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# Efficient tree methods for option pricing

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by

**Elisa Appolloni**

**Program Coordinator**
Prof. Dr. Maria B. Chiarolla

**Thesis Advisors**
Prof. Dr. Lucia Caramellino
Prof. Dr. Antonino Zanette

*To my family*

# Contents

# Introduction

The aim of this dissertation is the study of efficient algorithms based on lattice procedures for dealing with two relevant issues arising in the recent literature on option pricing: the pricing of complex barrier-type options and the pricing of options when the equity model takes into account a stochastic interest rate. This research is developed with a twofold perspective: first, we propose a good solution from a numerical point of view through the introduction of efficient lattice procedures and secondly, we study the theoretical aspects related to the tackled problems. Tree-based algorithms for option pricing are studied since the seminal work of Cox, Ross and Rubinstein ([28], 1979) and turn out to be very simple and fast to be implemented by a backward induction. An important characteristic which makes these procedures very appealing is that they easily include American-style features once the European case is treated and well set up. This makes lattice techniques widely used in the practice because although many progresses have been done in the development of exact formulas or other numerical procedures (Monte Carlo, finite differences, etc.) for European option prices, the American counterparts, that involve a control problem, are not so well-provided.

The mathematical background underlying the approximation of diffusion processes with tree methods is briefly developed in Chapter 1. Roughly speaking, we can recall such methods as follows. Let $X$ denote a diffusion process, that is the solution to the following stochastic differential equation (SDE):

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0. \tag{0.0.1}$$

For the sake of simplicity, we assume to be in the 1-dimensional case, so the drift coefficient $b$ and the diffusion coefficient $\sigma$ denote suitable functions and $B$ stands for a standard Brownian motion in $\mathbb{R}$. A tree-method consists in approximating the evolution of $X$ over a time interval $[0, T]$ by means of a suitable Markov chain: one fixes $0 = t_0 < t_1 < \cdots < t_n = T$ with $t_k = kh = kT/n$, $k = 1, \dots, n$, and construct a Markov chain $(X^h_k)_{k=0,\dots,n}$ such that, as $h \to 0$, for every $k$ then $X^h_k$ is "close to" $X_{t_k}$. More precisely, by setting $\bar{X}^h$ as the continuous-time process given by the linear interpolation in time of the Markov chain $X^h$, that is

$$\bar{X}^h_t = \frac{(k+1)h - t}{h} X^h_k + \frac{t - kh}{h} X^h_{(k+1)h}, \quad kh \le t < (k+1)h, \tag{0.0.2}$$

then the Markov chains $X^h$ are set in order that $(\bar{X}^h)_h$ converges in law on the space of the continuous paths over $[0, T]$ to the diffusion process $X$ solution to (0.0.1). But what about the Markov chain? It is built possibly in a "simple way", that is as a Markov process (in discrete time and space) running on a computationally simple lattice. We deal here with a binomial lattice: at each time-step $k$, the Markov process may do only two jumps, an up-jump or a down-jump, and the jump-probabilities are set in order to achieve the convergence, see Section 1.1.3. This means that the approximating process evolves on a very simple structure, see Figure 1.

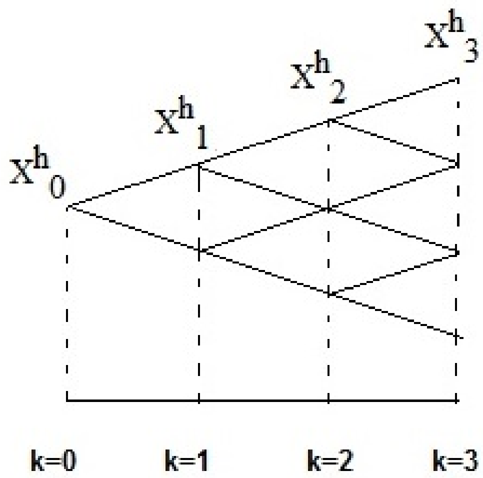


Figure 1: *Binomial tree with 3 time steps.*

We consider here the Black and Scholes model, which is either classical and still widely used in finance. This means that the underlying asset price process $(S_t)_{t\in[0,T]}$ evolves as the diffusion process (0.0.1) in which $b$ and $\sigma$ are chosen linear and non-affine that is

