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XXV CICLO

Game Theory Approach to Competitive Economic Dynamics

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE
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BY

Ilaria Poggio

Program Coordinator
Prof. Dr. Maria B. Chiarolla

Thesis Advisors
Prof. Dr. Lina Mallozzi
Dr. Arsen Palestini

To my father

Contents

Introduction

Game theory is a mathematical theory that studies models of conflict and cooperation between intelligent and rational decision-makers. Specifically it deals with all real-life situations in which rational people interact each-other, that is when an individual's single strategy depends on what other individuals choose to do. In this sense it should not be suprising that economics is the field in which game theorists develop their main ideas: the narrowness of economic world resources and the conflicts between countries to get them both create all the necessary ingredients for a game situation. In literature game theory's birth coincides with the book *Theory of Games and Economic Behavior* published in 1944 by the mathematician John Von Neumann and the economist Oskar Morgenstern (see [123]).

In game theory there is a classical distinction between non-cooperative games and cooperative games. In a non-cooperative game, player's agreements either do not occur or are not binding, even if pre-play communication between players is possible. In contrast, in cooperative game theory, player's commitments are binding and enforceable. In non-cooperative game theory the focus is mainly on individual behavior while in cooperative game the emphasis is on the group of coalitions of players and on how to divide the gains among coalitions.

This thesis deals both with non-cooperative and cooperative games in order to apply the mathematical theory to competitive dynamics arising from economics, particularly quantity competition in oligopolies and pollution reduction models in IEA (International Environmental Agreements). In Chapter 3 a new model of game is defined: the Bayesian multicriteria game. In our opinion this class of game is a very useful tool to model economic situations as Cournot duopoly game in which firms produce two different goods and a firm may have different production costs according to a given probability distribution. The new idea is to think that a firm can produce two (or more) kind of products. For example firms produce two types of mineral water: without bubbles and with bubbles or we can consider the diamond market which is tipically divided in two lines of production: one covering the luxury market and the other for an industrial use. This leads to optimize the different profits at the same time. On the other side it is naturally imagine that each firm profit can be affected by uncertainty: for example, the cost could be different depending on the used technology. We extend the definition of Bayesian game when the players have many objectives to optimize as defined by Shapley in [102] and investigate the existence of strong and weak Bayesian Pareto equilibria ([30] and [31]). In the special case of potential games ([74]) it is extended the result obtained in [94] to Bayesian multicriteria game. In general it is used a scalarization approach to obtain an existence theorem for weak and strong Bayesian Pareto equilibria (wBPE and sBPE for short, respectively). The existence of approximate equilibria for Bayesian games (see ([73]) is also discussed in the multicriteria case.

There is a field of game theory literature which deals with environmental issues, in particular a big number of contributions have been published on pollution reduction models in recent years, see for example [1], [2], [20], [29], [42], [46], [59], [70], [85], [116] and [127]. The typical issues analyzed in this literature are the incentive schemes of countries which sign a treaty and the stabilization of International Environmental Agreements. There are two main lines of thought. The first line of research, exemplified by [9], [10], [11], [22], [23], [35] and [45], sees the problem of designing (or signing) an IEA from the perspective of coalitions stability, a concept that has its root in the cartel problem in industrial organization literature. The stability of an IEA is ensured by two tests: the entry test that intends to see whether it is in interest of an already formed group of signatories to enlarge the IEA with new members; the exit test that intends to check whether it is in the interest of a player to remain in the coalition. The general message carried out in this literature is that only a small number of countries will end up signing an IEA, i.e. only a small stable coalition can emerge. This approach is also known as the **small coalition approach**. The second line of thought adopts cooperative game theory as the analytical framework. The allocation problem is solved following a two-step methodology. First, one computes the Pareto-optimal emission levels and second, one uses a solution concept based on cooperative game theory (Core, Shapley value, etc) to allocate each player his share of the total optimized cooperative payoff. The remaining issue is to find the right allocation function that guarantees the stability of the formed solution in the core sense. Contrary to the first approach, here the stability of the coalition is passive in the sense that the number of partecipating countries is exogenous. In other words, this approach supposes the existence of a large number of countries that are predisposed to sign the agreement, from which the naming **grand coalition approach** originates. (See for example [26], [27], [28], [41], [44], [51], [52], [65] and [96]).

Chapter 4 is devoted to illustrate a pollution-reduction model. In this chapter an application of Bayesian game is shown in the field of environmental economics. Specifically we apply the model of Bayesian oligopoly games to an environmental game where countries choose their optimal emissions strategy maximizing their own profits, having to take into account that their aggregate emissions amount to an environmental cost suffered by all of them. Here the type structure, which is about marginal gains and production function, is finite and partially ordered. Under some hypothesis the Bayesian game has a potential function and, in this way, it is simple to compute optimal pure strategies in classical examples: in this chapter we deal with three different models, whose respective payoffs were endowed with linear, linear-quadratic and linear-logarithmic cost functions.

The starting points are [5] and [18], which on their turn are related to [52] and [53]. In the above models, the involved countries aim at maximizing their utility functions by manoeuvring their emissions strategies, which affect both their revenues and the damage provoked by the polluting actions. The countries are differentiated based on these two crucial characteristics: marginal gains and marginal damage, the former expressing competitiveness and intensity of production, the latter involving the negative impact of the economic activities on the environment. Such double formulation of uncertainty is somewhat similar to the uncertainty in inverse demand functions and cost functions analyzed by [39] and [40] in their papers on the existence of Bayesian Cournot equilibria in duopolies. Differently from their approach, here the focus is on monotonicity with respect to the partial order of the type spaces rather than on existence and uniqueness of equilibria. In the second part, the environmental aspect is faced with a cooperative point of view. Chapter 6 proposes a new perspective on cooperative games, by assuming that the involved players are supposed to face a common damage. The agents can choose to make an agreement and form a coalition or to defect and face such damage individually.

When such disadvantage is modeled by a dynamic state variable evolving over time, cooperating and non-cooperating agents solve different optimization problems, but they all must take into account such state variable, as if it represented an externality in all their respective value functions. Even if we just consider the cooperative and static aspects of such a game, the externality has a key role in the worth of coalitions.

The approach relies on a class of cooperative games including an external effect, such that the characteristic value function is split in two parts: one of them is standard, the other one is affected by externality.

It is worth describing this new idea of externality, which basically differs from the previous characterizations in literature. Transferable utility games with positive externalities were defined by [99], which related such externality to an increase in pay-off for the players in a specific coalition when the remaining coalitions committed to merging. That is, in presence of a partition of the set of agents and of multiple coalitions, a group of players may enjoy a positive spillover originating from a merger of external coalitions rather than from a strategic choice.

On the other hand, the role of externality is played, and its amount is measured, by a different state variable, not directly depending on the possibly undertaken agreements. Loosely speaking, externalities arise in the same way as they do in standard dynamic oligopoly models (see [64]).

When we relate this idea to the welfare of a country dealing with an emission reduction strategy, we stress that the clean share of welfare is always positive, whereas the share including the pollution effect is negative, then the total welfare must be globally evaluated.

The tools which allowed us to study economic applications are discussed in the rest of the thesis. In particular the first part is devoted to non-cooperative games. Chapter 1 shows classical tools of non-cooperative game theory. More precisely we underline the distinction between finite games and non-finite games discussing Nash equilibria and approximate Nash equilibria. A section is dedicated to potential games: in such games, introduced in [83], the incentive of all players to change their strategy can be expressed using a single global function called the potential function. Section 1.5 deals with Bayesian games. Harsanyi in [58] introduces games with incomplete information. He proves the existence of Bayesian equilibria for the case when the pure strategy spaces are finite. Many aspects of Bayesian games have been studied in literature. Some of them regard the existence of equilibria in these games. Milgrom and Weber in [82] noted that the usual fixed point argument of Nash in [86] with the standard assumptions is not applicable in proving the existence of Bayesian equilibrium and hence introduced sufficient conditions for the existence. Balder in [6] and [7] generalized their result and Radner and Rosenthal in [98] presented sufficient conditions for the existence of pure strategy Bayesian equilibrium. Kim and Yannelis in [67] provide equilibrium existence results for Bayesian games with infinitely many agents. Reny in [100] generalizes Athey's and McAdams results in [4] and [76] respectively, on the existence of monotone pure strategy equilibria in Bayesian games. Mallozzi, Pusillo and Tijs in [73] consider situations where one of the players may have an infinite set of pure strategies, one criterion and a finite number of types and get an existence theorem of approximate equilibria. As for mixed strategies they are usually regarded as unappealing because they are not only hard to interpret, but also, considered as too complex for real players to use. Motivated by this view, game-theorists have provided several purification theorems that describe when mixed strategies can be replaced by equivalent pure strategies. Several purification results have been obtained for games with a large number of players, see for example Cartwright and Wooders in [24] and Carmona in [21]. As concerns the economics literature, Bayesian games play a key role: indeed several types of uncertainty are considered, and their implications on the provision of public goods are discussed. Gradstein in [54] assumes that consumers are uncertain about the contribution of other individuals. Under this uncertainty, the time dynamics of the private provision of public good is derived. Gradstein *et al.* in [55] reexamine Warr's neutrality of the provision of a public good with respect to income distribution (see [126]) in the context of uncertainty. In the model, uncertainty is about the consumers' income: each consumer knows her own endowment, but her information regarding the endowments of other consumers is incomplete. Keenan *et al.* in [66] examine the impact of increased uncertainty on the provision of the public good under a non-Nash response and symmetric equilibrium. Here again, uncertainty is about the response of other contributors to a contribution to the public good. In [61] the authors consider a public good economy with differential information regarding consumers income and preferences. The private information of each consumer is given by her information partition: that is, a consumer cannot distinguish between different states of nature that belong to the same element in her information partition. In [62] the authors apply the concept of information advantage in [38] to a model of a public good economy introduced in [61]: they consider a public good economy where the consumers' state-dependent utilities have a multiplicative structure. Also as regards Cournot oligopoly in [38] authors study the value of information: in an oligopoly where the market demand and the linear cost are uncertain, a firm with superior information obtains higher expected profits than a firm whose information is inferior. Einy *et al.* in [38] also present an example of a Cournot duopoly with quadratic costs where superior information is disadvantageous. Also in [32] and in [68] the authors show that in equilibrium a less informed firm earns higher expected profits than a more informed firm. Finally, the last section of Chapter 1 is devoted to supermodular games introduced by Topkis in [109] and very useful to describe, for example, oligopoly situations.

Chapters 2 deals with multicriteria games. In recent years, many authors have studied the game problem with vector payoffs, for example, see [3] and [14]. Although many concepts have been suggested to solve multicriteria games, the notion of Pareto equilibrium, introduced by Shapley in [102], is the most studied concept in game theory. In [125], Voorneveld *et al.* introduced the new concept of ideal Nash equilibrium for finite multicriteria games which has the best properties and Radjef and Fahem in [97] provide an existence theorem for this new solution concept. Patrone, Pusillo, Tijs in [94] link the concept of multicriteria game with that one of potential game. For some applications see for example [31].

Chapter 5, in the second part of thesis, provides the tools of cooperative game theory. In particular in this chapter TU-games are investigated.

Part I

Non Cooperative Approach

Chapter 1

Basic concepts on games and equilibria

The theory of non-cooperative games studies the behavior of agents in any situations where each agent's optimal choice may depend on her forecast of the choices of her opponents. In non-cooperative games the emphasis is mainly on the individual behavior.

1.1 Finite Games

Let us denote with $N = \{1, \dots, n\}$ the players' finite set of cardinality n.

Definition 1.1. *A **non-cooperative game** with a finite number n of players is a tuple $G = (N, X_1, \dots, X_n, u_1, \dots, u_n)$ where $\forall i \in N$,*

- *X_i is a non-empty set and represents the pure strategy space of player i;*
- *$u_i : X := \prod_{i \in N} X_i \to \mathbb{R}$ is the utility function of player i.*

If we also assume that $\forall i \in N$, X_i is a finite set, we say that G is a finite game, and we denote with Γ_{finite} the class of finite games.

Notation 1.1. *Take $i \in N$, we denote with (x_i, x_{-i}) the element belonging to X such that:*

- *$x_i \in X_i$;*
- *$x_{-i} \in \prod_{j \in N \setminus \{i\}} X_j =: X_{-i}$.*

Definition 1.2. *Let* $G = (N, X_1, \dots, X_n, u_1, \dots, u_n)$ *be a non-cooperative game. Player* i'*s strategy* $\widetilde{x}_i \in X_i$ *dominates strategy* $x_i \in X_i$ *if*

$$u_i(\widetilde{x}_i, x_{-i}) \geq u_i(x_i, x_{-i}) \quad \forall x_i \in X_i,\ \forall x_{-i} \in X_{-i}, \tag{1.1}$$

with a strict inequality for at least one $x_i \in X_i$. *A player* i'*s strategy is dominated if there exists at least another strategy which dominates it.*
Player i'*s strategy* $\widetilde{x}_i \in X_i$ *strictly dominates strategy* $x_i \in X_i$ *if*

$$u_i(\widetilde{x}_i, x_{-i}) > u_i(x_i, x_{-i}) \quad \forall x_i \in X_i,\ \forall x_{-i} \in X_{-i}, \tag{1.2}$$

The most important solution concept for non-cooperative games are Nash equilibria.

Definition 1.3. *Let* $G = (N, X_1, \dots, X_n, u_1, \dots, u_n)$ *be a non-cooperative game.*
A n-tuple $(\widetilde{x}_1, \dots, \widetilde{x}_n) \in X$ *is a* ***Nash Equilibrium*** *(NE for short) [**Strong Nash Equilibrium** (sNE for short)] for G if* $\forall i \in N$ *we have:*

$$u_i(\widetilde{x}_i, \widetilde{x}_{-i}) \geq [>]\, u_i(x_i, \widetilde{x}_{-i}) \qquad \forall x_i \in X_i. \tag{1.3}$$

We denote with $NE(G)\,[sNE(G)]$ the set of Nash equilibria [strong Nash equilibria] for G.

We can observe that the condition (1.3) is equivalent to say that $\forall i \in N$ we have:

$$u_i(\widetilde{x}_i, \widetilde{x}_{-i}) \geq \sup_{x_i \in X_i} u_i(x_i, \widetilde{x}_{-i}). \tag{1.4}$$

Nash equilibria are characterized as fixed points of particular correspondences called best response correspondences.

In general given two sets X, Y a correspondence from X to Y is a map associating to each element of X a subset of Y.

Definition 1.4. *Let* X, Y *be topological spaces and* $F : X \rightrightarrows Y$ *a correspondence. We say that:*

- F *is* ***upper hemicontinuous*** *(u.h.c. for short) in* $x \in X$ *if for any open neighbourhood* V *of* $F(x)$, *there exists a neighbourhood* U *of* x *such that* $F(x') \subseteq V\ \forall x' \in U$;

- *F is **lower hemicontinuous** (l.s.c. for short) in $x \in X$ if $\forall y \in F(x)$ and for any open neighbourhood V of y in Y, there exists a neighbourhood U of x in X such that $F(x') \cap V \neq \emptyset$ $\forall x' \in U$;*
- *F is **upper hemicontinuous** in X if it is upper hemicontinuous $\forall x \in X$;*
- *F is **lower hemicontinuous** in X if it is lower hemicontinuous $\forall x \in X$;*
- *F is **continuous** in $x \in X$ if it is upper hemicontinuous and lower hemicontinuous in $x \in X$;*
- *F is **continuous** in X if it is upper hemicontinuous and lower hemicontinuous in X.*

Recall that, given a correspondence $F : X \rightrightarrows Y$, we say that $k \in X$ is a fixed point for F if $k \in F(k)$.

For further explanations see [69].

Definition 1.5. *Let $G = (N, X_1, \ldots, X_n, u_1, \ldots, u_n)$ be a non-cooperative game. We define for all $i \in N$ the correspondence $R_i : X_{-i} \rightrightarrows X_i$ where*

$$R_i\left(\widetilde{x}_{-i}\right) = \arg\max_{x_i \in X_i} u_i\left(x_i, \widetilde{x}_{-i}\right) = \left\{\widetilde{x}_i \in X_i : u_i\left(\widetilde{x}_i, \widetilde{x}_{-i}\right) \geq u_i\left(x_i, \widetilde{x}_{-i}\right) \quad \forall x_i \in X_i\right\},$$

*that is the player i's **best reply** when the other players play $\widetilde{x}_{-i}$.*

Let us call

$$R : X \rightrightarrows X$$

the correspondence such that

$$R(x) = \left(R_1\left(x_{-1}\right), \ldots, R_n\left(x_{-n}\right)\right), \quad \forall x \in X.$$

Then R is said best reply for Nash equilibria of G.

The following theorem links fixed points and Nash equilibria.

Theorem 1.1. *Let $G = (N, X_1, \ldots, X_n, u_1, \ldots, u_n)$ be a non - cooperative game, and $\widetilde{x} \in X$ a strategy profile. Then*

- $\widetilde{x} \in NE(G)$ *iff* $\widetilde{x} \in R(\widetilde{x})$.

Proof. It follows from Definitions 1.3 and 1.5. □

Now let us consider the following example.

Example 1.1. *Let us consider the game* $G = (N, X_1, X_2, u_1, u_2) \in \Gamma_{finite}$ *with two players and payoffs' matrix given by Table1.1 in which* $X_1 = X_2 = \{T, B\}$ *are the finite strategy spaces of player I and II respectively. The utility functions* $u_1, u_2 : X_1 \times X_2 \longrightarrow \mathbb{R}$ *of player I and II respectively are defined in the following way:*

$$u_1(T,T) = 2 \; u_1(T,B) = 0 \; u_1(B,T) = 4 \; u_1(B,B) = 1;$$

$$u_2(T,T) = 2 \; u_2(T,B) = 4 \; u_2(B,T) = 0 \; u_2(B,B) = 1.$$

Table 1.1: Prisoner's dilemma

I \ II	T	B
T	2 2	0 4
B	4 0	**1 1**

We have that $NE(G) = \{(B, B)\}$.

In the Example 1.1 we have seen that there exists a unique NE. However the existence and uniqueness property are not ensured for this kind of solution as the following examples show. In particular such properties are not ensured for the class Γ_{finite}.

Example 1.2. *Let us consider, as in the previous example, the game* $G = (N, X_1, X_2, u_1, u_2) \in \Gamma_{finite}$ *with two players and payoffs' matrix given by Table1.2*

Table 1.2: Matching Pennies

I \ II	P	D
P	1 -1	-1 1
D	-1 1	1 -1

Here there are not Nash equilibria.

Example 1.3. *Let us consider, as in the previous example, the game* $G = (N, X_1, X_2, u_1, u_2) \in \Gamma_{finite}$ *with two players and payoffs' matrix given by Table1.3*

Table 1.3: Battle of the sexes

I \ II	T	B
T	**3 1**	0 0
B	0 0	**1 3**

We have that $NE(G) = \{(T,T), (B,B)\}$.

To get a result that ensures the existence of at least a NE we have to consider the mixed extensions of finite games.

Definition 1.6. *Take* $G = (N, X_1, \dots, X_n, u_1, \dots, u_n) \in \Gamma_{finite}$. *We define* ***mixed extension*** *of* G *the game* $\widetilde{G} = (N, \Delta(X_1), \dots, \Delta(X_n), \widetilde{u}_1, \dots, \widetilde{u}_n)$, *where for all* $i \in N$ *we have:*

- $\Delta(X_i) = \left\{ p_i \in \mathbb{R}^{|X_i|} : p_{ij} \geq 0 \quad \forall j = 1, \dots, |X_i|, \quad \sum_{j=1}^{|X_i|} p_{ij} = 1 \right\}$ *that is the probability space on* X_i*;*

- $\widetilde{u}_i : \Delta(X) := \prod_{i \in N} \Delta(X_i) \longrightarrow \mathbb{R}$ *defined in the following way:*

$$\widetilde{u}_i(p) = \sum_{k_1=1}^{|X_1|} \cdots \sum_{j=1}^{|X_i|} \cdots \sum_{k_n=1}^{|X_n|} p_{1_{k_1}} \cdots p_{i_j} \cdots p_{n_{k_n}} u_i \left(x_{1_{k_1}}, \dots, x_{i_j}, \dots, x_{n_{k_n}} \right),$$

where $p \in \Delta(X)$.

We denote with Γ_{mixed} the class of mixed extension of finite game.

Nash equilibria and best replies for mixed extensions of a finite game are defined in a similar way.

Let us consider the following results.

Definition 1.7. *Let* X, Y *be subsets of* $\mathbb{R}^n$ *and* $F : X \rightrightarrows Y$ *a correspondence. We say that* F *has a* ***closed graph*** *if the set* $\{(x,y) \in X \times Y : y \in F(x)\}$ *is a closet subset of* $X \times Y$.

Theorem 1.2. *Let K be a compact, convex and non-empty subset of $\mathbb{R}^n$ and $F : K \rightrightarrows K$ a corrispondence with closed graph and where $F(K)$ is a non-empty and convex set. Then there is a $x \in K$ such that $x \in F(x)$.*

Proof. See [69]. □

Theorem 1.3. *Let S,T be metric spaces and $f : S \times T \longrightarrow \mathbb{R}$ a continuous function. Then the correspondence $M : S \rightrightarrows T$ such that*

$$M(s) = \arg\max_{t \in T} f(s,t)$$

has closed graph.

Proof. See [69]. □

Let us consider the following definition:

Definition 1.8. *Let $X \subseteq \mathbb{R}^n$ be a convex set. A function $f : X \longrightarrow \mathbb{R}$ is said* ***quasi concave*** *if $\forall t \in \mathbb{R}$ the set $\{x \in X : f(x) \geq t\}$ is convex.*

Theorem 1.4. *Take $G = (N, X_1, \ldots, X_n, u_1, \ldots, u_n) \in \Gamma_{finite}$ then the mixed extension of G, $\widetilde{G}$ has a NE .*

Proof. For every $i \in N$ the set $R_i(\widetilde{p}_{-i})$ is non-empty since $\widetilde{u}_i$ is continuous and $\Delta(X_i)$ is compact, and it is convex since $\widetilde{u}_i$ is quasi-concave on $\Delta(X_i)$; R is upper hemicontinuous (that is equivalent to have closed graph since R is compact-valued), since each $\widetilde{u}_i$ is continuous. Thus by Theorem 1.2 R has a fixed point. □

Let us calculate Nash equilibria in mixed strategies of Example 1.2.

Example 1.4. *Let us consider the mixed extension $\widetilde{G} = (N, \Delta(X_1), \Delta(X_2), \widetilde{u}_1, \widetilde{u}_2)$ in Example 1.2.*

We can identify the mixed-strategy space of player I and II, $\Delta(X_1), \Delta(X_2)$ respectively as the *interval* $[0,1]$*. Let us call* $\underline{p} = (p, 1-p)$ *the mixed-strategy of player I and* $\underline{q} = (q, 1-q)$ *the mixed-strategy of player II. The utility functions in mixed-strategy* $\widetilde{u}_1, \widetilde{u}_2 : \Delta(X_1) \times \Delta(X_2) \longrightarrow \mathbb{R}$ *for player I and II respectively are defined in the following way:*

$$\widetilde{u}_1(\underline{p}, \underline{q}) = pq - p(1-q) - q(1-p) + (1-p)(1-q),$$

$$\widetilde{u}_2(\underline{p}, \underline{q}) = -pq + p(1-q) + q(1-p) - (1-p)(1-q).$$

Moreover

$$\begin{aligned} R_I(\underline{q}) &= argmax_{p\in[0,1]}\ pq - p(1-q) - q(1-p) + (1-p)(1-q) \\ &= argmax_{p\in[0,1]}\ p(4q-2) + 1 - 2q. \end{aligned}$$

Then

$$R_I(\underline{q}) = \begin{cases} \{1\} & \textit{if } q > \frac{1}{2} \\ \{0\} & \textit{if } q < \frac{1}{2} \\ [0,1] & \textit{if } q = \frac{1}{2} \end{cases}$$

Similarly

$$R_{II}(\underline{p}) = \begin{cases} \{0\} & \textit{if } p > \frac{1}{2} \\ \{1\} & \textit{if } p < \frac{1}{2} \\ [0,1] & \textit{if } p = \frac{1}{2} \end{cases}$$

Then $NE\left(\widetilde{G}\right) = \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}.$

We can note that Theorem 1.4 is only an existence-theorem and does not ensure the uniqueness as the following example shows.

Example 1.5. *Let us consider the mixed extension* $\widetilde{G} = (N, \Delta(X_1), \Delta(X_2), \widetilde{u}_1, \widetilde{u}_2)$ *in Example 1.3.*

We can identify the mixed-strategy space of player I and II, $\Delta(X_1), \Delta(X_2)$ *respectively as the interval* $[0,1]$*. Let us call* $\underline{p} = (p, 1-p)$ *the mixed-strategy of player I and* $\underline{q} = (q, 1-q)$ *the mixed-strategy of player II. The utility functions in mixed-strategy* $\widetilde{u}_1, \widetilde{u}_2 : \Delta(X_1) \times \Delta(X_2) \longrightarrow \mathbb{R}$ *for player I and II respectively are defined in the following way:*

$$\widetilde{u}_1(\underline{p}, \underline{q}) = 3pq + (1-p)(1-q),$$

$$\widetilde{u}_2(\underline{p}, \underline{q}) = pq + 3(1-p)(1-q).$$

Moreover

$$\begin{aligned} R_I(\underline{q}) &= argmax_{p\in[0,1]}\ 3pq + (1-p)(1-q) \\ &= argmax_{p\in[0,1]}\ p(4q-1) + 1 - q. \end{aligned}$$

Then

$$R_I(\underline{q}) = \begin{cases} \{1\} & \text{if } q > \frac{1}{4} \\ \{0\} & \text{if } q < \frac{1}{4} \\ [0,1] & \text{if } q = \frac{1}{4} \end{cases}$$

Similarly

$$R_{II}(\underline{p}) = \begin{cases} \{1\} & \text{if } p > \frac{3}{4} \\ \{0\} & \text{if} p < \frac{3}{4} \\ [0,1] & \text{if } p = \frac{3}{4} \end{cases}$$

In Figure1.1 the continuous line describes the graph of $R_I(\underline{q})$*, while the dotted line describes the graph of* $R_{II}(\underline{p})$*. The circles represent the Nash equilibria. Then* $NE\left(\widetilde{G}\right) = \left\{(0,0), \left(\frac{3}{4}, \frac{1}{4}\right), (1,1)\right\}$ *We can note that the equilibria* $(0,0), (1,1)$ *correspond to pure-equilibria* $(T,T), (B,B)$ *respectively.*

The last remark of Example 1.5 is true for all game $G \in \Gamma_{finite}$ but the viceversa does not hold as we can see from Example 1.5.

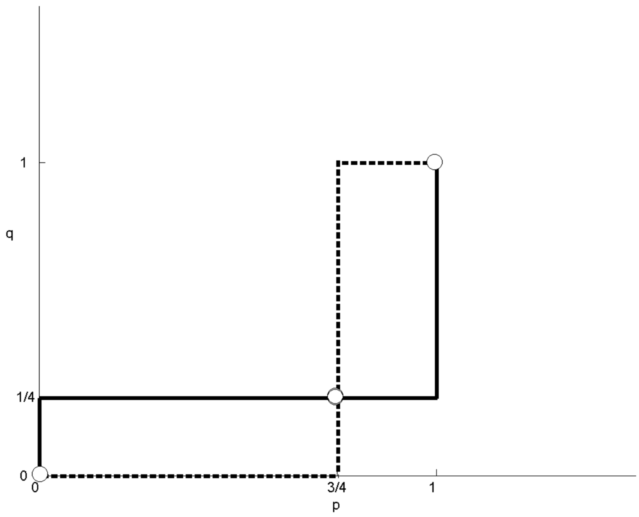


Figure 1.1: Nash Equilibria

1.2 Non-Finite Games

A non-cooperative game with non-finite strategy-spaces is called non-finite game. In particular the mixed-extention of a finite game is a non-finite game. In this sense we have a corollary of Theorem 1.4 for non-finite game in general.

Corollary 1.1. *A non-cooperative game* $G = (N, X_1, \ldots, X_n, u_1, \ldots, u_n)$ *has a NE if for each player* i :

- *the strategy set* X_i *is a non-empty, compact and convex subset of an Euclidean space;*
- *the payoff function* u_i *is continuous and quasi-concave in* x_i*.*

Proof. It follows from Theorem 1.4. □

1.2.1 Cournot-NE

A solution very similar to NE was first used by Cournot as early as 1838 in the framework of duopoly model. This model is considered rightly as one of the major classic examples of applied game theory in economics. In this model, the firms are supposed to choose simultaneously their volume of output. See [108].

Two firms produce and sell a homogeneous good. Let us call q_1 and q_2 the quantities produced by firm 1 and firm 2, respectively. To simplify matters, assume that there are not fixed costs and that marginal costs are constant and equal to c, so that the total cost is:

$$C_i = cq_i.$$

Firms face an inverse demand function given by:

$$P = \max\{a - Q, 0\},$$

where $Q = q_1 + q_2$, P is the price of the good and a is a positive constant and, in generally, it is assumed to be the reservation price of the homogeneous good. In order to avoid a corner solution assume that $a > c$.

Firms are supposed to choose simultaneously the quantities q_1 and q_2. In this model those variable are thus the players' strategies. The strategy sets of the player are identical and given by:

$$X_1 = X_2 = [0, a - c].$$

The players' payoff functions are here the profit functions of the firms:

$$\begin{aligned} u_1(q_1, q_2) &= P(q_1, q_2)\, q_1 - cq_1 \\ u_2(q_1, q_2) &= P(q_1, q_2)\, q_2 - cq_2. \end{aligned}$$

Or,more generally, after a clear change of notations:

$$u_i(q_i, q_j) = \begin{cases} [a - (q_i + q_j) - c]\, q_i & \text{if } \; 0 \leq q_i \leq a - c - q_j \\ \\ 0 \;\; \text{if} \;\; 2 - c - q_j \leq q_i \leq a - c. \end{cases} \tag{1.5}$$

If $\left(q_i^*, q_j^*\right)$ is a NE of this game, then $\forall i \in N$:

$$u_i\left(q_i^*, q_j^*\right) \geq u_i\left(q_i, q_j^*\right),$$

for all $q_i \in X_i$. Then for each player i, q_i^* must be a solution of:

$$\max_{q_i} u_i = [a - (q_i + q_j) - c]\, q_i.$$

It is easy to check that, by Corollary 1.1, in this game there always exists at least a NE.

With the assumption that $q_i^* < a - c$, the first-order conditions of this optimization problem are necessary and sufficient:

$$\frac{\partial u_i}{\partial q_i} = 0, \quad i = 1, 2,$$

which gives:

$$q_i^* = \frac{\left(a - q_j^* - c\right)}{2}, \quad i = 1, 2.$$

Solving this pair of equations leads finally to the outcome of the game:

$$q_1^* = q_2^* = \frac{a - c}{3}.$$

The Cournot duopoly model can be extended to the case in which there are many firms ($n > 2$): in this case we speak of Cournot oligopoly. In general the early literature on Cournot oligopoly has been concerned with three main issues: whether the model is quasi-competitive, i. e., industry output rises and price falls with additional firms (see for example [13], [49] and [78]); whether the model converges to perfect competition with an infinite number of firms (see [25], [49], [60], [78], [103] and [119]). The third issue concerning the question whether the equilibrium solution itself is dynamically stable (see [56], [89] and [105]).

This model has many variants in which cost structures, inverse demand and value of information change. For example in [12] authors consider a duopoly model with quadratic cost functions. They show existence and uniqueness of affine equilibrium strategies and that, in equilibrium, expected profits of firm i increase with the precision of its information and decrease with the precision of the rival's information. Novshek and Sonnenschein in [88] consider a duopoly model with constant costs and examine the incentives for the firms to acquire and release private information. Clarke in [33] considers an n-firm oligopoly model and shows that there is never a mutual incentive for all firms in the industry to share information unless they may cooperate on strategy once information has been shared.

Vives in [120] observed that more information can be undesirable in the setting of Cournot oligopoly. More recently papers about the value of information in this framework are [39] and [68].

1.3 Approximate Nash Equilibria

In this section we deal with a different concept solution.

We consider the following definition introduced by Tijs in [106].

Definition 1.9. *Let $G = (N, X_1, \ldots, X_n, u_1, \ldots, u_n)$ be a non-cooperative game and $\epsilon > 0$. Then $(\widetilde{x}_1, \ldots, \widetilde{x}_n) \in X$ is an* ***approximate Nash equilibrium*** *(ϵ-NE for short) for G if for each $i \in N$ we have:*

$$u_i\left(\widetilde{x}_i, \widetilde{x}_{-i}\right) \geq u_i\left(x_i, \widetilde{x}_{-i}\right) - \epsilon \qquad \forall x_i \in X_i. \tag{1.6}$$

We denote with $\epsilon - NE(G)$ the set of approximate Nash equilibria of G.

Obviously for $\epsilon = 0$ the set of approximate Nash equilibria is equal to the set of Nash equilibria. The condition (1.6) means that for each $i \in N$ we have:

$$u_i\left(\widetilde{x}_i, \widetilde{x}_{-i}\right) \geq \sup_{x_i \in X_i} u_i\left(x_i, \widetilde{x}_{-i}\right) - \epsilon. \tag{1.7}$$

Moreover if $\epsilon_1 < \epsilon_2$ then $\epsilon_1 - NE(G) \subseteq \epsilon_2 - NE(G)$ for each game G.

Example 1.6. *Let us consider $G = (N, X_1, X_2, u_1, u_2)$ with $X_1 = X_2 = \mathbb{R}$ and $u_1\left(x_1, x_2\right) = -u_2\left(x_1, x_2\right) = x_2^2 - x_1^2$. By condition 1.7 we have that, by fixing $\epsilon > 0$, the pair $(\widetilde{x}_1, \widetilde{x}_2) \in \mathbb{R}^2$ is a ϵ-NE if*

$$\begin{aligned} \widetilde{x}_2^2 - \widetilde{x}_1^2 \geq \sup_{x_1 \in \mathbb{R}} u_1\left(x_1, \widetilde{x}_2\right) - \epsilon &= \sup_{x_1 \in \mathbb{R}} \left(\widetilde{x}_2^2 - x_1^2\right) - \epsilon \\ &= \widetilde{x}_2^2 - \epsilon, \end{aligned}$$

and

$$\widetilde{x}_1^2 - \widetilde{x}_2^2 \geq \sup_{x_2 \in \mathbb{R}} u_2(\widetilde{x}_1, x_2) - \epsilon = \sup_{x_2 \in \mathbb{R}} \left(\widetilde{x}_1^2 - x_2^2\right) - \epsilon = \widetilde{x}_1^2 - \epsilon.$$

That is if $(\widetilde{x}_1, \widetilde{x}_2) \in [-\sqrt{\epsilon}, \sqrt{\epsilon}] \times [-\sqrt{\epsilon}, \sqrt{\epsilon}]$.

$\epsilon - NE(G)$ *is the square with center* $(0,0)$ *and side* $2\sqrt{\epsilon}$. *In particular* $NE(G) = \{(0,0)\}$.

Example 1.7. *Let* $G = (N, X_1, X_2, u_1, u_2)$ *be a non-cooperative game with* $X_1 = X_2 = \mathbb{R}$ *and* $u_1(x_1, x_2) = -u_2(x_1, x_2) = x_1 x_2$. *We have that:*

$$\sup_{x_1 \in \mathbb{R}} u_1(x_1, \widetilde{x}_2) - \epsilon = \sup_{x_1 \in \mathbb{R}} \widetilde{x}_2 x_1 - \epsilon < +\infty \Leftrightarrow \widetilde{x}_2 = 0,$$

and

$$\sup_{x_2 \in \mathbb{R}} u_2(\widetilde{x}_1, x_2) - \epsilon = \sup_{x_2 \in \mathbb{R}} -\widetilde{x}_1 x_2 - \epsilon < +\infty \Leftrightarrow \widetilde{x}_1 = 0.$$

So the unique $\epsilon - NE$ *is the pair* $(\widetilde{x}_1, \widetilde{x}_2) = \{(0,0)\}$. *In this case* $\epsilon - NE(G) = NE(G)$.

Next example shows that for some values of ϵ the existence of approximate Nash equilibria is not ensured.

Example 1.8. *Let us consider* $G = (N, X_1, X_2, u_1, u_2)$ *a non-cooperative game with* $X_1 = X_2 = \{-1, 1\}$ *and* $u_1(x_1, x_2) = -u_2(x_1, x_2) = x_1 x_2$, *(see 1.2). By condition (1.7) we have that, by fixing* $\epsilon > 0$, *the pair* $(\widetilde{x}_1, \widetilde{x}_2) \in \{-1, 1\} \times \{-1, 1\}$ *is a* ϵ*-NE if:*

$$\widetilde{x}_1 \widetilde{x}_2 \geq \max_{x_1 \in \{-1,1\}} u_1(x_1, \widetilde{x}_2) - \epsilon = \max_{x_1 \in \{-1,1\}} \widetilde{x}_2 x_1 - \epsilon = 1 - \epsilon,$$

and

$$-\widetilde{x}_1 \widetilde{x}_2 \geq \max_{x_2 \in \{-1,1\}} u_2(\widetilde{x}_1, x_2) - \epsilon = \max_{x_2 \in \{-1,1\}} -\widetilde{x}_1 x_2 - \epsilon = 1 - \epsilon.$$

then the pair $(\widetilde{x}_1, \widetilde{x}_2) \in \{-1,1\} \times \{-1,1\}$ *is a* ϵ*-NE iff* $\epsilon \geq 2$*. So if* $\epsilon < 2$*, there are not approximate Nash equilibria and, in particular* $NE(G) = \emptyset$*.*

From these examples we have seen that, as for Nash equilibria, neither the existence (see Example 1.8) nor the uniqueness (see Example 1.6) of approximate Nash equilibria is guaranteed. Also in this case there are existence theorems. We quickly show a result due to Tijs (see [106]), but many other papers have been written about this topic (see for example [16], [104]).

