega f_y^2 \right) \left(\int_\Omega g_y^2 \right) \mathrm{d}t \right. \\ &+ \left. \int_0^t \left(\int_\Omega f_y^2 \right) \left(\Big(\int_\Omega g_y^2 \Big) \Big(\int_\Omega g_{yyy}^2 \Big) \right)^{1/2} \mathrm{d}t \right]^{1/2} \\ &\leq t^{1/2} \left(\int_0^t a^2 \right)^{1/2} \left(\sup_t \int_\Omega f_y^2 \sup_t \int_\Omega g_y^2 \right)^{1/2} \\ &+ t^{1/4} \left(\int_0^t a^2 \right)^{1/2} \left(\sup_t \int_\Omega f_y^2 \right)^{1/2} \left(\left(\sup_t \int_\Omega g_y^2 \right) \left(\iint_{Q_t} g_{yyy}^2 \right) \right)^{1/4}. \end{aligned}$$

□

Proof. For notational convenience in this proof we set $\partial v = v_y$. We will use (u, v) to indicate the scalar product $\int_\Omega \partial u \partial v$. First of all we pass to a zero boundary condition for the unknown function at $y = 1$, by defining the function $\hat{v}(y,t) = v(y,t) - \varepsilon$: then (4.3.4) reads as

$$\begin{cases} \hat{v}_t - \dfrac{\dot{s}}{s} y \partial \hat{v} + \dfrac{1}{s^4} \partial(\widehat{m}(\hat{v}) \partial^3 \hat{v}) = 0 & \text{in } Q_T \\ \partial \hat{v} = \partial^3 \hat{v} = 0 & \text{at } (0,T) \times \{y = 0\} \\ \hat{v} = \partial^3 \hat{v} = 0 & \text{at } (0,T) \times \{y = 1\} \\ \hat{v}(0, y) = v_0(y), & \text{in } \Omega \end{cases} \tag{4.5.5}$$

where $\widehat{m}(\hat{v}) := m_{\delta,\sigma}(\hat{v} + \varepsilon)$. For notational convenience we remove all hats, except that on $\widehat{m}$, and we proceed by analyzing the following problem:

$$(\hat{P}) \begin{cases} v_t - \dfrac{\dot{s}}{s} y \partial v + \dfrac{1}{s^4} \partial(\widehat{m}(v) \partial^3 v) = 0 & \text{in } Q_T \\ \partial v = \partial^3 v = 0 & \text{at } (0,T) \times \{y = 0\} \\ v = \partial^3 v = 0 & \text{at } (0,T) \times \{y = 1\} \\ v = v_0 & \text{at } \{t = 0\} \times \Omega. \end{cases} \tag{4.5.6}$$

We set

$$H^1_*(\Omega) = \{v \in H^1(\Omega) \ : \ v(1) = 0\}$$

and

$$H^3_*(\Omega) = \{v \in H^3(\Omega) \ : \ v(1) = 0 \text{ and } \partial v(0) = 0\}$$

which take into account the essential boundary conditions. The spaces are equipped equivalent norms

$$||v||_{H^1_*(\Omega)} := ||\partial v||_{L^2(\Omega)}$$

and

$$||v||_{H^3_*(\Omega)} := ||\partial v||_{L^2(\Omega)} + ||\partial^3 v||_{L^2(\Omega)}$$

respectively. The Galerkin discretization consists in replacing the infinite-dimensional space $H^3_*(\Omega)$ with a finite-dimensional space V_N:

$$V_N \subset H^3_*(\Omega), \ dim V_N = N < \infty.$$

In order to define V_N, we now construct a suitable Hilbertian basis of $H^3_*(\Omega)$. To this aim, we wish to define a linear solution operator

$$T : H^3_*(\Omega) \longrightarrow H^3_*(\Omega)$$

with $T(g) = v$ solving the problem

$$\begin{cases} v + \partial^4 v = g & \text{in } \Omega \\ \partial v = \partial^3 v = 0 & \text{at } \{y = 0\} \\ v = \partial^3 v = 0 & \text{at } \{y = 1\}. \end{cases} \tag{4.5.7}$$

In order to do so, we formally multiply the equation in (4.5.7) by $-\partial^2 w$ with $w \in H^3_*(\Omega)$. After integrations by parts we obtain the following weak form

$$\int_\Omega \partial v \partial w \, \mathrm{d}y + \int_\Omega \partial^3 v \partial^3 w \, \mathrm{d}y = -\int_\Omega g \partial^2 w \, \mathrm{d}y. \qquad (4.5.8)$$

This naturally leads to define the linear continuous functional $L : H^3_*(\Omega) \to \mathbf{R}$ by

$$L(w) := -\int_\Omega g \partial^2 w \, \mathrm{d}y$$

and the bilinear operator $a : H^3_*(\Omega) \times H^3_*(\Omega) \to \mathbf{R}$ as follows:

$$a(v, w) := \int_\Omega \partial v \partial w \, \mathrm{d}y + \int_\Omega \partial^3 v \partial^3 w \, \mathrm{d}y.$$

So the variational equation (4.5.8) can be written in the abstract form of

$$a(v, w) = L(w) \quad \text{for all } w \in H^3_*(\Omega). \qquad (4.5.9)$$

Being an equivalent H^3-norm, it follows immediately that a is continuous and coercive. Therefore, by the Lax-Milgram theorem, for any $g \in H^3_*(\Omega)$ there exists a unique element $v \in H^3_*(\Omega)$ such that (4.5.9) holds. This implies that there exists a unique weak solution $v \in H^3_*(\Omega)$ of (4.5.7) in the weak sense (4.5.8). This allows us to define T as follows:

$$T(g) := v.$$

By a bootstrap argument, we in fact have $v \in C^\infty(\Omega)$: Indeed, since v, w and $g \in H^3_*(\Omega)$, from (4.5.8) we have

$$\int_\Omega \partial^3 v \partial^3 w \, \mathrm{d}y = \int_\Omega (v - g) \partial^2 w \, \mathrm{d}y.$$

Hence $\partial^3 v \in H^3(\Omega)$, which implies $v \in H^6(\Omega)$. Therefore $v \in C^5(\Omega)$, and iterating this argument, the C^∞-regularity is achieved. Integrating by parts (4.5.8), it holds

$$\int_\Omega (v + \partial^4 v - g) \partial^2 w \, \mathrm{d}y + [\partial^3 v(1) \partial^2 w(1) - \partial^3 v(0) \partial^2 w(0)] = 0. \qquad (4.5.10)$$

Choosing a suitably smooth test function $\varphi = \partial^2 w$, we have $v + \partial^4 v - g = 0$ a.e. in Ω, and consequently (4.5.10) implies that

$$\partial^3 v(1) \varphi(1) - \partial^3 v(0) \varphi(0) = 0$$

for all $\varphi \in C^\infty(Q_T)$. From the arbitrary of φ we deduce

$$\partial^3 v(0) = \partial^3 v(1) = 0. \qquad (4.5.11)$$

In contrast to the essential boundary conditions, the conditions in (4.5.11) follow from the variational equation (4.5.8), hence it is not necessary to impose them explicitly on v in the definition of the space (i.e., they are of "natural" type). We observe that T satisfies the following properties:

- T is self-adjoint:
  We let $w = T(f)$; integrating by parts we have:

$$\begin{aligned}
(T(g), f) &= \int \partial(T(g))\partial f = \int_\Omega \partial v \partial f = -\int_\Omega \partial^2 v f \\
&= -\int_\Omega \partial^2 v(w + \partial^4 w) = -\int_\Omega \partial^2 v\, w - \int_\Omega \partial^2 v\, \partial^4 w \\
&= -\int_\Omega \partial^2 w\, v - \int \partial^2 w \partial^4 v = -\int_\Omega (v + \partial^4 v)\partial^2 w = -\int_\Omega g\partial^2 w \\
&= \int_\Omega \partial g \partial w = \int_\Omega \partial(T(f))\partial g = (g, T(f)). \qquad (4.5.12)
\end{aligned}$$

- T is compact:
  Let $||g||_{H^3_*(\Omega)} \leq C$. From (4.5.7) and recalling that $v \in C^5(\Omega)$, we have the further conditions

$$\partial^4 v(1) = 0 \ \text{ and } \partial^5 v(0) = 0. \qquad (4.5.13)$$

  Multiplying the equation in (4.5.7) by $\partial^6 v$, integrating by parts, and using Hölder inequality, we obtain

$$\int_\Omega (\partial^3 v)^2 + \int (\partial^5 v)^2 = \int_\Omega \partial g \partial^5 v \leq \left(\int_\Omega (\partial^5 v)^2\right)^{1/2} \left(\int_\Omega (\partial g)^2\right)^{1/2}.$$

  Then

$$\int_\Omega (\partial^3 v)^2 + \int_\Omega (\partial^5 v)^2 \leq \int_\Omega (\partial g)^2.$$

  On the other hand, choosing $w = v$ in (4.5.8), it easily follows that

$$\int_\Omega (\partial v)^2 + \int_\Omega (\partial^3 v)^2 \leq \int_\Omega (\partial g)^2$$

  and since $v(1) = 0$, we conclude that

$$||T(g)||_{H^5(\Omega)} = ||v||_{H^5(\Omega)} \leq ||g||_{H^1_*(\Omega)} \leq ||g||_{H^3_*(\Omega)},$$

  hence T is compact.

T being a self-adjoint and compact operator in the Hilbert space $H^3_*(\Omega)$ we conclude that $H^3_*(\Omega)$ admits an Hilbertian basis, $\{\psi_k\}_{k=0}^\infty$, consisting of eigenfunctions of T [26, Theorem (VI.11)]:

$$T\psi_k = \lambda_k \psi_k, \quad \lambda_k \in \mathbb{R}. \qquad (4.5.14)$$

Substituting (4.5.14) in (4.5.7), ψ_k satisfies the following spectral problem

$$\begin{cases} -\mu_k \psi_k + \partial^4 \psi_k = 0, & \text{in } \Omega \\ \partial \psi_k = \partial^3 \psi_k = 0, & \text{at } \{y = 0\} \\ \psi_k = \partial^3 \psi_k = 0, & \text{at } \{y = 1\} \end{cases} \qquad (4.5.15)$$

where $\mu_k = \frac{1-\lambda_k}{\lambda_k}$. Multiplying the equation in (4.5.15) by $\partial^2\psi_k$ we have

$$\mu_k \int_\Omega (\partial\psi_k)^2 + \int_\Omega (\partial^3\psi_k)^2 = 0$$

which, from the coercivity, implies $\mu_k \geq 0$ for all $k \geq 0$. Let $V_N = span\{\psi_0, ..., \psi_N\}$. Without loss of generality, the eigenvalues are ordered so that $0 = \mu_0 < \mu_1 < ...$, and, after a suitable Gram-Schimdt orthonormalization process, the eigenfunctions are taken to be orthonormal in $H^1_*(\Omega)$, i.e.

$$(\psi_i, \psi_j) = \int_\Omega \partial\psi_i \partial\psi_j = \delta_{ij}. \tag{4.5.16}$$

Note that, from (4.5.15) and (4.5.16),

$$\psi_0 = \frac{\sqrt{3}}{2}(1 - y^2).$$

Fix now an integer N. Let $v^N(t, y)$ be an approximated solution belonging to V_N, namely

$$v^N(t, y) = \sum_{k=0}^{N} a_k^N(t)\psi_k(y) \quad \text{in } Q_T. \tag{4.5.17}$$

We want to select the unknown coefficients $a_k^N(t)$ $(0 \leq t \leq T, k = 0, ..., N)$ by plugging (4.5.17) into the problem (4.5.6). We first notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}(v^N, \psi_k) = \int_\Omega \partial v_t^N \partial\psi_k = \sum_{j,k=0}^{N} \dot{a}_j^N(t) \int_\Omega \partial\psi_j \partial\psi_k \overset{(4.5.16)}{=} \dot{a}_k^N(t). \tag{4.5.18}$$

Assuming for a moment that v^N is a solution of $(\hat{P})$, by integration by parts we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(v^N, \psi_k) &= -\sum_{j,k=0}^{N} \dot{a}_j^N(t) \int_\Omega \psi_j \partial^2\psi_k \,\mathrm{d}y = -\int_\Omega v_t^N \partial^2\psi_k \,\mathrm{d}y \\ &= \int_\Omega [-\frac{\dot{s}}{s} y\partial v^N + \frac{1}{s^4}\partial(\widehat{m}(v^N)\partial^3 v^N)]\partial^2\psi_k \,\mathrm{d}y \\ &= -\frac{\dot{s}}{s}\int_\Omega y\partial v^N \partial^2\psi_k \,\mathrm{d}y + \frac{1}{s^4}\int_\Omega \partial(\widehat{m}(v^N)\partial^3 v^N)\partial^2\psi_k \,\mathrm{d}y. \end{aligned} \tag{4.5.19}$$

From the boundary conditions and an integration by parts we obtain

$$\dot{a}_k^N(t) = \frac{\mathrm{d}}{\mathrm{d}t}(v^N, \psi_k) = -\frac{\dot{s}}{s}\int_\Omega y\partial v^N \partial^2\psi_k \,\mathrm{d}y - \frac{1}{s^4}\int_\Omega \widehat{m}(v^N)\partial^3 v^N \partial^3\psi_k \,\mathrm{d}y \tag{4.5.20}$$

on $0 \leq t \leq T$ for all $k = 0, ..., N$. The initial condition $v^N(0) = v^N(0, y) = v_0^N$ reads as

$$a_k^N(0) = (v_0^N, \psi_k) \quad k = 0, ..., N. \tag{4.5.21}$$

Collecting (4.5.18) and (4.5.20), we obtain the following initial value problem for the coefficients a_k^N:

$$\dot{a}_k^N(t) = -\frac{\dot{s}}{s}\int_\Omega y\partial v^N \partial^2\psi_k - \frac{1}{s^4}\int_\Omega \widehat{m}(v^N)\partial^3 v^N \partial^3 \psi_k \tag{4.5.22}$$

with initial condition

$$a_k^N(0) = (v_0^N, \psi_k) \tag{4.5.23}$$

for all $k = 0, ..., N$. Since $s \in C^1(0,T)$, the right hand side of (4.5.22) is locally Lipschitz with respect to $a^N = (a_0^N, ..., a_N^N)$. Therefore, according to the standard existence theory for ordinary differential equations, there exists $\tau > 0$ and a unique function $a^N(t) = (a_0^N(t), ..., a_N^N(t))$ satisfying (4.5.22) and (4.5.23) for $0 \le t \le \tau$. This leads to the local existence of a function v^N satisfying (4.5.21) and (4.5.20). These locally defined solutions can be extended to the whole time line as a consequence of the a priori estimates on $a_k^N(t)$, independent of N, that shall be proved in the next step. From the choice of the eigenfunctions ψ_k we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{(\partial v^N)^2}{2} &= \int_\Omega \partial v^N \partial v_t^N \overset{(4.5.17)}{=} \sum_{j,k=0}^N \int_\Omega a_j^N(t)\partial\psi_j \dot{a}_k^N(t)\partial\psi_k \\
&= \sum_{j,k=0}^N a_j^N(t)\dot{a}_k^N(t)\int_\Omega \partial\psi_j\partial\psi_k = \sum_{k=0}^N a_k^N(t)\dot{a}_k^N(t) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^N \frac{(a_k^N(t))^2}{2}.
\end{aligned} \tag{4.5.24}$$

