CORE
Metadata, citation and similar papers at core.ac.uk
Provided by Archivio della ricerca- Università di Roma La Sapienza

SAPIENZA
UNIVERSITÀ DI ROMA

Scuola Dottorale in Scienze Astronomiche, Chimiche, Fisiche, Matematiche e della Terra "Vito Volterra"

Dottorato di Ricerca in Matematica

Dipartimento di Matematica "Guido Castelnuovo"

Some problems concerning the pseudoeffective cone of blown-up surfaces and projectivized vector bundles

Candidate: Fulvio Di Sciullo, XXIV Ciclo
Thesis advisor: Prof. Angelo Felice Lopez

A thesis submitted for the degree of
Doctor of Philosophy in Mathematics

June 2012

Fulvio Di Sciullo
Some problems concerning the pseudoeffective cone
of blown-up surfaces and projectivized vector bundles.

Ph.D. thesis Sapienza – Univeristà di Roma

email: disciullo@mat.uniroma1.it
fulviodisciullo@gmail.com

ai miei maestri

Introduction

The study of divisors is a very powerful tool to achieve the understanding of the geometry of a projective variety X. From the second half of the past century, the sheaf-theoretic approach brought into light the importance of ample line bundles and, consequently, ample divisors; in the last decades, with the flowering of higher dimensional algebraic geometry, a number of notions of *positivity* appeared. The general picture has been successfully summarized in the book by Lazarsfeld [Laz04].
Once we focus on the numerical class of a divisor, where two divisors are said to be numerically equivalent if they have the same intersection behaviour with respect to all curves (we denote the intersection of a divisor D and a curve C with $D \cdot C$), we can consider the real vector space: $N^1(X) = (\mathrm{Pic}(X)/\equiv) \otimes \mathbb{R}$.
In this space there are several convex cones, each one corresponding to a different notion of positivity.

Definition. *Let us consider $D = \sum a_i D_i \in \mathrm{Div}_{\mathbb{R}}(X)$, $a_i \in \mathbb{R}$, $D_i \in \mathrm{Div}(X)$; we say that:*

1. *D is* ample *if $a_i > 0$ and D_i is ample for all i, that is $D_i \cdot \gamma > 0$ for all non-zero 1-cycle $\gamma \in \overline{\mathrm{NE}}(X)$ (that is the closure of the cone spanned by effective 1-cycles);*
2. *D is* big *if $a_i > 0$ and D_i is big for all i, that is the Kodaira dimension of D_i is $\dim X$.*

The classes of ample divisors span the open convex cone $\mathrm{Amp}(X) \subset N^1(X)$; its closure is $\mathrm{Nef}(X)$, the cone of *nef* divisors. Similarly, the classes of big divisors span the open convex cone $\mathrm{Big}(X) \subset N^1(X)$; its closure is $\overline{\mathrm{Eff}}(X)$, the cone of *pseudoeffective* divisors, the closure of the cone spanned by the classes of effective divisors.
Therefore we deal with two open convex cones and two closed convex cones; these cones fit, via inclusions, in the following picture:

$$\begin{tikzcd} \mathrm{Amp}(X) \arrow[r, hook] \arrow[d, hook] & \mathrm{Big}(X) \arrow[d, hook] \\ \mathrm{Nef}(X) \arrow[r, hook] & \overline{\mathrm{Eff}}(X). \end{tikzcd}$$

In this thesis we want to discuss some topics concerning the largest of these cones: the pseudoeffective cone.

The first problem we deal with is treated in Part I: *Influence of the Segre Conjecture on the Mori cone of blown-up surfaces* (Chapters 2-3); we consider a specific kind of surfaces and we want to describe the pseudoeffective cone that, since we are working with two dimensional varieties, does coincide with the Mori cone.

In particular, we study the influence of the generalization of a conjecture by Beniamino Segre on the shape of the Mori cone $\overline{\mathrm{NE}}(X)$ of a projective surface X obtained as the blow-up of a smooth surface at finitely many points.

Strictly related to this problem is the behaviour of linear systems of curves; although it is far to be fully understood, several conjectures can be stated in the hope of taming this situation.

The most important conjecture we want to deal with in the first part of the thesis has been stated in [Seg62] by Segre in the setting of linear systems on $\mathbb{P}^2$; the original statement can be found in the footnote at page 35.

Afterwards equivalent formulations were given by several authors: Harbourne in [Har86], Gimigliano in [Gim87] and Hirschowitz in [Hir89]; in literature these statements are known, after their authors, as SHGH Conjectures.

The truth of these conjectures would solve a number of central problems in the study of linear systems: for example, it would give a method to compute the dimension of planar linear systems and would imply the celebrated Nagata Conjecture.

In his work [dF10], Tommaso de Fernex points out how these conjectures can be translated in a more Mori-theoretic flavour.

Indeed, SHGH Conjectures on $\mathbb{P}^2$ do imply the so called (-1)-curve Conjecture on the blow-up of $\mathbb{P}^2$ at r general points: this conjecture says that the only curves with negative self-intersection are (-1)-curves. From information on linear system on $\mathbb{P}^2$, we get information on $\overline{\mathrm{NE}}(\mathrm{Bl}_r\,\mathbb{P}^2)$.

Moreover, De Volder and Laface in [DVL05] underline how Segre Conjecture can be easily stated for any surface and they consider its generalization to the case of generic $K3$ surfaces.

In light of these facts, we want to investigate how far we can go in the generalization of the Segre Conjecture to any surface; moreover, we are interested, as we said before, in its influence on the Mori cone of the blown-up surface.

In the second part, *Weak Zariski decomposition on projectivized vector bundles* (Chapters 4-5), we leave the world of conjectures on linear systems on surfaces and we focus on the pseudoeffective cone of a projectivized vector bundle.

The whole question began with Zariski and Fujita that, in [Zar62] and [Fuj79], proved the existence of the *Zariski decomposition* in the two dimensional case: for any pseudoeffective $\mathbb{R}$-divisor D on a smooth surface X, there exist P and N such that $D = P + N$, where P is nef and N in an effective divisor that is 0 or such that the intersection matrix of its components is negative definite, see Theorem 4.1.

In literature many attempts to generalize to higher dimension this kind of decomposition for pseudoeffective divisors can be found; some of the most relevant are the *Fujita-Zariski decomposition* and the *Cutkosky-Kawamata-Moriwaki-Zariski (CKM-Zariski for short) decomposition*; an account on these definitions can be found in [Bir09]. In both of them the decomposition of a pseudoeffective $\mathbb{R}$-divisor as a sum of a nef and an effective divisor is required together with an additional property.

A much weaker notion is the following (see Definition 4.2):

> **Definition.** *We say that a pseudoeffective divisor $D \in \mathrm{Div}_{\mathbb{R}}(X)$ on a normal variety X has a* weak Zariski decomposition (WZD for short) *if there exists a projective birational morphism $f : W \to X$ form a normal variety W such that*
>
> 1. *$f^*D = P + N$, where P, N are $\mathbb{R}$-divisors;*
> 2. *P is nef, N is effective.*

In his paper [Bir09], Caucher Birkar proves the following result, highlighting the relationships between this kind of decompositions and the theory of Minimal Models; we refer to his work to further details.

> **Theorem** (Birkar). *If the Log Minimal Model Program for $\mathbb{Q}$-factorial dlt pairs in dimension $d-1$ holds true and $(X/Z, B)$ is a log-canonical pair of dimension d, then the following are equivalent:*
>
> 1. *$K_X + B$ has a weak Zariski decomposition $/Z$,*
> 2. *$K_X + B$ birationally has a CKM-Zariski decomposition$/Z$,*
> 3. *$K_X + B$ birationally has a Fujita-Zariski decomposition$/Z$,*
> 4. *$(X/Z, B)$ has a log minimal model.*

The question about the existence of a weak Zariski decomposition for every pseudoeffective divisor follows from a question posed by Nakayama in [Nak04, Problem, page 4]; in the following we prove, in a number of meaningful situations, the existence of a weak Zariski decomposition for the elements of $\overline{\mathrm{Eff}}(X)$, where $X = \mathbb{P}(\mathcal{E})$ is the projectivization of a vector bundle $\mathcal{E}$ on a variety Z.
It is worth to say that the pseudoeffective cone of a projectivized vector bundle on a curve has been recently studied by Fulger in [Flg11]; we used some of his ideas to give our proof of the existence of a weak Zariski decomposition in that situation.

We can now go through the structure of this thesis. At first we say that the whole work is developed in the setting of complex numbers.

Basic concepts

In the first chapter, we introduce the notation and we recall basic notions about positivity in algebraic geometry (see [Laz04]); we present several properties, with original proofs, concerning cones associated to projective varieties paying special attention to the case of a surface S and to the positive cone:

$$\overline{\mathrm{Pos}}(S) = \{\alpha \in N^1(S) \mid \alpha^2 \geqslant 0, \alpha \cdot h \geqslant 0, h \text{ ample divisor}\}.$$

This cone will play a central role especially in the first part of the thesis.

Part I: Influence of the Segre Conjecture on the Mori cone of blown-up surfaces

The second chapter, introducing the main problem of the first part, is dedicated to a number of conjectures about linear systems and to the relations among them.
The goal of this chapter is the generalization to any surface of the conjecture by Beniamino Segre concerning linear system on $\mathbb{P}^2$.

Let us recall that a non empty linear system $\mathcal{L}$ on a surface X is said to be *special* (respectively *non special*) if $h^1(X, L) \neq 0$ (respectively $h^1(X, L) = 0$), where L is the line bundle associated to $\mathcal{L}$.
After excluding some situations, in order to assure these definitions make sense, we state, and we name it again after Segre, the following (see Conjecture 2.23).

Segre Conjecture. *Let Y be a smooth projective surface such that Y is either a K3 or $p_g(Y) = 0$ or it is a non simple abelian surface and let $X = \mathrm{Bl}_r Y$ be the blow-up of Y at r general points.*

If $\mathcal{L}$ is a non empty, non exceptional and reduced linear system on X, then $\mathcal{L}$ is non special.

The non speciality of a linear system $\mathcal{L}$ allows us to compute the dimension of $\mathcal{L}$; in this situation, indeed, it is the so-called *expected dimension*, $e(\mathcal{L}) = \max\{\chi(L) - 1, -1\}$.
The Segre Conjecture would imply, in the setting of blown-up surfaces, the so called (see [Har10]) *Bounded Negativity Conjecture*, saying that for any surface S, there exists an integer ν_S such that $C^2 \geqslant -\nu_S$ for any curve $C \subset S$.
The boundedness of the negativity influences the shape of the Mori cone of a blown-up surface; indeed if $X = \mathrm{Bl}_r Y$, the BN Conjecture would give the decomposition:

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{Pos}}(X) + \sum_{0 > C^2 \geqslant -\nu_X} R(C).$$

Moreover, the Segre Conjecture implies the boundedness form above of the arithmetic genus of curves in a blown-up surface; we state therefore the following (see Conjecture 2.28).

List Conjecture. Let $X = \mathrm{Bl}_r Y$ be a blown-up surface; then there exist $\nu_X, \pi_X \in \mathbb{N}$ such that for every curve $C \subset X$ with negative self-intersection, $C^2 \geqslant -\nu_X$, and $p_a(C) \leqslant \pi_X$.

Theorem 3.13, central point of Chapter 3, is the main result of the first part; looking for a decomposition of the Mori cone, we generalize a result by de Fernex (see [dF10]) on the shape of $\overline{\mathrm{NE}}(X)$: if the Segre Conjecture holds true and the number of blown-up points is large enough, then a non empty part of $\overline{\mathrm{NE}}(X)$ has to be circular.

Theorem. *Let $X = \mathrm{Bl}_r Y$ be the blow up of Y at r general points and let L be the pull-back of an ample divisor A on Y.*

Let us assume the existence of $\nu, \pi \in \mathbb{N}$ such that for any integral curve $C \subset X$ with negative self-intersection, we have

$$C^2 \geqslant -\nu \quad \text{and} \quad p_a(C) \leqslant \pi.$$

If r is large enough (explicit bound depending on π, ν and A), then there exists $s \in \mathbb{R}$ (explicit value, depending on A and ν) such that

$$\overline{\mathrm{NE}}(X)_{(K-sL)^{\geqslant 0}} = \overline{\mathrm{Pos}}(X)_{(K-sL)^{\geqslant 0}}.$$

In particular, this holds true if Segre Conjecture is verified and $r \gg 0$.

This result is, in a certain sense, sharp: in order to have a circular part, that is a part coinciding with $\overline{\mathrm{Pos}}(X)$, the number s can't be avoided. Indeed we prove (see Proposition 3.17 and Proposition 3.19) that in many meaningful situations, independently from any conjecture,

$$\overline{\mathrm{Pos}}(X)_{K_X^{\geqslant 0}} \subsetneq \overline{\mathrm{NE}}(X)_{K_X^{\geqslant 0}}.$$

Part II: Weak Zariski decomposition on projectivized vector bundles

Chapter 4 is dedicated to the presentation of the question we investigate in the second part of the thesis: the existence of a weak Zariski decomposition for pseudoeffective divisors on a normal projective variety X.
After the reduction of this problem to the extremal ray of the pseudoeffective cone, we focus on the specific case of projectivized vector bundles recalling and proving a number of useful properties.
Projectivized vector bundles are indeed an interesting class of varieties and they provide a very manageable tool to produce examples and counterexamples, see for example [Laz04, Example 1.5.1]. Thus, as first step in the direction of a general solution to the problem, we ask the following question.

> **Question.** *If $\mathcal{E}$ is a vector bundle on a variety Z, does a weak Zariski decomposition exist for every pseudoeffective divisor on $X = \mathbb{P}(\mathcal{E})$?*

The results giving a positive answer to the question, contained in Chapter 5, can be summarized in the following statement.

> **Theorem.** *Let $\mathcal{E}$ be a rank r vector bundle on a variety Z; setting $X = \mathbb{P}(\mathcal{E})$, there is a weak Zariski decomposition for every pseudoeffective class in $\overline{\mathrm{Eff}}(X)$ in the following cases:*
>
> 1. *Z is a curve;*
> 2. *$\mathcal{E}$ is completely decomposable as direct sum of r line bundles on a variety Z with Picar number $\rho(Z) = 1$;*
> 3. *Z is a Fano variety with Picard number $\rho(Z) = 1$, $\mathcal{E}$ is rank 2 vector bundle that is either unstable or semistable and non stable;*
> 4. *$\mathcal{E}$ is a Schwarzenberger bundle on $\mathbb{P}^2$ (important class of stable rank 2 bundles on $\mathbb{P}^2$);*
> 5. *$\mathcal{E}$ is the rank 2 stable vector bundle on $\mathbb{P}^3$ associated, via the Hartshorne-Serre correspondence, to the disjoint union of $s \geqslant 2$ lines in $\mathbb{P}^3$;*

These results come form a collaboration with Luis Solá Conde and Roberto Muñoz.
The case of curves is treated in Section 5.1 (see Theorem 5.17); our proof is different and independent form the one, already known, by Nakayama and it is based on some ideas of Fulger and the reduction of the problem to a vector bundle of smaller rank via the Harder-Narasimhan filtration of the bundle $\mathcal{E}$. Moreover we give in Proposition 5.18 a characterization of vector bunldes on a curve C such that a birational map f is required in order to have a WZD for every pseudoeffective class.
If $\mathcal{E}$ is fully decomposable, we prove in Proposition 5.24 that the effective cone is indeed a closed convex cone; thus in particular we have what we call a direct weak Zariski decomposition, that is a WZD without the birational map f.
In Section 5.3, using some results from [MOSC11], we prove that, in the situation of the third point of the Theorem above, the effective cone is closed. Moreover, referring again to [MOSC11], we point out that there are rank 2 vector bundles on $\mathbb{P}^2$ strictly related to the Nagata Conjecture.
The others positive answers are a direct consequence of the following result (see Proposition 5.36.

Proposition. *Let $\mathcal{E}$ be a rank 2 vector bundle on Z (the smallest twist with sections) such that $\rho(Z) = 1$; if for some a, b, $a \leqslant 0, a < b$ there exists a set $\mathcal{M} = \mathcal{M}(a, b)$ of smooth rational curves $C \subset Z$ dominating Z such that for every curve in $\mathcal{M}$*

$$\mathcal{E}|_C = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b),$$

then the effective cone of $\mathbb{P}(\mathcal{E})$ is closed. Moreover we can give, in terms of $C \in \mathcal{M}$, a description of the two rays of $\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}))$.

In order to prove the fourth and the fifth points in the Theorem, it is enough to construct the dominating set of curves $\mathcal{M}$ and verify the splitting type of the vector bundle on the curve $C \in \mathcal{M}$, see Proposition 5.42 and Fact 5.48.
It is worth to say that vector bundles in the fifth point of the Theorem are closely related to instantons constructed by the physicists (see [Har78a, Example 2.2]) and they are called *t'Hooft bundles* (see [BF01]).
Finally, in view of our general discussion, we recall that the effective cone is known to be closed if X is a Mori dream space; since, by [Gon10], a toric vector bundle on a toric variety is a Mori dream space, our question has a positive answer in also this situation.

Roma, June 2012

Fulvio Di Sciullo

Acknowledgments

My deep gratitude goes at first to my advisor Angelo Felice Lopez, for proposing me the problems I deal with in this thesis and for its firm support throughout this work; it would not have been possible to accomplish this dissertation without his wise guidance, his advice, his patience and his teachings of various kinds.
I would like to thank Luis Solá Conde and Roberto Muñoz for their kind hospitality during my stay in Madrid; they made possible a meaningful experience, both for life and Mathematics.
A special thank to some companions, namely Salvatore Cacciola, Lorenzo Di Biagio and Simone Marchesi, for sharing with me a long part of my mathematical path through many moments in different places and situations.
I would like to express my sincere gratitude to all PhD students and people I met in Rome, both in Sapienza and in Roma Tre.
Finally, a special mention goes to my cardinal friends Francesco, Federico and Marco, to Maura and to my whole family.

Contents

Chapter 1

Basic concepts

The first chapter of this thesis is devoted to recall some definitions and some basic facts about Mori theory. We refer, for positivity topics to the books by Lazarsfeld ([Laz04]); for Mori Theory we refer to the book by Debarre ([Deb01]) and the book by Kollár and Mori ([KM98]).
We will work over the field $\mathbb{C}$ of complex numbers.

1.1 Notation

Definition 1.1. A *scheme* is a separated algebraic scheme of finite type over $\mathbb{C}$. A *variety* is a reduced and irreducible scheme.

To fix the notation, we give the following definition.

Definition 1.2. A *Cartier divisor* on a variety X is a global section of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$; we denote the group of Cartier divisors with

$$\mathrm{Div}(X) = \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*).$$

We denote the group of $\mathbb{R}$-*Cartier* $\mathbb{R}$-*divisors* with

$$\mathrm{Div}_{\mathbb{R}}(X) = \mathrm{Div}(X) \otimes \mathbb{R}.$$

If D, D' are two divisors on a variety X; we denote by $D \equiv D'$ the numerical equivalence and by $D \sim D'$ the linear equivalence.

Definition 1.3. If X is a normal projective variety of dimension n, we have

1. $\mathrm{Num}(X) = \mathrm{Pic}(X)/\equiv$;
2. $N^1(X) = \mathrm{Num}(X) \otimes \mathbb{R}$;
3. $N_1(X) = (\{\text{1-cycles}\}/\equiv) \otimes \mathbb{R}$;
4. $\rho(X) = \dim_{\mathbb{R}} N^1(X) = \dim_{\mathbb{R}} N_1(X)$, the Picard number of X.

We will denote the intersection form between $N^1(X)$ and $N_1(X)$ with:

$$\begin{array}{cccc} \cdot : & N^1(X) \times N_1(X) & \to & \mathbb{R} \\ & (\delta, \gamma) & \mapsto & \delta \cdot \gamma. \end{array} \tag{1.1}$$

As well known, this pairing is symmetric, not degenerate and continuous.
If X is a projective variety and C is a curve, we will denote with $[C]$ its class in $N_1(X)$. Similarly if $D \in \mathrm{Div}_{\mathbb{R}}(X)$, $[D]$ will be its class in $N^1(X)$.
In the spaces $N^1(X)$ and $N_1(X)$ it is useful to consider subsets generated by classes of some particular divisors and curves.

Definition 1.4. Let X be a normal projective variety. We define in $N^1(X)$

1. $\mathrm{Nef}(X)$, the set spanned by classes of nef divisors;
2. $\mathrm{Amp}(X)$, the set spanned by ample classes;
3. $\mathrm{Big}(X)$, the set spanned by big classes;
4. $\mathrm{Eff}(X)$, the set spanned by effective classes;
5. $\overline{\mathrm{Eff}}(X)$, the set spanned by pseudoeffective classes.

Definition 1.5. Let X be a normal projective variety, we define in $N_1(X)$

1. $\mathrm{NE}(X)$, the set of classes in $N_1(X)$ generated by the effective 1-cycles;
2. $\overline{\mathrm{NE}}(X)$, the closure of $\mathrm{NE}(X)$ in $N_1(X)$ with respect to the Euclidean topology;

We can introduce another subset of $N_1(X)$; to this end we give the following.

Definition 1.6. Let X be a projective variety with $n = \dim X$.

1. We say that a curve $C \subset X$ is a *movable curve* if $C = C_0$ is a member of an algebraic family $\{C_t\}_{t \in S}$ such that $\bigcup_{t \in S} C_t = X$; the set spanned by the classes of movable curves in $N_1(X)$ is denoted by $\mathrm{ME}(X)$ and its closure by $\overline{\mathrm{ME}}(X)$.
2. A curve $C \subset X$ is said to be a *strongly movable curve* if there exist a birational map $\mu : X' \to X$ together with ample classes $\alpha_1, \dots, \alpha_{n-1}$ of X' such that

$$[C] = \mu_*(\alpha_1 \cdot \dots \cdot \alpha_{n-1}); \tag{1.2}$$

the set spanned by classes of strongly movable curves is denoted by $\mathrm{SME}(X)$ and its closure with $\overline{\mathrm{SME}}(X)$.

Moreover, we have the following definition.

Definition 1.7. If $D \in \mathrm{Div}_{\mathbb{R}}(X)$, we define:

$$D^{\perp} = \{\gamma \in N_1(X) \mid [D] \cdot \gamma = 0\};$$
$$D^{\geqslant 0} = \{\gamma \in N_1(X) \mid [D] \cdot \gamma \geqslant 0\}.$$

We can likewise define $D^{\leqslant 0}$, $D^{>0}$ and $D^{<0}$. Similar definitions can be given for a curve C and its class $[C] \in N_1(X)$.

We will denote with K_X the canonical divisor of the variety X.

1.2 Conology

Among the tools we have, the study of the sets introduced in Definition 1.4 and Definition 1.5 is undoubtedly one of the most powerful to achieve the understanding of projective varieties. As we will soon see, these sets are indeed cones in the real vector spaces $N_1(X)$ and $N^1(X)$; this section is dedicated to fix a number of useful properties of cones contained in a real vector space of finite dimension.

Definition 1.8. Let V be a finite dimensional $\mathbb{R}$-vector space (or $\mathbb{Q}$-vector space).

1. A subset $\mathcal{C} \subset V$ is a *cone* if $\mathcal{C}$ is closed under the positive scalar multiplication, that is if $x \in \mathcal{C}$ then $\lambda x \in \mathcal{C}$ for all $\lambda \in \mathbb{R}^{>0}$.
2. If $\mathcal{C} \subset V$ is a convex cone, the *dimension of* $\mathcal{C}$, $\dim \mathcal{C}$, is the dimension of the smallest vector subspace containing $\mathcal{C}$.
3. A closed and convex subcone $\mathcal{K} \subseteq \mathcal{C}$ is called *extremal face* of $\mathcal{C}$ if for all $u, v \in \mathcal{C}$ such that $u + v \in \mathcal{K}$, then $u, v \in \mathcal{K}$. A 1-dimensional extremal face is called an *extremal ray*.
4. If $x \in V$, the *ray generated by* x is
$$R(x) = \{\lambda x \in V, \text{ for all } \lambda \geqslant 0\}.$$
5. If $X \subset V$ is a subset, we denote with $\langle X \rangle$ the *convex cone spanned by* X:
$$\langle X \rangle = \left\{ \sum_{\text{finite}} \lambda_i x_i \mid \lambda_i \geqslant 0, x_i \in X \right\}.$$

Definition 1.9. If $\mathcal{C}$ is a convex cone, we denote with $\partial \mathcal{C}$ the *boundary* of $\mathcal{C}$.

Our goal is now to prove an useful fact allowing us to write elements in a cone as the sum of extremal rays.

Definition 1.10. Let $\mathcal{C}_i \subset \mathbb{R}^t$, $i \in I$ be cones, then the *convex hull* of $\mathcal{C}_i$ is

$$\langle \mathcal{C}_i \rangle^{\circ}_{i \in I} = \left\{ \sum_{\text{finite}} a_i \gamma_i \mid \gamma_i \in \mathcal{C}_i, a_i \geqslant 0, \sum a_i = 1 \right\}, \tag{1.3}$$

and the *sum* of $\mathcal{C}_i$ is

$$\sum_{i \in I} \mathcal{C}_i = \left\{ \sum_{\text{finite}} a_i \gamma_i \mid \gamma_i \in \mathcal{C}_i, a_i \geqslant 0 \right\}. \tag{1.4}$$

We have the following.

Fact 1.11. *Let $\mathcal{C}_i$, $i \in I$ be closed convex cones in $\mathbb{R}^t$, then*

$$\langle \mathcal{C}_i \rangle^{\circ}_{i \in I} = \sum_{i \in I} \mathcal{C}_i$$

Proof. We will prove the two inclusions; the $(\subseteq)$ is obvious from the definitions. Therefore we can focus on the second one. Let us take $0 \neq x \in \sum \mathcal{C}_i$; then we have $x = \sum a_i \gamma_i$, $a_i \geqslant 0$ and we can set $A = \sum a_i > 0$. Hence we can write

$$x = \sum \frac{a_i}{A} A\gamma_i, \quad \frac{a_i}{A} \geqslant 0$$

but since $\mathcal{C}_i$ are cones, $A\gamma_i \in \mathcal{C}_i$; moreover, since

$$\sum \frac{a_i}{A} = \frac{\sum a_i}{A} = 1,$$

we get $x \in \langle \mathcal{C}_i \rangle^\circ$. □

The former lemma allows us, speaking of cones, to confuse the concepts of sum and convex hull.
Before going on, let us fix the notation for segments.

Definition 1.12. If $a, b \in \mathbb{R}^t$, then (a, b) is the *open segment* and $[a, b]$ is the *closed segment* joining a and b.

Let us recall that if $\mathcal{C} \subset \mathbb{R}^t$ is a closed convex cone and F is an extremal face, then the *affine space* generated by F is the smallest linear space containing F:

$$\operatorname{aff}(F) = \left\{ \sum_{\text{finite}} a_i f_i \mid a_i \in \mathbb{R}, f_i \in F \right\}.$$

It is immediate to see that if $x \in \operatorname{aff}(F)$, then

$$x = \sum_{a_i > 0} a_i f_i - \sum_{a_i < 0} (-a_i) f_i,$$

and thus we can write

$$x = f_1 - f_2, \quad f_1, f_2 \in F.$$

Fact 1.13. *Let $\mathcal{C} \subset \mathbb{R}^t$ be a closed convex cone and let F be an extremal face of $\mathcal{C}$; then*

$$\operatorname{aff}(F) \cap \mathcal{C} = F. \tag{1.5}$$

Proof. If $y \in F$, then immediately we get $y \in \operatorname{aff}(F) \cap \mathcal{C}$ and the first inclusion is proved. In order to prove the other, let us consider $y \in \operatorname{aff}(F) \cap \mathcal{C}$; we have

$$y = f_1 - f_2 \quad f_1, f_2 \in F.$$

hence $f_2 = y + f_1$, and, since $f_2 \in F$, by extremality of the face F, we get $y \in F$. □

To prove Fact 1.19, we need to introduce some notation; for further details we refer to [Roc97, Section 18]. In particular, in the setting of convex sets $C \subset \mathbb{R}^t$, we recall that the *relative interior* $\operatorname{ri}(C)$ is the interior of C in the $\operatorname{aff}(F)$. Moreover, we have the following.

Definition 1.14. Let $C \subset \mathbb{R}^t$ be a convex set; a *face* is a convex subset $C' \subset C$ such that if $[x, y] \subset C$ is a closed segment with a point of (x, y) in C', then $x, y \in C'$.

We see at once that a face of a closed convex cone is indeed a subcone and we can show that, moreover, it is an extremal face. At first we recall the following result (see [Roc97, Theorem 18.1]).

Theorem 1.15. *Let C be a convex set, and let C' be a face of C. If D is a convex set in C such that* $\mathrm{ri}(D) \cap C' \neq \emptyset$*, then $D \subset C'$.*

Corollary 1.16. *If C' is a face of a convex set C, then $C' = C \cap \mathrm{cl}(C')$. In particular if C is closed, then C' is closed.*

Coming to closed convex cones, we have the following.

Fact 1.17. *Let K be a closed convex cone and let $C \subset K$ be a face (hence a subcone), then C is an extremal face of K.*

Proof. The face C is closed by Corollary 1.16. Let us now take $x, y \in K$ such that $x + y \in C$; since the midpoint of segment $[x, y]$ is $(x + y)/2 \in C$ and C is a subcone, then $x + y \in C$ and, by definition of face, we get $x, y \in C$. □

We have the following theorem (see [Roc97, Theorem 18.2]).

Theorem 1.18. *If C is a non empty convex subset and then C is the disjoint union of the relative interior of its faces.*

We are now ready to give the description of the boundary of a closed convex cone in terms of its extremal faces.

Fact 1.19. *Let $\mathcal{C}$ be a closed convex cone of maximal dimension in $\mathbb{R}^t$, then $\partial\mathcal{C}$ is the union of its extremal faces:*

$$\partial\mathcal{C} = \bigcup_{\dim F_i < t} F_i,$$

where F_i are extremal faces.

Proof. Let us consider $x \in F$, where F is an extremal face of dimension $n < t$; if x is not in the boundary, then there exists a ball centred in x of ray ε:

$$B = B_\varepsilon(x) \subset \mathrm{int}(\mathcal{C}).$$

Let us consider a point $z \notin F$; the line L joining x and z, by Fact 1.13, is such that $L \cap F = \{x\}$ and it does intersect ∂B in two points α and β. In particular we have that $\alpha, \beta \notin F$ and that, since the segment $[\alpha, \beta]$ is a diameter, $\beta - x = x - \alpha$. Thus we get $2x = \alpha + \beta$ and

$$x = \frac{1}{2}\alpha + \frac{1}{2}\beta.$$

Since α and β are outside F, this also applies to their own half; but $x \in F$ and this gives a contradiction with the extremality of F and hence $F \subset \partial\mathcal{C}$.
To prove the reverse inclusion, we see, by Theorem 1.18, that

$$\mathcal{C} = \bigsqcup \mathrm{ri}(F_i),$$

where F_i are faces and hence, by Fact 1.17, extremal faces. Now, by Fact 1.13, we have that $\mathcal{C}$ itself is the only extremal face of dimension t and moreover $\mathrm{ri}(\mathcal{C}) = \mathrm{int}(\mathcal{C})$; hence we can write

$$\mathcal{C} = \mathrm{int}(\mathcal{C}) \sqcup \left(\bigsqcup_{\dim F_i < t} \mathrm{ri}(F_i) \right).$$

Thus we get:

$$\partial\mathcal{C} = \bigsqcup_{\dim F_i < t} \mathrm{ri}(F_i) \subseteq \bigcup_{\dim F_i < t} F_i.$$

□

Here it is a couple of other interesting easy facts concerning extremal faces and extremal rays.

Fact 1.20. *Let us assume $\mathbb{R}^t$ is endowed with a scalar product. Let $\mathcal{C} \subset \mathbb{R}^t$ be a closed convex cone not containing lines through the origin and let $h = H^\perp$ be an hyperplane; if F is an extremal face of $\mathcal{C} \cap H^{\geqslant 0}$ such that $F \setminus \{0\} \subset H^{>0}$, then F is an extremal face of $\mathcal{C}$.*

Proof. Let us consider $x, y \in \mathcal{C} \setminus \{0\}$ such that $x + y \in F$; we want to show that $x, y \in F$ (that is, F is an extremal face of $\mathcal{C}$).
At first we claim that $x+y \in H^{>0}$. Indeed $x+y \in F \subset H^{\geqslant 0}$ and if it were $(x+y) \cdot H = 0$, since

$$F \setminus \{0\} \subset H^{>0},$$

we would have $x + y = 0$ and then $x = -y$. This is a contradiction since $\mathcal{C}$, and hence its subcone $\mathcal{C} \cap H^{\geqslant 0}$, does not contain lines passing by the origin.
Now, if $x, y \in H^{\geqslant 0}$, then they are in $\mathcal{C} \cap H^{\geqslant 0}$ and by extremality, both x, y lie in F and we have finished.
If $x, y \in H^{<0}$, we immediately get a contradiction since $(x + y) \cdot H < 0$.
Let us now suppose that $x \in H^{<0}$; then we have $y \in H^{>0}$ (we see that y can't be in $H^\perp$) and $x + y \in H^{>0}$. If we consider the continuous function $\lambda(t) = (x + ty) \cdot H$, then $\lambda(0) = x \cdot H < 0$ and $\lambda(1) = (x + y) \cdot H > 0$; therefore there exists t_0 such that

$$(x + t_0 y) \cdot H = 0 \quad \text{and} \quad 0 < t_0 < 1.$$

Writing

$$x + y = (x + t_0 y) + (1 - t_0)y,$$

we see that, since $(x+t_0 y)$ and $(1-t_0)y$ lie in $\mathcal{C} \cap H^{\geqslant 0}$, by extremality, $x + t_0 y \in F \cap H^\perp$. As before, this gives $x + t_0 y = 0$ and hence $x = -t_0 y$ that is a contradiction since there can't be lines through the origin. This last case, hence, does not occur and the proof is concluded. □

Fact 1.21. *Let $\mathcal{C}$ be a closed convex cone and F be an extremal face of $\mathcal{C}$; if R is an extremal face of F, then it is an extremal face of $\mathcal{C}$. In particular this applies to extremal rays.*

Proof. Let us consider $\alpha, \beta \in \mathcal{C}$ with $\alpha + \beta \in R$, we need to show that $\alpha, \beta \in R$: since $\alpha + \beta \in R \subseteq F$, by extremality of F in $\mathcal{C}$, we get $\alpha, \beta \in F$ and, by extremality of R in F, we see that $\alpha, \beta \in R$. □

We have now the interesting Lemma (see [Kol96][Lemma II.4.10.4]) that allows us to write the elements of our cones as positive linear combination of extremal rays.

Lemma 1.22. *Let $\mathcal{C} \subset \mathbb{R}^t$ be a closed convex cone of positive dimension which does not contain a line through the origin, then $\mathcal{C}$ is the convex hull of its extremal rays.*
More precisely, if $x \in \mathcal{C}$ then there exists $s \in \mathbb{N}$, such that $x \in \sum_{i=1}^{s} R_i$, where R_i are extremal rays of $\mathcal{C}$.
In particular, if $\dim \mathcal{C} \geqslant 2$, a closed convex cone not containing lines through the origin is the convex hull of its boundary.

Proof. We can assume that $\mathcal{C} \subset \mathbb{R}^t$ is a closed convex cone of dimension $t \geqslant 1$. In light of Fact 1.11, we want to prove that if $x \in \mathcal{C}$, then there exist an $s \in \mathbb{N}$ and $R_1, \dots . R_s$ extremal rays of $\mathcal{C}$ such that $x \in \sum R_i$.

We proceed by induction on t. The case $t = 1$ is obvious since a 1-dimensional convex cone has just one extremal ray. In the 2-dimensional case, the convex cone is spanned by two extremal rays and $x \in \mathcal{C}$ can be written as linear combination with non negative coefficients of the generators of the rays.
Consider now $\dim \mathcal{C} \geqslant 3$. By Fact 1.19, we can write $\partial\mathcal{C}$ as union of extremal faces.
Thus if $y \in \partial C$, we can fix an extremal face $y \in F$ of dimension n, for some n; since F is a closed convex cone of dimension $n < t$, by induction, y belongs to the sum of q extremal rays R''_k of F. By Fact 1.21 we have that these R''_k are indeed extremal rays of $\mathcal{C}$.
If $x \in \partial\mathcal{C}$ we have done; if $x \in \mathrm{int}(\mathcal{C})$, let us consider an hyperplane H passing through x and the origin. The cone $\mathcal{C} \cap H$ is a closed convex cone of dimension $t - 1$; by induction we have that

$$x \in \sum_{i=1}^{s'} R'_i,$$

where R'_i are extremal rays of $\mathcal{C} \cap H$ as a cone in $H \simeq \mathbb{R}^{t-1}$: we claim that R'_i must lie in $\partial\mathcal{C}$. Indeed, by contradiction, we could fix $y \in R'_i$ with $y \in \mathrm{int}\,\mathcal{C}$ and hence, in the topological space H, we would have $y \in \mathrm{int}(\mathcal{C} \cap H)$, but this can't be since y lies in an extremal ray of $\mathcal{C} \cap H$ and hence in $\partial(\mathcal{C} \cap H)$.
Hence $R'_i \subset \partial\mathcal{C}$: for each R'_i we can fix a generator y_i and since it lies in the boundary then we can write

$$y_i \in \sum_{j=1}^{s''} R_{ij}, \quad j = 1, \ldots, s'.$$

Thus we have that

$$x \in \sum_{i,j=1}^{s',s''} R_{ij},$$

and we have written x as a positive linear combination of at most $s = s's''$ extremal rays. $\square$

Cones on projective varieties

As pointed out before, in the following, we will consider essentially cones in the spaces $N_1(X)$ and $N^1(X)$. We have that the intersection pairing allows us to define a duality between cones.

Definition 1.23. Let $\mathcal{C}$ be a cone in $N^1(X)$, the *dual cone of* $\mathcal{C}$ is

$$\mathcal{C}^\vee = \{x \in N_1(X) \mid x \cdot y \geqslant 0, \text{for all } y \in \mathcal{C}\}. \tag{1.6}$$

We can similarly define the dual of a cone in $N_1(X)$.

We have the following useful lemma (see [Deb01, Lemma 6.7]); for simplicity's sake we state if for $N^1(X)$, but it is true whenever we have a non degenerated scalar product defining a duality between real vector spaces.

Lemma 1.24. *Let $\mathcal{C} \subset N^1(X)$ a closed convex cone.*

1. *$\mathcal{C} = \mathcal{C}^{\vee\vee}$;*

2. *$\mathcal{C}$ contains no lines through the origin $\iff$ $\mathcal{C}^\vee$ spans $N_1(X)$;*

3. *The interior of* $\mathcal{C}^\vee$ *is given by*

$$\{\gamma \in N_1(X) \mid \gamma \cdot c > 0 \text{ for any } c \in \mathcal{C} \setminus \{0\}\}.$$

Definition 1.25. Let X be a normal projective variety and consider a cone $\mathcal{C}$ in $N_1(X)$ (respectively in $N^1(X)$) and a divisor (respectively a curve) D. We denote the *D-positive part of $\mathcal{C}$* the subcone

$$\mathcal{C}_{D^{>0}} = \mathcal{C} \cap D^{>0}. \tag{1.7}$$

Similarly we can define $\mathcal{C}_{D^{\geqslant 0}}$, $\mathcal{C}_{D^{\leqslant 0}}$, $\mathcal{C}_{D^{<0}}$ and $\mathcal{C}_{D^{\perp}}$

Since in the following a great importance will be given to extremal rays, we fix the notation for rays.

Definition 1.26. Let X be a normal projective variety and $C \subset X$ is an integral curve with class $[C] \in N_1(X)$, we will denote $R(C)$ the ray generated by the class $[C]$ in $N_1(X)$ (see Definition 1.8.4).

Definition 1.27. Let X be a normal projective variety, R be a ray in $N_1(X)$ and D a divisor. We say that the ray R is *D-positive* (or *D-negative*) if $D \cdot \gamma > 0$ (respectively $D \cdot \gamma < 0$) for all $\gamma \in R$.

In the beginning of this chapter we defined a number of subsets of $N^1(X)$ and $N_1(X)$; as it is well-known, all of them are cones contained in a real vector space.
We have the following fact; see [Laz04] (for 1. 2. and 3.) and [BDPP04] (for 4.).

Fact 1.28. *Let X be a normal projective variety, then*

1. $\mathrm{Nef}(X)$ *is a closed convex cone in* $N^1(X)$ *and it is the dual of the convex cone* $\overline{\mathrm{NE}}(X)$*;*

2. $\mathrm{Amp}(X)$ *is an open convex cone in* $N^1(X)$ *and it is the interior of* $\mathrm{Nef}(X)$*;*

3. $\mathrm{Big}(X)$, $\mathrm{Eff}(X)$ *and* $\overline{\mathrm{Eff}}(X)$ *are convex cones and it holds:*

$$\mathrm{Big}(X) = \mathrm{int}(\overline{\mathrm{Eff}}(X)) \quad \text{and} \quad \overline{\mathrm{Eff}}(X) = \mathrm{cl}(\mathrm{Big}(X));$$

4. $\overline{\mathrm{ME}}(X)$ *is a closed convex cone and it coincides with* $\overline{\mathrm{SME}}(X)$.

We see that amplitude can be interpreted via cones (see [Laz04]).

Theorem 1.29 (Kleiman's Criterion)**.** *Let X be a projective variety and D an $\mathbb{R}$-divisor on X; then D is ample if and only if*

$$\overline{\mathrm{NE}}(X) \setminus \{0\} \subset D^{>0}.$$

The starting result of the whole theory is doubtless the Cone Theorem, whose first formulation was given in [Mor82] by Shigefumi Mori. This theorem gives a structure of the K_X-negative part of $\overline{\mathrm{NE}}(X)$.