Theorem 1.5. *Take* $G = (N, X_1, \ldots, X_n, u_1, \ldots, u_n) \in \Gamma_{finite}$ *such that for each player* i*,* u_i *is an upper bounded function on* $X = \prod_{i \in N} X_i$*, then for each* $\epsilon > 0$*, the mixed extension of* G*,* $\widetilde{G}$ *has* $\epsilon - NE$*.*

Proof. See [106]. □

1.4 Potential Games

Potential games were introduced by Monderer and Shapley in [83] and studied for example in [124]. A game is said a **potential game** if the incentive of all players to change their strategy can be expressed using a single global function called the **potential function**. This potential function provides the necessary information for the computation of the pure Nash equilibria. Thus a potential function is an economical way to summarize the information concerning pure Nash equilibria into a single function. Moreover, every finite game with a potential function has an equilibrium in pure strategies: since the strategy space is finite, the potential function achieves its maximum at a certain strategy profile.

Definition 1.10. *Let* $G = (N, X_1, \ldots, X_n, u_1, \ldots, u_n)$ *be a non-cooperative game. We say that* G *is a* ***potential game*** *if there is a function (called potential function)* $\Pi : X := \prod_{i \in N} X_i \longrightarrow \mathbb{R}$ *such that for each* $i \in N$*,* $x_i, y_i \in X_i$*,* $x_{-i} \in X_{-i}$ *we have:*

$$u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = \Pi(x_i, x_{-i}) - \Pi(y_i, x_{-i}) .$$

Table 1.4: A potential function

	T	B
T	2	4
B	4	5

Example 1.9. *Let us consider the Prisoner's dilemma game (see Example 1.1). It is a potential game, where a potential function* Π *is given in Table1.4.*

From Definition 1.10 it follows that a potential function is not unique: if Π *is a potential function for a game* G *also* $\Pi + c$*, with* $c \in \mathbb{R}$*, is a potential function for* G*. Then all potential functions of the Prisoner's dilemma game are* Π_k*, with* $k \in \mathbb{R}$*, given by Table1.5.*

Table 1.5: All potential functions

	T	B
T	2+k	4+k
B	4+k	5+k

Interesting classes of potential games are the coordination games and the dummy games.

Definition 1.11. *Let* $G = (N, X_1, \dots, X_n, u_1, \dots, u_n)$ *be a non-cooperative game. We say that* G *is a* ***coordination game*** *if* $u_i = u_j \ \forall i, j = 1, \dots, n$*. That is the utility functions are equal for each player.*

Definition 1.12. *Let* $G = (N, X_1, \dots, X_n, u_1, \dots, u_n)$ *be a non-cooperative game. We say that* G *is a* ***dummy game*** *if* $u_i(x_i, x_{-i}) = u_i(y_i, x_{-i}) \ \ \forall i = 1, \dots, n, \quad x_i, y_i \in X_i, \quad x_{-i} \in X_{-i}$*. That is player* i'*s strategy choice does not affect her payoff.*

A potential function for a coordination game $G = (N, X_1, \dots, X_n, u_1, \dots, u_n)$ is $\Pi = u_1$ while a potential function for a dummy game is the null function.

Not all finite games admit a potential function as we can conclude from:

Theorem 1.6. *Take* $G = (N, X_1, \ldots, X_n, u_1, \ldots, u_n) \in \Gamma_{finite}$ *a potential game,* Π *a potential function for* G, *and* $G_1 = (N, X_1, \ldots, X_n, \Pi, \ldots, \Pi)$ *a coordination game. Then*

i) $NE(G) = NE(G_1)$;

ii) *G has a NE.*

Proof. See [107]. □

By Theorem 1.6 the game in the Example 1.2 is not a potential game.

For non-finite games we have the following theorem:

Theorem 1.7. *Let* $G = (N, X_1, \ldots, X_n, u_1, \ldots, u_n)$ *be a potential game and* Π *an upper bounded potential function for* G. *Then* $\forall \epsilon > 0$, G *has* $\epsilon-$*NE.*

Proof. See [83]. □

The next results, dealing with differentiable games (i.e. such that their utility functions are differentiable) are well-known.

Lemma 1.1. *Let* $G = (N, X_1, \ldots, X_n, u_1, \ldots, u_n)$ *be a game in which for each player* i, X_i *are intervals of real numbers. Suppose the utility functions* u_i *are continuously differentiable* $\forall i \in N$, *and let* $\Pi : X \longrightarrow \mathbb{R}$. *Then* Π *is a potential function for* G *iff* Π *is continuously differentiable, and*

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial \Pi}{\partial x_i}, \ \forall i \in N.$$

Theorem 1.8. *Let* $G = (N, X_1, \ldots, X_n, u_1, \ldots, u_n)$ *be a game in which for each player* i, X_i *are intervals of real numbers. Suppose the utility functions* u_i *are twice continuously differentiable* $\forall i \in N$. *Then* G *is a potential game iff*

$$\frac{\partial^2 u_i}{\partial x_i \partial x_j} = \frac{\partial^2 u_j}{\partial x_i \partial x_j}, \ \forall i, j \in N. \tag{1.8}$$

Moreover, if the utility functions satisfy (1.8) and z *is an arbitrary (but fixed) strategy profile in* X, *then a potential function for* G *is given by*

$$\Pi(x) = \sum_{i \in N} \int_0^1 \frac{\partial u_i}{\partial x_i} (y(t)) (y_i)'(t) \, dt, \tag{1.9}$$

where $y : [0,1] \longrightarrow X$ *is a piecewise continuously differentiable path in* X *that connects* z *to* x.

Example 1.10. *Let* $G_c = (N, X_1, \ldots, X_n, u_1, \ldots, u_n)$ *be a Cournot oligopoly game in which there is a linear inverse demand function* $P = \max\{a - Q, 0\}$, *where* $Q = \sum_{i=1}^{n} q_i$, *and cost functions* $c_1, \ldots, c_n$ *with continuous derivatives. We take* $X_i = [0, +\infty)$ *and*

$$u_i(q_1, \ldots, q_n) = P(q_1, \ldots, q_n)\, q_i - c_i(q_i), \ \forall i \in N.$$

It is simple to prove that G_c *is a potential game with potential function*

$$\Pi(q_1, \ldots, q_n) = a \sum_{i=1}^{n} q_i - \sum_{i=1}^{n} q_i^2 - \sum_{1 \leq i < j \leq n} q_i q_j - \sum_{i=1}^{n} c_i(x_i).$$

1.5 Bayesian Games

We frequently wish to model situations in which some of the parties are not certain of the characteristics of some of the other parties. The model of a Bayesian game is designed for this purpose: indeed the case of perfect knowledge of payoffs is a simplifying assumption that may be a good approximation in some cases. A Bayesian game, or game with incomplete information, is a game in which, at the first point in time when the players can begin to plan their moves in the game, some players already have private information about the game that other players do not know. The initial private information that a player has at the first point in time is called the **type** of the player. The type of a player embodies any private information (more precisely, any information that is not common knowledge to all players) that is relevant to the player's decision making. This may include, in addition to the player's utility function, her beliefs about other players' utility functions, her beliefs about what other players believe her beliefs are, and so on.

To define a Bayesian game, see for example [50], we must specify a set of players N and, for each player $i \in N$, we must specify a set of possible actions A_i, a set of possible types T_i, a probability function p_i and a utility function u_i. We let $A = \prod_{i \in N} A_i$, $\quad T = \prod_{i \in N} T_i$. That is, A is the set of all possible profiles of actions that the players may use in the game, and T is the set of all possible profiles of types that the players may have in the game. For each player i, we let T_{-i} denote the set of all possible combinations of types for the players other than i. The probability function p_i must be a function from T_i into $\Delta(T_{-i})$, the set of probability distributions over T_{-i}. That is, for any possible type $t_i \in T_i$, the probability function must specify a probability distribution $p_i(\cdot|t_i)$ over the set T_{-i}, representing what player i would believe about the other players' types if her own type were t_i. Thus, for any $t_{-i} \in T_{-i}$, $p_i(t_{-i}|t_i)$ denotes the subjective probability that i would assign to the event that t_{-i} is the actual profile of types for the others players, if her own type were t_i.

For any player $i \in N$, the utility function u_i in the Bayesian game must be a function from $A \times T$ to $\mathbb{R}$.

These structures together define a **Bayesian game** G, so we may write

$$G = (N, A_1, \ldots, A_n, T_1, \ldots, T_n, p_1, \ldots, p_n, u_1, \ldots, u_n).$$

G is finite iff the sets N, A_i and T_i are finite $\forall i \in N$. When we study such a Bayesian game G, we assume that each player i knows the entire structure of the game and her own actual type in T_i and this fact is common knowledge among all the players in N. A strategy for a player i in the Bayesian game G is defined to be a function from her set of types T_i into her set of action A_i.

We say that beliefs $(p_i)_{i \in N}$ in a Bayesian game are consistent iff there is some common prior distribution over the set of type profile t such that each players' beliefs given her type are just the conditional probability distribution that can be computed from the prior distribution by Bayes's formula. That is, beliefs are consistent iff there exists some probability distribution $p \in \Delta(T)$ such that:

$$p_i(t_{-i}|t_i) = \frac{p(t_{-i}, t_i)}{p(t_i)} \quad \forall i \in N. \tag{1.10}$$

Because we consider in the following consistent beliefs under condition 1.10 we denote
$G = (N, A_1, \ldots, A_n, T_1, \ldots, T_n, p, u_1, \ldots, u_n)$ instead of
$G = (N, A_1, \ldots, A_n, T_1, \ldots, T_n, p_1, \ldots, p_n, u_1, \ldots, u_n)$.

A play of such a game proceeds as follows: before the types are announced each player i chooses a strategy $x_i \in A_i^{T_i}$[1]. If the type profile is $t = (t_1, \ldots, t_n)$ then player i's payoff is

[1] In general, given two sets X and Y, the notation X^Y indicates the set of functions from Y to X, that is $X^Y = \{f | f : Y \longrightarrow X\}$

$u_i\left(x_1(t_1), x_2(t_2), \ldots, x_n(t_n), t_1, \ldots, t_n\right)$.

The a priori expected payoff for player i when the players use strategies $x_1, \ldots, x_n$ respectively is a function $U_i : A_1^{T_1} \times \cdots \times A_n^{T_n} \longrightarrow \mathbb{R}$ such that

$$U_i\left(x_1, \ldots, x_n\right) = \sum_{t \in T} p(t) u_i\left(x_1(t_1), x_2(t_2), \ldots, x_n(t_n), t_1, \ldots, t_n\right),$$

being $p(t)$ the probability distribution of player i when her type profile is t.

Definition 1.13. *Let $G = (N, A_1, \ldots, A_n, T_1, \ldots, T_n, p, u_1, \ldots, u_n)$ be a Bayesian game. We say that a strategy profile $\widehat{x} = (\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n) \in A_1^{T_1} \times A_2^{T_2} \times \cdots \times A_n^{T_n}$ is a **Bayesian Nash equilibrium** (for short BNE) if $\forall i \in N$, $\forall x_i \in A_i^{T_i}$ we have*

$$U_i\left(\widehat{x}\right) \geq U_i\left(x_i, \widehat{x}_{-i}\right).$$

Example 1.11. *Let $G = (N, A_1, A_2, T_1, T_2, p, u_1, u_2)$ be a finite Bayesian game where:*

- $N = \{1, 2\}$;
- $A_1 = \{a_1, b_1\}$, $A_2 = \{a_2, b_2\}$;
- $T_1 = \left\{t_1^1\right\}$, $T_2 = \left\{t_2^1, t_2^2\right\}$.

The functions $u_1, u_2 : (A_1 \times A_2) \times (T_1 \times T_2) \longrightarrow \mathbb{R}$ are represented by the bimatrices Table 1.6 and Table 1.7.

Table 1.6: u_1

$1 \setminus 2.1$	a_2	b_2
a_1	1 2	0 1
b_1	0 4	1 3

The first one represents the case in which player 2's type is t_2^1, while the second one represents the case in which player 2's type is t_2^2. We can note that the player 1's payoffs are the same in both matrices.

Table 1.7: u_2

$1 \setminus 2.2$	a_2	b_2
a_1	1 1	0 3
b_1	0 2	1 3

We suppose that Nature extracts with probability $P \in [0,1]$ the type t_2^1 (obviously with probability $1-P$ the type t_2^2). So we have that $p\left(t_1^1, t_2^1\right) = P$ and $p\left(t_1^1, t_2^2\right) = 1-P$.

Then the values of U_1 and U_2 are given in the bimatrix Table 1.8, Now we want to compute the

Table 1.8: A priori expected payoff functions

$U_1 \setminus U_2$	a_2	b_2
a_1	1 P+1	0 3-2P
b_1	0 2P+2	1 3

Bayesian Nash equilibria for this game depending on P.

- *If $P \geq \frac{2}{3}$ $BNE = \{(a_1, a_2), (b_1, b_2)\}$;*

- *If $P < \frac{2}{3}$ $BNE = \{(b_1, b_2)\}$.*

Now we introduce the notion of mixed extension of a Bayesian game.

Definition 1.14. *Let $G = (N, A_1, \ldots, A_n, T_1, \ldots, T_n, p, u_1, \ldots, u_n)$ be a Bayesian game. Then the* ***mixed extension*** *of G is the Bayesian game $\widetilde{G} = \left(N, \widetilde{A}_1, \ldots, \widetilde{A}_n, T_1, \ldots, T_n, p, \widetilde{u}_1, \ldots, \widetilde{u}_n\right)$, where $\widetilde{A}_i$ is the family of probability measures (on the $\sigma-$algebra of all subsets of A_i) with finite support. Such probability measures are the form $\mu_i = \sum_{k=1}^{s} p_k e_{a_k}$ where $a_1, \ldots, a_s \in A_i$, $p_k \geq 0$, for all $k \in \{1, \ldots, s\}$ and $\sum_{k=1}^{s} p_k = 1$, where*

$$e_{a_k}(B) = \begin{cases} 1 & \text{if } B \subset A_i, a_k \in B \\ 0 & \text{if } B \subset A_i, a_k \notin B \end{cases},$$

Furthermore $\widetilde{u}_i(\mu_i, \ldots, \mu_n, t) = \int u_i(a_1, \ldots, a_n, t)\, d\mu_1(a_1) \ldots d\mu_n(a_n)$ for all $i \in N$ and $(\mu_1, \ldots, \mu_n) \in \widetilde{A} = \prod_{i \in N} \widetilde{A}_i$.

1.6 Supermodular Games

The class of supermodular games was introduced by [109] and further studied by [81], [110], [121] and [122]. Supermodular games are games in which each player's marginal utility of increasing her strategy rises with increases in her rivals' strategies. In such games the best response correspondences are increasing, so that the players' strategies are strategic complements. When there are two players, a change in variables allows this framework to also accomodate the case of decreasing best responses. Supermodular games are particularly well behaved: they have pure-strategy Nash equilibria. The upper bound of player i's Nash-equilibrium strategies exists and it is a best response to the upper bounds of her rivals' sets of Nash-equilibrium strategies, and similarly for the lower bounds. The simplicity of supermodular games makes convexity and differentiability assumptions unnecessary, although they are satisfied in many applications, for example in the Cournot duopoly.

Let us recall some definitions about supermodular games.

Definition 1.15. *A **partially ordered set (POSET)** is a set X on which there is a binary relation $\preceq$ that is reflexive, antisymmetric and transitive.*

Definition 1.16. *Let us consider a partially ordered set X and a subset Y of X.*

- *If $y \in X$ and $y \preceq x$ for each $x \in Y$, then y is a **lower bound** for Y;*
- *If $z \in X$ and $x \preceq z$ for each $x \in Y$, then z is an **upper bound** for Y.*

When the set of lower bounds of Y has a greatest element, then this greatest lower bound of Y is the **infimum** of Y in X.

When the set of upper bounds of Y has a least element, then this least upper bound of Y is the **supremum** of Y in X.

Definition 1.17.
- *If two elements x_1 and x_2 of a partially ordered set X have a supremum in X, it is called the **meet** of x_1 and x_2 and it is denoted by $x_1 \wedge x_2$;*
- *If two elements x_1 and x_2 of a partially ordered set X have a infimum in X, it is called the **join** of x_1 and x_2 and it is denoted by $x_1 \vee x_2$.*

Definition 1.18. *A partially ordered set that contains the join and the meet of each pair of its elements is a* ***lattice****. If a subset Y of a lattice X contains the join and the meet (with respect to X) of each pair of elements of Y, then Y is a* ***sublattice*** *of X.*

Remark 1.1. *The real line $\mathbb{R}$ with the natural ordering $\geq$ is a lattice with $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$ $\forall x, y \in \mathbb{R}$. Also $\mathbb{R}^n$, $(n > 1)$ with the usual partial order $\geq$ is a lattice with $x \vee y = (x_1 \vee y_1, \ldots, x_n \vee y_n)$ and $x \wedge y = (x_1 \wedge y_1, \ldots, x_n \wedge y_n)$, $\forall x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Any subset of $\mathbb{R}$ is a sublattice of $\mathbb{R}$, and a subset X of $\mathbb{R}^n$ is a sublattice of $\mathbb{R}^n$ if $\forall x, y \in X$ we have that $x \vee y,\ x \wedge y \in X$.*

Definition 1.19. *A supermodular game*

$$G = (N, X_1, \ldots, X_n, u_1, \ldots, u_n)$$

is a tuple where

- *$N = \{1, \ldots, n\}$ is a finite set of players;*
- *$\forall i \in N$, $X_i \subseteq \mathbb{R}^{m(i)}$ (for some $m(i) \in \mathbb{N}$) and $X_i \neq \emptyset$ is the strategy space of player i, $\boldsymbol{X} = \prod_{i \in N} X_i$ is the cartesian product of the strategy spaces;*
- *$u_i : \boldsymbol{X} \longrightarrow \mathbb{R}$ is the payoff function of player i;*
- *$\forall i \in N$, X_i is a sublattice of $\mathbb{R}^{m(i)}$;*
- *$\forall i \in N$, u_i have* ***increasing differences*** *on $\boldsymbol{X}$, i.e.*
  $\forall x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \boldsymbol{X}$ such that $x_i \geq y_i$, we have
  $$u_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - u_i(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n)$$
  $$\geq u_i(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) - u_i(y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n);$$
- *$\forall i \in N$, u_i is* ***supermodular*** *in the $i-th$ coordinate, i.e.*
  $\forall x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), z = (z_1, \ldots, z_n) \in \boldsymbol{X}$ we have
  $$u_i(z_1, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_n) + u_i(z_1, \ldots, z_{i-1}, y_i, z_{i+1}, \ldots, z_n)$$

$$\leq u_i(z_1, \ldots, z_{i-1}, x_i \wedge y_i, z_{i+1}, \ldots, z_n) + u_i(z_1, \ldots, z_{i-1}, x_i \vee y_i, z_{i+1}, \ldots, z_n).$$

Increasing differences point out that an increase in the strategies of player i's rivals raises the desirability of playing a high strategy for player i.

We can observe that supermodularity is automatically satisfied if for each $i \in N$, X_i is single-dimensional. We will need supermodularity in the case of multi-dimensional strategy spaces to prove that each player's best responses are increasing with her rivals' strategies.

For example the Cournot duopoly defined in Subsection 1.2.1 of Chapter 1 is a supermodular game.

From [110] we have the following propositions.

Proposition 1.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable function on $\mathbb{R}^n$, then f has increasing differences on $\mathbb{R}^n$ iff $\frac{\partial f}{\partial x_i}$ is increasing in x_j for each $i, j = 1, \ldots, n$ with $i \neq j$ and $\forall x = (x_1, \ldots, x_n)$.*

Proof. See [110]. □

Proposition 1.2. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a twice differentiable function on $\mathbb{R}^n$, then f has increasing differences on $\mathbb{R}^n$ iff $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$ for each $i, j = 1, \ldots, n$ with $i \neq j$ and $\forall x = (x_1, \ldots, x_n)$.*

Proof. See [110]. □

The following existence theorem is due to Topkis in [109].

Theorem 1.9. *Let $G = (N, X_1, \ldots, X_n, u_1, \ldots, u_n)$ be a supermodular game. If, for each $i \in N$, X_i is compact and u_i is upper hemicontinuous in x_i for each $x_{-i} \in X_{-i}$, then the set of pure-strategy Nash equilibria is nonempty and possesses greatest and least equilibrium points.*

Proof. See [109]. □

Chapter 2

Multicriteria games

Multicriteria (or multiobjective) optimization problems typically have conflicting objectives, and a gain in one objective is, sometimes, a loss for another. Therefore the definition of optimality is not obvious as in the one-criterion case. However in many settings, mathematical models involving more than one objective seem much more adherent to the real problems.

Formally, a multicriteria optimization problem can be formulated as

$$\text{Optimize } f_1(x), \ldots, f_r(x) \tag{2.1}$$

$$\text{subject to } x \in D,$$

where D denotes the feasible set of alternatives and $r \in \mathbb{N}$ the number of criterion functions $f_k : D \longrightarrow \mathbb{R},\ k = 1, \ldots, r$.

See for example [30], [31], [101] and [124].

In this chapter we study the situation in which there is not only a conflict between criteria, but there are, also, many optimization problems to solve simultaneously: that is we deal with **multicriteria games**.

In recent years, many authors have studied the game problem with vector payoffs, for example, see [3] and [14]. Although many concepts have been suggested to solve multicriteria games, the notion of Pareto equilibrium, introduced by Shapley in [102], is the most studied concept in game theory. In [125], Voorneveld *et al.* introduced the new concept of ideal Nash equilibrium for finite multicriteria games which has the best properties and Radjef and Fahem in [97] provide an existence theorem for this new solution concept. Patrone, Pusillo, Tijs in [94] link the concept of multicriteria game with that one of potential game. For some applications see for example [31].

2.1 Weak and Strong Pareto Equilibria

Definition 2.1. *A* ***non-cooperative multicriteria game*** *is a tuple* $G_m = (N, X_1, \ldots, X_n, u_1, \ldots, u_n)$ *where for each* $i \in N$

- *N is a finite set and represents the set of players ;*
- *X_i is a non-empty set and represents the pure-strategy space of player i;*
- *$u_i : X := \prod_{i \in N} X_i \to \mathbb{R}^m$ is the utility function of player i, where m is the number of objectives.*

Let us denote with Γ^m_{finite} the class of finite multicriteria games.

Remark 2.1. *We recall the partial order definition in $\mathbb{R}^m$. For all a, $b \in \mathbb{R}^m$, we say that:*

- $a = b$ *if* $a_i = b_i \quad \forall i = 1, \ldots, m$;
- $a \geqq b$ *if* $a_i \geq b_i \quad \forall i = 1, \ldots, m$;
- $a \geq b$ *if* $a_i \geq b_i \quad \forall i = 1, \ldots, m$, *and* $a \neq b$;
- $a > b$ *if* $a_i > b_i \quad \forall i = 1, \ldots, m$.

Definition 2.2. *Let $G_m = (N, X_1, \ldots, X_n, u_1, \ldots, u_n)$ be a multicriteria game. Then the strategy profile $\widetilde{x} \in X$ is*

- *a* ***weak Pareto equilibrium*** *for G_m if $\forall i \in N \quad \nexists x_i \in X_i$ such that*
$$u_i(x_i, \widetilde{x}_{-i}) > u_i(\widetilde{x}_i, \widetilde{x}_{-i});$$
- *a* ***strong Pareto equilibrium*** *for G_m if $\forall i \in N \quad \nexists x_i \in X_i$ such that*
$$u_i(x_i, \widetilde{x}_{-i}) \geq u_i(\widetilde{x}_i, \widetilde{x}_{-i}).$$

Let us denote with $wPE(G_m)$ and $sPE(G_m)$ weak and strong Pareto equilibria of G_m.

From Definition 2.2 we can note that, in one-criterion case, weak [strong] Pareto equilibria correspond to NE [sNE] respectively, for the game.

It is clear that a strong Pareto equilibrium is also a weak Pareto equilibrium but the viceversa does not hold as the following example shows.

Example 2.1. *Let us consider the finite bicriteria game* $G_2 = (N, X_1, X_2, u_1, u_2)$ *with payoff matrix given by Table2.1 where* $X_1 = \{T, B\}, X_2 = \{L, R\}$ *are the strategy spaces of player I and II respectively. The utility functions* $u_1, u_2 : X_1 \times X_2 \longrightarrow \mathbb{R}^2$ *of player I and II respectively are defined in the following way:*

$$u_1(T,L) = (3,4) \ u_1(T,R) = (4,3) \ u_1(B,L) = (3,5) \ u_1(B,R) = (1,2),$$

$$u_2(T,L) = (3,2) \ u_2(T,R) = (2,3) \ u_2(B,L) = (1,1) \ u_2(B,R) = (2,2).$$

We have that $wPE(G) = \{(T,L),(T,R)\}$ *while* $sPE(G) = \{(T,R)\}$*.Then* $sPE(G) \subset wPE(G)$*.*

Table 2.1: Weak Pareto Equilibria

I \ II	L	R
T	**(3, 4) (3, 2)**	**(4, 3) (2, 3)**
B	(3, 5) (1, 1)	(1, 2) (2, 2)

Pareto equilibria can be characterized as fixed points of best reply correspondences.

Definition 2.3. *Let* $G_m = (N, X_1, \ldots, X_n, u_1, \ldots, u_n)$ *be a multicriteria game.*

- *We define for each* $i \in N$ $wB_i : X_{-i} \rightrightarrows X_i$ *where*

$$wB_i(x_{-i}) = \{x_i \in X_i \mid \nexists y_i \in X_i : u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})\}.$$

Call $X := \prod_{i \in N} X_i$*, and define*

$$wB : X \rightrightarrows X$$

the correspondence such that

$$x \longmapsto \prod_{i \in N} wB_i(x_{-i})$$

that is

$$wB\,(x) = (wB_1\,(x_{-1})\,, \ldots, wB_n\,(x_{-n})), \quad \forall x \in X;$$

- *We define for each* $i \in N$ $sB_i : X_{-i} \rightrightarrows X_i$ *where*

$$sB_i(x_{-i}) = \{x_i \in X_i \mid \nexists y_i \in X_i : u_i(y_i, x_{-i}) \geq u_i(x_i, x_{-i})\}\,.$$

Define

$$sB : X \rightrightarrows X$$

the correspondence such that

$$x \longmapsto \prod_{i \in N} sB_i\,(x_{-i})\,,$$

that is

$$sB\,(x) = (sB_1\,(x_{-1})\,, \ldots, sB_n\,(x_{-n})), \quad \forall x \in X.$$

Then wB and sB are called **best reply correspondences for weak and strong Pareto equilibria in pure strategies**, respectively.

The mixed extension of a finite multicriteria game is defined in the same way of the one of one criterion game (see Definition 1.6) where utility functions are $\mathbb{R}^m$-valued. The class of mixed extension of a finite multicriteria game is denoted with Γ^m_{mixed}.

Weak and strong Pareto equilibria of the mixed extension $\widetilde{G}_m$ of a finite multicriteria game G_m are defined similarly and are denoted with $wPE(\widetilde{G}_m)$ and with $sPE(\widetilde{G}_m)$.

As concern the best reply correspondences for weak and strong Pareto equilibria in mixed strategies we have the following definition.

Definition 2.4. *Let* $\widetilde{G}_m = (N, \Delta(X_1), \ldots, \Delta(X_n), \widetilde{u}_1, \ldots, \widetilde{u}_n)$ *be the mixed extension of a multicriteria game.*

- *We define for each* $i \in N$ $wB_i : \Delta(X_{-i}) \rightrightarrows \Delta(X_i)$ *where*

$$\widetilde{wB}_i(x_{-i}) = \{x_i \in \Delta(X_i) \mid \nexists y_i \in \Delta(X_i) : \widetilde{u}_i(y_i, x_{-i}) > \widetilde{u}_i(x_i, x_{-i})\}.$$

Call $\Delta(X) := \prod_{i \in N} \Delta(X_i)$*, and define*

$$\widetilde{wB} : \Delta(X) \rightrightarrows \Delta(X)$$

the correspondence such that

$$x \longmapsto \prod_{i \in N} \widetilde{wB}_i(x_{-i}),$$

that is

$$\widetilde{wB}\,(x) = (\widetilde{wB}_1\,(x_{-1})\,, \ldots, \widetilde{wB}_n\,(x_{-n})) \quad \forall x \in \Delta\,(X)\,;$$

- *We define for each* $i \in N$ $\widetilde{sB}_i : \Delta(X_{-i}) \rightrightarrows \Delta(X_i)$ *where*

$$\widetilde{sB}_i(x_{-i}) = \{x_i \in \Delta(X_i) \mid \nexists y_i \in \Delta(X_i) : \widetilde{u}_i(y_i, x_{-i}) \geq \widetilde{u}_i(x_i, x_{-i})\}.$$

Define

$$\widetilde{sB} : \Delta(X) \rightrightarrows \Delta(X)$$

the correspondence such that

$$x \longmapsto \prod_{i \in N} \widetilde{sB}_i(x_{-i}),$$

that is

$$\widetilde{sB}\,(x) = (\widetilde{sB}_1\,(x_{-1})\,, \ldots, \widetilde{sB}_n\,(x_{-n})) \quad \forall x \in \Delta\,(X)\,.$$

Then $\widetilde{wB}$ and $\widetilde{sB}$ are called **best reply correspondences for weak and strong Pareto equilibria in mixed strategies**, respectively.

Fixed points of $\widetilde{wB}$ and $\widetilde{sB}$ are weak and strong Pareto equilibria of $\widetilde{G}_m$, respectively as stated by the next theorem.

Theorem 2.1. *Let* $\widetilde{G}_m = (N, \Delta(X_1), \ldots, \Delta(X_n), \widetilde{u}_1, \ldots, \widetilde{u}_n)$ *be the mixed extension of a finite multicriteria game, and let* $\bar{x} \in \Delta(X)$ *be a strategy profile. Then*

- $\widetilde{x} \in wPE(\widetilde{G}_m)$ *iff* $\widetilde{x} \in \widetilde{wB}(\bar{x})$;
- $\widetilde{x} \in sPE(\widetilde{G}_m)$ *iff* $\widetilde{x} \in \widetilde{sB}(\bar{x})$.

Proof. It immediately follows from Definition 2.2 and Definition 2.4. □

Let us consider the next example taken from [117] (see also [124]).

Example 2.2. *Consider a bicriteria game G_2 with two players: an ispector (player I) who has to decide whether or not to inspect a factory (player II) to check if its production is hygienical. Each player has two strategies and two objectives which can be summarized in Table2.2. Payoff functions*

Table 2.2: Strategies and Objectives

	Strategies	Objectives
Inspector (I)	• inspect (I)	• minimize inspection costs
	• non inspect(NI)	• guarantee an acceptable level of hygiene in production
Factory(II)	• hygienical (H)	• minimize production costs
	• non hygienical (NH)	• achieve some level of hygienical production

are given below in Table2.3. Here $c > 1$ denotes the penalty that is imposed if the inspected production fails to be hygienical.

Table 2.3: Payoffs functions

I \ II	H	NH
I	$(-1,1)\ (-1,1)$	$\left(c-1,\frac{1}{2}\right)\ (-c-1,1)$
NI	$(0,1)\ (-1,1)$	$(0,0)\ (0,0)$

The first coordinate of the payoff to player I denotes the negative costs of inspection, the second coordinate specifies satisfaction with the hygienical situation. The first coordinate for the factory depicts extra negative production costs, the second represents the hygiene satisfaction level.
We have that $wPE(G) = \{(I,H), (I,NH), (NI,H)\}$, while $sPE(G) = \{(NI,H)\}$.
As regards Pareto equilibria in mixed strategies, let $p \in [0,1]$ the probability of player I playing I and let $q \in [0,1]$ the probability of player II playing H. Let $\widetilde{u}_1$ and $\widetilde{u}_2$ be the utility functions in mixed strategies of player I and II respectively. Then we have that

$$\widetilde{u}_1(p,q) = \left(pc - p - pqc, \frac{1}{2}p - \frac{1}{2}pq + q\right),$$

$$\widetilde{u}_2(p,q) = (-p - pc + pqc + pq - q, p - pq + q).$$

The best reply correspondence for weak Pareto equilibria in mixed strategies for player I is

$$\widetilde{wB}_I(q) = \{\bar{p} \in [0,1] \mid \nexists p \in [0,1] : \widetilde{u}_1(p,q) > \widetilde{u}_1(\bar{p},q)\}.$$

So we have

$$\widetilde{wB}_I(q) = \begin{cases} \{1\} & \text{if } 0 \leq q < 1 - \frac{1}{c} \\ & \\ [0,1] & \text{if } 1 - \frac{1}{c} \leq q \leq 1 \end{cases}$$

Similarly for player II we have

$$\widetilde{wB}_{II}(p) = \begin{cases} \{1\} & \text{if } \frac{1}{c+1} < p < 1 \\ & \\ [0,1] & \text{if } 0 \leq p \leq \frac{1}{c+1} \cup p = 1 \end{cases}$$

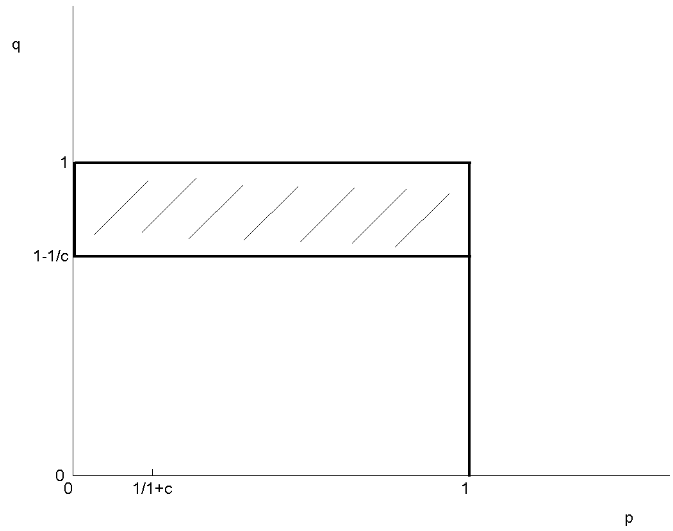


Figure 2.1: $\widetilde{wB}_I(q)$

Figures 2.1 and 2.2 represent best reply correspondences for weak Pareto equilibria in mixed strategies for player I and II respectively.
Then, as we can see in Figure2.3 we have that

$$wPE(\widetilde{G}_2) = \left(\left[0, \frac{1}{1+c}\right] \times \left[1 - \frac{1}{c}, 1\right]\right) \cup \left(\left(\frac{1}{c+1}, 1\right) \times \{1\}\right) \cup (\{1\} \times [0,1]).$$

Weak Pareto equilibria in mixed strategies occuring in this model are those in which there is full inspection (that is $p = 1$), those in which the factory produces in a hygienical way with probability $q = 1$ and those in which the chance upon inspection is small ($p \leq \frac{1}{c+1}$), but the production is neverthless hygienical with high probability ($q \geq 1 - \frac{1}{c}$). This last fact is due to the penalty $c > 1$ imposed if the checked production is not hygienical. Obviously if c increases the set of weak Pareto

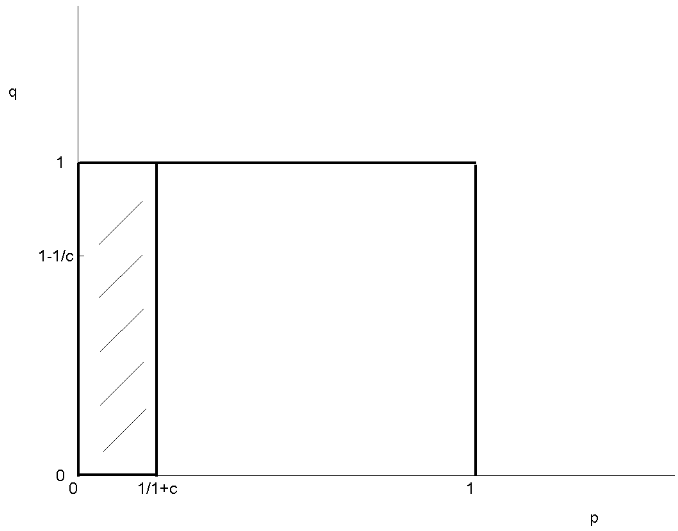


Figure 2.2: $\widetilde{wB_{II}}\,(p)$

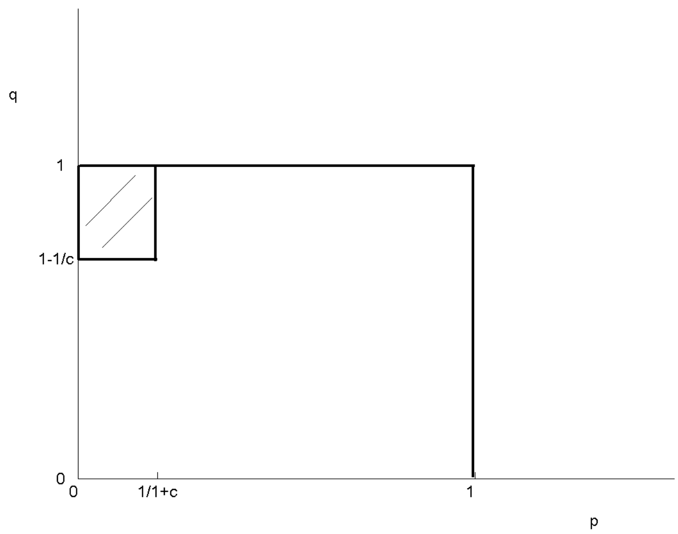


Figure 2.3: $wPE(\widetilde{G}_2)$

equilibria in mixed strategies shrinks because, from an interpretative point of view, the factory have to pay a higher penalty if the inspector finds that its production is not hygienical. From a mathematical point of view it is sufficient to observe that $\lim_{c\to+\infty} 1-\frac{1}{c}=1$ *and* $\lim_{c\to+\infty}\left(\frac{1}{c+1}\right)=0$.