Thus, in view of (4.5.23), we have

$$\sum_{k=0}^N \frac{(a_k^N(t))^2}{2} = \int_\Omega \frac{(\partial v^N)^2}{2}. \tag{4.5.25}$$

Integrations by parts in (4.5.22) lead to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{(\partial v^N)^2}{2} = -\frac{\dot{s}}{2s}(\partial v^N(1))^2 + \frac{\dot{s}}{2s}\int_\Omega (\partial v^N)^2 - \frac{1}{s^4}\int_\Omega \widehat{m}(v^N)(\partial^3 v^N)^2 \tag{4.5.26}$$

and integrating in time we have

$$\begin{aligned}
&\int_\Omega \frac{(\partial v^N(t))^2}{2} + \iint_{Q_t} \frac{1}{s^4}\widehat{m}(v^N)(\partial^3 v^N)^2 \\
&= \int_\Omega \frac{(\partial v_0^N)^2}{2} - \frac{1}{2}\int_0^t \frac{\dot{s}}{s}(\partial v^N(1))^2 + \frac{1}{2}\iint_{Q_t} \frac{\dot{s}}{s}(\partial v^N)^2.
\end{aligned} \tag{4.5.27}$$

In particular we can estimate

$$\begin{aligned}
\left|\int_0^t \frac{\dot{s}}{s}(\partial v^N(1))^2\right| &\overset{(4.5.3),(4.4.4)}{\le} C\int_0^t \left(\int_\Omega (\partial v^N)^2\right)^{3/4}\left(\int_\Omega (\partial^3 v^N)^2\right)^{1/4} \\
&\le C_\alpha \int_{Q_t} (\partial v^N)^2 + \alpha\int_{Q_t} (\partial^3 v^N)^2
\end{aligned} \tag{4.5.28}$$

for all $\alpha > 0$. Choosing α sufficiently small, and using again (4.5.3) and $m \in [\delta, \delta^{-1}]$, from (4.5.27) we obtain that

$$\int_\Omega \frac{(\partial v^N(t))^2}{2} + C^{-1} \iint_{Q_t} (\partial^3 v^N)^2 \leq \int_\Omega \frac{(\partial v_0^N)^2}{2} + C \iint_{Q_t} (\partial v^N)^2. \tag{4.5.29}$$

Hence a Gronwall argument yields

$$\int_\Omega \frac{(\partial v^N(t))^2}{2} + \iint_{Q_t} (\partial^3 v^N)^2 \leq C_T \quad \text{for all } t \in (0,T), \tag{4.5.30}$$

independently of N. Then (4.5.25) implies that

$$\sum_{k=0}^{N} \frac{(a_k^N(t))^2}{2} \leq C_T \quad \text{for all } t \in (0,T). \tag{4.5.31}$$

In particular from (4.5.30) we have

$$\|v^N\|_{L^\infty((0,T),H^1_*(\Omega))} \leq C \tag{4.5.32}$$

and

$$\|v^N\|_{L^2((0,T),H^3_*(\Omega))} \leq C. \tag{4.5.33}$$

The estimate in (4.5.31) allow to extend globally the solution to (4.5.22) to $(0,T)$ for an arbitrary $T > 0$. Our task now is to pass to the limit as $N \to \infty$. Given $\varphi \in L^2((0,T);H^1(\Omega))$, let $\psi \in L^2((0,T);H^3_*(\Omega))$ be defined by

$$\psi(t,y) := \int_y^1 \int_0^{y'} \varphi(t,y'')\,\mathrm{d}y''\,\mathrm{d}y' \tag{4.5.34}$$

so that $\partial\psi(0) = \psi(1) = 0$. Let P^N be the projection on the subspace V_N of $H^3_*(\Omega)$:

$$P^N\psi = \sum_{k=0}^{N} b_k\psi_k, \; b_k = (\psi,\psi_k). \tag{4.5.35}$$

Multiplying (4.5.20) by b_k, summing from 0 to N and integrating in time, it follows that

$$\iint_{Q_T} v_t^N \partial^2 P^N\psi \;=\; \iint_{Q_T} \frac{\dot{s}}{s} y \partial v^N \partial^2 P^N\psi + \iint_{Q_T} \frac{1}{s^4} \widehat{m}(v^N) \partial^3 v^N \partial^3 P^N\psi. \tag{4.5.36}$$

By (4.5.4), (4.5.20) and (4.5.30) we have that

$$\left| \iint_{Q_t} \frac{\dot{s}}{s} y \partial v^N \partial^2 P^N\psi \right| \leq C\|\psi\|_{L^2((0,T);H^3_*(\Omega))}. \tag{4.5.37}$$

After integrations by parts we have

$$\int_\Omega \partial^3\psi_j \partial^3\psi_k = -\int_\Omega \partial^2\psi_j \partial^4\psi_k = -\int_\Omega \partial^2\psi_j \mu_k \psi_k = \mu_k \int_\Omega \partial\psi_j \partial\psi_k = \mu_k \delta_{jk} \tag{4.5.38}$$

and by (4.5.35)

$$\begin{aligned}\int_\Omega (\partial^3 P^N \psi)^2 &= \int_\Omega \sum_{j,k=0}^N b_j b_k \partial^3\psi_j \partial^3\psi_k \\ &\overset{(4.5.38)}{=} \sum_{j=0}^N b_j^2\mu_j \le \sum_{j=0}^\infty b_j^2\mu_j = \int_\Omega (\partial^3\psi)^2. \end{aligned} \tag{4.5.39}$$

Thus

$$\begin{aligned}\left|\iint_{Q_t} \frac{1}{s^4}\widehat{m}(v^N)\partial^3 v^N \partial^3 P^N\psi\right| &\le C\left(\iint_{Q_t}\widehat{m}(v^N)(\partial^3 v^N)^2\right)^{1/2}\left(\iint_{Q_T}(\partial^3 P^N\psi)^2\right)^{1/2} \\ &\overset{(4.5.30)}{\le} C\left(\iint_{Q_T}(\partial^3 P^N\psi)^2\right)^{1/2} \\ &\overset{(4.5.39)}{\le} C\|\psi\|_{L^2((0,T);H^3_*(\Omega))}. \end{aligned} \tag{4.5.40}$$

Gathering (4.5.37) and (4.5.40) we obtain

$$\left|\iint_{Q_t} v_t^N \partial^2\psi\right| \le C\|\psi\|_{L^2((0,T);H^3_*(\Omega))} \quad \text{for all } \psi \in L^2((0,T);H^3_*(\Omega)) \tag{4.5.41}$$

and since $\partial^2\psi = -\varphi$

$$\left|\iint_{Q_t} v_t^N \varphi\right| \le C\|\varphi\|_{L^2((0,T);H^1(\Omega))} \quad \text{for all } \varphi \in L^2((0,T);H^1(\Omega)). \tag{4.5.42}$$

Hence

$$\|v_t^N\|_{L^2((0,T);(H^1(\Omega))')} \le C. \tag{4.5.43}$$

Collecting (4.5.32), (4.5.33), and (4.5.43), and using Simon compactness criterion (see Theorem 4.3 in Section 4.4), a subsequence (still indexed by N) can be selected in such a way that

$$v^N \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty((0,T);H^1_*(\Omega)), \tag{4.5.44}$$

$$v^N \rightharpoonup v \quad \text{in } L^2((0,T);H^3_*(\Omega)), \tag{4.5.45}$$

$$v^N \to v \quad \text{in } C([0,T);L^2(\Omega)), \tag{4.5.46}$$

$$v_t^N \rightharpoonup v_t \quad \text{in } L^2((0,T);(H^1_*(\Omega))'). \tag{4.5.47}$$

which in particular implies $(i)-(iii)$ of Proposition 4.1. We want now pass to the limit as $N \to \infty$ in the weak formulation (4.5.36). By (4.5.47), we have that as $N \to \infty$:

$$\iint_{Q_T} v_t^N \partial^2 P^N\psi = \iint_{Q_T} v_t^N\partial^2\psi \longrightarrow \int_0^T < v_t, \partial^2\psi > \overset{(4.5.34)}{=} -\int_0^T < v_t, \varphi > . \tag{4.5.48}$$

From (4.5.45) and the regularity of $\widehat{m}$ we have

$$\widehat{m}(v^N) \longrightarrow \widehat{m}(v) \quad \text{in } L^2(Q_T) \text{ as } N \to \infty. \tag{4.5.49}$$

Indeed

$$\iint_{Q_T} |\widehat{m}(v^N) - \widehat{m}(v)|^2 \le \iint_{Q_T} \sup |\widehat{m}'|^2 |v^N - v|^2 \le C \iint_{Q_T} |v^N - v|^2 \overset{(4.5.45)}{\longrightarrow} 0$$

as $N \to \infty$. Thus (4.5.49) and (4.5.45) allow to pass to the limit in the third integral of (4.5.36):

$$\iint_{Q_T} \frac{1}{s^4}\widehat{m}(v^N)\partial^3 v^N \partial^3 P^N \psi \to \iint_{Q_T} \frac{1}{s^4}\widehat{m}(v)\partial^3 v \partial^3 \psi \overset{(4.5.34)}{=} -\iint_{Q_T} \frac{1}{s^4}\widehat{m}(v)\partial^3 v \partial\varphi. \tag{4.5.50}$$

Now for the first term of the right hand side in (4.5.36) we have

$$\iint_{Q_T} \frac{\dot{s}}{s} y \partial v^N\, \partial^2 P^N \psi \overset{(4.5.44)}{\longrightarrow} \iint_{Q_T} \frac{\dot{s}}{s} y \partial v \partial^2 \psi \overset{(4.5.34)}{=} -\iint_{Q_T} \frac{\dot{s}}{s} y \partial v \varphi. \tag{4.5.51}$$

Combining (4.5.48) and (4.5.50) with (4.5.51) enables us to pass to the limit and obtain a weak solution in the following sense:

$$\int_0^T < v_t, \varphi >= \iint_{Q_T} \frac{\dot{s}}{s} y \partial v \varphi + \iint_{Q_T} \frac{1}{s^4}\widehat{m}(v)\partial^3 v \partial\varphi \tag{4.5.52}$$

for all $\varphi \in L^2((0,T), H^1(\Omega))$. It follows that the original function $v(y,t) = \hat{v}(y,t) + \varepsilon$ is a solution v to $(P_{\varepsilon,\delta,\sigma,s})$ in the sense of Proposition 4.1. □

In order to pass from Proposition 4.2 to Proposition 4.1, some useful a priori estimates are presented in the following lemma:

Lemma 4.2. *Let v be a solution to $(P_{\varepsilon,\delta,\sigma,s})$. Suppose $m_{\delta,\sigma} \in C^1(\mathbf{R}, [\delta, \delta^{-1}])$ and (4.5.3). Then the following a priori estimates hold*

$$\sup_t \int_\Omega \frac{v_y^2}{s}\, dy + \iint_{Q_t} \frac{\delta}{s^5} v_{yyy}^2 \le C \int_\Omega \frac{v_{0y}^2}{s_0}\, dy \quad \textit{for all } t < \frac{\delta}{Ck^4} =: T_{\delta,k} \tag{4.5.53}$$

$$\|v_t\|_{L^2((0,T);(H^1(\Omega))')} \le C(\delta, k, s_m). \tag{4.5.54}$$

Proof. Here and after $C > 1$ denotes a universal constant. Since $v \in L^2(0,T; H^3(\Omega)$, choosing $\varphi = -\frac{\partial^2 v}{s}$ in (4.5.2) we obtain on the left hand side:

$$-\int_0^t \frac{1}{s} < v_t, \partial^2 v >= \frac{1}{2s}\int_\Omega (\partial v)^2\Big|_0^t + \int_0^t \frac{\dot{s}}{2s^2}\int_\Omega v_y^2.$$

Therefore

$$\begin{aligned}
\int_\Omega \frac{v_y^2}{2s}\Big|_0^t &= -\int_0^t \frac{\dot s}{s^2}\int_\Omega y(\frac{v_y^2}{2})_y - \iint_{\Omega_t}\frac{1}{s^5}\widehat{m}(v)v_{yyy}^2 - \iint_{\Omega_t}\frac{v_y^2\dot s}{2s^2}\\
&= -\int_0^t \frac{\dot s}{s^2}\frac{v_y^2}{2}(1) + \iint_{\Omega_t}\frac{\dot s}{s^2}\frac{v_y^2}{2} - \iint_{\Omega_t}\frac{1}{s^5}m_{\delta,\sigma}(v)v_{yyy}^2 - \iint_{\Omega_t}\frac{v_y^2\dot s}{2s^2}\\
&= -\int_0^t \frac{\dot s}{s^2}\frac{v_y^2(1)}{2} - \iint_{\Omega_t}\frac{1}{s^5}m_{\delta,\sigma}(v)v_{yyy}^2\\
&\overset{(4.4.4),(4.3.1)}{\leq} \int_0^t \frac{|\dot s|}{2s^2}\left(\int_\Omega v_y^2\right)^{3/4}\left(\int_\Omega v_{yyy}^2\right)^{1/4} - \delta\iint_{\Omega_t}\frac{1}{s^5}v_{yyy}^2\\
&= \int_0^t \frac{|\dot s|}{(2\delta)^{1/4}}\left(\frac{1}{2s}\int_\Omega v_y^2\right)^{3/4}\left(\frac{\delta}{s^5}\int_\Omega v_{yyy}^2\right)^{1/4} - \delta\iint_{\Omega_t}\frac{1}{s^5}v_{yyy}^2. \quad (4.5.55)
\end{aligned}$$