Theorem 1.30 (Mori cone Theorem)**.** *Let X be a smooth projective variety of dimension n. Then there exists a countable set of curves C_i, $i \in I$ with $K_X \cdot C_i < 0$, such that we have the decomposition*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X^{\geqslant 0}} + \sum_{i \in I} R(C_i). \tag{1.8}$$

Moreover, we have that:

1. *this decomposition is minimal (in the sense that no smaller index set is sufficient to generate the cone);*
2. *for any small $\epsilon > 0$ and ample divisor H, there are just finitely many extremal rays in $(K_X + \epsilon H)^{<0}$;*
3. *the curves C_i are (possibly singular) reduced irreducible rational curves satisfying the condition*

$$-(n+1) \leqslant K_X \cdot C_i \leqslant -1.$$

Corollary 1.31. *The rays $R(C_i)$ given by the theorem are indeed extremal rays of $\overline{\mathrm{NE}}(X)$.*

Proof. Let us consider $R(C_{i_0})$ a ray as in the statement of Cone theorem; we want to show that it is an extremal ray. Consider $\gamma, \delta \in \overline{\mathrm{NE}}(X)$ such that $\gamma + \delta \in R(C_{i_0})$. The theorem gives the decomposition

$$\gamma = \gamma' + \sum a_i[C_i] \quad \text{and} \quad \delta = \delta' + \sum b_i[C_i], \tag{1.9}$$

with $\gamma', \delta' \in {K_X}^{\geqslant 0}$ and $a_i, b_i \geqslant 0$. Therefore we have

$$\gamma + \delta = (\gamma' + \delta') + \sum (a_i + b_i)[C_i],$$

but also $\gamma + \delta = \alpha[C_{i_0}]$ for some $\alpha \geqslant 0$. We have hence that

$$\gamma' + \delta' + \sum_{i \neq i_0} (a_i + b_i)[C_i] = (\alpha - a_{i_0} - b_{i_0})[C_{i_0}]. \tag{1.10}$$

Now if $(\alpha - a_{i_0} - b_{i_0}) \neq 0$, intersecting with an ample divisor, we get $(\alpha - a_{i_0} - b_{i_0}) > 0$ and hence we would have

$$[C_{i_0}] \in \overline{\mathrm{NE}}(X)_{K_X \geqslant 0} + \sum_{i \neq i_0} R(C_i),$$

and the index set I from the theorem wouldn't be minimal. Therefore it must be $(\alpha - a_{i_0} - b_{i_0}) = 0$. Fix an ample class h; intersecting equation (1.10) with h we get

$$(\gamma' + \delta') \cdot h + \sum_{i \neq i_0} (a_i + b_i)[C_i] \cdot h = 0.$$

Since $a_i, b_i, (a_i + b_i) \geqslant 0$ and h is an ample class, we must have $\gamma' \cdot h = \delta' \cdot h = 0$ and $a_i = b_i = 0$ for all $i \neq i_0$; therefore we have that, since h is an ample class, Kleiman Criterion (see Theorem 1.29) gives $\gamma' = \delta' = 0$.
Thus decomposition (1.9) gives $\gamma = a_{i_0}[C_{i_0}]$ and $\delta = b_{i_0}[C_{i_0}]$, hence $R(C_{i_0})$ is an extremal ray. □

Theorem 1.32 (Contraction Theorem)**.** *Let X be a smooth projective variety, for each extremal ray R in ${K_X}^{<0}$ there is a contraction morphism $\mathrm{cont}_R : X \to Z$, that is a morphism such that*

1. $\mathrm{cont}_R(C) = \text{point}$ *for an irreducible curve $C \subset X$ if and only if $[C] \in R$;*
2. $(\mathrm{cont}_R)_* \mathcal{O}_X = \mathcal{O}_Z$.

We now show an interesting and easy consequence of the Cone Theorem.

Fact 1.33. *Let X be a projective smooth variety; each extremal K_X-negative ray R is spanned by the class of an integral curve.*

Proof. Let γ be a generator of R (that is $R = R(\gamma)$). The Cone theorem gives the decomposition

$$\gamma = \alpha + \sum a_i[C_i], \quad \alpha \in K_X{}^{\geqslant 0}, \quad a_i > 0$$

with C_i rational irreducible curve. Now, since R is an extremal ray, if $\alpha \neq 0$, then there exists $a > 0$ such that $\gamma = a\alpha$; but then

$$0 > K_X \cdot \gamma = a\alpha \cdot K_X \geqslant 0$$

and hence we must have $\alpha = 0$. Again by extremality, we can conclude that there exists $b_i > 0$ such that $\gamma = b_i a_i [C_i]$; therefore we get

$$[C_i] = \frac{1}{a_i b_i}\gamma$$

and so $[C_i]$ generates R. □

Since in the following we will be interested in the study of the pseudoeffective cone, we have to cite a very important result by Boucksom, Demailly, Paun and Peternell (see [BDPP04]) relating $\overline{\mathrm{Eff}}(X)$ and the cone of movable curves.

Theorem 1.34 (BDPP)**.** *Let X be a projective variety; then*

$$\overline{\mathrm{Eff}}(X)^{\vee} = \overline{\mathrm{ME}}(X). \tag{1.11}$$

We can now see that Lemma 1.22 can be useful dealing with the Mori cone $\overline{\mathrm{NE}}(X)$: we need to show that it does not contain lines through the origin.

Fact 1.35. *The Mori cone $\overline{\mathrm{NE}}(X)$ of a projective variety does not contain lines through the origin. Moreover it is spanned by its extremal rays.*

Proof. We will prove that if γ and $-\gamma$ lie in $\overline{\mathrm{NE}}(X)$, then $\gamma = 0$. Since $\overline{\mathrm{NE}}(X)$ is the dual of $\mathrm{Nef}(X)$, we have that $\gamma \cdot \delta \geqslant 0$ and $-\gamma \cdot \delta \geqslant 0$ for every $\gamma \in \mathrm{Nef}(X)$; we immediately get $\gamma \cdot \delta = 0$ for every nef class. In particular, if α is an ample class, then $\gamma \cdot \alpha = 0$ and Kleiman's Criterion (Theorem 1.29) gives us $\gamma = 0$.
The last statement follows directly from Lemma 1.22. □

In the pseudoeffective case, we are able to give an analogous result.

Fact 1.36. *The pseudoeffective cone of a projective variety X does not contain lines through the origin. Moreover it is spanned by its extremal rays.*

Proof. We want to prove that for $F \in \mathrm{Div}_{\mathbb{R}}(X)$, if F and $-F$ are pseudoeffective, then $F \equiv 0$. We can easily perform the following reduction. Let us assume the fact is true in the smooth case; if X is not smooth, let us consider the resolution of singularities $\pi : \tilde{X} \to X$. We have that $\pi^*F \in \mathrm{Div}_{\mathbb{R}}(\tilde{X})$ and, since we are assuming F and $-F$ pseudoeffective, also π^*F and $-\pi^*F$ will be pseudoeffective. Now, $\tilde{X}$ is smooth and therefore $\pi^*F \equiv 0$. To show that $F \equiv 0$, let us consider a curve $C \subset X$; if $C' \subset \tilde{X}$ is a curve mapped to C, by projection formula, $F \cdot C$ is proportional to $\pi^*F \cdot C'$ that is zero because $\pi^*F \equiv 0$.
Thus we can assume X to be a smooth variety. We proceed by induction on $n = \dim X$; if $n = 2$, then $\overline{\mathrm{Eff}}(X) = \overline{\mathrm{NE}}(X)$ and, by Fact 1.35, we are done.

By induction, let us suppose $n \geqslant 3$.
Let $0 \neq \delta \in \overline{\mathrm{Eff}}(X)$ be a pseudoeffective class and H an ample effective divisor on X; by Bertini Theorem we can suppose it is smooth. Writing

$$\delta = \lim_i [D_i],$$

for effective divisors $D_i \in \mathrm{Div}_{\mathbb{R}}(X)$, we claim that $\delta|_H \in \overline{\mathrm{Eff}}(H)$. Indeed we have that $\delta|_H = (\lim_i[D_i])|_H$; by linearity and hence by continuity, we get

$$\delta|_H = \lim_i [D_i]|_H = \lim_i [D_i|_H].$$

Thus we need to show that $D_i|_H$ is an effective divisor on H for all i. Let us set:

$$D_i = \sum a_j D_i^j, \quad \text{with } D_i^j \text{ prime divisor and } a_j > 0.$$

If $D_i^j \neq H$, then the restriction to H is given by intersection and thus we get an effective divisor; if $D_i^j = H$, by ampleness $H|_H$ is still effective. Therefore we get $D_i|_H$ as a sum of effective divisors and $\delta|_H$ as a limit of effective class.
We claim now that $\delta|H \neq 0$, that is the restriction map $N^1(X) \to N^1(H)$ is injective.
Since $n \geqslant 3$, the exponential sequence and Lefschetz Hyperplane Theorem (see [Laz04, Theorem 3.1.17]) imply that

$$\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}(H)$$

is injective. Now, using a result by Kleiman, see [Laz04, 1.1.20], we can easily get an injective map $N^1(X)_{\mathbb{Z}} \hookrightarrow N^1(H)_{\mathbb{Z}}$ that, tensoring with $\mathbb{R}$, gives the required injection $N^1(X) \hookrightarrow N^1(H)$.
At this point, to prove that in $\overline{\mathrm{Eff}}(X)$ there are no lines passing by the origin, it is enough to show that if $0 \neq \delta \in \overline{\mathrm{Eff}}(X)$, then $-\delta \notin \overline{\mathrm{Eff}}(X)$.
If, by contradiction, $-\delta \in \overline{\mathrm{Eff}}(X)$, arguing as before, we should have $(-\delta)|_H \in \overline{\mathrm{Eff}}(H)$; since $(-\delta)|_H = -(\delta|_H)$, we would have $0 \neq \delta|_H \in \overline{\mathrm{Eff}}(H)$ and $-\delta|_H \in \overline{\mathrm{Eff}}(H)$, that is a contradiction because $\dim H = n - 1$. ☐

1.3 Cones on a surface

In this section we focus the two dimensional varieties; let us recall an useful lemma (see [Deb01, Lemma 6.2]) concerning extremal rays and their generating curves in the case of surfaces.

Lemma 1.37. *Let S be a smooth projective surface. Then:*

1. *if C is an integral curve on S such that $C^2 \leqslant 0$, then $[C] \in \partial\overline{\mathrm{NE}}(S)$;*
2. *if C is an integral curve such that $C^2 < 0$, then $[C]$ spans an extremal ray;*
3. *if r spans an extremal ray of $\overline{\mathrm{NE}}(S)$, then either $r^2 \leqslant 0$ or $\rho(S) = 1$;*
4. *if r spans an extremal ray R of $\overline{\mathrm{NE}}(S)$ and $r^2 < 0$, then R is spanned by a class of an irreducible curve.*

In the case of surfaces, contractions of extremal K_S-negative rays can be classified. The following result is an useful consequence of Cone Theorem in dimension 2.

Proposition 1.38. *Let S be a smooth surface and let R be a K_S-negative extremal ray of $\overline{\mathrm{NE}}(S)$. Then the contraction morphism*

$$\mathrm{cont}_R : S \to Z$$

exists and it is one of the following:

1. *Z is a smooth surface and $S = \mathrm{Bl}_P Z$ for a closed point P; in this case $\rho(Z) = \rho(S) - 1$.*
2. *Z is a smooth curve, S is a minimal ruled surface over Z and $\rho(S) = 2$.*
3. *Z is a point, we have that $\rho(S) = 1$ and $-K_S$ is ample and in fact $S \simeq \mathbb{P}^2$.*

Proof. See [KM98, Theorem 1.28]. □

In the case of a smooth projective surface S, the two vector spaces $N^1(S)$ and $N_1(S)$ are indeed the same space; we eventually shall denote it with $N(S)$. At first we recall a well-known fact: if C and D are distinct curves in S, then we have

$$C \cdot D \geqslant 0. \tag{1.12}$$

In the case of surfaces it is possible to consider the self-intersection of curves. We are mainly interested in the study of negative self-intersection curves. We have the following lemma.

Lemma 1.39. *Let $C \subset S$ be an integral curve such that $C^2 < 0$ and let $R = R(C)$ be the ray generated by $[C]$; then in R there are no other integral curves.*

Proof. If there were another integral curve C' (that is C' integral and distinct from C) with $[C'] \in R$, then there would exist $b > 0$ such that $[C'] = b[C]$; now:

$$C \cdot C' = bC \cdot C = bC^2 < 0,$$

but since C and C' are distinct curves, $C \cdot C' \geqslant 0$, and we get a contradiction. □

The integral curves with negative self-intersection can be described as follows.

Definition 1.40 (($-n, p$)-curves)**.** Let C be an integral curve in a smooth surface S.

1. C is said to be a $(-n, p)$-*curve* if $C^2 = -n$ and it has arithmetic genus $p_a(C) = p$; in particular a $(-n, 0)$-curve is a $(-n)$-curve.
2. A ray R in $\overline{\mathrm{NE}}(S)$ is said to be a $(-n, p)$-*ray* if $R = R(C)$ is generated by a $(-n, p)$-curve $C \subset S$.

We point out that the former definition of $(-n, p)$-ray is consistent in view of Lemma 1.39.

1.4 The positive cone of a surface

As pointed out before, in the case of surfaces the cones of divisors and the cones of curves can be compared and it is interesting to find these relationships. If S is a projective surface, the most studied cones are undoubtedly the Mori cone $\overline{\mathrm{NE}}(S)$ and the nef cone $\mathrm{Nef}(S)$, but there is another interesting cone: the positive cone; in this section we put in evidence some properties of this cone.
First of all, let us recall the following Lemma.

Lemma 1.41. *Let X be a normal projective variety; if $b \in \operatorname{Big} X$ e $f \in \overline{\operatorname{Eff}}(X)$, then $b + f \in \operatorname{Big}(X)$.*

Proof. Proposition 2.2.22(ii) from [Laz04] gives the existence of two classes $a \in \operatorname{Amp}(X)$ and $e \in \operatorname{Eff}(X)$ such that $b = a + e$. Since $\operatorname{Amp}(X)$ is an open cone, there exists a small disc $D_\epsilon(a) \subset \operatorname{Amp}(X)$. Let $e' \in \operatorname{Eff}(X)$ such that $\|f - e'\| < \epsilon$. We have therefore that

$$b + f = \underbrace{a + (f - e')}_{\in \operatorname{Amp}(X)} + \underbrace{e' + e}_{\in \operatorname{Eff}(X)} \in \operatorname{Big}(X),$$

since $a + (f - e') \in D_\epsilon(a)$ is ample and $e + e'$ is effective. □

Definition 1.42 (Positive cone)**.** Let S be a smooth projective surface and let $h \in \operatorname{Amp}(S)$. The *open positive cone of S* is

$$\operatorname{Pos}(S) = \left\{ x \in N^1(S) \mid x^2 > 0, x \cdot h > 0 \right\}. \tag{1.13}$$

The *positive cone of S* is

$$\overline{\operatorname{Pos}}(S) = \left\{ x \in N^1(S) \mid x^2 \geqslant 0, x \cdot h \geqslant 0 \right\}. \tag{1.14}$$

We can immediately see that our notations make sense and indeed $\overline{\operatorname{Pos}}(S)$ is the closure of $\operatorname{Pos}(S)$. Since the intersection form is continuous, we have that $\overline{\operatorname{Pos}}(S)$ is a closed cone. Consider now $x \in \overline{\operatorname{Pos}}(S)$ and $m \in \mathbb{N}$, we have that $x + \frac{1}{m} h \in \operatorname{Pos}(S)$. In fact

$$\begin{cases} \left(x + \frac{1}{m} h\right)^2 = x^2 + \frac{1}{m^2} h^2 + \frac{2}{m} x \cdot h > 0 \\ \left(x + \frac{1}{m} h\right) \cdot h = x \cdot h + \frac{1}{m} h^2 > 0. \end{cases}$$

Since obviously $x = \lim_{m \to +\infty} \left(x + \frac{1}{m} h\right)$, we get that x is in the closure of $\operatorname{Pos}(S)$.
Let us recall that the space $N(S)$ is a ρ-dimensional vector space that can be equipped with the Euclidean topology; by Hodge Index Theorem (see [Har77, Theorem V.1.9]), the intersection form is a bilinear form on $N(S)$ with signature $(1, \rho - 1)$ and the Sylvester theorem assures us the existence of a basis $\{e_1, \dots, e_\rho\}$ such that

$$\begin{array}{ll} e_1 = \frac{h}{\sqrt{h^2}} & e_1{}^2 = 1 \\ e_i{}^2 = -1 & \text{for } i = 2, \dots, \rho \\ e_i \cdot e_j = 0 & \text{for } 1 \leqslant i < j \leqslant \rho, \end{array} \tag{1.15}$$

and we have therefore that the intersection matrix is $\operatorname{diag}(1, -1, \dots, -1)$. We will use this basis to write the elements $x \in N(S)$ as $x = \sum_{i=1}^{\rho} x_i e_i$.

Definition 1.43. If S is a smooth projective surface, the *negative curve set* is

$$\operatorname{Neg}(S) = \{[C] \mid C \subset S \text{ integral curve such that } C^2 < 0\}, \tag{1.16}$$

where $[C]$ is the class of C in $N(S)$.

We want now to put in evidence some interesting properties of the positive cone $\overline{\operatorname{Pos}}(S)$ of a surface S.

Fact 1.44. *With the choices of (1.15), the positive cones have the following equations*

1. $\operatorname{Pos}(S) = \left\{ x \in N(S) \mid x_1 > 0, x_1 > \sum_{i=2}^{\rho} x_i{}^2 \right\}$;

2. $\overline{\mathrm{Pos}}(S) = \left\{x \in N(S) \mid x_1 \geqslant 0, x_1 \geqslant \sum_{i=2}^{\rho} x_i{}^2\right\}$.

Proof. If $x = \sum_{i=1}^{\rho} x_i e_i$, then we get

$$x \cdot h = x \cdot \sqrt{h^2} e_1 = \sqrt{h^2} x_1 \quad \text{and} \quad x^2 = x_1^2 - \sum_{i=2}^{\rho} x_i{}^2,$$

which gives immediately the first and the second claim of the fact. □

Before going on, we recall the following consequence of Hodge index theorem.

Remark 1.45. Let x be a real class in $N(S)$ and $h \in \mathrm{Amp}(S)$, then

$$x^2 h^2 \leqslant (x \cdot h)^2. \tag{1.17}$$

Indeed Hodge index theorem assures that the (1.17) holds for integer classes. If we deal with rational classes the inequality holds true since we can multiply for an appropriate integer and we can reduce to the integer case. If, in the end, x, h are real classes than the inequality holds true approximating x, h with rational classes and passing to the limit.

Fact 1.46. *If $x, y \in \overline{\mathrm{Pos}}(S)$, then $x \cdot y \geqslant 0$; moreover if $x \neq 0$ and $y \in \mathrm{Pos}(S)$ or $y \neq 0$ and $x \in \mathrm{Pos}(S)$, we have that $x \cdot y > 0$. In particular, the positive cone $\overline{\mathrm{Pos}}(S)$ is a convex cone.*

Proof. We will see two different proofs of this fact.
In the first we don't use coordinates; consider $x, y \in \overline{\mathrm{Pos}}(S)$. From Hodge Index Theorem we have that the intersection form is negative definite on $h^{\perp}$; therefore if $x \cdot h = 0$ we must have $x^2 \leqslant 0$; since by hypothesis $x^2 \geqslant 0$, we have $x = 0$ and $x \cdot y = 0 \geqslant 0$ (similarly if $y \cdot h = 0$).
Let us suppose $x \cdot h > 0$ and $y \cdot h > 0$. Consider the vectors

$$v = \frac{\sqrt{h^2}}{x \cdot h} x - \frac{1}{\sqrt{h^2}} h \quad \text{and} \quad w = \frac{\sqrt{h^2}}{y \cdot h} y - \frac{1}{\sqrt{h^2}} h$$

We can see that $v \cdot h = \sqrt{h^2} - \frac{h^2}{\sqrt{h^2}} = 0$ and similarly $w \cdot h = 0$, we have therefore that $v, w \in h^{\perp}$. Since the form is negative definite on $h^{\perp}$, if we define $v * w = -v \cdot w$, we get a scalar product on $h^{\perp}$. In particular, the Cauchy-Schwartz inequality holds and give

$$-v \cdot w = v * w \leqslant |v * w| \leqslant \|v\| \|w\|, \tag{1.18}$$

where $\|v\| = \sqrt{v * v} = \sqrt{-v \cdot v}$ and $\|w\| = \sqrt{w * w} = \sqrt{-w \cdot w}$. Now we have that

$$-v \cdot v = -\left(\frac{\sqrt{h^2}}{x \cdot h} x - \frac{1}{\sqrt{h^2}} h\right) \cdot \left(\frac{\sqrt{h^2}}{x \cdot h} x - \frac{1}{\sqrt{h^2}} h\right) = 1 - \frac{h^2 x^2}{(x \cdot h)^2}. \tag{1.19}$$

Using equation (1.17) we easly see that since $x^2 \geqslant 0$, then $-v \cdot v \leqslant 1$ and $-v \cdot v \geqslant 0$.
Therefore we have $0 \leqslant -v \cdot v \leqslant 1$ and similarly $0 \leqslant -w \cdot w \leqslant 1$. Thus we get $\|v\| \leqslant 1, \|w\| \leqslant 1$ and equation (1.18) implies

$$-v \cdot w \leqslant 1. \tag{1.20}$$

But we can compute

$$-v \cdot w = -\left(\frac{\sqrt{h^2}}{x \cdot h} x - \frac{1}{\sqrt{h^2}} h\right) \cdot \left(\frac{\sqrt{h^2}}{y \cdot h} y - \frac{1}{\sqrt{h^2}} h\right) = 1 - \frac{h^2}{(x \cdot h)(y \cdot h)} x \cdot y,$$

and equation (1.20) gives

$$1-\frac{h^2}{(x\cdot h)(y\cdot h)}x\cdot y\leqslant 1 \tag{1.21}$$

that is

$$\frac{h^2}{(x\cdot h)(y\cdot h)}x\cdot y\geqslant 0,$$

which gives $x\cdot y\geqslant 0$. The first part has therefore been shown. However, if $x\cdot y=0$ then there is equality in equation (1.21), and so in the (1.20). As a consequence we have that $\|v\|=1$, $\|w\|=1$ and the (1.19) implies that $x^2=y^2=0$. So if $x\neq 0$ and $y\in \mathrm{Pos}(S)$ (or $y\neq 0$ and $x\in\mathrm{Pos}(S)$) it follows that $x\cdot y>0$.
Before giving an other proof using coordinates, we have the following

Claim 1.47. For all $(x_2,\dots,x_\rho),(y_2,\dots,y_\rho)\in\mathbb{R}^{\rho-1}$, we have

$$\sqrt{\sum_{i=2}^{\rho}x_i^2}\sqrt{\sum_{i=2}^{\rho}y_i^2}\geqslant\sum_{i=2}^{\rho}x_iy_i. \tag{1.22}$$

This inequality is immediately verified if $\sum_{i=2}^{\rho}x_iy_i\leqslant 0$, therefore we can suppose that $\sum_{i=2}^{\rho}x_iy_i>0$ and we have that (1.22) is equivalent to

$$\left(\sum_{i=2}^{\rho}x_i^2\right)\left(\sum_{i=2}^{\rho}y_i^2\right)\geqslant\left(\sum_{i=2}^{\rho}x_iy_i\right)^2. \tag{1.23}$$

Now we have that

$$\left(\sum_{i=2}^{\rho}x_i^2\right)\left(\sum_{i=2}^{\rho}y_i^2\right)=\sum_{i=2}^{\rho}x_i^2y_i^2+\sum_{2\leqslant i<j\leqslant\rho}x_i^2y_j^2+\sum_{2\leqslant i<j\leqslant\rho}x_j^2y_i^2$$

whereas instead

$$\left(\sum_{i=2}^{\rho}x_iy_i\right)^2=\sum_{i=2}^{\rho}x_i^2y_i^2+2\sum_{2\leqslant i<j\leqslant\rho}x_iy_ix_jy_j.$$

To show (1.23) it is enough to prove that

$$\sum_{2\leqslant i<j\leqslant\rho}x_i^2y_j^2+\sum_{2\leqslant i<j\leqslant\rho}x_j^2y_i^2\geqslant 2\sum_{2\leqslant i<j\leqslant\rho}x_iy_ix_jy_j. \tag{1.24}$$

To see it, it is sufficient to see that

$$\begin{aligned}0\leqslant\sum_{2\leqslant i<j\leqslant\rho}(x_iy_j-x_jy_i)^2&=\sum_{2\leqslant i<j\leqslant\rho}(x_i^2y_j^2+x_j^2y_i^2-2x_iy_ix_jy_j)=\\ =\sum_{2\leqslant i<j\leqslant\rho}x_i^2y_j^2&+\sum_{2\leqslant i<j\leqslant\rho}x_j^2y_i^2-2\sum_{2\leqslant i<j\leqslant\rho}x_iy_ix_jy_j\ .\end{aligned}$$

And this gives the (1.22).
Now let us prove Fact 1.46 using coordinates.
Consider $x=x_1e_1+\ldots+x_\rho e_\rho$, and $y=y_1e_1+\ldots+y_\rho e_\rho$; we have that $x\cdot y=x_1y_1-\sum_{i=2}^{\rho}x_iy_i$. Since $x_1\geqslant 0, y_1\geqslant 0$ and $x,y\in\overline{\mathrm{Pos}}(S)$, we have that

$$x_1\geqslant\sqrt{\sum_{i=2}^{\rho}x_i^2}\quad\text{and}\quad y_1\geqslant\sqrt{\sum_{i=2}^{\rho}y_i^2}, \tag{1.25}$$

which, using (1.22), gives

$$x_1 y_1 \geqslant \sqrt{\sum_{i=2}^{\rho} x_i^2}\sqrt{\sum_{i=2}^{\rho} y_i^2} \geqslant \sum_{i=2}^{\rho} x_i y_i \tag{1.26}$$

Therefore we have shown the first part of Fact 1.46. If $x \cdot y = 0$ and $x \neq 0, y \neq 0$, then there is equality in (1.26) and so also in (1.25). Then $x^2 = y^2 = 0$. If $x \neq 0$ and $y \in \mathrm{Pos}(S)$ (or $y \neq 0$ and $x \in \mathrm{Pos}(S)$), it follows that $x \cdot y > 0$ which concludes the proof. □

Fact 1.48. *If $x \in \overline{\mathrm{Pos}}(S)$, $a \geqslant 0$, $y^2 \geqslant -a$ and $y \cdot h \geqslant 0$, then we have the following useful inequality:*

$$x \cdot y \geqslant \frac{(y \cdot h - \sqrt{(y \cdot h)^2 + ah})x \cdot h}{h^2}. \tag{1.27}$$

Proof. Since $x \in \overline{\mathrm{Pos}}(S)$ and $x_1 \geqslant 0$, we get

$$x_1 \geqslant \sqrt{\sum_{i=2}^{\rho} x_i^2} \quad \text{and} \quad \sqrt{y_1^2 + a} \geqslant \sqrt{\sum_{i=2}^{\rho} y_i^2}; \tag{1.28}$$

since $y_1 \geqslant 0$, using also (1.28) and (1.22), we get

$$\begin{aligned} x \cdot y = x_1 y_1 - \sum_{i=2}^{\rho} x_i y_i \geqslant y_1 \sqrt{\sum_{i=2}^{\rho} x_i^2} - \sum_{i=2}^{\rho} x_i y_i = \\ = \sqrt{\sum_{i=2}^{\rho} x_i^2}\left(\sqrt{y_1^2 + a} - \sqrt{y_1^2 + a} + y_1\right) - \sum_{i=2}^{\rho} x_i y_i \geqslant \\ \geqslant \sqrt{\sum_{i=2}^{\rho} x_i^2}\sqrt{\sum_{i=2}^{\rho} y_i^2} - \sum_{i=2}^{\rho} x_i y_i + \left(y_1 - \sqrt{y_1^2 + a}\right)\sqrt{\sum_{i=2}^{\rho} x_i^2} \\ \geqslant \left(y_1 - \sqrt{y_1^2 + a}\right) x_1. \end{aligned} \tag{1.29}$$

It is now enough to see that

$$\left(y_1 - \sqrt{y_1^2 + a}\right) x_1 = \frac{(y \cdot h - \sqrt{(y \cdot h)^2 + ah^2})x \cdot h}{h^2}$$

and the proof is concluded. □

Fact 1.49. *The definitions of $\overline{\mathrm{Pos}}(S)$ and $\mathrm{Pos}(S)$ do not depend on the choice of the ample class h.*

Proof. It is an immediate application of Fact 1.46. If $\overline{\mathrm{Pos}}(S)$ is defined using an other ample class h', we have to show that $x \cdot h' \geqslant 0$ for all $x \in \overline{\mathrm{Pos}}(S)$ (defined using h). Since the ample cone $\mathrm{Amp}(S) \subset \overline{\mathrm{Pos}}(S)$, Fact 1.46 gives in a moment that $x \cdot h' \geqslant 0$. □

Fact 1.50. *For all $y \in N(S)$, there exist $x \in \partial\overline{\mathrm{Pos}}(S)$ and $u \in \mathbb{R}$ such that $y = x + uh$.*

Proof. Let $y \in N(S)$ and consider

$$\Delta = (y \cdot h)^2 - h^2 y^2. \tag{1.30}$$

Hodge Index Theorem immediaely gives $\Delta \geqslant 0$; let t be the following solution of the equation $(y + Th)^2 = 0$:

$$t = \frac{-y \cdot h + \sqrt{\Delta}}{h^2}. \tag{1.31}$$

By definition we have that $(y + th)^2 = 0$ and since we get $(y + th) \cdot h = \sqrt{\Delta} \geqslant 0$, we have $y + th \in \overline{\text{Pos}}(S)$. Setting $u = -t$ and $x = y + th$, we conclude the proof. □

Consider now an interesting property of the positive cone: it is self-dual.

Proposition 1.51. *If S is a projective smooth surface, then*

$$\overline{\text{Pos}}(S) = \left(\overline{\text{Pos}}(S)\right)^{\vee}. \tag{1.32}$$

We will see three different proofs of this fact.

First proof of Proposition 1.51. We can see that Fact 1.46 gives the inclusion

$$\overline{\text{Pos}}(S) \subseteq \overline{\text{Pos}}(S)^{\vee}.$$

Consider now $y \in \overline{\text{Pos}}(S)^{\vee}$; since $h \in \overline{\text{Pos}}(S)$, we have that $y \cdot h \geqslant 0$. Take Δ as in (1.30) and t as in (1.31); we know that $y + th \in \overline{\text{Pos}}(S)$ and therefore $y \cdot (y + th) \geqslant 0$. If $\Delta = 0$, then $y^2 = \frac{(y \cdot h)^2}{h^2} \geqslant 0$ and therefore $y \in \overline{\text{Pos}}(S)$.
Suppose now $\Delta > 0$. We have

$$\begin{aligned} 0 \leqslant y \cdot (y + th) = y^2 + \frac{-(y \cdot h)^2 + y \cdot h\sqrt{\Delta}}{h^2} = \\ = \frac{y^2 h^2 - (y \cdot h)^2 + y \cdot h\sqrt{\Delta}}{h^2} = \frac{-\Delta + y \cdot h\sqrt{\Delta}}{h^2}, \end{aligned}$$

and therefore

$$\Delta \leqslant y \cdot h\sqrt{\Delta},$$

that is

$$\Delta \leqslant (y \cdot h)^2,$$

so

$$(y \cdot h)^2 - h^2 y^2 \leqslant (y \cdot h)^2,$$

which gives

$$h^2 y^2 \geqslant 0$$

that is

$$y^2 \geqslant 0.$$

Therefore we have that $y \in \overline{\text{Pos}}(S)$ and the proposition is proven. □

Now we see an other proof using coordinates.

Second proof of Proposition 1.51. As before, Fact 1.46 implies $\overline{\mathrm{Pos}}(S) \subseteq \overline{\mathrm{Pos}}(S)^\vee$. Let now $y \in \overline{\mathrm{Pos}}(S)^\vee$. Since $e_1 \in \overline{\mathrm{Pos}}(S)$, we have $0 \leqslant y \cdot e_1 = y_1$.
If $y_1 = 0$, since for $i \in \{2, \dots, \rho\}$, $e_1 \pm e_i \in \overline{\mathrm{Pos}}(S)$ (easy computation), we have $0 \leqslant y \cdot (e_1 \pm e_i) = \mp y_i$, which gives $y_i = 0$ for $i \in \{2, \dots, \rho\}$, and therefore $y = 0 \in \overline{\mathrm{Pos}}(S)$.
If $y_1 \neq 0$, let $v = (\frac{y_2}{y_1}, \dots, \frac{y_\rho}{y_1})$ be a vector in $\mathbb{R}^{\rho-1}$. If $v = 0$ then $y_i = 0$ for $i \in \{2, \dots, \rho\}$, which gives $y^2 = y_1^2 \geqslant 0$, that is $y \in \overline{\mathrm{Pos}}(S)$.
If $v \neq 0$, define $x = y_1 e_1 + \frac{1}{\|v\|}(y_2 e_2 + \dots y_\rho e_\rho)$. Now

$$x^2 = y_1^2 - \frac{1}{\|v\|^2}\sum_{i=2}^{\rho} y_i^2 = y_1^2\left(1 - \frac{1}{\|v\|^2}\sum_{i=2}^{\rho}\left(\frac{y_i}{y_1}\right)^2\right) = 0$$

which gives $x \in \overline{\mathrm{Pos}}(S)$ and therefore

$$\begin{aligned} 0 \leqslant y \cdot x &= y_1^2 - \frac{1}{\|v\|}\sum_{i=2}^{\rho} y_i^2 \\ &= y_1^2\left(1 - \frac{1}{\|v\|}\sum_{i=2}^{\rho}\left(\frac{y_i}{y_1}\right)^2\right) = y_1^2(1 - \|v\|). \end{aligned}$$

Thus we have

$$\|v\| \leqslant 1$$

which gives

$$\|v\|^2 \leqslant 1$$

that is

$$\sum_{i=2}^{\rho}\left(\frac{y_i}{y_1}\right)^2 \leqslant 1$$

which gives

$$y_1^2 \geqslant \sum_{i=2}^{\rho} y_i^2$$

and then $y \in \overline{\mathrm{Pos}}(S)$. □

We will now give a more intuitive and geometric proof of the Proposition. The following Lemma will be useful to understand the visual idea behind the proof.

Lemma 1.52. *Let γ be a class in $N(S)$, with $\rho(S) \geqslant 3$, such that $\gamma^2 < 0$, $\gamma \cdot h \geqslant 0$ and let us consider $0 \neq \alpha \in \overline{\mathrm{Pos}}(S)$; let L be the line joining α to γ, then*

$$L \cap \overline{\mathrm{Pos}}(S) = \{\alpha\} \quad \iff \quad \alpha^2 = \alpha \cdot \gamma = 0.$$

Proof. At first, we have that the line joining α to γ is $L(\alpha, \gamma) = \{\gamma_t \mid t \in \mathbb{R}\}$, where

$$\gamma_t = t\gamma + (1-t)\alpha. \tag{1.33}$$

We can hence compute:

$$\begin{aligned} {\gamma_t}^2 &= (\gamma^2 - 2\alpha\cdot\gamma + \alpha^2)t^2 - 2(\alpha^2 - \alpha\cdot\gamma)t + \alpha^2; \\ \gamma_t \cdot h &= t\gamma\cdot h + (1-t)\alpha\cdot h. \end{aligned} \tag{1.34}$$

We can now prove the two implications.

($\Leftarrow$) The intersection $L \cap \overline{\text{Pos}}(S)$ is given by γ_t such that ${\gamma_t}^2 \geqslant 0$ and $\gamma_t \cdot h \geqslant 0$.

By hypothesis we have $\alpha^2 = \alpha \cdot \gamma = 0$, thus from (1.34) we get ${\gamma_t}^2 = \gamma^2 t^2$ and and since $\gamma^2 < 0$, then $\gamma_t^2 \geqslant 0$ if and only if $t = 0$ that gives $\gamma_t = \alpha$ and the intersection is just the point α.

($\Rightarrow$) Suppose that $L \cap \overline{\text{Pos}}(S) = \{\alpha\}$; at first we claim that $\alpha^2 = 0$. Indeed, if it were $\alpha^2 > 0$, then for $t \to 0^+$ we can find γ_t such that

$$\begin{cases} \gamma_t \cdot h \geqslant 0 \\ {\gamma_t}^2 \geqslant 0. \end{cases} \tag{1.35}$$

This is true because

$$\lim_{t \to 0^+} \gamma_t \cdot h = \alpha \cdot h,$$

that is positive by Kleiman criterion and

$$\lim_{t \to 0^+} {\gamma_t}^2 = \alpha^2 > 0.$$

Using equations (1.35), we immediately see that $\gamma_t \in L \cap \overline{\text{Pos}}(S)$ and $\gamma_t \neq \alpha$ that is a contradiction; therefore we have $\alpha^2 = 0$.

Let us suppose, by contradiction, that $\alpha \cdot \gamma \neq 0$. Consider $0 < t < 1$; by definition of γ_t we see that $\gamma_t \cdot h \geqslant 0$. We claim that ${\gamma_t}^2 < 0$; if indeed ${\gamma_t}^2 \geqslant 0$, then $\gamma_t \in L \cap \overline{\text{Pos}}(S)$ and $\gamma_t = \alpha$ which is not our case.

Recalling that $\alpha^2 = 0$ we can calculate

$${\gamma_t}^2 = t((\gamma^2 - 2\alpha \cdot \gamma)t + 2\alpha \cdot \gamma) < 0; \tag{1.36}$$

since $t > 0$, then $(\gamma^2 - 2\alpha \cdot \gamma)t + 2\alpha \cdot \gamma < 0$ and passing to the limit for $t \to 0^+$, we get that $2\alpha \cdot \gamma \leqslant 0$, but since $\alpha \cdot \gamma \neq 0$ we have $\alpha \cdot \gamma < 0$.

Consider now $t \to 0^-$; as before $\gamma_t \cdot h \geqslant 0$ and ${\gamma_t}^2 < 0$; since now $t < 0$, equation (1.36) gives $\alpha \cdot \gamma > 0$, we get a contradiction and the lemma is proven.

$\square$

Remark 1.53. This Lemma essentially gives a visual way to find the orthogonal hyperplane in $N(S)$ corresponding to a class γ: if γ is outside $\overline{\text{Pos}}(S)$, $\gamma^\perp$ is simply the hyperplane passing through the intersection points of $\partial\overline{\text{Pos}}(S)$ with the tangent lines from γ. Furthermore, if we consider rays in $N(S)$ corresponding to curves with strictly negative self-intersection approaching to $\partial\overline{\text{Pos}}(S)$, we see, passing to the limit, that the orthogonal hyperplane corresponding to a ray in $R \subset \partial\overline{\text{Pos}}(S)$ is exactly the tangent hyperplane containing R. As we will see in 1.54, if $\gamma \in \partial\overline{\text{Pos}}(S)$, then $\overline{\text{Pos}}(X) \cap \gamma^\perp = R(\gamma)$.

We now give the last proof of Proposition 1.51: we get a description of $\overline{\text{Pos}}(S)$ (or its dual) in terms of hyperspaces corresponding to the rays in $\partial\overline{\text{Pos}}(S)$.

Proof of Proposition 1.51. We want to show that $\overline{\text{Pos}}(S) = \overline{\text{Pos}}(S)^\vee$; if R is a ray in $\partial\overline{\text{Pos}}(S)$, we will prove that

$$\overline{\text{Pos}}(S) = \bigcap_{R \subseteq \partial\overline{\text{Pos}}(S)} R^{\geqslant 0} = \overline{\text{Pos}}(S)^\vee. \tag{1.37}$$

Let us prove the first equality. If $\alpha \in \overline{\mathrm{Pos}}(S)$, then Fact 1.46 gives that, in particular, for any generator γ of any ray $R \subseteq \partial\overline{\mathrm{Pos}}(S)$, $\alpha\cdot\gamma \geqslant 0$ and therefore we have that $\alpha \in \bigcap R^{\geqslant 0}$. If $\alpha \in \bigcap R^{\geqslant 0}$ we have that for every generator γ of R, $\alpha\cdot\gamma \geqslant 0$. Now Fact 1.50 gives the decomposition $\alpha = x + uh$, with $x \in \partial\overline{\mathrm{Pos}}(S)$, $u \in \mathbb{R}$ and h the ample class. If $x = 0$ we have that $\alpha = uh$; if γ is a generator of R we have $\gamma\cdot\alpha \geqslant 0$ that is $uh\cdot\gamma \geqslant 0$ and since, by Kleiman's criterion, $h\cdot\gamma > 0$ we get $u \geqslant 0$, hence $\alpha \in \overline{\mathrm{Pos}}(S)$. If $x \neq 0$, x spans a ray in $\partial\overline{\mathrm{Pos}}(S)$. We can compute

$$0 \leqslant \alpha\cdot x = x^2 + uh\cdot x = uh\cdot x,$$

since by Kleiman's criterion $h\cdot x > 0$, we get, also in this situation, that $u \geqslant 0$. Now compute again:

$$\begin{aligned} \alpha\cdot h &= (x+uh)\cdot h = x\cdot h + uh^2 \geqslant 0,\\ \alpha^2 &= (x+uh)\cdot\alpha = \alpha\cdot x + u\alpha\cdot h \geqslant 0; \end{aligned}$$

and we have therefore that $\alpha \in \overline{\mathrm{Pos}}(S)$.
Let us prove the second equality in (1.37). If $\alpha \in \overline{\mathrm{Pos}}(S)^\vee$, we have that $\alpha\cdot\gamma \geqslant 0$ for all $\gamma \in \overline{\mathrm{Pos}}(S)$, in particular it is true for $\gamma \in \partial\overline{\mathrm{Pos}}(S)$ and therefore $\alpha \in \bigcap R^{\geqslant 0}$.
If $\alpha \in \bigcap R^{\geqslant 0}$, we need to show that $\alpha\cdot\gamma \geqslant 0$ for all $\gamma \in \overline{\mathrm{Pos}}(S)$; but this is true since we have seen that $\alpha \in \bigcap R^{\geqslant 0} = \overline{\mathrm{Pos}}(S)$. Then $\alpha \in \overline{\mathrm{Pos}}(S)$ and $\alpha\cdot\gamma \geqslant 0$ for all $\gamma \in \overline{\mathrm{Pos}}(S)$, again by Fact 1.46. ☐

We can now describe the behaviour of orthogonal hyperplanes corresponding to integral curves with non negative self-intersection: we have the following lemma.