Similar computations show

$$\widetilde{sB}_I(q) = \begin{cases} \{1\} & \text{if } 0 \le q \le 1 - \frac{1}{c} \\ [0,1] & \text{if } 1 - \frac{1}{c} < q < 1 \\ \{0\} & \text{if } q = 1 \end{cases}$$

and

$$\widetilde{sB}_{II}(p) = \begin{cases} \{1\} & \text{if } \frac{1}{c+1} \le p \le 1 \\ [0,1] & \text{if } 0 \le p < \frac{1}{c+1} \end{cases}$$

From which we have

$$sPE\left(\widetilde{G}_2\right) = \left(\{0\} \times \left(1 - \frac{1}{c}, 1\right]\right) \cup \left(\left(0, \frac{1}{c+1}\right) \times \left(1 - \frac{1}{c}, 1\right)\right).$$

In contrast to what happens for Nash equilibria, here it is not true that a Pareto equilibrium in pure strategy is also a Pareto equilibrium in mixed strategies.

Example 2.3. *Let us consider the finite bicriteria game G_2 with two players with utility functions given by the bimatrix in Table2.4.*

Table 2.4: Weak Pareto equilibria in pure strategies

I \ II	L	R
T	(2, 0) (4, 0)	(2, 0) (-1, -1)
M	(0, 2) (0, 4)	(0, 2) (-1, -1)
B	(0, 0) (0, 1)	(0, 0) (-1, -1)

Here player I has 3 pure strategies T, M and B, while player II has two pure strategies L and R. We can see that $wPE(G_2) = \{(T,L), (M,L), (B,L)\}$. Let us consider the mixed extension $\widetilde{G}_2 = (N, \Delta(X_1), \Delta(X_2), \widetilde{u}_1, \widetilde{u}_2)$ of the game G_2. Previously we can see that the strategy R is strongly dominated, so player II will assign probability 0 at the strategy R (and so 1 at the strategy L). Let $p = (p_1, p_2, p_3)$ where $p_3 = 1 - p_1 - p_2$ with $p_1, p_2 \in [0,1] = \Delta(X_1)$ such that p_1 is the probability that player I assigns to T, p_2 the probability for M and p_3 that one for B. By previous observation, player II has the unique mixed strategy $q = (1,0)$. Then if player I chooses T with probability $\frac{1}{2}$ and

M with probability $\frac{1}{2}$, we have that

$$\widetilde{u}_1\left(\left(\frac{1}{2},\frac{1}{2},0\right),(1,0)\right)=(1,1)>(0,0)=\widetilde{u}_1\left((0,0,1),(1,0)\right),$$

and

$$\widetilde{u}_2\left(\left(\frac{1}{2},\frac{1}{2},0\right),(1,0)\right)=(2,2)>(0,1)=\widetilde{u}_2\left((0,0,1),(1,0)\right).$$

We can see that the mixed strategy $((0,0,1),(1,0))$ corresponds to weak Pareto equilibrium in pure strategy (B,L). So the mixed strategy $((0,0,1),(1,0))$ is strongly dominated by the mixed strategy $\left(\left(\frac{1}{2},\frac{1}{2},0\right),(1,0)\right)$ and so it cannot be a weak Pareto equilibrium for $\widetilde{G}_2$.

2.2 Scalarization

Generally, there are many efficient points of a multicriteria problem. One of the most analyzed topics in multicriteria optimization is the scalarization of (2.1), namely how to build a scalar maximization problem, which leads one to find all the solutions of (2.1).

The classic scalarization (see [30] and [31]), which is called weight-method, consists in considering the following scalar maximum problem

$$\max_{x\in D}\sum_{k=1}^{r}\mu_k f_k(x), \tag{2.2}$$

where $\mu=(\mu_1,\ldots,\mu_r)\in\mathbb{R}^r_+,\ \sum_{k=1}^{r}\mu_k$.

Every solution of (2.2) is a weak Pareto solution of (2.1). Moreover, if, for a fixed weight-vector $\mu\geq 0$, (2.2) admits a unique solution, then it is a Pareto solution of (2.1). If $\mu_k>0,\ \forall k=1,\ldots,r$, then every solution of (2.2) is a strong Pareto solution of (2.1).

In the last part of this section, we are going to show the existence of weak and strong Pareto equilibria for mixed extensions of a finite multicriteria game. This result is proved through a particular kind of one criterion game which arises from the multicriteria game assigning non negative weights to different objectives. See for example [14].

In order to prove an existence theorem for Pareto equilibria we give the definition of scalarized game as follows [1].

Definition 2.5. *Let* $G_m = (N, X_1, \dots, X_n, u_1, \dots, u_n) \in \Gamma^m_{finite}$ *and let* $\lambda = (\lambda_1, \dots, \lambda_n)$ *where* $\lambda_i = (\lambda_{i1}, \dots, \lambda_{im}) \in \Delta_m$ *is a weight vector for player i' s objectives* $\forall i \in N$. *We define the weighted game*

$$G^\lambda = \left(N, X_1, \dots, X_n, v_1^\lambda, \dots, v_n^\lambda\right),$$

where $\forall i \in N,\ v_i^\lambda : X \longrightarrow \mathbb{R}$ *is defined in the following way:*

$$v_i^\lambda (x_1, \dots, x_n) \ = \ \sum_{k=1}^{m} \lambda_{ik} u_{ik} (x_1, \dots, x_n). \tag{2.3}$$

Theorem 2.2. *Let* $\widetilde{G}_m = (\Delta(X_1), \dots, \Delta(X_n), \widetilde{u}_1, \dots, \widetilde{u}_n) \in \Gamma^m_{mixed}$ *and we take* $\widetilde{x} \in \Delta(X)$. *Then*

- $\widetilde{x} \in wPE(\widetilde{G}_m)$ *iff for each* $i \in N$ *there exists a* $\lambda_i = (\lambda_{i1}, \dots, \lambda_{im}) \in \Delta_m$ *such that* $\widetilde{x} \in NE(\widetilde{G_\lambda})$;

- $\widetilde{x} \in sPE(\widetilde{G}_m)$ *iff for each* $i \in N$ *there exists a* $\lambda_i = (\lambda_{i1}, \dots, \lambda_{im}) \in \Delta^0_m$ *such that* $\widetilde{x} \in NE(\widetilde{G_\lambda})$

Proof. See [124]. □

2.3 Relation between Supermodular Multicriteria Games and Potential Multicriteria Games

In this section we want to extend the theory of supermodular games to potential multicriteria games.

Definition 2.6. *A supermodular multicriteria game*

$$G_m = (N, X_1, \dots, X_n, u_1, \dots, u_n)$$

is a tuple where

- $N = \{1, \dots, n\}$ *is a finite set of players;*

[1] Let us denote with Δ_n and with Δ_n^0 the following sets: $\Delta_n = \left\{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\right\}$, $\Delta_n^0 = \left\{x \in \mathbb{R}^n_{++} : \sum_{i=1}^n x_i = 1\right\}$.

- $\forall i \in N,\ \ X_i \subseteq \mathbb{R}^{m(i)}$*(for some* $m(i) \in \mathbb{N}$*) and* $X_i \neq \emptyset$ *is the strategy space of player* i, $\boldsymbol{X} = \prod_{i \in N} X_i$ *is the cartesian product of the strategy spaces;*

- $u_i : \boldsymbol{X} \longrightarrow \mathbb{R}^m$ *is the payoff function of player* i;

- $\forall i \in N,\ X_i$ *is a sublattice of* $\mathbb{R}^{m(i)}$;

- $\forall i \in N,\ u_i$ *have increasing differences on* $\boldsymbol{X}$, *i.e.*

  $\forall x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \boldsymbol{X}$ *such that* $x_i \geq y_i$, *we have*

$$u_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - u_i(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n)$$

$$\succeq u_i(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) - u_i(y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n);$$

- $\forall i \in N,\ u_i$ *is supermodular in the* $i-th$ *coordinate, i.e.*

  $\forall x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), z = (z_1, \ldots, z_n) \in \boldsymbol{X}$ *we have*

$$u_i(z_1, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_n) + u_i(z_1, \ldots, z_{i-1}, y_i, z_{i+1}, \ldots, z_n)$$

$$\preceq u_i(z_1, \ldots, z_{i-1}, x_i \wedge y_i, z_{i+1}, \ldots, z_n) + u_i(z_1, \ldots, z_{i-1}, x_i \vee y_i, z_{i+1}, \ldots, z_n).$$

Proposition 2.1. *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *be a twice differentiable function on* $\mathbb{R}^n$, *then* f *has increasing differences on* $\mathbb{R}^n$ *iff* $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq (0, \ldots, 0)$, *for each* $i, j = 1, \ldots, n$ *with* $i \neq j$.

Proof. It is a straight generalization of a result in [110] to vectorial functions. □

The following proposition is a generalization of Lemma 1 in [15].

Proposition 2.2. *Let* $G_m = (N, X_1, \ldots, X_n, u_1, \ldots, u_n)$ *be a potential multicriteria game with potential function* Π. *Then* $\forall i \in N$ *there exist functions* $f_{-i} : X_{-i} := \prod_{j \neq i} X_j \longrightarrow \mathbb{R}^m$ *such that*

$$u_i(x_i, x_{-i}) = \Pi(x_i, x_{-i}) + 2 f_{-i}(x_{-i}) \quad \forall x_i \in X_i,\ x_{-i} \in X_{-i}.$$

Proof. For each $x_i \in X_i$ and $x_{-i} \in X_{-i}$ let

$$f_{-i}(x_{-i}) = \frac{1}{2}\left[u_i(x_i, x_{-i}) - \Pi(x_i, x_{-i})\right].$$

Since Π is a potential function for G_m, $\forall x_i, y_i \in X_i$ and $\forall x_{-i} \in X_{-i}$, we have

$$u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = \Pi(x_i, x_{-i}) - \Pi(y_i, x_{-i})$$

or

$$\begin{aligned} u_i(x_i, x_{-i}) - \Pi(x_i, x_{-i}) &= u_i(y_i, x_{-i}) - \Pi(y_i, x_{-i}) \\ &= 2f_{-i}(x_{-i}). \end{aligned}$$

□

For each $i \in N$, the functions f_{-i} in Proposition 2.2 are called **separating functions**.

Let us consider a Cournot bicriteria game G_2 where the demand arises from a competitive market of two types of commodities.

Definition 2.7. *We define* $G_2 = (N, X, Y, u_1, u_2)$, *where*

- $N = \{1, 2\}$;
- $X = Y = \left[0, \frac{\delta}{2}\right]^2$ *where* δ *is a positive amount;*
- $u_i : X \times Y \longrightarrow \mathbb{R}^2,\ i = 1, 2$ *are the utility functions defined in the following way:*

Take $x = (x_1, x_2) \in X,\ y = (y_1, y_2) \in Y$, *and*

$$u_1(x, y) = (x_1(\delta - x_1 - y_1) - c_1, x_2(\delta - x_2 - y_2) - c_2),$$

and

$$u_2(x, y) = (y_1(\delta - x_1 - y_1) - c_1, y_2(\delta - x_2 - y_2) - c_2),$$

where $c_1, c_2 > 0$.

Remark 2.2. *We can note that* G_2 *is a particular case (more precisely is a deterministic case) of the Cournot game* Γ_c *defined in [75] and discussed in Chapter 3. Therefore* G_2 *is a potential game* *with potential function*

$$\begin{aligned}\Pi(x,y) &= \Big(-\left(x_1^2+x_2^2+y_1^2+y_2^2\right)-\frac{1}{2}\left(x_1y_1+x_1y_2+x_2y_1+x_2y_2\right)\\ &+\delta\left(x_1+y_1+x_2+y_2\right),\\ &-\left(x_1^2+x_2^2+y_1^2+y_2^2\right)-\frac{1}{2}\left(x_1y_1+x_1y_2+x_2y_1+x_2y_2\right)\\ &+\delta\left(x_1+y_1+x_2+y_2\right)\Big).\end{aligned}$$

As done in [15] for the scalar case, we put $\bar{x}=x$ and $\bar{y}=-y$ for each $x\in X,\ y\in Y$ and consider the game

$\bar{G}_2=\left(N,\bar{X},\bar{Y},\bar{u}_1,\bar{u}_2\right)$ where $\bar{X}=X,\ \bar{Y}=-Y=\left[-\frac{\delta}{2},0\right]^2$ and for each $\bar{x}\in\bar{X},\ \bar{y}\in\bar{Y}$,

$\bar{u}_1(\bar{x},\bar{y})=u_1(\bar{x},-\bar{y}),\ \bar{u}_2(\bar{x},\bar{y})=u_2(\bar{x},-\bar{y})$. The game $\bar{G}_2$ is strategically equivalent to game G_2. So we have that (x,y) is a weak [strong] Pareto equilibrium for G_2 iff $(x,-y)$ is a weak [strong] Pareto equilibrium for $\bar{G}_2$. In particular, since G_2 is a potential bicriteria game with potential function defined in Remark 2.2, also $\bar{G}_2$ is a potential bicriteria game with potential function $\bar{\Pi}$ given by $\bar{\Pi}(\bar{x},\bar{y})=\Pi(\bar{x},-\bar{y}),\ \forall\bar{x}\in\bar{X},\ \bar{y}\in\bar{Y}$.

Proposition 2.3. *Let G_2 be a Cournot bicriteria game defined as in Definition 2.7 and consider the game $\bar{G}_2$ as above. Then we have that $\bar{G}_2$ is a supermodular game.*

Proof. $\bar{X}$ and $\bar{Y}$ are sublattices of $\mathbb{R}^2$ because they are product of intervals. Moreover $\bar{u}_1$ and $\bar{u}_2$ have increasing differences properties on $\bar{X}\times\bar{Y}$, because by Proposition 2.1 we have

$$\frac{\partial^2\bar{u}_1}{\partial\bar{x}\partial\bar{y}}=(1,1)>(0,0),$$

and

$$\frac{\partial^2\bar{u}_2}{\partial\bar{x}\partial\bar{y}}=(1,1)>(0,0).$$

Finally it is simple to prove that $\bar{u}_1$ is supermodular in the first coordinate and $\bar{u}_2$ is supermodular in the second coordinate. □

Chapter 3

Bayesian Pareto Equilibria in Multicriteria Games

3.1 Introduction

Shapley in [102] introduced the concept of multicriteria games, that is games with vector payoffs, and studied their equilibrium points. Subsequently many papers have been published about this topics, as Borm, Tijs, van den Aarssen in [14] or Patrone, Pusillo, Tijs in [94]; in particular this last paper links the concept of multicriteria games with that of potential games introduced by Monderer and Shapley in [83]. On the other hand Harsanyi in [58] introduced games with incomplete information and he called them Bayesian games. In these games the players are not completely informed about the real-valued payoff function of the other players and there is an uncertainty about the characteristics of the players (or types). The existence of Bayesian Nash equilibria (BNE for short) in the case where the pure strategy spaces are finite have been proved in [58]. Later Kim and Yannelis in [67] proved existence theorems where the set of agent is an infinite set. Their model allows the individual's action set to depend on the states of nature and to be an arbitrary subset of an infinite dimensional space. Also Kitti and Mallozzi in [68] prove an existence theorem based on Corollary 1.1. Meirowitz in [79] shows that if type and action spaces are both non-empty, compact and convex subsets of a finite dimensional Euclidean space, agent utility functions are continuous in their type and action as well as the action of the other players, agent expected utility functions are strictly quasiconcave in the agent's action for every type, the set of rationalizable mappings from type to action have uniformly bounded slope and agent posterior beliefs are suitably continuous in their types, then BNE exist. Reny in [100] generalizes Athey's and McAdams results in [4] and [76] respectively, on the existence of monotone pure strategy equilibria in Bayesian games. Mallozzi, Pusillo and Tijs in [73] consider situations where one of the players may have an infinite set of pure strategies, one criterion and a finite number of types and get an existence theorem of approximate equilibria. Many Bayesian models have been studied recently, for example Einy *et al.* in [40] study conditions under which a Bayesian Cournot equilibrium exists and is unique, in an oligopoly, in [61] the authors prove the existence and uniqueness of a Bayesian Nash equilibrium in a public good economy with differential information regarding consumers income and preferences with incomplete information.

In this chapter we combine the concept of multicriteria games and the concept of Bayesian games: Bayesian multicriteria games are defined and some equilibrium concepts are discussed. Moreover we present the classical model of Cournot duopoly game in the sense of a Bayesian multicriteria game.

3.2 Bayesian Multicriteria Games (BMG)

A Bayesian multicriteria game is a tuple

$$\Gamma = (N, A_1, \dots, A_n, T_1, \dots, T_n, p, u_1, \dots, u_n)$$

(for short $\Gamma = (N, \mathbf{A}, \mathbf{T}, p, u)$) where

- $N = \{1, \dots, n\}$ is a finite set of players;
- $\forall i \in N$, the action space is a metric space A_i, $A = \prod_{i \in N} A_i$ is the cartesian product of the action spaces;
- $\forall i \in N$, $T_i \neq \emptyset$ is a finite set and represents the type space of player i, $T = \prod_{i \in N} T_i$ is the cartesian product of the type spaces;
- $\forall i \in N$, a strategy for player i is a function $x_i : T_i \longrightarrow A_i$;

- p is a probability distribution on the set T;
- $u_i : A \times T \longrightarrow \mathbb{R}^m$ is the payoff function which assigns to player i the payoff $u_i(a_1, \ldots, a_n, t_1, \ldots, t_n)$ given that the players $1, 2, \ldots, n$ have type $t_1, \ldots, t_n$ and choose actions $a_1, \ldots, a_n$ respectively.

Assume that it is common knowledge that each player i belongs to one of the possible types $t_i \in T_i$. Each player knows only her own type t_i. The beliefs embodied in the description of a type $t_i \in T_i$ must include subjective probability distributions over the sets $T^{-i} := \prod_{k \neq i} T_k$. These probabilities $p_i\left(t^{-i}|t_i\right)$ represent uncertainty about players' type against whom i is playing. If these players types are independent, then p_i is independent of t_i.

Recall that we assume that initially Nature draws a vector of types $(t_1, \ldots, t_n)$ according to the prior probability distribution p. Once Nature reveals t_i to player i, she is able to compute the belief $p_i(t_{-i}|t_i)$ using the Bayes' rule:

$$p_i(t_{-i}|t_i) = \frac{p(t_{-i}, t_i)}{p(t_i)}.$$

Of course, if players' types are independent, $p_i(t_{-i}|t_i)$ does not depend on t_i, but the belief is still derived from the prior distribution p.

A play of such a game proceeds as follows: before the types are announced each player i chooses a strategy $x_i \in A_i^{T_i}$.If the type profile is $t = (t_1, \ldots, t_n)$ then player i's payoff is
$u_i(x_1(t_1), x_2(t_2), \ldots, x_n(t_n), t_1, \ldots, t_n)$.

The a priori expected payoff for player i when the players use strategies $x_1, \ldots, x_n$ respectively is a function $U_i : A_1^{T_1} \times \cdots \times A_n^{T_n} \longrightarrow \mathbb{R}^m$ such that

$$U_i(x_1, \ldots, x_n) = \sum_{t \in T} p(t) u_i(x_1(t_1), x_2(t_2), \ldots, x_n(t_n), t_1, \ldots, t_n),$$

that is U_i is a vectorial sum of $\mathbb{R}^m$, being $p(t)$ the probability distribution of player i when her type is t_i.

Definition 3.1. *Let $\Gamma = (N, A, T, p, u)$ be a Bayesian multicriteria game. We say that a strategy profile $\widehat{x} = (\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n) \in A_1^{T_1} \times A_2^{T_2} \times \ldots A_n^{T_n}$ is a*

- ***Weak Bayesian Pareto equilibrium*** *(for short wBPE) for the game* Γ *if* $\forall i \in N$, $\nexists x_i \in A_i^{T_i}$ *such that*
$$U_i(x_i, \widehat{x}_{-i}) > U_i(\widehat{x});$$
- ***Strong Bayesian Pareto equilibrium*** *(for short sBPE) for the game* Γ *if* $\forall i \in N$, $\nexists x_i \in A_i^{T_i}$ *such that*
$$U_i(x_i, \widehat{x}_{-i}) \geq U_i(\widehat{x}).$$

We denote with $wBPE(\Gamma)$ and with $sBPE(\Gamma)$ the set of weak Bayesian Pareto equilibria and strong Bayesian Pareto equilibria respectively for Γ.

Remark 3.1. *If a strategy profile* $\widehat{x}$ *is a strong Bayesian Pareto equilibrium then it is a weak Bayesian Pareto equilibrium, but the viceversa does not hold. See the Example 3.1.*

Remark 3.2. *If in* Γ *the type spaces are trivial, i.e.* $|T_1| = |T_2| = \cdots = |T_n| = 1$ *then we can write* (N, A, u) *and we obtain a multicriteria game with complete information and* $wBPE$ *and* $sBPE$ *boil down to* wPE *(weak Pareto equilibria) and* sPE *(strong Pareto equilibria) respectively.*

A Bayesian multicriteria game is called a finite game if A_i are finite sets for all $i \in N$.

We defined a strategy of player i as a function $x_i : T_i \longrightarrow A_i$. For semplicity in the following examples, in place of $x_i(t_i)$ we will write the corresponding action in A_i.

Let us consider the following example:

Example 3.1. *Let* $\Gamma = (N, \boldsymbol{A}, \boldsymbol{T}, p, u)$ *be a finite Bayesian multicriteria game where:*

- $N = \{1, 2\};$
- $A_1 = \{a_1, b_1\},\ A_2 = \{a_2, b_2\};$
- $T_1 = \{t_1^1\},\ T_2 = \{t_2^1, t_2^2\}.$

The functions $u_1, u_2 : (A_1 \times A_2) \times (T_1 \times T_2) \longrightarrow \mathbb{R}^2$ *are represented by the bimatrices Table 3.1 and Table 3.2.*

Table 3.1: u_1

$1 \setminus 2$ type 1	a_2	b_2
a_1	(1,0) (2, 3)	(0, 1) (1, 0)
b_1	(0, 1) (4, 3)	(1, 0) (3, 1)

Table 3.2: u_2

$1 \setminus 2$ type 2	a_2	b_2
a_1	(1,0) (1, 2)	(0, 1) (3, 4)
b_1	(0, 1) (2, 3)	(1, 0) (3, 2)

The first table represents the case in which player 2's type is t_2^1*, whereas the second one represents the case in which player 2's type is* t_2^2*. In each entry of any bimatrix the first pair is the payoff of player 1, while the second one is the payoff of player 2. In this example each player has two criteria. We can note that the player 1's payoffs are the same in both matrices. In the first case* a_2 *is a dominant strategy for player 2.*

Now we suppose that player 2's type is t_2^1 *with probability* $P \in [0, 1]$ *and type is* t_2^2 *with probability* $1 - P$*. Then the a priori expected payoff functions* U_1 *and* U_2 *are given in the bimatrix Table 3.3, where* x_1, y_1 *are the strategies of player 1, while* x_2, y_2 *are the strategies of player 2. 3.3. Now we want to compute the strong and weak Bayesian Pareto equilibria for this game depending on* P*.*

Table 3.3: A priori expected payoff functions

$U_1 \setminus U_2$	x_2	y_2
x_1	(1,0) (P+1, P+2)	(0, 1) (3-2P, 4-4P)
y_1	(0, 1) (2P+2, 3)	(1, 0) (3, 2-P)

- *If* $P > \frac{2}{3}$ $wBPE = sBPE = \{(a_1, a_2), (b_1, a_2)\}$;
- *If* $\frac{1}{2} < P < \frac{2}{3}$ $wBPE = sBPE = \{(a_1, a_2), (a_1, b_2), (b_1, a_2)\}$;

- *If* $\frac{2}{5} < P < \frac{1}{2}$ $wBPE = sBPE = \{(a_1, a_2), (a_1, b_2), (b_1, a_2), (b_1, b_2)\}$;
- *If* $P < \frac{2}{5}$ $wBPE = sBPE = \{(a_1, b_2), (b_1, a_2), (b_1, b_2)\}$;
- *If* $P = \frac{2}{3}$ $wBPE = \{(a_1, a_2), (b_1, a_2)\}$ $sBPE = \{(b_1, a_2)\}$*;*
- *If* $P = \frac{2}{5}$ $wBPE = \{(a_1, b_2), (b_1, a_2), (b_1, b_2)\}$ $sBPE = \{(b_1, a_2), (b_1, b_2)\}$*;*
- *If* $P = \frac{1}{2}$ $wBPE = \{(a_1, a_2), (a_1, b_2), (b_1, a_2)\}$ $sBPE = \{(a_1, a_2), (a_1, b_2)\}$.

In the Example 3.1 there are many equilibria. Generally the existence of equilibria is not ensured. Indeed let us consider the following example:

Example 3.2. *Let* $\Gamma = (N, \boldsymbol{A}, \boldsymbol{T}, p, u)$ *be a finite Bayesian multicriteria game where:*

- $N = \{1, 2\}$;
- $A_1 = \{a_1, b_1\}$, $A_2 = \{a_2, b_2\}$;
- $T_1 = \{t_1^1\}$, $T_2 = \{t_2^1, t_2^2\}$.

The functions $u_1, u_2 : (A_1 \times A_2) \times (T_1 \times T_2) \longrightarrow \mathbb{R}^2$ *are represented by the bimatrices Table 3.4 and Table 3.5.*

Table 3.4: u_1

1\ 2 type 1	a_2	b_2
a_1	(2,3) (1, -1)	(4, 1) (2, 1)
b_1	(1, 2) (3, 2)	(5, 2) (2, 1)

Table 3.5: u_2

1 \ 2 type 2	a_2	b_2
a_1	(2,3) (2, -1)	(4, 1) (3, 0)
b_1	(1, 2) (1, 2)	(5, 2) (0, 0)

Also we suppose that player 1 assigns probability $P \in [0, 1]$ *to type* t_2^1*, and probability* $1 - P$ *to type* t_2^2*, so we have that* $p(t_1^1, t_2^1) = P$ *and* $p(t_1^1, t_2^2) = 1 - P$*. Then the values of* U_1 *and* U_2 *are given* *in the bimatrix Table 3.6.*

Table 3.6: A priori expected payoff functions

$U_1 \backslash U_2$	x_2	y_2
x_1	(2,3) (-P+2, -1)	(4, 1) (-P+3, P)
y_1	(1, 2) (2P+1, 2)	(5, 2) (2P, P)

It is easy to prove that there are not weak Bayesian Pareto equilibria or strong Bayesian Pareto equilibria. In effect we can note that $U_1(x_1, x_2) > U_1(y_1, x_2)$ *and* $U_1(y_1, y_2) > U_1(x_1, y_2)$. *So possibly Bayesian Pareto equilibria are* $(x_1, x_2), (y_1, y_2)$. *But, since* $U_2(x_1, y_2) = (3 - P, P) > (2 - P, -1) = U_2(x_1, x_2)$ *and* $U_2(y_1, x_2) = (2P + 1, 2) > (2P, P) = U_2(y_1, y_2)$, *for each* $P \in [0, 1]$, *they are not Bayesian Pareto equilibria.*

3.3 Bayesian Potential Multicriteria Games (BPMG)

In this section we define a Bayesian potential multicriteria game.

Definition 3.2. *Let* $\Gamma = (N, \boldsymbol{A}, \boldsymbol{T}, p, u)$ *be a Bayesian multicriteria game with a priori expected payoff* U_i *for all player* $i \in N$, *we say that* Γ *is a Bayesian potential multicriteria game (BPMG for short) if there exists a map* $\Pi : A_1^{T_1} \times \cdots \times A_n^{T_n} \longrightarrow \mathbb{R}^m$ *such that* $\forall i \in N$, $x_i, y_i \in A_i^{T_i}$ *and* $\forall x_{-i} \in A_{-i}^{T_{-i}} := \prod_{j \in N \setminus \{i\}} A_j^{T_j}$, *we have*

$$U_i(x_i, x_{-i}) - U_i(y_i, x_{-i}) = \Pi(x_i, x_{-i}) - \Pi(y_i, x_{-i}). \tag{3.1}$$

We call Π a potential function.

Example 3.3. *Let* $\Gamma = (N, \boldsymbol{A}, \boldsymbol{T}, p, u)$ *be a finite Bayesian multicriteria game where:*

- $N = \{1, 2\}$;
- $A_1 = \{a_1, b_1\}$, $A_2 = \{a_2, b_2\}$;
- $T_1 = \{t_1^1\}$, $T_2 = \{t_2^1, t_2^2\}$.

The functions $u_1, u_2 : (A_1 \times A_2) \times (T_1 \times T_2) \longrightarrow \mathbb{R}^2$ *are represented by the bimatrices Table 3.7 and Table 3.8.*

Table 3.7: u_1

$1 \setminus 2$ type 1	a_2	b_2
a_1	(3,4) (3, 2)	(4, 3) (2, 3)
b_1	(0, 5) (1, 1)	(1, 2) (0, 0)

The first one represents the case in which player 2's type is t_2^1*, whereas the second one represents the case in which player 2's type is* t_2^2*. We can note that the player 1's payoffs are the same in both matrices.*

The value of U_1 *and* U_2 *are given in the bimatrix Table 3.9.*

It is easy to see that a potential Π *for this game is given by the matrix Table 3.10.*

We can see that the strategies $(x_1, x_2), (x_1, y_2), (y_1, x_2)$ *are sBPE for* Γ*, for all* $P \in [0, 1]$*.*

Definition 3.3. *Let* $\Gamma = (N, \boldsymbol{A}, \boldsymbol{T}, p, u)$ *be a Bayesian multicriteria game, we say that* Γ *is a Bayesian coordination game if* $u_i = u_j \ \ \forall i \neq j$*.*

Clearly, if Γ is a Bayesian coordination game then $U_i = U_j \ \forall i \neq j$. For such a game there is a potential $\Pi : A_1^{T_1} \times \cdots \times A_n^{T_n} \longrightarrow \mathbb{R}^m$. Take, for example, $\Pi = U_1$.

Definition 3.4. *Let* $\Gamma = (N, \boldsymbol{A}, \boldsymbol{T}, p, u)$ *be a Bayesian multicriteria game, we say that* Γ *is a Bayesian dummy game if*

$$u_i\left(x_i\left(t_i\right), x_{-i}\left(t_j\right), t\right) = u_i\left(y_i\left(t_i\right), x_{-i}\left(t_j\right), t\right)$$

$\forall t \in T \ \ \forall i \in N, \ \forall x_i, y_i \in A_i^{T_i}, \ \ \forall x_{-i} \in A_{-i}^{T_{-i}} \ \ \forall j \neq i.$

Table 3.8: u_2

$1 \setminus 2$ type 2	a_2	b_2
a_1	(3,4) (6, 4)	(4, 3) (4, 4)
b_1	(0, 5) (2, 2)	(1, 2) (0, 0)

Table 3.9: A priori expected payoff functions

$U_1 \backslash U_2$	x_2	y_2
x_1	(3,4) (6-3P, 4-2P)	(4, 3) (4-2P, 4-P)
y_1	(0, 5) (2-P, 2-P)	(1, 2) (0, 0)

Table 3.10: A potential function

Π	x_2	y_2
x_1	(3,4)	(1+P, 4+P)
y_1	(0, 5)	(P-2, 3+P)

For a dummy game, since u_i does not depend on the i-th component, for each $i \in N$ we let $u_i(x_i, x_{-i}) = d_i(x_{-i}), \forall x_i, x_{-i}$ and we denote by $\Gamma^d = (N, \mathbf{A}, \mathbf{T}, p, d)$ a dummy game. If Γ is a Bayesian dummy game then $U_i(x_i, x_{-i}) = U_i(y_i, x_{-i}) \quad \forall i \in N, x_i, y_i \in A_i^{T_i}, \quad \forall x_{-i} \in A_{-i}^{T_{-i}}$, and it has a potential $\Pi : A_1^{T_1} \times \cdots \times A_n^{T_n} \longrightarrow \mathbb{R}^m$, which is $\Pi = (0, \ldots, 0)$.

Remark 3.3. *Let $\Gamma^j = \left(N, \boldsymbol{A}, \boldsymbol{T}, p, u^j\right)$ be a finite set of Bayesian potential multicriteria games with potential $\Pi^j \quad \forall j = 1, \ldots, k$, then the Bayesian multicriteria game $\Gamma = \left(N, \boldsymbol{A}, \boldsymbol{T}, p, \sum_{j=1}^k u^j\right)$ is a Bayesian potential multicriteria game with potential $\sum_{j=1}^k \Pi^j$.*

Remark 3.4. *By the decomposition theorem (see [16], [94]), a Bayesian multicriteria game $\Gamma = (N, \boldsymbol{A}, \boldsymbol{T}, p, u)$ has a potential function iff there exist a coordination game $\Gamma^\pi = (N, \boldsymbol{A}, \boldsymbol{T}, p, \pi)$ and a dummy game $\Gamma^d = (N, \boldsymbol{A}, \boldsymbol{T}, p, d)$ such that $U_i = \Pi + D_i \ \forall i \in N$, where with U_i, Π, D_i we denote the a priori expected functions of u_i, π, d_i, respectively. Moreover Π is a potential function of Γ.*

Theorem 3.1. *Let $\Gamma = (N, \boldsymbol{A}, \boldsymbol{T}, p, u)$ be a finite Bayesian potential multicriteria game, then there exists a strategy profile $\widehat{x} \in A_1^{T_1} \times \cdots \times A_n^{T_n}$ such that $\widehat{x}$ is a sBPE for Γ.*

Proof. Let $\Pi : A_1^{T_1} \times \cdots \times A_n^{T_n} \longrightarrow \mathbb{R}^m$ be a potential for Γ, with $\Pi = (\Pi_1, \ldots, \Pi_m)$ where

$$\Pi_k : A_1^{T_1} \times \cdots \times A_n^{T_n} \longrightarrow \mathbb{R} \quad \forall k = 1, \ldots, m.$$

Since Γ is a finite game, $argmax_{y \in A_1^{T_1} \times \cdots \times A_n^{T_n}} \sum_{k=1}^m \Pi_k(y) \neq \emptyset$. Take

$$x \in argmax_{y \in A_1^{T_1} \times \cdots \times A_n^{T_n}} \sum_{k=1}^m \Pi_k(y). \tag{3.2}$$

Then x is a $sBPE$ for Γ. Indeed, suppose that x is not a $sBPE$ for Γ, then there exists $i \in N$ and $y_i \in A_i^{T_i}$ such that $U_i\left(y_i, x_{-i}\right) \geq U_i\left(x_i, x_{-i}\right)$. But then $\Pi\left(y_i, x_{-i}\right) - \Pi\left(x_i, x_{-i}\right) \geq 0$, so $\sum_{k=1}^{m} \Pi_k\left(y_i, x_{-i}\right) - \Pi_k\left(x_i, x_{-i}\right) > 0$, which is in contradiction with condition (3.2). □

Clearly, by Remark 3.1, $\widehat{x}$ is also a wBPE for Γ.

Theorem 3.2. *Let $\Gamma = (N, \boldsymbol{A}, \boldsymbol{T}, p, u)$ be a Bayesian potential multicriteria game with potential* Π. *Then*

$$sBPE\left(\Gamma\right) = sBPE\left(\Gamma^{\pi}\right),$$

where $\Gamma^{\pi} = \left(N, \boldsymbol{A}, \boldsymbol{T}, p, \pi\right)$.

Proof. It follows from Remark 3.3. □

An economic application of the concept of Bayesian multicriteria game in particular case of one criterion is studied in [93].

3.3.1 Approximate Bayesian Pareto Equilibria

Definition 3.5. *Let $\Gamma = (N, \boldsymbol{A}, \boldsymbol{T}, p, u)$ be a Bayesian multicriteria game. We say that a strategy profile $\widehat{x} = \left(\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n\right) \in A_1^{T_1} \times A_2^{T_2} \times \cdots \times A_n^{T_n}$ is a*

- ***Approximate Bayesian Pareto equilibrium*** *(for short ϵBPE) for the game Γ if $\forall i \in N,\ \forall \epsilon > 0$, we have that $\widehat{x}_i \in \epsilon B\left(\widehat{x}_{-i}\right)$, where*

$$\epsilon B\left(\widehat{x}_{-i}\right) \quad = \quad \left\{x_i \in X_i | U_i\left(y_i, x_{-i}\right) \notin U_i\left(x_i, x_{-i}\right) + \epsilon\mathbb{R}_{+}^{m}\right\},$$

with $\epsilon\mathbb{R}_{+}^{m} = \mathbb{R}_{+}^{m} \setminus \left\{[0, \epsilon]^m\right\}$, where $\mathbb{R}_{+}^{m} = \left\{(x_1, \ldots, x_m)\, | x_i \geq 0\ i = 1, \ldots, m\right\}$.

We denote with $\epsilon BPE\left(\Gamma\right)$ the set of approximate Bayesian Pareto equilibria for Γ.