$$dS_t = rS_t dt + \sigma S_t dB_t, \quad S_0 = s_0 > 0, \tag{0.0.3}$$

where $r$ is the interest rate and $\sigma$ is the volatility parameter. For more details one can refer to Black and Scholes ([13], 1973) and Merton ([64], 1973). Notice that (0.0.3) says that we are writing the dynamics under the risk-neutral measure. We recall that (0.0.3) means that $(S_t)_{t\in[0,T]}$ evolves as a geometric Brownian motion and, roughly speaking, in a small time interval $\Delta t$, the percentage variation $\frac{\Delta S_t}{S_t}$ is approximately a Gaussian r.v. with mean $r\Delta t$ and variance $\sigma^2 \Delta t$.
Option prices with the underlying stock price process following the SDE (0.0.3) can be computed by using the simple tree method due to Cox, Ross and Rubinstein ([28], 1979), CRR in what follows. We briefly recall the procedure in [28]. We define $h = T/n$ and then

we build a binomial tree with $n$ time-steps of length $h$. We label $(0,0)$ the starting node that corresponds to the value $S_0 = s_0$. At time step $ih$, the discrete process may be located at one of the nodes $(i,j)$ corresponding to the values

$$S_{i,j} = s_0 e^{(2j-i)\sigma\sqrt{h}}, \quad i = 0,1,...,n, \quad j = 0,1,...,i. \tag{0.0.4}$$

Hence, starting from $S_{i,j}$ at time $ih$, the process may jump at time $(i+1)h$ to the value $S_{i+1,j+1}$ or to the value $S_{i+1,j}$ with probability $p$ and $1-p$ respectively, where $p$ is defined as

$$p = \frac{e^{rh} - d}{u - d}, \tag{0.0.5}$$

where $u = e^{\sigma\sqrt{h}} = d^{-1}$. We remark again that the advantage of this procedure is that both European and American prices can be easily calculated.
However financial derivatives have been becoming more and more sophisticated and this means that the standard implementation of the CRR binomial tree brings to further errors in the approximation of the Black and Scholes prices. This is the reason why it becomes important to set up "efficient tree schemes", that are tree methods which allow one to reduce the approximation errors.
In what follows, we briefly present the innovative contribution given in this thesis with respect to the existing literature on some complex option pricing topics, without leaving out the reasons for which they are intensively used in practice. More precisely, we deal with the following arguments:

1. double and multi-step double barrier options;

2. barrier options on discontinuous payoff functions.

Then, we build and study a more complex efficient tree for:

3. options in a model where the interest rate is no more constant and deterministic and follows the Cox, Ingersoll and Ross process ([27], 1985), CIR hereafter.

We now describe in details the content of this dissertation by means of three paragraphs that correspond to the three main chapters of this thesis.

# 1. Double and multi-step double barrier options

Barrier options are path-dependent options that become activated or nullified if the underlying asset price reaches certain levels. Double barrier options are characterized by two price levels that are located above (higher barrier) and below (lower barrier) the initial stock price. If we denote with $f$ a generic payoff function, then the payoff of a double barrier knock-out option with payoff $f$ is given by:

$$f(S_T)\mathbb{1}_{S_{\inf}>L, S_{\sup}<H}, \quad S_{\inf} = \inf_{t\in[0,T]} S_t \quad \text{and} \quad S_{\sup} = \sup_{t\in[0,T]} S_t, \tag{0.0.6}$$

where $L$ and $H$ denote the lower barrier and the higher barrier respectively. In particular we consider call and put options, i.e. $f(x) = \max(\theta(x - K), 0)$ with $\theta = 1$ for call options and $\theta = -1$ for put options, $K$ standing for the strike price.

Nowadays barrier options are frequently traded because they have some features that make them more attractive than standard or vanilla options. In fact they are less expensive than vanilla contracts because they can be knocked-out or knocked-in. Moreover, they are much more flexible than standard contracts because they allow to set knock-out or knock-in levels depending on the expectations and the needs of the customers. For example they are widely used in the foreign exchange markets. In fact barrier options are embedded in financial products used to allow for hedging strategies in order to protect the holder from a potential appreciation and/or depreciation of the foreign currency with respect to the domestic one. They are also standard ingredients of a variety of structured products of range-type and it is also possible to find a barrier-type structure for example in loans and forward contracts (for details see Wystup ([86], 2006)).