Lemma 1.54. *Let S be a smooth projective surface and let $0 \neq \gamma \in \overline{\mathrm{Pos}}(S)$.*

1. *If $\gamma^2 > 0$, then $\gamma^\perp \cap \overline{\mathrm{Pos}}(S) = \{0\}$;*
2. *if $\gamma^2 = 0$, then $\gamma^\perp \cap \overline{\mathrm{Pos}}(S) = R(\gamma)$.*

Proof. In the case $\gamma^2 > 0$, we have that $\gamma \in \mathrm{Pos}(S)$; let us suppose by contradiction that there exists $0 \neq x \in \gamma^\perp \cap \overline{\mathrm{Pos}}(S)$. Then by Fact 1.46, we have $x\cdot\gamma > 0$, a contradiction, since $x \in \gamma^\perp$.
Let us suppose that $\gamma^2 = 0$. We will show the two inclusions. If $x \in R(\gamma)$, we have $x = a\gamma$, for some $a \geqslant 0$; hence we get, for an ample class h,

$$\begin{aligned} x^2 &= a^2\gamma^2 = 0 \geqslant 0\\ x\cdot h &= a\gamma\cdot h \geqslant 0\\ x\cdot\gamma &= a\gamma\cdot\gamma = 0, \end{aligned}$$

and therefore $x \in \gamma^\perp \cap \overline{\mathrm{Pos}}(S)$.
Let us now consider x such that $x^2 \geqslant 0$, $x\cdot h \geqslant 0$ and $\gamma\cdot x = 0$; we claim that if $\gamma^2 = 0$, then $x^2 = 0$. Indeed, if $x^2 > 0$, then $x \in \mathrm{Pos}(S)$ and since $\gamma \neq 0$, by Fact 1.46, $x\cdot\gamma > 0$ and it is not our case.
Now we have $x^2 = 0$, $\gamma^2 = 0$ and hence $x, \gamma \in \partial\overline{\mathrm{Pos}}(S)$. We immediately see that for the segment S joining x to γ, we have by convexity that $S \subset \overline{\mathrm{Pos}}(S)$ and furthermore, it lies in $\partial\overline{\mathrm{Pos}}(S)$; indeed we have:

$$(t\gamma + (1-t)x)^2 = 0 \quad \text{for all } t \in [0,1],$$

Let us note that, since there exists a γ with $\gamma^2 = 0$, then $\rho(S) \geqslant 2$. Now, the equations of $\partial\overline{\mathrm{Pos}}(S)$ are

$$x_1^2 = \sum_{i=2}^{\rho} x_i^2, \quad x_1 \geqslant 0,$$

hence we have that there can't be any segment supported outside one of the rays generated by the elements of $\partial\overline{\mathrm{Pos}}(S)$. Thus, since $\gamma \in S$, the segment S is contained in $R(\gamma)$ and $x \in S \subset R(\gamma)$. □

As pointed out before, it is interesting to find relationships between different cones we can define on a surface S. The following proposition shows some inclusions.

Proposition 1.55. *Let S be a smooth projective surface, then*

1. $\mathrm{Amp}(S) \subseteq \mathrm{Pos}(S) \subseteq \mathrm{Big}(S)$*;*
2. $\mathrm{Nef}(S) \subseteq \overline{\mathrm{Pos}}(S) \subseteq \overline{\mathrm{NE}}(S)$.

Proof. It is obvious to see that $\mathrm{Amp}(S) \subseteq \mathrm{Pos}(S)$, and therefore, passing to closed cones, that $\mathrm{Nef}(S) \subseteq \overline{\mathrm{Pos}}(S)$.
Furthermore, if $x \in \overline{\mathrm{Pos}}(S)$, then Fact 1.46 and inclusion $\mathrm{Nef}(S) \subseteq \overline{\mathrm{Pos}}(S)$ give that $x \cdot n \geqslant 0$ for all $n \in \mathrm{Nef}(S)$, which gives $x \in \mathrm{Nef}(S)^\vee = \overline{\mathrm{NE}}(S)$ (see [Laz04, Prop.1.4.28]). This concludes the second part of the Proposition.
Consider now $x \in \mathrm{Pos}(S)$. For $0 < \varepsilon \ll 1$ we have that $x - \varepsilon h \in \mathrm{Pos}(S)$, in fact we can compute:

$$(x - \varepsilon h)^2 = x^2 + \varepsilon^2 h^2 - 2\varepsilon x \cdot h > 0$$
$$(x - \varepsilon h) \cdot h = x \cdot h - \varepsilon h^2 > 0.$$

Therefore, using the second part of the proposition, we get $f := x - \varepsilon h \in \overline{\mathrm{NE}}(S) = \overline{\mathrm{Eff}}(S)$; then $x = \varepsilon h + f \in \mathrm{Big}(S)$ by Lemma 1.41 and this concludes the proof. □

Proposition 1.56. *If S is a smooth projective surface, we have the following decompositions.*

1. *For any $y \in \overline{\mathrm{NE}}(S)$, there exist $p \in \mathrm{Nef}(S)$ and $n \in \mathrm{Eff}(S)$ such that $y = p + n$ and $p \cdot n = 0$.*
2. *We have*

$$\overline{\mathrm{NE}}(S) = \overline{\mathrm{Pos}}(S) + \sum_{[C] \in \mathrm{Neg}(S)} R(C) = \mathrm{Nef}(S) + \sum_{[C] \in \mathrm{Neg}(S)} R(C). \tag{1.38}$$

Proof. To see the first statement, let us consider $y \in \overline{\mathrm{NE}}(S)$; if $y = [D]$, where D is a real divisor on S, using [Laz04, Theorem 2.3.19] or [Bad01, Theorem 14.14], since the proof of the cited results holds true also for $\mathbb{R}$-divisors, we get that there is a Zariski decomposition for D:

$$D = P + N, \quad P \in \mathrm{Nef}(S), N \in \mathrm{Eff}(S),$$

the matrix of components of N is definite negative and $P \cdot \Gamma = 0$ for every component Γ of N.
If we now set $p = [P], n = [N]$ we have that $y = [D] = [P] + [N] = p + n$ with $p \in \mathrm{Nef}(S), n \in \mathrm{Eff}(S)$ and $p \cdot n = 0$, that is the first part of Proposition 1.56.

We now prove the other decomposition. We can see that Proposition 1.55 immediately gives

$$\overline{\mathrm{NE}}(S) \supseteq \overline{\mathrm{Pos}}(S) + \sum_{[C]\in \mathrm{Neg}(S)} R(C) \supseteq \mathrm{Nef}(S) + \sum_{[C]\in \mathrm{Neg}(S)} R(C). \tag{1.39}$$

Viceversa if $y \in \overline{\mathrm{NE}}(S)$, the first part of the proposition gives $y = p + n$ as above. In particular, since the matrix of the components of N is negative definite, for any component Γ of N, we have $\Gamma^2 < 0$.
It follows that

$$n = [N] \in \sum_{[C]\in \mathrm{Neg}(S)} R(C),$$

and, obviously

$$y = p + n \in \mathrm{Nef}(S) + \sum_{[C]\in \mathrm{Neg}(S)} R(C).$$

This gives that

$$\overline{\mathrm{NE}}(S) \subseteq \mathrm{Nef}(S) + \sum_{[C]\in \mathrm{Neg}(S)} R(C) \subseteq \overline{\mathrm{Pos}}(S) + \sum_{[C]\in \mathrm{Neg}(S)} R(C),$$

that concludes the proof.
We can also give a second simpler proof of the decomposition:

$$\overline{\mathrm{NE}}(S) = \overline{\mathrm{Pos}}(S) + \sum_{[C]\in \mathrm{Neg}(S)} R(C).$$

Let us take $\gamma \in \overline{\mathrm{NE}}(S)$; then, by Lemma 1.22, there exist finitely many classes γ_i generating an extremal ray of $\overline{\mathrm{NE}}(S)$ such that we can write:

$$\gamma = \sum a_i \gamma_i, \quad \text{with } a_i \geqslant 0.$$

We have now the decomposition:

$$\gamma = \sum_{\gamma_j{}^2 \geqslant 0} a_j \gamma_j + \sum_{\gamma_k{}^2 < 0} a_k \gamma_k, \quad \text{with } a_j \geqslant 0, a_k \geqslant 0.$$

Now $\gamma_j \in \overline{\mathrm{Pos}}(S)$ and by convexity (see Fact 1.46) we have that $\sum_{\gamma_j{}^2\geqslant 0} a_j\gamma_j \in \overline{\mathrm{Pos}}(S)$. If ${\gamma_k}^2 < 0$, by Lemma 1.37 there exists an integral curve C_k such that $\gamma_k = [C_k]$, ${C_k}^2 < 0$ and hence $C_k \in \mathrm{Neg}(S)$.
The reverse inclusion is obvious since $\overline{\mathrm{Pos}}(S) \subset \overline{\mathrm{NE}}(S)$, C is a curve and $\overline{\mathrm{NE}}(S)$ is a convex cone. □

We spend some words on a method we can use to visualize the cone $\mathrm{Nef}(S)$ using the $\overline{\mathrm{Pos}}(S)$ and the rays generated by curves in $\mathrm{Neg}(S)$. We a have seen that $\overline{\mathrm{NE}}(S)$ is given by the convex hull of $\overline{\mathrm{Pos}}(S)$ and the rays generated by classes of curves $[C_i] \in \mathrm{Neg}(S)$; we want now to picture $\mathrm{Nef}(S)$ inside $\overline{\mathrm{Pos}}(S)$.
Let us consider a curve $C_i \in \mathrm{Neg}(S)$ and the corresponding ray $R = R(C_i)$; Lemma 1.52 gives that $R^{\perp} \cap \partial\overline{\mathrm{Pos}}(S)$ is given exactly by the tangency points of lines joining R and $\partial\overline{\mathrm{Pos}}(S)$. The nef cone has to lie in the R-non negative part of $N(S)$ that is the opposite part of C_i with respect to $R^{\perp}$; we have therefore that each so called self-negative ray cuts out a slice of $\overline{\mathrm{Pos}}(S)$. Since $\mathrm{Nef}(S) \subseteq \overline{\mathrm{Pos}}(S)$, repeating this process for all curves in $\mathrm{Neg}(S)$ we get that $\mathrm{Nef}(S)$ can be pictured as in Figure 1.1.

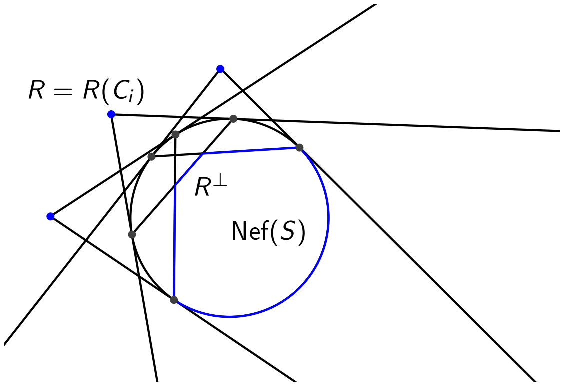

Figure 1.1: The Nef(S) and $R = R(C_i) \in \mathrm{Neg}(S)$ in the case $\rho(S) = 3$

Remark 1.57. Some words about Figure 1.1; since we deal with cones $\mathcal{C}$ of the real ρ-dimensional vector space $N(S)$, it is natural to picture the slice of $\mathcal{C}$ given by an hyperplane far from the origin. In particular, in our situation, we have fixed an orthonormal basis of $N(S)$ and we have seen that the positive cone has equations:

$$x_1 \geqslant 0, \quad x_1 \geqslant \sum_{i=2}^{\rho} {x_i}^2.$$

If we intersect $\overline{\mathrm{Pos}}(S)$ with the hyperplane $x_1 = 1$ we see that the section we get is circular. In Figure 1.1 we are obviously supposing that $\rho(S) = 3$ in order to have a 2-dimensional slice. For the sake of simplicity, we will usually draw pictures in the situation $\rho(S) = 3$.

1.5 Topological tricks

In this small section we state and prove a couple of easy lemmas of topological taste.

Lemma 1.58. *Let C, D be closed subsets of a topological space V. Then a Leibniz formula holds for the topological boundary:*

$$\partial(C \cap D) = (\partial C \cap D) \cup (C \cap \partial D). \tag{1.40}$$

Proof. Let us prove the two inclusions.

($\subseteq$) If $x \in \partial(C \cap D)$, since C, D are closed sets, we have $x \in C \cap D$. We want to show at first that $x \in \partial C$ or $x \in \partial D$; if, by contradiction, $x \notin \partial C$ neither $x \notin \partial D$, we would have that $x \in \mathrm{int}\, C$ and $x \in \mathrm{int}\, D$ and hence, there exist two neighbourhoods U_x and V_x of x such that $U_x \cap C^c = \emptyset$ and $V_x \cap D^c = \emptyset$.

Now, since $x \in \partial(C \cap D)$, we have that for every neighbourhood I_x of x, then $I_x \cap (C \cap D)^c \neq \emptyset$, that is

$$(I_x \cap C^c) \cup (I_x \cap D^c) \neq \emptyset; \tag{1.41}$$

setting $I_x = U_x \cap V_x$ we find a neighbourhood of x contradicting (1.41).

Since $x \in C \cap D$ and $x \in \partial C \cup \partial D$, we immediately get

$$x \in (\partial C \cap D) \cup (C \cap \partial D).$$

($\supseteq$) Let us now suppose that $x \in \partial C \cap D$; we have that for every neighbourhood U_x of x, $U_x \cap C \neq \emptyset$ and $U_x \cap C^c \neq \emptyset$; since $x \in D$ we have

$$U_x \cap (C \cap D) \neq \emptyset. \tag{1.42}$$

Consider now:

$$U_x \cap (C \cap D)^c = U_x \cap (C^c \cup D^c) = (U_x \cap C^c) \cup (U_x \cap D^c) \neq \emptyset, \tag{1.43}$$

since $U_x \cap C^c \neq \emptyset$.

From (1.42) and (1.43) we see that $x \in \partial(C \cap D)$ and the proof is concluded.

□

Lemma 1.59. *Let $C \subseteq T$ be a closed set of a topological space T and let $H \subset T$ be a topological subspace; then*

$$\partial_H(C \cap H) \subseteq \partial C \cap H, \tag{1.44}$$

where ∂_H denote the topological boundary in the space H.

Proof. At first we note that if $x \in H$, then an open neighbourhood $U_{x,H} \subseteq H$ of x can be obtained from an open $U_x \subset T$ such that $U_x \cap H = U_{x,H}$.
If $x \in \partial_H(C \cap H)$, then $x \in H$. Let us prove that $x \in \partial C$; by hypothesis we have that for every $U_{x,H}$:

$$\begin{aligned} \emptyset \neq & U_{x,H} \cap (C \cap H) = U_x \cap C \cap H \quad \text{and} \\ \emptyset \neq & U_{x,H} \cap (C \cap H)^c = U_x \cap H \cap (C^c \cup H^c) = U_x \cap H \cap C^c; \end{aligned}$$

in particular $U_x \cap C \neq \emptyset$ and $U_x \cap C^c \neq \emptyset$ and hence $x \in \partial C \cap H$. □

Part I

Influence of the Segre Conjecture on the Mori cone of blown-up surfaces

Chapter 2

Conjectures on linear systems

In the study of linear systems many questions remain still open; yet a number of conjectures can be stated to try to tame the behaviour of linear systems. This chapter is dedicated to some of these conjectures and the relationships among them.
In particular, in the case of surfaces, we deal with linear systems with given multiplicities at a finite number of points; these conjectures, as we will see, can be reformulated in a more Mori-theoretic flavour on the blow up of the surface at certain points.
This reformulation can be made following the spirit of equivalent conjectures of Segre, Harbourne, Gimigliano and Hirschowitz (see [Seg62], [Har86], [Gim87] and [Hir89]); we will expand this discussion in the following sections in order to get to the statement of the so called *Segre Conjecture*.

2.1 Nagata Conjecture and the plane case

In this section we focus on the $\mathbb{P}^2$ case and we stress the relationship between some classical conjectures and some interesting reformulations in terms of Mori theory. This relation has been recently sudied by several authors; we refer in particular to [dF10].
We want to consider linear systems of curves in $\mathbb{P}^2$ with assigned multiplicities at general or very general points $x_1, \dots, x_r \in \mathbb{P}^2$.
Let us recall that a point of a variety is said to be *general* if it is chosen in the complement of a closed subset and it is said to be *very general* if it is chosen in the complement of the countable union of preassigned proper closed subsets.
Nagata Conjecture (see [Nag59] or [Laz04, Remark 5.1.14]) is certainly one of the most renowned open problems in the study of planar linear system.

Conjecture 2.1 (Nagata Conjecture)**.** *Let $x_1, \dots, x_r \in \mathbb{P}^2$ be very general points; if $r \geqslant 10$, then*

$$\deg(D) \geqslant \frac{1}{\sqrt{r}} \sum_{i=1}^{r} \operatorname{mult}_{x_i}(D) \tag{2.1}$$

for every effective divisor D in $\mathbb{P}^2$.

A stronger bound is given in the following conjecture.

Conjecture 2.2 (see [dF10]). *Let $x_1, \dots, x_r \in \mathbb{P}^2$ be very general points; if $r \geqslant 10$, then*

$$\deg(D)^2 \geqslant \sum_{i=1}^{r} \mathrm{mult}_{x_i}(D)^2, \tag{2.2}$$

for every non rational integral curve D in $\mathbb{P}^2$.

Nagata Conjecture has been classically stated for the projective plane; we are interested in some generalization of this kind of statements for X, a smooth projective surface Y blown up at r general points. We want to study the relationship among the cone $\overline{\mathrm{NE}}(X)$, the positive cone and the curves with negative self-intersection.
In [dF10], the author states some conjectures with $Y = \mathbb{P}^2$.
We can now ask ourselves some conjecture-like problems: the first of them is about the positive cone $\overline{\mathrm{Pos}}(X)$ and K_X-extremal rays.

Problem 2.3. *Let Y be a smooth projective surface and consider $X = \mathrm{Bl}_r(Y)$ the blow up of Y at r very general points, then*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{Pos}}(X) + \sum R_i, \tag{2.3}$$

where the sum runs over all K_X-negative extremal rays of $\overline{\mathrm{NE}}(X)$.

The second, instead, involves curves with self-negative intersection.

Problem 2.4 ((−1)-Curves Conjecture). *Let $X = \mathrm{Bl}_r(Y)$ be the blow up of a smooth projective surface Y and let $C \subset X$ be an integral curve such that $C^2 < 0$, then C is a (-1)-curve.*

Remark 2.5. We just point out that in the case of surfaces, Mori theory gives that if X is a surface with $\rho(X) \geqslant 3$, then the extremal rays of $\overline{\mathrm{NE}}(X)$ spanned by K_X-negative curves are precisely those spanned by (-1)-curves. Indeed, since $\rho(X) \geqslant 3$, each K_X-negative ray (that can be contracted by Contraction Theorem) corresponds to a contraction of type (1) in Proposition 1.38 and it comes from a blow up at a point and therefore it is generated by a (-1)-curve. If viceversa, R is generated by a (-1)-curve C, an immediate computation using adjunction formula shows that $C \cdot K_X = -1$ and R is a K_X-negative ray.
In light of Proposition 1.38, we have that this happens either if we blow up at least 2 point, or if $Y \neq \mathbb{P}^2$ or Y is not a minimal ruled surface.

Remark 2.5 immediately gives that if either $r \geqslant 2$, or $Y \neq \mathbb{P}^2$ or Y is not minimal ruled, the decomposition in Problem 2.3 is equivalent to decomposition

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{Pos}}(X) + \sum_{C\ (-1)\text{-curve}} R(C). \tag{2.4}$$

We have the following fact.

Fact 2.6. *Problem 2.3 and Problem 2.4 are equivalent.*

Proof. Let us suppose Problem 2.3 and consider an integral curve $C \subset X$ such that $C^2 < 0$. Problem 2.3 implies that

$$[C] = \alpha + \sum_{i=1}^{s} a_i[C_i], \tag{2.5}$$

where $\alpha \in \overline{\mathrm{Pos}}(X)$, $a_i > 0$ and C_i is an irreducible curve in ${K_X}^{<0}$ spanning an extremal ray (see Fact 1.33). Since $C^2 < 0$, then by Lemma 1.37, $[C]$ spans an extremal ray R and, by extremality, we have that $\alpha \in R$ and $a_i[C_i] \in R$ for every i. So we have that $\alpha = a[C]$ for some $a \geqslant 0$ and therefore, since $0 \leqslant \alpha^2 = a^2C^2$ and $C^2 < 0$, then $a = 0$ and $\alpha = 0$. So we have $[C] = \sum a_i[C_i]$, but again by extremality we have that there exists a $b > 0$ such that $b[C] = a_1[C_1]$, and we have

$$a_1 \underbrace{C_1 \cdot K_X}_{<0} = bC \cdot K_X,$$

which gives $C \cdot K_X < 0$. Since $C^2 < 0$ and $C \cdot K_X < 0$, adjunction formula immediately gives $2p_a(C) - 2 < 0$. So we have $p_a(C) = 0$, $C^2 = -1$ and $C \cdot K_X = -1$, that is C is a (-1)-curve.
We will see two different proofs of the reverse implication.
First proof. Proposition 1.56 gives the decomposition:

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{Pos}}(X) + \sum_{[C] \in \mathrm{Neg}(X)} R(C);$$

since $[C] \in \mathrm{Neg}(X)$, then $C^2 < 0$; Problem 2.4 gives that C is a (-1)-curve and in particular, it spans a K_X-negative ray.
Second proof. Since $\overline{\mathrm{Pos}}(X) \subset \overline{\mathrm{NE}}(X)$, by convexity of $\overline{\mathrm{NE}}(X)$,

$$\overline{\mathrm{Pos}}(X) + \sum R_i \subset \overline{\mathrm{NE}}(X)$$

and we just need to prove the reverse inclusion. Consider $\gamma \in \overline{\mathrm{NE}}(X)$; we have by Lemma 1.22 that there exist finitely many $\gamma_i \in \overline{\mathrm{NE}}(X)$ such that $R(\gamma_i)$ is an extremal ray and there exist $a_i \geqslant 0$ such that

$$\gamma = \sum a_i \gamma_i.$$

We can now write

$$\gamma = \sum_{{\gamma_i}^2 \geqslant 0} a_i \gamma_i + \sum_{{\gamma_i}^2 < 0} a_i \gamma_i.$$

Now the first summand is in $\overline{\mathrm{Pos}}(X)$; since $R(\gamma_i)$ is an extremal ray with a generator such that ${\gamma_i}^2 < 0$, then by Lemma 1.37 we have that there exists an irreducible curve C_i spanning $R(\gamma_i)$ such that ${C_i}^2 < 0$. Problem 2.4 assures us that C_i is a (-1)-curve that, again by adjunction, is K_X-negative. □

Remark 2.7. We have seen that Problem 2.3 and Problem 2.4 are equivalent, but they shall immediately be false if Y contains integral curves C with $C^2 \leqslant -2$.
This fact is not so unusual and this is why we didn't use the term *Conjecture* in Problem 2.3 and 2.4.

It is interesting and useful to recall a step toward the proof of Problem 2.4 in the case of $Y = \mathbb{P}^2$, see [dF05, Proposition 2.4].

Proposition 2.8 ([dF05])**.** *If C is an integral rational curve on X with negative self-intersection, then C is a (-1)-curve.*

This proposition allows us to prove the following.

Proposition 2.9. *In the case of $Y = \mathbb{P}^2$, Problem 2.3 is equivalent to Conjecture 2.2.*

Proof. Let $D \subset \mathbb{P}^2$ be an integral non rational curve; let $\tilde{D}$ be its strict transform in $X = \mathrm{Bl}_r \mathbb{P}^2$. We have that

$$\tilde{D} \sim d\tilde{H} - \sum m_i E_i,$$

where $\tilde{H}$ is the pull back of the hyperplane $H \subset \mathbb{P}^2$ and we set $m_i = \mathrm{mult}_{x_i}(D)$. We have that $\tilde{D}^2 = d^2 - \sum m_i^2$ and we want to prove that it is non negative.
If $\tilde{D}^2 < 0$, Problem 2.4 would imply that $\tilde{D}$ is a (-1)-curve, hence rational, and we get a contradiction. Therefore we have

$$d^2 \geqslant \sum m_i^2.$$

To prove the other implication, consider a curve C such that $C^2 < 0$; if C is rational, then it is a (-1)-curve by Proposition 2.8. If C it is not rational, then C is the strict transform of a $D \subset \mathbb{P}^2$ with $C = \tilde{D}$. So in this case we have that

$$0 \leqslant \tilde{D}^2 = d^2 - \sum m_i^2 = C^2 < 0,$$

that is a contradiction, concluding the proof. □

2.2 Segre Conjecture

In the previous section we have stated some conjectures on the blow up X of a smooth projective surface Y at r very general points. In this section will study some conjectures about the Mori cone $\overline{\mathrm{NE}}(X)$ and the curves on X with negative self-intersection. In particular we do not want to limit ourselves to the projective plane, but we want to consider a smooth projective surface Y as general as possible.
The background problem we deal with, in general very difficult, is the dimension of a linear system of given degree and with fixed multiplicities in certain points.
This will be our setting: fix an ample divisor $H \subset Y$ and consider an integral curve $C \subset Y$ in the linear system $|dH|$, passing through r points $x_1, \dots, x_r$ with given multiplicities $m_i = \mathrm{mult}_{x_i}(C)$. If we denote $X = \mathrm{Bl}_r(Y)$ the blow up of Y at the r points, we see immediatly that the strict transform $\tilde{C}$ of C is in the linear system on X

$$\left| d\tilde{H} - \sum_{i=1}^{r} m_i E_i \right|, \tag{2.6}$$

where $\tilde{H}$ is the pull back of H and E_i are the exceptional divisors over x_i.
This is the situation we want to investigate:

- Y smooth projective surface over the field of complex numbers;
- $x_1, \dots, x_r$ general points on Y;
- $X = \mathrm{Bl}_r(Y) = \mathrm{Bl}_{x_1,\dots,x_r}(Y)$ the blow up of Y;
- C an integral curve on X, $|C|$ the reduced linear system associated to C and $L = \mathcal{O}_X(C)$ the associated line bundle.

In order to generalize the definition of special linear system, as we will soon see, we need to require that $h^2(X, L) = 0$. In this situation, indeed, we can give the following definition.

Definition 2.10. Let L be a line bundle on a smooth projective surface X with $h^2(X, L) = 0$; we call *virtual dimension* of the linear system $\mathcal{L}$ associated L the number

$$v(\mathcal{L}) = \chi(L) - 1,$$

and *expected dimension* the number

$$e(\mathcal{L}) = \max\{v(\mathcal{L}), -1\}.$$

Remark 2.11. Usual definitions of virtual dimension are consistent with our definition in the case of $K3$ surfaces (see [DVL05]).
Indeed, let Y denote a generic $K3$ surface, that is a $K3$ such that $\operatorname{Pic} Y = \langle H \rangle$ with $n = H^2$. Consider r points $x_1, \dots, x_r$ in general position on Y, and denote m_i the multiplicity at point x_i. Let $\mathcal{L} = \mathcal{L}^n(d; m_1, \dots, m_r)$ be the linear system of curves in $|dH|$ passing through the r points with the given multiplicities.
Now we denote with $\tilde{\mathcal{L}}$ the corresponding linear system in the blow up $X = \mathrm{Bl}_r(Y)$ and with L the associated line bundle on X.
In this situation the *virtual dimension* of the linear system $\mathcal{L}$ is classically defined by

$$v(\mathcal{L}) = \frac{d^2 n}{2} + 1 - \sum_i \frac{m_i(m_i+1)}{2}. \tag{2.7}$$

The following holds:

$$v(\mathcal{L}) = \chi(L) - 1. \tag{2.8}$$

Indeed, by Riemann-Roch Theorem, we have

$$\chi(L) - 1 = \chi(\mathcal{O}_Y) + \frac{1}{2} L \cdot (L - K_X) - 1;$$

Now since intersection does not change, if we consider an integral curve $C \in \mathcal{L}$, we get

$$\chi(L) - 1 = 1 - q + p_g(Y) + \frac{1}{2}C^2 - \frac{1}{2}C \cdot K_X - 1, \tag{2.9}$$

Now, since Y is a $K3$ surface, we have $q = 0$, $p_g(Y) = 1$ and $K_X \equiv \sum E_i$; the (2.9) becomes

$$\begin{aligned} \chi(L) - 1 &= 1 + \frac{1}{2}\left(nd^2 - \sum m_i^2\right) - \frac{1}{2}\left(\sum m_i\right) \\ &= \frac{d^2 n}{2} + 1 - \sum_i \frac{m_i(m_i+1)}{2} = v(\mathcal{L}). \end{aligned}$$

We now have the following interesting result.

Proposition 2.12. *Let L be a line bundle on a smooth projective surface X with associated linear system $\mathcal{L}$, such that $h^2(X, L) = 0$, then*

1. $\dim(\mathcal{L}) \geqslant e(\mathcal{L})$*;*

2. $\dim(\mathcal{L}) = e(\mathcal{L})$ *if and only if* $h^0(L)h^1(L) = 0$.

Proof. To prove the first fact, recall that we have $e(\mathcal{L}) = \max\{v(\mathcal{L}), -1\}$. In case $e(\mathcal{L}) = v(\mathcal{L})$, we have in particular that $e(\mathcal{L}) = \chi(L) - 1$ and we can calculate

$$\begin{aligned} \dim(\mathcal{L}) &= h^0(X, L) - 1 \\ &= \chi(L) - 1 + h^1(X, L) - \underbrace{h^2(X, L)}_{=0} \\ &\geqslant \chi(X, L) - 1 = e(\mathcal{L}). \end{aligned} \tag{2.10}$$

In the case $e(\mathcal{L}) = -1$ we immediately see that

$$\dim(\mathcal{L}) = h^0(X, L) - 1 \geqslant -1 = e(\mathcal{L}).$$

We want now to prove the second statement.

($\Leftarrow$) In the case $h^0(L) = 0$, we have that $\dim(\mathcal{L}) = h^0(L) - 1 = -1$ and we claim that also $e(\mathcal{L}) = -1$; since $e(\mathcal{L}) = \max\{\chi(L) - 1, -1\}$, if it was $e(\mathcal{L}) = \chi(L) - 1$, we would have that

$$\chi(L) - 1 = h^0(L) - h^1(L) + h^2(L) - 1 = -h^1(L) - 1 \geqslant -1,$$

hence $h^1(L) = 0$ and therefore $e(\mathcal{L}) = -1$.
In the case $h^1(L) = 0$ we get

$$-1 \leqslant \dim(\mathcal{L}) = h^0(L) - 1 = \chi(L) + h^1(L) - h^2(L) - 1 = \chi(L) - 1,$$

which gives $\chi(L) - 1 \geqslant -1$ and therefore $e(\mathcal{L}) = \chi(L) - 1 = \dim(\mathcal{L})$.

($\Rightarrow$) In the case $e(\mathcal{L}) = -1$ we have that $-1 = e(\mathcal{L}) = \dim(\mathcal{L}) = h^0(L) - 1$, which gives $h^0(L) = 0$ and in particular $h^0(L)h^1(L) = 0$. In the case $e(\mathcal{L}) = \chi(L) - 1$, we have $\chi(L) - 1 = e(\mathcal{L}) = \dim(\mathcal{L}) = h^0(L) - 1 = \chi(L) + h^1(L) - 1$; then $h^1(L) = 0$ and in particular $h^0(L)h^1(L) = 0$.

□

Definition 2.13. Let L be the line bundle associated to a linear system $\mathcal{L}$ on X with $h^2(X, L) = 0$, then we say that:

1. L is *special* (equivalently, $\mathcal{L}$ is special) if $\dim(\mathcal{L}) > e(\mathcal{L})$;
2. L is *non special* (equivalently, $\mathcal{L}$ is non special) if $\dim(\mathcal{L}) = e(\mathcal{L})$.

Question 2.14. Since we want to distinguish special and non special linear systems, we gave all these definitions in the case $h^2(L) = 0$; we would like to find out what kind of surfaces realize this condition. Let L be a line bundle on $X = \mathrm{Bl}_r Y$, $L \neq \mathcal{O}_X$ with associated linear system $\mathcal{L} \neq \emptyset$, when does $h^2(L) = 0$?

Remark 2.15. As will be clearer in the following, we want now to turn our gaze to surfaces Y with $p_g(Y) = 0$ or $K_Y \equiv 0$. It interesting to point out that these two cases cover a number of interesting surfaces; indeed, if $p_g(Y) = 0$ we get surfaces as the projective plane, Enriques and bielliptic surfaces and a number of surfaces of general type; if else $K_Y \equiv 0$ and $p_g(Y) \neq 0$, we have a fortiori that $K_Y \sim 0$ and hence Y has to be an Abelian or a $K3$ surface. We will discuss many examples in Section 2.4.
In the following the surface Y will be one of the following:

- a surface with $p_g(Y) = 0$;
- an abelian surface;
- a $K3$ surface.

In the spirit of generalizing the Segre Conjecture, we will be mainly interested in non exceptional linear systems.

Definition 2.16. We say that a linear system $\mathcal{L}$ on a $X = \mathrm{Bl}_r Y$ is *non exceptional* if there is a divisor in $\mathcal{L}$ such that its support is not contained in the exceptional locus of X.

The following Lemma gives an answer to Question 2.14.

Lemma 2.17. *Let Y be a smooth surface with either $p_g(Y) = 0$ or a $K3$ or an abelian surface; let us consider a line bundle L on $X = \mathrm{Bl}_r(Y)$ with associated linear system $\mathcal{L} \neq \emptyset$. We have the following.*

1. *If $\mathcal{L}$ is not exceptional, then $h^2(X, L) = 0$.*
2. *In the exceptional case, we have:*

$$h^2(X, L) = \begin{cases} 0 & \text{if } p_g(Y) = 0 \\ 0 & \text{if } Y \text{ abelian or } K3 \text{ surface and } \mathcal{L} \text{ non reduced} \\ 1 & \text{if } Y \text{ abelian or } K3 \text{ surface and } \mathcal{L} \text{ reduced.} \end{cases}$$

3. *If $\mathcal{L}$ is exceptional and reduced, then $h^2(X, L) = p_g(Y)$ and $h^2(X, L) = 1$ if Y is either a $K3$ or an abelian surface.*

Proof. In our situation we have that the line bundle L is associated to an effective divisor on X and we can write $L = \mathcal{O}_X(D)$, with

$$D = F + \sum a_i E_i, \quad a_i \in \mathbb{Z}, a_i \geqslant 0,$$

where $F \geqslant 0$ is a divisor without exceptional components.
By duality, we immediately get

$$\begin{aligned} h^2(X, L) = h^2(X, \mathcal{O}_X(D)) &= h^0(\mathcal{O}_X(K_X - D)) = \\ &= h^0(\mathcal{O}_X(\varphi^* K_Y + \sum E_i - F - \sum a_i E_i)). \end{aligned} \tag{2.11}$$

We can make a reduction to the reduced case: indeed we see at once that

$$M' := \varphi^* K_Y + \sum E_i - F - \sum a_i E_i \leqslant \varphi^* K_Y + \sum E_i - F - \sum_{a_i \leqslant 1} a_i E_i =: M.$$

Now, since $a_i \geqslant 0$, we have that

$$M = \varphi^* K_Y - F + \sum_{j \in J} E_j, \quad \text{for some } J = \{j_1, \dots, j_k\} \subseteq \{1, \dots, r\}.$$

Thus, since $h^0(\mathcal{O}_X(M')) \leqslant h^0(\mathcal{O}_X(M))$, if we show that $h^0(\mathcal{O}_X(M)) = 0$, then we have the vanishing also in the non reduced case.
We can use the following easy fact.

Claim 2.18. Let E be an exceptional prime divisor and D a divisor; if $D \cdot E \leqslant 0$, then

$$h^0(\mathcal{O}_X(D+E)) = h^0(\mathcal{O}_X(D)).$$

Proof of the Claim. From the exact sequence

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D+E) \to \mathcal{O}_E((D+E)|_E) \to 0,$$

we get, since $E \simeq \mathbb{P}^1$ and $D \cdot E \leqslant 0$,

$$h^0(\mathcal{O}_E((D+E)|_E)) = h^0(\mathcal{O}_{\mathbb{P}^1}(\underbrace{D \cdot E - 1}_{\leqslant 0})) = 0.$$

□

Thanks to Claim 2.18 we will be able to drop, one by one, all the exceptional components $E_j, j \in J$ in M. Thus we want to use Claim 2.18 with

$$D = \varphi^* K_Y - F - E_{j_1} - \cdots - E_{j_i} \quad \text{and} \quad E = E_{j_{i+1}}$$

for $0 \leqslant i \leqslant k-1$. Thus we get

$$(\varphi^* K_Y - F - E_{j_1} - \cdots - E_{j_i}) \cdot E_{j_{i+1}} = \varphi^* K_Y \cdot E_{j_{i+1}} - F \cdot E_{j_{i+1}} - 0 \leqslant 0,$$

since $\varphi^* K_Y \cdot E_{j_{i+1}} k = 0$ and $F \cdot E_{j_{i+1}} \geqslant 0$ because F has no exceptional components.
We can therefore apply the Claim 2.18 for all $i = 0, \ldots, k-1$ and we finally get

$$\begin{aligned} h^0(\mathcal{O}_X(M)) &= h^0(\mathcal{O}_X(\varphi^* K_Y - F)) \\ &= h^0(\varphi_* \mathcal{O}_X(\varphi^* K_Y - F)) \\ &= h^0(\mathcal{O}_Y(K_Y) \otimes \varphi_* \mathcal{O}_X(-F)). \end{aligned} \tag{2.12}$$

Let us now suppose $\mathcal{L}$ exceptional: in this situation $F = 0$.
We see that $h^0(\mathcal{O}_X(M)) = p_g(Y)$ and hence if $p_g(Y) = 0$ we have $h^2(X, L) = 0$.
If otherwise Y is either a $K3$ or an abelian surface, since $K_Y \sim 0$, we get $h^0(\mathcal{O}_X(M)) = 1$ and hence $h^2(X, L) \leqslant 1$. In this situation we have

$$h^2(L) = h^0(\mathcal{O}_X(\sum E_i - \sum a_i E_i));$$

if L is reduced, then $a_i = 0, 1$ and by Claim 2.18 we get $h^2(L) = h^0(\mathcal{O}_X) = 1$.
If L is not reduced, then there exist some $a_j \geqslant 2$; hence, again by Claim 2.18, we get

$$\begin{aligned} h^2(L) &= h^0(\mathcal{O}_X(\sum E_i - \sum_{i: a_i = 1} E_i - \sum_{j: a_j \geqslant 2} a_j E_j)) \\ &= h^0(\mathcal{O}_X(\sum_{i \in I \subseteq \{1, \ldots, r\}} E_i - \sum_{j: a_j \geqslant 2} a_j E_j)) = \\ &= h^0(\mathcal{O}_X(-\sum_{j: a_j \geqslant 2} a_j E_j)) = 0. \end{aligned}$$

In the non exceptional case we have $F \neq 0$. Let us consider at first the case $p_g(Y) = 0$.
We have the exact sequence

$$0 \to \mathcal{O}_Y(K_Y) \otimes \varphi_* \mathcal{O}_X(-F) \to \mathcal{O}_Y(K_Y),$$

which, since $h^0(\mathcal{O}_Y(K_Y)) = 0$, gives $h^0(\mathcal{O}_Y(K_Y)\otimes\varphi_*\mathcal{O}_X(-F)) = 0$ and hence $h^2(X, L) = 0$.
In the case of abelian or $K3$ surface, we have $K_Y \sim 0$ and hence from the (2.12), since $F \neq 0$ is effective, we get

$$h^0(\varphi_*\mathcal{O}_X(-F)) = h^0(\mathcal{O}_X(-F)) = 0,$$

concluding the proof. □

Remark 2.19. In light of Question 2.14, we want to consider line bundles L with $h^2(X, L) = 0$. Now, Lemma 2.17 tells us that, in the case of surfaces with geometric genus 0, $K3$ or abelian, this always happens if L is a line bundles associated to non exceptional linear systems $\mathcal{L} \neq \emptyset$.
The exceptional case, since the dimension of the linear system is obviously zero, is not interesting: henceforth, we will focus on non exceptional linear system.

The list of ingredients to state the Segre Conjecture is now ready and we are getting closer to the goal of this section.
The original Segre Conjecture[1], stated in the setting of planar linear system, can be easily stated for any surface; in [DVL05], the authors state the Segre Conjecture for a generic $K3$ surface.

Conjecture 2.20 (see [DVL05])**.** *Let Y be a generic $K3$ surface and let $\mathcal{L}$ be a non empty and reduced linear system on Y, then $\mathcal{L}$ is non special.*

In the following, we are interested in the generalization of this conjecture to other surfaces.
As pointed out at the beginning of this section, the study of linear systems of divisors on Y with multiplicities at certain points can be easily translated in the study of linear systems on the blown up surface X.
We have seen that, in order to ensure that the definition of special linear system makes sense, we have to ask $h^2(X, L) = 0$ and Lemma 2.17 provides an answer.

Problem 2.21 (Segre Problem)**.** *Mimicking Conjecure 2.20, we can ask ourselves a sort of* Segre Problem *for all surfaces as in Remark 2.15.*

($\star$) Let $X = \mathrm{Bl}_r Y$ a blown up surface at r general points; let us suppose $h^2(X, L) = 0$ for all line bundles L associated to a non exceptional and non empty linear system $\mathcal{L}$. If moreover $\mathcal{L}$ is reduced, then $\mathcal{L}$ is non special.

Remark 2.22. We called *Problem* the statement ($\star$) because we will soon see in Section 2.4 that this can't be true for a number of surfaces.
To this end it may be worth to recall that an abelian variety is called *simple* if it does not contain any non trivial abelian subvarieties. In the non simple case, we have, moreover, that if $C \subset Y$ is an elliptic curve contained in an abelian surface, then Y does contain another curve C' such that Y is isogenous to $C \times C'$ (see [BL04, Poincaré's complete reducibility theorem]).

[1] From [Seg62]:

> [...]*Questi ed altri esempi consimili portano a fare ritenere probabile che*
>
> *Affinché un sistema lineare completo Σ di curve piane, dotato di un numero finito di punti base assegnati in posizione generica ed avente dimensione virtuale $d-1$, sia sovrabbondante (e quindi effettivo, cioè di dimensione $\delta \geqslant 0$) è necessario (ma, come risulta da esempi, non sufficiente) ch'esso possegga qualche componente fissa multipla.*

We will soon see in Section 2.4 that a statement like ($\star$) can't be true for non simple abelian surfaces and some remarkable cases of surfaces Y with $p_g(Y) = 0$, like Enriques or bielliptic surfaces.

We are now ready to state our formulation of Segre Conjecture.

Conjecture 2.23 (Segre Conjecture)**.** *Let Y be either a $K3$ surface or a simple abelian surface or a surface with $p_g(Y) = 0$ and let $\varphi : X \to Y$ be the blow up at $x_1, \dots, x_r$, general points of Y.*
If $\mathcal{L}$ is a non exceptional, non empty and reduced linear system on X, then $\mathcal{L}$ is non special.

2.3 List Conjecture

We want now to interpret Conjecture 2.23 in terms of integral curves on the surface X in order to get some informations about $\overline{\mathrm{NE}}(X)$.
In particular we are interested in curves with negative self-intersection (see Definition 1.40) and we would like to locate the rays generated by these curves.
What can we say about curves with negative self-intersection? There are some important conjectures about these topics. In particular, we recall the following conjecture, stated by its author in [Har10, Section 1].

Definition 2.24. We say that a smooth surface S has *bounded negativity* if there exists an integer ν_S such that $C^2 \geqslant -\nu_S$ for each integral curve $C \subset S$.