Definition 3.6. *For $f : A_1^{T_1} \times \cdots \times A_n^{T_n} \longrightarrow \mathbb{R}$, we define*

$$\begin{aligned} & argsup^{\epsilon}_{x \in A_1^{T_1} \times \cdots \times A_n^{T_n}} f\left(x\right) \\ = & \left\{ y \in A_1^{T_1} \times \cdots \times A_n^{T_n} | f\left(y\right) \geq \sup_{x \in A_1^{T_1} \times \cdots \times A_n^{T_n}} f\left(x\right) - \epsilon \right\}. \end{aligned}$$

Example 3.4. *Consider the Example 3.2 in which there are not weak (strong) Bayesian Pareto equilibria and let us compute the ϵBPE of the game Γ. With trivial computations we get:*

If $P > \frac{1}{2}$,

(1) $1 < \epsilon \leq 2 - P,\ \epsilon BPE\,(\Gamma) = \{(a_1, b_2)\,,(b_1, a_2)\}\,;$

(2) $2 - P < \epsilon \leq P + 1,\ \epsilon BPE\,(\Gamma) = \{(a_1, b_2)\}\,;$

(3) $\epsilon > P + 1,\ \epsilon BPE\,(\Gamma) = \emptyset.$

If $P < \frac{1}{2}$,

(1) $1 < \epsilon \leq P + 1,\ \epsilon BPE\,(\Gamma) = \{(a_1, b_2)\,,(b_1, a_2)\}\,;$

(2) $P + 1 < \epsilon \leq 2 - P,\ \epsilon BPE\,(\Gamma) = \{(b_1, a_2)\}\,;$

(3) $\epsilon > 2 - P,\ \epsilon BPE\,(\Gamma) = \emptyset.$

If $P = \frac{1}{2}$,

(1) $1 < \epsilon < \frac{3}{2},\ \epsilon BPE\,(\Gamma) = \{(a_1, b_2)\,,(b_1, a_2)\}\,;$

(2) $\epsilon > \frac{3}{2},\ \epsilon BPE\,(\Gamma) = \emptyset;$

(3) $\epsilon = \frac{3}{2},\ \epsilon BPE\,(\Gamma) = \{(a_1, a_2)\,,(a_1, b_2)\,,(b_1, a_2)\,,(b_1, b_2)\}\,.$

- *If* $\epsilon < 1,\ \epsilon BPE\,(\Gamma) = \emptyset;$
- *If* $\epsilon = 1,\ \epsilon BPE\,(\Gamma) = \{(a_1, a_2)\,,(a_1, b_2)\,,(b_1, a_2)\,,(b_1, b_2)\}\,.$

The following theorem extends the Theorem 3.1 in [73] to multicriteria games, and to Bayesian games the Theorem 3.6 in [94].

Theorem 3.3. *Let $\Gamma = (N, \boldsymbol{A}, \boldsymbol{T}, p, u)$ be a Bayesian potential multicriteria game. Suppose that the potential function is upper bounded, then there exists a strategy profile*
$\widehat{x} \in A_1^{T_1} \times \cdots \times A_n^{T_n}$ *such that $\widehat{x}$ is a ϵBPE for Γ, for all $\epsilon > 0$.*

Proof. Let $\Pi : A_1^{T_1} \times \cdots \times A_n^{T_n} \longrightarrow \mathbb{R}^m$ be a potential for Γ, with $\Pi = (\Pi_1, \ldots, \Pi_m)$ where

$$\Pi_k : A_1^{T_1} \times \cdots \times A_n^{T_n} \longrightarrow \mathbb{R} \ \forall k = 1, \ldots, m.$$

Since Π is upper bounded, $argsup^{\epsilon}_{y \in A^T} \sum_{k=1}^m \Pi_k (y) \neq \emptyset$. Take

$$x \in argsup^{\epsilon}_{y \in A_1^{T_1} \times \cdots \times A_n^{T_n}} \sum_{k=1}^{m} \Pi_k (y) . \tag{3.3}$$

Then x is a ϵBPE for Γ. Indeed, suppose that x is not a ϵBPE for Γ, then let be $i \in N$ and $y_i \in A_i^{T_i}$ such that $U_i (y_i, x_{-i}) \in U_i (x_i, x_{-i}) + \mathbb{R}^m_{+,\epsilon}$. But then

$$\Pi (y_i, x_{-i}) - \Pi (x_i, x_{-i}) = U_i (y_i, x_{-i}) - U_i (x_i, x_{-i}) \in \mathbb{R}^m_{+,\epsilon},$$

so $\sum_{k=1}^m \Pi_k (y_i, x_{-i}) - \Pi_k (x_i, x_{-i}) > \epsilon$, which is in contradiction with condition (3.3). □

Remark 3.5. *Let* $\Gamma = (N, \boldsymbol{A}, \boldsymbol{T}, p, u)$ *be a Bayesian potential multicriteria game with potential* Π. *Then* $\forall \epsilon \geq 0$, *we have*

$$\epsilon BPE (\Gamma) = \epsilon BPE (\Gamma^{\pi}) ,$$

where $\Gamma^{\pi} = (N, \boldsymbol{A}, \boldsymbol{T}, p, \pi)$.

3.4 Existence results: the scalarization approach

In the setting of Section 3.2, we consider the Bayesian multicriteria game Γ, and as in [68], we use the following additional assumptions on the Bayesian game Γ. For every player i :

(A1) $u_i (\cdot, t)$ is continuous $\forall t \in T \ \ \forall i \in N$.

(A2) A_i is a compact and convex set, and $u_i (\cdot, a_{-i}, t)$ are quasiconcave $\forall a_{-i} \in A_{-i}$, $\forall t \in T$, $\forall i \in N$.

In the following we apply the most popular existing methods for solving multiobjective optimizations problems to Bayesian multicriteria games. See for example [80].

In order to prove an existence theorem for Bayesian Pareto equilibria (see Definition 3.1) we give the definition of scalarized game as follows

Definition 3.7. *Let* $\Gamma = (N, A_1, \ldots, A_n, T_1, \ldots, T_n, p, u_1, \ldots, u_n)$ *be a Bayesian multicriteria game and let* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *where* $\lambda_i = (\lambda_{i1}, \ldots, \lambda_{im}) \in \Delta_m$ *is a weight vector for player i' s objectives* $\forall i \in N$. *We define the weighted game*

$$\Gamma^\lambda = \left(N, A_1, \ldots, A_n, T_1, \ldots, T_n, p, v_1^\lambda, \ldots, v_n^\lambda\right),$$

where $\forall i \in N,\ v_i^\lambda : A \times T \longrightarrow \mathbb{R}$ *is defined in the following way:*

$$v_i^\lambda (a_1, \ldots, a_n, t_1, \ldots, t_n) = \sum_{k=1}^{m} \lambda_{ik} u_{ik} (a_1, \ldots, a_n, t_1, \ldots, t_n). \tag{3.4}$$

Remark 3.6. *We recall that the inner product, denoted by* $\langle \cdot, \cdot \rangle$ *of two vectors* $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ *is a real number given*

$$\langle a, b \rangle = \sum_{i=1}^{n} a_i b_i.$$

Remark 3.7. *The a priori expected payoff* $V_i^{\lambda_i}$ *for player i can be written in this way:*

$$\begin{aligned} V_i^{\lambda_i} (x_1, \ldots, x_n) &= \sum_{t \in T} \sum_{k=1}^{m} \lambda_{ik} u_{ik} (x_1 (t_1), \ldots, x_n (t_n)) p (t) \\ &= \sum_{k=1}^{m} \left[\lambda_{ik} \sum_{t \in T} u_{ik} (x_1 (t_1), \ldots, x_n (t_n)) p (t) \right] \\ &= \langle \lambda_i, U_i (x_1, \ldots, x_n) \rangle. \end{aligned}$$

The next theorem links the Bayesian Pareto equilibria for the game Γ to the Bayesian Nash equilibria for the game Γ^λ.

Theorem 3.4. *Let* $\Gamma = (N, \boldsymbol{A}, \boldsymbol{T}, p, u)$ *be a Bayesian multicriteria game which satisfies Assumptions (A1) and (A2), such that* $A_1^{T_1} \times \cdots \times A_n^{T_n}$ *is a convex subset. Let* $\widehat{x} \in A_1^{T_1} \times \cdots \times A_n^{T_n}$, *then*

- $\widehat{x}$ *is a wBPE for* Γ *iff for all* $i \in N$ *exists* $\tilde{\lambda}_i \in \Delta_m$ *such that* $\widehat{x}$ *is a BNE for* $\Gamma^{\tilde{\lambda}}$;

Proof. Let $\widehat{x}$ be a wBPE for Γ then $\forall i \in N,\ \nexists x_i \in A_i^{T_i}$ such that $U_i (x_i, \widehat{x}_{-i}) > U_i (\widehat{x})$. Then $U_i (\widehat{x})$ is weak Pareto optimal in $\mathbb{R}^m$. By Theorem 10.1 pages 117-118 in [124] and for the convexity of $\mathbb{R}^m$ and of $A_1^{T_1} \times \cdots \times A_n^{T_n}$ we have that $\forall i \in N,\ \exists \tilde{\lambda}_i \in \Delta_m$ such that $\forall z \in A_1^{T_1} \times \cdots \times A_n^{T_n}$ $V_i^{\tilde{\lambda}_i} (\widehat{x}) \geq V_i^{\tilde{\lambda}_i} (z)$

because of the Remark 3.7. It follows that $\widehat{x}$ is a BNE for $\Gamma^{\tilde{\lambda}}$.

On the other hand, we suppose by contradiction that $\widehat{x}$ is not a $wBPE$ of Γ. Then $\exists i \in N, \exists y_i \in A_i^{T_i}$ such that

$$U_i\left(y_i, \widehat{x}_{-i}\right) > U_i\left(\widehat{x}\right). \tag{3.5}$$

For hypothesis $\exists \tilde{\lambda}_i \in \Delta_m$ such that

$$V_i^{\tilde{\lambda}_i}\left(\widehat{x}\right) \geq V_i^{\tilde{\lambda}_i}\left(y_i, \widehat{x}_{-i}\right). \tag{3.6}$$

So the conditions (3.5) and (3.6) lead to a contradiction.

□

The next corollary ensures the existence of Bayesian Nash equilibria for the weighted game Γ^{λ}.

Corollary 3.1. *If v_i^{λ} is continuous, A_i is a compact and convex set and $v_i^{\lambda}\left(\cdot, a_{-i}, t\right)$ are quasiconcave $\forall a_{-i} \in A_{-i}$ $\forall t \in T$, $\forall i \in N$, then Γ^{λ} possesses a Bayesian equilibrium $\forall \lambda_i \in \Delta_m$.*

Proof. See Proposition 1 in [68]. □

Finally, Corollary 3.2 ensures the existence of $wBPE$ for a Bayesian multicriteria game.

Corollary 3.2. *If v_i^{λ} is continuous, A_i is a compact and convex set and $v_i^{\lambda}\left(\cdot, a_{-i}, t\right)$ are quasiconcave $\forall a_{-i} \in A_{-i}$ $\forall t \in T$, $\forall i \in N$, and if $A_1^{T_1} \times \cdots \times A_n^{T_n}$ is a convex subset then*

- *$\forall i \in N$, $\exists \tilde{\lambda}_i \in \Delta_m$ such that $\widehat{x}^{\tilde{\lambda}}$ is a $wBPE$ for the multicriteria game Γ.*

Proof. It follows from Theorem 3.4. □

A question of interest in economics is how the optimal choice changes as a parameter changes. We know that, $\forall i \in N$, if $V_i^{\lambda_i}$ is twice continuously differentiable with respect to x_i, given a BNE profile $\widehat{x} = \left(\widehat{x}_1, \ldots, \widehat{x}_n\right)$ it must satisfy $\forall i \in N$ the condition

$$\frac{\partial}{\partial x_i} V_i^{\lambda_i}\left(\widehat{x}\right) = 0 \tag{3.7}$$

and $\forall j = 1, \ldots, m$

$$\frac{\partial^2}{\partial x_i^2} V_i^{\lambda_{ij}} (\widehat{x}) < 0 \tag{3.8}$$

So we have $\forall i = 1, \ldots, n \ \ \forall j = 1, \ldots, m,$

$$\frac{\partial^2}{\partial x_i^2} V_i^{\lambda_i} (\widehat{x})\, dx_i + \frac{\partial^2}{\partial x_i \partial \lambda_{ij}} V_i^{\lambda_i} (\widehat{x})\, d\lambda_{ij} = 0 \tag{3.9}$$

In our case we have $\forall i = 1, \ldots, n \ \ \forall j = 1, \ldots, m,$

$$\begin{aligned} \frac{dx_i}{d\lambda_{ij}} &= -\frac{\sum_{t\in T} \frac{\partial}{\partial x_i} u_{ij} \left(\widehat{x}_1 (t_1), \ldots, \widehat{x}_n (t_n)\right) p(t)}{\sum_{t\in T} \sum_{k=1}^{m} \lambda_{ik} \frac{\partial^2}{\partial x_i^2} u_{ik} \left(\widehat{x}_1 (t_1), \ldots, \widehat{x}_n (t_n)\right) p(t)} \\ &= -\frac{\sum_{t\in T} \frac{\partial}{\partial x_i} u_{ij} \left(\widehat{x}_1 (t_1), \ldots, \widehat{x}_n (t_n)\right) p(t)}{\sum_{k=1}^{m} \left[\lambda_{ik} \sum_{t\in T} \frac{\partial^2}{\partial x_i^2} u_{ik} \left(\widehat{x}_1 (t_1), \ldots, \widehat{x}_n (t_n)\right) p(t)\right]} \\ &= -\frac{\frac{\partial}{\partial x_i} U_{ij} \left(\widehat{x}_1, \ldots, \widehat{x}_n\right)}{\left\langle \lambda_i, \frac{\partial^2}{\partial x_i^2} U_i \left(\widehat{x}_1, \ldots, \widehat{x}_n\right)\right\rangle}, \end{aligned}$$

so the sign of this derivative is given by the sign of $\frac{\partial}{\partial x_i} U_{ij} \left(\widehat{x}_1, \ldots, \widehat{x}_n\right).$

3.5 An economic application: the Cournot duopoly

Let us consider a Cournot game Γ_c with incomplete information on production costs and where the demand arises from a competitive market of two types of commodity.

Then, we define $\Gamma_c = (N, A_1, A_2, T_1, T_2, P, u_1, u_2)$, where

- $N = \{1, 2\}$;
- $A_1 = A_2 = \left[0, \frac{\delta}{2}\right]^2$ where δ is a positive amount. We denote with $A = A_1 \times A_2$ the Cartesian product of the action spaces of firms;
- $T_1 = \left\{t_1^1\right\}, T_2 = \left\{t_2^1, t_2^2\right\}$ are the type finite set. We denote with $T = T_1 \times T_2$ the Cartesian product of the type spaces of firms;
- $P \in [0, 1]$ is the probability that firm 2's type is t_2^1;

- $u_i : A \times T \longrightarrow \mathbb{R}^2,\ i = 1, 2$ are the utility functions defined in the following way:

Call $a = (a_1, a_2) \in A_1,\ b = (b_1, b_2) \in A_2$, and

$$u_1(a, b, t) = (f_1(a, b), f_2(a, b)),$$

where

- $f_1(a, b) = a_1(\delta - a_1 - b_1) - c_1;$
- $f_2(a, b) = a_2(\delta - a_2 - b_2) - c_2.$

where $c_1, c_2 > 0$.

$$u_2(a, b, t) = \begin{cases} u_2^1(a, b, t) & \text{if the type is } t_2^1 \\ \\ u_2^2(a, b, t) & \text{if the type is } t_2^2, \end{cases}$$

where with u_2^1, u_2^2 we denote the utility function of player 2's type is t_2^1, t_2^2, respectively.

In particular

$$u_2^1(a, b, t) = \left(g_1^1(a, b), g_2^1(a, b)\right),$$

where

- $g_1^1(a, b) = b_1(\delta - a_1 - b_1) - kb_1^2;$
- $g_2^1(a, b) = b_2(\delta - a_2 - b_2) - kb_2^2.$

$$u_2^2(a, b, t) = \left(g_1^2(a, b), g_2^2(a, b)\right)$$

where

- $g_1^2(a, b) = b_1(\delta - a_1 - b_1) - k;$
- $g_2^2(a, b) = b_2(\delta - a_2 - b_2) - k.$

where $k > 0$. Let $(x, y) = (x_1, x_2, y_1, y_2) \in A_1^{T_1} \times A_2^{T_2}$ be the a priori expected payoff function of firm 1 is $U_1 : A_1^{T_1} \times A_2^{T_2} \longrightarrow \mathbb{R}^2$ such that

$$U_1(x, y) = (F_1(x, y), F_2(x, y))$$

where

- $F_1(x,y) = x_1(\delta - x_1 - y_1) - c_1$;
- $F_2(x,y) = x_2(\delta - x_2 - y_2) - c_2$.

and the a priori expected payoff function of firm 2 is $U_2 : A_1^{T_1} \times A_2^{T_2} \longrightarrow \mathbb{R}^2$ such that

$$U_2(x,y) = \left(PG_1^1(x,y) + (1-P)\,G_1^2(x,y)\,, PG_2^1(x,y) + (1-P)\,G_2^2(x,y)\right) =: (G_1(x,y)\,, G_2(x,y))\,.$$

where

- $G_1^1(x,y) = y_1(\delta - x_1 - y_1) - ky_1^2$;
- $G_2^1(x,y) = y_2(\delta - x_2 - y_2) - ky_2^2$;
- $G_1^2(x,y) = y_1(\delta - x_1 - y_1) - k$;
- $G_2^2(x,y) = y_2(\delta - x_2 - y_2) - k$.

We want to find Bayesian Pareto equilibria for Γ_c.

Let us define the weighted Cournot game $\Gamma_c^\lambda = \left(N, A_1, A_2, T_1, T_2, P, v_1^\lambda, v_2^\lambda\right)$, with $\lambda \in [0,1]$, where $v_i : A \times T \longrightarrow \mathbb{R},\ i = 1,2$ are defined in the following way:

$$v_1^\lambda(a,b,t) = \lambda f_1(a,b) + (1-\lambda)\,f_2(a,b)\,;$$

$$v_2^\lambda(a,b,t) = \begin{cases} \lambda g_1^1(a,b) + (1-\lambda)\,g_2^1(a,b) & \text{if the type is } \ t_2^1 \\ \\ \lambda g_1^2(a,b) + (1-\lambda)\,g_2^2(a,b) & \text{if the type is } \ t_2^2, \end{cases}$$

The weighted Cournot game Γ_c^λ satisfies the assumptions of Corollary 3.1 because A_1, A_2 are convex and compact sets, $v_1^\lambda(\cdot,t)$ and $v_2^\lambda(\cdot,t)$ are continuous functions because they are linear combinations of continuous functions for each $t \in T$, $v_1^\lambda(\cdot,y)$ and $v_2^\lambda(x,\cdot)$ are concave functions $\forall y \in A_2^{T_2}$, and $\forall x \in A_1^{T_1}$, respectively, because they are linear combinations of concave functions. Then Γ_c^λ possesses a Bayesian Nash equilibrium $\forall \lambda \in [0,1]$.

If $\lambda \in [0,1]$, denoting by V_1^λ and V_2^λ the a priori expected payoff functions for firms 1 and 2 respectively, we have for all $P \in [0,1]$,

$$\frac{\partial V_1^\lambda}{\partial x}(x,y) \quad = \quad \lambda(\delta - 2x_1 - y_1) + (1-\lambda)(\delta - 2x_2 - y_2)\,;$$

and

$$\frac{\partial V_2^{\lambda}}{\partial y}(x,y) = \lambda\left(-2Pky_1+\delta-x_1-2y_1\right)+(1-\lambda)\left(-2Pky_2+\delta-x_2-2y_2\right).$$

from which we obtain the following weighted quantities:

$$\begin{aligned}\lambda x_1+(1-\lambda)x_2 &= \frac{\delta(1+2Pk)}{3+4Pk}\\ \lambda y_1+(1-\lambda)y_2 &= \frac{\delta}{3+4Pk}\end{aligned}$$

from which we have $\forall\lambda\in(0,1]$ the Bayesian Nash equilibria of Γ_c^{λ}

$$\left(\left(\frac{1}{\lambda}\left[\frac{\delta(1+2Pk)}{3+4Pk}-(1-\lambda)x_2\right],x_2\right),\left(\frac{1}{\lambda}\left[\frac{\delta}{3+4Pk}-(1-\lambda)y_2\right],y_2\right)\right), \tag{3.10}$$

and for $\lambda=0$, we get the following Bayesian Nash equilibria

$$\left(\left(x_1,\frac{\delta(1+2Pk)}{3+4Pk}\right),\left(y_1,\frac{\delta}{3+4Pk}\right)\right). \tag{3.11}$$

3.5.1 The contraction approach

The contraction approach is based on showing that the best reply map

$$T(x,y)=\left(argmax_{x\in A_1^{T_1}}V_1^{\lambda}(x,y),argmax_{y\in A_2^{T_2}}V_2^{\lambda}(x,y)\right)$$

is a contraction. Then there is a unique fixed point of T, that is, unique Bayesian Nash equilibrium for Γ_c^{λ} according to the Banach fixed-point theorem [1]. Namely we have to prove that T is a contraction on $A_1^{T_1}\times A_2^{T_2}$. For this a sufficient condition is that (see [122]):

$$\frac{\partial^2 V_1^{\lambda}}{\partial x^2}+\left|\frac{\partial^2 V_1^{\lambda}}{\partial x\partial y}\right|<0; \tag{3.12}$$

[1]

Theorem 3.5. *Let (X,d) be a non-empty complete metric space. Let $T:X\longrightarrow X$ be a contraction mapping on X, i.e.: there is a nonnegative real number $h<1$ such that*

$$d(T(x),T(y))\le h\cdot d(x,y)$$

for all $x,y\in X$. Then the map T admits one and only one fixed-point $x^*\in X$.

and

$$\frac{\partial^2 V_2^\lambda}{\partial y^2} + \left| \frac{\partial^2 V_2^\lambda}{\partial y \partial x} \right| < 0, \tag{3.13}$$

for each $x \in A_1^{T_1},\ y \in A_2^{T_2}$ and $\lambda \in [0,1]$.

It easy to see that both conditions (3.12) and (3.13) are satisfied, so the weighted Cournot game Γ_c^λ admits a unique Bayesian Nash equilibrium. Conditions (3.12) and (3.13) also ensure dominance solvability of Γ_c^λ [2] and, consequentely, global stability.[3]

Suppose we want to determine how the optimal a priori expected payoff functions V_1^λ, and V_2^λ respond to changes in the parameter $\lambda \in [0,1]$.

It easy to see with comparative statics techniques that $\forall \lambda \in (0,1]$ the unique Bayesian Nash equilibrium for Γ_c^λ is

$$(x^*, y^*) \quad = \quad \left(\left(\frac{\delta\,(1+2Pk)}{3+4Pk}, \frac{\delta\,(1+2Pk)}{3+4Pk} \right), \left(\frac{\delta}{3+4Pk}, \frac{\delta}{3+4Pk} \right) \right). \tag{3.14}$$

In particular by Theorem 3.4 we proved that (x^*, y^*) in (3.14) is a wBPE for Γ_c, and for the uniqueness it is also a sBPE for Γ_c.

In the case $\lambda = 0$ the previous technique cannot be used because $x_1^*(\lambda)$ and $y_1^*(\lambda)$ defined in (3.10) are not differentiable at $\lambda = 0$. Indeed the game Γ_c^λ is reduced to be a one-criterion game with Bayesian Nash equilibium $(x^*, y^*) = \left(\frac{\delta(1+2Pk)}{3+4Pk}, \frac{\delta}{3+4Pk} \right)$.

3.5.2 The potential approach

From another point of view, we deal with the game Γ_c as a Bayesian potential bicriteria game.

Consider the following lemma

Lemma 3.1. *The Bayesian bicriteria game Γ_c is a Bayesian potential bicriteria game with the*

[2] A game is **dominance solvable** if the set remaining after iterated elimination of strictly dominated strategies is a singleton (see [84]).

[3] A BNE is **globally stable** if for any initial position the system converges to it.

following potential function:

$$\begin{aligned}\Pi(x,y) &= \Big(-\left(x_1^2+x_2^2\right)-(1+Pk)\left(y_1^2+y_2^2\right)-\frac{1}{2}\left(x_1y_1+x_1y_2+x_2y_1+x_2y_2\right)\\ &\quad+\delta\left(x_1+y_1+x_2+y_2\right),\\ &\quad-\left(x_1^2+x_2^2\right)-(1+Pk)\left(y_1^2+y_2^2\right)-\frac{1}{2}\left(x_1y_1+x_1y_2+x_2y_1+x_2y_2\right)\\ &\quad+\delta\left(x_1+y_1+x_2+y_2\right)\Big)\\ &:= \left(\Pi_1(x,y),\Pi_2(x,y)\right).\end{aligned}$$

Proof. Consider a two-person Bayesian bicriteria game with a priori expected payoff functions

$$\begin{aligned}W_1(x,y) &= \Big(-q_{11}^1(P)(x\cdot e_1)^2-2q_{12}^1(P)(x\cdot e_1)(y\cdot e_1)+2\theta_1^1(x\cdot e_1)+h_1^1(y,P),\\ &\quad -q_{11}^2(P)(x\cdot e_2)^2-2q_{12}^2(P)(x\cdot e_2)(y\cdot e_2)+2\theta_1^2(x\cdot e_2)+h_1^2(y,P)\Big)\end{aligned}$$

$$\begin{aligned}W_2(x,y) &= \Big(-q_{22}^1(P)(y\cdot e_1)^2-2q_{21}^1(P)(x\cdot e_1)(y\cdot e_1)+2\theta_2^1(y\cdot e_1)+h_2^1(x,P),\\ &\quad -q_{22}^2(P)(y\cdot e_2)^2-2q_{21}^2(P)(x\cdot e_2)(y\cdot e_2)+2\theta_2^2(y\cdot e_2)+h_2^2(x,P)\Big).\end{aligned}$$

From a straightforward generalization of Lemma 6 in [111] to two criteria, we have that it is a Bayesian potential bicriteria game iff $q_{12}(P)=q_{21}(P)\ \ \forall P\in[0,1]$. A Bayesian potential function Π is such that

$$\begin{aligned}\Pi(x,y) &= -q_{11}(P)(x\cdot x)-q_{22}(P)(y\cdot y)\\ &\quad -q_{12}(P)(x\cdot y)-q_{12}(P)(x\cdot e_1)(y\cdot e_2)-q_{12}(P)(x\cdot e_2)(y\cdot e_1)\\ &\quad +(2,2)(\theta_1\cdot x)+(2,2)(\theta_2\cdot y),\end{aligned}$$

where $\{e_1,e_2\}$ is the canonical basis of $\mathbb{R}^2$, and

$$q_{11}(P)=\left(q_{11}^1(P),q_{11}^2(P)\right),$$

$$q_{22}(P)=\left(q_{22}^1(P),q_{22}^2(P)\right),$$

$$q_{12}(P)=\left(q_{12}^1(P),q_{12}^2(P)\right)$$

are aleatory vectors,

$$\theta_1 = \left(\theta_1^1, \theta_1^2\right),$$

$$\theta_2 = \left(\theta_2^1, \theta_2^2\right)$$

are constant vectors and

$$h_1 : Y \times [0,1] \longrightarrow \mathbb{R}^2,$$

$$h_2 : X \times [0,1] \longrightarrow \mathbb{R}^2.$$

In our case we have:

(i) $q_{12}(P) = q_{21}(P) = \left(\frac{1}{2}, \frac{1}{2}\right);$

(ii) $q_{11}(P) = (1,1), q_{22}(P) = (1 + Pk, 1 + Pk);$

(iii) $\theta_1 = \theta_2 = \left(\frac{\delta}{2}, \frac{\delta}{2}\right);$

(iv) $h_1(y, P) = (-c_1, -c_2), h_2(x, P) = (-k(1-P), -k(1-P)).$

From (i) we have that Γ_c is a BPMG, and it is easy to compute the potential function. □

Since $\Pi(x,y) \leq \left(2\delta^2, 2\delta^2\right)$, the potential Π is an upper bounded function and for any $\epsilon > 0$ Γ_c has an ϵBPE (by Theorem 3.3). In particular, in our setting, if $\widehat{y} \in argmax_{x\in A_1^{T_1}} \Pi_1(x,y)$, and $\widehat{x} \in argmax_{y\in A_2^{T_2}} \Pi_2(x,y)$ (that is $(\widehat{x}, \widehat{y}) \in A_1^{T_1} \times A_2^{T_2}$ is a sBPE for $\Gamma_c^\pi = (N, A_1, A_2, T_1, T_2, P, \pi)$, then, for Theorem 3.2, is a sBPE for Γ_c, and, obviously, also a wBPE for Γ_c.

In our example we have

$$\begin{aligned} argmax_{x\in A_1^{T_1}} \Pi_1(x,y) &= \left\{\left(-\frac{1}{4}(y_1+y_2) + \frac{1}{2}\delta, -\frac{1}{4}(y_1+y_2) + \frac{1}{2}\delta\right)\right\}, \\ argmax_{y\in A_2^{T_2}} \Pi_2(x,y) &= \left\{\left(-\frac{1}{4(1+Pk)}(x_1+x_2) + \frac{1}{2(1+Pk)}\delta,\right.\right. \\ &\quad \left.\left. -\frac{1}{4(1+Pk)}(x_1+x_2) + \frac{1}{2(1+Pk)}\delta\right)\right\}. \end{aligned}$$

So it is easy to see that

$$\left(\left(\frac{\delta(1+2Pk)}{3+4Pk}, \frac{\delta(1+2Pk)}{3+4Pk}\right), \left(\frac{\delta}{3+4Pk}, \frac{\delta}{3+4Pk}\right)\right), \tag{3.15}$$

is a sBPE for Γ_c, as we proved in the first part of the model.

We studied the class of Bayesian multicriteria games establishing the existence of approximate, weak and strong Bayesian Pareto equilibria in the case of Bayesian potential multicriteria games. In a general case, by using the scalarization approach, we showed the existence of weak Bayesian Pareto equilibria. Moreover we gave an economic example which modelize the classical game of Cournot duopoly in the case of incomplete information and in which both firms have two objectives to optimize.

There are many topics for further research. First of all this model of Bayesian multicriteria game can be applied to environmental games in which countries have to take into account many objectives and it is realistic to suppose incomplete information about political strategies of opponent countries. The interesting case of supermodular games ([15], [94]) could be also considered in order to obtain some further results about the Bayesian multicriteria games.

Chapter 4

A Bayesian Potential Game to Illustrate Heterogeneity in Cost/Benefit Characteristics

4.1 Introduction

We are going to propose an idea to model heterogeneity of agents in games where the differences among players' behaviours can be outlined by considering two main features, basically relying on benefit and cost.

The chosen approach comes from Bayesian game theory, where the definition and the employment of suitable types for players lead to a naturally asymmetric structure. Specifically, we will select an asymmetric game where the type structure is finite and investigate the related Bayesian potential game. Potential games are quite a useful tool to simplify the determination of Nash equilibria and achieve a number of properties of the optimal strategies of the game. Specifically, potential functions of games collect all the relevant information in a unique function, whereas in Bayesian games every player only owns and takes into account information on her type in her payoff structure.

Actually, my analysis takes into account a specific kind of payoff, whose features are very common across oligopoly games, i.e. the typical form of profit equalling the difference between gains, where the information about benefit type is, and costs, where the information about cost type is. We assume that the cost functions include the contributions of all agents, as if all strategies caused a damage to all agents, even if at different levels. Such a formulation can be applied to an environmental game, where firms (or countries) choose their optimal emissions strategy maximizing their own profits, having to take into account that their aggregate emissions amount to an environmental cost suffered by all of them. We find it evident that when the involved agents are countries, even more than firms, the issue of modeling heterogeneity assumes high priority.

The Bayesian game approach seems to be suitable, in that it permits a complex characterization of types which may reflect countries' economic attitudes, productive characteristics and even propensity to cause environmental damage.

In our opinion, the distinction among types usually employed in Bayesian game theory responds to the commitment to manage the complexity and the heterogeneity of such a framework, in which all countries are endowed with such different prerogatives. Furthermore, some contributions appeared in recent literature taking into account the issue of monotonicity of pure-strategy equilibria in Bayesian games (in particular, [76], [118] and [100]) in setups where the type spaces are partially ordered probability spaces. In this chapter, we intend to develop a procedure to check monotonicity in our framework, where the type spaces are discrete and consisting of a finite number of types, because the techniques employed by [118] and [100] cannot be applied.

After constructing the Bayesian game and establishing a suitable preference order on the type spaces, we proceed to take into examination the Bayesian potential structure of the game. Additive separability in the strategic variables of the environmental cost functions makes such structure quite simple to deal with and provides some clear properties for the Nash equilibria.

In particular in this chapter we underline that the potential of the original game can be explicitly calculated and decomposed in the sum of the aggregate revenue and the aggregate cost, giving rise to an equivalent game where information on the probability distributions of all types is collected. Moreover the cost structure emerging from the formula of the potential function provides necessary and sufficient conditions to ensure monotonicity of the pure strategies, in compliance with the partial order established on the type spaces. Finally, monotonicity and feasibility of the pure strategies are shown and proved in some very basic examples. The first example involves a unique pure strategy in the original strategy space, in the second one the pure strategy is unique as well but the strategy space must be restricted in order to ensure monotonicity, whereas the third example, which relies on a non-standard payoff structure, does not have a globally concave potential function and then requires a different kind of analysis.

4.2 The setup of the model

In standard environmental games, N countries choose their optimal emissions strategies $e_i \geq 0$, in order to maximize their profit functions. Typically, their emissions contribute to increase the total stock of pollution.

An aggregate dynamic variable is usually employed denoting the stock of pollution produced by the accumulation of all countries'emissions: P. P causes damage to the environment and affects countries'payoffs negatively, behaving as a negative environmental externality whose effect differs across countries, as if each player had to bear a specific environmental cost depending on the aggregate stock of emissions. In [5], [18] and in analogous models, such heterogeneity is modeled by employing asymmetric marginal damages across countries and asymmetric marginal revenues. Our approach will be different, in that we will assume that individual environmental cost will be determined by each country's own type. Specifically, $P = f_k(e_1, \ldots, e_n)$, assuming that the production function $f_k(\cdot)$, corresponding to type k, is such that $f_k(0, \ldots, 0) = 0$, and is increasing with respect to each emission variable, i.e.:

$$\frac{\partial f_k}{\partial e_j} > 0, \qquad\qquad \forall\ j = 1, \ldots, n.$$

Such production function $f_k(\cdot)$ expresses the effect of the accumulation of the pollution stock on a country whose type is k. Thus, we can look upon it either as an environmental cost for the country or as the marginal contribution of each country with type k to the aggregate pollution stock.

Generally, the Bayesian games are endowed with a structure which may be suitable to model heterogeneity of agents. In this case, we are going to take into account two basic elements of heterogeneity, possibly different across countries: the marginal gains from emissions and the individual contributions to damage caused by the accumulation of pollution. In both cases, we will consider a low and a high level, labeled by indexes L and H. Hence, we intend to rely on 4 different types (an example of such a discrete type structure can be found in [24]).

The main characteristics of our framework can be summarized as follows:

- $N = \{1, \dots, n\}$, where $n < \infty$, is the set of players, i.e. countries which aim to maximize their payoffs;
- $b_L, b_H \in \mathbb{R}_+$ are parameters indicating all the possible marginal gains from emissions; $b_L < b_H$, then b_L is suitable to define the action spaces of countries;
- the action spaces of countries are called E_i (standing for emissions) and are all equal compact intervals, i.e.:
$$E_1 = E_2 = \ldots = E_n = E = [0, 2b_L];$$
- the finite sets representing the type spaces of players are all equal to
$$T = \{(b_L,\ f_L(\cdot)),\ (b_L,\ f_H(\cdot)),\ (b_H,\ f_L(\cdot)),\ (b_H,\ f_H(\cdot))\},$$
equipped with the partial ordering $\succcurlyeq$ which will be established in Definition 4.3, and T^n is the Cartesian product of n copies of T. Note that this type structure allows each involved country to belong to 4 alternative types;
- the common prior belief of the agents is represented by a discrete probability measure having full support on each finite type space T. Each probability distribution is designed in such a way that the i-th country has probability p_i^{jk} to belong to type $(b_j,\ f_k(\cdot)) \in T$; namely, the following properties are supposed to hold:
$$p_i^{jk} \geq 0, \qquad \forall\ i = 1, \dots, n,\ k, j \in \{L, H\},$$
$$p_i^{LL} + p_i^{LH} + p_i^{HL} + p_i^{HH} = 1 \qquad \forall\ i = 1, \dots, n.$$

- if the i-th country belongs to type $(b_j,\ f_k(\cdot))$, her payoff function reads as follows:

$$e_i\left(b_j - \frac{e_i}{2}\right) - f_k\left(e_1, \dots, e_n\right);$$

hence, the i-th utility function is given by $u_i : E^n \times T^n \longrightarrow \mathbb{R}$ such that:

$$u_i(e_1, \dots, e_n, t_1, \dots, t_n) = \begin{cases} u_i^{LL}(e_1, \dots, e_n, t_1, \dots, t_n) & \text{if } t_i = (b_L,\ f_L(\cdot)) \\ u_i^{LH}(e_1, \dots, e_n, t_1, \dots, t_n) & \text{if } t_i = (b_L,\ f_H(\cdot)) \\ u_i^{HL}(e_1, \dots, e_n, t_1, \dots, t_n) & \text{if } t_i = (b_H,\ f_L(\cdot)) \\ u_i^{HH}(e_1, \dots, e_n, t_1, \dots, t_n) & \text{if } t_i = (b_H,\ f_H(\cdot)) \end{cases}, \tag{4.1}$$

where

$$u_i^{jk}(e_1, \dots,\ e_n,\ t_1, \dots,\ t_n) = e_i\left(b_j - \frac{e_i}{2}\right) - f_k\left(e_1, \dots, e_n\right);$$

- the a priori expected payoff function of country i is $U_i : (E^T)^n \longrightarrow \mathbb{R}$ such that:

$$\begin{aligned} U_i(x_1, \dots, x_n) \quad = \quad & p_i^{LL}u_i^{LL}(x_1\left(t_1\right), \dots, x_n\left(t_n\right), t_1, \dots, t_n) \\ & + p_i^{LH}u_i^{LH}(x_1\left(t_1\right), \dots, x_n\left(t_n\right), t_1, \dots, t_n) \\ & + p_i^{HL}u_i^{HL}(x_1\left(t_1\right), \dots, x_n\left(t_n\right), t_1, \dots, t_n) \\ & + p_i^{HH}u_i^{HH}(x_1\left(t_1\right), \dots, x_n\left(t_n\right), t_1, \dots, t_n), \end{aligned} \tag{4.2}$$

where $x = (x_1, \dots,\ x_n) \in (E^T)^n$ are the strategic variables depending on the type profile $(t_1, \dots,\ t_n) \in T^n$ assigned to the n countries by Nature [1].