Using Hölder and Joung inequalities, we obtain

$$\begin{aligned}
\sup_t \int_\Omega \frac{v_y^2}{2s}\,\mathrm{d}y &\leq \frac{1}{(2\delta)^{1/4}}\left(\int_0^t \dot s^2\right)^{1/2}\left(\int_0^t\left(\int_\Omega\frac{v_y^2}{2s}\right)^{3/2}\left(\int_\Omega\frac{\delta}{s^5}v_{yyy}^2\right)^{1/2}\mathrm{d}t\right)^{1/2} - \iint_{Q_t}\frac{\delta}{s^5}v_{yyy}^2\,\mathrm{d}y\\
&\leq \frac{k}{(2\delta)^{1/4}}\left(\sup_t\int_\Omega\frac{v_y^2}{2s}\right)^{3/4}\left(\int_0^t\left(\int_\Omega\frac{\delta v_{yyy}^2}{s^5}\right)^{1/2}\mathrm{d}t\right)^{1/2} - \iint_{Q_t}\frac{\delta}{s^5}v_{yyy}^2\,\mathrm{d}y\\
&\leq \frac{kt^{1/4}}{(2\delta)^{1/4}}\left(\sup_t\int_\Omega\frac{v_y^2}{2s}\right)^{3/4}\left(\iint_{Q_t}\frac{\delta}{s^5}v_{yyy}^2\,\mathrm{d}y\mathrm{d}t\right)^{1/4} - \iint_{Q_t}\frac{\delta}{s^5}v_{yyy}^2\,\mathrm{d}y\\
&\leq C\frac{k^{4/3}t^{1/3}}{\delta^{1/3}}\sup_t(\int_\Omega\frac{v_y^2}{2s}\,\mathrm{d}y) - \frac{1}{2}\iint_{Q_t}\frac{\delta}{s^5}v_{yyy}^2\,\mathrm{d}y. \quad (4.5.56)
\end{aligned}$$

Now if $\dfrac{k^4t}{\delta} < C^{-1}$ then (4.5.53) is recovered. We now show (4.5.54):

$$\begin{aligned}
\left|\int_0^T < v_t,\varphi > \,\mathrm{d}t\right| &= \left|\iint_{Q_T} v_t\varphi\right|\\
&\leq \left|\iint_{Q_T}\frac{\dot s}{s}yv_y\varphi\right| + \left|\iint_{Q_T}\frac{1}{s^4}m_{\delta,\sigma}(v)v_{yyy}\varphi_y\right| = I_1 + I_2. \quad (4.5.57)
\end{aligned}$$

We note that

$$
\begin{aligned}
I_1 \quad &\leq \quad \int_0^T \frac{|\dot{s}|}{\sqrt{s}} \left(\left(\int_\Omega \frac{v_y^2}{s} \right)^{1/2} \left(\int_\Omega \varphi^2 \right)^{1/2} \right) \mathrm{d}t \\
&\leq \quad \frac{1}{\sqrt{s_m}} \left(\int_0^T \dot{s}^2 \right)^{1/2} \left(\int_0^T \left(\int_\Omega \varphi^2 \right) \left(\int_\Omega \frac{v_y^2}{s} \right) \right)^{1/2} \\
&\leq \quad \frac{k}{\sqrt{s_m}} \left(\sup_t \int_\Omega \frac{v_y^2}{s} \right)^{1/2} \left(\iint_{Q_T} \varphi^2 \right)^{1/2} \\
&\overset{(4.5.53)}{\leq} \quad \left(\iint_{Q_T} \varphi^2 \right)^{1/2} \qquad (4.5.58)
\end{aligned}
$$

and that

$$
\begin{aligned}
I_2 \quad &\leq \quad \left| \iint_{Q_T} \frac{1}{s^4} m_{\delta,\sigma}(v) v_{yyy} \varphi_y \, \mathrm{d}y \right| \\
&\leq \quad \left(\iint_{Q_T} \frac{1}{s^3} m_{\delta,\sigma}(v) \varphi_y^2 \right)^{1/2} \left(\iint_{Q_T} \frac{1}{s^5} m_{\delta,\sigma}(v) v_{yyy}^2 \right)^{1/2} \\
&\leq \quad \left(\frac{1}{\delta s_m^3} \right)^{1/2} \left(\iint_{Q_T} \varphi_y^2 \right)^{1/2} \left(\iint_{Q_T} \frac{\delta}{s^5} v_{yyy}^2 \right)^{1/2} \\
&\overset{(4.5.53)}{\leq} \quad C \|\varphi\|_{L^2([0,T);H^1(\Omega))}. \qquad (4.5.59)
\end{aligned}
$$

Inserting (4.5.58) and (4.5.59) in (4.5.57), (4.5.54) follows. □

The last task is to extend by a density argument Proposition 4.2 to the case of s satisfying (4.5.1), which will leads us to Proposition 4.1.

Proof of Proposition 4.1. Let $s_n \in C^1(0,T)$ be such that $s_n \geq s_m$ and

$$
s_n \longrightarrow s \quad \text{in } H^1(0,T) \quad \text{as } n \to \infty. \qquad (4.5.60)
$$

By Hölder inequality

$$
|s_n(t) - s(t)| \leq \int_0^t |\dot{s}_n - \dot{s}| \leq \left(\int_0^T |\dot{s}_n - \dot{s}|^2 \right)^{1/2} t^{1/2} \leq o_n(1) t^{1/2}
$$

so that

$$
s_n \longrightarrow s \quad \text{uniformly in } (0,T) \quad \text{as } n \to \infty. \qquad (4.5.61)
$$

Let v_n be the solution of $(P_{\varepsilon,\delta,\sigma,s_n})$ obtained in Proposition 4.2. From (4.5.53) and (4.5.54) we have, respectively,

$$
\sup_t \int_\Omega \frac{v_{ny}^2}{s_n} \, \mathrm{d}y + \iint_{Q_t} \frac{\delta}{s_n^5} v_{nyyy}^2 \leq C \int_\Omega \frac{v_{0ny}^2}{s_0} \, \mathrm{d}y \leq C \int_\Omega \frac{v_{0y}^2}{s_0} \, \mathrm{d}y \qquad (4.5.62)
$$

for all $t < T_{\delta,k}$ and

$$\|v_{nt}\|_{L^2((0,T_{\delta,k});(H^1_*(\Omega))')} \leq C. \tag{4.5.63}$$

It follows from (4.5.62) and (4.5.63) that for a subsequence (still denoted as v_n) we have, as $n \to \infty$:

$$v_n \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty((0,T_{\delta,k}); H^1(\Omega)), \tag{4.5.64}$$

$$v_n \rightharpoonup v \quad \text{in } L^2((0,T_{\delta,k}); H^3(\Omega)), \tag{4.5.65}$$

$$v_{nt} \rightharpoonup v_t \quad \text{in } L^2((0,T_{\delta,k}); (H^1(\Omega))'). \tag{4.5.66}$$

From the regularity given in (4.5.64) and (4.5.66), by Simon compactness criterion (see Theorem 4.3 in Section 4.4) we have:

$$v_n \to v \quad \text{in } C([0,T_{\delta,k}); L^2(\Omega)) \tag{4.5.67}$$

as $n \to \infty$, which, in particular, implies (i); both (ii) and (iii) are given by (4.5.65) and the continuity of the trace operator. Our last task is to pass to the limit as $n \to \infty$ in the weak formulation

$$\int_0^{T_{\delta,k}} < v_{nt}, \varphi >= \iint_{Q_{T_{\delta,k}}} \frac{\dot{s}_n}{s_n} y v_{ny} \varphi + \iint_{Q_{T_{\delta,k}}} \frac{1}{s_n^4} m_{\delta,\sigma}(v_n) v_{nyyy} \varphi_y \tag{4.5.68}$$

for all $\varphi \in L^2((0,T_{\delta,k}); H^1(\Omega))$. Firstly from (4.5.66) and (4.5.67) we obtain

$$\int_0^{T_{\delta,k}} < v_{nt}, \varphi > \longrightarrow \int_0^{T_{\delta,k}} < v_t, \varphi > \quad \text{as } n \to \infty. \tag{4.5.69}$$

Then, by definition (4.3.1), it follows that $m_{\delta,\sigma}$ is globally Lipschitz in $\mathbb{R}$, namely

$$\sup |m'_{\delta,\sigma}| \leq C \tag{4.5.70}$$

which together with (4.5.67) leads to

$$m_{\delta,\sigma}(v_n) \longrightarrow m_{\delta,\sigma}(v) \quad \text{in } L^2(Q_{T_\delta}) \tag{4.5.71}$$

as $n \to \infty$ as proved in (4.5.49). Hence combining (4.5.71) with (4.5.65) and (4.5.61), implies that as $n \to \infty$

$$\iint_{Q_{T_\delta}} \frac{1}{s_n^4} m_{\delta,\sigma}(v_n) v_{nyyy} \varphi_y \longrightarrow \iint_{Q_{T_\delta}} \frac{1}{s^4} m_{\delta,\sigma}(v) v_{yyy} \varphi_y. \tag{4.5.72}$$

Finally using (4.5.61) and (4.5.60) combined with (4.5.64) we have

$$\iint_{Q_{T_\delta}} \frac{\dot{s}_n}{s_n} y v_{ny} \varphi \longrightarrow \iint_{Q_{T_\delta}} \frac{\dot{s}}{s} y v_y \varphi \quad \text{as } n \to \infty. \tag{4.5.73}$$

Collecting (4.5.69), (4.5.72) and (4.5.73), we obtain (4.5.2) and Proposition 4.1 follows. □

4.6 A fixed Point result

In this section we prove:

Proposition 4.3. *For any $\varepsilon, \delta, \sigma > 0$ there exists a solution (s, v) to problem $(P_{\varepsilon,\delta,\sigma})$ (see* (4.3.2)*) in the sense of Definition 4.1 for T sufficiently small.*

Proof. Let $k \geq 1$ and $T > 0$ to be chosen later, and fix $s_m \in (0, \frac{s_0}{2}]$. We set

$$S_T = \{s \in H^1(0,T) : \|\dot{s}\|_{L^2} \leq k,\ s(0) = s_0,\ s \geq s_m\}.$$

Given $s \in S_T$, let v be the solution of $(P_{\varepsilon,\delta,\sigma,s})$ given in Proposition 4.1. We write $f \lesssim g$, resp. $f \ll g$, if a constant $C \geq 1$ independent of k and of $T < T_{\delta,k}$ (may depend on δ, v_0, s_m, s_0, ε, d, Lipschitz constant of $m_{\delta,\sigma}$) exists such that $f \leq Cg$, resp. $Cf \leq g$. The a-priori bounds translate into:

$$(4.5.53) \quad \Rightarrow \quad \sup_t \int_\Omega v_y^2 + \iint_{Q_t} v_{yyy}^2 \lesssim 1, \tag{4.6.1}$$

$$(4.5.54) \quad \Rightarrow \quad \|v_t\|_{L^2((0,t);(H^1_\varepsilon(\Omega))')} \lesssim 1. \tag{4.6.2}$$

We observe that

$$\begin{aligned}
\int_0^t (v_y(t,1))^4 \,\mathrm{d}t \quad &\overset{(4.4.4)}{\lesssim} \quad \int_0^t \left(\int_\Omega v_y^2\right)^{3/2} \left(\int_\Omega v_{yyy}^2\right)^{1/2} \\
&\leq \quad \left(\sup_t \int_\Omega v_y^2\right)^{3/2} \left(\iint_{Q_t} v_{yyy}^2\right)^{1/2} t^{1/2} \overset{(4.6.1)}{\leq} Ct^{1/2} && (4.6.3) \\
&\overset{(4.6.1)}{\lesssim} \quad 1 \quad \text{for } T \ll 1. && (4.6.4)
\end{aligned}$$

Hence it is well defined:

$$\tilde{s}(t) = s_0 + d \int_0^t \left(\frac{v_y^2(\tau,1)}{s^2(\tau)} - \theta_S^2 \right) \mathrm{d}\tau =: F(s). \tag{4.6.5}$$

In addition

$$\begin{aligned}
\int_0^t \dot{s}^2 \quad &\overset{(4.6.5)}{\lesssim} \quad 1 + \int_0^t (v_y(t,1))^4 \mathrm{d}t \\
&\overset{(4.6.4)}{\lesssim} \quad 1 \leq k^2 \quad \text{for } T \ll 1 && (4.6.6)
\end{aligned}$$

for k sufficiently large and, consequently

$$\tilde{s}(t) \geq s_0 - d\theta_S^2 t - Cdt^{1/2} \geq \frac{s_0}{2} \quad \text{for } T \ll 1.$$

Therefore the inclusion $F(S_T) \subseteq S_T$ holds. From now on k is fixed once for all and $\lesssim$, $\ll$ also include dependence on k. We claim that if T is small enough, then F is a contraction in S_T, i.e. there exists $L < 1$ such that

$$\|\dot{\tilde{s}}_1 - \dot{\tilde{s}}_2\|_{L^2(0,T)} \leq L\|\dot{s}_1 - \dot{s}_2\|_{L^2(0,T)} \tag{4.6.7}$$

for all $s_1, s_2 \in S_T$. Let (s_1, u) and (s_2, v) be two pairs. Define $s = s_1 - s_2$, $w = u - v$, $\tilde{s} = \tilde{s}_1 - \tilde{s}_2$. We have:

$$\begin{aligned}
\int_0^t \dot{\tilde{s}}^2 \, \mathrm{d}t \quad &\lesssim \quad \int_0^t \left(\frac{u_y^2(1)}{s_1^2} - \frac{v_y^2(1)}{s_2^2} \right)^2 \, \mathrm{d}t \\
&\lesssim \quad \int_0^t \left(\frac{u_y^2(1)}{s_1^2} - \frac{v_y^2(1)}{s_1^2} \right)^2 \, dt + \int_0^t v_y^4(1) \left(\frac{1}{s_1^2} - \frac{1}{s_2^2} \right)^2 \, \mathrm{d}t \\
&\lesssim \quad \int_0^t (u_y^2(1) + v_y^2(1))^2 w_y^2(1) \, \mathrm{d}t + \sup_t s^2 \int_0^t v_y^4(1) \, \mathrm{d}t \\
&\overset{(4.6.3)}{\lesssim} \quad t^{1/4} \left(\int_0^t w_y^4(1) \, \mathrm{d}t \right)^{1/2} + t^{3/2} \int_0^t \dot{s}^2
\end{aligned} \tag{4.6.8}$$

where in the last inequality we have used

$$s^2 = \left(\int_0^t |\dot{s}| \right)^2 \lesssim t \left(\int_0^t \dot{s}^2 \right).$$