Conjecture 2.25 (Bounded Negativity Conjecture)**.** *Every smooth surface S in characteristic 0 has bounded negativity.*

Remark 2.26. It is known that the Bounded Negativity Conjecture is false in positive characteristic: see, for example, [Har10, Remark I.2.2]; it may be worth to point out that recent attempts (see [BHK$^+$11]) by several authors to produce counterexamples in characteristic 0 have not been successful: the bounded negativity conjecture remains still open.

Nevertheless the conjecture holds true for a meaningful class of surfaces.

Fact 2.27. *Bounded Negativity Conjecture holds true for smooth projective surfaces S with $-K_S$ pseudoeffective. In particular it holds for $K3$ surfaces, Enriques surfaces and abelian surfaces.*

Proof. Our goal is the existence of a positive integer ν such that $C^2 \geqslant -\nu$ for all integral curve $C \subset S$.
Let us consider an integral curve C with $C^2 < 0$. By adjunction we have

$$C^2 = 2p_a(C) - 2 - K_S \cdot C;$$

hence if $-K_S \cdot C \geqslant 0$ we have $C^2 \geqslant -2$. If else $-K_S \cdot C < 0$, let us consider the Zariski decomposition of the pseudoeffective divisor $-K_S$:

$$-K_S = P + N,$$

with $P \in \mathrm{Nef}(S)$ and $N = \sum_{i=1}^{s} a_i E_i$ an effective $\mathbb{Q}$-divisor. We claim that there are just finitely many integral curves such that $-K_S \cdot C < 0$, indeed we have

$$-K_S \cdot C = P \cdot C + N \cdot C < 0,$$

and since we must have $(\sum a_i E_i) \cdot C < 0$, then C has to be one of the $E_1, \dots, E_s$.
Hence we finally get

$$-\nu = \min\{-2, {E_1}^2, \dots, {E_s}^2\}. \tag{2.13}$$

□

We have seen that negativity holds for certain surfaces, nevertheless it is not clear what happens blowing up some points on those surfaces. This is the case we will focus on.
It seems interesting to state the following conjecture, a natural generalization of the (-1)-Curve Conjecture (see Problem 2.4).

Conjecture 2.28 (List Conjecture)**.** *Let $C \subset X = \mathrm{Bl}_r Y$ be a non exceptional integral curve such that $C^2 < 0$, then there exist a positive number $\nu = \nu_X$ and a non negative integer $\pi = \pi_X$ such that C is a $(-n, p)$-curve for some $1 \leqslant n \leqslant \nu$ and $0 \leqslant p \leqslant \pi$ (that is there is a list of possible $(-n, p)$-curves).*

In the spirit of Fact 2.27, we have a similar result for the arithmetic genus of integral curves with negative self-intersection.

Fact 2.29. *Let S be a smooth projective surface with $-K_S$ pseudoeffective. Then there exists an integer $\pi \geqslant 0$ such that for every integral curve C with $C^2 < 0$, we have $p_a(C) \leqslant \pi$, that is, the arithmetic genus is bounded from above.*

Proof. Let us consider an integral curve C with $C^2 < 0$. By adjunction, we immediately get

$$2p_a(C) = C^2 + C \cdot K_S + 2 < C \cdot K_S + 2,$$

and hence

$$p_a(C) < \frac{1}{2} C \cdot K_S + 1.$$

Therefore for any curve C with $C \cdot K_S \leqslant 0$ we get $p_a(C) < 1$, that is $p_a(C) = 0$. If $C \cdot K_S > 0$ that is $-K_S \cdot C < 0$, we have, as in the proof of Fact 2.27, that C must be one of the components of the effective part $N = \sum_{i=1}^{s} a_i E_i$ in the Zariski decomposition of the anticanonical divisor $-K_S$.
Summarizing, we have that if C is an integral curve with negative self-intersection, then $p_a(C) \leqslant \pi$ for

$$\pi = \max\{0, p_a(E_1), \dots, p_a(E_s)\}.$$

□

Immediately we get the following.

Proposition 2.30. *Let $X = \mathrm{Bl}_r Y$ be a smooth projective surface with $-K_X$ pseudoeffective, then the List Conjecture (Conjecture 2.28) holds true.*

Proof. See Fact 2.27 and Fact 2.29. □

Remark 2.31. In the proof of Fact 2.27 and of Fact 2.29, we used, as essential ingredient, the Zariski decoposition. It is worth to point out that we did not use the whole power of the Zariski decomposition: we just used the existence of the so called *weak Zariski decomposition*:

$$-K_S = P + N, \quad \text{with} \quad P \in \mathrm{Nef}(S), N \in \mathrm{Eff}(S). \tag{2.14}$$

The weak Zariski decompostion will be one of the main topic of the second part of this thesis; we refer in particular to the beginning of Chapter 4 for further details.

We can now see that it is possible, starting from a surface Y with $-K_Y$ such that the decomposition (2.14) holds (equivalently if $-K_Y$ is pseudoeffective), to produce a blown up surface $X = \mathrm{Bl}_r Y$ satisfying the List conjecture.
Let us take Y and consider its blow up $\varphi : X \to Y$ at r general points $x_1, \dots, x_r$. If $-K_Y = P + N$ as in the (2.14), since by generality the components of N do not pass though the blown up points, we get

$$-K_X = \varphi^* P - \sum_{i=1}^{r} E_i + \varphi^* N.$$

Remark 2.32. In view of Remark 2.31 and Proposition 2.30, to verify the List Conjecture, it is enough to show that

$$\varphi^* P - \sum_{i=1}^{r} E_i \in \mathrm{Nef}(X). \tag{2.15}$$

Thus the List Conjecture holds true if $\varphi^* P$ is sufficiently positive and the number of blown-up points is small.
This seems to be a very interesting fact in the spirit of our main result (see Theorem 3.13): if we are able to find a smooth projective surface Y with a weak Zariski decomposition for $-K_Y$ with $\varphi^* P$ sufficiently positive with respect to r, then our main result, independently from any conjecture, is true on $X = \mathrm{Bl}_r Y$.

Remark 2.33. In the particular case of $Y = \mathbb{P}^2$, if L is the class of a line, the decomposition is $-K_Y = 3L$ and hence, on $X = \mathrm{Bl}_r Y$, we get $-K_X = 3\varphi^* L - \sum_{i=1}^{r} E_i$. As well known, this is nef for $r = 1, \dots, 9$. In this situation Fact 2.27 and Fact 2.29 give for the List Conjecture the bounds: $\nu = 2$ and $\pi = 0$.

Remark 2.34. In the following we will be interested in the case the bounds ν and π depend only on the surface Y; in principle, indeed, ν and π may depend on the number of points we want to blow up.

We can now see how Conjecture 2.23 implies Conjecture 2.28 and allows us to find explicit bounds on the negativity and on the arithmetic genus depending only on the blown up surface Y.

Proposition 2.35. *Let $C \subset X = \mathrm{Bl}_r Y$ be a non exceptional integral curve on a smooth blown up surface X, such that $C^2 < 0$; let us suppose Conjecture 2.23 (or, more generally, the Segre Problem).*

1. *It holds*
$$-1 \geqslant C^2 \geqslant p_a(C) - \chi(\mathcal{O}_Y) \geqslant -\chi(\mathcal{O}_Y), \tag{2.16}$$
and in particular $\chi(\mathcal{O}_Y) \geqslant 1$;

2. *Conjecture 2.28 holds true with*

$$\nu = \chi(\mathcal{O}_Y); \qquad \pi = \chi(\mathcal{O}_Y) - 1. \tag{2.17}$$

Proof. Let us consider the linear system $|C|$, associated to $C \subset X$, non exceptional, integral curve such that $C^2 < 0$; Conjecture 2.23 implies that the system is non special, and since it is non empty, we get that

$$\chi(\mathcal{O}_X(C)) \geqslant 1.$$

Riemann-Roch theorem gives

$$\chi(\mathcal{O}_X) + \frac{1}{2}C^2 - \frac{1}{2}C \cdot K_X = \chi(\mathcal{O}_Y) + \frac{1}{2}C^2 - \frac{1}{2}C \cdot K_X \geqslant 1, \tag{2.18}$$

since $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y)$ is a birational invariant. Set $\chi = \chi(\mathcal{O}_X)$ and $p = p_a(C)$. We get

$$C^2 - C \cdot K_X \geqslant 2 - 2\chi.$$

Adjunction formula gives $C \cdot K_X = 2p - 2 - C^2$ and so we have

$$C^2 - 2p + 2 + C^2 \geqslant 2 - 2\chi \quad \Rightarrow \quad C^2 - p \geqslant -\chi.$$

Recalling that $C^2 \leqslant -1$ we get

$$-1 \geqslant C^2 \geqslant p - \chi \geqslant -\chi, \tag{2.19}$$

that is the (2.16). This condition allows us to find immediatly the bounds

$$\begin{cases} C^2 \geqslant -\chi(\mathcal{O}_Y) \\ p_a(C) \leqslant \chi(\mathcal{O}_Y) - 1. \end{cases} \tag{2.20}$$

□

2.4 Special cases of the Segre Problem

In this section we will study the behaviour of the Segre Problem (see Remark 2.21) in some special cases; in particular, using elliptic fibrations, we will easily show that the Segre Problem must have a negative answer if the blown-up surface Y is abelian, Enriques or bielliptic.
At first, in the case of $\chi(\mathcal{O}_Y) \leqslant 0$, we have the following fact.

Fact 2.36. *Let Y be a smooth projective surface with $\chi(\mathcal{O}_Y) \leqslant 0$ and either $p_g(Y) = 0$ or Y is an abelian surface; suppose that Conjecture 2.23 holds true for X, the blow up of Y at r general points. If an integral curve $C \subset X$ is such that $C^2 < 0$, then C is exceptional.*

Proof. From the (2.16), we see in particular that

$$-1 \geqslant C^2 \geqslant -\chi(\mathcal{O}_Y).$$

If $\chi(\mathcal{O}_Y) \leqslant 0$, Conjecture 2.23 implies that there can't be non exceptional curves with negative self-intersection. □

We now focus on some special cases.

Projective Plane

Let $Y = \mathbb{P}^2$ be the projective plane; since $\chi(\mathcal{O}_{\mathbb{P}^2}) = 1$, in view of (2.17), Conjecture 2.28 is expected to hold with $\nu = 1$ and $\pi = 0$.
Therefore we have that if C is an irreducible and reduced non exceptional curve such that $C^2 < 0$, than C is a (-1)-curve; since exceptional curves are (-1)-curves, Conjecture 2.28 says that on the blow up of the plane at r general points, the only integral curves with negative self-intersection are (-1)-curves.
Hence we recover the so-called (-1)-Curve Conjecture 2.4 (see [dF10, Conjecture 1.1]).

General type

Fact 2.36 shows that in view of Segre Problem, the interesting cases are essentially given by surfaces Y with $\chi(\mathcal{O}_Y) \geqslant 1$ and either $p_g(Y) = 0$ or $K3$ or simple abelian surfaces. This situation covers also the interesting class of surfaces of general type with $p_g(Y) = 0$; this class is not empty in view of an important theorem from Castelnuovo (see [Bea96, Theorem X.4]).

Theorem 2.37 (Castelnuovo)**.** *Let Y be a non-ruled surface; then $\chi(\mathcal{O}_Y) \geqslant 0$. Moreover, if Y is of general type, then $\chi(\mathcal{O}_Y) > 0$.*

These surfaces of general type could be an interesting class to study.

Surfaces with a fibration and easy counterexamples

We want now to see some easy counterexamples of the Segre Problem. We have the following result.

Fact 2.38. *Let Y be a surface with a base point free pencil $\mathcal{V}$ of curves of arithmetic genus g; if for the strict transform $\tilde{C} \subset X = \mathrm{Bl}_r\, Y$ of a general curve in the pencil we have*

$$\chi(\mathcal{O}_Y) \neq \dim|\tilde{C}| + g + 1, \tag{2.21}$$

then the Segre problem has a negative answer for $X = \mathrm{Bl}_r\, Y$.
In particular, in the $p_g(Y) = 0$ case, this holds true if $g > 0$ or $q > 0$, where g is the genus of the curves in the pencil and q is the irregularity of Y.

Proof. The base point free pencil $\mathcal{V}$ determines a morphism $\psi_{\mathcal{V}} : Y \to \mathbb{P}^1$, whose fibres are exactly the genus g curves of $\mathcal{V}$. Thus for $C \in \mathcal{V}$, we immediately get: $C^2 = 0$.
Let us focus on $X = \mathrm{Bl}_r\, Y$ and let us consider the general points $x_1, \dots, x_r$; since we have a fibration, there exists a fibre $C = F_1$ passing through x_1. By generality and by Bertini theorem, we can suppose that C is a smooth curve and therefore

$$\begin{aligned} m_1 &= \mathrm{mult}_{x_1}(C) = 1 \\ m_i &= \mathrm{mult}_{x_i}(C) = 0 \quad \text{for } i = 2, \dots, r. \end{aligned}$$

Moreover, we can also see that the curve C is irreducible and hence integral; thus its strict transform $\tilde{C} = \varphi^* C - E_1$ is an integral, smooth and non exceptional curve with $\tilde{C}^2 = -1$.
Let us suppose, by contradiction, that the Segre problem holds true. For $L = \mathcal{O}_X(\tilde{C})$, Segre problem gives

$$\dim(|L|) = \max\{\chi(L) - 1, -1\};$$

since $|L| \neq \emptyset$, then

$$\dim(|L|) = \chi(L) - 1. \tag{2.22}$$

By Riemann-Roch theorem and adjunction, we get

$$\begin{aligned}\chi(L) &= \chi(\mathcal{O}_X) + \frac{1}{2}(\tilde{C}^2 - \tilde{C} \cdot K_X) \\ &= \chi(\mathcal{O}_Y) + \tilde{C}^2 - g + 1.\end{aligned}$$

Now, by equation (2.22), we get

$$\chi(\mathcal{O}_Y) = \dim(|L|) + g + 1, \tag{2.23}$$

a contradiction with our ad hoc hypothesis.
The former equation, since $\dim(|L|) \geqslant 0$, gives the bound $\chi(\mathcal{O}_Y) \geqslant g + 1$.
In the $p_g(Y) = 0$ case, since $\chi(\mathcal{O}_Y) = 1 - q$, this becomes $g + q \leqslant 0$ and hence

$$q = g = 0.$$

Thus, whenever $g > 0$ or $q > 0$, the Segre problem has a negative answer. □

Let us recall that a surface Y has an *elliptic fibration* (see [BPV84]) if there exists a proper connected morphism $Y \to C$ to an algebraic curve C such that the general fibre is a smooth elliptic curve.
In the light of this, it is immediate to state the following fact.

Fact 2.39. *Let Y be either an Enriques or a bielliptic surface, then the Segre Problem for $X = \mathrm{Bl}_r\, Y$ has a negative answer.*

Proof. It is well known that if Y is an Enriques or a bielliptic surface, then Y has an elliptic fibration (see [BPV84]). Let us take the elliptic fibre $C = F$ passing through the first blowing-up point x_1. We immediately see that $C^2 = 0$; by generality of the points, we can suppose that:

$$m_1 = 1; \quad m_i = 0, \text{ for } i = 2, \dots, r.$$

If we denote by $\tilde{C}$ its strict transform, we get

$$\tilde{C} = \varphi^* C - E_1,$$

which gives $\tilde{C}^2 = -1$. Now, since $\tilde{C}$ is a non exceptional curve with negative self-intersection, if Segre Problem had a positive answer, Proposition 2.35 would give

$$-1 \geqslant \tilde{C}^2 \geqslant 1 - \chi(\mathcal{O}_Y),$$

hence $\chi(\mathcal{O}_Y) \geqslant 2$, that is a contradiction. □

Fact 2.40. *Let Y be a not simple abelian surface, then the Segre Problem for $X = \mathrm{Bl}_r\, Y$ has a negative answer.*

Proof. It is enough to prove that there is an elliptic curve passing through a general point of Y, but since Y is abelian, if there is such a curve through $x \in Y$, then, by translation, there is a curve through every point.

Now Y is not simple, hence it is isogenous to $C \times C'$ and we can suppose that the first general point to blow up x_1 lies on the smooth elliptic curve $C \subset Y$.
By adjunction formula, since $K_Y \equiv 0$, we get $C^2 = 0$ and since C is smooth, by generality, we have

$$m_1 = 1; \quad m_i = 0, \text{ for } i = 2, \ldots, r.$$

Thus, mimicking the proof of Fact 2.39, we get the required contradiction with the Segre Problem. $\square$

Chapter 3

The shape of the Mori cone

This chapter is dedicated to prove the main result (see Theorem 3.13) of the first part of the thesis. We will see as the Segre Conjecture implies that a slice of the Mori cone of the blow-up surface has to coincide with the positive cone.

3.1 Negative part

In this section we want to study the decomposition of $\mathrm{Neg}(X)$, the set of negative self-intersection curves (see Definition 1.43), of a smooth projective surface X. In particular, this decomposition allows us to study the structure of the Mori cone $\overline{\mathrm{NE}}(X)$.
As pointed out before, we are interested in integral curves C with negative self-intersection, negativity bounded from below and arithmetic genus bounded from above (see Conjecture 2.28 and Remark 2.34).
We have the following fact.

Fact 3.1. *Let X be a smooth projective surface and let L' be a finite, subset*

$$L' \subset ((-\infty, -1] \cap \mathbb{Z}) \times ([0, +\infty) \cap \mathbb{Z})\,;$$

we say that the integral curve $C \subset X$ is in the list L' if $(C^2, p_a(C)) \in L'$ and we denote

$$L = \left\{[C] \mid (C^2, p_a(C)) \in L'\right\} \subset N(X).$$

Then the following are equivalent:

1. *for all integral curve $C \subset X$ such that $C^2 < 0$ we have that $[C] \in L$;*

2. *we have the decomposition*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{Pos}}(X) + \sum_{[C] \in L} R(C). \tag{3.1}$$

Proof. To show that the first implies the second, it is enough to recall from Proposition 1.56 that we have the decomposition

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{Pos}}(X) + \sum_{[C]\in \mathrm{Neg}(X)} R(C). \tag{3.2}$$

Now we claim that $\mathrm{Neg}(X) = L$; it is obvious that $L \subseteq \mathrm{Neg}(X)$ and our hypothesis gives the reverse inclusion. We have therefore that decomposition (3.2) is the same of equation (3.1).

Our goal is now to prove the reverse implication. Consider an integral curve C such that $C^2 < 0$; from (3.1) we get the decomposition:

$$[C] = \alpha + \sum_{i\in I} b_i[C_i], \tag{3.3}$$

where $\alpha \in \overline{\mathrm{Pos}}(X)$, $b_i > 0$ and $[C_i] \in L$. Now, since $C^2 < 0$, by Lemma 1.37, $[C]$ spans the extremal ray $R(C)$. By extremality we have that $\alpha \in R(C)$ and so there exists a real number $a \geqslant 0$ such that $\alpha = a[C]$. We immediately get

$$0 \leqslant \alpha^2 = a^2[C]^2 \leqslant 0,$$

that gives $a^2[C]^2 = 0$ and so $a = 0$ and $\alpha = 0$.

Again by extremality, we also have that $[C] \in R(C_i)$ for all $i \in I$; but since by Lemma 1.39 in such a ray there can't be two distinct integral curves, the decomposition (3.3) has only a summand with $b_{i_0} = 1$, $C = C_{i_0}$ and in particular $[C] \in L$. □

Remark 3.2. Since we are considering blown-up surfaces at r points, we can't avoid the exceptional curves $E_1, \dots, E_r \subset X$. In light of this, it is immediate to see that the first claim in Proposition 3.1 is thus equivalent to Conjecture 2.28. In particular Segre Conjecture (Conjecture 2.23) implies the decomposition given in (3.1).

$K3$ surfaces

The case of $K3$ surfaces has been considered in [DVL05]; in this paper the authors state the Segre Conjecture for a generic $K3$ surface, that is Y is a $K3$ and $\mathrm{Pic}(Y) = \mathbb{Z}[H]$, for an ample divisor H on Y.

In this subsection we want to study in more details how Segre Conjecture forces the structure of the negative part of $\overline{\mathrm{NE}}(X)$.

Let Y be a $K3$ surface and $X = \mathrm{Bl}_r\, Y$ the blow up at r very general points; let us recall that $\chi(\mathcal{O}_Y) = 2$.

Let us suppose Conjecture 2.23 holds true; on one hand for non exceptional curves $C \subset X$, Proposition 2.35 gives the bounds

$$-1 \geqslant C^2 \geqslant p_a(C) - 2 \geqslant -2; \tag{3.4}$$

on the other hand, if $C \subset X$ is an exceptional integral curve, then it is smooth, rational and $C^2 = -1$.

Now, since we are supposing Segre conjecture, the inequality (3.4) gives that the curves with negative self-intersection has to be in the list

$$\{(-1,0), (-1,1), (-2,0), (-2,1)\}.$$

Our goal is now to refine this list of the peculiar case of $K3$ surfaces.

As pointed out before there always are the exceptional curves C that are $(-1,0)$-curves. We will call a curve like this a *curve of kind I*.
Suppose now that C is not exceptional, and consider the case $C^2 = -2$, we immediately get from (3.4) that $p_a(C) = 0$ and C is a $(-2,0)$-curve; from adjunction formula $2p_a(C) - 2 = C^2 + C \cdot K_X$, we get that $C \cdot K_X = 0$.
We see therefore that adjunction prevents the existence of $(-2,1)$-curves.
Where does a curve like this come from? Since $K_X = E_1 + \cdots + E_r$, we have that

$$C \cdot K_X = C \cdot E_1 + \cdots + C \cdot E_r = 0,$$

since C is not exceptional, then $C \cdot E_i$ must be non negative and therefore we have $C \cdot E_i = 0$ for all i. Let $\Gamma = \varphi(C)$ be the image in Y; since C does not intersect the exceptional divisors, $C = \tilde{\Gamma} = \varphi^*\Gamma$. In particular, we have that $\Gamma^2 = (\varphi^*\Gamma)^2 = C^2 = -2$ and, since $K_Y = 0$, $p_a(\Gamma) = 0$.
To summarize, if C is a $(-2,0)$-curve on X, then $C \cdot K_X = 0$ and there exists a curve Γ on Y that is a $(-2,0)$-curve not passing through the blown-up points P_i. We will call such a curve C a *curve of kind II*.
Suppose now that C is not exceptional and $C^2 = -1$, then (3.4) gives us two possible values of $p_a(C)$.
If $C^2 = -1$ and $p_a(C) = 0$, adjunction gives

$$2p_a(C) - 2 = C^2 + C \cdot K_X \quad \Rightarrow \quad -2 = -1 + C \cdot K_X \quad \Rightarrow \quad C \cdot K_X = -1.$$

But this would give $C \cdot E_1 + \cdots + C \cdot E_r = -1$ that is impossible, since C is not exceptional and $C \cdot E_i \geqslant 0$ for all i.
Consider now C a $(-1,1)$-curve; adjunction gives $C \cdot K_X = 1$ and we get

$$C \cdot E_1 + \cdots + C \cdot E_r = 1.$$

This means that there exists an i such that $C \cdot E_i = 1$ and $C \cdot E_j = 0$ for all $j \neq i$. Let $\Gamma = \varphi(C)$; we have $\varphi^*\Gamma = \tilde{\Gamma} + E_i = C + E_i$, that gives $C = \tilde{\Gamma} = \varphi^*\Gamma - E_i$. Let us compute the self-intersection:

$$\Gamma^2 = (\varphi^*\Gamma)^2 = (C + E_i)^2 = C^2 + 2C \cdot E_i + {E_i}^2 = -1 + 2 - 1 = 0,$$

and by adjunction we get $p_a(\Gamma) = 1$.
Summarizing, we have that if C is a $(-1,1)$-curve on X, then $C \cdot K_X = -1$ and $C = \tilde{\Gamma}$, where Γ is a $(0,1)$-curve on Y passing through P_i for some i with multeplicity 1 and not passing through the others blown up points. We will call C a *curve of kind III*.
To summarize once again, we have that the Segre Conjecture implies that the curves with negative self-intersection on a blown up $K3$ surface are of one of this kind:

kind	I	II	III
$(C^2, p_a(C))$	(-1,0)	(-2,0)	(-1,1)
$C \cdot K_X$	-1	0	1
$\varphi(C)$	point	Γ	Γ
$(\Gamma^2, p_a(\Gamma))$		(-2,0)	(0,1)
		$\text{mult}_{P_i}(\Gamma) = 0$	$\text{mult}_{P_i} = 1, mult_{P_j} = 0$
		for all i	for all $j \neq i$.

In the $K3$ surface case, Proposition 3.1 gives the following fact.

Fact 3.3. *Let $X = \mathrm{Bl}_r Y$ be the blow up of a $K3$ surface; if the Segre Conjecture holds true, then the list in Proposition 3.1 is given by curves of kind I, II or III and we have the decomposition:*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{Pos}}(X) + \sum_{C_i \text{ of kind } I} R(C_i) + \sum_{C_j \text{ of kind } II} R(C_j) + \sum_{C_k \text{ of kind } III} R(C_k).$$

In the case of generic $K3$, we have the following fact.

Fact 3.4. *If Y is a generic $K3$ surface; suppose Segre Conjecture holds for $X = \mathrm{Bl}_r Y$, then if C is an irreducible curve such that $C^2 < 0$, then it is an exceptional (-1)-curve.*

Proof. Since Y is generic, then $\mathrm{Pic}(Y) = \mathbb{Z}[h]$ and $\overline{\mathrm{NE}}(Y) = R(h)$ is simply the ray generated by h. Therefore for every curve on Y we have $C^2 > 0$, hence on X there can't be curves of kind II or III. □

3.2 Goal and warm-up

In [dF10], the author shows that assuming Segre Conjecture, then the Mori cone of the projective plane blown up at sufficiently many points is circular in some half space depending on a certain divisor. In this section we generalize this result to any blown-up surface X satisfying certain properties.
In view of the conjectures we stated and recalled in Chapter 2, it is reasonable to expect the following.

Conjecture 3.5 (Circular part conjecture)**.** *Let X be the blow up of an smooth algebraic surface at r (eventually large) general points. Then there exists an $\mathbb{R}$-divisor D on X such that*

$$\overline{\mathrm{NE}}(X)_{D\geqslant 0} = \overline{\mathrm{Pos}}(X)_{D\geqslant 0}. \tag{3.5}$$

We will show in Theorem 3.13 how, if r is sufficiently large, this conjecture can be derived as a consequence of the List Conjecture (see Conjecture 2.28). In particular this is a consequence of Segre Conjecture, hence Conjecture 3.5 would follow from Segre.
In order to assure all the definitions in the former sections make sense, we work with a smooth projective surface Y with either $p_g(Y) = 0$ or Y a $K3$ surface or an abelian surface and let $X = \mathrm{Bl}_r Y$ be the blow up at the general points $x_1, \dots, x_r$.
In the spirit of generalizing [dF10], we want now to locate the $(-n, p)$-rays of X (see Definition 1.40). As well known, since we have negative self-intersection, these rays are extremal rays of the Mori cone (see Lemma 1.37).
Let us consider an ample divisor A on Y and let $L = \varphi^* A$ be the nef pullback to X.
We put in evidence some useful and immediate computations.

Fact 3.6. *It holds:*

1. $K_X = \varphi^* K_Y + E_1 + \cdots + E_r$;
2. ${K_X}^2 = {K_Y}^2 - r$;
3. $L^2 = (\varphi^* A)^2 = A^2 > 0$;
4. $K_X \cdot L = (\varphi^* K_Y + \sum E_i) \cdot L = A \cdot K_Y$.

We will denote with $K = K_X$ the canonical divisor of X.
Let us recall that we have shown in Proposition 2.35 that if C is a non exceptional $(-n, p)$-curve on X, Segre Conjecture implies the following bounds:

$$\chi(\mathcal{O}_Y) \geqslant 1; \qquad n^2 \geqslant -\chi(\mathcal{O}_Y); \qquad p \leqslant \chi(\mathcal{O}_Y) - 1. \tag{3.6}$$

We want to find conditions on the (eventually large) number r of points to blow up in order to describe the Mori cone $\overline{\mathrm{NE}}(X)$ of the blown-up surface X in terms of the positive cone $\overline{\mathrm{Pos}}(X)$.
In fact we have

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{Pos}}(X) + \sum_i R_i, \tag{3.7}$$

where the sum runs over all rays $R_i = R(C_i)$, for some integral curve $C_i \subset X$ such that $C_i^2 < 0$. Here is our strategy to locate $(-n, p)$-rays:

1. fix a curve C generating a $(-n, p)$-ray;
2. find a $s = s(n, p) \in \mathbb{R}$ such that $R(C) \subset \overline{\mathrm{Pos}}(X) + R(K - sL)$.

Performing our program we will find some conditions involving the number r of points to obtain the existence of solutions in certain inequalities. Since we will pretend bounded negativity and bounded arithmetic genus, in order to avoid accumulation phenomena, we will have to impose just finitely many inequalities.
Let us first prove the fact in the case of $(-1, p)$-curves.

Proposition 3.7 ($(-1, p)$-case)**.** *Let Y be an smooth projective surface and $X = \mathrm{Bl}_r Y$ the blow up of Y at r general points. If R is a $(-1, p)$-ray generated by a curve C, supposing that*

$$\begin{cases} r > {K_Y}^2 + 1 - \frac{(A\cdot K_Y)^2}{A^2} \\ r \geqslant {K_Y}^2 + 1 + 4A^2p^2 - 4(A\cdot K_Y)p \qquad \textit{if } p > \frac{A\cdot K_Y}{2A^2}, \end{cases} \tag{3.8}$$

then there exists

$$s_1 = \frac{A\cdot K_Y + \sqrt{(A\cdot K_Y)^2 - A^2{K_Y}^2 + A^2r - A^2}}{A^2}, \tag{3.9}$$

such that

$$R(C) \subset \overline{\mathrm{Pos}}(X) + R(K - s_1 L). \tag{3.10}$$

Proof. As first step we want to find a positive solution for t of the equation

$$(tC - (K - sL))^2 = 0, \tag{3.11}$$

where C is the $(-1, p)$-curve generating R. To ensure the existence of solutions of (3.11) we need $\Delta \geqslant 0$. From (3.11) we get:

$$t^2C^2 - 2tC\cdot(K - sL) + (K - sL)^2 = 0,$$

and we have

$$\frac{\Delta}{4} := [C\cdot(K - sL)]^2 - C^2(K - sL)^2;$$

since C is a $(-1, p)$-curve, by adjunction formula, we get

$$2p - 2 = C^2 + C\cdot K,$$

that gives

$$C \cdot K = 2p - 2 + 1 = 2p - 1.$$

So we would like to have

$$(2p - 1 - sC \cdot L)^2 + (K - sL)^2 \geqslant 0.$$

It is enough to require the existence of an s such that

$$\begin{cases} (K - sL)^2 = -1 \\ (2p - 1 - sC \cdot L)^2 \geqslant 1. \end{cases} \tag{3.12}$$

We have

$$\begin{aligned} (K - sL)^2 = -1 \quad &\Leftrightarrow \quad K^2 - 2sK \cdot L + s^2 L^2 = -1 \\ &\Leftrightarrow \quad s^2 L^2 - 2s(K \cdot L) + (K^2 + 1) = 0. \end{aligned}$$

Using the computations in Fact 3.6, the equation becomes

$$A^2 s^2 - 2sA \cdot K_Y + K_Y^2 - r + 1 = 0;$$

it has solutions if

$$\frac{\Delta_1}{4} := (A \cdot K_Y)^2 - A^2 K_Y^2 + A^2 r - A^2 \geqslant 0,$$

that is if

$$r \geqslant {K_Y}^2 + 1 - \frac{(A \cdot K_Y)^2}{A^2}.$$

Since in the following we will need the strict positivity of this discriminant, our first numerical condition on the number of points to blow up is:

$$r > {K_Y}^2 + 1 - \frac{(A \cdot K_Y)^2}{A^2}. \tag{3.13}$$

In this situation we can take

$$s_1 = \frac{A \cdot K_Y + \sqrt{\Delta_1/4}}{A^2}; \tag{3.14}$$

let us note as s_1 does not depend on the specific curve C, but just, as will be clearer in the next proposition, on the value of C^2.
Now we can fix $s = s_1$ as in (3.14); we want to check also the second inequality in (3.12). We immetiately see that it is enough that

$$2p - 1 - sC \cdot L \leqslant -1. \tag{3.15}$$

Now, if $C \cdot L = 0$, then C is a curve contracted by $X \to Y$ and so is one of the exceptional divisors E_i; in particular we have $p = 0$ and so the inequality holds.
If else $C \cdot L > 0$, the condition (3.15) is equivalent to

$$2p \leqslant sC \cdot L \quad \Leftrightarrow \quad s \geqslant \frac{2p}{C \cdot L},$$

and since $C \cdot L \geqslant 1$, it is enough that

$$s \geqslant 2p. \tag{3.16}$$

This is true when

$$s = \frac{A \cdot K_Y + \sqrt{\Delta_1/4}}{A^2} \geqslant 2p,$$

which gives

$$A \cdot K_Y + \sqrt{\Delta_1/4} \geqslant 2A^2 p,$$

that is

$$\sqrt{\Delta_1/4} \geqslant 2A^2 p - A \cdot K_Y. \tag{3.17}$$

If the right hand side is non positive, that is when $p \leqslant A \cdot K_Y/2A^2$, the inequality (3.17) holds and we have no other conditions to impose.
If otherwise $p > A \cdot K_Y/2A^2$, we get

$$\Delta_1/4 \geqslant (2A^2 p - A \cdot K_Y)^2,$$

that gives

$$(A \cdot K_Y)^2 - A^2 K_Y^2 + A^2 r - A^2 \geqslant 4(A^2)^2 p^2 + (A \cdot K_Y)^2 - 4A^2(A \cdot K_Y)p;$$

thus we get:

$$r \geqslant {K_Y}^2 + 1 + 4A^2 p^2 - 4(A \cdot K_Y)p. \tag{3.18}$$

Hence, we have the two conditions:

$$\begin{cases} r > {K_Y}^2 + 1 - \frac{(A \cdot K_Y)^2}{A^2} \\ r \geqslant {K_Y}^2 + 1 + 4A^2 p^2 - 4(A \cdot K_Y)p \qquad \text{if } p > \frac{A \cdot K_Y}{2A^2}. \end{cases}$$

In this situation, (3.12) holds true and we have solutions of $(tC - (K - sL))^2 = 0$. This last equation becomes

$$t^2 + 2tC \cdot (K - sL) + 1 = 0;$$

one of the two solutions is

$$t_0 = -C \cdot (K - sL) + \sqrt{\Delta/4} = -(2p - 1 - sC \cdot L) + \sqrt{\Delta/4}. \tag{3.19}$$

Thanks to the choice we did in (3.15) we have that $t_0 \geqslant 1 > 0$ and so we get a positive solution of (3.11).
Now we have that $\alpha = t_0 C - (K - sL)$ satisfies $\alpha^2 = 0$. In order to prove that $\alpha \in \overline{\mathrm{Pos}}(X)$, we need to check that $\alpha \cdot h \geqslant 0$ for some h ample.
In our situation, we have a fixed curve $C \subset X$ and we have produced $t_0 \geqslant 1$ depending only on C; we will consider a class h on X of the form

$$h = L - \sum \delta_i E_i, \quad \delta_i > 0;$$

it is well known that for small δ_i, h is ample. We will fix the δ_i after a formal computation; it may be worth to recall that if we get $\alpha \cdot h \geqslant 0$ for an ample class h, then $\alpha \cdot h' \geqslant 0$ for any other ample class h' (see Section 1.4).

A little remark: thanks to our condition, we have that $\sqrt{\Delta_1/4} > 0$. We can now proceed to the formal computation of $\alpha \cdot h$:

$$
\begin{aligned}
&[t_0 C - (K - sL)] \cdot [L - \sum \delta_i E_i] \\
&= t_0 C \cdot L - t_0 C \cdot \sum \delta_i E_i - (K - sL) \cdot L + (K - sL) \cdot (\sum \delta_i E_i) \\
&= t_0 C \cdot L - t_0 \sum \delta_i E_i \cdot C - K \cdot L + sL^2 + K \cdot \sum \delta_i E_i - \underbrace{L \cdot \sum \delta_i E_i}_{=0} \\
&= \underbrace{t_0 C \cdot L}_{\geqslant 0} - t_0 \sum \delta_i E_i \cdot C - K_Y \cdot A + sA^2 + \underbrace{(\varphi^* K_Y) \cdot \sum \delta_i E_i}_{=0} - \sum \delta_i \qquad (3.20) \\
&= \underbrace{t_0 C \cdot L}_{\geqslant 0} - t_0 \sum \delta_i E_i \cdot C - K_Y \cdot A + \frac{(K_Y \cdot A) + \sqrt{\Delta_1/4}}{A^2} A^2 - \sum \delta_i \\
&= \underbrace{t_0 C \cdot L}_{\geqslant 0} - t_0 \sum \delta_i E_i \cdot C + \underbrace{\sqrt{\Delta_1/4}}_{>0} - \sum \delta_i .
\end{aligned}
$$

Now, since $t_0 C \cdot L + \sqrt{\Delta_1/4} > 0$ and $E_i \cdot C$ depends only on C, for any C, we can fix small δ_i for which $\alpha \cdot h \geqslant 0$. From the remark we did before, this product is thus non negative for any ample class.

We have hence that for positive t_0, $\alpha = t_0 C - (K - sL) \in \overline{\mathrm{Pos}}(X)$. Therefore $t_0 C \in \overline{\mathrm{Pos}}(X) + (K - sL)$ and so, since t_0 is positive,

$$R(C) \subset \overline{\mathrm{Pos}}(X) + R(K - sL).$$

□

Our goal is now to prove of a similar fact in the general case of $(-n, p)$-curves.