From now on, we will call $\Gamma = (N,\ E,\ T,\ p,\ u)$ the game at hand.

4.2.1 Main characteristics of the game

Generally, we can formulate the conditions under which a Bayesian Nash equilibrium (BNE for short) is implicitly determined:

Proposition 4.1. *Let $\Gamma = (N,\ E,\ T,\ p,\ u)$ be a Bayesian game with a priori expected utility functions U_i, $i = 1, \dots, n$. Call W the largest open set with subset topology containing $(E^T)^n$. If $\forall\ i \in N$ the following additional assumptions hold:*

[1] We will take into account pure strategies only, i.e. measurable functions $x_i : T \longrightarrow E$, as in the definition in [100], p. 508.

(A1) $U_i \in C^2(W)$, *and* $F_H, F_L \in C^2(W)$;

(A2) $\frac{\partial U_i}{\partial x_i}(x_1, \ldots, x_n) \geq 0$;

(A3) $\frac{\partial^2 F_H}{\partial x_i^2}(x_1, \ldots, x_n) \geq 0$, $\frac{\partial^2 F_L}{\partial x_i^2}(x_1, \ldots, x_n) \geq 0$.

where F_L *and* F_H *are the a priori expected payoff functions of* f_L *and* f_H, *respectively.*

Then the strategy profile $\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_n) \in (E^T)^n$, *where* $\forall i \in N$

$$\widehat{x}_i = \sum_{j,k \in \{L,H\}} p_i^{jk} \left[b_j - \frac{\partial F_k}{\partial x_i}(\widehat{x}) \right] \tag{4.3}$$

is a candidate strategy to be a BNE for Γ.

Proof. Let $x = (x_1, \ldots, x_n) \in W$. We have that $\forall\, i \in N$,

$$\begin{aligned} U_i(x) \quad = \quad & p_i^{LL} \left[x_i \left(b_L - \frac{x_i}{2} \right) - F_L(x) \right] + p_i^{LH} \left[x_i \left(b_L - \frac{x_i}{2} \right) - F_H(x) \right] \\ & + p_i^{HL} \left[x_i \left(b_H - \frac{x_i}{2} \right) - F_L(x) \right] + p_i^{HH} \left[x_i \left(b_H - \frac{x_i}{2} \right) - F_H(x) \right]. \end{aligned} \tag{4.4}$$

If Assumption (A1) holds, we have

$$\frac{\partial U_i}{\partial x_i}(x) = -x_i + \sum_{j,k \in \{L,H\}} p_i^{jk} \left[b_j - \frac{\partial F_k}{\partial x_i}(x) \right]$$

and

$$\frac{\partial^2 U_i}{\partial x_i^2}(x) = -1 - \sum_{j,k \in \{L,H\}} p_i^{jk} \frac{\partial^2 F_k}{\partial x_i^2}(x)$$

is negative by Assumption (A3).

Therefore, $\forall\, i \in N$, if $\widehat{x}_i$ is solution to the n equations $\frac{\partial U_i}{\partial x_i}(x) = 0$, i.e. (4.3) hold, then the strategy profile $\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_n)$ is a candidate strategy to be a BNE for Γ. □

4.3 The Bayesian potential game

In compliance with the standard notation employed in Bayesian game theory, in this subsection we will use the following symbols, possibly indexed (see also [75]): N for the set of players, E for the action spaces, T for the type spaces, p for the probability distribution, u for the payoff functions, U

for the a priori expected utility functions, Γ for the Bayesian game and Π for the potential functions. Furthermore, we will denote with E^T the space of all functions from T to E. In [75] the authors prove existence theorem for Pareto equilibria in Bayesian potential multicriteria games. We adapt these results to the case in which there is one criterion as in Remark **??**.

We are going to construct a BPG by exploiting the additivity argument stated in Remark 3.3. Consider two different Bayesian games, the former representing the revenue contribution and the latter representing the cost contribution. Each one of them is endowed with the same type spaces and the same probability distributions, that is:

$$\Gamma_1 = \left(N,\ E,\ T,\ p,\ g_i^j\right) \qquad \text{and} \qquad \Gamma_2 = \left(N,\ E,\ T,\ p,\ h_i^k\right),$$

$\forall\ i \in N,\ j,k \in \{L, H\}$, where

$$g_i^j\left(e_1,\dots,e_n,t_1,\dots,t_n\right) = e_i\left(b_j - \frac{e_i}{2}\right),$$

$$h_i^k\left(e_1,\dots,e_n,t_1,\dots,t_n\right) = -f_k\left(e_1,\dots,e_n\right).$$

$\forall\ i \in N$, the a priori expected payoff functions respectively are:

$$G_i\left(x\right) = \left(p_i^{LL} + p_i^{LH}\right)\left[x_i\left(b_L - \frac{x_i}{2}\right)\right] + \left(p_i^{HL} + p_i^{HH}\right)\left[x_i\left(b_H - \frac{x_i}{2}\right)\right], \tag{4.5}$$

$$H_i\left(x\right) = -\left(p_i^{LL} + p_i^{HL}\right)F_L\left(x\right) - \left(p_i^{LH} + p_i^{HH}\right)F_H\left(x\right). \tag{4.6}$$

Before getting to calculate the potential functions, we remind the readers that a necessary and sufficient condition for the existence of a potential function Π is stated in [83]: given the payoff functions U_i, a potential function exists if and only if

$$\frac{\partial^2 U_i}{\partial x_i \partial x_j} = \frac{\partial^2 U_j}{\partial x_i \partial x_j} \tag{4.7}$$

for all $i \neq j$. The conditions (4.7) hold for $U_i = G_i$ as in (4.5), whereas for H_i as in (4.6) we need the following result:

Proposition 4.2. *Γ_2 is a potential game if and only if $\forall\ i \neq j \in N$ we have*

$$p_i^{LL} + p_i^{HL} = p_j^{LL} + p_j^{HL},$$

$$p_i^{LH} + p_i^{HH} = p_j^{LH} + p_j^{HH}.$$

Proof. It suffices to write down (4.7) for H_i:

$$\left(p_i^{LL} + p_i^{HL}\right) \frac{\partial^2 F_L(x)}{\partial x_i \partial x_j} + \left(p_i^{LH} + p_i^{HH}\right) \frac{\partial^2 F_H(x)}{\partial x_i \partial x_j} =$$

$$= \left(p_j^{LL} + p_j^{HL}\right) \frac{\partial^2 F_L(x)}{\partial x_i \partial x_j} + \left(p_j^{LH} + p_j^{HH}\right) \frac{\partial^2 F_H(x)}{\partial x_i \partial x_j}$$

if and only if for all $i \neq j$ we have:

$$p_i^{LL} + p_i^{HL} = p_j^{LL} + p_j^{HL} \qquad \text{and} \qquad p_i^{LH} + p_i^{HH} = p_j^{LH} + p_j^{HH}.$$

□

We are going to assume some kind of suitable separability in the variables of the production functions $F_L(\cdot)$ and $F_H(\cdot)$. For example, referring to [8], we can consider forms of payoffs taken from standard games in normal form.

Definition 4.1. *An $n-$ player normal form game $\Gamma = (N, X_1, \ldots, X_n, f_1, \ldots, f_n)$ is called a* ***partially separable game*** *if for any $i \in N$ there exist two functions $f_i^i : X_i \longrightarrow \mathbb{R}$ and $f_i^{-i} : \prod_{k \neq i} X_k \longrightarrow \mathbb{R}$ such that*

$$f_i(x) = f_i^i(x_i) + f_i^{-i}(x_{-i}).$$

The separable games, introduced in [8], are a particular case of partially separable games introduced in [95] in which $\forall i \in N,\ f_i^{-i} = 0$. Accordingly, we can employ production functions of the following kind:

$$F_L(x) = \sum_{j=1}^{n} \left[\Phi_{jL}(x_j) + \widetilde{\Phi}_{(-j)L}(x_{-j})\right],$$

$$F_H(x) = \sum_{j=1}^{n} \left[\Phi_{jH}(x_j) + \widetilde{\Phi}_{(-j)H}(x_{-j})\right].$$

Hypothesizing that for all $j = 1, \dots, n$, $\Phi_{jL}, \Phi_{jH} \in C^2\left(E^T\right)$ and $\widetilde{\Phi}_{(-j)L}, \widetilde{\Phi}_{(-j)H} \in C^2\left(\left(E^T\right)^{n-1}\right)$ and that the conditions on weights stated in Proposition 4.2 hold, then Γ_2 is a Bayesian potential partially separable game (see [95] for the non-Bayesian case). In our case, we will rely on a stronger form of separability, i.e. $\widetilde{\Phi}_{(-j)L}\left(x_{-j}\right) = 0$ and $\widetilde{\Phi}_{(-j)H}\left(x_{-j}\right) = 0$ for all $j = 1, \dots, n$.

Proposition 4.3. *If $F_L(x) = \sum_{j=1}^{n} \Phi_{jL}(x_j)$, and $F_H(x) = \sum_{j=1}^{n} \Phi_{jH}(x_j)$, and for all $j = 1, \dots, n$, Φ_{jL}, $\Phi_{jH} \in C^2\left(E^T\right)$, the Bayesian games Γ_1 and Γ_2 are BPGs, whose potential functions are as follows:*

$$P_1(x) = -\sum_{j=1}^{n} \frac{1}{2} x_j^2 + \sum_{j=1}^{n} \left[\left(b_L - b_H\right)\left(p_j^{LL} + p_j^{LH}\right) + b_H\right] x_j, \tag{4.8}$$

$$P_2(x) = -\sum_{j=1}^{n} \left(p_j^{LL} + p_j^{HL}\right) \Phi_{jL}(x_j) - \sum_{j=1}^{n} \left(p_j^{LH} + p_j^{HH}\right) \Phi_{jH}(x_j). \tag{4.9}$$

Proof. It is immediate to check that $\dfrac{\partial G_i}{\partial x_i} = \dfrac{\partial P_1}{\partial x_i}$, and that $\dfrac{\partial H_i}{\partial x_i} = \dfrac{\partial P_2}{\partial x_i}$. Furthermore, Proposition 4.2 is verified, hence (4.8) and (4.9) are potential functions for Γ_1 and Γ_2. □

Corollary 4.1. *If $F_L(x) = \sum_{j=1}^{n} \Phi_{jL}(x_j)$, and $F_H(x) = \sum_{j=1}^{n} \Phi_{jH}(x_j)$, and for all $j = 1, \dots, n$, $\Phi_{jL}, \Phi_{jH} \in C^2\left(E^T\right)$, the Bayesian game $\Gamma = (N,\ E,\ T,\ p,\ u)$, where $u_i^{jk} = g_i^j + h_i^k$, is a BPG whose potential function is given by:*

$$\Pi(x) = -\frac{1}{2}\sum_{j=1}^{n} x_j^2 + b_H \sum_{j=1}^{n} x_j + \sum_{j=1}^{n} \left[\left(b_L - b_H\right) x_j - \Phi_{jL}(x_j)\right] p_j^{LL} +$$

$$+ \sum_{j=1}^{n} \left[\left(b_L - b_H\right) x_j - \Phi_{jH}(x_j)\right] p_j^{LH} - \sum_{j=1}^{n} \Phi_{jL}(x_j) p_j^{HL} - \sum_{j=1}^{n} \Phi_{jH}(x_j) p_j^{HH}. \tag{4.10}$$

Proof. It immediately follows from Proposition 4.3 and Remark 3.3. Summing the potentials and finally rearranging terms yields:

$$\Pi(x) = P_1(x) + P_2(x) = -\sum_{j=1}^{n} \frac{1}{2} x_j^2 + \sum_{j=1}^{n} \left[\left(b_L - b_H\right)\left(p_j^{LL} + p_j^{LH}\right) + b_H\right] x_j +$$

$$-\sum_{j=1}^{n} \left(p_j^{LL} + p_j^{HL}\right) \Phi_{jL}(x_j) - \sum_{j=1}^{n} \left(p_j^{LH} + p_j^{HH}\right) \Phi_{jH}(x_j) =$$

$$= -\frac{1}{2}\sum_{j=1}^{n} x_j^2 + b_H \sum_{j=1}^{n} x_j + \sum_{j=1}^{n} \left[(b_L - b_H)\, x_j - \Phi_{jL}(x_j) \right] p_j^{LL} +$$

$$+ \sum_{j=1}^{n} \left[(b_L - b_H)\, x_j - \Phi_{jH}(x_j) \right] p_j^{LH} - \sum_{j=1}^{n} \Phi_{jL}(x_j) p_j^{HL} - \sum_{j=1}^{n} \Phi_{jH}(x_j) p_j^{HH}.$$

□

Remark 4.1. *The explicit calculation of Bayesian potential functions has already been investigated in some recent papers (for example, [111]), and in particular, it turns out to be simple when the structure of the game is linear-quadratic. If we consider a Bayesian game* Γ *with a priori expected payoff functions:*

$$\widetilde{U}_i\,(x) = -q_{ii}\,(p)\, x_i^2 - 2\sum_{j \neq i} q_{ij}\,(p)\, x_i x_j + 2\theta_i\,(p)\, x_i + h_i\,(x_{-i}, p)\,,$$

where q_{ij} *and* θ_i *are sufficiently regular functions of the probability distributions* p *and* h_i *is a sufficiently regular function of* x_{-i} *and* p*, then by Lemma 6 in [111], we have that* Γ *is a BPG iff* $q_{ij}\,(p) = q_{ji}\,(p)\ \ \forall\ i, j \in N$ *with* $i \neq j$*, and for all* p*. A Bayesian potential function for* Γ *is given by*

$$\Pi\,(x) = -x^t Q\,(p)\, x + 2\theta\,(p)^t\, x,$$

where $Q\,(p) = [q_{ij}\,(p)]_{n \times n}$*,* $\theta\,(p) = [\theta_1\,(p)\,, \ldots, \theta_n\,(p)]^t$ *and* ^{t} *denotes the transpose of a matrix.*

Note that (4.10) can easily be decomposed in the sum $R(x) + \Delta(x)$, i.e. the sum of aggregate revenue and aggregate damage. $R(x) = -\frac{1}{2}\sum_{j=1}^{n} x_j^2 + b_H \sum_{j=1}^{n} x_j$ has no aleatory features, whereas $\Delta(x) = \sum_{j=1}^{n} \mathbb{E}_j[\Delta_j(x_j)]$, where $\mathbb{E}_j[\cdot]$ is the expectation operator when its variable is distributed according to p_j, and the determinations of $\Delta_j(x_j)$ are:

$$\Delta_j(x_j,\ t_j) = \begin{cases} (b_L - b_H)\, x_j - \Phi_{jL}(x_j) & \text{if } \ t_j = (b_L,\ f_L(\cdot)) \\ (b_L - b_H)\, x_j - \Phi_{jH}(x_j) & \text{if } \ t_j = (b_L,\ f_H(\cdot)) \\ -\Phi_{jL}(x_j) & \text{if } \ t_j = (b_H,\ f_L(\cdot)) \\ -\Phi_{jH}(x_j) & \text{if } \ t_j = (b_H,\ f_H(\cdot)) \end{cases}.$$

The economic intuition behind the formulation $R(x) + \Delta(x)$ can be further refined: in particular, $R(x)$ represents the aggregate revenue of all agents if they all shared the same high marginal revenue type H, whereas $\Delta(x)$ can be expressed as follows:

$$\Delta(x) = \sum_{j=1}^{n} \mathbb{E}_j[\Delta_j(x_j)] = \sum_{j\in\{1,\dots,n\},\ s\in\{L,H\}} p_j^{sL}[(b_s - b_H)\, x_j - \Phi_{jL}(x_j)] +$$

$$+ \sum_{j\in\{1,\dots,n\},\ s\in\{L,H\}} p_j^{sH}[(b_s - b_H)\, x_j - \Phi_{jH}(x_j)],$$

where the two sums indicate the aggregate damage, or cost, for all agents due to their marginal gains'levels. The former measures the aggregate losses of agents with low production functions based on their marginal revenue type, whereas the latter does the same for the agents with high production functions. $\Delta(x)$ can be thought of as the function quantifying the loss in payoff caused by the fact that they fail to belong to the high marginal revenue type.

The following proposition provides necessary and sufficient conditions for Π to admit a maximum point, i.e. a BNE for Γ (by Theorem 3.2 in the special case of one criterion).

Proposition 4.4. *If there exists a unique point $x^* = (x_1^*, \dots, x_n^*) \in (0,\ 2b_L)^n$ such that for all $j = 1, \dots, n$ the following first order conditions hold:*

$$-x_j^* + (b_L - b_H)\left(p_j^{LL} + p_j^{LH}\right) + b_H - \left(p_j^{LL} + p_j^{HL}\right)\Phi'_{jL}(x_j^*) - \left(p_j^{LH} + p_j^{HH}\right)\Phi'_{jH}(x_j^*) = 0, \quad (4.11)$$

and $\Phi''_{jL}(x_j) \geq 0$ and $\Phi''_{jH}(x_j) \geq 0$ for any $x = (x_1, \dots, x_n)$ in an open neighbourhood of x^* for all $j = 1, \dots, n$, then x^* is a BNE for Γ.

Proof. The necessary conditions for the maximization of (4.10) are the n equations (4.11). Because all the second order partial mixed derivatives are zero, the sufficient conditions are given by:

$$\frac{\partial^2 \Pi}{\partial x_i^2} = -1 - \left(p_j^{LL} + p_j^{HL}\right)\Phi''_{jL}(x_j^*) - \left(p_j^{LH} + p_j^{HH}\right)\Phi''_{jH}(x_j^*),$$

which are strictly negative if $\Phi''_{jL}(x_j) \geq 0$ and $\Phi_{jH}(x_j) \geq 0$ for all $j = 1, \dots, n$. Then x^* is the unique maximizer for Π and consequently a BNE for Γ. □

Note that Proposition 4.4 only takes into consideration a unique equilibrium point and assumes it not to be on the boundary of the domain, hence it essentially concerns globally concave potential functions. In Section 4.4, we will also examine an example in which global concavity of Π is not verified.

4.3.1 Monotonicity of strategies

In this subsection, we are going to establish a preference order on our type spaces, whereby we will be able to define monotonicity of strategies. We recall for reader's convenience the following definition.

Definition 4.2. *Let A be a nonempty set. A* ***partial preorder*** $\succcurlyeq$ *on A is a reflexive and transitive relation. A* ***total preorder*** *is a reflexive, transitive and total relation. An antisymmetric partial preorder $\succcurlyeq$ is called* ***partial order****, if $\succcurlyeq$ is also a total relation, we call it a* ***total order****.*

Let Θ be a compact subset of $\mathbb{R}^m$ containing the origin and consider the following set

$$\begin{aligned} \mathcal{F}(\Theta) &= \{f : \Theta \longrightarrow \mathbb{R} \mid f(0,\ldots,0) = 0, \\ & \left. f \in C^2(\Theta),\ \frac{\partial f}{\partial x_j}(c) \geq 0,\ \forall\, c \in \Theta,\ \forall\, j = 1,\ldots,n.\right\}. \end{aligned} \tag{4.12}$$

We want to construct a partial order $\succcurlyeq$ on the set $\mathbb{R}_+ \times \mathcal{F}(\Theta)$ in the following way:

Definition 4.3. *Let $(\alpha,\ f),\ (\beta,\ g) \in \mathbb{R}_+ \times \mathcal{F}(\Theta)$, we say that*

$$(\alpha, f) \succcurlyeq (\beta, g)$$

iff

$$\alpha \geq \beta \qquad \text{and} \qquad \frac{\partial f(c)}{\partial x_j} \leq \frac{\partial g(c)}{\partial x_j} \quad \forall j = 1,\ldots,n. \tag{4.13}$$

It is easy to verify that the relation in Definition 4.3 is a partial order. Namely, (4.13) establishes a preference on pairs, leading to a preference on types, which is based on a double prerogative: a larger benefit parameter and a lower maximum cost. Note that because Θ is compact and f and g are both continuous, the sides of both inequalities involve finite values.

Remark 4.2. *The partial order $\succcurlyeq$ defined in Definition 4.3 on the set $\mathbb{R}_+ \times \mathcal{F}([0,\ 2b_L]^n)$ induces the following partial order on type set T :*

$$(b_H, f_L(\cdot)) \succcurlyeq (b_H, f_H(\cdot)) \succcurlyeq (b_L, f_H(\cdot))$$

$$(b_H, f_L(\cdot)) \succcurlyeq (b_L, f_L(\cdot)) \succcurlyeq (b_L, f_H(\cdot)),$$

then $(T, \succcurlyeq)$ *is a poset.*

Definition 4.4. *A strategy for player* i, $x_i : T \longrightarrow A$, *is monotone if* $t_i' \succcurlyeq t_i$ *implies* $x_i(t_i') \geq x_i(t_i)$ *for all* $t_i', t_i \in T$.

The following theorem intends to characterize the monotonicity of BNEs in our framework. Intuitively, it relies on the fact that the preference order on $\mathbb{R}_+ \times \mathcal{F}([0,\ 2b_L]^n)$ is reproduced in the FOCs of our maximization problem.

Theorem 4.1. *Let* $\Gamma = (N,\ E,\ T,\ p,\ u)$ *a Bayesian game such that* $BNE(\Gamma) \neq \emptyset$.
If $F_L(x) = \sum_{j=1}^{n} \Phi_{jL}(x_j)$, *and* $F_H(x) = \sum_{j=1}^{n} \Phi_{jH}(x_j)$, *and for all* $j = 1, \ldots, n$, $\Phi_{jL}, \Phi_{jH} \in \mathcal{F}([0,\ 2b_L]^n)$, Γ *admits a monotone BNE* x^* *with respect to the partial order* $\succcurlyeq$ *on the set* $\mathbb{R}_+ \times \mathcal{F}([0,\ 2b_L]^n)$ *iff* $\forall\ j = 1, \ldots, n$, *we have:*

$$\frac{\partial \Phi_{jL}(x_j)}{\partial x_j} \leq \frac{\partial \Phi_{jH}(x_j)}{\partial x_j}, \tag{4.14}$$

for all $x_j \in (0,\ 2b_L)$.

Proof. Rearranging the j-th FOC (4.11) yields:

$$x_j^* = (b_L - b_H)\left(p_j^{LL} + p_j^{LH}\right) + b_H - \left(p_j^{LL} + p_j^{HL}\right)\Phi'_{jL}(x_j^*) - \left(p_j^{LH} + p_j^{HH}\right)\Phi'_{jH}(x_j^*),$$

leading to the following implicit definitions of the pure strategies:

$$\begin{aligned}
x_j^*(b_H, f_L(\cdot)) &= b_H - \Phi'_{jL}(x_j^*) \\
x_j^*(b_H, f_H(\cdot)) &= b_H - \Phi'_{jH}(x_j^*) \\
x_j^*(b_L, f_H(\cdot)) &= b_L - \Phi'_{jH}(x_j^*) \\
x_j^*(b_L, f_L(\cdot)) &= b_L - \Phi'_{jL}(x_j^*)
\end{aligned}$$

and it follows immediately that if (4.14) holds for all players, the preference order defined in Definition 4.3 and in Remark 4.2 is satisfied by $x^* \in BNE(\Gamma)$. □

What follows is a criterion to ensure the feasibility of monotone pure strategies:

Proposition 4.5. *If x^* is a monotone pure strategy of Γ and if the following inequalities hold:*

$$\begin{cases} x_j^* \left(b_H, f_L\left(\cdot\right)\right)) < 2b_L \\ x_j^* \left(b_L, f_H\left(\cdot\right)\right) > 0, \end{cases}$$

then $x^* \in [0,\ 2b_L]^n$.

Proof. It directly follows from monotonicity of x^* and from the strategy spaces of the problem. □

We can also prove another standard property of oligopoly games:

Proposition 4.6. *If $x_j^*(b_k, \cdot)$ is a monotone pure strategy of Γ for player j, the j-th consumer surplus CS_j is monotone in $x_j^*(b_k, \cdot)$ irrespective of j's type.*

Proof. The j-th consumer surplus at equilibrium is the area of the triangle between the inverse demand function and its level corresponding to the equilibrium $x_j^*(b_k, \cdot)$, i.e.:

$$CS_j(x_j^*(b_k, \cdot)) = \frac{1}{2} x_j^*(b_k, \cdot) \left[b_k - \left(b_k - \frac{x_j^*(b_k, \cdot)}{2} \right) \right] = \frac{(x_j^*(b_k, \cdot))^2}{4},$$

then $CS_j(\cdot)$ is strictly monotone in x_j^* and does not depend on types. □

In the next section, we will investigate different kinds of functions $\Phi_{jL}(\cdot)$ and $\Phi_{jH}(\cdot)$, calculate the related potential functions explicitly, and determine the Nash equilibrium structures of the related BPGs.

4.4 Examples with different production functions

Case A

Consider an elementary case of Bayesian game where both production functions are linear, although having different slopes. Heterogeneity relies on the different individual effects from emissions, i.e.:

$$F_L\left(x\right) = \sum_{j=1}^{n} x_j \tag{4.15}$$

and

$$F_H\left(x\right) = \sum_{j=1}^{n} \delta_j x_j, \text{ with } \delta_j > 1 \ \ \forall\ j \in N. \tag{4.16}$$

Calling Γ_L the Bayesian game with F_L as in (4.15) and F_H defined as in (4.16), it is simple to prove the following:

Lemma 4.1. *Γ_L is a BPG with the following potential function:*

$$\Pi_L(x) = -\frac{1}{2}\sum_{j=1}^{n} x_j^2 + b_H \sum_{j=1}^{n} x_j +$$

$$+\sum_{j=1}^{n}(b_L - b_H - 1)\,x_j p_j^{LL} + \sum_{j=1}^{n}(b_L - b_H - \delta_j)\,x_j p_j^{LH} - \sum_{j=1}^{n} x_j p_j^{HL} - \sum_{j=1}^{n} \delta_j x_j p_j^{HH}. \qquad (4.17)$$

Proof. The conditions (4.7) hold for all $i, j = 1, \ldots, n$, then Γ_L is a BPG. (4.17) immediately follows from the application of (4.10). □

To determine the pure-strategy equilibria, it suffices to describe the FOCs:

$$\frac{\partial \Pi_L}{\partial x_i} = 0 \iff x_i = b_H + (b_L - b_H - 1)p_i^{LL} + (b_L - b_H - \delta_j)p_i^{LH} - p_i^{HL} - \delta_i p_i^{HH},$$

for all $i = 1, \ldots, n$. By Theorem 4.1, the pure strategies are monotone with respect to the poset $(T, \succcurlyeq)$. Namely, the i-th optimal strategies based on all possible type realizations are:

$$\begin{aligned}
\widehat{x}_i(b_H, f_L(\cdot)) &= b_H - 1\\
\widehat{x}_i(b_H, f_H(\cdot)) &= b_H - \delta_i\\
\widehat{x}_i(b_L, f_H(\cdot)) &= b_L - \delta_i\\
\widehat{x}_i(b_L, f_L(\cdot)) &= b_L - 1
\end{aligned}$$

and the verification of monotonicity is straightforward. Note that in this specific case they exactly correspond to marginal cost levels for all agents. In order to check the feasibility of such strategies, we have to state some suitable parametric conditions, in compliance with Proposition 4.5. $\widehat{x} \in [0,\ 2b_L]^n$ if:

$$\begin{cases} b_H - 1 < 2b_L \\ b_L - \delta_i > 0 \end{cases} \iff b_L > \max\left\{\delta_1, \delta_2, \ldots, \delta_n, \frac{b_H - 1}{2}\right\}.$$

Case B

A similar technique can be applied to a Bayesian game played by agents endowed with even more heterogenous types. In particular, a game in which the pollution production functions $f_k(\cdot)$ have different structures, for example the one corresponding to the lowest sensitivity to the stock of pollution which is linear and the one corresponding to the highest sensitivity which is quadratic.

In such a case, we will have:

$$F_L(x) = \sum_{j=1}^{n} \gamma_j x_j, \text{ with } \gamma_j > 1 \ \ \forall j \in N. \tag{4.18}$$

and

$$F_H(x) = \sum_{j=1}^{n} x_j^2. \tag{4.19}$$

Call Γ_{LQ} the Bayesian game with F_L as in (4.18) and F_H as in (4.19).

Lemma 4.2. *The Bayesian game Γ_{LQ} is a BPG with the following potential function:*

$$\Pi_{LQ}(x) = -\frac{1}{2}\sum_{j=1}^{n} x_j^2 + b_H \sum_{j=1}^{n} x_j +$$

$$+\sum_{j=1}^{n} [b_L - b_H - \gamma_j] x_j p_j^{LL} + \sum_{j=1}^{n} \left[(b_L - b_H) x_j - x_j^2\right] p_j^{LH} - \sum_{j=1}^{n} \gamma_j x_j p_j^{HL} - \sum_{j=1}^{n} x_j^2 p_j^{HH}. \tag{4.20}$$

Proof. (4.7) hold for all $i, j = 1, \dots, n$, then Γ_{LQ} is a BPG. The application of (4.10) yields (4.20). □

Note that in Γ_{LQ} the hypotheses of Theorem 4.1 are not verified in the whole strategy space $[0,\ 2b_L]^n$. In fact,

$$\frac{\partial \Phi_{iL}(x_i)}{\partial x_i} \leq \frac{\partial \Phi_{iH}(x_i)}{\partial x_i} \iff x_i \geq \frac{\gamma_i}{2},$$

hence we have to restrict the strategy space to $\Theta := \prod_{j=1}^{n} \left[\frac{\gamma_j}{2},\ 2b_L\right]$, which is nonempty if and only if $b_L > \max\left\{\frac{\gamma_1}{4}, \dots, \frac{\gamma_n}{4}\right\}$.

Because the structure of Π_{LQ} is linear-quadratic, it admits a unique maximizer as well, i.e.

$$\widehat{x}_i = \frac{[b_L - b_H - \gamma_i] p_i^{LL} + [b_L - b_H] p_i^{LH} - \gamma_i p_i^{HL}}{1 + 2p_i^{LH} + 2p_i^{HH}}, \tag{4.21}$$

so the profile strategy $\widehat{x}$ is a feasible and monotonic BNE for Γ_{LQ} if $\widehat{x} \in \prod_{j=1}^{n} \left[\frac{\gamma_j}{2},\ 2b_L\right]$. The i-th optimal strategies based on all types are as follows:

$$\begin{aligned}
\widehat{x}_i\left(b_H, f_L\left(\cdot\right)\right) &= b_H - \gamma_i \\
\widehat{x}_i\left(b_H, f_H\left(\cdot\right)\right) &= \frac{b_H}{3} \\
\widehat{x}_i\left(b_L, f_H\left(\cdot\right)\right) &= \frac{b_L}{3} \\
\widehat{x}_i\left(b_L, f_L\left(\cdot\right)\right) &= b_L - \gamma_i
\end{aligned}$$

and adapting the hypotheses of Proposition 4.5, the sufficient conditions for feasibility are given by:

$$\begin{cases} b_H - \gamma_i < 2b_L \\ \dfrac{b_L}{3} > \dfrac{\gamma_j}{2} \end{cases} \quad \Longleftrightarrow \quad b_L > \max\left\{\frac{b_H - \gamma_1}{2}, \ldots, \frac{b_H - \gamma_n}{2}, \frac{3\gamma_1}{2}, \ldots, \frac{3\gamma_n}{2}\right\}.$$

Case C

In this example we will deal with a potential having two stationary points. The damage functions are the following:

$$F_L\left(x\right) = \sum_{j=1}^{n} \log\left(x_j + 1\right), \tag{4.22}$$

and

$$F_H\left(x\right) = \sum_{j=1}^{n} x_j. \tag{4.23}$$

Call Γ_{LogL} the Bayesian game with F_L as in (4.22) and F_H as in (4.23).

Lemma 4.3. *The Bayesian game Γ_{LogL} is a BPG endowed with the following potential function:*

$$\Pi_{LogL}\left(x\right) = -\frac{1}{2}\sum_{j=1}^{n} x_j^2 + b_H \sum_{j=1}^{n} x_j +$$

$$+\sum_{j=1}^{n}\left[\left(b_L - b_H\right)x_j - \log(x_j+1)\right]p_j^{LL} + \sum_{j=1}^{n}(b_L - b_H - 1)x_j p_j^{LH} - \sum_{j=1}^{n}\log(x_j+1)p_j^{HL} - \sum_{j=1}^{n} x_j p_j^{HH}. \tag{4.24}$$

Proof. The formula (4.24) follows from the application of (4.10). □

The FOCs of this model read as:

$$\frac{\partial \Pi_{LogL}}{\partial x_i} = -x_i + b_H + \left[b_L - b_H - \frac{1}{x_i+1} \right] p_i^{LL} + (b_L - b_H - 1) p_i^{LH} - \frac{p_i^{HL}}{x_i+1} - p_i^{HH} = 0,$$

for all $i = 1, \ldots, n$.

The explicit computation of the related BNE deserves some detailed comments. To begin with, the conditions (4.14) hold because $\frac{1}{x_j+1} \leq 1$ for all $x_j \in [0,\ 2b_L]$. On the other hand, $\Pi_{LogL}(x)$ is not globally concave: in fact, $\frac{\partial^2 \Pi_{LogL}}{\partial x_i^2} \geq 0$ in the whole interval $[0,\ 2b_L]$ if and only if $p_i^{LL} = p_i^{HL} = 0$. Hence, the two unique pure strategies than can be immediately determined are:

$$\begin{aligned} \widehat{x}_i \left(b_H, f_H(\cdot)\right) &= b_H - 1 \\ \widehat{x}_i \left(b_L, f_H(\cdot)\right) &= b_L - 1, \end{aligned}$$

$\forall\ i \in N$. The remaining strategies must be deduced from a second degree equation. For example:

$$-\widehat{x}_i \left(b_H, f_L(\cdot)\right) + b_H - \frac{1}{\widehat{x}_i \left(b_H, f_L(\cdot)\right) + 1} = 0 \iff$$

$$\iff \widehat{x}_i \left(b_H, f_L(\cdot)\right) = \frac{b_H - 1 + \sqrt{b_H^2 + 2b_H - 3}}{2},$$

after discarding the negative root. Analogously, we have:

$$-\widehat{x}_i \left(b_L, f_L(\cdot)\right) + b_L - \frac{1}{\widehat{x}_i \left(b_L, f_L(\cdot)\right) + 1} = 0 \iff$$

$$\iff \widehat{x}_i \left(b_L, f_L(\cdot)\right) = \frac{b_L - 1 + \sqrt{b_L^2 + 2b_L - 3}}{2}.$$

Both of them are real for $b_L > 1$. As far as monotonicity is concerned, we note that all the inequalities

$$\widehat{x}_i \left(b_H, f_L(\cdot)\right) > \widehat{x}_i \left(b_H, f_H(\cdot)\right) > \widehat{x}_i \left(b_L, f_H(\cdot)\right),$$

$$\widehat{x}_i \left(b_H, f_L(\cdot)\right) > \widehat{x}_i \left(b_L, f_L(\cdot)\right) > \widehat{x}_i \left(b_L, f_H(\cdot)\right)$$

are simply verified for $b_L > 1$.

Finally, to ensure feasibility, we have to check the hypotheses of Proposition 4.5 to achieve a suitable parametric condition:

$$\begin{cases} \dfrac{b_H - 1 + \sqrt{b_H^2 + 2b_H - 3}}{2} < 2b_L \\ \\ b_L - 1 > 0 \end{cases} \iff b_L > \max\left\{ \frac{b_H - 1 + \sqrt{b_H^2 + 2b_H - 3}}{4},\ 1 \right\}.$$

In this chapter we proposed a new approach to model heterogeneity in oligopoly games, based on a 2-dimensional finite type structure, separately indicating benefit and cost characteristics of agents. In our opinion, an environmental n-countries game in which the revenue is a linear-quadratic function of emissions and the cost is a production function of the aggregate emissions is an appropriate application for our technique. We established a suitable preference order on the type spaces of the Bayesian game under consideration and subsequently took into examination the Bayesian potential structure of the game. We exploited additive separability in the strategic variables of the environmental cost functions to ensure the existence of a potential for the model. Such potential is relevant in that all information on the probability distributions of all types is collected. We found out that the cost structure emerging from the formula of the potential function provides necessary and sufficient conditions to ensure monotonicity of the pure strategies, in compliance with the partial order established on the type spaces. Finally, we applied our results to some different models, whose respective payoffs were endowed with linear, linear-quadratic and linear-logarithmic cost functions. The first model involves a unique pure strategy in the original strategy space, in the second one the pure strategy is unique as well but the strategy space must be restricted in order to ensure monotonicity, whereas the third model, relying on a non-standard payoff structure, does not have a globally concave potential function and then requires a different kind of analysis.