A closed-form formula for pricing European double barrier call and put options when the underlying process follows the SDE (0.0.3) exists when the two barriers are exponential functions of the time (see Kunitomo and Ikeda ([56], 1992)). In the special case in which the barriers are constant, the price can be derived by using techniques involving the Laplace transform of the option price (see Geman and Yor ([40], 1996)). This issue has also been approached by authors such as Kolkiewicz ([55], 1997), Sidenius ([73], 1998) and Pelsser ([67], 2000). Quasi-analytical expressions for American options are presented in Gao and Subrahmanyam ([36], 2000), but we remark that here just a single barrier is considered.

Since in all the previous cases it is not possible to price American-style double barrier call and put options, numerical methods have been examined in the literature. It is known that a naive application of the standard CRR binomial tree may lead to a very slow convergence if the barrier is not chosen at a sufficiently small distance with respect to a layer of nodes of the tree (see Boyle and Lau ([16], 1994)). Then a possible solution is to set the algorithm such that the barrier lies exactly on the lattice, as in Ritchken ([72], 1995), Cheuk and Vorst ([22], 1996), Figlewski and Gao ([34], 1999), Gaudenzi and Lepellere ([37], 2006) and Gaudenzi and Zanette ([39], 2009). However, all the previous lattice methods are only able to deal with a single barrier. The first attempt that considers the possibility of efficiently pricing double barrier options with a tree method is due to Dai and Lyuu ([29], 2010). They are able to construct a binomial mesh by choosing the time step such that both the lower barrier and the upper barrier are exactly on two nodes of the tree at maturity (see Section 2.4 for details). However their method is not able to deal with the "near barrier problem", that consists in a failure of the computational procedure (unless one drastically increases the number of time steps of the algorithm) when the initial asset price is close to one of the barriers.

Then we introduce a new algorithm, called the Binomial Interpolated Lattice (BIL hereafter), that is based on the Dai-Lyuu idea of defining the time step $\Delta t$ such that the barriers are exactly matched at maturity $T$ with two layer of nodes in the lattice. As explained in Section 2.5, the time step $\Delta t$ is obliged to take some specific values in order to match both the lower barrier $L$ and the higher barrier $H$ and this implies that $\frac{T}{\Delta t} \notin \mathbb{N}$. In order to arrive close to time 0, we add two further steps of length $\Delta t$ and so we get a fictitious time $t_0 < 0$ and a time $t_1 > 0$, see Figure 2.