Note that as in (4.6.4)

$$\int_0^t w_y^4(1) \, \mathrm{d}t \lesssim \left(\sup_t \int_\Omega w_y^2 \right)^{3/2} \left(\iint_{Q_t} w_{yyy}^2 \right)^{1/2} t^{1/2}. \tag{4.6.9}$$

Hence (4.6.8) turns into

$$\int_0^t \dot{\tilde{s}}^2 \, \mathrm{d}t \leq \left(\sup_t \int_\Omega w_y^2 \right)^{3/4} \left(\iint_{Q_t} w_{yyy}^2 \right)^{1/4} t^{1/2} + t^{3/2} \int_0^t \dot{s}^2. \tag{4.6.10}$$

We will now bound the energy of w in terms of $\dot{s}$. We formally write the equation for the difference as follows :

$$w_t - \frac{\dot{s}_1}{s_1} y u_y + \frac{\dot{s}_2}{s_2} y v_y + \frac{1}{s_1^4}(m_{\delta,\sigma}(u) u_{yyy})_y - \frac{1}{s_2^4}(m_{\delta,\sigma}(v) v_{yyy})_y = 0. \tag{4.6.11}$$

After few calculations,

$$\begin{gathered}
w_t - \frac{\dot{s}_1}{s_1} y w_y + \left(\frac{\dot{s}_2}{s_2} - \frac{\dot{s}_1}{s_1} \right) y v_y + \frac{1}{s_1^4}(m_{\delta,\sigma}(u)(u_{yyy} - v_{yyy}))_y + \\
+ \frac{1}{s_1^4}(m_{\delta,\sigma}(u) v_{yyy})_y - \frac{1}{s_1^4}(m_{\delta,\sigma}(v) v_{yyy})_y + \frac{1}{s_1^4}(m_{\delta,\sigma}(v) v_{yyy})_y - \frac{1}{s_2^4}(m_{\delta,\sigma}(v) v_{yyy})_y = 0.
\end{gathered} \tag{4.6.12}$$

So w formally solves the following equation

$$\begin{aligned} w_t \quad - \quad & \frac{\dot s_1}{s_1} y w_y + \frac{\dot s_2 - \dot s_1}{s_1} y v_y + \dot s_2 \left(\frac{1}{s_2} - \frac{1}{s_1}\right) y v_y + \frac{1}{s_1^4}(m_{\delta,\sigma}(u) w_{yyy})_y + \\ + \quad & \frac{1}{s_1^4}((m_{\delta,\sigma}(u) - m_{\delta,\sigma}(v)) v_{yyy})_y + \left(\frac{1}{s_1^4} - \frac{1}{s_2^4}\right)(m_{\delta,\sigma}(v) v_{yyy})_y = 0 \quad \text{in } (0,1). \end{aligned} \tag{4.6.13}$$

We translate (4.6.13) into the weak formulation by testing (4.5.2) with w_{yy}. We obtain

$$\begin{aligned} \int_\Omega \frac{w_y^2}{2}\,\mathrm{d}y \Big|_0^t \quad = \quad & -\iint_{Q_t} \frac{\dot s_1}{s_1} y w_y w_{yy} + \iint_{Q_t} \frac{\dot s_2 - \dot s_1}{s_1} w_{yy} y v_y + \iint_{Q_t} \dot s_2 \left(\frac{1}{s_2} - \frac{1}{s_1}\right) w_{yy} y v_y \\ - \quad & \iint_{Q_t} \frac{1}{s_1^4} m_{\delta,\sigma}(u) w_{yyy}^2 - \iint_{Q_t} \frac{1}{s_1^4}(m_{\delta,\sigma}(u) - m_{\delta,\sigma}(v)) w_{yyy} v_{yyy} \\ - \quad & \iint_{Q_t} \left(\frac{1}{s_1^4} - \frac{1}{s_2^4}\right) m_{\delta,\sigma}(v) w_{yyy} v_{yyy}. \end{aligned} \tag{4.6.14}$$

Our aim is now to obtain an estimate of the form

$$LHS := \sup_t \int_\Omega \frac{w_y^2}{2}\,\mathrm{d}y + \iint_{Q_t} w_{yyy}^2 \le R, \tag{4.6.15}$$

with the remainder terms in R which may be absorbed on the left hand side. We have for the first term in (4.6.14)

$$\left| \iint_{Q_t} \frac{\dot s_1}{s_1} y w_y w_{yy} \right| \overset{(4.5.4)}{\lesssim} t^{1/4}(LHS) \le l(LHS) \quad \text{for } T \ll 1 \tag{4.6.16}$$

where l is a small universal constant (say $l = 1/1000$) fixed once for all. For the others terms in (4.6.14) (except for the fourth one, which is our anchor) we have:

$$\begin{aligned} \left| \iint_{Q_t} \frac{\dot s}{s_1} y v_y w_{yy} \right| \quad & \overset{(4.5.4),(4.6.1)}{\lesssim} \quad t^{1/4} \left(\int_0^t \dot s^2 \right)^{1/2} (LHS)^{1/2} \\ & \le \quad l \int_0^t \dot s^2 + l(LHS) \quad \text{for} \quad T \ll 1. \end{aligned} \tag{4.6.17}$$

$$\begin{aligned} \left| \iint_{Q_t} \dot s_2 \left(\frac{1}{s_2} - \frac{1}{s_1}\right) y v_y w_{yy} \right| \quad & \overset{(4.5.4)}{\lesssim} \quad t^{1/4} \left(\sup_t s \right) (LHS)^{1/2} \\ & \le \quad t^{3/4} \left(\int_0^t \dot s^2 \right)^{1/2} (LHS)^{1/2} \\ & \le \quad l \int_0^t \dot s^2 + l(LHS) \quad \text{for} \quad T \ll 1. \end{aligned} \tag{4.6.18}$$

$$\left|\iint_{Q_t}\left(\frac{1}{s_1^4}-\frac{1}{s_2^4}\right)m_{\delta,\sigma}(v)w_{yyy}v_{yyy}\right| \lesssim \left(\sup_t s\right)\Big(\iint_{Q_t} v_{yyy}^2\Big)^{1/2}\Big(\iint_{Q_t} w_{yyy}^2\Big)^{1/2}$$

$$\overset{(4.6.1)}{\lesssim} \left(\sup_t s\right)\Big(\iint_{Q_t} w_{yyy}^2\Big)^{1/2}$$

$$\lesssim t^{1/2}\left(\int_0^t \dot{s}^2\right)^{1/2}\Big(\iint_{Q_t} w_{yyy}^2\Big)^{1/2}$$

$$\overset{(4.5.4)}{\leq} l\int_0^t \dot{s}^2 + l(LHS) \quad \text{for} \quad T \ll 1. \quad (4.6.19)$$

Since m is Lipschitz,

$$\left|\iint_{Q_t}\frac{1}{s_1^4}(m_{\delta,\sigma}(u)-m_{\delta,\sigma}(v))w_{yyy}v_{yyy}\right| \lesssim \sup_{t,y}|m_{\delta,\sigma}(u)-m_{\delta,\sigma}(v)|\left(\iint_{Q_t} v_{yyy}^2\right)^{1/2}\left(\iint_{Q_t} w_{yyy}^2\right)^{1/2}$$

$$\overset{(4.6.1)}{\lesssim} \left(\sup_{t,y}|w|\right)(LHS)^{1/2}. \quad (4.6.20)$$

Noting that

$$\sup_y |w| \leq w(1) + \int_y^1 |w_y| \leq \left(\int_\Omega w_y^2\right)^{1/2}, \quad (4.6.21)$$

taking the sup in t, we obtain

$$\sup_{t,y}|w| \leq \left(\sup_t \int_\Omega w_y^2\right)^{1/2}. \quad (4.6.22)$$

Therefore (4.6.20) turns into

$$\left|\iint_{Q_t}\frac{1}{s_1^4}(m_{\delta,\sigma}(u)-m_{\delta,\sigma}(v))w_{yyy}v_{yyy}\right| \lesssim (LHS). \quad (4.6.23)$$

Unfortunately, however, this is not enough to absorb on the left hand side. Hence we need a bound on $\sup_{t,y}|w|$ which depends on $\dot{s}$. To do this, we use w as test function in (4.5.2), obtaining as before

$$\int_\Omega \frac{w^2}{2}\,\mathrm{d}y\Big|_0^t = \iint_{Q_t}\frac{\dot{s}_1}{s_1}yww_y + \iint_{Q_t}\frac{\dot{s}_1-\dot{s}_2}{s_1}ywv_y + \iint_{Q_t}\dot{s}_2\left(\frac{1}{s_1}-\frac{1}{s_2}\right)ywv_y \quad (4.6.24)$$

$$+ \iint_{Q_t}\frac{1}{s_1^4}m_{\delta,\sigma}(u)w_{yyy}w_y + \iint_{Q_t}\frac{1}{s_1^4}(m_{\delta,\sigma}(u)-m_{\delta,\sigma}(v))v_{yyy}w_y$$

$$+ \iint_{Q_t}\left(\frac{1}{s_1^4}-\frac{1}{s_2^4}\right)m_{\delta,\sigma}(v)v_{yyy}w_y.$$

Setting

$$\sup_t \int_\Omega w^2 = R_1$$

we now estimate the terms in R_1 in a similar fashion as those in R:

$$\begin{aligned}
\left|\iint_{Q_t} \frac{\dot{s}_1}{s_1} yww_y\right| &\lesssim t^{1/2}\left(\sup_t \int_\Omega w^2\right)^{1/2}\left(\sup_t \int_\Omega w_y^2\right)^{1/2} \\
&\lesssim t^{1/2}\left(\sup_t \int_\Omega w^2\right) + t^{1/2}\left(\sup_t \int_\Omega w_y^2\right) \\
&\lesssim t^{1/2}\left(\sup_t \int_\Omega w^2\right) + t^{1/2} LHS \qquad (4.6.25)
\end{aligned}$$

$$\begin{aligned}
\left|\iint_{Q_t} \frac{\dot{s}_1 - \dot{s}_2}{s_1} ywv_y\right| &\overset{(4.6.1)}{\lesssim} t^{1/2}\left(\sup_t \int_\Omega w^2\right)^{1/2}\left(\int_0^t \dot{s}^2\right)^{1/2} \\
&\lesssim t^{1/2}\left(\sup_t \int_\Omega w^2\right) + t^{1/2}\int_0^t \dot{s}^2 \qquad (4.6.26)
\end{aligned}$$

$$\begin{aligned}
\left|\iint_{Q_t} \dot{s}_2\left(\frac{1}{s_1} - \frac{1}{s_2}\right) ywv_y\right| &\overset{(4.6.1)}{\lesssim} t^{1/2}\left(\sup_t \int_\Omega w^2\right)^{1/2} \sup_t s \\
&\lesssim t\left(\int_0^t \dot{s}^2\right) + t\left(\sup_t \int_\Omega w^2\right) \qquad (4.6.27)
\end{aligned}$$

$$\begin{aligned}
\left|\iint_{Q_t} \frac{1}{s_1^4} m_{\delta,\sigma}(u) w_{yyy} w_y\right| &\lesssim t^{1/2}\left(\iint_{Q_t} w_{yyy}^2\right)^{1/2}\left(\sup_t \int_\Omega w_y^2\right)^{1/2} \\
&\overset{(4.5.4)}{\lesssim} t^{1/2}(LHS) \qquad (4.6.28)
\end{aligned}$$

$$\begin{aligned}
|\iint_{Q_t} \frac{1}{s_1^4}(m_{\delta,\sigma}(u) - m_{\delta,\sigma}(v)) v_{yyy} w_y| &\lesssim \left(\sup_{t,y} |w|\right)\left(\iint_{Q_t} v_{yyy}^2\right)^{1/2} t^{1/2}\left(\sup_t \int_\Omega w_y^2\right)^{1/2} \\
&\overset{(4.6.22),(4.6.1)}{\lesssim} t^{1/2}(LHS) \qquad (4.6.29)
\end{aligned}$$

$$\begin{aligned}
\left|\iint_{Q_t} (\frac{1}{s_1^4} - \frac{1}{s_2^4}) m_{\delta,\sigma}(v) v_{yyy} w_y\right| &\lesssim \left(\sup_t s\right)\left(\iint_{Q_t} v_{yyy}^2\right)^{1/2} t^{1/2}\left(\sup_t \int_\Omega w_y^2\right)^{1/2} \\
&\overset{(4.6.1)}{\lesssim} t\left(\int_0^t \dot{s}^2\right)^{1/2}(LHS)^{1/2} \\
&\lesssim t\left(\int_0^t \dot{s}^2\right) + t\,(LHS). \qquad (4.6.30)
\end{aligned}$$

For $t \ll 1$, $\frac{1}{2} - Ct^{1/2} \geq \frac{1}{4}$. Hence, collecting (4.6.25)-(4.6.30) in (4.6.24) and absorbing on the left-hand side we conclude

$$\sup_t \int_\Omega w^2 \lesssim t^{1/2}\left(\int_0^t \dot{s}^2\right) + t^{1/2}(LHS). \tag{4.6.31}$$