Part II

Cooperative Approach

Chapter 5

TU-Games: an overview

While in non-cooperative game theory, we focus on the individual players' strategies and their influence on payoffs, and try to predict what strategies players will choose (equilibrium concept), in cooperative game theory, we abstract from individual players' strategies and instead focus on the coalition players may form. We assume each coalition may attain some payoffs, and then we try to predict which coalitions will form (and hence the payoffs agents obtain).

In its classical interpretation, a TU-game describes a situation in which the players in every coalition S of N can cooperate to form a feasible coalition and earn its worth.

Solutions of TU-games are divided in two types: set solutions and point solutions.

As concerns the set solutions, the first option when searching for a solution to a cooperative game is the core. In the core no coalition of agents ends up preferring to stay alone to that resulting from overall cooperation. However, the possible emptiness is a serious limitation of the core concept. A rich class of TU-games with a nonempty core is the class of convex games. For a convex game the Shapley value appears to be a core selector. Simply and well-known examples of convex games are the so-called unanimity games that create a basis in the game space.

Shapley value is a well-knowed point solution. Axiomatizations of the Shapley value can be found in [36], [37], [72], [112], [113], [114], [115], [128]. Another point solution is the Banzhaf-Coleman index which, for example, is studied in [112] and [63].

5.1 Preliminary Definitions

Definition 5.1. *Let N be a finite set with cardinality n and let $v : 2^N \longrightarrow \mathbb{R}$ be a map such that $v(\emptyset) = 0$. The ordered pair (N, v) is a* ***side-payment game*** *or* ***trasferable utility game (TU-game)****.*

We can interpret N as the set of players, and $S \in 2^N$ is a coalition of player. v is called characteristic function of the game.
We denote with Γ the class of TU - games and with G^n the set of all characteristic functions v, corresponding to a TU - game (N, v).

Notation 5.1. *If $S, T \in 2^N$, the inclusion $S \subseteq T$, means that each player of coalition S is a player of the coalition T. In particular $\emptyset \subset T \quad \forall T \in 2^N$.*

Definition 5.2. *A TU-game $(N, v) \in \Gamma$ is* ***cohesive*** *if*

$$v(N) \geq \sum_{i=1}^{k} v(S_i) \qquad \forall \{S_1, \ldots, S_k\} \quad \text{partition of } N$$

Most of TU-games derived from practical situations have the superadditivity property.

Definition 5.3. *A TU-game $(N, v) \in \Gamma$ is* ***superadditive*** *if*

$$v(S \cup T) \geq v(S) + v(T) \qquad \forall S, T \in 2^N, \text{with} \quad S \cap T = \emptyset.$$

In a superadditive game the value of the union of two disjoint coalitions is at least as large as the sum of the values of the subcoalition separately.
The following definition is less interesting because there is no convenience to cooperation.

Definition 5.4. *A TU-game $(N, v) \in \Gamma$ is* ***additive [subadditive]*** *if*

$$v(S \cup T) = v(S) + v(T) \quad [v(S \cup T) \leq v(S) + v(T)] \qquad \forall S, T \in 2^N, \text{with} \quad S \cap T = \emptyset.$$

It is easy to show that a superadditive game is also a cohesive game:

Proposition 5.1. *Let $(N, v) \in \Gamma$ be a superadditive game, then (N, v) is a cohesive game.*

Proof. Let $\{S, T\}$ be a partition of N. Then

$$v(N) = v(S \cup T) \geq v(S) + v(T).$$

□

We can observe that an additive game is both superadditive and subadditive. Moreover by Proposition 5.1 it is also a cohesive game. However, a cohesive game is not necessarily a superadditive game as shown by the next example.

Example 5.1. *We consider the game (N, v) where $N = \{1, 2, 3\}$ and $v : 2^N \longrightarrow \mathbb{R}$ is defined as follows:*

$$v(S) = \begin{cases} 0 \textit{ if } S = \emptyset \\ 1 \textit{ if } |S| = 1 \\ \frac{3}{2} \textit{ if } |S| = 2 \\ 5 \textit{ if } S = N \end{cases}$$

Note that this is a cohesive game, but it is not a superadditive game because if we consider $S = \{1\}$ and $T = \{2\}$, we have that $v(S \cup T) = \frac{3}{2} < 1 + 1 = v(S) + v(T)$.

Definition 5.5. *A TU-game $(N, v) \in \Gamma$ is **convex [strictly convex]** if*

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T) \quad [v(S \cup T) + v(S \cap T) > v(S) + v(T)] \quad \forall S, T \in 2^N.$$

We can see that a convex game is also a superadditive game:

Proposition 5.2. *Let $(N, v) \in \Gamma$ be a convex game, then (N, v) is a superadditive game.*

Proof. We take $S, T \in 2^N$ such that $S \cap T = \emptyset$. Then

$$\begin{aligned} v(S \cup T) + v(S \cap T) &= v(S \cup T) + v(\emptyset) \\ &= v(S \cup T) + 0 \\ &\geq v(S) + v(T). \end{aligned}$$

□

However, the viceversa is not true as shown by the next example.

Example 5.2. *We consider the game* (N, v) *where* $N = \{1, 2, 3\}$ *and* $v : 2^N \longrightarrow \mathbb{R}$ *is defined as follows:*

$$v(S) = \begin{cases} 0 \ \textit{if} \ S = \emptyset \\ \frac{1}{4} \ \textit{if} \ |S| = 1 \\ \frac{3}{2} \ \textit{if} \ |S| = 2 \\ \frac{7}{4} \ \textit{if} \ S = N \end{cases}$$

We can notice that this is a superadditive game, but it is not a convex game because if we consider $S = \{1, 2\}$ *and* $T = \{1, 3\}$, *we have that* $v(S \cup T) + v(S \cap T) = \frac{7}{4} + \frac{1}{4} < \frac{3}{2} + \frac{3}{2} = v(S) + v(T)$.

Definition 5.6. *A TU-game* $(N, v) \in \Gamma$ *is* ***concave [strictly concave]*** *if*

$$v(S \cup T) + v(S \cap T) \leq v(S) + v(T) \quad [v(S \cup T) + v(S \cap T) < v(S) + v(T)] \quad \forall S, T \in 2^N.$$

We can observe that $(N, v) \in \Gamma$ is [strictly] convex iff $(N, -v) \in \Gamma$ is [strictly] concave.

Definition 5.7. *A TU-game* $(N, v) \in \Gamma$ *is* ***monotonic [strictly monotonic]*** *if*

$$v(S) \leq v(T) \quad [v(S) < v(T)] \quad \forall S, T \in 2^N \textit{with} \ \ S \subset T.$$

Definition 5.8. *A TU-game* $(N, v) \in \Gamma$ *is* ***essential*** *if*

$$v(N) > \sum_{i \in N} v(\{i\})$$

It is inessential otherwise.

Proposition 5.3. *Let* $(N, v) \in \Gamma$ *be a superadditive game such that* $v(S) \geq 0 \ \ \forall S \in 2^N$; *then* (N, v) *is a monotonic game.*

Proof. We take $S, T \in 2^N$ such that $S \subset T$. We can observe that $T = S \cup (T \setminus S)$. Then

$$\begin{aligned} v(T) &= v(S \cup (T \setminus S)) \\ &\geq v(S) + v(T \setminus S). \end{aligned}$$

Then $v(T) - v(S) \geq v(T \setminus S) \geq 0$. □

In particular, because of Proposition 5.2 the following statement holds.

Corollary 5.1. *If* $(N, v) \in \Gamma$ *is a convex game such that* $v(S) \geq 0 \quad \forall S \in 2^N$, *then it is also a monotonic game.*

We can note that an additive game is an inessential game.

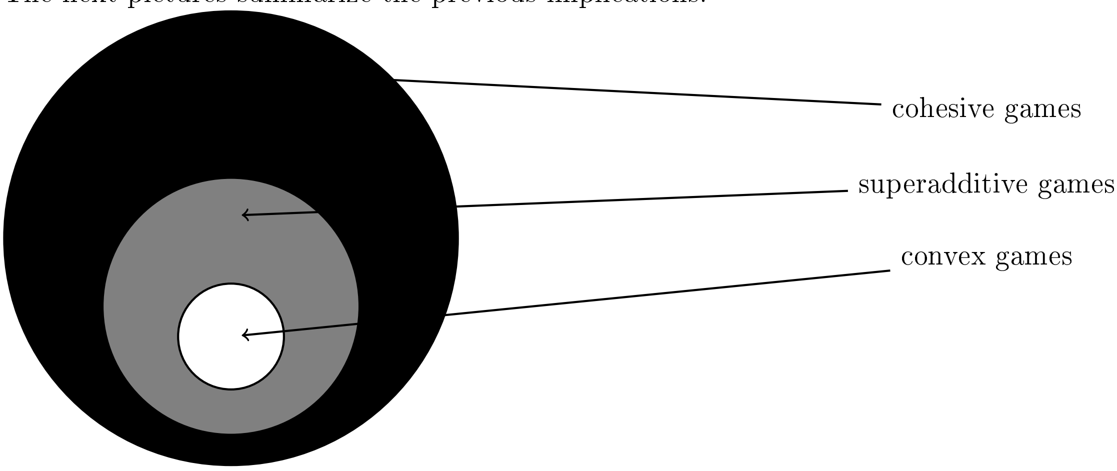
Proposition 5.4. *Let* $(N, v) \in \Gamma$ *be a monotonic game then* $v(S) \geq 0 \quad \forall S \in 2^N$.

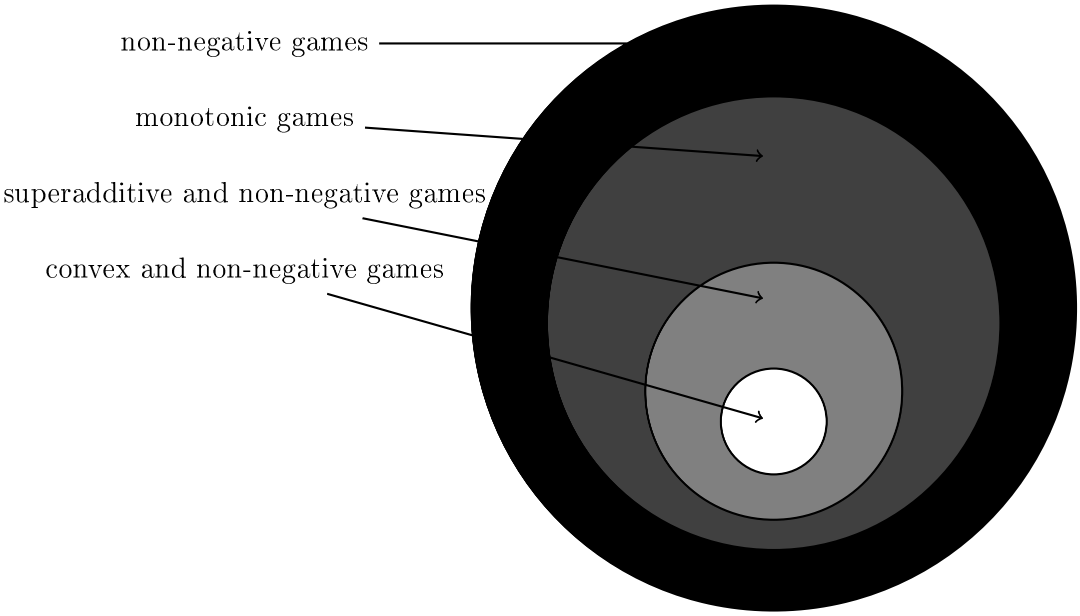
Proof. We have that $\emptyset \subset T \quad \forall T \in 2^N$. Then, by monotonicity, we have

$$0 = v(\emptyset) \leq v(T).$$

□

The next pictures summarize the previous implications.





We can also show that a convex game is not necessarily a monotonic game as the next example shows.

Example 5.3. *We consider the game (N, v) where $N = \{1, 2\}$ and $v : 2^N \longrightarrow \mathbb{R}$ is defined as follows:*

$$v(S) = \begin{cases} 0 \text{ if } S = \emptyset \\ -6 \text{ if } S = \{1\} \\ -4 \text{ if } S = \{2\} \\ -5 \text{ if } S = N \end{cases}$$

It is easy to show that this is a convex game, but it is not a monotonic game because if we consider $S = \{2\}$ and $T = \{1, 2\}$, we have that $S \subset T$ but $-4 = v(S) > v(T) = -5$.

Moreover, because of Proposition 5.2, the previous example shows that superadditivity condition is not sufficient for monotonicity.

We can also observe that the game of Example 5.3 is essential, so we conclude that convexity and essentiality are not sufficient conditions for monotonicity. (Obviously also superadditivity and essentiality are not sufficient conditions for monotonicity).

Now we show that monotonicity and essentiality are not sufficient conditions for superadditivity. We consider the following example.

Example 5.4. *We consider the game* (N, v) *where* $N = \{1, 2, 3\}$ *and* $v : 2^N \longrightarrow \mathbb{R}$ *is defined as follows:*

$$v(S) = \begin{cases} 0 \text{ if } S = \emptyset \\ \frac{1}{3} \text{ if } |S| = 1 \\ \frac{7}{4} \text{ if } |S| = 2 \\ 2 \text{ if } S = N \end{cases}$$

We can see that this is a monotonic and essential game, but it is not a superadditive game because if we consider $S = \{1\}$ *and* $T = \{2, 3\}$, *we have that*

$$\begin{aligned} v(S \cup T) &= v(N) \\ &= 2 \\ &< v(S) + v(T) \\ &= v(\{1\}) + v(\{2, 3\}) \\ &= \frac{1}{3} + \frac{7}{4} \\ &= \frac{25}{12}. \end{aligned}$$

Moreover, since this game is not cohesive, we can conclude that monotonicity and essentiality are not sufficient conditions for cohesivity.
In particular it follows that monotonicity and essentiality are not sufficient conditions for convexity.

Namely we can observe that monotonicity is not a sufficient condition for superadditivity (and for convexity).

Definition 5.9. *A TU-game* $(N, v) \in \Gamma$ *is said to be* ***constant -sum*** *if*

$$v(S) + v(N \setminus S) = v(N) \quad \forall S \in 2^N, \text{with } S \subset N.$$

Proposition 5.5. *Let $(N, v) \in \Gamma$ be an additive game then (N, v) is a constant- sum game.*

Proof. We take $S, T \in 2^N$ such that $T = N \setminus S$. Then

$$v(S) + v(T) = v(S) + v(N \setminus S) = v(N)$$

$\square$

It is easy to show that the opposite does not hold.

The most important problem for a TU-game (N, v) is how to divide the profits among the players. Indeed there is not an unique rule: the theory does not tell us which solution we have to choose, but it describes the property of solutions, highlighting positive and negative aspects.

For further details see [91] (pages 212-233) and [107] (pages 60-66).

The next subsections are devoted to describe some kind of solutions.

5.2 Imputation and Core

Definition 5.10. *Take $(N, v) \in \Gamma$. A vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is called* ***allocation****. If $\sum_{i \in N} x_i = v(N)$ then the allocation x is called* ***pre-imputation****. A pre imputation x such that $x_i \geq v(\{i\}) \ \ \forall i \in N$, is called* ***imputation****.*

A pre-imputation is a distribution of $v(N)$ among players. The condition $\sum_{i \in N} x_i = v(N)$ is an efficient condition or also called *collective rationality*. On the other hand the condition $x_i \geq v(\{i\})$ is called *individual rationality*.

The set of imputations of the TU-game (N, v) is denoted by $I(v)$.

From another point of view we can think of the imputation as a correspondence

$I : G^n \rightrightarrows \mathbb{R}^n$ where

$$I(v) = \left\{ x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N), x_i \geq v(\{i\}) \right\}.$$

Proposition 5.6. *Let $(N, v) \in \Gamma$ be a cohesive game then $I(v) \neq \emptyset$.*

Proof. Because (N, v) is a cohesive game, in particular we have

$$v(N) \geq \sum_{i=1}^{n} v(\{i\}).$$

Then $\exists \epsilon_i \geq 0 \ \ \forall i = 1, \ldots, n$ such that

$$v(N) = \sum_{i=1}^{n} (v(\{i\}) + \epsilon_i).$$

We can define the vector $x = (x_1, \ldots, x_n)$ with $x_i = v(\{i\}) + \epsilon_i \ \ \forall i = 1, \ldots, n$.

Obviously $x \in I(v)$. □

Naturally, it might happen that $I(v) = \emptyset$ as shown by the following example.

Example 5.5. *We consider the game (N, v) where $N = \{1, 2, 3\}$ and $v : 2^N \longrightarrow \mathbb{R}$ is defined as follows:*

$$v(S) = \begin{cases} 0 \ \textit{if} \ S = \emptyset \\ 1 \ \textit{if} \ |S| = 1 \\ \frac{3}{2} \ \textit{if} \ |S| = 2 \\ 2 \ \textit{if} \ S = N \end{cases}$$

Definition 5.11. *Take $(N, v) \in \Gamma$, let $x, y \in I(v)$ be two imputations, and let S be a coalition. We say x **dominates** y **through** S $(x \vdash_S y)$ if*

- $x_i > y_i \quad \forall i \in S$;
- $\sum_{i \in S} x_i \leq v(S)$.

*We say x **dominates** y $(x \vdash y)$ if there is a coalition S such that $x \vdash_S y$.*

Definition 5.12. *Take (N, v), $(N, w) \in \Gamma$. We say that they are **isomorphic** if there exists a bijection $f : I(v) \to I(w)$ such that*

$$x \vdash_S y \Leftrightarrow f(x) \vdash_S f(y) \quad \forall x, y \in I(v), \ S \subset N.$$

It may be difficult to tell whether two games are isomorphic in this sense. We have, however, the following criterion:

Definition 5.13. *Take $(N, v), (N, w) \in \Gamma$. We say that they are* ***S-equivalent*** *if there exist a positive number r and n real constants $\alpha_1, \ldots, \alpha_n$ such that*

$$v(S) = rw(S) + \sum_{i \in S} \alpha_i \quad \forall S \subset N.$$

Essentially, if two games are S-equivalent, we can obtain one from the other simply by performing a linear transformation on the utility space of the several players. It is easy to prove that S-equivalent implies isomorphism:

Theorem 5.1. *If (N, v), $(N, w) \in \Gamma$ are S-equivalent, they are isomorphic.*

Proof. See [91] (page 216) □

It is obvious that S-equivalence is, indeed, an equivalence relation. It is interesting to choose one particular game from each equivalent class.

Definition 5.14. *A TU-game $(N, v) \in \Gamma$ is said to be in* ***0 normalization*** *if*

- $v(\{i\}) = 0 \quad \forall i \in N$.

Theorem 5.2. *If $(N, w) \in \Gamma$ is a TU-game, it is S-equivalent to exactly one game in 0 normalization.*

Proof. It suffices to take $v(S) = w(S) - \sum_{i \in S} w(\{i\})$. In fact (N, v) is S-equivalent to (N, w) and it is also 0 normalized. □

Definition 5.15. *A TU-game $(N, v) \in \Gamma$ is said to be in* ***(0,1) normalization*** *if*

- $v(\{i\}) = 0 \quad \forall i \in N$;
- $v(N) = 1$.

Theorem 5.3. *If $(N, w) \in \Gamma$ is an essential game, it is S-equivalent to exactly one game in $(0, 1)$ normalization.*

Proof. It is sufficient to take $v(S) = \frac{w(S)-\sum_{i\in S} w(\{i\})}{w(N)-\sum_{i\in N} w(\{i\})}$. Indeed (N, v) is S-equivalent to (N, w) and it is also $(0, 1)$ normalized. □

There are two special types of games which are of interest:

Definition 5.16. *Take $(N, v) \in \Gamma$. We say that it is **symmetric** if $v(S)$ depends only on the number of elements in S.*

Definition 5.17. *Take $(N, v) \in \Gamma$ in $(0, 1)$ normalization. We say that it is **simple** if, for each $S \subset N$, we have either $v(S) = 0$ or $v(S) = 1$.*

We can note that a game is simple if its $(0, 1)$ normalization is simple.

In a simple game, a coalition S is said to be a **winning coalition** if $v(S) = 1$ and a **losing coalition** if $v(S) = 0$. So in a simple game every coalition is either winning or losing.

We can note that if $(N, v) \in \Gamma$ is a superadditive, simple game then every subset of a losing coalition is losing, and every superset of a winning coalition is winning.

We can interpret a simple game in the following way: players are members of legislature or members of the board of directors of a corporation, etc. In such games, a proposed bill or decision is either passed or rejected. Those subsets of the players that can approve bills without outside help are called winning coalitions while those that cannot are called losing coalitions.

Typical examples of simple games $(N, v) \in \Gamma$ are

- the **majority rule game** where $v(S) = 1$ if $|S| > n/2$, and $v(S) = 0$ otherwise;
- the **unanimity game** where $v(S) = 1$ if $S = N$ and $v(S) = 0$ otherwise;
- the **dictator game** where $v(S) = 1$ if $\{1\} \in S$ and $v(S) = 0$ otherwise.

We introduce a concept of solution, the core, that selects the imputations which have an other rationality property.

Definition 5.18. *Take $(N, v) \in \Gamma$. The* **core** *of the game is a vector $x = (x_1, \ldots, x_n) \in I(v)$ such that*

$$\sum_{i \in S} x_i \geq v(S) \qquad \forall S \in 2^N \setminus \{\emptyset\} .$$

The set of core elements of the TU-game (N, v) is denoted by $C(v)$.

We can observe that $C(v)$ is a convex set.

From an other point of view we can think the core as a correspondence $C : G^n \rightrightarrows \mathbb{R}^n$ where

$$C(v) = \left\{ x \in I(v) \mid \sum_{i \in S} x_i \geq v(S) \right\}$$

Obviously $C(v) \subseteq I(v)$. Consequely $I(v) = \emptyset$, implies $C(v) = \emptyset$. But also for a superadditive game it can happen that $C(v) = \emptyset$ as shown by the next example.

Example 5.6. *We consider the game (N, v) where $N = \{1, 2, 3\}$ and $v : 2^N \longrightarrow \mathbb{R}$ is defined as follows:*

$$v(S) = \begin{cases} 0 \ \textit{if} \ S = \emptyset \\ 0 \ \textit{if} \ |S| = 1 \\ 1 \ \textit{if} \ |S| \geq 2. \end{cases}$$

It is easy to show that this is a superdditive game, but $C(v) = \emptyset$.

Moreover we have

Theorem 5.4. *Let $(N, v) \in \Gamma$ be an essential and constant-sum game, then $C(v) = \emptyset$.*

Proof. See [91] (page 220). □

The concept of the core is useful as a measure of stability. As a solution concept, it presents a set of imputations without distinguishing one point of the set as preferable to another. Indeed, the core may be empty.

For further details see [91] (pages 212-233) and [107] (pages 67-73).

5.3 Indices of Power

Here we deal with the concept of a value. In this approach, one tries to assign to each game in coalitional form a unique vector of payoffs, called the value. The i-th entry of the value vector may be considered as a measure of the value or power of the i-th player in the game. Alternatively, the value vector may be thought of as an arbitration outcome of the game decided upon by some fair and impartial arbiter.

Definition 5.19. *A function $\phi : G^n \to \mathbb{R}^n$ is called **value**.*

Here $\phi_i(v)$ represents the worth or value of player $i \in N$ in the TU-game (N, v).

In this section we introduce two values: the **Shapley value** and the **Banzhaf-Coleman index**, and their variations.

To define Shapley value we give an axiomatic treatment.

Definition 5.20. *Take $(N, v) \in \Gamma$ and let $\sigma : N \to N$ be a permutation of the set N. Then, by $(N, \sigma v)$ we mean the TU-game such that,*

$$\sigma v(S) = v(\sigma(S)) \quad \forall S \in 2^N.$$

Definition 5.21. *Take $(N, v) \in \Gamma$ we called **dummy player** a player $i \in N$ such that*

$$v(S \cup \{i\}) = v(S) + v(\{i\}) \quad \forall S \in 2^N \text{with } i \notin S.$$

Definition 5.22. *Take $(N, v) \in \Gamma$ we called **null player** a player $i \in N$ such that*

$$v(S \cup \{i\}) = v(S) \quad \forall S \in 2^N \text{with } i \notin S.$$

Definition 5.23. *Take $(N, v) \in \Gamma$ we called **symmetric players** two players $i, j \in N$ such that*

$$v(S \cup \{i\}) = v(S \cup \{j\}) \quad \forall S \in 2^N \text{with } i, j \notin S.$$

Axioms 5.1. *Take $(N, v),\ (N, w) \in \Gamma$, then a value $\phi : G^n \to \mathbb{R}^n$ satisfies the next axioms if*

Axiom 1 ***(efficiency)*** $\sum_{i \in N} \phi_i(v) = v(N)$;

Axiom 2 ***(anonimity)*** $\phi_{\sigma(i)}(\sigma v) = \phi_i(v) \quad \forall \sigma : N \to N$ *permutation;*

Axiom 3 ***(dummy player)*** $\phi_i(v) = v(\{i\}) \quad \forall i \in N$ *dummy player;*

Axiom 4 ***(additivity)*** $\phi_i(v + w) = \phi_i(v) + \phi_i(w) \quad \forall i \in N.$

Theorem 5.5. *There is a unique value $\phi : G^n \to \mathbb{R}^n$ that satisfies efficiency, anonimity, dummy player and additivity.*

Proof. For the proof see [91] (pages 262-265). □

This unique value is called **Shapley value** and denoted with φ. Given $(N, v) \in \Gamma$, it is explicitly defined as follows

$$\varphi_i(v) = \sum_{\substack{i \in S \\ S \subseteq N}} \frac{(s-1)!(n-s)!}{n!} \left[v(S) - v(S \setminus \{i\}) \right], \tag{5.1}$$

where s denotes the cardinality of S.

It can be seen that

$$\sum_{\substack{i \in S \\ S \subseteq N}} \frac{(s-1)!(n-s)!}{n!} = 1.$$

We can note that the Shapley value is not only additive but also satisfies the following stronger axiom of linearity.

Axioms 5.2. *Take $(N, v),\ (N, w) \in \Gamma$, then a value $\phi : G^n \to \mathbb{R}^n$ satisfies the next axioms if*

Axiom 5 ***(linearity)*** $\phi_i(av + bw) = a\phi_i(v) + b\phi_i(w) \quad \forall i \in N$ *and* $\quad \forall a, b \in \mathbb{R}.$

We can also give an alternative definition of Shapley value. First of all we give the following definitions.

Definition 5.24. *Let $(N, v) \in \Gamma$ be a TU-game, take $i \in N$ and let $\sigma : N \longrightarrow N$ be a permutation. Take $j \in N$ such that $i = \sigma(j)$. A* ***marginal contribution of*** *$i \in N$* ***to the coalition*** *$\{\sigma(1), \ldots, \sigma(j-1)\}$ is the number*

$$m_i^{\sigma}(v) = v(\{\sigma(1), ..., \sigma(j)\}) - v(\{\sigma(1), ..., \sigma(j-1)\}).$$

We denote with $m^{\sigma}(v)$ the vector of $\mathbb{R}^n$ with component $m_i^{\sigma}(v)$.

Definition 5.25. *Let $(N, v) \in \Gamma$ be a TU-game and let $\sigma : N \longrightarrow N$ be a permutation. A **set of predecessor of** $i \in N$ **in** σ is the set*

$$P_\sigma(i) = \left\{ j \in N \mid \sigma^{-1}(j) < \sigma^{-1}(i) \right\}.$$

We can observe that, with the previous definitions $m_i^\sigma(v) = v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i))$. Then we have the following definition:

Definition 5.26. *Let $(N, v) \in \Gamma$ be a TU-game, then the Shapley value is defined in the following way:*

$$\varphi_i(v) = \frac{1}{n!} \sum_{\sigma \in \pi(N)} m_i^\sigma(v), \tag{5.2}$$

where $\pi(N)$ is the set of permutations of N.

Theorem 5.6. *Let $(N, v) \in \Gamma$ be a convex game, then $\varphi(v) \in C(v)$.*

Proof. Take $S \subseteq N$ and $\sigma : N \to N$ a permutation. Let $i_1, \ldots, i_k, \ldots, i_s$ be the elements of S in the order in which there are in σ.

So by definition of marginal contribution and for superadditivity of (N, v), we have

$$m_{i_k}^\sigma(v) = v(P_\sigma(i_k) \cup \{i_k\} - v(P_\sigma(i_k))) \geq v(\{i_k\}),$$

and

$$\sum_{k=1}^{n} m_{i_k}^\sigma(v) = v(N).$$

Moreover, for the convexity of (N, v), we have

$$\sum_{k=1}^{s} m_{i_k}^\sigma(v) \geq \sum_{k=1}^{s} v(P_\sigma(i_k) \cup \{i_k\}) - v(P_\sigma(i_k)) = v(S)$$

That is $m^\sigma(v) \in C(v)$. Moreover, by convexity of $C(v)$ and for Definition 5.26, we have $\varphi(v) \in C(v)$.

□

From previous theorem follows the next claims.

Corollary 5.2. *Let $(N, v) \in \Gamma$ be a superadditive game, then $\varphi(v) \in I(v)$.*

Corollary 5.3. *Let $(N, v) \in \Gamma$ be a convex game, then $C(v) \neq \emptyset$.*

There are also many others axiomatizations of the Shapley value. For example, let us consider the following axioms:

Axioms 5.3. *Take $(N, v), (N, w) \in \Gamma$, then a value $\phi : G^n \to \mathbb{R}^n$ satisfies the next axioms if*

Axiom 6 ***(null player)*** $\phi_i(v) = 0 \quad \forall i \in N$ *null player;*

Axiom 7 ***(fairness)*** $\phi_i(v + w) - \phi_i(v) = \phi_j(v + w) - \phi_j(v) \quad \forall i, j \in N$ *symmetric players.*

Then, in [113] the author proves the next theorem.

Theorem 5.7. *There is a unique value $\phi : G^n \to \mathbb{R}^n$ that satisfies efficiency, null player and fairness. This value is the Shapley value.*

Proof. See [113]. □

For further details see [112] and [128] where the authors characterize the Shapley value on the class of monotonic games.

One of the principal difficulties with the Shapley value is that its computation generally requires the sum of a very large number of terms. Thus, even when the characteristic function is easy to define, evaluation may require a prohibitive amount of work. Recourse is therefore frequently had to multilinear extension (MLE) of the game.

Definition 5.27. *Let $(N, v) \in \Gamma$ be a TU-game, then the* ***multilinear extension*** *of (N, v) is defined in the following way:*

$$h(x_1, \ldots, x_n) = \sum_{S \subset N} \left(\prod_{i \in S} x_i \prod_{i \notin S} (1 - x_i) \right) v(S), \tag{5.3}$$

with $x_i \in [0, 1] \quad \forall i \in N$.

The following property relates the MLE to the Shapley value.

Theorem 5.8. *Let $(N, v) \in \Gamma$ be a TU-game, and h its multilinear extension, then*

$$\varphi_i(v) = \int_0^1 \frac{\partial h}{\partial x_i}(t, \dots, t)dt \quad \forall i \in N. \tag{5.4}$$

For the proof see [91] (page 270) and [107] (page 94), while for more classical results see [91] (pages 261-280) and [107] (pages 85-95).

We can observe that if we take the subclass of essential simple games we have that $v(N) = 1$, for any (N, v) in this subclass and hence the sum of two simple games (N, v) and (N, w) does not belong to this subclass. Therefore, this has motivated several authors to introduce an alternative version of the additive axiom, see [36], [37] and [72].

In the following, as in [115], we denote the class of simple games by Γ^s and its subclass of essential, monotonic simple games by Γ^{sm}, and we denote by G_s^n, and G_{sm}^n the set of all characteristic functions v, corresponding to a TU- game $(N, v) \in \Gamma^s$ and Γ^{sm}, respectively. The alternative version of the additivity axiom in case of simple games makes use of the concept of the maxgame and the mingame of two games.

Definition 5.28. *Take (N, v), $(N, w) \in \Gamma$, then the* ***maxgame*** *of (N, v) and (N, w) is denoted (N, z_{vw}^+) where*

$$z_{vw}^+ = \max\left[v(S), w(S)\right] \quad \forall S \subseteq N,$$

and the ***mingame*** *of (N, v) and (N, w) is denoted (N, z_{vw}^-) where*

$$z_{vw}^- = \min\left[v(S), w(S)\right] \quad \forall S \subseteq N.$$

Clearly, when both $(N, v), (N, w) \in \Gamma^s$, then also $(N, z_{vw}^+), (N, z_{vw}^-) \in \Gamma^s$.

Moreover, we have the following lemma.

Lemma 5.1. *Let (N, v), $(N, w) \in \Gamma^s$ be two simple games. Then it holds that*

$$v(S) + w(S) = z_{vw}^+(S) + z_{vw}^-(S) \quad \forall S \subseteq N.$$

Proof. It follows immediately by Definition 5.28. □

Dubey in [36] stated the next additivity axiom for a value.

Axioms 5.4. *Take $(N,v),(N,w) \in \Gamma^s$, then a value $\phi : G_s^n \to \mathbb{R}^n$ satisfies the next axiom if*

Axiom 8 ***(minmax additivity)*** $\phi_i(z_{vw}^+) + \phi_i(z_{vw}^-) = \phi_i(v) + \phi_i(w) \quad \forall i \in N.$

The next theorem states the Shapley value is the unique value satisfying the axioms of efficiency, anonimity, dummy player and minmax additivity.

Theorem 5.9. *There is a unique value $\phi : G_s^n \to \mathbb{R}^n$ that satisfies efficiency, anonimity, dummy player and minmax additivity.*

Proof. See [36]. □

This unique value is called **Shapley-Shubik index**. Given $(N,v) \in \Gamma^s$, it is explicitly defined as follows

$$\varphi_i(v) = \sum_{\substack{i \in S \\ S \subseteq N}} \frac{(s-1)!(n-s)!}{n!}, \tag{5.5}$$

where s denotes the cardinality of S, and where the summation is taken over all winning coalition S such that $S \setminus \{i\}$ is not winning.

Another index of power has been suggested by Banzhaf and Coleman.

Definition 5.29. *Let $(N,v) \in \Gamma$ be a TU-game . We define the* ***Banzhaf-Coleman index*** *as*

$$\chi_i(v) = \sum_{\substack{i \in S \\ S \subseteq N}} \left(\frac{1}{2}\right)^{n-1} [v(S) - v(S \setminus \{i\})].$$

There is a certain relation between the Shapley value and the Banzhaf-Coleman index: both give averages of player i's marginal contributions $v(S) - v(S \setminus \{i\})$. The difference lies in the weighting coefficients used: for the Shapley value, these varied according to the size of S; for the Banzhaf-Coleman index, they are all equal.

It follows immediately by definition that the Banzhaf-Coleman index satisfies anonimity, dummy player, linearity and in particular, additivity. We can note that the Banzhaf-Coleman index does not satisfy the efficiency axiom. Infact we can consider the following example.

Example 5.7. *Let us consider the following simple game* $(N, v) \in \Gamma$ *and let us calculate the Banzhaf-Coleman index.* $N = \{1, 2, 3\}$ *and*

$$v(S) = \begin{cases} 1 \text{ if } |S| \geq 2 \\ 0 \text{ otherwise }. \end{cases}$$

We have $\chi(v) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

Axiomatizations of the Banzhaf-Coleman index on the class of simple games have been given in [37] and in [72]. In the latter paper an axiomatization is also given for the general case by using a monotonicity property with respect to the amalgamation of two players to one player. In [57] is provided an axiomatization for the Banzhaf-Coleman index on the class of TU-games Γ by using an axiom of neutral collusion, besides the standard axioms of dummy player, anonimity and linearity.

We can consider the following definition.

Definition 5.30. *Let* $(N, v) \in \Gamma$ *be a TU-game and let* $i, j \in N$ *be two different players. Take* $\zeta = \{i, j\}$ *a* ***reduced game*** *is a TU-game* $(N \setminus \{j\}, v_\zeta) \in \Gamma$ *where*

$$v_\zeta(S) = v(S) \quad \text{and} \quad v_\zeta(S \cup \{\zeta\}) = v(S \cup \{\zeta\}) \quad \forall S \subseteq N \setminus \{\zeta\}.$$

Clearly $(N \setminus \{j\}, v_\zeta)$ is a TU-game with $n - 1$ players obtained by amalgamating the players i and j in the game (N, v) into one player ζ.

In this way we can revisit the efficiency property in the following axiom.

Axioms 5.5. *Take* $(N, v) \in \Gamma$, *then a value* $\phi : G^n \to \mathbb{R}^n$ *satisfy the next axiom if*

Axiom 9 ***(2- efficiency)*** $\phi_i(v) + \phi_j(v) = \phi_\zeta(v_\zeta) \quad \forall i \neq j \in N.$

Now we introduce the axioms that characterize the Banzhaf-Coleman index.