Proposition 3.8 ($(-n, p)$-case)**.** *Let Y be an algebraic projective smooth surface and $X = \mathrm{Bl}_r Y$ the blow up of Y at r general points. If R is an $(-n, p)$-ray, $n \geqslant 2$, generated by a curve C, setting $q = 2p + n - 1$, and supposing that*

$$
\begin{cases}
r \geqslant {K_Y}^2 + \frac{1}{n} - \frac{(A \cdot K_Y)^2}{A^2} \\
r \geqslant {K_Y}^2 + \frac{1}{n} + A^2 q^2 - 2(A \cdot K_Y) q \quad \text{if } q > \frac{A \cdot K_Y}{A^2},
\end{cases}
\qquad (3.21)
$$

then there exists

$$s_n = \frac{A \cdot K_Y + \sqrt{(A \cdot K_Y)^2 - A^2 {K_Y}^2 + A^2 r - A^2/n}}{A^2} \qquad (3.22)$$

such that

$$R(C) \subset \overline{\mathrm{Pos}}(X) + R(K - s_n L). \qquad (3.23)$$

Proof. As before, we want to find a positive solution of the equation

$$(tC - (K - sL))^2 = 0, \qquad (3.24)$$

where C is the $(-n, p)$-curve generating R. To ensure the existence of solutions of (3.24) we need $\Delta \geqslant 0$. The (3.24) become

$$t^2 C^2 - 2tC \cdot (K - sL) + (K - sL)^2 = 0$$

and we have

$$\frac{\Delta}{4} := [C \cdot (K - sL)]^2 - C^2(K - sL)^2;$$

since C is a $(-n, p)$-curve, by adjunction formula, we get

$$2p - 2 = C^2 + C \cdot K \quad \Rightarrow \quad C \cdot K = 2p + n - 2.$$

So we would like

$$(2p + n - 2 - sC \cdot L)^2 + n(K - sL)^2 \geqslant 0.$$

To have this, it is enough require the existence of an s such that

$$\begin{cases} (K - sL)^2 = -\frac{1}{n} \\ (2p + n - 2 - sC \cdot L)^2 \geqslant 1. \end{cases} \tag{3.25}$$

We have

$$\begin{aligned} (K - sL)^2 = -\frac{1}{n} \quad &\Leftrightarrow \quad K^2 - 2sK \cdot L + s^2 L^2 = -\frac{1}{n} \\ &\Leftrightarrow \quad s^2 L^2 - 2s(K \cdot L) + \left(K^2 + \frac{1}{n}\right) = 0. \end{aligned}$$

Using the computations in Fact 3.6, the equation becomes

$$A^2 s^2 - 2sA \cdot K_Y + K_Y^2 - r + \frac{1}{n} = 0;$$

it has solutions if

$$\frac{\Delta_n}{4} := (A \cdot K_Y)^2 - A^2 K_Y^2 + A^2 r - \frac{A^2}{n} \geqslant 0;$$

and this is true if

$$r \geqslant {K_Y}^2 + \frac{1}{n} - \frac{(A \cdot K_Y)^2}{A^2}. \tag{3.26}$$

We have hence found the first numerical condition on the number of points to blow up. In this situation we can take

$$s_n = \frac{A \cdot K_Y + \sqrt{\Delta_n/4}}{A^2}, \tag{3.27}$$

Now we fix $s = s_n$ as in (3.27) and we want to check also the second inequality in (3.25). It is enough to see that

$$2p + n - 2 - sC \cdot L \leqslant -1. \tag{3.28}$$

Now, we have that $C \cdot L \geqslant 1$ since C can't be contracted, therefore (3.28) is equivalent to

$$2p \leqslant sC \cdot L - n + 1 \quad \Leftrightarrow \quad s \geqslant \frac{2p + n - 1}{C \cdot L}.$$

Since $C \cdot L \geqslant 1$ it is enough that

$$s \geqslant 2p + n - 1. \tag{3.29}$$

This is true when

$$s = \frac{A \cdot K_Y + \sqrt{\Delta_n/4}}{A^2} \geqslant 2p + n - 1,$$

that is when

$$A \cdot K_Y + \sqrt{\Delta_n/4} \geqslant A^2(2p+n-1) \qquad \text{that is}$$
$$\sqrt{\Delta_n/4} \geqslant A^2(2p+n-1) - A \cdot K_Y.$$

If the right hand side is negative this is true and we have no other conditions; otherwise, that is when

$$q := 2p+n-1 \geqslant \frac{A \cdot K_Y}{A^2},$$

we get

$$\Delta_n/4 \geqslant (A^2 q - A \cdot K_Y)^2$$

that is

$$A^2 r \geqslant A^2 K_Y^2 + \frac{1}{n} A^2 - (A \cdot K_Y)^2 + (A \cdot K_Y)^2 + (A^2)^2 q^2 - 2A^2(A \cdot K_Y)q,$$

which gives

$$r \geqslant {K_Y}^2 + \frac{1}{n} + A^2 q^2 - 2(A \cdot K_Y)q. \tag{3.30}$$

So the conditions (3.26) and (3.30) we found, give exactly the inequalities (3.21) in the statement of the proposition.
In this situation we have solutions of $(tC-(K-sL))^2 = 0$ and this last equation becomes:

$$nt^2 + 2t(2p+n-2-sC \cdot L) + \frac{1}{n} = 0.$$

One of the solutions is

$$t_0 = \frac{-(2p+n-2-sC \cdot L) + \sqrt{(2p+n-2-sC \cdot L)^2 - 1}}{n}. \tag{3.31}$$

Thanks to the choice we did in (3.28) we have that $t_0 \geqslant 1/n > 0$ and so we have a positive solution of (3.24).
Now we have that $\alpha = t_0 C - (K - sL)$ satisfies $\alpha^2 = 0$. In order to prove that $\alpha \in \overline{\mathrm{Pos}}(X)$, we need to check that $\alpha \cdot h \geqslant 0$ for some h ample.
As in the former proposition, we want to fix some small $\delta_i > 0$ in order to have $h = L - \sum \delta_i E_i$ an ample class such that $\alpha \cdot h \geqslant 0$ (and hence $\alpha \cdot h' \geqslant$ for any other ample class h').
We can formally compute $\alpha \cdot h$:

$$\begin{aligned}
&[t_0 C - (K - sL)] \cdot [L - \textstyle\sum \delta_i E_i] \\
=& t_0 C \cdot L - t_0 C \cdot \textstyle\sum \delta_i E_i - (K - sL) \cdot L + (K - sL) \cdot (\sum \delta_i E_i) \\
=& t_0 C \cdot L - t_0 \textstyle\sum \delta_i E_i \cdot C - K \cdot L + sL^2 + K \cdot \sum \delta_i E_i - \underbrace{L \cdot \textstyle\sum \delta_i E_i}_{=0} \\
=& \underbrace{t_0 C \cdot L}_{>0} - t_0 \textstyle\sum \delta_i E_i \cdot C - K_Y \cdot A + sA^2 + \underbrace{(\varphi^* K_Y) \cdot (\textstyle\sum \delta_i E_i)}_{=0} - \sum \delta_i \qquad (3.32) \\
=& \underbrace{t_0 C \cdot L}_{>0} - t_0 \textstyle\sum \delta_i E_i \cdot C - K_Y \cdot A + \dfrac{(K_Y \cdot A) + \sqrt{\Delta_n/4}}{A^2} A^2 - \sum \delta_i \\
=& \underbrace{t_0 C \cdot L}_{>0} - t_0 \textstyle\sum \delta_i E_i \cdot C + \underbrace{\sqrt{\Delta_n/4}}_{\geqslant 0} - \sum \delta_i.
\end{aligned}$$

Now, since $t_oC \cdot L + \sqrt{\Delta_n/4} > 0$, we can fix some small δ_is (eventually depending on C) such that $\alpha \cdot h$ is positive.
Again we have that for positive t_0, $\alpha = t_0C - (K - sL) \in \overline{\mathrm{Pos}}(X)$. Therefore $t_0C \in \overline{\mathrm{Pos}}(X) + (K - sL)$ and so,

$$R(C) \subset \overline{\mathrm{Pos}}(X) + R(K - sL).$$

□

Now, in view of what we pointed out at the beginning of this section, we will suppose that List Conjecture holds true with bounds depending only on the surface Y and not on the number of blown up points.
Therefore, in this situation, if C is an integral curve with negative self-intersection, then

$$-1 \geqslant C^2 \geqslant -\nu \quad \text{and} \quad 0 \leqslant p_a(C) \leqslant \pi, \tag{3.33}$$

for some non negative integers ν and π depending only on Y.
We want to satisfy the inequalities in (3.8) and (3.21) for every $n = 1, \dots, \nu$ and $p = 0, \dots, \pi$; these inequalities are verified if

$$r > {K_Y}^2 + 1 - \frac{(A \cdot K_Y)^2}{A^2}, \tag{3.34}$$

and, in the case $q := 2\pi + \nu - 1 > \frac{A \cdot K_Y}{A^2}$,

$$r \geqslant {K_Y}^2 + 1 + A^2(2\pi + \nu - 1)^2 - 2(A \cdot K_Y)(2\pi + \nu - 1). \tag{3.35}$$

Now we see that we can simplify our conditions; we claim that (3.35) implies (3.34). Indeed we have

$$(A^2q - A \cdot K_Y)^2 \geqslant 0,$$

which easily gives

$$A^2q - 2(A \cdot K_Y)q \geqslant -\frac{(A \cdot K_Y)^2}{A^2},$$

and hence the required implication.
Summarizing, we get

$$\begin{cases} r > {K_Y}^2 + 1 - \frac{(A \cdot K_Y)^2}{A^2} & \text{if} \quad q \leqslant \frac{A \cdot K_Y}{A^2} \\ r \geqslant {K_Y}^2 + 1 + A^2q^2 - 2(A \cdot K_Y)q & \text{if} \quad q > \frac{A \cdot K_Y}{A^2}. \end{cases} \tag{3.36}$$

If we can fix an r verifying all these inequalities, we have that

$$\begin{aligned} (-1, p) &\rightsquigarrow R(C) \subset \overline{\mathrm{Pos}}(X) + R(K - s_1L) \\ (-2, p) &\rightsquigarrow R(C) \subset \overline{\mathrm{Pos}}(X) + R(K - s_2L) \\ &\vdots \qquad\qquad \vdots \\ (-\nu, p) &\rightsquigarrow R(C) \subset \overline{\mathrm{Pos}}(X) + R(K - s_\nu L) \end{aligned} \tag{3.37}$$

where

$$\begin{aligned} s_1 &= \frac{(A \cdot K_Y) + \sqrt{(A \cdot K_Y)^2 - A^2K_Y^2 + A^2r - A^2}}{A^2} \\ s_2 &= \frac{(A \cdot K_Y) + \sqrt{(A \cdot K_Y)^2 - A^2K_Y^2 + A^2r - A^2/2}}{A^2} \\ &\vdots \\ s_\nu &= \frac{(A \cdot K_Y) + \sqrt{(A \cdot K_Y)^2 - A^2K_Y^2 + A^2r - A^2/\nu}}{A^2}. \end{aligned} \tag{3.38}$$

It is obvious to point out that s_1 is the smallest and s_ν is the largest:

$$s_1 < s_2 < \cdots < s_\nu.$$

This seems interesting because of the following fact:

Fact 3.9. *In our situation, if $s \geqslant t$ we have*

$$(K - sL)^\perp \cap L^{\geqslant 0} \subset (K - tL)^{\geqslant 0} \cap L^{\geqslant 0},$$

in particular, since L is nef, this is true intersecting with $\overline{\mathrm{NE}}(X)$ instead of $L^{\geqslant 0}$.

Proof. Let $\gamma \in (K - sL)^\perp \cap L^{\geqslant 0}$, we get

$$\gamma \cdot K - s\gamma \cdot L = 0 \quad \Rightarrow \quad \gamma \cdot K = s\gamma \cdot L.$$

Now we have

$$\gamma \cdot (K - tL) = \gamma \cdot K - t\gamma \cdot L = s\gamma \cdot L - t\gamma \cdot L = (s - t)\gamma \cdot L \geqslant 0.$$

□

3.3 Pictures

In this section we want to study the general picture of our situation. We focus on the vector space $N(X)$ of numerical classes of curves (and divisors). In particular we want to describe how rays generated by curves with negative self-intersection behave with respect to the positive cone $\overline{\mathrm{Pos}}(X)$. For the sake of semplicity, we will draw pictures supposing $\rho(X) = 3$.
In Section 1.4 we fixed a basis of $\{e_1, \dots, e_\rho\}$ of $N(X)$ with e_1 parallel to an ample class h; in this basis the positive cone has equations $\{x_1 \geqslant 0, {x_1}^2 \geqslant \sum_{i=2}^{\rho} {x_i}^2\}$.
As usual, we want to picture a slice of the cones intersecting with a hyperplane away from the origin. We fix the hyperplane $\Pi = (x_1 = 1)$.
Consider now a ray $R \subset N(X)$ generated by a class γ such that $\gamma^2 < 0$; in this situation we will say that R is a self-negative ray.
We will focus on X with $\rho(X) = 3$.

Case I: $R \cdot h > 0$ **(convex hull type).**
Let us suppose that $R \cdot h > 0$; we have that R intersects the hyperplane Π in a point (that we call again R) and thanks to Lemma 1.52 we have that we can find the locus $R^\perp$ in Π by drawing the tangent lines joining the point R and $\overline{\mathrm{Pos}}(X) \cap \Pi$. The convex cone $\overline{\mathrm{Pos}}(X) + R$ can be drawn by considering in the hyperplane Π the convex hull of $\overline{\mathrm{Pos}}(X)$ and R; moreover, since $\gamma^2 < 0$, then $R \in R^{<0}$. The 3-dimensional picture is given in Figure 3.1 .

Case II: $R \cdot h < 0$ **(shade type).**
In this situation we have that the opposite ray $(-R)$ intersects the hyperplane Π. We have that, since $R^\perp = (-R)^\perp$, the orthogonal hyperplane is obtained with the usual construction applied to $(-R)$ instead of R. We immediately have $(-R) \in R^{>0}$ and we can see that $\overline{\mathrm{Pos}}(X) + R$ intersects Π in the shade of $\overline{\mathrm{Pos}}(X)$ from $-R$, imagining $-R$ as source of light: see Figure 3.2.

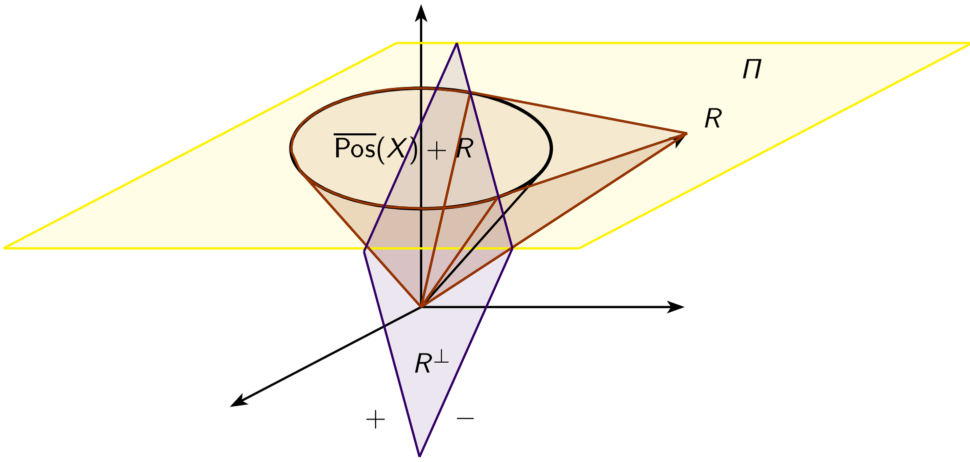

Figure 3.1: $\overline{\mathrm{Pos}}(X) + R$, when $R \cdot h > 0$

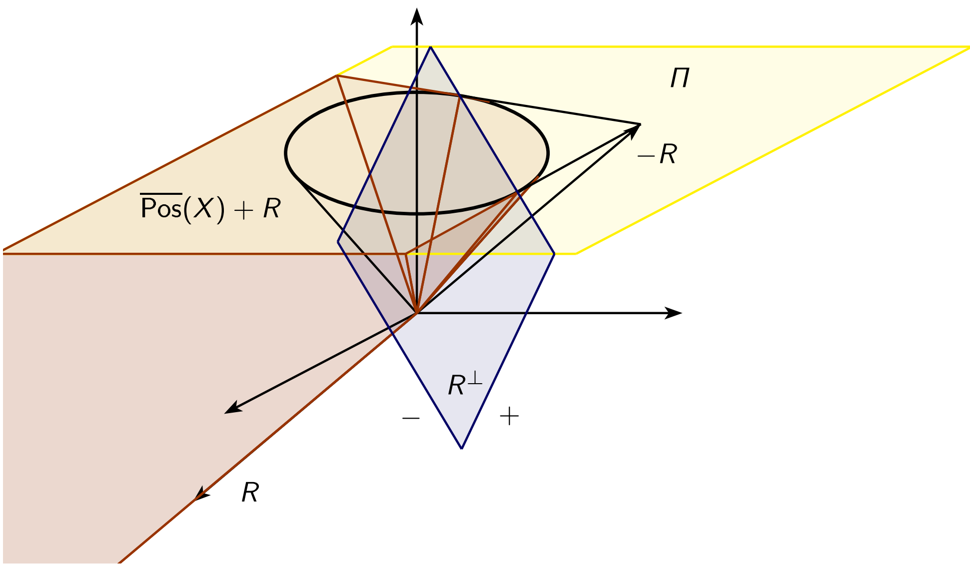

Figure 3.2: $\overline{\mathrm{Pos}}(X) + R$, when $R \cdot h < 0$

Case III: $R \cdot h = 0$ **(cylinder type).**
In the case of $R \cdot h = 0$, we will have that $R^\perp$ passes through the origin of the circle $\overline{\mathrm{Pos}}(X) \cap \Pi$, that is the point $(1, 0, 0) \in N(X)$.
In this situation the cone $\overline{\mathrm{Pos}}(X) + R$ is given in Π as the union of the circle $\overline{\mathrm{Pos}}(X) \cap \Pi$ and the strip over the diameter in the direction of R: see Figure 3.3.

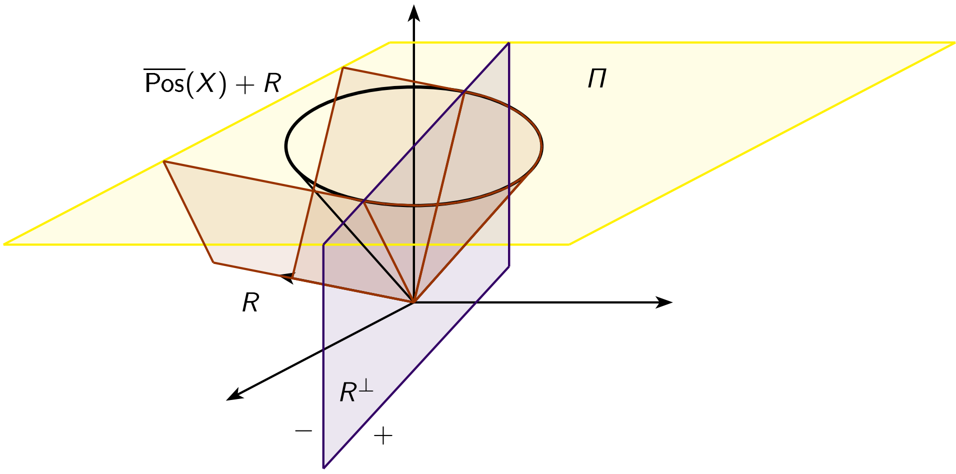

Figure 3.3: $\overline{\mathrm{Pos}}(X) + R$, when $R \cdot h = 0$

3.4 Circular part and main result

In the former sections we dealt with rays generated by $K - sL$, for some $s \in \mathbb{R}$. Does this rays intersect Π?

Fact 3.10. *In our situation if the conditions* (3.34) *and* (3.35) *on the number of points to blow up are verified, is it possible to find an ample class in the form* $h = L - \sum_i \delta_i E_i$ *for some sufficiently small coefficients* δ_i*, such that* $(K - sL) \cdot h < 0$ *for all* $s = s_n$ *as in* (3.38).

Proof. Let us compute.

$$
\begin{aligned}
&(K - sL) \cdot (L - \sum_i \delta_i E_i) \\
=&(K - sL) \cdot L - (K - sL) \cdot (\sum_i \delta_i E_i) \\
=&K_Y \cdot A - sA^2 - \sum_i \delta_i K \cdot E_i + s \sum_i \delta_i L \cdot E_i \\
=&K_Y \cdot A - sA^2 - \sum_i \delta_i (\varphi^* K_Y) \cdot E_i - \sum_i \delta_i (\sum_j E_j) \cdot E_i \\
=&K_Y \cdot A - sA^2 + \sum_i \delta_i \\
=&K_Y \cdot A - \frac{K_Y \cdot A + \sqrt{\Delta_n/4}}{A^2} A^2 + \sum_i \delta_i;
\end{aligned}
$$

thus we get

$$-\sqrt{\Delta_n/4} + \sum_i \delta_i,$$

that is negative since $\Delta_n/4 > 0$ and $\sum_i \delta_i$ is small. $\square$

We have therefore that the ray generated by $(K - sL)$ does not intersect Π, but the ray $R(-(K - sL))$ does and we are in the situation of Case II (Figure 3.2).
We are now getting closer to our main result; we need some preliminary results.

Fact 3.11. *For all $t \neq s \in \mathbb{R}$ we have that*

$$((K - sL)^{\perp} \cap \operatorname{Pos}(X)) \cap ((K - tL)^{\perp} \cap \operatorname{Pos}(X)) = \emptyset.$$

Proof. Consider γ in the intersection, then

$$(K - sL) \cdot \gamma = 0 = (K - tL) \cdot \gamma,$$

that is $(t - s)L \cdot \gamma = 0$, but since $t \neq s$ this means $L \cdot \gamma = 0$, but this is impossible since L is nef and $L^{\perp}$, by Lemma 1.54, lies outside $\operatorname{Pos}(X)$. $\square$

We are now able to give the following proposition.

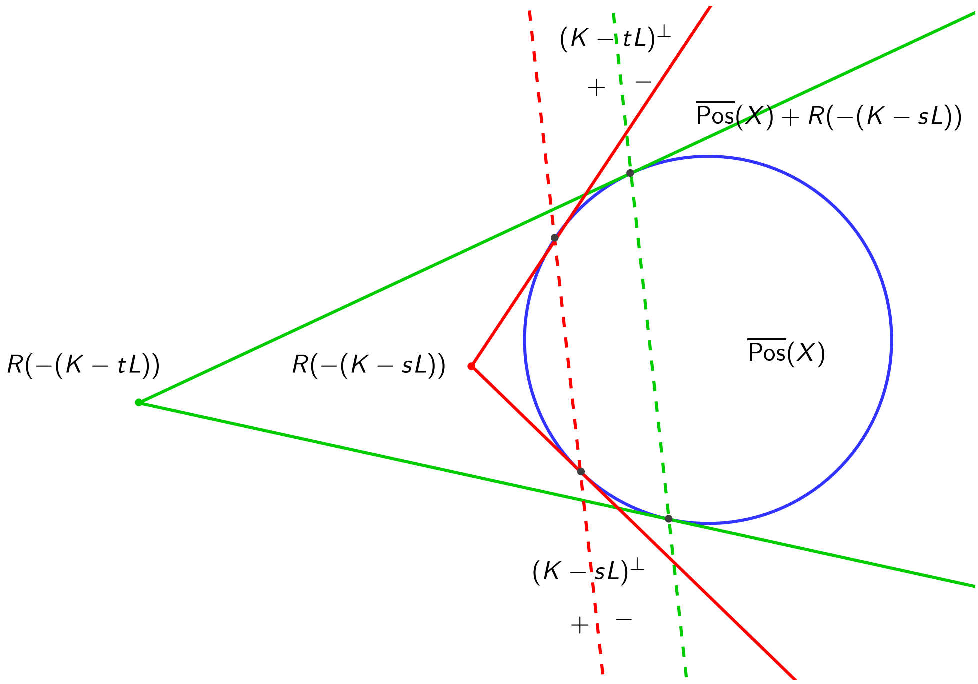

Figure 3.4: The positive cone $\overline{\operatorname{Pos}}(X)$ and the behaviour of $R(-(K - sL))$

Proposition 3.12. *If $s \geqslant t$, then*

$$\overline{\operatorname{Pos}}(X) + R(K - tL) \subset \overline{\operatorname{Pos}}(X) + R(K - sL). \tag{3.39}$$

In particular, if C is a $(-n, p)$-curve, for some $0 < n \leqslant \nu$ and $0 \leqslant p \leqslant \pi$, then

$$R(C) \subset \overline{\operatorname{Pos}}(X) + R(K - s_{\nu}L).$$

Proof. Let us consider $\gamma \in \overline{\mathrm{Pos}}(X) + R(K - tL)$; then we can write

$$\gamma = \alpha + a(K - tL), \quad \alpha \in \overline{\mathrm{Pos}}(X), a \geqslant 0.$$

We have

$$\begin{aligned}\gamma &= \alpha + a(K - sL + sL - tL) \\ &= \alpha + a(s - t)L + a(K - sL) \in \overline{\mathrm{Pos}}(X) + R(K - sL),\end{aligned}$$

since $s \geqslant t$, L is nef and hence, by convexity, it lies in $\overline{\mathrm{Pos}}(X)$.
Recalling the results of Proposition 3.7 and Proposition 3.8, since $s_1 < s_2 < \cdots < s_\nu$, we immediately get the second statement. □

In the case of $\rho(X) = 3$, the situation of Proposition 3.12 can be pictured as in Figure 3.4. In particular we see that as $s = s_n$ grows, the ray $R(-(K - sL))$ gets closer to the boundary of $\overline{\mathrm{Pos}}(X)$.
We are now ready to state our main result. We prove that Conjecture 3.5 is true if the List Conjecture is true with bounds depending only on Y and the number of points r is sufficiently large. In particular this is true if the Segre Conjecture is true.

Theorem 3.13. *Let $\varphi : X \to Y$ be the blow up at a set of r general points of a smooth projective surface Y. Let A be an ample divisor on Y and $L = \varphi^* A$. Let us suppose that:*

1. *there exist two integer numbers $\nu = \nu_X$ and $\pi = \pi_X$ such that the List Conjecture (Conjecture 2.28) holds on X with bounds for $(-n, p)$-curves given by $1 \leqslant n \leqslant \nu$ and $0 \leqslant p \leqslant \pi$; this is verified, for example, if Segre Conjecture holds true on X (see Proposition 2.35) or if $-K_X$ is pseudoeffective (see Proposition 2.30).*

2. *the following inequalities, with $q = 2\pi + \nu - 1$, hold:*

$$\begin{cases} r > {K_Y}^2 + 1 - \frac{(A \cdot K_Y)^2}{A^2} & \text{if} \quad q \leqslant \frac{A \cdot K_Y}{A^2} \\ r \geqslant {K_Y}^2 + 1 + A^2 q^2 - 2(A \cdot K_Y)q & \text{if} \quad q > \frac{A \cdot K_Y}{A^2}. \end{cases} \tag{3.40}$$

Then there exists $s = s_\nu \in \mathbb{R}$,

$$s = \frac{(A \cdot K_Y) + \sqrt{(A \cdot K_Y)^2 - A^2 {K_Y}^2 + A^2 r - A^2/\nu}}{A^2}, \tag{3.41}$$

such that

$$\overline{\mathrm{NE}}(X)_{(K - sL) \geqslant 0} = \overline{\mathrm{Pos}}(X)_{(K - sL) \geqslant 0}. \tag{3.42}$$

That is, Conjecture 3.5 holds with $D = K - sL$.
In particular, conditions 1. *and* 2. *are verified, for $r \gg 0$, if the bounds ν, π depend only on Y.*

Proof. It is obvious to see that $\rho(X) \geqslant 2$; since case $\rho(X) = 2$ is trivial, we will suppose henceforth that $\rho(X) \geqslant 3$.
Let us consider an extremal ray R of $\overline{\mathrm{NE}}(X)$ spanned by the class of an integral curve C with negative self-intersection. Since we are assuming the List Conjecture, we have that $C^2 \geqslant -\nu$, for some integer ν. By Proposition 3.12, we have that

$$R = R(C) \subset \overline{\mathrm{Pos}}(X) + R(K - sL),$$

where $s = s_\nu$ is the real number constructed in Proposition 3.7 or in Proposition 3.8.

As pointed out before, s_ν is the largest of the $s_1, \dots, s_\nu$ and the corresponding ray $R(-(K - s_\nu L))$ intersecting the hyperplane slice Π is the closest to the boundary of $\overline{\mathrm{Pos}}(X)$ (see, in the case $\rho(X) = 3$, Figure 3.4).
We want now to consider the nef cone of X; we have seen that $\mathrm{Nef}(X)$ can be obtained (in the slice Π) by cutting out from $\overline{\mathrm{Pos}}(X)$ the spherical portion corresponding to the curves with negative self-intersection (see Figure 1.1).
Since $R = R(C)$ is an extremal ray of $\overline{\mathrm{NE}}(X)$, it intersects the hyperplane slice Π and therefore the locus $R^\perp$ is obtained by drawing the tangent lines from R to $\overline{\mathrm{Pos}}(X)$ as in Lemma 1.52. Let us consider

$$G = R^\perp \cap \overline{\mathrm{Pos}}(X).$$

Claim 3.14. $G \subseteq (K - sL)^{\leqslant 0}$.

Proof of the Claim. Let us take $\gamma \in G$ and $0 \neq \delta \in R$; in particular $\gamma \cdot R = 0$ and since $R \subset \overline{\mathrm{Pos}}(X) + R(K - sL)$, we can write

$$\delta = \alpha + a(K - sL),$$

with $\alpha \in \overline{\mathrm{Pos}}(X)$ and $a > 0$ (indeed if $a = 0$, then $\delta^2 \geqslant 0$). We can compute

$$0 = \gamma \cdot \delta = \gamma \cdot \alpha + a\gamma \cdot (K - sL),$$

which gives

$$a\gamma \cdot (K - sL) = -\gamma \cdot \alpha,$$

that is non positive, since $\gamma, \alpha \in \overline{\mathrm{Pos}}(X)$ and by Fact 1.46 $\gamma \cdot \alpha \geqslant 0$. □

The situation of Claim 3.14 is pictured (if $\rho(X) = 3$) in the Figure 3.5: the fact that $R(C) \subset \overline{\mathrm{Pos}}(X) + R(K - sL)$ forces the facet $G = R^\perp \cap \overline{\mathrm{Pos}}(X)$ to be contained in $(K - sL)^{\leqslant 0}$.

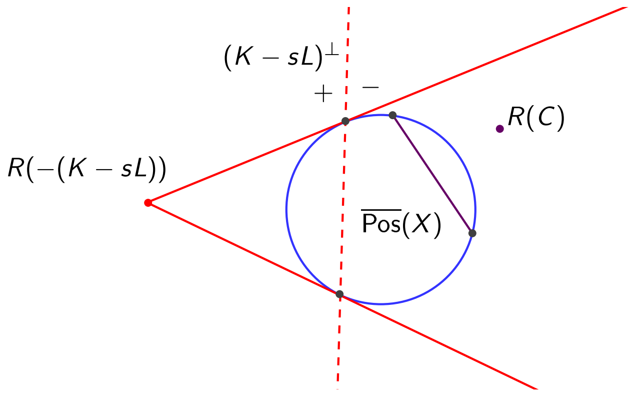

Figure 3.5: Proving Claim 3.14

Now a well-known theorem by Campana and Peternell gives a description of the shape of $\partial\,\mathrm{Nef}(X)$, see, for example [Laz04, Theorem 1.5.28]:

$$\partial\,\mathrm{Nef}(X) \subseteq \partial\overline{\mathrm{Pos}}(X) \cup \left(\bigcup_i H_i\right), \tag{3.43}$$

where H_i are hyperplanes.

These hyperplanes H_i are the orthogonal hyperplanes $C_i^\perp$ corresponding to classes of integral curves C_i with $C_i^2 < 0$. Indeed, let us suppose by contradiction that $C_i^2 \geqslant 0$. Then $[C_i] \in \overline{\text{Pos}}(X)$ and we have two possible cases (see Lemma 1.54): if $C_i^2 = 0$, then $[C_i] \in \partial\overline{\text{Pos}}(X)$, thus we get $C_i^\perp \cap \overline{\text{Pos}}(X) = R(C_i)$ and hence, since $\text{Nef}(X) \subseteq \overline{\text{Pos}}(X)$, we do not add anything in equation (3.43); otherwise, if $C_i^2 > 0$, then $C_i^\perp \cap \overline{\text{Pos}}(X) = \{0\}$, hence $C_i^\perp \cap \partial \text{Nef}(X) = \{0\}$ and again we do not have anything else to add.

Claim 3.15. $\partial \text{Nef}(X)_{(K-sL)^{>0}} = \partial\overline{\text{Pos}}(X)_{(K-sL)^{>0}}$.

Proof of the Claim. We will prove the two inclusions.

($\subseteq$) By Claim 3.14 we have that if $\beta \in \partial \text{Nef}(X)$ is supported on an hyperplane, then $\beta \in (K - sL)^{\leqslant 0}$; hence

$$\partial \text{Nef}(X) \cap (K - sL)^{>0} \subseteq \left(\partial\overline{\text{Pos}}(X) \cap (K - sL)^{>0}\right)$$

($\supseteq$) Let us take $0 \neq \alpha \in \partial\overline{\text{Pos}}(X) \cap (K - sL)^{>0}$; we want to show that α is in the boundary of $\text{Nef}(X)$.

It is enough to show that $\alpha \in \text{Nef}(X)$: indeed in this situation, since $0 \neq \alpha \in \partial\overline{\text{Pos}}(X)$, then $\alpha^2 = 0$, it can't lie in the interior of $\text{Nef}(X)$ and hence it is in the boundary. Let us show that $\alpha \in \text{Nef}(X)$.

Suppose by contradiction that α is not nef; then there exists a class of an integral curve C such that $\alpha \cdot C < 0$.

It easy to see that $C^2 < 0$, indeed if it were $C^2 \geqslant 0$, since $C \cdot h > 0$, then $C \in \overline{\text{Pos}}(X)$ and, by Fact 1.46, we would have $\alpha \cdot C \geqslant 0$ that is a contradiction.

Therefore we have $C^2 < 0$ (and hence it is a $(-n, p)$-curve for some n, p) and $\alpha \in C^{\leqslant 0} \cap \overline{\text{Pos}}(X)$.

Let us set $G = C^\perp \cap \overline{\text{Pos}}(X)$; as in Claim 3.14 we get $G \subseteq (K - sL)^{\leqslant 0}$.

We have also that $(K - sL) \cdot C < 0$: this is a consequence of the construction of s, see equation (3.15) and (3.28) in the proof of Proposition 3.7 and of Proposition 3.8: indeed, since $s = s_\nu \geqslant s_n$ and $C \cdot L \geqslant 0$, from (3.15) and (3.28) we get

$$(K - sL) \cdot C = 2p + n - 2 - sC \cdot L \leqslant 2p + n - 2 - s_n C \cdot L \leqslant -1 < 0.$$

Thus we get

$$G + R(C) \subset (K - sL)^{\leqslant 0}; \tag{3.44}$$

indeed if $x \in G + R(C)$, then

$$x = g + b[C], \quad g \in G; b \geqslant 0,$$

and

$$x \cdot (K - sL) = g \cdot (K - sL) + bC \cdot (K - sL) \leqslant 0.$$

We now claim the following:

$$C^{\leqslant 0} \cap \overline{\text{Pos}}(X) \subseteq G + R(C). \tag{3.45}$$

To prove it, let us take $0 \neq \beta \in C^{\leqslant 0} \cap \overline{\text{Pos}}(X)$; since we are dealing with cones, we can suppose that

$$\beta \cdot h = C \cdot h, \tag{3.46}$$

where h is an ample class we used to define $\overline{\mathrm{Pos}}(X)$.

Indeed, if $\beta \cdot h \neq C \cdot h$, we can consider $\beta' = a\beta$, for some $a > 0$ in order to have $\beta' \cdot h = C \cdot h$; if we prove that $a\beta \in G + R(C)$, then we have

$$\beta \in \frac{1}{a}(G + R(C)) = G + R(C).$$

Now we have $\beta \in C^{\leqslant 0}$; if $\beta \cdot C = 0$, then $\beta \in G \subset G + R(C)$ and we are done.

If $\beta \cdot C < 0$, we claim that

$$\beta \cdot C - C^2 > 0. \tag{3.47}$$

Indeed, since $\beta \in \overline{\mathrm{Pos}}(X)$, we have

$$\begin{aligned} 0 \leqslant \beta^2 &= \beta \cdot (\beta - C + C) = (\beta - C) \cdot \beta + \beta \cdot C \\ &= (\beta - C) \cdot (\beta - C + C) + \beta \cdot C \\ &= (\beta - C)^2 + (\beta - C) \cdot C + \beta \cdot C; \end{aligned}$$

this gives

$$\beta \cdot C - C^2 = (\beta - C) \cdot C \geqslant -\beta \cdot C - (\beta - C)^2.$$

We claim that this is positive since $\beta \cdot C < 0$ and $(\beta - C)^2 < 0$; to prove this last inequality we see that if it were $(\beta - C)^2 \geqslant 0$, then since $\beta \cdot h = C \cdot h$, we would have $(\beta - C) \cdot h = 0$ and hence $\beta - C \in \overline{\mathrm{Pos}}(X)$ and $\beta = C$, but this can't be since $\beta^2 \geqslant 0$ and $C^2 < 0$. Therefore the (3.47) is proved.

Let us consider the line L joining C and β:

$$L : \quad t\beta + (1 - t)C, \quad t \in \mathbb{R}.$$

We look for the intersection of L and $C^{\perp}$; we get:

$$0 = (t\beta + (1 - t)C) \cdot C = t(\beta \cdot C - C^2) + C^2,$$

which produces

$$t = \frac{-C^2}{\beta \cdot C - C^2}; \tag{3.48}$$

we see that thanks to the (3.47), since $C^2 < 0$ and $\beta \cdot C < 0$, we have $t > 1$. Let $\gamma \in C^{\perp}$ be the point in L corresponding to this $t > 1$. We claim that $\gamma \in \overline{\mathrm{Pos}}(X)$; indeed we have

$$\begin{aligned} &0 = \gamma \cdot C = t\beta \cdot C + (1 - t)C^2 \quad \Rightarrow \\ &(1 - t)C^2 = -t\beta \cdot C; \end{aligned}$$

and therefore, since $t > 1$:

$$\begin{aligned} \gamma^2 &= t^2\beta^2 + (1 - t)^2C^2 + 2t(1 - t)\beta \cdot C \\ &= t^2\beta^2 - t(1 - t)\beta \cdot C + 2t(1 - t)\beta \cdot C \\ &= t^2\beta^2 + t(1 - t)\beta \cdot C \geqslant 0; \end{aligned}$$

since $\beta \cdot h = C \cdot h$, we get

$$\begin{aligned} \gamma \cdot h &= t\beta \cdot h + (1 - t)C \cdot h \\ &= t\beta \cdot h + \beta \cdot h - t\beta \cdot h = \beta \cdot h \geqslant 0. \end{aligned}$$

Therefore we have $\gamma \in \overline{\mathrm{Pos}}(X) \cap C^{\perp} = G$. This is our situation: if $\beta \in C^{<0} \cap \overline{\mathrm{Pos}}(X)$, then we have

$$\gamma = t\beta + (1-t)C, \quad \gamma \in G, t > 1;$$

this gives

$$t\beta = \gamma + (t-1)C \quad \Rightarrow \quad \beta = \frac{1}{t}\gamma + \frac{t-1}{t}C \in G + R(C),$$

and this concludes the proof of the (3.45).

Now, since $\alpha \in C^{\leqslant 0} \cap \overline{\mathrm{Pos}}(X)$, using (3.45) and (3.44), we get

$$\alpha \in C^{\leqslant 0} \cap \overline{\mathrm{Pos}}(X) \subseteq G + R(C) \subseteq (K - sL)^{\leqslant 0},$$

but that is a contradiction since $\alpha \in (K - sL)^{>0}$.

□

Claim 3.16. $\mathrm{Nef}(X)_{(K-sL)^{\geqslant 0}} = \overline{\mathrm{Pos}}(X)_{(K-sL)^{\geqslant 0}}$.

Proof of the Claim. We have, by Claim 3.15, that

$$\partial\,\mathrm{Nef}(X) \cap (K - sL)^{>0} = \partial\overline{\mathrm{Pos}}(X) \cap (K - sL)^{>0}. \tag{3.49}$$

At first we show the following:

$$\partial\,\mathrm{Nef}(X) \cap (K - sL)^{\geqslant 0} = \partial\overline{\mathrm{Pos}}(X) \cap (K - sL)^{\geqslant 0}. \tag{3.50}$$

Indeed, from equation (3.49), taking the closure, we get

$$\mathrm{cl}(\partial\,\mathrm{Nef}(X) \cap (K - sL)^{>0}) = \mathrm{cl}(\partial\overline{\mathrm{Pos}}(X) \cap (K - sL)^{>0});$$

now, since $\mathrm{int}(\partial\overline{\mathrm{Pos}}(X)) = \emptyset$, by Lemma 1.58 (Leibniz formula for closed sets), we get

$$\begin{aligned}
&\mathrm{cl}(\partial\overline{\mathrm{Pos}}(X) \cap (K - sL)^{>0}) = \\
&\mathrm{int}(\partial\overline{\mathrm{Pos}}(X) \cap (K - sL)^{>0}) \cup \partial(\partial\overline{\mathrm{Pos}}(X) \cap (K - sL)^{>0}) = \\
&\emptyset \cup \partial(\partial\overline{\mathrm{Pos}}(X) \cap (K - sL)^{>0}) = \\
&(\partial\overline{\mathrm{Pos}}(X) \cap (K - sL)^{>0}) \cup (\partial\overline{\mathrm{Pos}}(X) \cap (K - sL)^{\perp}) = \\
&\partial\overline{\mathrm{Pos}}(X) \cap (K - sL)^{\geqslant 0}.
\end{aligned}$$

Now we have

$$\begin{aligned}
&\partial\overline{\mathrm{Pos}}(X) \cap (K - sL)^{\geqslant 0} = \mathrm{cl}(\partial\,\mathrm{Nef}(X) \cap (K - sL)^{>0}) \subseteq \\
&\mathrm{cl}(\partial\,\mathrm{Nef}(X)) \cap \mathrm{cl}((K - sL)^{>0}) = \partial\,\mathrm{Nef}(X) \cap (K - sL)^{\geqslant 0}.
\end{aligned}$$

Thus we have proved one of the two inclusions in (3.50), let us show the other.

Let us take $x \in \partial\,\mathrm{Nef}(X) \cap (K - sL)^{\geqslant 0}$. If it is in $(K - sL)^{>0}$, then it is in $\partial\overline{\mathrm{Pos}}(X)$ by Claim 3.15. Hence we can suppose $x \in (K - sL)^{\perp}$ and, by contradiction, $x \in \mathrm{Pos}(X)$; by the result of Campana and Peternell (see (3.43)), we have therefore that $x \in C^{\perp}$ for some C with $C^2 < 0$ and thus

$$x \in C^{\perp} \cap (K - sL)^{\perp} \cap \mathrm{Pos}(X).$$

Two different cases may arise. Let us suppose that $C^\perp = (K-sL)^\perp$; since C and $(K-sL)$ do determine the same orthogonal hyperplane, they must be parallel, but since $C \cdot h > 0$ and $(K-sL) \cdot h < 0$, then there exists $a > 0$ such that $aC = -(K-sL)$, which gives, by (3.15) and (3.28),

$$0 < -(K-sL) \cdot C = aC^2 < 0,$$

that is a contradiction and hence this case does not occur.
If $C^\perp \neq (K-sL)^\perp$, since the origin and x lie in both of them, they are not parallel and thus they intersects in a linear subspace of dimension $\rho(X) - 2$. Now, since

$$x \in C^\perp \cap (K-sL)^\perp \cap \operatorname{Pos}(X),$$

by dimension reasons, there will be an y' in $C^\perp \cap \operatorname{Pos}(X) \cap (K-sL)^{>0}$. Indeed since $C^\perp \neq (K-sL)^\perp$ we can fix $y \in C^\perp \cap \operatorname{Pos}(X)$ such that $y \notin (K-sL)^\perp$; if $y \in (K-sL)^{>0}$ then we get a contradiction with Claim 3.14; if $y \in (K-sL)^{<0}$, then if we consider the point $y' \in L(y,x)$:

$$y' = tx + (1-t)y \quad \text{with } t = 1+\varepsilon \quad \text{for } 0 < \varepsilon \ll 1,$$

we get $y' \in C^\perp \cap \operatorname{Pos}(X) \cap (K-sL)^{>0}$ that is a contradiction with Claim 3.14.
Thus we have the (3.50) and, by subtracting equation (3.49), we immediately see that

$$\partial \operatorname{Nef}(X) \cap (K-sL)^\perp = \partial \overline{\operatorname{Pos}}(X) \cap (K-sL)^\perp. \tag{3.51}$$

Now we claim that

$$\operatorname{Nef}(X) \cap (K-sL)^\perp = \overline{\operatorname{Pos}}(X) \cap (K-sL)^\perp. \tag{3.52}$$

One of the two inclusion is obvious. To prove the other, let us take $x \in \overline{\operatorname{Pos}}(X) \cap (K-sL)^\perp$; if $x \in \partial\overline{\operatorname{Pos}}(X)$, then by equation (3.51), we are done; if otherwise $x \in \operatorname{Pos}(X) \cap (K-sL)^\perp \subset (K-sL)^\perp$, by Lemma 1.22 it is in the convex hull of its boundary as a closed cone in $(K-sL)^\perp$ and we can write

$$x = \sum \gamma_i, \quad \gamma_i \in \partial_{(K-sL)^\perp}(\overline{\operatorname{Pos}}(X) \cap (K-sL)^\perp) \subseteq \partial\overline{\operatorname{Pos}}(X) \cap (K-sL)^\perp,$$

where the last inclusion is a consequence of Lemma 1.59.
Now equation (3.51) allows us to write

$$x = \sum \gamma_i, \quad \gamma_i \in \partial \operatorname{Nef}(X) \cap (K-sL)^\perp,$$

then $x \in \operatorname{Nef}(X) \cap (K-sL)^\perp$ and equation (3.52) is proved.
In our situation we can use Lemma 1.58 and thus, from equations (3.50) and (3.52), we get:

$$\begin{aligned}
\partial(\operatorname{Nef}(X) \cap (K-sL)^{\geqslant 0}) &= (\partial \operatorname{Nef}(X) \cap (K-sL)^{\geqslant 0}) \cup (\operatorname{Nef}(X) \cap (K-sL)^\perp) \\
&= (\partial\overline{\operatorname{Pos}}(X) \cap (K-sL)^{\geqslant 0}) \cup (\overline{\operatorname{Pos}}(X) \cap (K-sL)^\perp) \\
&= \partial(\overline{\operatorname{Pos}}(X) \cap (K-sL)^{\geqslant 0}).
\end{aligned}$$

Since we have two closed and convex cones not containing lines with the same boundary, again by Lemma 1.22, their convex hull is the same and the claim is proved. □

We are now getting closer to the conclusion: our goal is a sort of *dual statement* of Claim 3.16.
At first let us prove that

$$\overline{\mathrm{NE}}(X) \cap (K - sL)^{\perp} = \overline{\mathrm{Pos}}(X) \cap (K - sL)^{\perp}. \tag{3.53}$$

Since $\overline{\mathrm{Pos}}(X) \subseteq \overline{\mathrm{NE}}(X)$, one of the two inclusion is obvious.
For the other inclusion, let us suppose, by contradiction that there exists $\gamma \in \overline{\mathrm{NE}}(X) \cap (K - sL)^{\perp}$ with $\gamma^2 < 0$.
If we consider the rays outgoing from γ and tangent to $\overline{\mathrm{Pos}}(X)$, we see that, since $(K - sL)^2 < 0$ (see the proof of Proposition 3.8), by Lemma 1.52 and Remark 1.53, there are rays in both $(K - sL)^{<0}$ and $(K - sL)^{>0}$ side. If indeed they were all in the $\geqslant 0$ side (or in the $\leqslant 0$ side), then $(K - sL)^{\perp}$ would be tangent to $\overline{\mathrm{Pos}}(X)$ and this is not the case.
Thus we can fix two tangent rays intersecting $\partial\overline{\mathrm{Pos}}(X)$ in α and β such that:

$$\alpha, \beta \in \gamma^{\perp}; \quad \alpha^2 = \beta^2 = 0; \quad \alpha \in (K - sL)^{>0}; \quad \beta \in (K - sL)^{<0}. \tag{3.54}$$

We point out that since $\alpha \in (K - sL)^{>0}$ and $\beta \in (K - sL)^{<0}$, then α and β are not proportional and thus the segment $[\alpha, \beta]$ can't be contained in $\partial\overline{\mathrm{Pos}}(X)$ and therefore the open segment (α, β) does lie in $\mathrm{Pos}(X)$ (see the proof of Fact 1.54).
If we intersect the segment (α, β) with $(K - sL)^{\perp}$, we get

$$0 = (t\alpha + (1 - t)\beta) \cdot (K - sL),$$

which gives

$$\bar{t} = \frac{-\beta \cdot (K - sL)}{\alpha \cdot (K - sL) - \beta \cdot (K - sL)} \in (0, 1),$$

and $\bar{t}$ produces $y \in (\alpha, \beta)$ with $y \in \mathrm{Pos}(X)$. Since $\alpha, \beta \in \gamma^{\perp}$, we get at once:

$$y \in \gamma^{\perp} \cap (K - sL)^{\perp} \cap \mathrm{Pos}(X). \tag{3.55}$$

Since y is in the interior of $\overline{\mathrm{Pos}}(X)$ and γ in the exterior, then immediately we find an $x \in (y, \gamma)$ such that $x \in \partial\overline{\mathrm{Pos}}(X)$, that is $x^2 = 0$.
We immediately see that

$$x \cdot (K - sL) = ty \cdot (K - sL) + (1 - t)\gamma \cdot (K - sL) = 0,$$

and hence $x \in \overline{\mathrm{Pos}}(X)_{(K-sL)^{\geqslant 0}}$; but if we compute

$$x \cdot \gamma = ty \cdot \gamma + (1 - t)\gamma^2 < 0,$$

we see that, since $\gamma \in \overline{\mathrm{NE}}(X)$, then x can't be a nef class and this is a contradiction with Claim 3.16.
We want now to prove that:

$$\overline{\mathrm{NE}}(X)_{(K-sL)^{\geqslant 0}} = \overline{\mathrm{Pos}}(X)_{(K-sL)^{\geqslant 0}}.$$

Since $\overline{\mathrm{Pos}}(X) \subseteq \overline{\mathrm{NE}}(X)$ we have that one of the two inclusions is obvious.
In order to prove the other, suppose, by contradiction, that there exists $x \in \overline{\mathrm{NE}}(X) \cap (K - sL)^{\geqslant 0}$ such that $x \notin \overline{\mathrm{Pos}}(X)$.
We claim that it is possible to assume $x = [C]$ for some integral curve with $C^2 < 0$.