Now, by interpolation, we have

$$\begin{aligned} \sup_y |w|^2 &\leq \left(\int_\Omega w^2\right)^{1/2}\left(\int_\Omega w_y^2\right)^{1/2} + \int_\Omega w^2 \\ &\leq l^2\left(\int_\Omega w_y^2\right) + \frac{1}{l^2}\left(\int_\Omega w^2\right). \end{aligned} \tag{4.6.32}$$

Taking the sup in t

$$\begin{aligned} \sup_{t,y} |w|^2 &\overset{(4.6.31),(4.6.32)}{\leq} l^2(LHS) + \frac{1}{l^2}t^{1/2}\left(\int_0^t \dot{s}^2 + (LHS)\right) \\ &\lesssim l^2(LHS) + l^2\left(\int_0^t \dot{s}^2 + (LHS)\right) \quad \text{for } T \ll 1 \end{aligned} \tag{4.6.33}$$

Therefore, from (4.6.20),

$$\left|\iint_{Q_t} \frac{1}{s_1^4}(m_{\delta,\sigma}(u) - m_{\delta,\sigma}(v))w_{yyy}v_{yyy}\right| \overset{(4.6.33)}{\lesssim} l(LHS) + l\int_0^t \dot{s}^2. \tag{4.6.34}$$

Combining now (4.6.16)-(4.6.19) and (4.6.34) into (4.6.14) and since $l \ll 1$, we obtain the desired estimate of the form (4.6.15). More precisely (4.6.15) reduces to:

$$\sup_t \int_\Omega \frac{w_y^2}{2}\,\mathrm{d}y + \iint_{Q_t} w_{yyy}^2 \leq 4l\int_0^t \dot{s}^2.$$

Hence (4.6.10) reads as

$$\int_0^t \dot{\hat{s}}^2\,\mathrm{d}t \leq t^{1/2}\int_0^t \dot{s}^2 \tag{4.6.35}$$

i.e. the contractivity (4.6.7) for $t \ll 1$. Applying Banach Fixed-Point Theorem, there exists a unique fixed point $s \in S_T$ such that

$$F(s) = s$$

that is

$$\dot{s} = d\left(\frac{v_y^2(1)}{s^2} - \theta_S^2\right) \quad \text{in } L^2(0,T)$$

and the boundary condition is recovered. □

4.7 A-priori estimates and global existence for the approximating problems

Given a solution to $(P_{\varepsilon,\delta,\sigma})$, in the sense of Definition 4.1, we have (choosing $\varphi = s$ as test function in (4.5.2))

$$\begin{aligned}\int_\Omega sv\,\mathrm{d}y\Big|_0^t &= \iint_{Q_t}(\dot{s}v + sv_t)\\ &= \iint_{Q_t}\dot{s}v + \iint_{Q_t}\dot{s}yv_y\\ &= \iint_{Q_t}\dot{s}v + \int_0^t \dot{s}[yv]_0^1 - \iint_{Q_t}\dot{s}v\\ &= \int_0^t \dot{s}v(1) = \varepsilon\int_0^t \dot{s}. \end{aligned}\tag{4.7.1}$$

Therefore

$$\int_\Omega sv\,\mathrm{d}y = \int_\Omega s_0v_0\,\mathrm{d}y + \varepsilon(s(t) - s_0). \tag{4.7.2}$$

We are now ready to exploit the dissipative structure of the problem, obtaining the following a-priori bounds.

Lemma 4.3. *Let $\varepsilon > 0$, $\delta > 0$, $\sigma > 0$, $s_0 > 0$, and $v_{0\varepsilon} \in H_0^1(\Omega)$ such that $\int_\Omega v_0 > 0$. Then a positive constant C, depending only on $\|v_{0\varepsilon}\|_{H^1}$ and s_0, exists such that any solution (s, v) of $(P_{\varepsilon,\delta,\sigma})$ in the sense of Definition 4.1 satisfies for all $t \in (0, T)$:*

$$s(t) \geq C^{-1} \tag{4.7.3}$$

$$\sup_t \int_\Omega v_y^2 \leq C, \tag{4.7.4}$$

$$\int_0^t \int_\Omega m_{\delta,\sigma}(v)v_{yyy}^2 \leq C, \tag{4.7.5}$$

$$\int_0^t \dot{s}^2 \leq C, \tag{4.7.6}$$

$$\int_0^t \left(\frac{v_y^2(t,1)}{s^2} - \theta_S^2\right)^2 \leq C, \tag{4.7.7}$$

$$\|v_t\|_{L^2((0,T);(H^1(\Omega))')} \leq C, \tag{4.7.8}$$

$$\|s\|_\infty \leq \begin{cases} C & \text{if } \theta_S > 0\\ C(1+\sqrt{t}) & \text{if } \theta_S = 0.\end{cases} \tag{4.7.9}$$

Proof. Let v be a solution of $(P_{\varepsilon,\delta,\sigma})$ in the sense of Definition 4.1. Testing (4.3.3) with $-\frac{v_{yy}}{s}$ and arguing as in the proof of Lemma 4.2 we obtain

$$\int_\Omega \frac{v_y^2}{2s}\Big|_0^t = -\int_0^t \dot{s}\frac{v_y^2(t,1)}{2s^2} - \iint_{Q_t} \frac{1}{s^5} m_{\delta,\sigma}(v) v_{yyy}^2. \tag{4.7.10}$$

Note that

$$\int_\Omega \frac{s\theta_S^2}{2}\Big|_0^t = \int_0^t \int_\Omega \dot{s}\frac{\theta_S^2}{2}. \tag{4.7.11}$$

Hence, recalling (4.1.10),

$$E(v)\Big|_0^t = \frac{1}{2}\int_\Omega (\frac{v_y^2}{s} + s\theta_S^2)\,\mathrm{d}y\Big|_0^t = -\int_0^t \frac{\dot{s}}{2}\left(\frac{v_y^2|_{y=1}}{2s^2} - \theta_S^2\right) - \iint_{Q_t} \frac{1}{s^5} m_{\delta,\sigma}(v) v_{yyy}^2 \tag{4.7.12}$$

and since v satisfies the contact-angle condition, we conclude that

$$E(v)\Big|_0^t = -\frac{1}{2d}\int_0^t \dot{s}^2 - \iint_{Q_t} \frac{1}{s^5} m_{\delta,\sigma}(v) v_{yyy}^2 \tag{4.7.13}$$

as long as v is defined, i.e. for $t < T$. As long as it is defined (in particular, $s(t) > 0$), we also have

$$\begin{aligned} v(y,t) &= \varepsilon + \int_1^y v_y \le \varepsilon + \sqrt{s(t)}\left(\int_0^1 \frac{v_y^2}{s(t)}\right)^{1/2} \\ &\overset{(4.7.13)}{\le} \varepsilon + C\sqrt{s}, \end{aligned} \tag{4.7.14}$$

where C depends only on $||v_0||_{H^1}$ and s_0. On the other hand, it follows from (4.7.2) that

$$s(t)\int_\Omega v = s_0 \int_\Omega v_{\varepsilon 0} + \varepsilon(s(t) - s_0) \ge s_0 \int_\Omega (v_{\varepsilon 0} - \varepsilon) > 0 \tag{4.7.15}$$

provided $\int_\Omega v_{\varepsilon 0} > \varepsilon$. Combining (4.7.14) and (4.7.15),

$$s(t)(\varepsilon + C\sqrt{s(t)}) \ge C^{-1}$$

which implies that

$$s(t) \ge C^{-1}. \tag{4.7.16}$$

Using (4.7.16) into (4.7.13) and arguing as in the proof of Proposition 4.1, we obtain (4.7.3)–(4.7.9). ◻

We now show that inequalities (4.7.3)-(4.7.9) yield a uniform control on a suitable Hölder norm of v in Q_T.

Lemma 4.4. *Let $\varepsilon > 0$, $\delta > 0$, $\sigma > 0$, $s_0 > 0$, and $v_0 \in H^1_0(\Omega)$ such that $\int_\Omega v_0 > 0$. Then a positive constant C, depending only on $\|v_{0\varepsilon}\|_{H^1}$ and s_0, exists such that any solution (s, v) of $(P_{\varepsilon,\delta,\sigma})$ in the sense of Definition 4.1 satisfies:*

$$\begin{aligned} |v(t,y_1) - v(t,y_2)| &\leq C|y_1 - y_2|^{1/2} \quad \textit{for all } y_1, y_2 \in \Omega, t \in [0,T] \qquad (4.7.17)\\ |v(t_1,y) - v(t_2,y)| &\leq C|t_1 - t_2|^{1/8} \quad \textit{for all } y \in \Omega, t_1, t_2 \in (0,T) \qquad (4.7.18)\\ |v(t,y)| &\leq C \quad \textit{in } Q_T. \qquad (4.7.19) \end{aligned}$$

Proof. From (4.7.4) it follows that

$$\begin{aligned} |v(t,y_1) - v(t,y_2)| &\leq \int_{y_1}^{y_2} |v_y(t,\xi)| \,\mathrm{d}\xi \leq \left(\int_{y_1}^{y_2} |v_y|^2\right)^{1/2} |y_1 - y_2|^{1/2} \\ &\leq \left(\sup_t \int_\Omega |v_y|^2\right)^{1/2} |y_1 - y_2|^{1/2} \overset{(4.7.4)}{\leq} C|y_1 - y_2|^{1/2}. \qquad (4.7.20) \end{aligned}$$

Therefore (4.7.17) is achieved. (4.7.19) follows immediately from Poincare inequality, $v(t,1) = \varepsilon$, (4.7.4) and the embedding $H^1(\Omega) \subset L^\infty(\Omega)$. For the Hölder continuity in time we consider a non-negative cut-off function $\varphi \in C_c^\infty(\mathbb{R})$ such that

$$\mathrm{supp}(\varphi) \subset (-2,2) \quad \text{and} \quad \int_{\mathbb{R}} \varphi(s)\,\mathrm{d}s = 1,$$

and we set $\varphi_\delta(y) = \delta^{-1}\varphi(\delta^{-1}y)$, for some $\delta > 0$ to be chosen later. We have

$$\begin{aligned} |v(t_2,\overline{y}) - v(t_1,\overline{y})| &\leq \int_\Omega \varphi_\delta(y - \overline{y})|v(t_2,\overline{y}) - v(t_2,y)| \,\mathrm{d}y \\ &+ \left|\int_\Omega \varphi_\delta(y - \overline{y})(v(t_2,y) - v(t_1,y))\right| \mathrm{d}y \\ &+ \int_\Omega \varphi_\delta(y - \overline{y})|v(t_1,y) - y(t_1,\overline{y})| \,\mathrm{d}y =: I_1 + I_2 + I_3. \qquad (4.7.21) \end{aligned}$$

For the first and the third terms we have

$$\begin{aligned} I_1 + I_3 &\overset{(4.7.17)}{\leq} C \int_\Omega \varphi_\delta(y - \overline{y})|\overline{y} - y|^{1/2} \,\mathrm{d}y \\ &= C \int_\Omega \varphi\left(\frac{y - \overline{y}}{\delta}\right)|y - \overline{y}|^{\frac{1}{2}} \,\mathrm{d}\left(\frac{y - \overline{y}}{\delta}\right) \\ &= C \int_\Omega \varphi(z)(\delta z)^{\frac{1}{2}} \,\mathrm{d}z \\ &\leq C\delta^{1/2}. \qquad (4.7.22) \end{aligned}$$

For the second term we have

$$\begin{aligned}
I_2 &= \left|\int_\Omega \varphi_\delta(y-\overline{y})(v(t_2,y)-v(t_1,y))\right| \\
&= \left|\int_\Omega \varphi_\delta(y-\overline{y})\left(\int_{t_1}^{t_2} v_t(\tau,y)\,\mathrm{d}\tau\right)\right| = \left|\int_{t_1}^{t_2}\int_\Omega \varphi_\delta(y-\overline{y})v_t(\tau,y)\,\mathrm{d}x\mathrm{d}\tau\right| \\
&\overset{(4.3.3)}{\leq} \left|\int_{t_1}^{t_2}\int_\Omega \varphi_\delta(y-\overline{y})\frac{\dot{s}}{s}yv_y\right| + \left|\int_{t_1}^{t_2}\int_\Omega \varphi_{\delta y}(y-\overline{y})\frac{1}{s^4}m_{\delta,\sigma}(v)v_{yyy}\right| \\
&=: I_2' + I_2''. \qquad (4.7.23)
\end{aligned}$$

We note that

$$\begin{aligned}
I_2' &= \left|\int_{t_1}^{t_2}\int_\Omega \varphi_\delta(y-\overline{y})\frac{\dot{s}}{s}yv_y\right| = \left|\int_{t_1}^{t_2}\int_\Omega \frac{\dot{s}}{s}\frac{1}{\delta}\varphi\left(\frac{y-\overline{y}}{\delta}\right)yv_y\right| \leq \\
&\overset{(4.7.3)}{\leq} C\delta^{-1}\left(\int_0^t \dot{s}^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int_\Omega v_y^2\right)^{1/2} \\
&\overset{(4.7.4)}{\leq} C\delta^{-1}|t_1-t_2|^{1/2} \qquad (4.7.24)
\end{aligned}$$

and

$$\begin{aligned}
I_2'' &= \left|\int_{t_1}^{t_2}\int_\Omega \frac{1}{s^4}\varphi_{\delta y}(y-\overline{y})m_{\delta,\sigma}(v)v_{yyy}\right| \\
&\overset{(4.7.3)}{\leq} C\left(\int_{t_1}^{t_2}\int_\Omega m_{\delta,\sigma}(v)v_{yyy}^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int_\Omega m_{\delta,\sigma}(v)(\varphi_{\delta y}(y-\overline{y}))^2\right)^{1/2} \\
&\overset{(4.7.5),(4.7.19)}{\leq} C\left(\int_{t_1}^{t_2}\int_\Omega \delta^{-4}\varphi_y^2\right)^{1/2} \leq C\delta^{-2}\left(\int_{t_1}^{t_2}\int_\Omega \varphi_y^2\right)^{1/2} \\
&\leq C\delta^{-2}|t_1-t_2|^{1/2}|\mathrm{supp}\ \varphi_\delta|^{1/2} = \mathrm{C}\delta^{-3/2}|\mathrm{t}_1-\mathrm{t}_2|^{1/2} \qquad (4.7.25)
\end{aligned}$$

which imply

$$I_2 \leq C\delta^{-1}|t_1-t_2|^{1/2} + C\delta^{-3/2}|t_1-t_2|^{1/2}. \qquad (4.7.26)$$

Combining (4.7.23) and (4.7.26) in (4.7.21), we obtain

$$\begin{aligned}
|v(t_2,\overline{y})-v(t_1,\overline{y})| &\leq C(\delta^{-3/2}|t_1-t_2|^{1/2} + \delta^{-1}|t_1-t_2|^{1/2} + \delta^{1/2}) \\
&\leq C(\delta^{-3/2}|t_1-t_2|^{1/2} + \delta^{1/2}). \qquad (4.7.27)
\end{aligned}$$

Minimizing the right-hand side of (4.7.27) with respect to δ yields (4.7.18) . Indeed, setting $\Delta t = t_1 - t_2$ we consider the function

$$f(\delta) = \delta^{-3/2}(\Delta t)^{1/2} + \delta^{1/2}.$$

Deriving with respect to δ we obtain

$$f'(\delta) = -\frac{3}{2}\delta^{-5/2}(\Delta t)^{1/2} + \frac{1}{2}\delta^{-1/2}$$

so that the minimizer is

$$\delta_{\min} \sim (\Delta t)^{1/4}.$$

Therefore

$$f(\delta_{\min}) = f((\Delta t)^{1/4}) \sim (\Delta t)^{1/8}$$

and (4.7.18) follows. □

We are now ready to prove the following.