Axioms 5.6. *Take* $(N, v) \in \Gamma$, *then a value* $\phi : G^n \to \mathbb{R}^n$ *satisfy the next axiom if*

Axiom 10 ***(equal treatment)*** $\phi_i(v) = \phi_j(v) \quad \forall i, j \in N$ *symmetric players;*

Axiom 11 ***(marginal contributions)*** *Take* $(N,v),\ (N,w) \in \Gamma$. *If for some player* $i \in N$ *we have*

$$v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S) \quad \forall S \subseteq N \setminus \{i,j\}\,, \text{ then } \phi_i(v) = \phi_i(w).$$

Theorem 5.10. *There is a unique value* $\phi : G^n \to \mathbb{R}^n$ *that satisfies 2-efficiency, dummy player, equal treatment and marginal contributions. This value is the Banzhaf-Coleman index.*

Proof. See [87]. □

Also for the Banzhaf-Coleman index there is a characterization through the multilinear extension, see for example [90].

Theorem 5.11. *Let* $(N,v) \in \Gamma$ *be a TU-game, and* h *its multilinear extension, then*

$$\chi_i(v) = \frac{\partial h}{\partial x_i}\left(\frac{1}{2}, \dots, \frac{1}{2}\right) \qquad \forall i \in N. \tag{5.6}$$

Proof. See [91] (pages 294-297). □

See also [92] for a reformulation of the Banzhaf-Coleman index.

For a simple game, we can define also the **Normalized Banzhaf-Coleman index**. First of all we formalize a concept already discussed.

Definition 5.31. *Let* $(N,v) \in \Gamma^s$ *be a simple game [in* $(0,1)$ *normalization], a* ***swing*** *for player* $i \in N$ *is a set* $S \subset N$ *such that* $i \in S$, S *wins and* $S \setminus \{i\}$ *loses.*

Definition 5.32. *Let* $(N,v) \in \Gamma^s$ *be a simple game [in* $(0,1)$ *normalization], and let* θ_i *be the number of swings for player* $i \in N$. *We define the* ***Normalized Banzhaf-Coleman index on simple games*** *as*

$$\beta_i(v) = \frac{\theta_i}{\sum_{j \in N} \theta_j}.$$

Clearly, we can also define the Normalized Banzhaf-Coleman index on the more general class of TU-games.

Definition 5.33. *Let $(N, v) \in \Gamma$ be a TU-game. We define the* ***Normalized Banzhaf-Coleman index*** *as*

$$\beta_i(v) = \eta(v) \cdot \chi_i(v) \quad \forall i \in N,$$

where

$$\eta(v) = \frac{v(N)}{\sum_{j \in N} \chi_j(v)}.$$

The Normalized Banzhaf-Coleman index (on simple games and on TU- games in general) is efficient and it satisfies anonimity. It does not satisfy linearity and dummy player property.

In [112] there is a characterization of the Normalized Banzhaf-Coleman index on the class of monotonic games.

There is a large class of simple games called **weighted voting games**.

Definition 5.34. *A TU-game $(N, v) \in \Gamma$ is a* ***weighted voting game*** *if*

$$v(S) = \begin{cases} 1 \text{ if } \sum_{i \in S} w_i > q \\ 0 \text{ if } \sum_{i \in S} w_i \leq q, \end{cases}$$

for some non-negative numbers w_i, called the ***weights****, and some positive number q, called the* ***quota****. If $q = \frac{1}{2} \sum_{i \in S} w_i$, this is called a* ***weighted majority game****.*

Example 5.8. *The elections of Ireland held February 2011 are a 7-players weighted voting game (N, v). The Ireland Parliament has 166 seats so the quota q is fixed to 84, that is the majority plus one. The weights are the seats which each party get. The winner coalitions are:*

$$\{FG, LP\}, \ \{FG, FF\}, \ \{FG, I\}, \ \{FG, SF\}.$$

All the coalitions containing the previous four coalitions are winning. Moreover the following coalitions are winning

$$\{LP, FF, I, SF\}, \ \{LP, FF, I, SF, SP\}, \ \{LP, FF, I, SF, PBP\}, \ \{LP, FF, I, SF, SP, PBP\}.$$

Let us compute the number of swings (θ), divided per cardinality of coalitions (s), of players:

Now we can calculate the three power indices: Shapley value (φ), Banzhaf-Coleman index (χ), Normalized Banzhaf-Coleman index (β).

$\varphi_{FG}(v) = 0.6,\ \varphi_{LP}(v) = 0.1\ \varphi_{FF}(v) = 0.1\ \varphi_I(v) = 0.1\ \varphi_{SF}(v) = 0.1\ \varphi_{SP}(v) = 0\ \varphi_{PBP}(v) = 0.$

$\chi_{FG}(v) = 0.875\ \chi_{LP}(v) = 0.125\ \chi_{FF}(v) = 0.125\ \chi_I(v) = 0.125\ \chi_{SF}(v) = 0.125\ \chi_{SP}(v) = 0\ \chi_{PBP}(v) = 0.$

$\beta_{FG}(v) = \frac{56}{88}\ \beta_{LP}(v) = \frac{8}{88}\ \beta_{FF}(v) = \frac{8}{88}\ \beta_I(v) = \frac{8}{88}\ \beta_{SF}(v) = \frac{8}{88}\ \beta_{SP}(v) = 0\ \beta_{PBP}(v) = 0.$

Table 5.1 represents the three power indices: Shapley value (φ), Banzhaf-Coleman index (χ), Normalized Banzhaf-Coleman index (β).

Table 5.1: Elections of Ireland 2011

Parties	**Seats**	φ	χ	β
FG	76	0.6	0.875	$\frac{56}{88}$
LP	37	0.1	0.125	$\frac{8}{88}$
FF	20	0.1	0.125	$\frac{8}{88}$
I	15	0.1	0.125	$\frac{8}{88}$
SF	14	0.1	0.125	$\frac{8}{88}$
SP	2	0	0	0
PBP	2	0	0	0

Example 5.9. *The elections of Finland held April 2011 are a 9-players weighted voting game* (N, v). *The Finnish Parliament has* 200 *seats so the quota q is fixed to* 101, *that is the majority plus one. The weights are the seats which each party get.*

Table 5.2 represents the three power indices: Shapley value (φ), Banzhaf-Coleman index (χ), Normalized Banzhaf-Coleman index (β).

Table 5.2: Elections of Finland 2011

Parties	**Seats**	φ	χ	β
KOK	44	$0,221428571428571$	$0,44921875$	$\frac{115}{527}$
SDP	42	$0,2$	$0,40234375$	$\frac{103}{527}$
PS	39	$0,176190476190476$	$0,35546875$	$\frac{91}{527}$
KESK	35	$0,147619047619047$	$0,29296875$	$\frac{75}{527}$
VAS	14	$9,64285714285705 \cdot 10^{-2}$	$0,20703125$	$\frac{53}{527}$
VIHR	10	$6,19047619047615 \cdot 10^{-2}$	$0,13671875$	$\frac{35}{527}$
SFP	9	$5,47619047619044 \cdot 10^{-2}$	$0,12109375$	$\frac{31}{527}$
KD	6	$3,69047619047616 \cdot 10^{-2}$	$0,08203125$	$\frac{21}{527}$
RA	1	$4,76190476190458 \cdot 10^{-3}$	$0,01171875$	$\frac{3}{527}$

Chapter 6

A new perspective on cooperative games

In the following chapter we are going to propose a new perspective on cooperative games, by assuming that the involved players are supposed to face a common damage. The agents can choose to make an agreement and form a coalition or to defect and face such damage individually.

When such disadvantage is modeled by a dynamic state variable evolving over time, cooperating and non-cooperating agents solve different optimization problems, but they all must take into account such state variable, as if it represented an externality in all their respective value functions. Even if we just consider the cooperative and static aspects of such a game, the externality has a key role in the worth of coalitions.

The approach we will develop relies on a class of cooperative games including an external effect, such that the characteristic value function is split in two parts: one of them is standard, the other one is affected by externality.

It is worth describing our idea of externality, which basically differs from the previous characterizations in literature. Transferable utility games with positive externalities were defined by [99], which related such externality to an increase in pay-off for the players in a specific coalition when the remaining coalitions committed to merging. That is, in presence of a partition of the set of agents and of multiple coalitions, a group of players may enjoy a positive spillover originating from a merger of external coalitions rather than from a strategic choice.

In our case, on the other hand, the role of externality is played, and its amount is measured, by

a different state variable, not directly depending on the possibly undertaken agreements. Loosely speaking, in our setting externalities arise in the same way as they do in standard dynamic oligopoly models (see [64]).
When we relate this idea to the welfare of a country dealing with an emission reduction strategy, we stress that the clean share of welfare is always positive, whereas the share including the pollution effect is negative, then the total welfare must be globally evaluated.
In recent years, a growing interest has been devoted to dynamic models of pollution abatement (on which [64] is quite an exhaustive survey), within both cooperative and non-cooperative frameworks. In particular, the design and the modeling of International Environmental Agreements (IEA, from now on) have been extensively and critically discussed in [46]. Just to cite some recent examples, [48] investigated stability of coalitions to form IEAs empirically, [77] examined IEAs from the viewpoint of evolutionary game theory, whereas [71] concentrated on the cooperative dynamic allocation of total costs incurred by countries.
Our starting points are [5] and [18], which on their turn are related to [52] and [53]. We are going to arrange a theoretical setup building on their model, investigate a wider set of properties for such game structures, and finally carry out a coalitional power assessment in detail.

Substantially, this game relies on a cooperative structure which is generated as agents play the strategies of a dynamic optimization game of pollution reduction. In this game each country commits to maximizing her welfare either joining an IEA or refusing to join it, during the process of accumulation of an aggregate stock of pollution. All the countries implement their optimal emissions strategies, which differ between signatory and non-signatory countries. At a given level of pollution, an aggregate welfare must be shared among them, according to the quantity of cooperating nations and their related characteristics.

Note that this structure of game differs from the so-called *global emission games* (a very accurate description of which can be found in [45], Chapter 9), which are basically conceived in a static framework. The principal difference is in the structure of the damage cost function: in the global emission games it is an increasing convex function of the aggregate emissions, whereas in our framework it is a function of the stock of pollution, which incorporates the emissions in its evolutionary dynamics. We are going to characterize this game as an extended cooperative game affected by negative externality. Explicit formulas for the Banzhaf-Coleman index and the Shapley value (which belongs to the core of the game) will be reckoned and subsequently applied to a numerical framework exposing the countries' actual achievement of the aggregate welfare. We will also discuss the effects of the countries' marginal contributions to pollution on their share of welfare.

6.1 Extended cooperative games

We aim to extend the standard definition of cooperative games on a finite player set building on the dependence on a second argument, a real variable. Let N be a finite set with cardinality n, and consider a subset $D \subseteq [0, +\infty)$. Let $\bar{v} : 2^N \times D \longrightarrow \mathbb{R}$ be a map such that $\bar{v}(\emptyset, \cdot) = 0$.

Definition 6.1. *The ordered pair $(N, \bar{v})$ is an **extended cooperative game (ECG for short)**.*

We will denote with $\bar{\Gamma}$ the class of the extended cooperative games and with $\bar{G}^n$ the set of all characteristic functions $\bar{v}$, corresponding to an extended cooperative game $(N, \bar{v}) \in \bar{\Gamma}$.

We are going to investigate the properties of the extended cooperative games, particularly taking into account the effect of this non-negative variable whenever it negatively affects the values of coalitions. Note that, in the simplest case, Definition 6.1 reduces to one-parameter families of cooperative games, notwithstanding that we intend to make it fit to a possibly much wider class of games. The second argument of an ECG is allowed to denote the state level in a dynamic optimization problem when some agents make a coalition agreement. Under such circumstances, $\bar{v}(\cdot)$ is the optimal value function solving a discrete Bellman equation and its arguments respectively represent the set of cooperating agents and the problem's state variable.

The next definition characterizes the suitable class of games on which we will focus our attention. Specifically, we will denote with $\bar{v}_{NE}(\cdot)$ the characteristic function incorporating such negative externality.

Definition 6.2. *Given* $(N, \bar{v}_{NE}) \in \bar{\Gamma}$ *such that*

$$\bar{v}_{NE}(S,P) = w(S) - g(P)u(S)$$

$\forall S \in 2^N \setminus \{\emptyset\}\ \ \forall P \in D \subseteq [0,+\infty)$, *and*

$$\bar{v}_{NE}(\emptyset, P) = 0$$

$\forall P \in D \subseteq [0,+\infty)$, *where*

- $w(S) > 0 \quad \forall S \in 2^N \setminus \{\emptyset\}, w(\emptyset) = 0;$
- $u(S) > 0 \quad \forall S \in 2^N \setminus \{\emptyset\}, u(\emptyset) = 0;$
- $g(P) \geq 0\ \ \forall P \in D$, *g strictly increasing;*

we call such a game ***extended cooperative game with negative externalities (ECGWNE for short)***.

Call $\bar{\Gamma}_{NE}$ the class of ECGWNE, and $\bar{G}^n_{NE}$ the set of all characteristic functions $\bar{v}_{NE}$, corresponding to an ECGWNE $(N, \bar{v}_{NE})$. For the sake of simplicity, consider the non-restrictive assumption $D = [0, +\infty)$.

Proposition 6.1. *Let* $(N, \bar{v}_{NE}) \in \bar{\Gamma}_{NE}$. *Then the pair* $(N, \bar{v}_{NE})$ *is a nonnegative ECGWNE iff*

$$P \in \left[0, \min_{S \in 2^N \setminus \{\emptyset\}} \left\{ g^{-1}\left(\frac{w(S)}{u(S)}\right)\right\}\right].$$

Proof. By hypothesis, $\forall S \in 2^N \setminus \{\emptyset\}$ and $\forall P \in [0, +\infty)$, we have

$$\bar{v}_{NE}(S) = w(S) - g(P)u(S)$$

which is nonnegative if and only if

$$g(P) \leq \frac{w(S)}{u(S)}.$$

Since g is strictly increasing, then it's invertible. Taking the minimum interval with respect to all possible coalitions, we obtain:

$$P \leq \min_{S \in 2^N \setminus \{\emptyset\}} \left\{ g^{-1}\left(\frac{w(S)}{u(S)}\right)\right\}.$$

□

We can employ the same definitions of Chapter 5 for the convexity of an ECGWNE.

Proposition 6.2. *Let $(N, \bar{v}_{NE}) \in \bar{\Gamma}_{NE}$ be a nonnegative ECGWNE and let $w(\cdot)$ be convex and $u(\cdot)$ be strictly convex. Then the pair $(N, \bar{v}_{NE})$ is a convex ECGWNE iff*

$$P \in \left[0, \min\left\{\min_{S,\ T\in 2^N}\left\{g^{-1}\left(\frac{w(S\cup T)+w(S\cap T)-w(S)-w(T)}{u(S\cup T)+u(S\cap T)-u(S)-u(T)}\right)\right\},\right.\right.$$
$$\left.\left.\min_{S\in 2^N\setminus\{\emptyset\}}\left\{g^{-1}\left(\frac{w(S)}{u(S)}\right)\right\}\right\}\right],$$

where S, T are not both empty sets.

Proof. By hypothesis, $\forall S, T \in 2^N$ (not both empty), and $\forall P \in [0, +\infty)$, we have

$$\begin{aligned} & \bar{v}_{NE}(S\cup T, P) + \bar{v}_{NE}(S\cap T, P) - \bar{v}_{NE}(S, P) - \bar{v}_{NE}(T, P) \\ = \quad & w(S\cup T) + w(S\cap T) - w(S) - w(T) - g(P)\left[u(S\cup T) + u(S\cap T) - u(S) - u(T)\right], \end{aligned}$$

which is nonnegative if and only if

$$g(P) \leq \frac{w(S\cup T)+w(S\cap T)-w(S)-w(T)}{u(S\cup T)+u(S\cap T)-u(S)-u(T)}.$$

Since g is strictly increasing, then it's invertible. Taking the minimum interval with respect to all possible coalitions, we obtain:

$$P \leq \min_{S,\ T\in 2^N}\left\{g^{-1}\left(\frac{w(S\cup T)+w(S\cap T)-w(S)-w(T)}{u(S\cup T)+u(S\cap T)-u(S)-u(T)}\right)\right\}.$$

By intersecting such interval with the domain achieved in Proposition 6.1 for nonnegativity, we complete the proof. □

Remark 6.1. *In Proposition 6.2, note that if we replace the assumption of strict convexity of $u(\cdot)$ with the standard modularity assumption:*

$$u(S\cup T) + u(S\cap T) - u(S) - u(T) = 0$$

for all $S,\ T\ \in 2^N$, then the convexity of $\bar{v}_{NE}(S, P)$ holds for all $P \in D$.

6.1.1 Power indices for extended cooperative games

We are going to arrange the foremost solution concepts of cooperative games for ECGWNE.

Definition 6.3. *Given* $(N, \bar{v}) \in \bar{\Gamma}$*, the* ***Shapley value*** *of* $\bar{v}(S,P)$ *is the vector* $\varphi(\bar{v}(S,P)) = (\varphi_1(\bar{v}(S,P)),\ \varphi_2(\bar{v}(S,P)), \ldots, \varphi_n(\bar{v}(S,P))) \in \mathbb{R}^n$ *such that:*

$$\varphi_i(\bar{v}(S,P)) = \sum_{\substack{i \in S \\ S \subseteq N}} \frac{(s-1)!(n-s)!}{n!} \left[(\bar{v}(S,P)) - (\bar{v}(S \setminus \{i\}, P) \right], \tag{6.1}$$

$\forall\ i = 1, \ldots, n.$

In (6.1) s denotes the cardinality of each coalition S.

The computation of the Shapley value might require the sum of a very large number of terms. In the following, we will employ the formulas related to the multilinear extension (MLE) of the game:

Definition 6.4. *Given* $(N, \bar{v}) \in \bar{\Gamma}$*, then the* ***multilinear extension (or MLE) of*** $(N, \bar{v})$ *is defined as follows:*

$$h(x_1, \ldots, x_n) = \sum_{S \subset N} \left(\prod_{i \in S} x_i \prod_{i \notin S} (1 - x_i) \right) \bar{v}(S,P), \tag{6.2}$$

where $x_i \in [0,1] \quad \forall\ i = 1, \ldots, n.$

The MLE of a cooperative game $(N, \bar{v})$ is related to the Shapley value by the following alternative formulation:

$$\varphi_i(\bar{v}) = \int_0^1 \frac{\partial h}{\partial x_i}(t, \ldots, t)dt, \qquad \forall\ i \in N. \tag{6.3}$$

From now on, call h the MLE of an ECGWNE.

Theorem 6.1. *Let* $(N, \bar{v}_{NE}) \in \bar{\Gamma}_{NE}$ *be an ECGWNE. Then*

$$\varphi_i(\bar{v}_{NE}) = \varphi_i(w) - g(P)\varphi_i(u), \qquad \forall\ i \in N. \tag{6.4}$$

Proof. Let a and b be the multilinear extensions of (N, w) and (N, u), respectively. Then, Definitions 6.2 and 6.4 imply:

$$h(x_1, \ldots, x_n) = a(x_1, \ldots, x_n) - g(P)b(x_1, \ldots, x_n)$$

where $x_i \in [0,1] \quad \forall\, i = 1, \ldots, n$.

Consequently, (6.3) implies:

$$\begin{aligned}\varphi_i(w) - g(P)\varphi_i(u) &= \int_0^1 \frac{\partial a}{\partial x_i}(t,\ldots,t)dt - g(P)\int_0^1 \frac{\partial b}{\partial x_i}(t,\ldots,t)dt \\ &= \int_0^1 \frac{\partial\,(a - g(P)b)}{\partial x_i}(t,\ldots,t)dt \\ &= \int_0^1 \frac{\partial h}{\partial x_i}(t,\ldots,t)dt = \varphi_i(\bar{v}_{NE}).\end{aligned}$$

□

The following two corollaries simply follow from Propositions 6.1 and 6.2 and from Theorem 6.1.

Corollary 6.1. *If $(N, \bar{v}_{NE})$ is a nonnegative ECGWNE, then:*

$$\varphi_i(w) - g(\widetilde{P})\varphi_i(u) \le \varphi_i(\bar{v}_{NE}) \le \varphi_i(w) - g(0)\varphi_i(u),$$

$\forall\, i \in N$, where

$$\widetilde{P} = \min_{S \in 2^N \setminus \{\emptyset\}} \left\{ g^{-1}\left(\frac{w\,(S)}{u\,(S)}\right)\right\}.$$

Corollary 6.2. *If $(N, \bar{v}_{NE})$ is a nonnegative and convex ECGWNE, then:*

$$\varphi_i(w) - g(\widehat{P})\varphi_i(u) \le \varphi_i(\bar{v}_{NE}) \le \varphi_i(w) - g(0)\varphi_i(u),$$

$\forall\, i \in N$, where

$$\widehat{P} = \min\left\{ \min_{S,\ T \in 2^N} \left\{ g^{-1}\left(\frac{w\,(S\cup T) + w\,(S\cap T) - w\,(S) - w\,(T)}{u\,(S\cup T) + u\,(S\cap T) - u\,(S) - u\,(T)}\right)\right\},\ \widetilde{P}\right\}.$$

Another index of power has been suggested by Banzhaf and Coleman.

Definition 6.5. *Given $(N, \bar{v}) \in \bar{\Gamma}$, the* ***Banzhaf - Coleman index*** *of $\bar{v}(S,P)$ is the vector $\chi(\bar{v}(S,P)) = (\chi_1(\bar{v}(S,P)),\ \chi_2(\bar{v}(S,P)), \ldots, \chi_n(\bar{v}(S,P))) \in \mathbb{R}^n$ such that:*

$$\chi_i(\bar{v}(S,P)) = \sum_{\substack{i \in S \\ S \subseteq N}} \left(\frac{1}{2}\right)^{n-1} [\bar{v}(S,P) - \bar{v}(S \setminus \{i\}, P)],$$

$\forall\, i = 1, \ldots, n$.

The Banzhaf-Coleman index can be characterized via the MLE:

$$\chi_i(\bar{v}) = \frac{\partial h}{\partial x_i}\left(\frac{1}{2},\ldots,\frac{1}{2}\right), \qquad \forall\ i \in N. \tag{6.5}$$

Theorem 6.2. *Let $(N, \bar{v}_{NE}) \in \bar{\Gamma}_{NE}$ be an ECGWNE. Then*

$$\chi_i(\bar{v}_{NE}) = \chi_i(w) - g(P)\chi_i(u) \quad \forall i \in N \tag{6.6}$$

Proof. As in the proof of Theorem 6.1, we have:

$$\begin{aligned}
\chi_i(w) - g(P)\chi_i(u) &= \frac{\partial a}{\partial x_i}\left(\frac{1}{2},\ldots,\frac{1}{2}\right) - g(P)\frac{\partial b}{\partial x_i}\left(\frac{1}{2},\ldots,\frac{1}{2}\right)\\
&= \frac{\partial\left(a - g(P)b\right)}{\partial x_i}\left(\frac{1}{2},\ldots,\frac{1}{2}\right)\\
&= \frac{\partial h}{\partial x_i}\left(\frac{1}{2},\ldots,\frac{1}{2}\right) = \chi_i(\bar{v}_{NE}).
\end{aligned}$$

□

The estimates provided by Corollaries 6.1 and 6.2 can also be applied to $\chi(\bar{v}_{NE})$.

The core of an ECGWNE can be synthetically redefined too, and it will turn out to be a key tool for our next application. Suppose that $P^* \in D$ is a fixed level of P, such that Proposition 6.1 holds.

Definition 6.6. *Given $(N, \bar{v}_{NE}) \in \bar{\Gamma}_{NE}$, and P^* such that $\bar{v}_{NE} \geq 0$, the following set of vectors:*

$$\mathcal{C}_{P^*}(\bar{v}_{NE}) = \left\{(x_1,\ldots,x_n) \in \mathbb{R}^n \ :\ \sum_{j=1}^{n} x_j = w(N) - g(P^*)u(N),\right. \tag{6.7}$$

$$\left.\sum_{k\in S} x_k \geq w(S) - g(P^*)u(S),\ \forall\ S \in 2^N\right\} \tag{6.8}$$

is the ***core of*** $\bar{v}_{NE}$ ***at*** P^*.

Different levels of externality entail disjoint cores, as we show in the following:

Proposition 6.3. *If $P_1 \neq P_2$, then $\mathcal{C}_{P_1}(\bar{v}_{NE}) \cap \mathcal{C}_{P_2}(\bar{v}_{NE}) = \emptyset$.*

Proof. If at least one of the two cores is empty, the proof is trivial. Suppose they are both nonempty. By Definition 6.2, $P_2 > P_1$ implies $g(P_2) > g(P_1)$. Therefore, if $z = (z_1, \ldots, z_n) \in \mathcal{C}_{P_1}(\bar{v}_{NE})$, then

$$z_1 + z_2 + \ldots + z_n = w(N) - g(P_1)u(N) > w(N) - g(P_2)u(N),$$

then $z \notin \mathcal{C}_{P_2}(\bar{v}_{NE})$, hence the two cores have no common imputations. □

6.2 Pollution-Control Game

We want to give some practical applications of the class of games defined in Section 6.1. The first one is based on environmental issues. Taking inspiration from [5] we consider a set N of n players, called countries in the sequel, involved in a pollution-control game. Denote by e_{jt} the carbon emissions arising from the production activity of player j at the instant of time $t \in [0, +\infty)$. Assume that the net revenues derived from player j' s production activity in a given period are given by the following concave function of his emission $R(\cdot)$ such that

$$R(e_{jt}) = e_{jt}\left(b_j - \frac{1}{2}e_{jt}\right),$$

where b_j is a strictly positive parameter denoting the emission level at which the revenue attains its maximum. The stock of pollution, which causes damage to the environment, evolves according to

$$P_t = P_{t-1}(1-\delta) + \sum_{j \in N} e_{jt},$$

with P_0 given, where $\delta \in (0, 1)$ is the absorption rate by Mother Nature. Players suffer an environmental damage arising from global pollution, which is assumed linear in the pollution stock, and given by

$$D_j(P_t) = d_j P_t,$$

where $d_j > 0$ is the constant marginal damage. The total discounted welfare over an infinite horizon of player j is then given by

$$W_j = \sum_{t=0}^{+\infty} \beta^t \left(e_{jt}\left(b_j - \frac{e_{jt}}{2}\right) - d_j P_t\right),$$

such that

$$P_t = P_{t-1}(1-\delta) + \sum_{j \in N} e_{jt} \quad \text{with} \quad P_0 \quad \text{given},$$

where $\beta \in (0,1)$ is the one-period discount factor assumed common to all players.

Our purpose is the design and implementation of an International Environmental Agreement (IEA), that is a mechanism allocating to each country a collectively suitable emissions policy.

6.2.1 Different Approaches

Small coalition approach

Assume that a set of players, identified as 'signatory countries', decide to join an IEA, according to which their production activity is decided by maximizing the aggregate welfare of the coalition. We denote by S the set of signatory countries, with cardinality s. The remaining players, identified as 'non-signatory countries' act individually, that is, each of them decides her production activity by maximizing her individual welfare, and we denote by $\bar{S}$ the set of non-signatories, with cardinality $n-s$.

As part of the agreement, we assume that each signatory country has to punish a non-signatory for its irresponsible behavior with a punishment proportional to the level of pollution, reflecting an environmental concern increasing with pollution stock. The non-environmental cost incurred by a non-signatory punished by s signatories when the pollution stock is P_t is thus given by $s\alpha P_t$. We also suppose that punishing itself has a cost, which is proportional to the punishment αP_t imposed to the $n-s$ non -signatory countries, so that each signatory incurs a non-environmental cost given by $(n-s)\tau\alpha P_t$, where $\tau \geq 0$. As a consequence, the welfare of a signatory country $j \in S$ in time period t when the number of non-signatories is $n-s$ is given by

$$W_t^S(e_{jt}, P_t, s) = e_{jt}\left(b_j - \frac{1}{2}e_{jt}\right) - d_j P_t - (n-s)\tau\alpha P_t,$$

and the welfare of a non-signatory $j \in \bar{S}$ is given by

$$W_t^{\bar{S}}(e_{jt}, P_t, s) = e_{jt}\left(b_j - \frac{1}{2}e_{jt}\right) - d_j P_t - \alpha s P_t.$$

Notice that we implicitly assume that all countries are punished in the same way, and that the cost of punishment is the same for all countries.

Now, to solve the dynamic emissions game, we assume that, for a given fixed set of signatories, countries optimize their welfare by taking into account the evolution of the pollution stock. The total discounted welfare of players is maximized over an infinite horizon, where $\beta \in (0,1)$ is the one-period discount factor assumed common to all players. The welfare optimization problem for a signatory country $j \in S$ is thus given by

$$\max_{(e_j)_{j\in S}} W^S = \sum_{j\in S}\sum_{t=0}^{+\infty} \beta^t \left(e_{jt}\left(b_{jt} - \frac{e_{jt}}{2}\right) - P_t\left(d_j + \tau\alpha\left(n-s\right)\right)\right)$$

s.t.

$$P_t = P_{t-1}\left(1-\delta\right) + \sum_{i\in S} e_{it} + \sum_{k\in \bar{S}} e_{kt}, \quad \text{with} \quad P_0 \quad \text{given,}$$

where e_{jt} is the emissions of country j during period t and e_j denotes the sequence of emissions $\{e_{jt}\}_{t\in[0,+\infty)}$. In the same way, the welfare optimization problem for a non-signatory country $j \in \bar{S}$ is

$$\max_{(e_j)_{j\in \bar{S}}} W^{\bar{S}} = \sum_{t=0}^{+\infty} \beta^t \left(e_{jt}\left(b_{jt} - \frac{e_{jt}}{2}\right) - P_t\left(d_j + \alpha s\right)\right)$$

s.t.

$$P_t = P_{t-1}\left(1-\delta\right) + \sum_{i\in S} e_{it} + \sum_{k\in \bar{S}} e_{kt}, \quad \text{with} \quad P_0 \quad \text{given.}$$

As in [18], we use a dynamic programming formulation where the state variable is P, that is, the pollution stock level in the preceding time period. We obtain a Nash equilibrium in stationary feedback strategies between the group of signatories, acting as a single player, and the non-signatories, acting as $n-s$ individual players, where $s \in [0,n]$. The case where $s=0$ corresponds to a Nash equilibrium between all players, or fully non-cooperative outcome, while the case where $s=n$ corresponds to the optimization of the total welfare of all players, or fully cooperative outcome (grand coalition). We call k the constant representing the combined effect of the discount factor and the natural pollution decay, that is,

$$k = \frac{1}{1-\beta(1-\delta)} > 1.$$

For the set of signatory countries, denoted with $E^{\bar{S}}$ the emissions of non-signatory countries, that is $E^{\bar{S}} = \sum_{k\in\bar{S}} e_k$, the value function $V^S\left(P; E^{\bar{S}}\right)$ represents the optimal total welfare of the group, given $E^{\bar{S}}$, and it satisfies

$$\begin{aligned} V^S\left(P; E^{\bar{S}}\right) &= \max_{(e_j)_{j\in S}} \left\{ \sum_{j\in S} e_j \left(b_j - \frac{e_j}{2}\right) \right. \\ &\quad - \left(d_j + \tau\alpha\left(n-s\right)\right) \left(P\frac{k-1}{k\beta} + \sum_{j\in S} e_j + E^{\bar{S}}\right) \\ &\quad \left. + \beta V^S\left(P\frac{k-1}{k\beta} + \sum_{j\in S} e_j + E^{\bar{S}}; E^{\bar{S}}\right)\right\} \end{aligned} \tag{6.9}$$

Proposition 6.4. *The value function (6.9) of a signatory country is linear in P. The optimal reaction of signatory countries is independent of the level of pollution and of the defectors' strategy and it is given by*

$$e_j^S = b_j - ks\tau\alpha\left(n-s\right) - k\sum_{j\in S} d_j,$$

assuming non-negative emissions.

Proof. Assume that $V^S\left(P;E^{\bar{S}}\right)=h^S-m^SP$. Then we have

$$\begin{aligned}
V^S\left(P;E^{\bar{S}}\right) &= \max_{(e_j)_{j\in S}}\left\{\sum_{j\in S}e_j\left(b_j-\frac{e_j}{2}\right)\right.\\
&\quad-\left(d_j+\tau\alpha\left(n-s\right)\right)\left(P\frac{k-1}{k\beta}+\sum_{j\in S}e_j+E^{\bar{S}}\right)\\
&\quad\left.+\beta V^S\left(P\frac{k-1}{k\beta}+\sum_{j\in S}e_j+E^{\bar{S}};E^{\bar{S}}\right)\right\}\\
&= \max_{(e_j)_{j\in S}}\left\{\sum_{j\in S}e_j\left(b_j-\frac{e_j}{2}\right)-\left(d_j+\tau\alpha\left(n-s\right)\right)\left(P\frac{k-1}{k\beta}+\sum_{j\in S}e_j+E^{\bar{S}}\right)\right.\\
&\quad\left.+\beta\left(h^S-m^S\left(P\frac{k-1}{k\beta}+\sum_{j\in S}e_j+E^{\bar{S}}\right)\right)\right\}\\
&=: \max_{(e_j)_{j\in S}}v_1\left(e_j\right).
\end{aligned}$$

Differentiating with respect to emissions yields:

$$\frac{d}{de_j}v_1\left(e_j\right)=\sum_{j\in S}b_j-\sum_{j\in S}e_j-s\left(\sum_{j\in S}d_j+\tau\alpha\left(n-s\right)\right)-s\beta m^S.$$

$$\frac{d}{de_j}\left[\sum_{j\in S}b_j-\sum_{j\in S}e_j-s\left(\sum_{j\in S}d_j+\tau\alpha\left(n-s\right)\right)-s\beta m^S\right]=-s<0.$$

So that the first order conditions are necessary and sufficient, provided that the solutions is interior. The FOC are satisfied at

$$e_j^S=b_j-\sum_{j\in S}d_j-\tau\alpha\left(n-s\right)-\beta m^S.$$

Replacing e_j^S in (6.9) and placing $c^S=\sum_{j\in S}d_j+\tau\alpha\left(n-s\right)$, we obtain

$$\begin{aligned}
V^S\left(P;E^{\bar{S}}\right) &= \frac{1}{2}\sum_{j\in S}\left(b_j-c^S-\beta m^S\right)\left(b_j+c^S+\beta m^S\right)-c^S\sum_{j\in S}\left(b_j-c^S-\beta m^S\right)-c^SE^{\bar{S}}\\
&\quad+\beta h^S-\beta m^S\sum_{j\in S}\left(b_j-c^S-\beta m^S\right)-\beta m^SE^{\bar{S}}-P\frac{k-1}{k\beta}\left(c^S+\beta m^S\right)\\
&= h^S-m^SP.
\end{aligned}$$

so that

$$\begin{aligned} m^S &= c^S \frac{k-1}{\beta} \\ h^S &= \frac{1}{1-\beta}\left[\sum_{j\in S}\left(b_j - c^S - \beta c^S \frac{k-1}{\beta}\right)\left(\frac{1}{2}\sum_{j\in S}\left(b_j + c^S + \beta c^S\frac{k-1}{\beta}\right) - c^S - \beta c^S \frac{k-1}{\beta}\right)\right. \\ &\left. -E^{\bar{S}}\left(c^S + \beta c^S \frac{k-1}{\beta}\right)\right] \end{aligned}$$

The optimal emissions of a signatory country are therefore:

$$e_j^S = b_j - ks\tau\alpha\,(n-s) - k\sum_{j\in S} d_j.$$

□

In the same way, the value function $V_j^{\bar{S}}$ of a non-signatory country represents its optimal total welfare, given P and the emissions of the other players, denoted by E^{S+}, and it satisfies

$$\begin{aligned} V_j^{\bar{S}}\left(P; E^{S+}\right) &= \max_e \left\{ e\left(b_j - \frac{e}{2}\right) - \left(P\frac{k-1}{k\beta} + E^{S+} + e\right)(d_j + \alpha s) \right. \\ &\left. + \beta V_j^{\bar{S}}\left(P\frac{k-1}{k\beta} + E^{S+} + e; E^{S+}\right)\right\}, \end{aligned} \tag{6.10}$$

Proposition 6.5. *The value function (6.9) of a defector country is linear in P. The optimal reaction of non-member countries is independent of the level of pollution and of the other players' strategy and it is given by*

$$e_j^{\bar{S}} = b_j - k\,(d_j + \alpha s)\,,$$

assuming non-negative emissions.