Indeed, since $x \in \overline{\mathrm{NE}}(X) \cap (K - sL)^{\geqslant 0}$, we can write (see Lemma 1.22):

$$x = \sum_{i=1}^{s} \gamma_i, \quad \gamma_i \in R_i,$$

where R_i are extremal rays of $\overline{\mathrm{NE}}(X) \cap (K - sL)^{\geqslant 0}$.
We have, by Lemma 1.58, that

$$\begin{aligned} R_i &\subset (\partial\overline{\mathrm{NE}}(X) \cap (K - sL)^{\geqslant 0}) \cup (\overline{\mathrm{NE}}(X) \cap (K - sL)^{\perp}) \\ &= (\partial\overline{\mathrm{NE}}(X) \cap (K - sL)^{\geqslant 0}) \cup (\overline{\mathrm{Pos}}(X) \cap (K - sL)^{\perp}) \end{aligned}.$$

Now, since $x^2 < 0$ there must exist at least one of the $R_i \subset (\partial\overline{\mathrm{NE}}(X) \cap (K - sL)^{\geqslant 0})$ with $\gamma_i^2 < 0$ (if it were $\gamma_i^2 \geqslant 0$ for all i, then $x^2 \geqslant 0$).
In particular we have that this R_i is indeed an extremal ray of $\overline{\mathrm{NE}}(X)$, since if it was not, then R_i should be in $(\overline{\mathrm{NE}}(X) \cap (K - sL)^{\perp})$ and this can't be (see equation (3.53) and Fact 1.20).
Thus we have that there exists an extremal ray R_i generated by γ_i with $\gamma_i^2 < 0$ and hence, by Lemma 1.37, we can suppose that R_i is spanned by the class of an integral curve C.
Therefore we have:

$$C \cdot (K - sL) \geqslant 0 \quad \text{and} \quad C^2 < 0. \tag{3.56}$$

Now, as in Claim 3.14, setting $G = C^{\perp} \cap \mathrm{Pos}(X)$, we get

$$G \subset (K - sL)^{\leqslant 0}.$$

We point out that $G \neq \emptyset$ set: see Remark 1.53.
Thus we can fix a $\gamma \in G$; let us note that $\gamma \neq C$ because $\gamma^2 \geqslant 0$. Since $\gamma \cdot (K - sL) \leqslant 0$ and $C \cdot (K - sL) \geqslant 0$, the segment joining C to γ does intersect $(K - sL)^{\perp}$: the line

$$L(C, \gamma) : \quad \lambda(t) = t\gamma + (1 - t)C, \quad t \in \mathbb{R},$$

intersects $(K - sL)^{\perp}$ in $\bar{\lambda} = \lambda(\bar{t})$ for some $0 < \bar{t} \leqslant 1$; we have, indeed $\bar{t} > 0$: if it were $\bar{t} = 0$, then $\lambda(0) = C$ and it would be a contradiction with (3.53).
We see that $\bar{\lambda} \cdot C \leqslant 0$, indeed

$$\bar{\lambda} \cdot C = \bar{t} \underbrace{\gamma \cdot C}_{=0} + \underbrace{(1 - \bar{t})C^2}_{\leqslant 0} \leqslant 0.$$

Furthermore, we claim that this $\bar{\lambda} \in \mathrm{Pos}(X)$; if indeed it were $\bar{\lambda}^2 \leqslant 0$, then

$$\bar{t}\gamma = \bar{\lambda} + (\bar{t} - 1)C,$$

and hence

$$\bar{t}^2\gamma^2 = \bar{\lambda}^2 + (\bar{t} - 1)^2 C^2 + 2(\bar{t} - 1)\bar{\lambda} \cdot C \leqslant 0,$$

but this is a contradiction since $\gamma^2 > 0$ and $\bar{t} \neq 0$.
Let us set $\lambda_{\varepsilon} = \lambda(\bar{t} - \varepsilon)$, for some $0 < \varepsilon \ll 1$; we claim that $\lambda_{\varepsilon} \in (K - sL)^{\geqslant 0}$, indeed:

$$\begin{aligned} \lambda_{\varepsilon} \cdot (K - sL) &= [(\bar{t} - \varepsilon)\gamma + (1 - \bar{t} + \varepsilon)C] \cdot (K - sL) \\ &= [\bar{t}\gamma + (1 - \bar{t})C] \cdot (K - sL) + (-\varepsilon\gamma + \varepsilon C) \cdot (K - sL) \\ &= 0 + \varepsilon[C \cdot (K - sL) - \gamma \cdot (K - sL)] \geqslant 0. \end{aligned}$$

Now, since ε is small, we have that

$$\lambda_\varepsilon \in \overline{\mathrm{Pos}}(X) \cap (K - sL)^{\geqslant 0} = \mathrm{Nef}(X) \cap (K - sL)^{\geqslant 0};$$

in particular λ_ε is nef; on the other side, we immediately get

$$\begin{aligned} C \cdot \lambda_\varepsilon &= [(\bar{t} - \varepsilon)\gamma + (1 - \bar{t} + \varepsilon)C] \cdot C \\ &= (\bar{t} - \varepsilon)\underbrace{\gamma \cdot C}_{=0} + \underbrace{(1 - \bar{t} + \varepsilon)}_{>0}\underbrace{C^2}_{<0} < 0, \end{aligned}$$

that is a contradiction. Hence the $(K - sL)^{\geqslant 0}$-part of the $\overline{\mathrm{NE}}(X)$ cone must coincide with $\overline{\mathrm{Pos}}(X)_{(K-sL)^{\geqslant 0}}$ and the theorem is proved. □

3.5 Strict inclusion conditions

We have now seen that, assuming some conjectures, if a sufficiently large number of points are blown up, then the Mori cone $\overline{\mathrm{NE}}(X)$ does coincide with the positive cone in the $(K - sL)^{\geqslant 0}$ part. Our goal is now to show that, independently from any conjecture, the restriction of the positive cone to $K^{\geqslant 0}$ can't coincide with the restriction of $\overline{\mathrm{NE}}(X)$.

Proposition 3.17. *Let $X = \mathrm{Bl}_r Y$ be the blow up at r general points of a smooth projective surface Y and A be an ample divisor. Let us suppose one of the following holds true.*

$$(A) \quad \begin{cases} r \leqslant {K_Y}^2 + 1 - \frac{(A \cdot K_Y)^2}{A^2} \\ A \cdot K_Y > 0 \\ A^2 < (A \cdot K_Y)^2; \end{cases} \qquad (B) \quad \begin{cases} r > {K_Y}^2 + 1 - \frac{(A \cdot K_Y)^2}{A^2} \\ r \leqslant {K_Y}^2 + 1 \\ A \cdot K_Y > 0 \end{cases}$$

$$(C) \quad \begin{cases} r > 0 \\ {K_Y}^2 < 0; \end{cases} \qquad (D) \quad \begin{cases} r > {K_Y}^2 + 1 \\ {K_Y}^2 \geqslant 0; \end{cases}$$

Then, for a fixed (-1)-curve C, there exists $\alpha \in \overline{\mathrm{Pos}}(X)$ such that

$$\begin{cases} \alpha^2 = 0 \\ \alpha \cdot h \geqslant 0 \\ \alpha \cdot C \leqslant 0 \\ \alpha \cdot K > 0. \end{cases} \tag{3.57}$$

Moreover, we get:

$$\overline{\mathrm{Pos}}(X)_{K^{\geqslant 0}} \subsetneq \overline{\mathrm{NE}}(X)_{K^{\geqslant 0}}. \tag{3.58}$$

Proof. Let us fix on the blown-up surface X an exceptional curve $C = E_i$ for some i, generating a $(-1, 0)$-ray $R = R(C)$.
At first we prove the last implication: if equations (3.57) hold true, then we have the strict inclusion of (3.58). Let us set

$$\gamma = C + \lambda\alpha, \quad \lambda \gg 1.$$

Since $\alpha \in \overline{\mathrm{Pos}}(X) \subseteq \overline{\mathrm{NE}}(X)$, then $\gamma \in \overline{\mathrm{NE}}(X)$; on the other side

$$\gamma^2 = (C + \lambda\alpha)^2 = C^2 + 2\lambda C \cdot \alpha < 0,$$

which gives $\gamma \notin \overline{\mathrm{Pos}}(X)$. Now, since $\lambda \gg 1$ and $\alpha \cdot K > 0$, we also get

$$(C + \lambda\alpha) \cdot K = -1 + \lambda\alpha \cdot K > 0.$$

Hence $\gamma \in \overline{\mathrm{NE}}(X)_{K>0}$ and $\gamma \notin \overline{\mathrm{Pos}}(X)_{K>0}$.
We now look for an α of this kind:

$$\alpha = tC - (K - sL), \quad \text{with } t, s \in \mathbb{R}, \tag{3.59}$$

and show the existence of t, s in order to fulfill conditions (3.57). First of all, we need

$$\alpha^2 = (tC - (K - sL))^2 = 0. \tag{3.60}$$

To ensure the existence of solutions for t of (3.60), we require

$$\Delta_t := (C \cdot (K - sL))^2 + (K - sL)^2 \geqslant 0, \tag{3.61}$$

that, by adjunction and by Fact 3.6, becomes

$$s^2A^2 - 2sA \cdot K_Y + {K_Y}^2 + 1 - r \geqslant 0. \tag{3.62}$$

We have now two different cases according to the sign of the discriminant of the former inequality:

$$\Delta_s := (A \cdot K_Y)^2 - A^2({K_Y}^2 + 1 - r). \tag{3.63}$$

Thus we have

$$\text{Case } \Delta_s \geqslant 0: \quad \begin{cases} r \geqslant {K_Y}^2 + 1 - \frac{(A\cdot K_Y)^2}{A^2} \\ s \leqslant \frac{A\cdot K_Y - \sqrt{\Delta_s}}{A^2} \quad \vee \quad s \geqslant \frac{A\cdot K_Y + \sqrt{\Delta_s}}{A^2}. \end{cases} \tag{3.64}$$

$$\text{Case } \Delta_s < 0: \quad \begin{cases} r < {K_Y}^2 + 1 - \frac{(A\cdot K_Y)^2}{A^2} \\ \forall s \in \mathbb{R}. \end{cases} \tag{3.65}$$

With this conditions on s, $\Delta_t \geqslant 0$ and, among the solutions of (3.60), we pick

$$t = 1 + \sqrt{\Delta_t}. \tag{3.66}$$

We now impose $\alpha \cdot h \geqslant 0$, for an ample class $h = L - \sum \delta_j E_j$. An easy computation, since $0 < \delta_j \ll 1$, shows that

$$\alpha \cdot h = [tE_i - (K - sL)] \cdot [L - \sum \delta_j E_j] = t\delta_i - A \cdot K_Y + sA^2 - \sum \delta_j \geqslant 0,$$

if and only if $(sA^2 - A \cdot K_Y) > 0$, that is

$$s > \frac{A \cdot K_Y}{A^2}. \tag{3.67}$$

Now, the case $\Delta_s = 0$ in (3.64) can be associated to equation (3.65) and these two conditions, together with (3.67), become

$$\begin{cases} r \leqslant {K_Y}^2 + 1 - \frac{(A\cdot K_Y)^2}{A^2} \\ s > \frac{A\cdot K_Y}{A^2}, \end{cases} \tag{3.68}$$

and

$$\begin{cases} r > {K_Y}^2 + 1 - \frac{(A \cdot K_Y)^2}{A^2} \\ s \geqslant \frac{A \cdot K_Y + \sqrt{\Delta_s}}{A^2}. \end{cases} \tag{3.69}$$

We have now found conditions ensuring $\alpha \in \overline{\mathrm{Pos}}(X)$; let us check that $\alpha \cdot C \leqslant 0$.

$$\alpha \cdot C = tC^2 - (K - sL) \cdot C = -t + 1,$$

that is not positive since we set $t = 1 + \sqrt{\Delta_t} \geqslant 1$. To prove (3.57) it is left to deal with $\alpha \cdot K$.

$$\begin{aligned} \alpha \cdot K &= (tC - (K - sL)) \cdot (\varphi^* K_Y + \sum E_j) \\ &= -t - {K_Y}^2 + r + sA \cdot K_Y. \end{aligned}$$

Hence the condition to impose is

$$r - {K_Y}^2 - 1 + sA \cdot K_Y > \sqrt{\Delta_t}. \tag{3.70}$$

At the end, we get two different systems of inequalities for s:

$$\begin{cases} r \leqslant {K_Y}^2 + 1 - \frac{(A \cdot K_Y)^2}{A^2} \\ s > \frac{A \cdot K_Y}{A^2} \\ r - {K_Y}^2 - 1 + sA \cdot K_Y > \sqrt{\Delta_t}, \end{cases} \tag{3.71}$$

and

$$\begin{cases} r > {K_Y}^2 + 1 - \frac{(A \cdot K_Y)^2}{A^2} \\ s \geqslant \frac{A \cdot K_Y + \sqrt{\Delta_s}}{A^2} \\ r - {K_Y}^2 - 1 + sA \cdot K_Y > \sqrt{\Delta_t}. \end{cases} \tag{3.72}$$

The hypothesis in the statement of the proposition are exactly the conditions ensuring the existence of solutions for s in (3.71) and (3.72). □

Remark 3.18. To solve (3.71) and (3.72), we used the computational system Wolfram Alpha (`http://www.wolframalpha.com/`). Setting

$$\begin{cases} x = A \cdot K_Y \\ y = A^2 \\ z = {K_Y}^2 + 1, \end{cases}$$

the solutions of (3.71) are given by the string:

```
Reduce[{r <= -(x^2/y) + z, r > 0, y > 0, s > x/y,
r + s x - z > Sqrt[-r - 2 s x + s^2 y + z]}, s]
```

and the solutions of (3.72) by

```
Reduce[{r > -(x^2/y) + z, r > 0, y > 0,
s >= (x + Sqrt[x^2 - y (-r + z)])/y,
r + s x - z > Sqrt[-r - 2 s x + s^2 y + z]}, s]
```

An easy refinement of the computed solution gives the result.

We can now give a similar statement in the case of an interesting geometrical hypothesis.

Proposition 3.19. *Let $X = \mathrm{Bl}_r Y$ be the blow up at $r \geqslant 2$ general points of a projective surface Y; let us suppose that for an ample divisor A on Y the inequality*

$$A \cdot K_Y + \sqrt{A^2(r-1)} > 0 \tag{3.73}$$

holds true. Then

$$\overline{\mathrm{Pos}}(X)_{K\geqslant 0} \subsetneq \overline{\mathrm{NE}}(X)_{K\geqslant 0}. \tag{3.74}$$

In particular this is true if Y is a non uniruled surface.

Proof. In light of Proposition 3.17, we just have to show, for a fixed (-1)-curve, the existence of an α such that the conditions (3.57) are satisfied.
To fulfill conditions (3.57), we can fix $C = E_r$, the last exceptional curve on X, and we can consider an ample divisor A on Y. We look for an α in the form

$$\alpha = \varphi^* A + \sum_{i=1}^{r} a_i E_i \quad \text{with } a_i \in \mathbb{R}.$$

We impose $\alpha \cdot C = 0$, which gives $a_r = 0$ and hence we can write

$$\alpha = \varphi^* A + \sum_{i=1}^{r-1} a_i E_i.$$

The $\alpha^2 = 0$ condition gives

$$\alpha^2 = A^2 - \sum_{i=1}^{r-1} {a_i}^2 = 0 \quad \Rightarrow \quad A^2 = \sum_{i=1}^{r-1} {a_i}^2. \tag{3.75}$$

We see that in order to verify the last equation, we need $r \geqslant 2$.
Let us consider

$$a_i = -\sqrt{\frac{A^2}{r-1}}, \quad \text{for } i = 1, \dots, r-1; \quad a_r = 0.$$

We get:

$$\alpha^2 = A^2 - \sum_{i=1}^{r-1} \frac{A^2}{r-1} = A^2 - (r-1)\frac{A^2}{r-1} = 0.$$

To show that this $\alpha \in \overline{\mathrm{Pos}}(X)$ we still have to show that $\alpha \cdot h \geqslant 0$ for an appropriate ample class h: for some $0 < \delta_i \ll 0$ we can compute

$$\begin{aligned}\alpha \cdot h &= \left(\varphi^* A - \sum_{i=1}^{r-1} \sqrt{\frac{A^2}{r-1}} E_i\right) \cdot \left(\varphi^* A - \sum_{i=1}^{r} \delta_i E_i\right)\\ &= A^2 - \sum_{i=1}^{r-1} \sqrt{\frac{A^2}{r-1}} \delta_i,\end{aligned}$$

that is positive for small δ_i, since $A^2 > 0$.

Let us compute $\alpha \cdot K$:

$$
\begin{aligned}
\alpha \cdot K &= \left(\varphi^* A + \sum_{i=1}^{r-1} a_i E_i\right) \cdot \left(\varphi^* K_Y + \sum_{i=1}^{r} E_i\right) \\
&= \varphi^* A \cdot \varphi^* K_Y - \sum_{i=1}^{r-1} a_i \\
&= A \cdot K_Y - \sum_{i=1}^{r-1} \left(-\sqrt{\frac{A^2}{r-1}}\right),
\end{aligned}
\tag{3.76}
$$

that is positive from conditions (3.73).
In the non uniruled case, we have in particular that K_Y is a pseudoeffective divisor, hence $A \cdot K_Y \geqslant 0$ and condition (3.73) is immediately satisfied. □

Remark 3.20. We have that in the case $Y = \mathbb{P}^2$, Proposition 3.17 and Proposition 3.19 give the same result. Thus if we blow up $r > 10$ points, then

$$\overline{\mathrm{Pos}}(X)_{K \geqslant 0} \subsetneq \overline{\mathrm{NE}}(X)_{K \geqslant 0}$$

and we have recovered the same results of [dF10].

Part II

Weak Zariski decomposition on projectivized vector bundles

Chapter 4

The problem

The problem we want to deal with in this part of the thesis is the existence of some kind of Zariski decomposition for a smooth projective variety X; as it is well-known, it is a central problem in Algebraic Geometry, following [Bir09], we can say that it all began with Zariski (see [Zar62]) and was then refined by Fujita (see [Fuj79]).

Theorem 4.1 (Zariski decomposition)**.** *Let D be a pseudoeffective $\mathbb{R}$-divisor on a smooth surface X; then there exist two $\mathbb{R}$-divisors P and N, such that:*

1. *$D = P + N$;*
2. *P is nef and N is effective;*
3. *$N = 0$ or the intersection matrix of the components $\{C_i\}$ of N is negative definite;*
4. *$P \cdot C_i = 0$ for any i.*

In literature we can find a number of attempts to generalize this kind of decomposition to higher dimensional varieties, for example Fujita-Zariski decomposition and the so called CKM-Zariski decomposition (after Cutkosty, Kawamata and Moriwaki).

4.1 Statement and reduction

Now, we want to deal with the weakest of this kind of decomposition; we refer to [Bir09] for account of its relationship with the minimal model conjecture.

Definition 4.2 (Weak Zariski decomposition)**.** Let $D \in \mathrm{Div}_{\mathbb{R}}(X)$ be an $\mathbb{R}$-divisor on a normal variety X; we say that D has a *weak Zariski decomposition (WZD)* if there exists a projective birational morphism $f : W \to X$ from a normal variety W, such that

1. $f^*D = P + N$, where P, N are $\mathbb{R}$-divisors;
2. P is nef and N is effective.

Remark 4.3. We will see in Section 5.1 a concrete case where the birational map has to be considered in order to ensure the existence of this decomposition. Since in many meaningful situations this map is not required, we can also introduce a stronger form of decomposition.

Definition 4.4 (Direct Weak Zariski decomposition)**.** In the same setting of Definition 4.2, we say that D has a *direct weak Zariski decomposition (DWZD)* if it has a WZD with $f = \mathrm{id}$, that is, without the birational modification.

The natural question to ask is thus the following.

Question 4.5. Let X be a normal variety. Does a (direct) weak Zariski decomposition exist for every pseudoeffective divisor?

It is worth to point out that the existence of such a decomposition for any pseudoeffective divisor is strictly related to the existence of the decomposition for extremal rays of $\overline{\mathrm{Eff}}(X) \subset N^1(X)$.

Fact 4.6. *Question 4.5 has a positive answer if for every extremal ray $R_i = R(D_i)$ of the pseudoeffective cone $\overline{\mathrm{Eff}}(X)$ there exists a birational map $f_i : W_i \to X$ such that the class of $f_i^* D_i$ is nef or effective.*

Proof. By Fact 1.36 we have that the cone $\overline{\mathrm{Eff}}(X)$ does not contain lines through the origin; thus it is the convex hull of its extremal rays. Thus for a fixed divisor D we can find finitely many pseudoeffective classes $[D_1], \ldots, [D_s]$, generating extremal rays such that $[D] = \sum_{i=1}^{s} [D_i]$.
Now we know by hypothesis that for each of these divisors D_i there exists $f_i : W_i \to X$ with $[f_i^* D_i]$ nef or effective; since the varieties W_i are all in the same birational class, we can consider the common resolution W of all the maps $W_i \dashrightarrow W_j$:

$$\begin{tikzcd}
 & & W \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
W_1 \arrow[drr, "f_1"'] & \cdots \quad W_i \arrow[r, dashed] \arrow[dr, "f_i"] & & W_j \arrow[dl, "f_j"'] \quad \cdots & W_s \arrow[dll, "f_s"] \\
 & & X & &
\end{tikzcd} \tag{4.1}$$

In W, since the pull-back of a nef or effective class is still a nef or effective class, we have that the pull-back of each D_i is a nef or an effective class.
Now, setting $f : W \to X$, without loss of generality we can suppose, for some $t \in \{0, \ldots, s\}$, that $[f^* D_i]$ is nef for $i = 1, \ldots, t$ and that $[f^* D_i]$ is effective for $i = t+1, \ldots, s$ and thus we can write

$$f^* D = f^* \left(\sum D_i \right) = \sum_{i=1}^{t} f^* D_i + \sum_{i=t+1}^{s} f^* D_i, \tag{4.2}$$

that gives a weak Zariski decomposition of D. □

Remark 4.7. In the direct case we have that the existence of a direct weak Zariski decomposition for any pseudoeffective divisor in X is indeed equivalent to the effectiveness or nefness of any generator of an extremal ray.

We have also the following characterization of Question 4.5.

Fact 4.8. *Question 4.5 has a positive answer if and only if there is a weak Zariski decoposition for every extremal ray of the pseudoeffective cone.*

Proof. If there is a positive answer, then a WZD does exist in particular for extremal rays. Viceversa, let us consider a pseudoeffective divisor D whose class lies in $\overline{\mathrm{Eff}}(X)$; then, by Lemma 1.22 we can write $D = \sum_{i=1}^{s} D_i$, where D_i is a generator of an extremal ray; by hypothesis we have that there exists a birational map such that $f_i^* D_i = P_i + N_i$, for $i = 1, \dots, s$ where P_i gives a nef class and N_i an effective class. Now considering a common resolution $f : W \to X$ as in (4.1), we get

$$f^*(D) = \sum_{i=1}^{s} f^*(D_i) = \sum_{i=1}^{s} P_i + \sum_{i=1}^{s} N_i,$$

that is a weak Zariski decomposition of D. □

Thus we have reduced the problem in understanding the generators of extremal rays in the pseudoeffective cone $\overline{\mathrm{Eff}}(X)$.
We want now to introduce another important cone; as we will soon see it lies between the nef and the pseudoeffective cone.

Definition 4.9. Let X be a smooth projective variety; the *movable cone* $\overline{\mathrm{Mov}}(X) \subseteq N^1(X)$ of *movable divisors* is the closed convex cone spanned by classes of divisors without fixed component (that is such that the base locus has codimension at least 2).

An important decomposition can be produced for pseudoeffective divisors in terms of movable divisors; we refer to [Nak04, Chapter III] for further details. First of all we have the following chain of inclusions.

Fact 4.10. *Let X be a smooth projective variety, then we have the following inclusions:*

$$\mathrm{Nef}(X) \subseteq \overline{\mathrm{Mov}}(X) \subseteq \overline{\mathrm{Eff}}(X). \tag{4.3}$$

Nakayama proves that is it indeed possible to write a pseudoeffective divisor as a sum of an effective and a movable divisor.

Theorem 4.11. *Let D be a pseudoeffective $\mathbb{R}$-divisor over a smooth projective variety X, then there exists the so called σ-decomposition:*

$$D = P_\sigma(D) + N_\sigma(D), \tag{4.4}$$

where $P_\sigma(D)$ lies in $\overline{\mathrm{Mov}}(X)$ and $N_\sigma(D)$ is effective.

In this paper we try to give an answer to Question 4.5 in the particular case of projectivized vector bundle.

Question 4.12. Let us consider a rank $r \geqslant 2$ vector bundle $\mathcal{E}$ on a smooth projective variety Z and let

$$X = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} Z$$

be the projectivized vector bundle over Z with the bundle map π.
Does a (D)WZD exists for any pseudoeffective divisor on X?

Remark 4.13. The existence of a Zariski decomposition for projectivized vector bundles on a curve has been already investigated by Nakayama; in particular in [Nak04, Section IV.3], the author proves that every pseudoeffective $\mathbb{R}$-divisor on a projectivized vector bundle on a non singular curve has a Zariski decomposition.
In the following, see Theorem 5.17, we will prove a weaker result using different and simpler techniques. In particular the proof we present is independent from the argument of Nakayama and it is based on some ideas by Fulger, see [Flg11].

4.2 On projectivized vector bundles

Let us fix the notations about projectivised vector bundles.

Definition 4.14. If V is a vector space, then we define

$$\mathbb{P}(V) := \mathbf{P}(V^*) = (V^* \setminus \{0\})/\sim, \tag{4.5}$$

where $\sim$ is the equivalence given by parallelism and $\mathbf{P}$ denotes the usual projectivization of vector spaces.
If Z is a projective algebraic variety and $\mathcal{E}$ is a rank $r \geqslant 2$ vector bundle on Z, we define the *projectivization* of $\mathcal{E}$ as

$$\mathbb{P}(\mathcal{E}) := \mathbf{P}(\mathcal{E}^*). \tag{4.6}$$

Remark 4.15. We want now to give a description of the projectivization of a vector bundle $\mathcal{E}$ on Z; an element $x \in \mathbb{P}(\mathcal{E})$ is given by a pair

$$x = (z_x, [\varphi_x]), \tag{4.7}$$

where $z_x \in Z$ and $[\varphi_x] \in \mathbb{P}(\mathcal{E}_{z_x}) = \mathbf{P}(\mathcal{E}^*_{z_x})$ is the equivalence class of

$$\varphi_x \in \mathcal{E}^*_{z_x} \setminus \{0\}.$$

If we consider the one dimensional vector space generated by φ_x, we get

$$\langle \varphi_x \rangle \subseteq \mathcal{E}^*_{z_x}$$

that, by duality, gives the surjection

$$\mathcal{E}_{z_x} \to \langle \varphi_x \rangle^* \to 0. \tag{4.8}$$

Thus a point of $\mathbb{P}(\mathcal{E})$ is essentially given by a point $z \in Z$ and a line bundle quotient of the fibre $\mathcal{E}_z$ of $\mathcal{E}$ at z.
Moreover we have the natural projection map:

$$\begin{array}{rccc} \pi : & \mathbb{P}(\mathcal{E}) & \longrightarrow & Z \\ & x & \longmapsto & z_x. \end{array} \tag{4.9}$$

Remark 4.16. The tautological line bundle on $\mathbb{P}(\mathcal{E})$ is the line bundle defined, on each point $x \in \mathbb{P}(\mathcal{E})$, by

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)_x := \langle \varphi_x \rangle^*. \tag{4.10}$$

This line bundle defines the surjection

$$\pi^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \to 0; \tag{4.11}$$

indeed for every point $x \in \mathbb{P}(\mathcal{E})$ we have

$$(\pi^*\mathcal{E})_x \to (\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))_x \to 0,$$

that, since $(\pi^*\mathcal{E})_x = \mathcal{E}_{\pi(x)}$ and by (4.10), is equivalent to (4.8).

Fact 4.17 (Maps to $\mathbb{P}(\mathcal{E})$)**.** *Let $\mathcal{E}$ be a vector bundle of rank r on a variety Z; let Y be a variety and let $p : Y \to Z$ be a map. Then there is a one to one correspondence between line bundle quotients of the form*

$$p^*\mathcal{E} \to \mathcal{L} \to 0, \tag{4.12}$$

and maps $f : Y \to \mathbb{P}(\mathcal{E})$ over Z. Under this correspondence $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Proof. As a first step, we want to build a line bundle quotient starting from the commutative diagram

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathbb{P}(\mathcal{E}) \arrow[dl, "\pi"] \\ & Z. & \end{tikzcd} \tag{4.13}$$

By equation (4.11), we get

$$p^*\mathcal{E} = f^*\pi^*\mathcal{E} \to f^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \to 0; \tag{4.14}$$

setting $\mathcal{L}_f := f^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ we get a line bundle on Y. Let us consider $y \in Y$, from the former equation we get

$$\mathcal{E}_{p(y)} = (p^*\mathcal{E})_y \to (\mathcal{L}_f)_y \to 0,$$

that, by duality, gives the inclusion $(\mathcal{L}_f)^*_y \subseteq \mathcal{E}^*_{p(y)}$. Since $(\mathcal{L}_f)^*_y$ is a one dimensional vector space, we can write

$$(\mathcal{L}_f)^*_y = \langle \varphi^f_y \rangle. \tag{4.15}$$

We claim that the map f is given by

$$f(y) = (p(y), [\varphi^f_y]). \tag{4.16}$$

Indeed, let us assume $f(y) = (y, [\varphi_{f(y)}])$; by equation (4.14) we get a surjection over Y

$$p^*\mathcal{E} \to \mathcal{L}_f \to 0,$$

that, for every $y \in Y$ gives

$$\mathcal{E}_{p(y)} = (p^*\mathcal{E})_y \to (\mathcal{L}_f)_y \to 0.$$

By (4.10) and by its definition, we get that

$$(\mathcal{L}_f)_y = (f^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))_y = (\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))_{f(y)} = \langle \varphi_{f(y)} \rangle^*,$$

Thus by (4.15) and duality we get

$$(\mathcal{L}_f)_y = \langle \varphi_{f(y)} \rangle^* = \langle \varphi^f_y \rangle^*$$

and therefore we have $[\varphi^f_y] = [\varphi_{f(y)}]$ and the (4.16) is proven.
We want now to construct the map starting from the line bundle quotient. Let us consider the map $p : Y \to Z$ and the line bundle quotient on Y

$$p^*\mathcal{E} \to \mathcal{L} \to 0.$$

For every $y \in Y$ we have

$$\mathcal{E}_{p(y)} = (p^*\mathcal{E})_y \to \mathcal{L}_y \to 0;$$

thus, by duality we get $(\mathcal{L}_y)^* \subseteq (\mathcal{E}_{p(y)})^*$; since $(\mathcal{L}_y)^*$ is one dimensional, we consider a generator $\varphi_y \in \mathcal{E}^*_{p(y)} \setminus \{0\}$ such that

$$(\mathcal{L}_y)^* = \langle \varphi_y \rangle \tag{4.17}$$

and we set

$$f(y) := (p(y), [\varphi_y]). \tag{4.18}$$

Under this map we have, by (4.10) and by (4.17), that

$$(f^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))_y = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)_{f(y)} = \langle \varphi_y \rangle^* = \mathcal{L}_y.$$

Therefore we get $f^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \simeq \mathcal{L}$. $\square$

As an application of the former result, we prove the following well-known result concerning quotient vector bundles.

Fact 4.18 (Restriction to quotient subbundles)**.** *Let $\mathcal{E}, \mathcal{F}$ be two vector bundles of rank $r > 1$ and $r' < r$ on a projective variety Z; if there is a surjection $\mathcal{E} \to \mathcal{F} \to 0$, then:*

1. *$\mathbb{P}(\mathcal{F}) \subseteq \mathbb{P}(\mathcal{E})$;*
2. *$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{\mathbb{P}(\mathcal{F})} = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$;*
3. *The fibre of $\mathbb{P}(\mathcal{E})$ restricted to $\mathbb{P}(\mathcal{F})$ is a fibre of $\mathbb{P}(\mathcal{F})$.*

Proof. Let us set $p : \mathbb{P}(\mathcal{F}) \to Z$ and $\pi : \mathbb{P}(\mathcal{E}) \to Z$; since $\mathcal{E} \to \mathcal{F}$ is surjective and by equation (4.11), we get

$$p^*\mathcal{E} \twoheadrightarrow p^*\mathcal{F} \to \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \to 0. \tag{4.19}$$

Thus, setting $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ we have a surjection

$$p^*\mathcal{E} \to \mathcal{L} \to 0; \tag{4.20}$$

by Fact 4.17, we get a map $f : \mathbb{P}(\mathcal{F}) \to \mathbb{P}(\mathcal{E})$ such that $f^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$.
In order to prove the first two statements of the Fact, it is enough to show that f is the inclusion.
To this end, let us consider $y = (p(y), [\varphi_y]) \in \mathbb{P}(\mathcal{F})$; by construction, see equation (4.16), we have that

$$f(y) = (p(y), [\varphi_y^f]),$$

where φ_y^f is a generator of $\mathcal{L}_y^*$; considering equation (4.20) on y we get

$$\mathcal{E}_{p(y)} \to (\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))_y \to 0,$$

that, by (4.10) becomes

$$\mathcal{E}_{p(y)} \to \langle \varphi_y \rangle^* \to 0.$$

Thus we get

$$\langle \varphi_y \rangle = \mathcal{L}_y^* = \langle \varphi_y^f \rangle;$$

in particular $[\varphi_y] = [\varphi_y^f]$ and we have proved that, since f takes $(p(y), [\varphi_y])$ to itself, it is the inclusion.
The third statement follows immediately by set theory and the definition of fibre. □

We refer to [Har77] for the following well-known fact.

Fact 4.19. *Let $X = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} Z$ be a projectivized vector bundle on a smooth variety Z.*

1. *More algebraically, $\mathbb{P}(\mathcal{E})$ can be described as*

$$\mathbb{P}(\mathcal{E}) = Proj_{\mathcal{O}_Z}\left(\bigoplus_{m \geqslant 0} S^m\mathcal{E}\right).$$

2. *We have that*

$$\mathrm{Pic}(X) \simeq \mathrm{Pic}(Z) \times \mathbb{Z} = \{\mathcal{O}_X(a) \otimes \pi^*M \mid a \in \mathbb{Z}, M \in \mathrm{Pic}(Y)\} \tag{4.21}$$

and thus $\rho(X) = \rho(Z) + 1$.

3. We have

$$\pi_*\mathcal{O}_X(a) = \begin{cases} S^a\mathcal{E} & a \geqslant 0 \\ 0 & a < 0. \end{cases} \tag{4.22}$$

4. If $\mathcal{E}'$ is a vector bundle on Z, then

$$\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\mathcal{E}') \iff \mathcal{E}' \simeq \mathcal{E} \otimes \mathcal{L}, \text{ for some } \mathcal{L} \in \mathrm{Pic}(Z); \tag{4.23}$$

moreover, we have that

$$\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^*\mathcal{L}, \tag{4.24}$$

In view of what we proved in Fact 4.18, we have that in the particular case of a quotient bundle of rank $r-1$ (where $r = \mathrm{rk}\,\mathcal{E}$), we have that the projectivized quotient bundle gives a divisor inside $\mathbb{P}(\mathcal{E})$; let us compute the associated line bundle.
First of all, a short lemma.

Lemma 4.20. *Let $\mathcal{E}$ be vector bundle of rank $r > 1$ on a smooth projective variety Z, $\pi : X = \mathbb{P}(\mathcal{E}) \to Z$. If $\mathcal{M}$ is a line bundle on X such that $\mathcal{M}|_F \simeq \mathcal{O}_F$ for every fibre, then*

$$\mathcal{M} \simeq \pi^*(\pi_*\mathcal{M}).$$

Proof. By Fact 4.19.2, we have that $\mathcal{M} \simeq \mathcal{O}_X(a) \otimes \pi^*\mathcal{G}$; restricting to F we get

$$\mathcal{M}|_F \simeq \mathcal{O}_F \simeq \mathcal{O}_X(a)|_F \otimes \pi^*\mathcal{G}|_F, \simeq \mathcal{O}_X(a)|_F$$

and thus $a = 0$ and $\mathcal{M} \simeq \pi^*\mathcal{G}$. By projection formula we see that $\pi_*\mathcal{M} \simeq \pi_*\pi^*\mathcal{G} \simeq \mathcal{G}$, and therefore $\mathcal{M} \simeq \pi^*(\pi_*\mathcal{M})$. □

We can now prof the following fact.

Fact 4.21. *Let $\mathcal{E}$ be a rank $r \geqslant 2$ vector bundle on a smooth projective variety Z and let $\mathcal{F}$ be a rank $r-1$ quotient bundle with kernel $\mathcal{L}$:*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0. \tag{4.25}$$

Then, setting $X = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} Z$ and $Y = \mathbb{P}(\mathcal{F})$, we have that

$$\mathcal{O}_X(Y) \simeq \mathcal{O}_X(1) \otimes \pi^*(\mathcal{L}^\vee). \tag{4.26}$$

Proof. From the exact sequence

$$0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0,$$

tensorizing by $\mathcal{O}_X(1)$, we get

$$0 \to \mathcal{O}_X(-Y) \otimes \mathcal{O}_X(1) \to \mathcal{O}_X(1) \to \mathcal{O}_X(1) \otimes \mathcal{O}_Y \to 0. \tag{4.27}$$

We see that $\mathcal{O}_X(1) \otimes \mathcal{O}_Y$ is just the restriction of $\mathcal{O}_X(1)$ to $\mathbb{P}(\mathcal{F})$, that, by Fact 4.18, is $\mathcal{O}_Y(1)$.

Claim 4.22. Pushing forward via π, we get the exact sequence

$$0 \to \pi_*(\mathcal{O}_X(1) \otimes \mathcal{O}_X(-Y)) \to \mathcal{E} \to \mathcal{F} \to 0. \tag{4.28}$$

Proof of the Claim. Since π on Y does coincide with the projection $\mathbb{P}(\mathcal{F}) \to Z$, from Remark 4.19 we immediately get the second and the third term. To show that the last term is 0, we prove that

$$R^1\pi_*(\mathcal{O}_X(1) \otimes \mathcal{O}_X(-Y)) = 0.$$

Indeed, by a well-known fact (see [Har77, III. Ex 11.8]), it is enough to check that, for any fibre F of π,

$$H^1(F, (\mathcal{O}_X(1) \otimes \mathcal{O}_X(-Y))|_F) = 0.$$

To see that, we have that the restriction of the tautological bundle to a fibre gives the class of an hyperplane in $F = \mathbb{P}^{r-1}$; similarly $\mathcal{O}_Y(1)|_F$ is the fibre of $\mathbb{P}(\mathcal{F}) \to Z$ and hence is a $\mathbb{P}^{r-2}$ inside a $\mathbb{P}^{r-1}$. Thus $(\mathcal{O}_X(1) \otimes \mathcal{O}_X(-Y))|_F \simeq \mathcal{O}_{\mathbb{P}^{r-1}}$ and its first cohomology vanishes. □

Coming back to the proof of the Fact, we have that

$$\mathcal{L} = \ker(\mathcal{E} \to \mathcal{F}) \simeq \pi_*(\mathcal{O}_X(1) \otimes \mathcal{O}_X(-Y)).$$

Since $(\mathcal{O}_X(1) \otimes \mathcal{O}_X(-Y))|_F \simeq \mathcal{O}_F$, by Lemma 4.20, we get

$$\mathcal{O}_X(1) \otimes \mathcal{O}_X(-Y) \simeq \pi^*\mathcal{L},$$

that is

$$\mathcal{O}_X(Y) \simeq \mathcal{O}_X(1) \otimes \pi^*(\mathcal{L}^\vee).$$

□

Chapter 5

Positive Answers

In this chapter we give a positive answer to Question 4.12 in a number of meaningful situations.