Proposition 4.4. *Let* $\varepsilon > 0$, $\delta > 0$, $\sigma > 0$, $s_0 > 0$, *and* $v_0 \in H^1(\Omega)$ *such that* $\int_\Omega v_0 > 0$. *Then there exists a pair* (s, v) *which solves* $(P_{\varepsilon,\delta,\sigma})$ *in* $(0, T)$ *for all* $T < \infty$ *in the sense of Definition 4.1. Furthermore, estimates* (4.7.3)–(4.7.9) *and* (4.7.17)-(4.7.19) *hold true.*

Proof. By Proposition 4.3, there exists a pair (s, v) which solves $(P_{\varepsilon,\delta,\sigma})$ in the sense of Definition 4.1 up to a certain $T > 0$, which we assume w.l.o.g. to be maximal. If by contradiction $T < \infty$, by (4.7.6), (4.7.9), (4.7.4) and (4.7.19) we may find a subsequence $t_n \to T$ such that $s(t_n) \to s(t)$ and $v(t_n, y) \rightharpoonup v_T(y)$ in $H^1_\varepsilon(\Omega)$. We may therefore apply Proposition 4.3 with initial datum $v_T(y)$ and $s_T(0) = s(T)$, obtaining a solution $(s_T(t), v_T)$ in $Q_{T'}$ for some $T' > 0$. But then

$$\tilde{s}(t) = \begin{cases} s(t) & t < T \\ s_T(t-T) & t \in (T, T+T') \end{cases} \qquad \tilde{v}(t,y) = \begin{cases} v(t,y) & t < T \\ v_T(t-T,y) & t \in (T, T+T') \end{cases}$$

would solve $(P_{\varepsilon,\delta,\sigma})$ in $(0, T + T')$, in cotradiction with the maximality of T. □

4.8 The limit $\delta \to 0$: Entropy estimates and positive solutions of approximating problems

The aim of this section is to pass to the limit as $\delta \to 0$, obtaining *positive* solutions of the approximating problems $(P_{\varepsilon,\sigma}) = (P_{0,\varepsilon,\sigma})$. Crucial to this aim is the following entropy-type estimate:

Lemma 4.5. *Let* $\delta, \varepsilon, \sigma > 0$, $v_0 \in H^1(\Omega)$ *non-negative with* $\int_\Omega v_0 > 0$, $s_0 > 0$, *and let* v *be a global solution of Problem* $(P_{\varepsilon,\delta,\sigma})$ *as given by Proposition 4.4. Then positive constants* $C \geq 1$ *and* $C(\varepsilon, T) \geq 1$ *exist such that*

$$\sup_{t \leq T} \int_\Omega G_{\sigma,\delta}(v(t)) + C^{-1} \iint_{Q_T} v_{yy}^2 \leq C(\varepsilon, T) \quad \text{for all } T < \infty. \tag{4.8.1}$$

Proof. We introduce the functions

$$G_{\sigma,\delta}(\tau) = \int_\tau^A \int_{\tau'}^A \frac{1}{m_{\sigma,\delta}(\tau'')}\, \mathrm{d}\tau'' \mathrm{d}\tau', \tag{4.8.2}$$

where $A > \|v_{\sigma,\delta}\|$ (A is uniform in view of (4.7.19)), so that

$$G''_{\sigma,\delta} = \frac{1}{m_{\sigma,\delta}}. \tag{4.8.3}$$

Using $sG'_{\sigma,\delta}(v)$ as test function in (4.3.3) we obtain

$$\begin{aligned}
\left[s\int_0^1 G_{\sigma,\delta}(v)\right]_0^t &= \int_0^t < v_t, sG'_{\sigma,\delta}(v) > + \iint_{Q_t} \dot{s}G_{\sigma,\delta}(v) \\
&= \iint_{Q_t} \dot{s}yv_yG'_{\sigma,\delta}(v) + \iint_{Q_t} \frac{1}{s^3}G''_{\sigma,\delta}(v)v_y m_{\sigma,\delta}(v)v_{yyy} + \iint_{Q_t} \dot{s}G_{\sigma,\delta}(v) \\
&\overset{(4.8.3)}{=} \int_0^t \dot{s}\left[yG_{\sigma,\delta}(v)\right]_0^1 - \iint_{Q_t} \dot{s}G_{\sigma,\delta}(v) + \iint_{Q_t} \frac{1}{s^3}v_y v_{yyy} + \iint_{Q_t} \dot{s}G_{\sigma,\delta}(v) \\
&= \int_0^t \dot{s}G_{\sigma,\delta}(\varepsilon) + \int_0^t \frac{1}{s^3}\left[v_y v_{yy}\right]_0^1 - \iint_{Q_t} \frac{1}{s^3}v_{yy}^2.
\end{aligned} \tag{4.8.4}$$

Since $m_{\sigma,\delta}(\varepsilon) \geq \varepsilon^n$ we have

$$G_{\sigma,\delta}(\varepsilon) = \int_\varepsilon^A \int_\tau^A \frac{\mathrm{d}r}{m_{\sigma,\delta}(r)} \leq \frac{(A-\varepsilon)^2}{m_{\sigma,\delta}(\varepsilon)} \leq \frac{A^2}{\varepsilon^n}. \tag{4.8.5}$$

Therefore, recalling (4.7.6) and (4.7.3), we obtain

$$\int_\Omega G_{\sigma,\delta}(v)\Big|_0^t + C^{-1}\iint_{Q_t} v_{yy}^2 \leq C(\varepsilon, t) + C\int_0^t \left[|v_y v_{yy}|\right]_0^1, \tag{4.8.6}$$

where in this proof C denotes a generic universal constant. In order to estimate the other boundary term, we recall the boundary condition $\dot{s}(t) = d\left(\frac{v_y^2(t,1)}{s^2} - \theta_S^2\right)$. Hence we have

$$|v_y(t,1)| = \sqrt{\left(\frac{\dot{s}(t)}{d} + \theta_S^2\right)s^2(t)} \quad \text{a.e. in } L^2(0,T).$$

Therefore (we drop time-dependence for notational convenience):

$$\begin{aligned}
\int_0^t |v_y(1)v_{yy}(1)| &\leq \int_0^t \frac{\sqrt{(\frac{\dot{s}}{d} + \theta_S^2)s^2}}{v^2(1)} v^2(1)|v_{yy}(1)| \\
&\leq \int_0^t \frac{\sqrt{(\frac{\dot{s}}{d} + \theta_S^2)s^2}}{\varepsilon^2} \|v^2 v_{yy}\|_{L^\infty(\Omega)} \\
&\overset{(4.7.9)}{\lesssim} \frac{1}{\varepsilon^2}\left(\int_0^t (1+\dot{s}^2)\right)^{1/4}\left(\int_0^t \left\|v^2 v_{yy}\right\|_{L^\infty(\Omega)}^{4/3}\right)^{3/4} \\
&\overset{(4.7.6)}{\lesssim} \frac{1}{\varepsilon^2}\left(\int_0^t \left\|v^2 v_{yy}\right\|_{L^\infty(\Omega)}^{4/3}\right)^{3/4}.
\end{aligned}$$

We observe that

$$
\begin{aligned}
|v^2 v_{yy}|_{L^\infty(\Omega)} &\le \left|\fint_\Omega v^2 v_{yy}\right| + \int_\Omega \left|(v^2 v_{yy})_y\right| \\
&= \left|\fint_\Omega v^2 v_{yy}\right| + \int_\Omega 2v_y v v_{yy} + \int_\Omega v^2 v_{yyy} \\
&\le \left(\int_\Omega v^4\right)^{1/2}\left(\int_\Omega v_{yy}^2\right)^{1/2} + \left(\int_\Omega v^2 v_y^2\right)^{1/2}\left(\int_\Omega v_{yy}^2\right)^{1/2} + \left(\int_\Omega v^4 v_{yyy}^2\right)^{1/2} \quad (4.8.7)
\end{aligned}
$$

Therefore, recalling the uniform bounds in Proposition 4.4,

$$
\int_0^t |v_y(1)v_{yy}(1)| \lesssim \frac{1}{\varepsilon^2}\left(\int_0^t \left(\int_\Omega v_{yy}^2\right)^{2/3} + \left(\int_\Omega v_{yyy}^2 v^4 \,\mathrm{d}t\right)^{2/3}\right)^{3/4}. \tag{4.8.8}
$$

We recall once again (see (4.7.19)) that the solutions satisfy $\|v\|_\infty \le C$. Since

$$
m_{\sigma,\delta}(v) \le C^{-1} v^4 \quad \text{for all } |v| \le C,
$$

in fact we have

$$
\begin{aligned}
\int_0^t |v_y(1)v_{yy}(1)| &\lesssim \frac{1}{\varepsilon^2}\left(\int_0^t \left(\int_\Omega v_{yy}^2\right)^{2/3} + \left(\int_\Omega m_{\sigma,\delta}(v) v_{yyy}^2\right)^{2/3}\right)^{3/4} \\
&\le \frac{1}{\varepsilon^2}\left(t^{1/3}\left(\iint_{Q_t} v_{yy}^2\right)^{2/3} + t^{1/3}\left(\iint_{Q_t} m_{\sigma,\delta}(v) v_{yyy}^2\right)^{2/3}\right)^{3/4} \\
&\le \frac{t^{1/4}}{\varepsilon^2}\left(\iint_{Q_t} v_{yy}^2\right)^{1/2} + \frac{t^{1/4}}{\varepsilon^2}\left(\iint_{Q_t} m_{\sigma,\delta}(v) v_{yyy}^2\right)^{1/2}. \qquad (4.8.9)
\end{aligned}
$$

Using Young's inequality and the uniform bounds of Proposition 4.4 we conclude that

$$
\int_0^t |v_y(1)v_{yy}(1)| \le C(\varepsilon, T) + C^{-1}\iint_{Q_t} v_{yy}^2. \tag{4.8.10}
$$

Plugging (4.8.10) into (4.8.6) we conclude that

$$
\int_\Omega G_{\sigma,\delta}(v(t)) + C^{-1}\iint_{Q_t} v_{yy}^2 \le C(\varepsilon, T) + \int_\Omega G_{\sigma,\delta}(v_{0\varepsilon})
$$

and since $\varepsilon \le v_{0\varepsilon} \le C$, the proof is complete. □

We are now ready to pass to the limit as $\delta \to 0$. Namely, we will prove the following:

Proposition 4.5. *Let $\sigma, \varepsilon > 0$. For any non-negative $v_0 \in H^1(\Omega)$ with $\int_\Omega v_0 > 0$, a pair of functions (s, v) exists which solves Problem $(P_{\varepsilon,0,\sigma})$ in $(0, T)$, for all $T > 0$, in the sense of Definition 4.1. Furthermore*

$$
v > 0 \ \text{ in } \ \overline{Q}_T
$$

and v satisfies the estimates in Proposition 4.4.

Proof. Let v_δ be a global solution of $(P_{\varepsilon,\delta,\sigma})$ as given by Proposition 4.4, and let $T > 0$. In view of (4.7.17)-(4.7.19), the Ascolì-Arzelà theorem allows to select a subsequence (still indexed by δ) such that

$$v_\delta \longrightarrow v \ \text{ in } \ C^{\frac{1}{2},\frac{1}{8}}([0,T) \times \overline{\Omega}) \quad \text{as } \ \delta \to 0. \tag{4.8.11}$$

The right-hand side of (4.8.1) is uniformly bounded with respect to δ. Therefore $v_{\delta yy} \rightharpoonup v_{yy}$ in $L^2(Q_T)$. Passing to the limit in (4.8.1) and using lower semi-continuity we see that

$$\sup_{t \le T} \int_\Omega G_\sigma(v(t)) + \iint_{Q_T} v_{yy}^2 < \infty. \tag{4.8.12}$$

Since $G_\sigma(v) \sim v^{-2}$ as $v \to 0$, the Hölder continuity of v implies that $v > 0$ in $\overline{Q_T}$ for all $T > 0$. Because of this bound, the problem becomes essentially a uniformly parabolic one, and it is therefore straightforward to pass to the limit as $\delta \to 0$ and complete the proof, as done in the proof of Proposition 4.1. We only note that (v) holds: since v_t and v are bounded in $L^2((0,T);(H^1(\Omega))')$, resp. $L^2((0,T);H^3(\Omega))$, uniformly with respect to δ, Simon's compactness criterion (see Theorem 4.3 in Section 4.4) implies that $v_\delta \to v$ strongly in $L^2((0,T);H^2(\Omega))$, hence $v_{\delta y}|_{y=1} \to v_y|_{y=1}$ in $L^2(0,T)$ by the continuous embedding $H^1(\Omega) \subset L^2(\partial\Omega)$. □

Remark 4.1. *We observe that v_{yy} may be used as test function in* (4.3.3). *Therefore, arguing as in the proof of* (4.7.13), *the energy estimate continues to hold* as an equality*:*

$$\frac{1}{2}\int_\Omega \frac{v_{\sigma y}^2}{s}(t)\,\mathrm{d}y + \frac{1}{2d}\int_0^t \dot{s}^2 + \iint_{Q_T} \frac{1}{s^5} m_\sigma(v_\sigma) v_{\sigma yyy}^2 \,\mathrm{d}y\,\mathrm{d}t = \frac{1}{2}\int_\Omega \frac{v_{0\sigma y}^2}{s_0}\,\mathrm{d}y. \tag{4.8.13}$$

4.9 The limit $\sigma \to 0$: Proof of the main result

We let $\varepsilon = \sigma$. The aim of this section is to let $\sigma \to 0$ in $(P_{\sigma,\sigma})$ and thus prove Theorem 4.1.