Proof. Assume that $V_j^{\bar{S}}\left(P; E^{S+}\right) = h^{\bar{S}} - m^{\bar{S}}P$. Then we have:

$$\begin{aligned} V_j^{\bar{S}}\left(P; E^{S+}\right) &= \max_e \left\{ e\left(b_j - \frac{e}{2}\right) - \left(P\frac{k-1}{k\beta} + E^{S+} + e\right)(d_j + \alpha s) \right. \\ &\left. + \beta V_j^{\bar{S}}\left(P\frac{k-1}{k\beta} + E^{S+} + e; E^{S+}\right)\right\} \\ &= \max_e \left\{ e\left(b_j - \frac{e}{2}\right) - \left(P\frac{k-1}{k\beta} + E^{S+} + e\right)(d_j + \alpha s) \right. \\ &\left. + \beta\left(h^{\bar{S}} - m^{\bar{S}}\left(P\frac{k-1}{k\beta} + E^{S+} + e; E^{S+}\right)\right)\right\} \\ &=: \max_e v_2\,(e)\,. \end{aligned}$$

Differentiating with respect to emissions yields:

$$\frac{d}{de}v_2(e) = b_j - e - d_j - \alpha s - \beta m^{\bar{S}}.$$

$$\frac{d}{de}\left[b_j - e - d_j - \alpha s - \beta m^{\bar{S}}\right] = -1 < 0.$$

So that the first order conditions are necessary and sufficient, provided that the solutions is interior. The FOC are satisfied at

$$e_j^{\bar{S}} = b_j - d_j - \alpha s - \beta m^{\bar{S}}.$$

Replacing $e_j^{\bar{S}}$ in (6.10) and placing $c^{\bar{S}} = d_j + \alpha s$, we obtain

$$\begin{aligned} V^S\left(P; E^{S+}\right) &= \frac{1}{2}\left(b_j - c^{\bar{S}} - \beta m^{\bar{S}}\right)\left(b_j + c^{\bar{S}} + \beta m^{\bar{S}}\right) - c^{\bar{S}}\left(b_j - c^{\bar{S}} - \beta m^{\bar{S}} + E^{S+}\right) \\ &\quad + \beta h^{\bar{S}} - \beta m^{\bar{S}}\left(b_j - c^{\bar{S}} - \beta m^{\bar{S}} + E^{S+}\right) - P\frac{k-1}{k\beta}\left(c^{\bar{S}} + \beta m^{\bar{S}}\right) = h^{\bar{S}} - m^{\bar{S}}P \end{aligned}$$

so that

$$\begin{aligned} m^{\bar{S}} &= c^{\bar{S}}\frac{k-1}{\beta} \\ h^S &= \frac{1}{1-\beta}\left[\left(b_j - c^{\bar{S}} - \beta c^{\bar{S}}\frac{k-1}{\beta}\right)\left(\frac{1}{2}\left(b_j + c^{\bar{S}} + \beta c^{\bar{S}}\frac{k-1}{\beta}\right) - c^{\bar{S}} - \beta c^{\bar{S}}\frac{k-1}{\beta}\right)\right. \\ &\quad \left. - E^{S+}\left(c^{\bar{S}} + \beta c^{\bar{S}}\frac{k-1}{\beta}\right)\right]. \end{aligned}$$

The optimal emissions of a defector country are therefore:

$$e_j^{\bar{S}} = b_j - k\left(d_j + \alpha s\right).$$

□

Combining these results, the equilibrium strategy vector is given by $\left(e_j^S, e_k^{\bar{S}}\right)_{j\in S, k\in\bar{S}}$ and the total emissions at equilibrium when the set of signatory countries is S are

$$T^S = \sum_{j=1}^{n} b_j - k\left(\sum_{j\in\bar{S}} d_j + s\sum_{j\in S} d_j + s\alpha\left(s\tau+1\right)\left(n-s\right)\right),$$

from which we obtain the steady-state of the pollution stock corresponding to coalition S

$$P^S = \frac{T^S k\beta}{k\beta - k + 1}.$$

Finally, the total discounted welfare, over an infinite horizon, of player j, according to his status, when the set of signatories is S and the pollution stock is P, is given by

$$\begin{aligned} W_j^S(P) &= \frac{b_j^2 - k^2\left(\sum_{k\in S} d_k + \tau\alpha(n-s)s\right)^2}{2(1-\beta)} \\ &\quad - \left(d_j + \tau\alpha(n-s)\right)k\left(P\frac{k-1}{k\beta} + \frac{T^S}{1-\beta}\right). \end{aligned} \tag{6.11}$$

$$W_j^{\bar{S}}(P) = \frac{b_j^2 - k^2\left(d_j + \alpha s\right)^2}{2(1-\beta)} - k\left(d_j + \alpha s\right)\left(P\frac{k-1}{k\beta} + \frac{T^S}{1-\beta}\right). \tag{6.12}$$

To conclude, we point out that the assumption that the damage function is linear in P makes the emission strategies of all players independent of the stock of pollution, but this does not mean that the emissions of the players are necessarily constant in time, because they depend on the number of signatories, which could depend on the stock of pollution.

To check for the stability of a coalition, we use a dynamic version of the equilibrium concept introduced in [34]. Thus, internal stability of a coalition S is achieved at P if no signatory country would increase its total discounted welfare by deciding to quit the coalition, that is

$$W_j^S(P) \geq W_j^{\bar{S}+j}(P) \quad \forall j \in S.$$

Similarly, external stability of a coalition S is achieved at P if no non-signatory country would increase its total discounted welfare by deciding to join the coalition, that is

$$W_j^{\bar{S}}(P) > W_j^{S+j}(P) \quad \forall j \in \bar{S}.$$

A general result in static games with identical players is that these stability conditions can only be satisfied by very small coalitions if no additional mechanism is provided in the agreement. For instance, in the quadratic cost/linear damage case, which corresponds to our model, they can only be satisfied by coalitions of two members. In a dynamic setting with identical players, Breton *et al.* in [18] show that the addition of a punishment mechanism in the agreement allows to obtain stable coalitions where $s \in n\left(\frac{\tau}{1+\tau}, 1\right]$, depending on the value of α and of P_t.

Grand coalition approach

In the grand coalition approach to IEAs, it is assumed that a group of countries has already agreed to participate in a joint agreement. The design of a stable IEA then reduces to finding a way to distribute the benefits of cooperation to the members of the coalition, which is acceptable to all players. In opposition to what is assumed in the small coalition approach, the grand coalition approach implies that the agreement collapses if at least one player defects from it. In this sense this approach is also called the cooperative one in opposition to the small coalition approach, called non-cooperative approach.

If one interprets the characteristic function $v(S)$, defined in the Chapter 5, as the payoff that a coalition S of countries can secure when they sign an environmental treaty, then it is clear that its actual value depends on the environmental strategies (or behavior) of the left-out-players (LOP, for short), i.e., $N \setminus S$. A first option is to assume that $v(S)$ is given as a Nash equilibrium payoff of the non-cooperative game played between S and the LOP acting individually. In that case, each characteristic function value involves computing a Nash equilibrium of a non-cooperative game with $n - s + 1$ players. This approach is often referred to as PNE (for Partial Nash Equilibrium), or γ -characteristic function. A second option is to assume that the LOP also form a coalition, and, consequently, $v(S)$ is defined as a Nash equilibrium payoff of the non-cooperative game between S and $N \setminus S$. The advantage of this approach with respect to the previous one is that each equilibrium problem now involves finding a Nash equilibrium of a two-player non-cooperative game, however, it may not lead to a superadditivity characteristic function. For an example in the framework of IEAs see [17]. A third possibility is to follow [123] and suppose that the LOP form an anti-coalition whose sole aim is to minimize the payoff of coalition S, which transforms the computation of $v(S)$ into the simple problem of finding a solution to a zero-sum game.

In the context of IEAs, the PNE approach seems to be the most attractive. Indeed, there is no reason to believe that, if some countries decide to form a coalition to tackle an environmental problem, then necessarily the remaining players will design a parallel treaty, and even less a treaty aiming at minimizing the welfare of the environmentally responsible countries. Further, the fact that this approach leads to a superadditive characteristic function is definitely an interesting feature. The only drawback is that it is heavily demanding in terms of computation: with n players, one needs to solve $2^n - 1$ equilibrium problems. To reduce the computational burden, Petrosjan and Zaccour in [96] propose a characteristic function where the LOP stick to their Nash strategies as determined in the fully non-cooperative game, i.e., when each player acts alone. The advantage of this approach is that only one equilibrium problem has to be solved, and it only remains to solve the optimization problem of each possible coalition.

The PNE approach corresponds to the assumption used for the solution of the emission game in the small coalition approach: the members of a coalition S decide to join an IEA, according to which their production activity is decided by maximizing the aggregate welfare of the coalition. The remaining left-out-players act individually, that is, each of them decides its production activity by maximizing his individual welfare. The solution of the emissions game can be retrieved from the solution of the dynamic programs (6.9) and (6.10) by setting α to 0. At P, the characteristic function assigns to each of the possible subsets $S \subseteq N$ the total discounted welfare of coalition S over an infinite horizon, that is

$$\begin{aligned}\widehat{v}(S,P) &= \frac{\sum_{j\in S} b_j^2 - sk^2\left(\sum_{j\in S} d_j\right)^2}{2\,(1-\beta)} \\ &\quad -k\left[P\frac{k-1}{k\beta} + \frac{\sum_{j\in N} b_j - k\left(\sum_{l\in\bar{S}} d_l + s\sum_{j\in S} d_j\right)}{1-\beta}\right]\sum_{j\in S} d_j.\end{aligned}$$

6.2.2 Main Features

Our aim is to study the main features of this characteristic function.

Remark 6.2. *It's necessary to check the positivity of $\widehat{v}(S,P)$, subject to the constraints for the optimal emissions obtained in Propositions 6.4 and 6.5, which entail the following lower bound for the sum of all b_j, for each coalition $S \in 2^N$:*

$$\sum_{j\in N} b_j \geq k\left[s\sum_{i\in S} d_i + \sum_{l\in\bar{S}} d_l\right]. \tag{6.13}$$

Theorem 6.3. *If*

$$\max_S \left\{ k \left[s \sum_{i\in S} d_i + \sum_{l\in\bar{S}} d_l \right] \right\} \le \sum_{j\in N} b_j < \min_{S\neq\emptyset} \left\{ \frac{\sum_{j\in S} b_j^2}{2k\sum_{j\in S} d_j} + \frac{sk}{2}\sum_{j\in S} d_j + k\sum_{l\in\bar{S}} d_l \right\} \tag{6.14}$$

holds, then the pair $(N, \widehat{v})$ *is a nonnegative ECGWNE for all* $P \in [0, +\infty)$.

Proof. Recalling the notation in Section 6.1, we can define:

$$\widehat{v}(S,P) = \widehat{w}(S) - \widehat{g}(P)\widehat{u}(S),$$

where

$$\begin{aligned}
\widehat{w}(S) &= \frac{\sum_{j\in S} b_j^2 - sk^2\left(\sum_{j\in S} d_j\right)^2}{2(1-\beta)} - k\left[\frac{\sum_{j\in N} b_j - k\left(\sum_{j\in\bar{S}} d_j + s\sum_{j\in S} d_j\right)}{1-\beta}\right]\sum_{j\in S} d_j,\\
\widehat{u}(S) &= \frac{k-1}{\beta}\sum_{j\in S} d_j,\\
\widehat{g}(P) &= P.
\end{aligned}$$

Obviously, if $S=\emptyset$ then $\widehat{v}(\emptyset, P) = 0$. Moreover $\widehat{g}(P)$ is nonnegative in $[0, +\infty)$ and strictly increasing, and $\widehat{u}(S)$ is positive $\forall S \neq \emptyset$. After extending (6.13) to all $S \subseteq N$, we are going to assess the positivity of $\widehat{w}(S)$:

$$\begin{aligned}
\widehat{w}(S) &= \frac{\sum_{j\in S} b_j^2 - sk^2\left(\sum_{j\in S} d_j\right)^2}{2(1-\beta)} - \frac{k\sum_{j\in S} d_j}{1-\beta}\left[\sum_{j\in N} b_j - k\left(\sum_{l\in\bar{S}} d_l + s\sum_{j\in S} d_j\right)\right]\\
&= \frac{\sum_{j\in S} b_j^2 + sk^2\left(\sum_{j\in S} d_j\right)^2}{2(1-\beta)} - \frac{k\sum_{j\in S} d_j \sum_{j\in N} b_j}{1-\beta} + \frac{k^2\sum_{j\in S} d_j\sum_{l\in\bar{S}} d_l}{1-\beta} > 0,
\end{aligned}$$

if the condition (6.14) holds. □

Proposition 6.6. *The game* $(N, \widehat{v})$ *is convex* $\forall\ P \in [0, +\infty)$.

Proof. By Proposition 6.2 and Remark6.1, since

$$\begin{aligned}
&\widehat{u}(S\cup T) + \widehat{u}(S\cap T) - \widehat{u}(S) - \widehat{u}(T)\\
&= \frac{k-1}{\beta}\left(\sum_{j\in S\cup T} d_j + \sum_{j\in S\cap T} d_j - \sum_{j\in S} d_j - \sum_{j\in T} d_j\right) = 0,
\end{aligned}$$

then if $\widehat{w}$ is convex $\forall\ P \in [0,+\infty)$, convexity is ensured for all nonnegative P, hence $(N,\widehat{v})$ is a convex ECGWNE for all suitable P such that Proposition 6.2 holds. If we denote respectively with s,t,h the cardinality of $S,\ T,\ S\cap T$, we have

$$\begin{aligned}
\widehat{w}(S\cup T)+\widehat{w}(S\cap T)-\widehat{w}(S)-\widehat{w}(T) &= \frac{k^2}{2(1-\beta)}\left[(s+t-h-2)\left(\sum_{j\in S\cup T} d_j\right)^2+(h-2)\left(\sum_{j\in S\cap T} d_j\right)^2+\right. \qquad (6.15)\\
&\quad \left.+(2-s)\left(\sum_{j\in S} d_j\right)^2+(2-t)\left(\sum_{j\in T} d_j\right)^2\right]\\
&= \frac{k^2}{2(1-\beta)}\left\{s\left[\left(\sum_{j\in S\cup T} d_j\right)^2-\left(\sum_{j\in S} d_j\right)^2\right]\right.\\
&\quad +t\left[\left(\sum_{j\in S\cap T} d_j\right)^2-\left(\sum_{j\in T} d_j\right)^2\right]\\
&\quad +h\left[\left(\sum_{j\in S\cap T} d_j\right)^2-\left(\sum_{j\in S\cup T} d_j\right)^2\right]+2\left[\left(\sum_{j\in S} d_j\right)^2\right.\\
&\quad \left.\left.+\left(\sum_{j\in T} d_j\right)^2-\left(\sum_{j\in S\cup T} d_j\right)^2-\left(\sum_{j\in S\cap T} d_j\right)^2\right]\right\}\\
&\geq \frac{k^2}{2(1-\beta)}\left\{h\left[\left(\sum_{j\in S\cup T} d_j\right)^2-\left(\sum_{j\in S} d_j\right)^2+\left(\sum_{j\in S\cap T} d_j\right)^2-\left(\sum_{j\in T} d_j\right)^2\right]\right.\\
&\quad \left.+2\left[-\left(\sum_{j\in S\cup T} d_j\right)^2+\left(\sum_{j\in S} d_j\right)^2-\left(\sum_{j\in S\cap T} d_j\right)^2+\left(\sum_{j\in T} d_j\right)^2\right]\right\}\\
&= \frac{k^2(h-2)}{2(1-\beta)}\left[\left(\sum_{j\in S\cup T} d_j\right)^2+\left(\sum_{j\in S\cap T} d_j\right)^2-\left(\sum_{j\in S} d_j\right)^2-\left(\sum_{j\in T} d_j\right)^2\right]\\
&= \frac{k^2(h-2)}{2(1-\beta)}\left[\left(\sum_{j\in S\cap T} d_j-\sum_{j\in S} d_j\right)\left(\sum_{j\in S\cap T} d_j+\sum_{j\in S} d_j\right)\right.\\
&\quad \left.+\left(\sum_{j\in S\cup T} d_j-\sum_{j\in T} d_j\right)\left(\sum_{j\in S\cup T} d_j+\sum_{j\in T} d_j\right)\right],
\end{aligned}$$

and since

$$\sum_{j\in S} d_j+\sum_{j\in T} d_j=\sum_{j\in S\cup T} d_j+\sum_{j\in S\cap T} d_j,$$

then (6.16) is equal to

$$=\frac{k^2(h-2)}{2(1-\beta)}\left[\left(\sum_{j\in T} d_j-\sum_{j\in S\cup T} d_j\right)\left(\sum_{j\in S\cap T} d_j+\sum_{j\in S} d_j-\sum_{j\in S\cup T} d_j-\sum_{j\in T} d_j\right)\right],$$

which is positive because its two factors are both negative, consequently $\widehat{w}(\cdot)$ is convex and the proof is complete. □

The convexity of $(N,\widehat{v})$ ensures that its Shapley value belongs to its core (see Definition 5.18). Moreover

Corollary 6.3. *If $n \geq 2$, the game $(N,\widehat{v})$ is an essential ECGWNE $\forall P \in [0,+\infty)$.*

Proof. We have that the game $(N,\widehat{w})$ is an essential game. Indeed

$$\begin{aligned}\widehat{w}(N)-\sum_{j\in N}\widehat{w}(\{j\}) &= \frac{-nk^2\left(\sum_{j\in N}d_j\right)^2+k^2\sum_{j\in N}d_j^2}{2\,(1-\beta)} \\ &\quad +\frac{nk^2\left(\sum_{j\in N}d_j\right)^2}{1-\beta}-\frac{k^2\left(\sum_{j\in N}d_j\right)^2}{1-\beta} \\ &= \frac{k^2(n-2)\left(\sum_{j\in N}d_j\right)^2}{2\,(1-\beta)}>0\end{aligned}$$

Moreover

$$\widehat{u}(N)-\sum_{j\in N}\widehat{u}(\{j\}) \;=\; 0$$

The thesis follows because $\widehat{v}$ is an affine trasformation of $\widehat{w}$ and $\widehat{u}$. □

Remark 6.3. *The nonnegative ECGWNE $(N,\widehat{v})$ is a monotonic game $\forall P \in [0,+\infty)$.*

6.2.3 Welfare allocation among players

First of all we consider the following notation:

Notation 6.1. *Let $(N,\widehat{v})$, be an extended cooperative game. We indicate respectively with $\varphi_i^{(n)}(\widehat{v})$ and with $\chi_i^{(n)}(\widehat{v})$ the Shapley value and the Banzhaf - Coleman index of $(N,\widehat{v})$ when the number of players is n.*

(Case 1) Symmetric players In this case we assume symmetry by positing $d_i = d_j = d$ and $b_i = b_j = b$ for all $i,j \in N$.

Proposition 6.7. *When agents are symmetric, the Shapley value for the extended cooperative game $(N,\widehat{v})$ $\forall i \in N$ $\forall n \geq 2$ is*

$$\varphi_i^{(n)}(\widehat{v}) = \frac{(ndk-b)^2}{2(1-\beta)} - P\left[d\frac{k-1}{\beta}\right].$$

Proof. We proceed by induction on the number of players. We calculate the Shapley value using the multilinear extension. For $n = 2$ we obtain

$$\varphi_i^{(2)}(\widehat{v}) \quad = \quad \frac{(2dk-b)^2}{2(1-\beta)} - P\left[d\frac{k-1}{\beta}\right].$$

Supposing that the thesis is true for n, we are going to prove it for $n+1$.

The inductive hypothesis implies that

$$\varphi_i^{(n-1)}(\widehat{v}) = \frac{((n-1)dk-b)^2}{2(1-\beta)} - P\left[d\frac{k-1}{\beta}\right]$$

and

$$\varphi_i^{(n)}(\widehat{v}) = \frac{(ndk-b)^2}{2(1-\beta)} - P\left[d\frac{k-1}{k\beta}\right],$$

and so

$$\varphi_i^{(n)}(\widehat{v}) - \varphi_i^{(n-1)}(\widehat{v}) = \frac{d^2k^2\,(2n-1) - 2dkb}{2\,(1-\beta)}.$$

Consequently, it suffices to show that

$$\varphi_i^{(n+1)}(\widehat{v}) - \varphi_i^{(n)}(\widehat{v}) = \frac{d^2k^2\,(2(n+1)-1) - 2dkb}{2\,(1-\beta)}$$

Then

$$\varphi_i^{(n)}(\widehat{v}) + \frac{d^2k^2\,(2(n+1)-1) - 2dkb}{2\,(1-\beta)} =$$

$$= \frac{(ndk-b)^2}{2(1-\beta)} - P\left[d\frac{k-1}{\beta}\right] + \frac{d^2k^2\,(2(n+1)-1) - 2dkb}{2\,(1-\beta)} =$$

$$= \frac{((n+1)dk-b)^2}{2(1-\beta)} - P\left[d\frac{k-1}{k\beta}\right] = \varphi_i^{(n+1)}(\widehat{v}).$$

□

Remark 6.4. *Proposition 6.7 maintains that the Shapley value is constant across players under symmetry, providing an alternative formula for the Shapley value:*

$$\varphi_i^{(n)}(\widehat{v}) = \frac{\widehat{v}(N, P)}{n} \qquad \forall i \in N, \qquad \forall n \geq 2 \tag{6.16}$$

Since the Banzhaf - Coleman index is equal for all players, we can state the following:

Proposition 6.8. *If $d_j = d$ and $b_j = b \quad \forall j \in N$, then we have*

$$\chi_i^{(n)}(\widehat{v}) \leq \varphi_i^{(n)}(\widehat{v}) \qquad \forall i \in N, \qquad \forall n \geq 2 \tag{6.17}$$

Proof. Since Banzhaf-Coleman index doesn't satisfy the efficiency axiom (see [87]), whereas the Shapley value does, we have

$$\sum_{i \in N} \chi_i^{(n)}(\widehat{v}) \leq \widehat{v}(N, P) = \sum_{i \in N} \varphi_i^{(n)}(\widehat{v}).$$

Moreover, since none of the indices depend on players, we have

$$n\chi_i^{(n)}(\widehat{v}) \leq \widehat{v}(N, P) = n\varphi_i^{(n)}(\widehat{v}) \qquad \forall i \in N,$$

trivially proving the assertion.

□

We can note that in the relation (6.17) the strict inequality does not hold because, for example, for $n = 2$, $\chi_i^{(n)}(\widehat{v}) = \varphi_i^{(n)}(\widehat{v}) \quad \forall i \in N$.

Proposition 6.8 means that for all agents, the allocation corresponding to the Shapley value is preferable to the one corresponding to the Banzhaf-Coleman index.

(Case 2) Non Symmetric players

We can calculate some indices of power for the extended cooperative game $(N, \widehat{v})$, in the general case.

Proposition 6.9. *The Shapley value for the extended cooperative game* $(N, \widehat{v})$ $\forall i \in N \quad \forall n \geq 2$ *is:*

$$\varphi_i^{(n)}(\widehat{v}) = \frac{k^2}{3(1-\beta)} \left[(n-2)\, d_i \sum_{\substack{j=1 \\ j\neq i}}^{n} d_j + \sum_{\substack{j,k=1 \\ j<k \\ j\neq i}}^{n} d_j d_k \right] + \frac{k^2}{4(1-\beta)} \left[(n-1)\, d_i^2 + \sum_{\substack{j=1 \\ j\neq i}}^{n} d_j^2 \right] +$$

$$+ \frac{b_i^2 - k^2 d_i^2}{2\,(1-\beta)} - \frac{k}{1-\beta} \left[\sum_{j=1}^{n} b_j - k \sum_{j=1}^{n} d_j \right] d_i - P\left[d_i \frac{k-1}{\beta} \right]. \qquad (6.18)$$

Proof. By induction on the number of players, we use Theorem 6.1 to calculate the Shapley value. For $n = 2$ we have

$$\varphi_1^{(2)}(\widehat{v}) = \frac{k^2(3d_1^2 + d_2^2) + 4k^2 d_1 d_2 + 2b_1^2 - 4kd_1(b_1 + b_2)}{4(1-\beta)} - P\left[d_1 \frac{k-1}{\beta} \right],$$

$$\varphi_2^{(2)}(\widehat{v}) = \frac{k^2(d_1^2 + 3d_2^2) + 4k^2 d_1 d_2 + 2b_2^2 - 4kd_2(b_1 + b_2)}{4(1-\beta)} - P\left[d_2 \frac{k-1}{\beta} \right].$$

For $n = 3$ we have

$$\begin{aligned} \varphi_1^{(3)}(\widehat{v}) &= \frac{k^2\left(12d_1^2 + 3d_2^2 + 3d_3^2 + 16d_1d_2 + 16d_1d_3 + 4d_2d_3\right)}{12(1-\beta)} \\ &+ \frac{6b_1^2 - 12kd_1(b_1 + b_2 + b_3)}{12(1-\beta)} - P\left[d_1 \frac{k-1}{\beta} \right]. \end{aligned}$$

Similarly

$$\begin{aligned} \varphi_2^{(3)}(\widehat{v}) &= \frac{k^2\left(3d_1^2 + 12d_2^2 + 3d_3^2 + 16d_1d_2 + 4d_1d_3 + 16d_2d_3\right) + 6b_2^2}{12(1-\beta)} \\ &- \frac{12kd_2(b_1 + b_2 + b_3)}{12(1-\beta)} - P\left[d_2 \frac{k-1}{\beta} \right]. \end{aligned}$$

$$\begin{aligned} \varphi_3^{(3)}(\widehat{v}) &= \frac{k^2\left(3d_1^2 + 3d_2^2 + 12d_3^2 + 4d_1d_2 + 16d_1d_3 + 16d_2d_3\right) + 6b_3^2}{12(1-\beta)} \\ &- \frac{12kd_3(b_1 + b_2 + b_3)}{12(1-\beta)} - P\left[d_3 \frac{k-1}{\beta} \right]. \end{aligned}$$

Supposing that the thesis is true for n, we prove it for $n+1$. By the inductive hypothesis we have that:

$$\varphi_i^{(n-1)}(\widehat{v}) = \frac{k^2}{3(1-\beta)}\left[(n-3)\,d_i\sum_{\substack{j=1\\j\neq i}}^{n-1} d_j + \sum_{\substack{j,k=1\\j<k\\j\neq i}}^{n-1} d_j d_k\right] + \frac{k^2}{4(1-\beta)}\left[(n-2)\,d_i^2 + \sum_{\substack{j=1\\j\neq i}}^{n-1} d_j^2\right] +$$

$$+\frac{b_i^2-k^2d_i^2}{2\,(1-\beta)} - \frac{k}{1-\beta}\left[\sum_{j=1}^{n-1} b_j - k\sum_{j=1}^{n-1} d_j\right] d_i - P\left[d_i\frac{k-1}{\beta}\right],$$

$$\varphi_i^{(n)}(\widehat{v}) = \frac{k^2}{3(1-\beta)}\left[(n-2)\,d_i\sum_{\substack{j=1\\j\neq i}}^{n} d_j + \sum_{\substack{j,k=1\\j<k\\j\neq i}}^{n} d_j d_k\right] + \frac{k^2}{4(1-\beta)}\left[(n-1)\,d_i^2 + \sum_{\substack{j=1\\j\neq i}}^{n} d_j^2\right] +$$

$$+\frac{b_i^2-k^2d_i^2}{2\,(1-\beta)} - \frac{k}{1-\beta}\left[\sum_{j=1}^{n} b_j - k\sum_{j=1}^{n} d_j\right] d_i - P\left[d_i\frac{k-1}{\beta}\right].$$

Therefore

$$\varphi_i^{(n)}(\widehat{v}) - \varphi_i^{(n-1)}(\widehat{v}) = \frac{k^2}{3\,(1-\beta)}\left[d_id_n + d_n\sum_{\substack{j=1\\j\neq i}}^{n-1} d_j\right] + \frac{k^2}{4\,(1-\beta)}\left[d_i^2+d_n^2\right] - \frac{k}{1-\beta}\left[b_n - kd_n\right] d_i.$$

We have to show that

$$\varphi_i^{(n+1)}(\widehat{v}) - \varphi_i^{(n)}(\widehat{v}) = \frac{k^2}{3\,(1-\beta)}\left[d_id_{n+1} + d_{n+1}\sum_{\substack{j=1\\j\neq i}}^{n} d_j\right] + \frac{k^2}{4\,(1-\beta)}\left[d_i^2+d_{n+1}^2\right] - \frac{k}{1-\beta}\left[b_{n+1} - kd_{n+1}\right] d_i.$$

Then

$$\varphi_i^{(n)}(\widehat{v}) + \frac{k^2}{3\,(1-\beta)}\left[d_id_{n+1} + d_{n+1}\sum_{\substack{j=1\\j\neq i}}^{n} d_j\right] + \frac{k^2}{4\,(1-\beta)}\left[d_i^2+d_{n+1}^2\right] - \frac{k}{1-\beta}\left[b_{n+1} - kd_{n+1}\right] d_i$$

$$= \frac{k^2}{3(1-\beta)}\left[(n-2)\,d_i\sum_{\substack{j=1\\j\neq i}}^{n} d_j + d_i d_{n+1} + \sum_{\substack{j,k=1\\j<k\\j\neq i}}^{n} d_j d_k + d_{n+1}\sum_{\substack{j=1\\j\neq i}}^{n} d_j\right]$$

$$+\frac{k^2}{4(1-\beta)}\left[(n-1)\,d_i^2 + d_i^2 + \sum_{\substack{j=1\\j\neq i}}^{n} d_j^2 + d_{n+1}^2\right] + \frac{b_i^2 - k^2 d_i^2}{2\,(1-\beta)}$$

$$-\frac{k}{1-\beta}\left[\sum_{j=1}^{n} b_j + b_{n+1} - k\left(\sum_{j=1}^{n} d_j + d_{n+1}\right)\right] d_i - P\left[d_i\frac{k-1}{\beta}\right]$$

$$= \frac{k^2}{3(1-\beta)}\left[(n-1)\,d_i\sum_{\substack{j=1\\j\neq i}}^{n+1} d_j + \sum_{\substack{j,k=1\\j<k\\j\neq i}}^{n+1} d_j d_k\right] + \frac{k^2}{4(1-\beta)}\left[n d_i^2 + \sum_{\substack{j=1\\j\neq i}}^{n+1} d_j^2\right]$$

$$+\frac{b_i^2 - k^2 d_i^2}{2\,(1-\beta)} - \frac{k}{1-\beta}\left[\sum_{j=1}^{n+1} b_j - k\sum_{j=1}^{n+1} d_j\right] d_i - P\left[d_i\frac{k-1}{\beta}\right] = \varphi_i^{(n+1)}(\widehat{v}).$$

□

Moreover, we have a similar formula for the Banzhaf - Coleman index.

Proposition 6.10. *The Banzhaf - Coleman index for the extended cooperative game* $(N, \widehat{v})$ $\forall i \in N \quad \forall n \geq 2$ *is*

$$\chi_i^{(n)}(\widehat{v}) = \frac{k^2}{4(1-\beta)}\left[(n-2)\,d_i\sum_{\substack{j=1\\j\neq i}}^{n} d_j + \sum_{\substack{j,k=1\\j<k\\j\neq i}}^{n} d_j d_k\right] + \frac{k^2}{4(1-\beta)}\left[(n-1)\,d_i^2 + \sum_{\substack{j=1\\j\neq i}}^{n} d_j^2\right] +$$

$$+\frac{b_i^2 - k^2 d_i^2}{2\,(1-\beta)} - \frac{k}{1-\beta}\left[\sum_{j=1}^{n} b_j - k\sum_{j=1}^{n} d_j\right] d_i - P\left[d_i\frac{k-1}{\beta}\right]. \tag{6.19}$$

Proof. It is similar to the proof of Proposition 6.9. □

Propositions 6.9 and 6.10 imply that, also in the non-symmetric case, the Banzhaf - Coleman index is suboptimal respect to Shapley value, i.e.

$$\chi_i^{(n)}(\widehat{v}) \leq \varphi_i^{(n)}(\widehat{v}) \quad \forall i \in N \quad \forall n \geq 2.$$

In particular, by (6.18) and (6.19), the difference between the two measures is given by:

$$\varphi_i^{(n)}(\widehat{v}) - \chi_i^{(n)}(\widehat{v}) = \frac{k^2}{12(1-\beta)} \left[(n-2)\, d_i \sum_{\substack{j=1 \\ j \neq i}}^{n} d_j + \sum_{\substack{j,k=1 \\ j<k \\ j \neq i}}^{n} d_j d_k \right].$$

Hence, they are equal for $n = 2$ players.

6.2.4 A numerical simulation

As in [5], we use the MERGE model to calibrate the parameters of the game. MERGE is a Model for Evaluating the Regional and Global Effects of GHG reductions. In MERGE, the world is divided into nine geopolitical regions: Canada, Australia and New Zealand (CANZ); China; Eastern Europe and the former Soviet Union (EEFSU); India; Japan; Mexico and OPEC (MOPEC); USA; Western Europe (WEUR) and the rest of the world (ROW). The data are borrowed from [5].

Table 6.1: Model parameter values

	b_j	d_j (low)	d_j (high)	β	k	P
USA	1759	0.358	0.429	0.95	15.095	390000
WEUR	993	0.310	0.4345	0.95	15.095	390000
Japan	318	0.143	0.1495	0.95	15.095	390000
CANZ	293	0.053	0.0685	0.95	15.095	390000
EEFSU	919	0.056	0.134	0.95	15.095	390000
China	985	0.216	0.807	0.95	15.095	390000
India	334	0.063	0.1915	0.95	15.095	390000
MOPEC	751	0.100	0.231	0.95	15.095	390000
ROW	1202	0.288	0.717	0.95	15.095	390000

With data of Table 6.1, the game $(N, \widehat{v})$ satisfies the hypothesis of Theorem 6.3, and so it's an ECGWNE.

In the following two tables, we are going to show the Shapley values and the Banzhaf-Coleman indices for this game in the two different cases corresponding to distinct levels for d_j.

Table 6.2: Numerical solution (low case)

	$\varphi^{(9)}$ (Shapley)	$\chi^{(9)}$ (Banzhaf)
USA	28062244	28060838
WEUR	7368204.666	7366899.333
Japan	-137234.3214	-138108.2105
CANZ	434146.1666	433557.5263
EEFSU	7997129.5	7996530.5
China	7966384.666	7965306.5
India	610759.9444	610137.8
MOPEC	4837516	4836773.8
ROW	12132794.5	12131538.66

Table 6.3: Numerical solution (high case)

	$\varphi^{(9)}$ (Shapley)	$\chi^{(9)}$ (Banzhaf)
USA	27506247	27501994
WEUR	6381780	6377499
Japan	-180148.8947	-182755.8823
CANZ	317080.4761	315016.8571
EEFSU	7378603.3333	7376098
China	3233134.3333	3227223.1667
India	-412872.8889	-415749.8571
MOPEC	3794584.25	3791460.6667
ROW	8701525.5	8695950

The above tables suggest us a reduction effect of the individual welfare for higher values of d_i, namely the sensitivity of the power indices with respect to the coefficients d_i can be assessed more accurately, by taking into account the first order derivatives of (6.18) and (6.19):

Proposition 6.11. *There exist n threshold values*

$$\mathcal{P}_i^* = \frac{\beta(1-\beta)}{k(k-1)}\left[k(n+1)\left(\frac{\sum_{j\neq i} d_j}{3} + \frac{d_i}{2}\right) - \sum_{j=1}^{n} b_j\right]$$

such that for any $P > \mathcal{P}_i^*$ the Shapley value of the i-th country decreases as its marginal contribution to the stock of pollution increases.

Proof. Deriving (6.18) with respect to d_i we have:

$$\frac{\partial \varphi_i^{(n)}(\widehat{v})}{\partial d_i} = \frac{k^2}{1-\beta}\left(\frac{n+1}{3}\sum_{j\neq i} d_j + \frac{n+1}{2} d_i\right) - \frac{k}{1-\beta}\sum_{j=1}^{n} b_j - \frac{k-1}{\beta} P$$

which is negative for

$$P > \frac{\beta(1-\beta)}{k(k-1)}\left[k(n+1)\left(\frac{\sum_{j\neq i} d_j}{3} + \frac{d_i}{2}\right) - \sum_{j=1}^{n} b_j\right] := \mathcal{P}_i^*.$$

□

Consequently, the value

$$\mathcal{P}^* := \max_{i\in N} \mathcal{P}_i^*$$

is the level such that, for any $P > \mathcal{P}^*$, each country's individual welfare decreases as its marginal contribution to externality increases.

Also in this case, we can prove an analogous assertion on the Banzhaf-Coleman index.

Remark 6.5. *In our numerical example we have the following threshold values for P in the two cases:*

Table 6.4: Threshold values

	$\mathcal{P}^*$ (low case)	$\mathcal{P}^*$ (high case)
USA	-1.6667	-1.6485
WEUR	-1.6668	-1.6485
Japan	-1.6678	-1.6501
CANZ	-1.6683	-1.6505
EEFSU	-1.6683	-1.6502
China	-1.6674	-1.6464
India	-1.6682	-1.6498
MOPEC	-1.6680	-1.6496
ROW	-1.6670	-1.6469

We can note that in our simulation such values are slightly different from one another, and that given their negativity, the Shapley value and the Banzhaf-Coleman index of all countries decrease as their marginal contributions to the stock of pollution increase, for all positive levels of P.

This chapter is divided in two parts: in the first part we introduced the class of extended cooperative games with negative externality in order to contribute to the modeling of TU-games in which the coalitional payoff is affected by an undesired and inevitable effect. We showed that such games may have good properties, such as nonnegativity and convexity. Moreover, we proved several features for classical concept solutions of cooperative games belonging to this new class.

In the second part the traditional model of emissions reduction game is seen as an application of the theory elaborated in the first part: in particular we obtained two closed form formulas for the Shapley value and the Banzhaf-Coleman index. In a concluding numerical simulation we employed such formulas to calculate the welfare of countries inside the MERGE model.

Taking into account asymmetrical countries, we found that the welfare depends on each country's specific prerogatives, and the suboptimality of the Banzhaf-Coleman index with respect to the Shapley value suggests that, the latter is the preferred solution because it is also efficient in core-sense.

The reduction effect of the individual welfare for higher values of countries' marginal contributions to the pollution stock suggests that power indices are decreasing functions not just of pollution but also of the propensity to pollute as soon as pollution outnumbers a certain threshold.

In general, we stress that the theory approached in Section 6.1 can also be exploited to investigate other kinds of games where players maximize their payoffs while facing a common damage. Future developments of such findings might concern games either with more complex accumulation dynamics (i.e., nonlinear) or with more than one externality variable, for example introducing different types of damages for different geographical areas.

Moreover, a deeper discussion on the stability of the computed solution concepts in this framework has still to be entirely developed.

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