5.1 The case of curves

In this section we deal with the case of projectivized vector bundles on a curve. As pointed out before, our proof is independent from the work of Nakayama (see [Nak04]).
As a first step we focus on the computation of the pseudoeffective cone $\overline{\mathrm{Eff}}(X)$, where

$$X = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C$$

is the projectivization of a rank r vector bundle $\mathcal{E}$ on a smooth projective curve C.
The achievement of this computation is due to the author of [Flg11]; we recall it here for clarity's sake.
Since we are dealing with curves, by Fact 4.19 we have $\rho(X) = 2$ and thus the shape of cones in $N^1(X)$ is easy to handle. To fix notations, we will denote with $\xi = [\mathcal{O}_X(1)]_\equiv$ the numerical class of the tautological line bundle and with $f = [F]_\equiv$ the numerical class of a fibre. We immediately get, again by Fact 4.19, that $N^1(X)$ is a 2 dimensional real vector space with bases $\{\xi, f\}$.
Studying cones associated to X, we are interested also in the space of curves; since $\dim N_1(X) = 2$, we must fix two numerically independent classes of 1-cycles. A natural choice is:

$$\sigma := \xi^{r-1} \qquad \text{and} \qquad L := [\text{line on } F]_\equiv.$$

We will show in the following that σ, L are numerically independent and hence they form a basis of $N_1(X)$.
We can easily compute the intersection form involving these numerical classes.

Fact 5.1 (Intersection form)**.** *Let $X = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C$ be as before, then the intersection form of $N^1(X)$ is given by:*

$$(1) \quad f^i \cdot \xi^{r-i} = 0, \quad 2 \leqslant i \leqslant r; \quad (2) \quad f \cdot \xi^{r-1} = 1; \quad (3) \quad \xi^r = \deg \mathcal{E}. \tag{5.1}$$

Proof. (1) Since f is the class of a fibre, two of them are disjoint and thus we get to the conclusion.
(2) Let us compute:

$$f \cdot \xi^{r-1} = (\xi|_F)^{r-1} = (\mathcal{O}_{P^{r-1}}(1))^{r-1} = 1.$$

(3) As well-known (see [Har77, Appendix A]) we have

$$\sum_{i=0}^{r}(-1)^i \pi^* c_i(\mathcal{E}) \cdot \xi^{r-i} = 0,$$

thus we get

$$\xi^r = \sum_{i=1}^{r}(-1)^{i+1}\pi^* c_i(\mathcal{E}) \cdot \xi^{r-i} = \pi^* c_1(\mathcal{E}) \cdot \xi^{r-1} = \deg(\mathcal{E}). \tag{5.2}$$

□

We can easily show the following fact.

Fact 5.2 (Intersection pairing)**.** *The intersection pairing between $N^1(X)$ and $N_1(X)$ is ruled by:*

$$\begin{array}{llll} (1) & \xi \cdot \sigma = \deg \mathcal{E} & (2) & \xi \cdot L = 1 \\ (3) & f \cdot \sigma = 1 & (4) & f \cdot L = 0. \end{array} \tag{5.3}$$

Moreover we have

$$(5) \quad \xi^{r-2} \cdot F = L \qquad (6) \quad \xi^{r-1-i} \cdot F^i = 0, \quad \textit{for } i \geqslant 2. \tag{5.4}$$

Proof. (1) Direct consequence of Fact 5.1. (2) $\xi \cdot L = \xi|_F \cdot L = \mathcal{O}_{\mathbb{P}^{r-1}}(1) \cdot L = 1$. (3) $f \cdot \sigma = f \cdot \xi^{r-1} = 1$ by Fact 5.1. (4): $f \cdot L = 0$ (change the fibre).
(5) Let us compute:

$$f \cdot \xi^{r-2} = (\xi|_F)^{r-2} = (\mathcal{O}_{\mathbb{P}^{r-1}}(1))^{r-2} = L.$$

(6) Whenever at least two fibres are intersected, we get 0. □

In order to understand cones associated to X, we see at once the following.

Fact 5.3. *Let $X = \mathbb{P}(\mathcal{E}) \to C$ as above, then: (1) $f \in \partial\overline{\mathrm{Eff}}(X)$; (2) $f \in \partial \operatorname{Nef}(X)$.*

Proof. First of all we see that f is an effective (and hence pseudoeffective) class. Moreover we claim that it can't lie in the interior of $\overline{\mathrm{Eff}}(X)$: indeed, since it is the pullback of a point of C, we have that $f \in \operatorname{Nef}(X)$, but since $f^r = 0$, it is a nef and not big class. Therefore $f \in \partial\overline{\mathrm{Eff}}(X)$ and, since it is not ample, $f \in \partial \operatorname{Nef}(X)$. □

Let us now recall some important facts about semistability and about the Hader-Narasimhan filtration.

Definition 5.4 (Slope and semistability)**.** Let $\mathcal{E}$ be a vector bundle on the smooth projective curve C; the *slope* of $\mathcal{E}$ is the number

$$\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}. \tag{5.5}$$

We say that $\mathcal{E}$ is *semistable* if

$$\mu(\mathcal{F}) \leqslant \mu(\mathcal{E}) \tag{5.6}$$

for every subbundle $\mathcal{F} \subset \mathcal{E}$ of $\mathrm{rk}\,\mathcal{F} \geqslant 1$ and that $\mathcal{E}$ is *stable* if (5.6) is a strict inequality. We say that $\mathcal{E}$ is *unstable* if it is not semistable.

A very important tool in this matter is certainly the Harder-Narasimhan filtration.

Proposition 5.5 (Harder-Narasimhan filtration). *If $\mathcal{E}$ is a rank r vector bundle on a smooth projective curve C, then there exists a canonical defined filtration:*

$$0 = \mathcal{E}_k \subset \mathcal{E}_{k-1} \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0 = \mathcal{E} \tag{5.7}$$

by subbundles such that each successive quotient $\mathcal{Q}_i := \mathcal{E}_{i-1}/\mathcal{E}_i$ is semistable and such that:

$$\mu(\mathcal{Q}_k) > \cdots > \mu(\mathcal{Q}_2) > \mu(\mathcal{Q}_1). \tag{5.8}$$

Notation 5.6. For the Harder-Narasimhan filtration of a degree d and rank r vector bundle $\mathcal{E}$ as in (5.7), we set

$$r_i := \mathrm{rk}(\mathcal{Q}_i); \qquad d_i := \deg(\mathcal{Q}_i); \qquad \mu_i := \mu(\mathcal{Q}_i). \tag{5.9}$$

We have the following result (see [Laz04, Theorem 6.4.15]).

Theorem 5.7 (Hartshorne's theorem). *A vector bundle $\mathcal{E}$ on a curve C is nef (resp. ample) if and only if $\mathcal{E}$ and every quotient bundle of $\mathcal{E}$ has non negative (resp. strictly positive) degree.*

This allows to give a description of the nef cone of X

Proposition 5.8 (Miyaoka). *If $X = \mathbb{P}(\mathcal{E})$ and $\mu_1 = \mu(\mathcal{Q}_1)$ is the slope of the first quotient in Harder-Narasimhan filtration, then*

$$\mathrm{Nef}(X) = \langle \xi - \mu_1 f, f \rangle. \tag{5.10}$$

Proof. It is essentially an application of Theorem 5.7. The proof uses the setting of $\mathbb{Q}$-twisted bundle (see [Laz04, Section 6.2]); for further details see [Flg11, Lemma 2.1]. □

The semistable case is completely worked out by the following result (see [Flg11, Lemma 2.2].

Fact 5.9. *If $\mathcal{E}$ is a rank r semistable vector bundle on C with slope $\mu = \mu(\mathcal{E})$, then*

$$\overline{\mathrm{Eff}}(X) = \langle \xi - \mu f, f \rangle. \tag{5.11}$$

Proof. In view of Fact 5.3, we can focus on the second ray of $\overline{\mathrm{Eff}}(X)$. Since in the Harder-Narasimhan filtration there is just an element, then $\mu_1 = \mu$; moreover, by Fact 5.1, we can easily compute

$$(\xi - \mu f)^r = \xi^r - r\mu\xi^{r-1}f = \deg \mathcal{E} - r\frac{\deg \mathcal{E}}{r} = 0.$$

Now, by Proposition 5.8, $\xi - \mu f$ is nef, but since its top self-intersection vanishes, then it can't be big and thus it is in the boundary of $\overline{\mathrm{Eff}}(X)$ and therefore it is the generator of the second ray. □

Remark 5.10. In the semistable case, we have that $\mu = \mu_1$ and thus we have just proved that, in this situation,

$$\overline{\mathrm{Eff}}(X) = \mathrm{Nef}(X);$$

therefore, in view of Fact 4.6, Question 4.12 has a positive answer.
Moreover the author of [Flg11] proves (see Proposition 1.5) the following. A vector bundle $\mathcal{E}$ is semistable if and only if $\overline{\mathrm{Eff}}^i(X) = \mathrm{Nef}^i(X)$ for any $i \in \{1, \ldots, r-1\}$, where $\overline{\mathrm{Eff}}^i(X)$ and $\mathrm{Nef}^i(X)$ are the cones of pseudoeffective (resp. nef) cycles of dimension i.

It may be worth to recall a more complete result relating semistability, nefness and pseudoeffectiveness: see [Miy87, Theorem 3.1] and [BHR06, Theorem 2.3] for an account on the proof.

Theorem 5.11. *Let $\mathcal{E}$ be a vector bundle on a smooth projective curve C of rank r and slope $\mu = \mu(\mathcal{E})$, then, setting $X = \mathbb{P}(\mathcal{E})$ the following are equivalent.*

1. *$\mathcal{E}$ is semistable;*
2. *$\xi - \mu f$ is nef;*
3. *$\mathrm{Nef}(X) = \langle \xi - \mu f, f \rangle$;*
4. *$\overline{\mathrm{NE}}(X) = \langle (\xi - \mu f)^{r-1}, (\xi - \mu f)^{r-2} \cdot f \rangle$*
5. *every effective divisor in X is nef.*

Let us now focus on the unstable case; since $\mathcal{E}$ is not stable, then the Harder-Narasimhan filtration is not trivial:

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{Q}_1 \to 0.$$

We have the following (see [Flg11, Proposition 1.3]).

Fact 5.12. *Let $\mathcal{E}$ be an unstable rank r vector bundle on a curve; let us suppose that the first quotient in Harder-Narasimhan filtration is such that $rk(\mathcal{Q}_1) = r - 1$. Then*

$$\overline{\mathrm{Eff}}(X) = \langle [\mathbb{P}(\mathcal{Q}_1)], f \rangle. \tag{5.12}$$

Proof. By Fact 4.21 we have that $[\mathbb{P}(\mathcal{Q}_1)] = \xi + (d_1 - d)f$ and by Fact 4.18 it is effective and thus pseudoeffective. To check that it does lie in the boundary of $\overline{\mathrm{Eff}}(X)$, let us consider the nef class $\xi - \mu_1 f$ and compute:

$$\begin{aligned} \mathbb{P}(\mathcal{Q}_1) \cdot (\xi - \mu_1 f)^{r-1} &= (\xi + (d_1 - d)f) \cdot (\xi - \mu_1 f)^{r-1} \\ &= (\xi + (d_1 - d)f) \cdot (\xi^{r-1} - \mu_1 (r-1)\xi^{r-2} \cdot f) = \\ &= \xi^r - \mu_1 (r-1)\xi^{r-1} \cdot f + (d_1 - d)\xi^{r-1} \cdot f = \\ &= d - \frac{d_1}{r-1}(r-1) + d_1 - d = 0. \end{aligned}$$

Therefore, by Lemma 1.24.3, $[\mathbb{P}(\mathcal{Q}_1)]$ can't be in the interior and thus it spans the second extremal ray of $\overline{\mathrm{Eff}}(X)$. □

Remark 5.13. By Fact 4.18 we have that $\mathbb{P}(\mathcal{Q}_1) \subset \mathbb{P}(\mathcal{E})$ and it is thus effective. Therefore, whenever the first quotient $\mathcal{Q}_1$ in the Harder-Narasimhan filtration of an unstable vector bundle of rank r has a rank $r - 1$, then Question 4.12 has a positive answer.

Now, we deal with an unstable vector bundle $\mathcal{E}$ such that its first Harder-Narasimhan quotient has rank smaller than $r - 1$.
We refer again to [Flg11] for further details and generalization to lower dimensional cycles.

Situation. Let us consider the short exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{Q} \to 0, \tag{5.13}$$

where $\mathcal{E}$ is a rank r vector bundle and we may assume that its quotient $\mathcal{Q}$ is semistable of rank $s < r-1$ (so that $rk(\mathcal{E}') > 1$) and slope $\mu' = \mu(\mathcal{Q})$; we need a construction relating $\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E})$ and $\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}'))$.
Although there is not a morphism from $\mathbb{P}(\mathcal{E})$ to $\mathbb{P}(\mathcal{E}')$, we can consider the linear projection

$$p : \mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(\mathcal{Q}) \to \mathbb{P}(\mathcal{E}').$$

Setting $X = \mathbb{P}(\mathcal{E})$ and $Y = \mathbb{P}(\mathcal{E}')$, this projection can be seen as a rational map $X \dashrightarrow Y$ that can be resolved by blowing $\mathbb{P}(\mathcal{Q})$ up and thus getting $\eta : \tilde{X} = \mathrm{Bl}_{\mathbb{P}(\mathcal{Q})} X \to Y$.
We have the following commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\eta"] \arrow[d, "B"'] & \mathbb{P}(\mathcal{E}') = Y \arrow[d, "\rho"] \\ X = \mathbb{P}(\mathcal{E}) \arrow[r, "\pi"] \arrow[ur, dashed, "p"] & C \end{tikzcd} \tag{5.14}$$

We define the following map

$$\begin{array}{rccc} \mathrm{cone}: & \overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}')) & \longrightarrow & N^1(\mathbb{P}(\mathcal{E})) \\ & [D] & \longmapsto & [B_*\eta^*D]. \end{array} \tag{5.15}$$

This map is well defined, indeed B is birational and since η is flat, it follows that cone$[D]$ does not depend on D, but just on its numerical class, see [Flt98, Section 1.7] for further details.
Moreover, it has a meaningful geometrical description: to the divisor D whose class lies in $\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}'))$, it is associated the class of $\overline{p^{-1}(D)}$, that is the cone over Z with center $\mathbb{P}(\mathcal{Q})$.
As we report in the following, it turns out that the triple $(\tilde{X}, Y, \eta)$ can be described as a projective bundle on Y with fibre $\mathbb{P}^s$, see [Flg11, Proposition 2.4].

Lemma 5.14. *With above notations we have the following.*

1. *There exists a locally free sheaf $\mathcal{F}$ on Y such that $\tilde{X} \simeq \mathbb{P}_Y(\mathcal{F})$ and $\eta : \mathbb{P}_Y(\mathcal{F}) \to Y$ is its bundle map.*

2. *Let ξ' be the class of $\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)$, f' the class of a fibre of ρ, γ the class of $\mathcal{O}_{\mathbb{P}_Y(\mathcal{F})}(1)$ and $\tilde{E}$ the class of the exceptional divisor of B. Then we have:*

$$\gamma = B^*\xi; \qquad \eta^*\xi' = B^*\xi - \tilde{E}; \qquad \eta^*f' = B^*f. \tag{5.16}$$

3. *We have that*

$$\tilde{E} \cdot B^*(\xi - \mu' f)^s = 0. \tag{5.17}$$

4. *Still denoting by $\tilde{E}$ its support, if $j : \tilde{E} \to \tilde{X}$ is the canonical inclusion, then*

$$\tilde{E} \cdot N^1(\tilde{X}) = j_* N(\tilde{E}) \quad \text{as subset of } N^1(\tilde{X}).$$

Using this Lemma it is possible the prove that the cone map is indeed pseudeoffective (that is the image of the pseudoeffective cone lies in the pseudoeffective cone of the target set) and, moreover, it provides an isomorphism between the two pseudoeffective cones. The following result is a special case of [Flg11, Lemma 2.7].

Fact 5.15. *The map* $\mathrm{cone} = B_*\eta^*|_{\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}'))}$ *is indeed an isomorphism:*

$$\mathrm{cone}: \quad \overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}')) \xrightarrow{\simeq} \overline{\mathrm{Eff}}(\mathcal{E}).$$

Proof. Between the abstract groups $N^1(X)$ and $N^1(Y)$ there is a natural isomorphism

$$\varphi: \quad a\xi + bf \longmapsto a\xi' + bf';$$

we will prove that this induces the isomorphism between the pseudoeffective divisors. Let us understand the geometric behaviour of φ. We can define the unrestricted coning construction $U : N^1(Y) \to N^1(X)$, setting

$$U(c) = B_*\eta^*c.$$

We claim that U is pseudoeffective; if $c \in \overline{\mathrm{Eff}}(Y)$, then η^*c is pseudoeffective in the blow-up and so it is its push forward.
We immediately have that $U = \varphi^{-1}$; indeed, using (5.16):

$$\begin{aligned} U(a\xi' + bf') &= B_*(a(B^*\xi - \tilde{E})) + B_*(bB^*f) = \\ &= a(B_*B^*\xi - B_*\tilde{E}) + bB_*B^*f = a\xi + bf. \end{aligned}$$

To conclude the proof, if we construct an inverse V of U that is also pseudoeffective, then we are done because we have $V = \varphi$ and an isomorphism of the two pseudoeffective cones.
Setting $\delta = B^*(\xi - \mu'f)^s$, we define $V : N^1(X) \to N^1(Y)$ by

$$V(k) = \eta_*(\delta \cdot B^*k).$$

We claim that $\eta_*\delta = [Y]$; indeed, by the first equation of (5.16), we can compute:

$$\begin{aligned} \eta_*\delta &= \big(B^*(\xi - \mu'f)^s\big) = \eta_*\big(B^*(\xi^s - \mu'\xi^{s-1} \cdot f)\big) = \\ &= \eta_*\big(B^*(\xi^s) - \mu'B^*(\xi^{s-1} \cdot f)\big) = \\ &= \eta_*\big((B^*\xi)^s - \mu'B^*(\xi^{s-1} \cdot f)\big) = \\ &= \eta_*\gamma^s - \mu'\eta_*(B^*(\xi^{s-1} \cdot f)). \end{aligned}$$

Now we have that a classical result, see [Flt98, Proof of Propositon 3.1], gives that $\eta_*\gamma^s = [Y]$; to conclude the proof of what we claimed to be true, we need to show that the second summand has to vanish. To this end, we see that the cycle $\xi^{s-1} \cdot f$ has codimension s in $\mathbb{P}(\mathcal{E})$; more precisely, in each fibre $f \simeq \mathbb{P}^{r-1}$, it is given by intersecting $s-1$ times an hyperplane class and thus, in each fibre, it is the class σ of $\mathbb{P}^{r-1-s+1} = \mathbb{P}^{r-s} \subset \mathbb{P}^{r-1}$. Now, since by blowing up and pushing forward via η_*, the fibre f is sent to $f' \simeq \mathbb{P}^{r-s-1}$, the $(r-s)$-dimensional cycle $B^*\sigma$ has to be mapped by η to something of smaller dimension. Therefore, by definition of push forward, (see, for example, [Flt98, Section 1.4]), we see that it vanishes.
We can now compute, by (5.16), (5.17) and projection formula, that

$$\begin{aligned} V(a\xi + bf) &= \eta_*(B^*(a\xi + bf) \cdot \delta) = \eta_*([\eta^*(a\xi' + bf') + a\tilde{E}] \cdot \delta) = \\ &= (a\xi' + bf') \cdot \eta_*\delta = (a\xi' + bf') \cdot [Y] = a\xi' + bf'. \end{aligned}$$

Thus $V = U^{-1} = \varphi$. Let us prove that V is pseudoeffective: for an effective class k in X, we have that $B^*k = \tilde{k} + x\tilde{E}$ for some number x. Now $\tilde{k}$ is the strict transform of k and it is still effective; since $\delta \cdot \tilde{E} = 0$, we have that

$$V(k) = \eta_*(\delta \cdot (\tilde{k} + x\tilde{E})) = \eta_*(\delta \cdot \tilde{k});$$

but now δ is given by intersecting nef classes, thus $\delta \cdot \tilde{k}$ is pseudoeffective and so it is $\eta_*(\delta \cdot \tilde{k})$. □

Remark 5.16. We have proved that the cone map is pseudoeffective; moreover, in view of Question 4.12, we can say that it preserves effectiveness: if the second ray of $\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}'))$ is effective so it is for the second ray of $\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}))$.

The answer in the curve case

We now come back to our goal: our main tool to give an answer to Question 4.12 will be the Harder-Narasimhan filtration.
As we said before, this would follow from what Nakayama proves in [Nak04].

Theorem 5.17. *Let $\mathcal{E}$ be a rank r vector bundle on a smooth projective curve C, then every pseudoeffective divisor on $X = \mathbb{P}(\mathcal{E})$ has a weak Zariski decomposition.*

Proof. In the semistable case we have, by Remark 5.10, that $\overline{\mathrm{Eff}}(X) = \mathrm{Nef}(X)$ and we are done. So let us move to the unstable case; in this situation we have a non trivial Harder-Narasimhan filtration of $\mathcal{E}$:

$$0 = \mathcal{E}_k \subset \mathcal{E}_{k-1} \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}.$$

If the first quotient $\mathcal{Q}_1$ has rank $r-1$, then by Fact 5.12 we are done; otherwise Fact 5.15 provides the isomorphism

$$\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E})) \simeq \overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}_1))$$

and we can start the procedure; as we suggested in Remark 5.13, if at a certain point we find a quotient $\mathcal{Q}_j$ such that $\mathrm{rk}(\mathcal{Q}_j) = \mathrm{rk}(\mathcal{E}_{j-1}) - 1$, then

$$\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E})) \simeq \overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}_1)) \simeq \cdots \simeq \overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}_{j-1})) = \langle [\mathbb{P}(\mathcal{Q}_j)], f_j \rangle. \tag{5.18}$$

Thus in this situation, see Remark 5.16, the second ray of $\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}))$ is effective and we have a weak Zariski decomposition; moreover, since we do not need a birational transformation of X, we have a direct WZD.
If there exists such a $\mathcal{Q}_j$, then the vector bundle $\mathcal{E}_j$ in the Harder-Narasimhan filtration has to be a line bundle and this $\mathcal{E}_j$ is indeed the last bundle in the filtration.
Thus we are reduced to consider the last step of the filtration:

$$0 \subset \mathcal{E}_k \subset \mathcal{E}_{k-1} \subset \cdots$$

with $\mathrm{rk}(\mathcal{E}_k) > 1$. Now, since it is the last bundle in the filtration, it has to be semistable. Since $\mathcal{E}_k$ is semistable, then the nef cone does coincide with the pseudoeffective cone and, by the cone map, we have the following:

$$\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E})) \simeq \overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}_1)) \simeq \cdots \simeq \overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}_k)) = \mathrm{Nef}(\mathbb{P}(\mathcal{E}_k)). \tag{5.19}$$

Since our situation, for some quotient $\mathcal{Q}$, is:

$$0 \to \mathcal{E}_k \to \mathcal{E} \to \mathcal{Q} \to 0,$$

mimicking (5.14), we have the commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\eta"] \arrow[d, "B"'] & \mathbb{P}(\mathcal{E}_k) \arrow[d, "\rho"] \arrow[dl, dashed, "\text{cone}"'] \\ \mathbb{P}(\mathcal{E}) \arrow[r, "\pi"] & C. \end{tikzcd} \tag{5.20}$$

Let us set, for a suitable pseudoeffective class D,

$$\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E})) = \langle D, f \rangle$$

and, for a nef class D_k,

$$\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}_k)) = \langle D_k, f_k \rangle.$$

Now, since f is an effective class, in view of Fact 4.8, we need to produce a weak Zariski decomposition for D. However, it is worth to point out that for a class in the interior of $\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}))$, since it is big, it is effective and thus we trivially have a direct weak Zariski decomposition.
Focusing on D, we see that via the cone map we have

$$\text{cone} : D_k \longmapsto B_* \eta^* D_k = D.$$

We will show that the class of $B^*(D)$ can be written as the sum of an effective and a nef class, producing a weak Zariski decomposition.
To this end, confusing for a moment the divisor and its class, we have that by Fact 5.9,

$$D_k = \xi_k + b f_k, \quad \text{for some } b,$$

via the cone map we have

$$D = \xi + bf.$$

Now since D_k is nef, its pullback

$$\eta^* D_k = \eta^* \xi_k + b \eta^* f_k$$

is nef, too. By (5.16) (Lemma 5.14), we can have:

$$\begin{aligned} B^*(D) = B^*(\xi + bf) &= (\eta^* \xi_k + \tilde{E}) + b\eta^* f_k \\ &= \eta^* \xi_k + b\eta^* f_k + \tilde{E} \\ &= \eta^* D_k + \tilde{E}, \end{aligned}$$

that, since B is birational, gives a weak Zariski decomposition for D. $\square$

In the final part of the proof of Theorem 5.17, we used a birational transformation to ensure the existence of weak Zariski decomposition; in the following we give a description of the situation where this transformation can't be avoided.
We have the following result.

Proposition 5.18. *Let $\mathcal{E}$ be an unstable rank r vector bundle on a curve C and let $\mathcal{F}$ be the last non-zero term in the Harder-Narasimhan filtration of $\mathcal{E}$.*
A direct weak Zariski decomposition does not exist for every pseudoeffective divisor on $X = \mathbb{P}(\mathcal{E})$ if and only if $\mathrm{rk}(\mathcal{F}) \geqslant 2$ and $\mathrm{Eff}(\mathbb{P}(\mathcal{F}))$ is not a closed cone.

Proof. Let us suppose that a direct weak Zariski decomposition does not exist; by the proof of Theorem 5.17, we are reduced to this case:

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{Q} \to 0,$$

where $\mathrm{rk}(\mathcal{F}) \geqslant 2$; we know that

$$\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E})) \simeq \overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{F})) = \mathrm{Nef}(\mathbb{P}(\mathcal{F})) \tag{5.21}$$

and we can set

$$\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E})) = \langle D, f \rangle; \qquad \mathrm{Nef}(\mathbb{P}(\mathcal{F})) = \langle D', f' \rangle.$$

To prove the first implication, let us suppose, by contradiction, that $\mathrm{Eff}(\mathbb{P}(\mathcal{F}))$ is closed; now, by (5.21), we have that D' gives an effective class and therefore, since the cone map sends effective classes to effective classes, D is effective and a direct weak Zariski decomposition does exist.
Viceversa, we want to prove that if D' is not effective then D is neither nef nor effective (and thus we need a birational transformation to make it nef). We see at once that D can't be nef: indeed if it were nef, then we would have $\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E})) = \mathrm{Nef}(\mathbb{P}(\mathcal{E}))$, but this is not possible since $\mathcal{E}$ is unstable (see Theorem 5.11). We have also that D is not effective; indeed we have that $D = \mathrm{cone}(D')$, but since the inverse of cone (see the proof of Theorem 5.17) sends effective to effective, then also D' should be effective. □

Example 5.19. Let us consider the semistable rank 2 vector bundle $\mathcal{U}$ on a curve C of genus $g \geqslant 2$ of [Laz04, Example 1.5.1]; as it is well known, $\mathbb{P}(\mathcal{U})$ has a non closed effective cone. Setting

$$\mathcal{E} = \mathcal{U} \oplus \mathcal{N},$$

for a line bundle $\mathcal{N}$ with $\deg(\mathcal{N}) < 0$, we have that the Harder-Narasimhan filtration of $\mathcal{E}$ is

$$0 \subset \mathcal{U} \subset \mathcal{E}.$$

By Proposition 5.18, a birational transformation of the threefold $X = \mathbb{P}(\mathcal{E})$ is required in order to get a weak Zariski decomposition.

5.2 Completely decomposable vector bundles

In this section we prove that a direct weak Zariski decomposition always exists for a pseudoeffective divisor on $X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a completely decomposable rank r vector bundle on a variety with Picard number one or on a curve.

The case of curves

As a first step we put in evidence what happens in the case of curves; this could be seen as an application of results from the former section or, as we sill see, it can be easily directly shown.

Fact 5.20. *Let $\mathcal{E}$ be a completely decomposable rank r vector bundle on a smooth projective curve C:*

$$\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r, \quad \deg(\mathcal{L}_i) = a_i, \quad a_1 \leqslant \cdots \leqslant a_r. \tag{5.22}$$

Setting $X = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C$ we have

$$\overline{\mathrm{Eff}}(X) = \langle \xi - a_r f, f \rangle. \tag{5.23}$$

Proof. We show the two inclusions. At first we have that $f = [F]$ is effective and hence pseudoeffective. To see that $\xi - a_r f \in \overline{\mathrm{Eff}}(X)$, we see that, since $\deg \mathcal{L}_r = a_r$, $\xi - a_r f \equiv \xi - \pi^*\mathcal{L}_r$; if we show that $\xi - \pi^*\mathcal{L}_r$ has sections, then we are done. Let us compute:

$$\begin{aligned} h^0(X, \xi - \pi^*\mathcal{L}_r) = h^0(C, \pi_*(\xi - \pi^*\mathcal{L}_r)) &= h^0(C, \mathcal{E} \otimes (\mathcal{L}_r)^\vee) = \\ &= h^0(C, (\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r) \otimes (\mathcal{L}_r)^\vee) = \\ &= h^0(C, (\mathcal{L}_1 \otimes \mathcal{L}_r^\vee) \oplus \cdots \oplus \mathcal{O}_C) > 0. \end{aligned}$$

To prove the reverse inclusion, if $x\xi + yf \in \overline{\mathrm{Eff}}(X)$, then, for the movable class $L \in \overline{\mathrm{ME}}(X)$ (the line on a fibre), $(x\xi + yF) \cdot L = x \geqslant 0$.
Let us now consider an integer effective class: $a\xi + bf$, with $a, b \in \mathbb{Z}$, $a \geqslant 0$. We have that $[a\xi + bf] \in \mathrm{Eff}(X)$ and thus there exists an effective divisor $E \equiv a\xi + bf$; hence for a suitable $\mathcal{M} \in \mathrm{Pic}(C)$ we have $E \sim \mathcal{O}_X(a) \otimes \pi^*\mathcal{M}$. Therefore we get

$$0 < h^0(X, \mathcal{O}_X(a) \otimes \pi^* M) = h^0(C, S^a\mathcal{E} \otimes \mathcal{M}),$$

that easily gives

$$\sum_{\substack{j_1 + \cdots + j_r = a \\ j_1 \geqslant 0, \ldots, j_r \geqslant 0}} h^0(C, \mathcal{L}_1^{j_1} \otimes \cdots \otimes \mathcal{L}_r^{j_r} \otimes \mathcal{M}) \neq 0. \tag{5.24}$$

Thus one of the summands must be non negative; we have that there exist

$$j_1 \geqslant 0, \ldots, j_r \geqslant 0 \quad \text{such that} \quad \sum_{j=1}^r j_i = a, \quad \sum_{j=1}^r j_i a_i + b \geqslant 0.$$

Thus we get

$$b \geqslant -\sum_{i=1}^r j_i a_i \geqslant -a a_r. \tag{5.25}$$

Now we need to pass from integer to real classes; setting $V := \langle \xi - a_r f, f \rangle$, we claim that $\mathrm{Eff}(X) = V$ (and thus, since V is closed, $\overline{\mathrm{Eff}}(X) = V$). We have seen that f and $\xi - a_r f$ are effective and thus $V \subseteq \mathrm{Eff}(X)$. Viceversa let $E = \sum e_i [E_i] \in \mathrm{Eff}(X)$ with E_i integral divisors and $e_i \in \mathbb{R}_{\geqslant 0}$; then by (5.25), $E_i \in \mathbb{Z}_{\geqslant 0}[F] + \mathbb{Z}_{\geqslant 0}[\xi - a_r F]$ and thus $E \in V$. □

After a direct proof, we can also prove Fact 5.20 as a consequence of Theorem 5.17.

Second proof of Fact 5.20. In order to use Theorem 5.17 we need to understand the Harder-Narasimhan filtration of $\mathcal{E}$. Since it is decomposable, at each step the maximal destabilizing subsheaf is obtained by elimination of all the summands of smallest degree.

Putting together all summands of same degree, we can write

$$\mathcal{E} = \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_s,$$

for some $s \leqslant r$, where

$$\begin{cases} \mathcal{N}_j = \mathcal{M}_j^{\oplus r_j} & j = 1, \dots, s \quad \text{for some } \mathcal{M}_j \in \mathrm{Pic}(C) \\ \deg \mathcal{M}_j = b_j & a_1 = b_1 < \cdots < b_s = a_r \\ \sum_{j=1}^{s} r_j = r. \end{cases}$$

Thus the Harder-Narasimhan filtration is:

$$0 \quad \subset \quad \mathcal{N}_r \quad \subset \quad \mathcal{N}_{r-1} \oplus \mathcal{N}_r \quad \subset \quad \cdots \quad \subset \quad \mathcal{E}. \tag{5.26}$$

Therefore, setting

$$\mathcal{E}_i = \mathcal{N}_{i+1} \oplus \cdots \oplus \mathcal{N}_s, \quad \text{for } i = 0, \dots, s-1,$$

we have that $\mathrm{rk}(\mathcal{E}_i) > 1$ for $i < s-1$; following the cone construction, we produce the chain of isomorphisms

$$\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E})) \simeq \overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{N}_2 \oplus \cdots \oplus \mathcal{N}_s)) \simeq \cdots \simeq \overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{N}_{s-1} \oplus \mathcal{N}_s)), \tag{5.27}$$

and we must distinguish two cases.
If $r_s = 1$, we have the following sequence:

$$0 \to \mathcal{M}_s \to \mathcal{N}_{s-1} \oplus \mathcal{M}_s \to \mathcal{N}_{s-1} \to 0;$$

by Fact 5.12, we get

$$\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{N}_{s-1} \oplus \mathcal{N}_s)) = \langle [\mathbb{P}(\mathcal{N}_{s-1})], f_{s-1} \rangle.$$

Now the result follows since $\deg(\mathcal{M}_s) = b_s = a_r$ and, by Fact 4.21, we have

$$[\mathbb{P}(\mathcal{N}_{s-1})] = \xi_{s-1} - a_r f_{s-1},$$

where ξ_{s-1} and f_{s-1} are the suitable classes corresponding to the tautological line bundle and to the fibre.
If else $r_s > 1$, we have that $\mathcal{N}_s$ is semistable and, from (5.27), we get $\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{N}_{s-1} \oplus \mathcal{N}_s)) = \overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{N}_s))$; by Fact 5.9, the second ray of this cone is generated by

$$\xi_s - \mu(\mathcal{N}_s) f_s = \xi_s - \frac{\deg \mathcal{N}_s}{r_s} f_s = \xi_s - \frac{r_s b_s}{r_s} f_s = \xi_s - a_r f_r,$$

where ξ_s and f_s are the suitable classes corresponding to the tautological line bundle and to the fibre. $\square$

The cone of movable curves

Furthermore, in this decomposable situation it is easy to compute the movable cone $\overline{\mathrm{ME}}(X) \subset N_1(X)$; this can be immediately worked out by Theorem 1.34: since $\overline{\mathrm{Eff}}(X) = \langle \xi - a_r f, f \rangle$, it is enough to ask (see Fact 5.2):

$$\begin{cases} (x\sigma + yL) \cdot F = x \geqslant 0 \\ (x\sigma + yL) \cdot (\xi - a_r f) = x \deg \mathcal{E} - x a_r + y \geqslant 0, \end{cases}$$

which gives

$$x \geqslant 0, \qquad y \geqslant -\left(\sum_{i=1}^{r-1} a_i\right) x.$$

Thus the movable cone is given by:

$$\overline{\mathrm{ME}}(X) = \langle \sigma - \sum_{i=1}^{r-1} a_i L, L \rangle. \tag{5.28}$$

In view of Definition 1.6, it may be worth to understand where the ray $R(\sigma - \sum_{i=1}^{r-1} a_i L)$ comes from: we want to explicitly find a birational map $\varphi : X' \to X$ and nef classes $\delta_1, \dots, \delta_{r-1}$ such that

$$\sigma - \sum_{i=1}^{r-1} a_i L = \varphi_*(\delta_1 \cdot \ldots \cdot \delta_{r-1}) \tag{5.29}$$

For a better understanding of the strategy, let us proceed by a sort of induction on the rank. In the rank 2 case, we have that $\overline{\mathrm{ME}}(X)$ is generated by $R(\sigma - a_1 L) = R(\xi - a_1 F)$ that is nef and thus (5.29) is satisfied with $\delta_1 = \xi - a_1 f$.
If $r = 3$ we immediately get $(\xi - a_1 f) \cdot (\xi - a_2 f) = \sigma - (a_1 + a_2)L$ and more generally, by Fact 5.2 we can compute:

$$(\xi - a_1 f) \cdot (\xi - a_2 f) \cdot \ldots \cdot (\xi - a_{r-1} f) = \sigma - \sum_{i=1}^{r-1} a_i L. \tag{5.30}$$

But now we see that, since $a_i \geqslant a_1$, $\xi - a_i f$ is not nef for $i \geqslant 2$ and we do not have the required classes. To "resolve" this lack of nefness, a suitable birational modification will be necessary; to this end it is worth to point out that $\xi - a_i f$ is an effective class and thus $|\xi - a_i F| \neq \emptyset$. With some abuse of notation, in this computation, we denote with ξ the line bundle $\mathcal{O}_X(1)$.
This is our strategy: at each step, to resolve the non nefness of $\xi - a_i f$, we blow up the base locus $\mathrm{Bs}(\xi - a_i F)$ and we will subtract the exceptional locus to the strict transform of $\xi - a_i F$.
Let us consider the rank 3 case. Let $X_1 := X$ and $B_2 := \mathrm{Bs}(\xi - a_2 F)$; we can consider

$$X_2 = \mathrm{Bl}_{B_2} \xrightarrow{\mu_2} X_1,$$

with exceptional divisor E_2. We have hence that

$$\mu_2^*(\xi - a_2 F) - E_2 \qquad \text{and} \qquad \mu_2^*(\xi - a_1 F)$$

are nef classes and

$$\begin{aligned}(\mu_2)_*\big(\mu_2^*(\xi - a_1 f) \cdot (\mu_2^*(\xi - a_2 f) - E_2)\big) = \\ =(\xi - a_1 f) \cdot (\xi - a_2 f) = \sigma - (a_1 + a_2)L.\end{aligned}$$

If $r > 3$, at each step, if we have

$$\varphi_k : X_k = \mathrm{Bl}_{B_k} X_{k-1} \xrightarrow{\mu_k} X_{k-1} \xrightarrow{\varphi_{k-1}} X_1 = X,$$

we set

$$\begin{cases} B_{k+1} := \mathrm{Bs}((\varphi_k^{-1})_*(\xi - a_{k+1} f)) \\ X_{k+1} := \mathrm{Bl}_{B_{k+1}} X_k \xrightarrow{\mu_{k+1}} X_k \\ \varphi_{k+1} := \varphi_k \circ \mu_{k+1} : X_{k+1} \to X. \end{cases}$$

Thus, setting $\varphi = \varphi_{r-1}$ we finally get:

$$\begin{aligned}
&\varphi_*\Big([\mu_{r-1}^* \cdots \mu_3^* \mu_2^*(\xi - a_1 f)] \cdot [\mu_{r-1}^* \cdots \mu_3^*(\mu_2^*(\xi - a_2 f) - E_2)] \cdot \\
&\quad \cdot [\mu_{r-1}^* \cdots \mu_4^*(\mu_3^*((\varphi_2^{-1})_*(\xi - a_3 f)) - E_2)] \cdot \quad \cdots \quad \cdot \\
&\quad \cdot [\mu_{r-1}^*((\varphi_{r-2}^{-1})_*(\xi - a_{r-1} f)) - E_{r-1}] \Big) = \\
&\quad = (\xi - a_1 f) \cdot (\xi - a_2 f) \cdot \ldots \cdot (\xi - a_{r-1} f) = \sigma - \sum_{i=1}^{r-1} a_i L.
\end{aligned} \tag{5.31}$$

Remark 5.21. We can run this sort of strategy whenever for a $\gamma \in \overline{\mathrm{ME}}(X)$ we have $\gamma = \gamma_1 \cdot \ldots \cdot \gamma_{r-1}$ for some effective classes γ_i. At each step we blow up the base locus of γ_i and we subtract the corresponding exceptional divisor to build a nef divisor.

Higher dimensional case

We can now focus on the higher dimensional case; more precisely, we are in the following setting.

Notation 5.22. If Z is a smooth projective variety with $\mathrm{Pic}(Z) = \mathbb{Z}[H_Z]$, we consider a rank r vector bundle on Z

$$\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_r, \tag{5.32}$$

where $\mathcal{L}_i$ is a line bundle such that in $\mathcal{L}_i \simeq \mathcal{O}_Z(a_i H_Z)$.
We have that $N^1(\mathbb{P}(\mathcal{E}))$ is a real two dimensional vector space generated by the class ξ of the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and by the class of the pull-back of H_Z.

First of all, we can prove a general fact about one of the extremal rays of the cones that is a generalization of Fact 5.3.

Fact 5.23. *Let Z be a smooth projective variety such that $\mathrm{Pic}(Z) = \mathbb{Z}[H_Z]$ for an ample divisor H_Z on Z; let $\mathcal{E}$ be a rank r vector bundle on Z. Setting $\pi : X = \mathbb{P}(\mathcal{E}) \to Z$, then $H = \pi^* H_Z$ spans an extremal ray of $\overline{\mathrm{Eff}}(X)$, $\overline{\mathrm{Mov}}(X)$ and $\mathrm{Nef}(X)$.*

Proof. Since H_Z is ample we have that its pull-back is nef but not ample and thus it lies in the boundary of $\mathrm{Nef}(X)$. In order to see that H is in $\partial\overline{\mathrm{Eff}}(X)$, it is enough to show that it is not big; but this is true since the top self intersection of H is 0. Concerning the movable cone, since it lies between $\mathrm{Nef}(X)$ and $\overline{\mathrm{Eff}}(X)$, then H spans an extremal ray of $\overline{\mathrm{Mov}}(X)$ too. □

We can now focus on the second ray.