Proof of Theorem 4.1. Let $\tilde{v}_\sigma$ be a global solution of $(P_{\sigma,\sigma})$ with initial datum $v_{0\sigma}$, as given in Proposition 4.5, let $I = (-1,1)$, $Q_t = (0,t) \times I$, and let

$$v_\sigma(t,y) = \begin{cases} \tilde{v}_\sigma(t,y) & \text{if } \ y \in [0,1] \\ \tilde{v}_\sigma(t,-y) & \text{if } \ y \in [-1,0). \end{cases}$$

Note that we have $v_\sigma \in L^2_{loc}([0,\infty);H^3(I))$ since $(\tilde{v}_\sigma)_y|_{y=0} = 0$. In the course of the proof C will denote a generic positive constant independent of σ. In view of (4.7.17)-(4.7.19), the Ascolì-Arzelà theorem allows to select a subsequence (still indexed by σ) such that

$$v_\sigma \longrightarrow v \ \text{ in } \ C^{\frac{1}{2},\frac{1}{8}}([0,T] \times \overline{I}) \quad \text{for all } T > 0 \text{ as } \ \sigma \to 0. \tag{4.9.1}$$

In particular, we also have that

$$m_\sigma(v_\sigma) \longrightarrow m(v) \text{ uniformly in } [0,T]\times\overline{I} \quad \text{for all } T>0 \text{ as } \sigma\to 0. \tag{4.9.2}$$

Bounds (4.7.4), (4.7.3), (4.7.6), and (4.7.8) imply, respectively, that (for a subsequence)

$$v_\sigma \overset{*}{\rightharpoonup} v \text{ in } L^\infty((0,T);H^1(I)) \quad \text{for all } T>0 \text{ as } \sigma\to 0, \tag{4.9.3}$$

$$s_\sigma \rightharpoonup s \text{ in } H^1((0,T)) \quad \text{for all } T>0 \text{ as } \sigma\to 0, \tag{4.9.4}$$

$$s_\sigma \to s>0 \text{ uniformly in } (0,T) \quad \text{for all } T>0 \text{ as } \sigma\to 0, \tag{4.9.5}$$

and

$$v_{\sigma t} \rightharpoonup v \quad \text{in } L^2((0,T);(H^1(I))') \quad \text{as } \sigma\to 0. \tag{4.9.6}$$

We recall (see (4.7.5)) that

$$\iint_{Q_T} m_\sigma(v_\sigma)v^2_{\sigma yyy} \le C \tag{4.9.7}$$

for all $T>0$. We want now to prove that the weak formulation (4.1.12) holds, passing to the limit as $\sigma\to 0$ in

$$\int_0^T <v_{\sigma t},\varphi> \,\mathrm{d}t = \iint_{Q_T} \frac{\dot{s}_\sigma}{s_\sigma} y v_{\sigma y}\varphi + \iint_{Q_T} \frac{1}{s_\sigma^4} m_\sigma(v_\sigma)v_{\sigma yyy}\varphi_y \tag{4.9.8}$$

for all $\varphi\in L^2((0,T),H^1(I))$. It follows from (4.9.6) that

$$\int_0^T <v_{\sigma t},\varphi> \,\mathrm{d}t \longrightarrow \int_0^T <v_t,\varphi> \,\mathrm{d}t \quad \text{as } \sigma\to 0. \tag{4.9.9}$$

From (4.9.3), (4.9.4), and (4.9.5) we easily see that

$$\iint_{Q_T} \frac{\dot{s}_\sigma}{s_\sigma} y v_{\sigma y}\varphi \longrightarrow \iint_{Q_T} \frac{\dot{s}}{s} y v_y\varphi \quad \text{as } \sigma\to 0. \tag{4.9.10}$$

Finally, we show that

$$J_\sigma = \iint_{Q_T} \frac{1}{s_\sigma^4} m_\sigma(v_\sigma)v_{\sigma yyy}\varphi_y \longrightarrow \iint_{\{v>0\}} \frac{1}{s^4} m(v)v_{yyy}\varphi_y \quad \text{as } \sigma\to 0. \tag{4.9.11}$$

For (4.9.11), we use the argument in [13], which is nowadays standard for thin-film equations. Given a compact set $K \Subset \{v>0\}$, by (4.9.1) we have $\min_K v > 0$. By the uniform convergence (4.9.1) we in fact have

$$v_\sigma \ge \frac{1}{2}\min_K v \quad \text{in } K$$

for $\sigma<\sigma(K)$. Since m_σ is increasing, it follows from (4.9.7) that

$$\iint_K v^2_{\sigma yyy} \le C(K) \quad \text{for } \sigma<\sigma(K). \tag{4.9.12}$$

Hence a subsequence $\sigma_n \to 0$ (depending on K) exists such that

$$v_{\sigma_n yyy} \rightharpoonup f \quad \text{in } L^2(K).$$

Given any sequence v_{σ_n} with this property, for all $\varphi \in C_c^\infty(K)$ we have

$$\iint_K \varphi (v_{\sigma_n})_{yyy} = - \iint_K \varphi_{yyy} v_{\sigma_n}$$

and passing to the limit as $n \to \infty$ we can identify $f = v_{yyy}$ in $L^2(K)$. Therefore the whole sequence converges to v_{yyy} in K, and the arbitrariness of K implies that

$$v_{\sigma yyy} \rightharpoonup v_{yyy} \quad \text{in } L^2_{loc}(\{v > 0\}) \quad \text{as } \sigma \to 0. \tag{4.9.13}$$

For a fixed $\eta > 0$, we split J_σ as follows:

$$J_\sigma = \iint_{\{v \geq \eta\}} \frac{1}{s_\sigma^4} m_\sigma(v_\sigma) v_{\sigma yyy} \varphi_y + \iint_{\{v<\eta\}} \frac{1}{s_\sigma^4} m_\sigma(v_\sigma) v_{\sigma yyy} \varphi_y = J'_\sigma + J''_\sigma. \tag{4.9.14}$$

From (4.9.13), (4.9.5), and (4.9.2) we obtain

$$J'_\sigma = \iint_{\{v \geq \eta\}} \frac{1}{s_\sigma^4} m_\sigma(v_\sigma) v_{\sigma yyy} \varphi_y \overset{\sigma \to 0}{\longrightarrow} \iint_{\{v \geq \eta\}} \frac{1}{s^4} m(v) v_{yyy} \varphi_y. \tag{4.9.15}$$

By Hölder inequality, and since $v_\sigma < 2\eta$ in $\{v < \eta\}$ for $\sigma < \sigma(\eta)$ sufficiently small, we have

$$\begin{aligned} \left| J''_\sigma \right| = \left| \iint_{\{v<\eta\}} \frac{1}{s_\sigma^4} m_\sigma(v_\sigma) v_{\sigma yyy} \varphi_y \right| &\overset{(4.7.3)}{\leq} C \left(\iint_{Q_T} m_\sigma(v_\sigma) v_{\sigma yyy}^2 \right)^{1/2} \left(\iint_{\{v<\eta\}} m_\sigma(v_\sigma) \varphi_y^2 \right)^{1/2} \\ &\overset{(4.7.5)}{\leq} C \left(\sup_{v_\sigma \in (0,2\eta)} |m_\sigma(v)| \right)^{1/2} \left(\iint_{Q_T} \varphi_y^2 \right)^{1/2}. \end{aligned}$$

Therefore

$$\limsup_{\sigma \to 0} \left| J''_\sigma \right| \leq o_\eta(1) \quad \text{as } \eta \to 0.$$

Hence, passing to the limit in (4.9.14) as $\sigma \to 0$, recalling (4.9.15) and using the arbitrariness of η we conclude that

$$\iint_{Q_T} \frac{1}{s_\sigma^4} m_\sigma(v_\sigma) v_{\sigma yyy} \varphi_y \longrightarrow \iint_{\{v>0\}} \frac{1}{s^4} m(v) v_{yyy} \varphi_y \quad \text{as } \sigma \to 0. \tag{4.9.16}$$

Combining (4.9.9), (4.9.10) and (4.9.16) we pass to the limit as $\sigma \to 0$ in (4.9.8) and (4.1.12) is recovered. Finally, the energy estimate is an immediate consequence of (4.8.13) and lower semi-continuity. □

Appendix A

The Burgers tensor

A.0.1 The discrete viewpoint

Plasticity crystal arise in response to the motion of dislocations, and dislocation-induces defectiveness of a crystal may be characterized by the **Burgers vector**, a geometric quantity that measures the closure failure of circuits in the atomic lattice. Both dislocations and they accompanying Burgers vector are microscopic quantities: There are no dislocations in a continuum theory. Even so, the microscopic definition of the Burgers vector may be lifted, almost without change,to form a macroscopic kinematical concept appropriate to a continuous body undergoing plastic deformation. Consider a two-dimensional crystal lattice as displayed in the following figures:

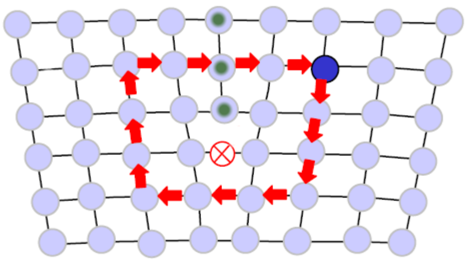

Figure 1.1: A closed path in a lattice with a dislocation at the point marked ⊗

In Fig. 1.2 it is shown the deformed lattice with a dislocation at the point marked with the symbol ⊗, while Fig. 1.1 shows the undeformed defect-free crystal lattice. Consider a clockwise closed circuit, the *Burgers circuit*, with starting and ending lattice point the purple one, that lies in the deformed lattice and surrounds the dislocation. Then, because of the presence of the dislocation, the same circuit in the undeformed defect-free lattice starts

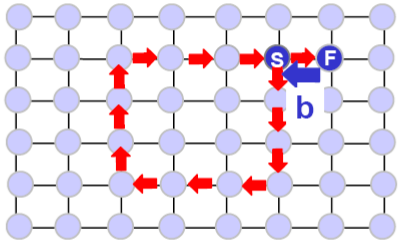

Figure 1.2: A undeformed defect-free crystal lattice

at point S and ends at F, and is therefore not closed. The vector $\mathbf{b}$ closing the circuit in Fig. 1.2 and directed from the end point F to the starting point S is called the *Burgers vector*.

A.0.2 The continuum viewpoint

In formulating the constitutive equation for the free energy ψ, we not only consider the standard dependence on the elastic strain $\mathbf{E}^{\mathrm{e}}$, but we also consider a dependence of ψ on $\nabla\mathbf{E}^{\mathrm{p}}$ via dependence on the Burger tensor

$$\mathbf{G} := \mathrm{curl}\mathbf{E}^{\mathrm{p}}$$

which is a measure of the *macroscopic Burger vector*.

Assume that Γ is the boundary curve on a smooth oriented surface S in the body, with unit norma $\mathbf{e}$ for S. Because by $\mathbf{H}^{\mathrm{p}}$ represents the distortion of the lattice due to the formation of dislocations, the corresponding integration around Γ in the distorted lattice is represented through Stokes' Theorem by the integral

$$\mathbf{b}(\Gamma) = \int_{\Gamma} \mathbf{H}^{\mathrm{p}}\, \mathrm{d}\mathbf{X} = \int_{S} (\mathrm{curl}\mathbf{H}^{\mathrm{p}})^{T}\mathbf{e}\, \mathrm{d}A. \tag{A.0.1}$$

This integral is nonzero, as the plastic distortion $\mathbf{H}^{p}$ is not the gradient of a vector field, and we associate the vector measure

$$(\mathrm{curl}\mathbf{H}^{p})^{T}\mathbf{e}\, \mathrm{d}A$$

with the *Burgers vector* corresponding to the boundary curve of the surface-element $\mathbf{e}\, \mathrm{d}A$. Thus, in this sense the tensor field

$$\mathbf{G} = \mathrm{curl}\mathbf{H}^{\mathrm{p}} \tag{A.0.2}$$

which we refer to as the **Burgers tensor**, provides a local characterization of the Burgers vector. Specifically, $\mathbf{G}^{T}\mathbf{e}$ provides a measure of the (local) Burgers vector for the plain Π

with unit normal $\mathbf{e}$, and may be viewed as the local Burgers vector, per unit area, for those dislocations lines that pierce Π. Since $\mathrm{curl}\nabla\mathbf{u} = 0$,

$$\mathbf{G} = -\mathrm{curl}\mathbf{H}^{\mathrm{e}}$$

a relation often referred as the *fundamental equation of the continuos theory of disloctions*. The relation (A.0.2) seems most relevant to theories of plasticity involving plastic-strain gradients.

Bibliography

[1] M. Abramowitz and I. A. Stegun, editors. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. Dover, 1964.

[2] A. Acharya and J. Bassani. Lattice incompatibility and a gradient theory of crystal plasticity. *J. Mech. Phys. Solids*, 48(8):1565 – 1595, 2000.

[3] E. C. Aifantis. On the microstructural origin of certain inelastic models. *J. Eng. Mat. Tech.*, 106(4):326–330, 1984.

[4] M. Amar, M. Chiricotto, L. Giacomelli, and G. Riey. Mass-constrained minimization of a one-homogenous functional arising in strain-gradient plasticity. *Submitted*.

[5] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford Mathematical Monographs. Oxford University Press, Oxford, United Kingdom, 2000.