Proposition 5.24. *Let Z be a smooth projective variety with $\mathrm{Pic}(Z) = \mathbb{Z}[H_Z]$, where H_Z is an ample generator of the Picard group; let $\mathcal{E}$ be a completely decomposable rank r vector bundle*

$$\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r, \qquad \mathcal{L}_i \simeq \mathcal{O}_Z(a_i H_Z), \qquad a_1 \leqslant \cdots \leqslant a_r. \tag{5.33}$$

Then for $X = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} Z$, setting $H = \pi^* H_Z$, we have

$$\overline{\mathrm{Eff}}(X) = \mathrm{Eff}(X) = \langle \xi - a_r H, H \rangle. \tag{5.34}$$

Proof. One of the two rays, as we have seen in Fact 5.23, is generated by H. Let us focus now on the second ray and let us suppose it is generated by D. If we consider a very general movable curve $C \subset Z$ (that is very general in the covering algebraic family, see Definition 1.6), we have the surjection of vector bundles

$$\mathcal{E} \to \mathcal{E}|_C \to 0.$$

Thus we have $Y := \mathbb{P}(\mathcal{E}|_C) \subset X$ and we want restrict a pseudoeffective divisor $D \in \overline{\mathrm{Eff}}(X)$ to Y.

Claim 5.25. If $D \in \overline{\mathrm{Eff}}(X)$, then $D|_Y \in \overline{\mathrm{Eff}}(Y)$.

To prove the claim we can consider at first the effective case. Without loss of generality, we can suppose D a prime divisor; thus, if $D|_Y = D \cap \mathbb{P}(\mathcal{E}|_C)$ were not effective, then we would have

$$\mathbb{P}(\mathcal{E}|_C) \subseteq D,$$

but, since C is very general and movable, this can't be. If D is pseudoeffective we can write $D = \lim_n D_n$, with D_n effective classes; by linearity we have that the restriction is a continuous function $N^1(X) \to N^1(Y)$, therefore

$$D|_Y = (\lim_{n\to\infty} D_n)|_Y = \lim_{n\to\infty} (D_n|_Y),$$

and we have written $D|_Y$ as limit of effective classes. To be precise we should check that $D_n|_Y$ is effective for every D_n, but this is the case. Indeed we have that C is movable and thus $Y = \mathbb{P}(\mathcal{E}|_C)$ covers X; now, since D_n are countable many and C is general, we can pick C such that

$$C \nsubseteq \bigcup_n \mathrm{Supp}(D_n).$$

Claim 5.26. Let $D = \xi + xH$ be the generator of the second ray of $\overline{\mathrm{Eff}}(X)$, then $x \geqslant -a_r$

Indeed, we have

$$D|_Y = \xi|_Y + xH|_Y = \xi_C + x\deg(C)f_C,$$

where ξ_C and f_C are the generators of the projectivized vector bundle $\mathcal{E}|_C$ on the curve C. Now, the restriction of vector bundle $\mathcal{E}$ to C is

$$\mathcal{E}|_C = \mathcal{L}'_1 \oplus \cdots \oplus \mathcal{L}'_r, \qquad \deg(\mathcal{L}'_i) = a_i \deg(C),$$

since $D|_Y$ is a pseudoeffective divisor of $\mathbb{P}(\mathcal{E}|_C)$, by Fact 5.20,

$$x\deg(C) \geqslant -a_r \deg(C),$$

that gives Claim 5.26.

Thus we have seen that $\overline{\mathrm{Eff}}(X) \subseteq \langle \xi - a_r H, H\rangle$. To the reverse inclusion, we will see that $\xi - a_r H$ is an effective class; indeed the exact sequence

$$0 \to \mathcal{L}_r \to \mathcal{E} \to \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_{r-1} \to 0,$$

provides the existence of the map $\mathcal{O} \to \mathcal{E} \otimes \mathcal{L}_r^\vee$ and thus of an element in $H^0(C, \mathcal{E} \otimes \mathcal{L}_r^\vee)$. Now it happens that the numerical class of the line bundle $\mathcal{O}_X(1) \otimes \pi^*\mathcal{L}_r^\vee$ is exactly $\xi - a_r H$. Since

$$H^0(X, \mathcal{O}_X(1) \otimes \pi^*\mathcal{L}_r^\vee) = H^0(C, \mathcal{E} \otimes \mathcal{L}_r^\vee) \neq 0,$$

we have that $\xi - a_r H \in \mathrm{Eff}(X)$ and the proof is concluded. □

Remark 5.27. In the case of completely decomposable vector bundles on varieties with Picard number one, we have that the second ray of the pseudoeffective cone is indeed effective; we have thus proved the existence of a direct weak Zariski decomposition for any pseudoeffective class in $\overline{\mathrm{Eff}}(X)$.

5.3 Rank 2 on Fano

In this section we give an account of some results of [MOSC11] on projectivized rank 2 vector bundles on a Fano variety. For a Fano variety Z of Picard number one and a rank 2 vector bundle $\mathcal{E}$ on Z, we give a positive answer to Question 4.12 if $\mathcal{E}$ is unstable or if it is semistable but not stable (that is if $\mathcal{E}$ is not strictly stable). Moreover, we see that in this situation we do not need a birational transformation in order to ensure the existence of a weak Zariski decomposition for any pseudoeffective divisor on $X = \mathbb{P}(\mathcal{E})$.

Notation 5.28. We consider a smooth irreducible Fano variety Z with Picard number one and we fix an ample generator H_Z so that $\mathrm{Pic}(Z) = \mathbb{Z}[H_Z]$.
If $\mathcal{E}$ is a rank 2 vector bundle on Z, we set $\pi : X = \mathbb{P}(\mathcal{E}) \to Z$; we have thus that $N^1(X)$ is a two dimensional vector space generated by $\xi = \xi_{\mathbb{P}(\mathcal{E})}$, the class associated to the tautological line bundle $\mathcal{O}_X(1)$, and the class H of π^*H_Z;
With abuse of notation, we will denote with ξ the numerical class and the divisor associated to $\mathcal{O}_X(1)$; similarly H will denote the numerical class and the divisor associated to π^*H_Z.
We can suppose that $\mathcal{E}$ is a normalized vector bundle so that its determinant is given by $\det\mathcal{E} = \mathcal{O}_Z(c_1(\mathcal{E})H_Z)$, where $c_1(\mathcal{E})$ is its first Chern class and $c_1 = c_1(\mathcal{E}) = 0, 1$ (see Fact 4.19.4).

In the curve and decomposable cases worked out in previous sections, when dealing with $N^1(X)$, we considered the basis given by the tautological class and by the pull-back of the generator of the base; in this setting, it turns out to be far more convenient the use of the relative anticanonical class instead of the tautological class.
From the canonical bundle formula for projectivized vector bundles (see [BS95]):

$$K_{\mathbb{P}(\mathcal{E})} = -\,\mathrm{rk}(\mathcal{E})\xi + \pi^*(K_Z + \det\mathcal{E}), \tag{5.35}$$

we immediately get the formula relating the *relative anticanonical divisor* with the base $\{\xi, H\}$ of $N^1(X)$:

$$-K_{\mathrm{rel}} = 2\xi - c_1 H. \tag{5.36}$$

This relation shows that with the basis $\{-K_{\mathrm{rel}}, H\}$ of $N^1(X)$, we take intrinsically account of the degree of the vector bundle $\mathcal{E}$: this is therefore a wise choice in order to simplify computations and notations.
The stability properties of the rank two vector bundle we want to consider are successfully detected by β, the smallest integer such that $\mathcal{E}(\beta) = \mathcal{E} \otimes \mathcal{O}_Z(\beta H_Z)$ has non-zero global sections. We have indeed that for such a β, there is the injective map

$$0 \to \mathcal{O}_Z(-\beta H_Z) \to \mathcal{E}; \tag{5.37}$$

since the slope of $\mathcal{O}_Z(-\beta H_Z)$ is $-\beta$ and the slope of $\mathcal{E}$ is $c_1/2$, we have that $\mathcal{E}$ is

$$\begin{array}{rl} \text{stable} & \text{if } \beta > -\frac{c_1}{2} \\ \text{semistable} & \text{if } \beta \geqslant -\frac{c_1}{2} \\ \text{semistable not stable} & \text{if } \beta = -\frac{c_1}{2} \\ \text{unstable} & \text{if } \beta < -\frac{c_1}{2}. \end{array} \tag{5.38}$$

Now in order to investigate the answer to Question 4.12, since we are dealing with two dimensional cones sharing one of the two extremal rays (see Fact 5.23), the shape of these cones is essentially ruled by the slope of the second ray, since each ray different from $R(H)$ is generated by $-K_{\mathrm{rel}} + xH$ for a suitable $x \in \mathbb{R}$, we can give the following definition.

Definition 5.29. Whenever we have a vector bundle on a variety of Picard number one, with notation as above, we define the *slopes* τ and ρ of $\mathrm{Nef}(X)$ and $\overline{\mathrm{Eff}}(X)$ (see Fact 5.23) by setting:

$$\begin{aligned} \mathrm{Nef}(X) &= \langle -K_{\mathrm{rel}} + \tau H, H\rangle \\ \overline{\mathrm{Eff}}(X) &= \langle -K_{\mathrm{rel}} + \rho H, H\rangle \end{aligned} \tag{5.39}$$

Remark 5.30. It is a direct consequence of Fact 4.10 that $\tau \geqslant \rho$.

Let us now recall a list of useful results from [MOSC11, Section 3]; the pair $(X, \mathcal{E})$ denote a rank 2 vector bundle $\mathcal{E}$ on a Fano variety Z and $X = \mathbb{P}(\mathcal{E})$.

Lemma 5.31. *Let $\mathcal{E}$ be a vector bundle on Z as in Notation 5.28; if $\mathcal{E}$ is semistable, then $\rho \geqslant 0$.*

Let us recall the following lemma, see [MOSC11, Lemma 3.2]

Lemma 5.32. *Let $\mathcal{E}$ be a vector bundle on Z as in Notation 5.28 and let $\mathcal{M}$ be a dominating family of rational curves in Z of degree μ and $\mathcal{M}^t, t \geqslant 0$ is its dominating subscheme of curves with splitting type $((c_1+t)\mu/2, (c_1-t)/\mu/2)$.*
If $b < t$, then every linear system of the form $|k(-K_{rel}+bH)|$ contains a fixed component F, where F is the closure of the union of the strict transforms in $\mathbb{P}(\mathcal{E})$ of the curves in $\mathcal{M}^t$.

We have now that linear systems of a certain kind have to contain a fixed component lying in the second pseudoeffective ray (see [MOSC11, Proposition 3.5].

Proposition 5.33. *Let $\mathcal{E}$ be a vector bundle on Z as in Notation 5.28; a non empty linear system of the form*

$$|k(-K_{rel} + bH)|, \quad \text{with } k, b \in \mathbb{Q}, b < 0,$$

contains a fixed component F numerically proportional to $-K_{rel} + \rho H$.
Moreover, if $\mathcal{E}$ is unstable, then $\rho = 2\beta + c_1$.

Proof. We can consider a covering family of rational curves and we can set F as in Lemma 5.32; since we are supposing $b < 0$, we have, again by Lemma 5.32, that F is a base component of $|k(-K_{\mathrm{rel}} + bH)|$.
We have that whenever we are in this situation, either F or $k(-K_{\mathrm{rel}} + bH) - F$ has to be numerically proportional to $-K_{\mathrm{rel}} + cH$, for some c such that $\rho \leqslant c \leqslant 0$. Indeed if we write

$$k(-K_{\mathrm{rel}} + bH) = F + [k(-K_{\mathrm{rel}} + bH) - F],$$

we have that the numerical class of one of the two summands has to lie in a ray under the ray spanned by $-K_{\mathrm{rel}} + bH$; therefore, since it has still to be pseudoeffective, its class has to lie in $R(-K_{\mathrm{rel}} + cH)$, for $\rho \leqslant c \leqslant b < 0$.
Now, if the class of F does not lie in $R(-K_{\mathrm{rel}}+cH)$, then we can apply the same procedure to $|k(-K_{\mathrm{rel}} + bH) - F|$ that again will have F as base component. After a finite number of steps we get that

$$|k'(-K_{\mathrm{rel}} + b'H)| = |k(-K_{\mathrm{rel}} + bH) - sF|$$

does not contain F as base component; but since b' is negative, this is a contradiction with Lemma 5.32.
Thus we can assume that $[F] \in R(-K + cH)$. With the same procedure as before, we get that every multiple sF of F has F in its base locus; in particular $|sF|$ has dimension zero and therefore F can't be big and its class has to lie in the second ray of the pseudoeffective cone that is, by definition, $R(-K_{\mathrm{rel}} + \rho H)$.
Let us focus on the second part. If $\mathcal{E}$ is unstable, then by (5.38) we have that $2\beta + c_1 < 0$; thus, by what we saw before, the non empty linear system

$$|1/2(-K_{\mathrm{rel}} + (2\beta + c_1)H)| = |1/2(2\xi - c_1 H + 2\beta H + c_1 H)| = |\xi + \beta H|$$

has a fixed component $F \equiv j(-K_{\mathrm{rel}} + \rho H)$. Since $|\xi + \beta H|$ consists of irreducible unisecant divisors, this can only happen when $F \in |\xi + \beta H|$. Hence

$$-K_{\mathrm{rel}} + (2\beta + c_1)H \in R(-K_{\mathrm{rel}} + \rho H),$$

that gives $\rho = 2\beta + c_1$. □

We are now ready to answer to Question 4.12.

Fact 5.34. *If $\mathcal{E}$ is unstable, then the effective cone is closed; in particular, a direct weak Zariski decomposition exists for every pseudoeffective divisor of $X = \mathbb{P}(\mathcal{E})$.*

Proof. Since, by Proposition 5.33 $\rho = 2\beta + c_1$, we have that the second ray of the pseudoeffective cone is indeed generated by

$$-K_{\mathrm{rel}} + \rho H = 2\xi - c_1 H + 2\beta H + c_1 H$$

that is proportional to $\xi + \beta H$; by definition of β, this is gives an effective class and the Fact is proven. □

Fact 5.35. *If $\mathcal{E}$ is semistable but not stable, then the effective cone is closed; in particular, a direct weak Zariski decomposition exists for every pseudoeffective divisor of $X = \mathbb{P}(\mathcal{E})$.*

Proof. Since $\mathcal{E}$ is semistable but not stable, by (5.38), we have that $\beta = -c_1/2$; since they are integers and $c_1 = 0$ or $c_1 = 1$, we get $\beta = c_1 = 0$ that immediately gives $-K_{\mathrm{rel}} = 2\xi$.
Now from equation (5.37) we see that $h^0(X, \mathcal{O}_X(1)) = h^0(Z, \mathcal{E}) \neq 0$; therefore ξ, and thus $-K_{\mathrm{rel}}$ is an effective class. This means that $\rho \leqslant 0$, but since we are in the semistable case, by Lemma 5.31, se get $\rho = 0$.
Thus the second ray of $\overline{\mathrm{Eff}}(X)$ is spanned by $-K_{\mathrm{rel}}$ that, as we pointed out, is effective. □

In order to try to work the stable case out, we have the following result; although we state it in considerable generality, we will see in the following section how it can be concretely used to give a positive answer to Question 4.12.
Let us recall that a normalized vector bundle $\mathcal{E}$ on a projective variety with Picard number one, is the smallest twist with sections, that is $h^0(\mathcal{E}) \neq 0$ and $h^0(\mathcal{E}(-1)) = 0$.

Proposition 5.36. *Let $\mathcal{E}$ be a normalized rank 2 vector bundle on a projective variety Z with $\mathrm{Pic}(Z) = \mathbb{Z}[H_Z]$; let us suppose that, for two integers a, b with $a < b$ and $a \leqslant 0$, there exists a set $\mathcal{M} = \mathcal{M}(a, b)$ of smooth rational curves $C \subset Z$ such that*

1. *the set spanned by $C \in \mathcal{M}$ dominates Z;*
2. *for any $C \in \mathcal{M}$, we have*

$$\mathcal{E}|_C = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b).$$

Then the effective cone of $\mathbb{P}(\mathcal{E})$ is closed; in particular, a weak Zariski decomposition does exist for every pseudoeffective divisor on $\mathbb{P}(\mathcal{E})$.
Moreover if R_2 is the second ray of $\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}))$, that is the ray different from the ray generated by H, the pullback of H_Z (see Fact 5.23), we have two possibilities:

1. *R_2 is generated by ξ;*
2. *if $\tilde{C}$, $C \in \mathcal{M}(a, b)$, is the minimal section of the ruled surface $Y_C := \mathbb{P}(\mathcal{E}|_C) \subset \mathbb{P}(\mathcal{E})$, we have that*
$$\Gamma := \overline{\bigcup_{C \in \mathcal{M}} \tilde{C}}$$
has one irreducible codimension one component that generates R_2.

Proof. Let us consider the surjection $\mathcal{E} \to \mathcal{E}|_C \to 0$; by Fact 4.18, we get:

$$Y_C := \mathbb{P}(\mathcal{E}|_C) \subset \mathbb{P}(\mathcal{E}), \quad \text{and} \quad \xi|_{Y_C} = \xi',$$

where ξ' is the class of the tautological divisor associated to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b))$. Since

$$(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)) \otimes \mathcal{O}_{\mathbb{P}^1}(-b) = \mathcal{O}_{\mathbb{P}^1}(a-b) \oplus \mathcal{O}_{\mathbb{P}^1}, \tag{5.40}$$

we have that

$$\xi|_{Y_C} = \xi' = \tilde{C} + bf; \tag{5.41}$$

by (5.40) and since $a < b$, with the notation of [Har77, Section V.2], Y_C is the rational ruled surface of invariant $e = b - a$.
Let us consider the pseudoeffective cone of $\mathbb{P}(\mathcal{E})$; since $\mathcal{E}$ is normalized, ξ is effective and we can assume

$$\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E})) = \langle H, \xi - \delta H \rangle, \quad \text{for some } \delta \geqslant 0. \tag{5.42}$$

If $\delta = 0$, we have that the second extremal ray is generated by ξ and we are done.
Thus let us assume that $\delta > 0$ and let us consider

$$\Gamma = \overline{\bigcup_{C \in \mathcal{M}} \tilde{C}}. \tag{5.43}$$

Since $\mathrm{Eff}(\mathbb{P}(\mathcal{E}))$ is spanned by the classes of prime effective divisors, we can assume that for some $k \in \mathbb{N}$ and $\epsilon \in \mathbb{Q}^{>0}$, there exists a prime effective divisor D such that

$$[D] = k(\xi - \epsilon H), \tag{5.44}$$

with $0 < \epsilon \leqslant \delta$ and $[D]$ lies in the region between $R(\xi)$ and $R(\xi - \delta H)$.
Let us focus for a moment on the restriction of D to Y_C; we can consider a general $C \in \mathcal{M}$ and we can thus suppose that $D|_{Y_C}$ is effective; moreover we have that

$$D|_{Y_C} \equiv k(\tilde{C} + (b - \epsilon t)f), \quad \text{where } t = C \cdot H_Z. \tag{5.45}$$

Since $\tilde{C} \subset Y_C$ and $a \leqslant 0$, we see that:

$$D \cdot \tilde{C} = D|_{Y_C} \cdot \tilde{C} = k(a - b + b - \epsilon t) = k(a - \epsilon t) < 0; \tag{5.46}$$

in particular we have that for every general C, $\tilde{C} \subset D$; therefore we get $\Gamma \subseteq D$. Thus we get $\dim \Gamma \leqslant \dim D = \dim Z$; but since there is a surjective map $\Gamma \to Z$ we get $\dim \Gamma \geqslant \dim Z$, we have that $\dim \Gamma = \dim D$.
Hence we see that there are irreducible components of Γ of dimension $\dim D$; since these components are inside D, that is a prime divisor, we have that there is just one component, say W, such that $W = D$.
Moreover we have seen that a prime divisor whose class is in the region $\langle \xi, \xi - \delta H\rangle$ (not lying in $R(\xi)$) is forced to be W.
To conclude the proof, we show that

$$\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E})) = \langle H, \xi - \delta H\rangle = \langle H, [W]\rangle. \tag{5.47}$$

Indeed one of the inclusions is obvious; to prove the other let us assume, by contradiction, that there exists a divisor $F \in \mathrm{Eff}(\mathbb{P}(\mathcal{E})) \setminus \langle H, [W]\rangle$; writing $F = \sum f_i F_i$, with $f_i > 0$ and F_i prime divisors, we have that there exists an i such that

$$[F_i] \in \overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E})) \setminus \langle H, [W]\rangle \subseteq \overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E})) \setminus \langle H, \xi\rangle,$$

arguing as before, we must have $F_i = W$, that is a contradiction. □

Corollary 5.37. *In the situation of the Proposition 5.36, if* $\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E})) = \langle H, \xi\rangle$ *and* $a < 0$, *then* $R(\xi)$ *is generated by an irreducible component of the closure of the union of* $\tilde{C}$.

Proof. Since ξ is effective, there is an effective divisor D such that $[D] = \xi$. Restricting it to Y_C, we get $D|_{Y_C} \equiv \tilde{C} + bf$; since

$$D \cdot \tilde{C} = D|_{Y_C} \cdot \tilde{C} = a - b + b = a < 0,$$

we have that the union of $\tilde{C} \subset D$. Arguing as before, an there is an irreducible component generating the ray. □

Corollary 5.38. *In the situation of Proposition 5.36, if* $\delta > 0$ *and* $W|_{Y_C}$ *is irreducible, then*

$$\overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E})) = \langle H, t\xi - bH\rangle.$$

Proof. By the proposition, $W \equiv k(\xi - \epsilon H)$, we have that

$$W|_{Y_C} \equiv k(\tilde{C} + (b - \epsilon t)f);$$

since it is irreducible and $b - \epsilon t < b - a$, by [Har77, Corollary V.2.18] we have that it must be $\tilde{C}$. Hence we get $k = 1$ and $b - \epsilon t = 0$, that gives

$$W \equiv \xi - \frac{b}{t}H,$$

concluding the proof. □

5.4 Schwarzenberger bundles on the plane

In this section we focus on stable vector bundles in $\mathbb{P}^2$; in particular, as consequence of Proposition 5.36, we will give a positive answer to Question 4.12 in some meaningful situations.
We start considering an important set of stable rank 2 vector bundles on $\mathbb{P}^2$, the Schwarzenberger bundles; we refer to [Val00] for further details. The author of [Sch61] proved that any rank 2 vector bundle on $\mathbb{P}^2$ can be obtained from an invertible sheaf $\mathcal{L}$ over a smooth surface S that is a double covering of $\mathbb{P}^2$ and, in particular, he focuses on coverings ramified along a smooth conic.
Let us consider the following construction: let $F \subset \mathbb{P}^2 \times \mathbb{P}^{2\vee}$ be the incidence variety between points and lines of $\mathbb{P}^2$ and let $C \subset \mathbb{P}^{2\vee}$ be a smooth conic; setting $S := F \cap (\mathbb{P}^2 \times C)$, we have the diagrams

$$\begin{tikzcd} F \arrow[r, "\bar{q}"] \arrow[d, "\bar{p}"'] & \mathbb{P}^{2\vee} \\ \mathbb{P}^2 & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S \arrow[r, "q"] \arrow[d, "p"'] & C \\ \mathbb{P}^2 & \end{tikzcd}$$

We can thus give the definition.

Definition 5.39. Referring to the former construction, we say that a *Schwarzenberger vector bundle* associated to the conic C is the rank 2 vector bundle on $\mathbb{P}^2$

$$\mathcal{E}_{n,C} := (p_* q^* \mathcal{O}_{\mathbb{P}^1}(n)). \tag{5.48}$$

Remark 5.40. Let us recall some useful facts about Schwarzenberger bundles:

1. for *small* degrees, we have
$$\mathcal{E}_{0,C} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1), \quad \mathcal{E}_{1,C} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}, \quad \mathcal{E}_{2,C} = \Omega^{\vee}_{\mathbb{P}^2}(-1);$$
2. we have: $\mathcal{E}_{-i,C} = \mathcal{E}_{i,C}(-i)$;

In view of our goal, the existence of the weak Zariski decomposition, since for small values of n these bundles are decomposable, by Proposition 5.24 and Remark 5.40, we can suppose that $n \geqslant 2$.
In order to use our Proposition 5.36 to answer to Question 4.12, we must find a set of curves with constant splitting type; the following result provide what we need.

Fact 5.41. *If $C^\vee \subset \mathbb{P}^2$ is the conic dual to C, then for any tangent line L to $C^\vee$, we have the splitting*
$$\mathcal{E}_{n,C}|_L = \mathcal{O}_{\mathbb{P}^1}(n-1) \oplus \mathcal{O}_{\mathbb{P}^1}.$$

Proof. See the proof of [Val00, Remarque 1.3]. □

Therefore we can prove the following result.

Proposition 5.42. *Let $\mathcal{E}_{n,C}$ be a Schwarzenberger bundle on $\mathbb{P}^2$, then the effective cone of $X = \mathbb{P}(\mathcal{E}_{n,C})$ is closed and thus there exists a direct weak Zariski decomposition for every pseudoeffective divisor.*

Proof. The result follows from Proposition 5.36 where the dominating set $\mathcal{M}(a,b) = \mathcal{M}(0, n-1)$ is given by lines tangent to the dual conic $C^\vee$. □

5.5 Stable rank 2 bundles on the projective space

Following [Har78b], we focus now on the case of a stable vector bundle $\mathcal{E}$ of rank 2 on the projective space $\mathbb{P}^3$. Our main tool will be the well-known Hartshorne-Serre correspondence.
Under this correspondence, to the pair $(\mathcal{E}, s)$, where $\mathcal{E}$ is a rank 2 vector bundle on $\mathbb{P}^3$ and $0 \neq s \in H^0(\mathbb{P}^3, \mathcal{E})$, is associated a locally complete intersection curve $Y \subset \mathbb{P}^3$ (see [Har78b, Theorem 1.1]), where Y is the scheme of zeros of the section s.
In particular, we have the following exact sequence:

$$0 \to \det \mathcal{E}^\vee \to \mathcal{E}^\vee \to \mathcal{J}_Y \to 0, \tag{5.49}$$

where $\mathcal{J}_Y$ is the ideal sheaf defining $Y \subset \mathbb{P}^3$.
It may be worth of interest to point out the following result, relating the numerical invariants of Y and of $\mathcal{E}$ (see [Har78b, Proposition 2.1]).

Proposition 5.43. *Let $(\mathcal{E}, s)$ be a pair, where $\mathcal{E}$ is a rank 2 vector bundle on $\mathbb{P}^3$ and $s \in H^0(\mathbb{P}^3, \mathcal{E})$, corresponding to a curve Y of degree d and arithmetic genus p_a, then:*

$$d = c_2(\mathcal{E}); \qquad \text{and} \qquad 2p_a - 2 = c_2(\mathcal{E})(c_1(\mathcal{E}) - 4). \tag{5.50}$$

Now, focusing on rank 2 stable vector bundles, we have the following characterization, see [Har78b, Proposition 3.1].

Proposition 5.44. *Let $(\mathcal{E}, s)$ be the pair consisting of a vector bundle on $\mathbb{P}^3$ and $s \in H^0(\mathbb{P}^3, \mathcal{E})$ corresponding to a curve Y; $\mathcal{E}$ is stable (respectively semistable) if and only if*

1. *$c_1(\mathcal{E}) > 0$ (respectively $c_1(\mathcal{E}) \geqslant 0$);*
2. *Y is not contained in any surface of degree smaller or equal than $\frac{1}{2}c_1(\mathcal{E})$ (respectively smaller than $\frac{1}{2}c_1(\mathcal{E})$).*

Thus is it easily possible to construct examples of stable vector bundles starting from curves (eventually reducible) of $\mathbb{P}^3$.
Here we recall three examples exposed in [Har78b, Section 3]; the stability of the bundle associated to the curve follows from Proposition 5.44.

Example 5.45 ($r \geqslant 2$ skew lines)**.** Let us consider Y as the union of r disjoint lines in $\mathbb{P}^3$ and $\mathcal{E}$ the associated rank 2 vector bundle. By (5.50), we easily get the invariants of the bundle: $c_1 = c_1(\mathcal{E}) = 2 > 0$ and $c_2 = c_2(\mathcal{E}) = r$; since, for $r \geqslant 2$, Y is not contained in a plane, we have that $\mathcal{E}$ is stable.

Example 5.46 ($r \geqslant 2$ disjoint conics)**.** Let $Y \subset \mathbb{P}^3$ be the union of r disjoint conics. We hae that Y corresponds to a bundle $\mathcal{E}$ with $c_1 = 3$ and $c_2 = 2r$. If $r \geqslant 2$, the curve is not contained in any plane and thus $\mathcal{E}$ is stable.

Example 5.47 (Plane cubic and elliptic curve)**.** Let Y be the disjoint union of a non singular plane cubic curve and a non singular elliptic space curve of degree $r \geqslant 4$. We have that Y corresponds to a bundle with $c_1 = 4$ and $c_2 = r + 3$. The elliptic curve is not contained in a plane and thus Y is not contained in any surface of degree 2 and $\mathcal{E}$, again is stable.

Let us consider a rank 2 stable vector bundle $\mathcal{E}$ associated to a curve $Y \subset \mathbb{P}^3$; by Proposition 5.44 and the stability of $\mathcal{E}$, we have $c_1(\mathcal{E}) > 0$; since $\det \mathcal{E}^\vee = \mathcal{O}_{\mathbb{P}^3}(-c_1)$, $\mathcal{E}^\vee = \mathcal{E}(-c_1)$, by tensoring equation (5.49), we get the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{E} \to \mathcal{J}_Y(c_1) \to 0. \tag{5.51}$$

Since $\mathcal{E}$ has a section, this gives at once, in view of the pseudoeffective cone of $\mathbb{P}(\mathcal{E})$, that the tautological class $\xi = \xi_{\mathbb{P}(\mathcal{E})}$ is effective.
Let us consider the restriction of $\mathcal{E}$ to a degree t rational curve C meeting Y in exactly $e = \#\{Y \cap C\}$ points. By construction a section of $\mathcal{E}$ vanishes on Y, thus restricting (5.51) to the curve $C \simeq \mathbb{P}^1$, we get

$$0 \to \mathcal{O}_{\mathbb{P}^1}(e) \to \mathcal{E}|_C \to \mathcal{O}_{\mathbb{P}^1}(-e + tc_1) \to 0. \tag{5.52}$$

In particular, if $2e - tc_1 > -2$, we get the splitting:

$$\mathcal{E}|_C = \mathcal{O}_{\mathbb{P}^1}(e) \oplus \mathcal{O}_{\mathbb{P}^1}(-e + tc_1). \tag{5.53}$$

Indeed in order to get the splitting, it is enough to show the vanishing of the group $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(-e + tc_1), \mathcal{O}_{\mathbb{P}^1}(e))$.
Now, since

$$\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(-e + tc_1), \mathcal{O}_{\mathbb{P}^1}(e)) = H^1(\mathcal{O}_{\mathbb{P}^1}(2e - tc_1)),$$

if $-2e + tc_1 - 2 < 0$ we get the vanishing.
With this splitting, we can solve our problem in the case of Example 5.45.

Fact 5.48. *Let $\mathcal{E}$ be the bundle over $\mathbb{P}^3$ associated to $r \geqslant 2$ disjoint lines in $\mathbb{P}^3$ as in Example 5.45; then $\mathrm{Eff}(\mathbb{P}(\mathcal{E}))$ is closed and thus a direct weak Zariski decomposition exists for every pseudoeffective divisor on $\mathbb{P}(\mathcal{E})$.*
Moreover, if $r \geqslant 3$, then $\mathrm{Nef}(\mathbb{P}(\mathcal{E})) \neq \overline{\mathrm{Eff}}(\mathbb{P}(\mathcal{E}))$.

Proof. Let us consider the set of lines C meeting Y in $e = 2$ points; since $c_1(\mathcal{E}) = 2$, from (5.53), we get the splitting $\mathcal{E}|_C = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}$.
Therefore, since the lines of Y are skew, we get a dominating set of lines $\mathcal{M}(0, 2)$ with constant splitting type; the result follows from Proposition 5.36.
To prove the second statement, if C is a line meeting 3 lines of Y, the (5.52) becomes

$$0 \to \mathcal{O}_{\mathbb{P}^1}(3) \to \mathcal{E}|_C \to \mathcal{O}_{\mathbb{P}^1}(-1) \to 0,$$

giving $\mathcal{E}|_C = \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ that is not nef. Thus the tautological class ξ associated to $\mathcal{E}$ can't be nef and the two cones are different. □

5.6 Further remarks

In this section we point out that, as it is well-known, Question 4.5 has a positive answer in the general setting of Mori dream spaces. For clearness' sake we recall a couple of gereral facts about the theory of MDS and we refer to [HK00] for further details.
Before the definition of Mori dream space, we recall an important class of birational transformation appearing in this setting.

Definition 5.49. A *small $\mathbb{Q}$-factorial modification (SQM)* of a normal variety X is a birational map $f : X \dashrightarrow X'$ such that X' is a projective $\mathbb{Q}$-factorial variety and f is an isomorphism in codimension 1.

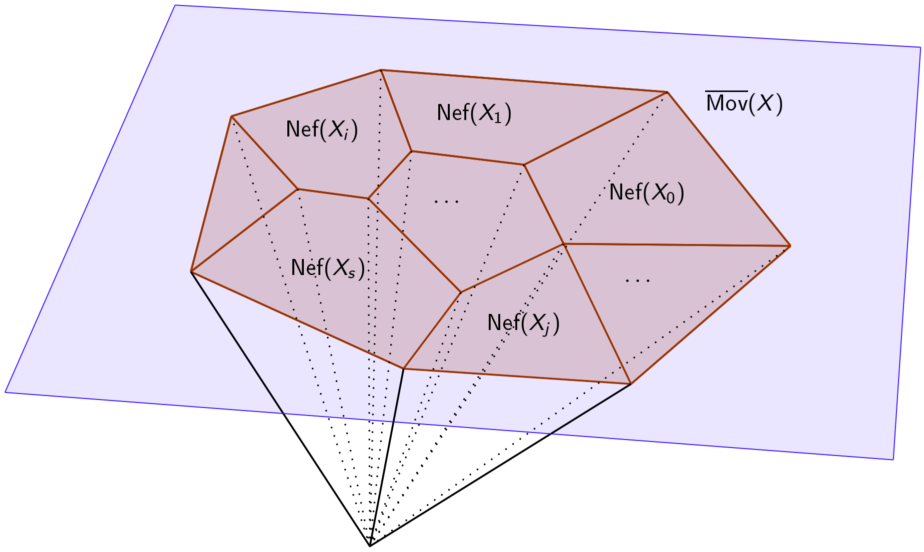

Figure 5.1: The decomposition of $\overline{\mathrm{Mov}}(X)$ in movable chambers

The most important examples of SQMs are undoubtedly *flips*.
We point out that, since a SQM is an isomorphism in codimension 1, then it does not affect the divisors and thus $N^1(X) = N^1(X')$.

Definition 5.50 (Mori dream spaces)**.** A normal variety X is said to be a *Mori dream space (MDS)* if the following hold:

1. X is $\mathbb{Q}$-factorial and $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}}$;
2. the $\mathrm{Nef}(X)$ cone is the convex hull of a finite number of extremal rays spanned by semiample line bundles;
3. there is a finite collection of SQMs $f_i : X \dashrightarrow X_i$, $i = 0, \ldots, s$ such that each X_i satisfies condition 2. and, setting $X_0 = X$ and $f_0 = \mathrm{id}$, we have the decomposition

$$\overline{\mathrm{Mov}}(X) = \bigcup_{i=0}^{s} f_i^*(\mathrm{Nef}(X_i)). \tag{5.54}$$

Remark 5.51. The decomposition of the movable cone given by equation (5.54) can be pictured as in Figure 5.1. The subsets $f_i^*(\mathrm{Nef}(X_i))$ are called *movable chambers* of $\overline{\mathrm{Mov}}(X)$; since we are considering isomorphisms in codimension 1, it make sense to picture the movable chambers in the same space.
Moreover it can be proved (see [HK00, Proposition 1.11]) that two adjacent chambers are related by a flip.

Remark 5.52. If can be shown, see [HK00, Proposition 1.11(2)], that the pseudoeffective cone of a Mori Dream Space is the convex hull of finitely many effective divisors. In particular, we have that the effective cone is closed and thus the answer to Question 4.12 is obviously positive.

Remark 5.53. In view of our general discussion about projectivized variety, we have to say that has been recently shown by González in [Gon10] that the projecivized of a rank two toric vector bundle on a toric variety is indeed a Mori dream space.
Therefore we give a positive answer to Question 4.12 in the case of a toric variety with a rank 2 toric bundle. Let us recall what a toric bundle is.
To this end we have to define the *geometric vector bundle* associated to the vector bundle $\mathcal{E}$ on Z; it is the variety

$$\mathbb{V}(\mathcal{E}) := \mathrm{Spec} \bigoplus_{m \geqslant 0} S^m \mathcal{E}^\vee. \tag{5.55}$$

Now if Z is a toric variety, then we have the associated torus T and we say that a *toric vector bundle* $\mathcal{E}$ on Z is a vector bundle $\mathcal{E}$ together with an action of the torus T on the variety $\mathbb{V}(\mathcal{E})$, such that the projection $\varphi : \mathbb{V} \to Z$ is equivariant and T acts linearly on the fibres of φ.

List of Figures

Bibliography

[Bad01] Lucian Badescu, *Algebraic surfaces*, Universitext, Springer-Verlag, New York, 2001, Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author.

[BDPP04] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Paun, and Thomas Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, 2004, arXiv:math/0405285v1 [math.AG].

[Bea96] Arnaud Beauville, *Complex algebraic surfaces*, second ed., London Mathematical Society Student Texts, vol. 34, Cambridge University Press, Cambridge, 1996, Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid.

[BF01] Valentina Beorchia, Davide Franco, *On the moduli space of 't Hooft bundles*, Annali dell'Università di Ferrara **47** (2001), 253–268.

[BHK+11] Thomas Bauer, Brian Harbourne, Andreas L. Knutsen, Alex Küronya, Stefan Müller-Stach, Xavier Roulleau, Tomasz Szemberg, *Negative curves on algebraic surfaces*, 2012, arXiv:1109.1881v2 [math.AG].

[BHR06] Ugo Bruzzo and Daniel Hernández Ruipérez, *Semistability vs. nefness for (Higgs) vector bundles*, Differential Geom. Appl. **24** (2006), no. 4, 403–416.

[Bir09] Caucher Birkar, *On the existence of log minimal models and weak zariski decomposition*, 2009, arXiv:0907.5182v1 [math.AG].

[BL04] Christina Birkenhake and Herbert Lange, *Complex abelian varieties*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 302, Springer-Verlag, Berlin, 2004.

[BPV84] Wolf P. Barth, Chris A. M. Peters, and Antonius Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 4, Springer-Verlag, Berlin, 1984.

[BS95] Mauro C. Beltrametti and Andrew J. Sommese, *The adjunction theory of complex projective varieties*, de Gruyter Expositions in Mathematics, vol. 16, Walter de Gruyter & Co., Berlin, 1995.

[Deb01] Olivier Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, New York, 2001.

[dF05] Tommaso de Fernex, *Negative curves on very general blow-ups of* $\mathbb{P}^2$, Projective varieties with unexpected properties, Walter de Gruyter GmbH & Co. KG, Berlin, 2005, pp. 199–207.

[dF10] ______, *On the Mori cone of blow-ups of the plane*, 2010, arXiv:1001.5243v2 [math.AG].

[DVL05] Cindy De Volder and Antonio Laface, *Linear systems on generic* $K3$ *surfaces*, Bull. Belg. Math. Soc. Simon Stevin **12** (2005), no. 4, 481–489.

[Fuj79] Takao Fujita, *On Zariski problem*, Proc. Japan Acad. Ser. A Math. Sci. **55** (1979), no. 3, 106–110.

[Flt98] William Fulton, *Intersection theory*, 2 ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 2, Springer-Verlag, 1998.

[Flg11] Mihai Fulger, *The cones of effective cycles on projective bundles over curves*, Math. Z. **269** (2011), no. 1-2, 449–459.

[Gim87] Alessandro Gimigliano, *On linear systems of plane curves*, ProQuest LLC, Ann Arbor, MI, 1987, Thesis (Ph.D.)–Queen's University (Canada).

[Gon10] José Luis González, *Projectivized rank two toric vector bundles are Mori dream spaces*, 2010, arXiv:1001.0838v2 [math.AG].

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[Har78a] ______, *Stable vector bundles and instantons*, Communications in Mathematical Physics **59** (1978), 1–15.

[Har78b] ______, *Stable vector bundles of rank 2 on* $\mathbb{P}^3$, Mathematische Annalen **238** (1978), 229–280.

[Har86] Brian Harbourne, *The geometry of rational surfaces and Hilbert functions of points in the plane*, Proceedings of the 1984 Vancouver conference in algebraic geometry (Providence, RI), CMS Conf. Proc., vol. 6, Amer. Math. Soc., 1986, pp. 95–111.

[Har10] ______, *Global aspects of the geometry of surfaces*, Ann. Univ. Paedagog. Crac. Stud. Math. **9** (2010), 5–41.

[Hir89] André Hirschowitz, *Une conjecture pour la cohomologie des diviseurs sur les surfaces rationnelles génériques*, J. Reine Angew. Math. **397** (1989), 208–213.

[HK00] Yi Hu and Sean Keel, *Mori dream spaces and GIT*, Michigan Math. J. **48** (2000), 331–348, Dedicated to William Fulton on the occasion of his 60th birthday.

[KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, vol. 134, Cambridge University Press, Cambridge, 1998.

[Kol96] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996.

[Laz04] Robert Lazarsfeld, *Positivity in Algebraic Geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, 49, Springer-Verlag, Berlin, 2004.

[Miy87] Yoichi Miyaoka, *The Chern classes and Kodaira dimension of a minimal variety*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 449–476.

[MOSC11] Roberto Muñoz, Gianluca Occhetta, and Luis Solá Conde, *On rank 2 vector bundles on Fano manifolds*, 2011, arXiv:1104.1490v3 [math.AG].

[Mor82] Shigefumi Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math. (2) **116** (1982), no. 1, 133–176.

[Nag59] Masayoshi Nagata, *On the* 14*-th problem of Hilbert*, Amer. J. Math. **81** (1959), 766–772.

[Nak04] Noboru Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004.

[Roc97] R. Tyrrell Rockafellar, *Convex analysis*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997, Reprint of the 1970 original, Princeton Paperbacks.

[Sch61] R. L. E. Schwarzenberger, *Vector bundles on the projective plane*, Proc. London Math. Soc. (3) **11** (1961), 623–640.

[Seg62] Beniamino Segre, *Alcune questioni su insiemi finiti di punti in geometria algebrica.*, Atti Convegno Internaz. Geometria Algebrica (Torino, 1961), Rattero, Turin, 1962, pp. 15–33.

[Val00] Jean Vallès, *Fibrés de Schwarzenberger et coniques de droites sauteuses*, Bull. Soc. Math. France **128** (2000), no. 3, 433–449.

[Zar62] Oscar Zariski, *The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface*, Ann. of Math. (2) **76** (1962), 560–615.