[6] L. Anand, M. E. Gurtin, S. P. Lele, and C. Gething. A one-dimensional theory of strain-gradient plasticity: formulation, analysis, numerical results. *J. Mech. Phys. Solids*, 53(8):1789–1826, 2005.

[7] F. Andreu, V. Caselles, and J. M. Mazon. A strongly degenerate quasilinear equation: the elliptic case. *Ann. Scuola Norm. Pisa Cl. Sci.*, (5) III:555–587, 2004.

[8] G. Anzellotti. Pairings between measures and bounded functions and compensated compactness. *Ann. Mat. Pura Appl.*, (4) 135:293–318, 1983.

[9] L. Bardella. A comparison between crystal and isotropic strain gradient plasticity theories with accent on the role of the plastic spin. *Euro. J. Mech. - A/Solids*, 28(3):638 – 646, 2009.

[10] E. Beretta, M. Bertsch, and R. Dal Passo. Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation. *Arch. Rational Mech. Anal.*, 129(2):175–200, 1995.

[11] F. Bernis. Finite speed of propagation and continuity of the interface for thin viscous flows. *Advances in Differential Equations*, 1(3):337368, 1996.

[12] F. Bernis. Finite speed of propagation for thin viscous flows when 2≥n¡3. *CR Acad Sci Paris SrI Math*, 322:1169–1174, 1996.

[13] F. Bernis and A. Friedman. Higher order nonlinear degenerate parabolic equations. *J. Differential Equations*, 83(1):179–206, 1990.

[14] F. Bernis, L. A. Peletier, and S. M. Williams. Source type solutions of a fourth order nonlinear degenerate parabolic equation. *Nonlinear Anal.*, 18(3):217–234, 1992.

[15] A. L. Bertozzi, M. P. Brenner, T. F. Dupont, and L. P. Kadanoff. Singularities and similarities in interface flows. *Trends and perspectives in applied mathematics*, 100:155–208, 1994.

[16] A. L. Bertozzi and M. Pugh. The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions. *Comm. Pure Appl. Math.*, 49(2):85–123, 1996.

[17] M. Bertsch, R. Dal Passo, S. H. Davis, and L. Giacomelli. Effective and microscopic contact angles in thin film dynamics. *European J. Appl. Math.*, 11(2):181–201, 2000.

[18] M. Bertsch, R. Dal Passo, H. Garcke, and G. Grün. The thin viscous flow equation in higher space dimensions. *Adv. Differential Equations*, 3(3):417–440, 1998.

[19] M. Bertsch, R. Dal Passo, L. Giacomelli, and G. Tomassetti. A nonlocal and fully nonlinear degenerate parabolic system from strain-gradient plasticity. *Discrete Contin. Dyn. Syst. Ser. B*, 15(1):15–43, 2011.

[20] M. Bertsch, L. Giacomelli, and G. Karali. Thin-film equations with “partial wetting” energy: existence of weak solutions. *Phys. D*, 209(1-4):17–27, 2005.

[21] S. Boatto, L. P. Kadanoff, and P. Olla. Traveling-wave solutions to thin-film equations. *Phys. Rev. E (3)*, 48(6):4423–4431, 1993.

[22] D. Bonn, J. Eggers, J. Indekeu, J. Meunier, and E. Rolley. Wetting and spreading. *Rev. Mod. Phys.*, 81(2):739–805, 2009.

[23] G. Borino and C. Polizzotto. Thermodynamically consistent residual-based gradient plasticity theory and comparisons. *Modelling and Simulation in Materials Science and Engineering*, 15(1):S23–S35, 2007.

[24] B. Botteron and P. Marcellini. A general approach to the existence of minimizers of one-dimensional non-coercive integrals of the calculus of variations. *Ann. I.H.P.*, 8 C (2):197–223, 1991.

[25] G. Bouchitté, I. Fonseca, and L. Mascarenhas. Relaxation of variational problems under trace constraints. *Nonlinear Anal.*, 49(2, Ser. A: Theory Methods):221–246, 2002.

[26] H. Brezis. *Analyse fonctionnelle*. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master’s Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[27] F. Brock. *Rearrangments and applications to symmetry problems in PDE*, volume VI (160) of *Handbook of Differential Equations: Stationary Partial Differential Equations*. Elsevier/North-Holland, Amsterdam, 2007.

[28] R. Buckingham, M. Shearer, and A. Bertozzi. Thin film traveling waves and the Navier slip condition. *SIAM J. Appl. Math.*, 63(2):722–744, 2002.

[29] P. Cermelli and M. Gurtin. On the characterization of geometrically necessary dislocations in finite plasticity. *J. Mech. Phys. Solids*, 49(7):1539–1568, Jul 2001.

[30] M. Chiricotto. Droplet spreading under contact-line friction: existence of weak solutions. *In preparation*.

[31] M. Chiricotto and L. Giacomelli. Droplets spreading with contact-line friction: lubrication approximation and traveling wave solutions. *Communications in Applied and Industrial Mathematics*, 2(2), 2011.

[32] M. Chiricotto and L. Giacomelli. Scaling laws for droplets spreading under contact-line friction. *Comm. Math. Sci.*, to appear.

[33] M. Chiricotto, L. Giacomelli, and G. Tomassetti. Torsion in strain-gradient plasticity: energetic scale effects. *SIAM J. Appl. Math.*, to appear.

[34] R. G. Cox. The dynamics of the spreading of liquids on a solid surface. 1. Viscous-flow. *J. Fluid Mech.*, 168:169–194, 1986.

[35] B. Dacorogna. *Direct methods in the calculus of variations*, volume 78 of *Applied Mathematical Sciences*. Berlin, 1989.

[36] G. Dal Maso. Integral representation on BV(Ω) of Γ-limits of variational integrals. *Manuscripta Math.*, (4) 30:387–416, 1980.

[37] G. Dal Maso. *An introduction to* Γ*-convergence*, volume 8 of *Progress in Nonlinear Differential Equations and their Applications*. Boston, 1993.

[38] R. Dal Passo, H. Garcke, and G. Grün. On a fourth-order degenerate parabolic equation: global entropy estimates, existence, and qualitative behavior of solutions. *SIAM J. Math. Anal.*, 29(2):321–342 (electronic), 1998.

[39] P. G. de Gennes. Wetting - Statics and dynamics. *Rev. Mod. Phys.*, 57(3):827–863, 1985.

[40] O. Dillon and J. Kratochvil. A strain gradient theory of plasticity. *Int. J. Solids Structures*, 6(12):1513 – 1533, 1970.

[41] E. B. Dussan V. and S. H. Davis. On the motion of fluid-fluid interface along a solid surface. *J. Fluid Mech.*, 65:71–95, 1974.

[42] F. Ebobisse and P. Neff. Existence and uniqueness for rate-independent infinitesimal gradient plasticity with isotropic hardening and plastic spin. *Math. Mech. Solids*, 15(6):691–703, 2010.

[43] P. Ehrhard and S. H. Davis. Nonisothermal spreading of liquid drops on horizontal plates. *J. Fluid Mech.*, 229:365–388, 1991.

[44] C. M. Elliott and H. Garcke. On the Cahn-Hilliard equation with degenerate mobility. *SIAM J. Math. Anal.*, 27(2):404–423, 1996.

[45] R. Ferreira and F. Bernis. Source-type solutions to thin-film equations in higher dimensions. *European J. Appl. Math.*, 8(5):507–524, 1997.

[46] N. Fleck and J. Hutchinson. Strain gradient plasticity. volume 33 of *Advances in Applied Mechanics*, pages 295 – 361. Elsevier, 1997.

[47] N. Fleck, G. Muller, M. Ashby, and J. Hutchinson. Strain gradient plasticity: Theory and experiment. *Acta Metallurgica et Materialia*, 42(2):475–487, 1994.

[48] E. Gagliardo. Ulteriori proprietà di alcune classi di funzioni in più variabili. *Ricerche Mat.*, 8:24–51, 1959.

[49] A. Garroni, G. Leoni, and M. Ponsiglione. Gradient theory for plasticity via homogenization of discrete dislocations. *J. Eur. Math. Soc. (JEMS)*, 12(5):1231–1266, 2010.

[50] L. Giacomelli and H. Knüpfer. A free boundary problem of fourth order: Classical solutions in weighted hlder spaces. *Communications in Partial Differential Equations*, 35(11):2059–2091, 2010.

[51] L. Giacomelli, H. Knüpfer, and F. Otto. Smooth zero-contact-angle solutions to a thin-film equation around the steady state. *J. Differential Equations*, 245(6):1454–1506, 2008.

[52] L. Giacomelli and F. Otto. Droplet spreading: intermediate scaling law by PDE methods. *Comm. Pure Appl. Math.*, 55(2):217–254, 2002.

[53] L. Giacomelli and F. Otto. Rigorous lubrication approximation. *Interfaces Free Bound.*, 5(4):483–529, 2003.

[54] A. Giacomini and L. Lussardi. Quasi-static evolution for a model in strain gradient plasticity. *SIAM J. Math. Anal.*, 40(3):1201–1245, 2008.

[55] K. B. Glasner. Variational models for moving contact lines and the quasi-static approximation. *European J. Appl. Math.*, 16(6):713–740, 2005.

[56] H. P. Greenspan. On the motion of a small viscous droplet that wets a surface. *J. Fluid Mech.*, 84:125–143, 1978.

[57] G. Grün. Degenerate parabolic differential equations of fourth order and a plasticity model with non-local hardening. *Z. Anal. Anwendungen*, 14(3):541–574, 1995.

[58] P. Gudmundson. A unified treatment of strain gradient plasticity. *J. Mech. Phys. Solids*, 52(6):1379 – 1406, 2004.

[59] M. E. Gurtin. The linear theory of elasticity. In S. Flügge, editor, *Handbuch der Physik*, volume VIa/2. Springer Verlag, 1972.

[60] M. E. Gurtin. A gradient theory of small-deformation isotropic plasticity that accounts for the burgers vector and for dissipation due to plastic spin. *J. Mech. Phys. Solids*, 52(11):2545 – 2568, 2004.

[61] M. E. Gurtin and L. Anand. A theory of strain-gradient plasticity for isotropic, plastically irrotational materials. Part I: Small deformations. *J. Mech. Phys. Solids*, 53(7):1624–1649, 2005.

[62] M. E. Gurtin, E. Fried, and L. Anand. *The Mechanics and Thermodynamics of Continua*. Cambridge University Press, 2010.

[63] P. J. Haley and M. J. Miksis. The effect of the contact line on droplet spreading. *J. Fluid Mech.*, 223:57–81, 1991.

[64] W. Han and B. D. Reddy. *Plasticity. Mathematical theory and numerical analysis*. Springer-Verlag, New York, 1999.

[65] L. M. Hocking. The spreading of a thin drop by gravity and capillarity. *Q. J. Mech. Appl. Math.*, 36:55–69, 1983.

[66] L. M. Hocking. Rival contact-angle models and the spreading of drops. *J. Fluid Mech.*, 239:671–681, 1992.

[67] C. Huh and L. E. Scriven. Hydrodynamic model of steady movement of a solid/liquid/fluid contact line. *J. Colloid Interface Sci.*, 35:85–101, 1971.

[68] B. Kawohl and F. Schuricht. Dirichlet problems for the 1-Laplace operator, including the eigenvalue problem. *Commun. Contemp. Math.*, 9(4):515–543, 2007.

[69] H. Knüpfer. Well-posedness for the navier slip thin-film equation in the case of partial wetting. *Comm. Pure Appl. Math.*, 64(9):1263–1296, 2011.

[70] H. Knüpfer and N. Masmoudi. Darcy flow on a plate with prescribed contact angle — well-posedness and lubrication approximation. *Preprint*, 2010.

[71] Q. Ma and D. R. Clarke. Size dependent hardness of silver single crystals. *Journal of Materials Research*, 10(04):853–863, 1995.

[72] A. Mielke. Evolution of rate-independent systems. In *Evolutionary equations. Vol. II*, Handb. Differ. Equ., pages 461–559. Elsevier/North-Holland, Amsterdam, 2005.

[73] L. Nirenberg. On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa (3)*, 13:115–162, 1959.

[74] L. Nirenberg. An extended interpolation inequality. *Ann. Scuola Norm. Sup. Pisa (3)*, 20:733–737, 1966.

[75] A. Oron, S. H. Davis, and S. G. Bankoff. Long-scale evolution of thin liquid films. *Rev. Mod. Phys.*, 69:931–980, 1997.

[76] F. Otto. Lubrication approximation with prescribed nonzero contact angle. *Comm. Partial Differential Equations*, 23(11-12):2077–2164, 1998.

[77] L. A. Peletier. *The porous media equation*, volume 6 of *Surveys Reference Works Math.* Pitman, Boston, Mass., 1981.

[78] T. Z. Qian, X. P. Wang, and P. Sheng. Molecular scale contact line hydrodynamics of immiscible flows. *Phys. Rev. E*, 68(1, Part 2):016306, 2003.

[79] B. D. Reddy, F. Ebobisse, and A. McBride. Well-posedness of a model of strain gradient plasticity for plastically irrotational materials. *Int. J. Plasticity*, 24:55–73, 2008.

[80] W. Ren and W. E. Boundary conditions for the moving contact line problem. *Physics of fluids*, 19:022101, 2007.

[81] W. Ren, D. Hu, and W. E. Continuum models for the contact line problem. *Physics of fluids*, 22:102103, 2010.

[82] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[83] N. Stelmashenko, M. Walls, L. Brown, and Y. Milman. Microindentations on w and mo oriented single crystals: An stm study. *Acta Metallurgica et Materialia*, 41(10):2855 – 2865, 1993.

[84] J. Stlken and A. Evans. A microbend test method for measuring the plasticity length scale. *Acta Materialia*, 46(14):5109 – 5115, 1998.

[85] L. H. Tanner. The spreading of silicone oil drops on horizontal surfaces. *Journal of Physics D: Applied Physics*, 12(9):1473, 1979.

[86] P. A. Thompson and S. M. Troian. A general boundary condition for liquid flow at solid surfaces. *Nature*, 389(6649):360–362, 1997.

[87] J. L. Vázquez. *The porous medium equation*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2007. Mathematical theory.