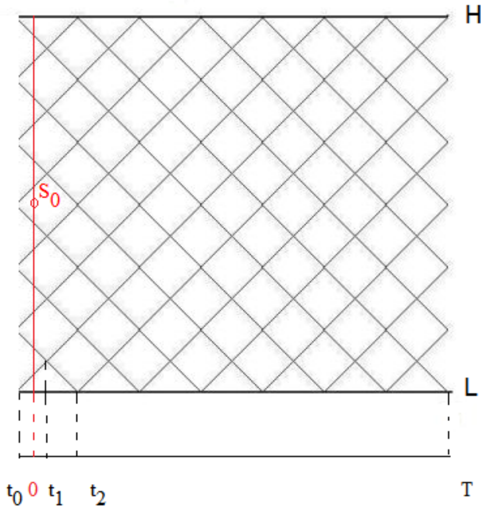


Figure 2: *Binomial interpolated lattice mesh.*

Since we do not know a priori if the initial price $s_0$ is a point of the lattice (and in general it is not), the approximating option price at $(0, s_0)$ is provided by suitable interpolations in time and in space involving the prices which are computed by the standard backward induction at times $t_0$ and $t_2$. We also notice that the prices at $t_0 < 0$ have not a financial meaning, but from the mathematical point of view we are supposing to extend by continuity the price function for negative times and this allows us to set up the computational procedure.
The proposed algorithm turns out to be efficient: it gives accurate results for every value of the initial asset price. In particular, numerical experiments show that it provides precise double barrier option prices when compared to the Kunitomo and Ikeda closed-form formula ([56], 1992) and also to the finite difference method of Zvan, Forsyth and Vetzal ([87], 2000). We also remark that the values of the delta, vega and gamma computed by using a finite difference approximation on the prices given by the lattice turn out to be comparable with the ones obtained with the finite differences approximation in [87].
Our method can also be applied to the case in which the interest rate and the volatility (this is indeed the more interesting case) are piecewise constant functions of time.
From a theoretical point of view we provide the speed of convergence of the algorithm by using PDE techniques as in Gobet ([42], 2001). For every $n \in \mathbb{N}$, where $n$ denotes the number of time steps of the tree, we define the approximation error as follows:

$$\mathrm{Err}_{BIL}(n) = \mathrm{price}_{BIL}(n) - \mathrm{price}_{BS}, \qquad (0.0.7)$$

where $\mathrm{price}_{BIL}(n)$ denotes the option price value computed by using our algorithm and $\mathrm{price}_{BS}$ is the Black and Scholes formula that gives the continuous option price. Then we get that

$$\text{Err}_{BIL}(n) = o\left(\frac{1}{n^\alpha}\right), \quad \forall\, \alpha \in (0,1),$$

that essentially means that the algorithm is of order $\frac{1}{n}$.
Moreover our lattice procedure represents an efficient way to solve the "near barrier" problem occurring in Dai-Lyuu ([29], 2010) and the algorithm is robust because it is not affected by the choice of the input parameters as it follows from our numerical results (Section 2.8).
However, in their standard form, double barrier options are not so flexible because the contractual barriers are assumed to be constant during all the option lifetime. For this reason, 2-step and more generally multi-step double barrier options, i.e. options in which the barriers evolve in time as piecewise constant functions, have been introduced (see Guillaume ([43], 2010)).
To be precise, a regular $m$-step double barrier option is an option in which the lifetime $[0, T]$ is divided into $m$ intervals $[T_i, T_{i+1}]$, for $i = 0, 1, ..., m-1$, and at each time interval a constant lower barrier $L_i$ and a constant higher barrier $H_i$ are contractually associated. For example a regular $m$-step double knock-out option with payoff function $f$, has this payoff at maturity provided that the underlying asset price stays in $(L_i, H_i)$ in every interval $[T_i, T_{i+1}]$, otherwise it expires worthless or provides a contractual rebate. We recall here that the rebate is the amount paid to the holder if the option expires worthless.
These options turn out to be innovative products because they allow to adjust the barrier levels according to the investor's level of risk-aversion and so they combine together cost saving and protection. For example, if it is known or expected at the contract inception time that some events that can affect the risk of knocking-out will occur during the option lifetime, then investors may want to widen the level barriers as a form of protection accepting a moderate increase in the hedging cost. Instead, if the investors anticipate the volatility of the underlying asset price to decrease during a certain period and then need for a lower protection, they can decide to narrow the barriers in order to reduce the hedging costs. Those possibilities are not allowed in the standard double barrier case. In fact, if one has no choice and is constrained to hold a double barrier option with wide barriers, one may be over-protected during some periods and the hedging costs might be relatively high to the needs. Specularly, if one holds a double barrier option with too narrow barriers one might be under-protected and the risk exposure will be too high. As remarked before, the levels of the barriers are contractually specified and this could represent a limit of these products. However, as far as we know, no one in the literature has ever treated the problem of pricing options with random barriers and it is not an issue we consider in this thesis.
Multi-step double barrier options are also introduced in order to manage the danger of "sudden death", that happens when the option is knocked-out at the first passage time on a knocked-out level. For a detailed discussion from a financial point of view and an analysis of the advantages of multi-step double barrier options one can refer to Guillaume ([43], 2010).
The valuation of these contracts is thus an important and practical question. If in the European case a closed-form formula for call and put options exists when the number of

steps in which the option lifetime interval is divided is equal to two (i.e. for 2-step double barrier options, see Guillaume ([43], 2010) for details), no analytical formulas exist in the more general case (multi-step double barrier options). Moreover, we stress again that barrier options typically include American features and there are no formulas in this case even for 2-step double barrier options.
We develop here an extension of the BIL lattice procedure to this more general case. In Section 2.7 we see that it is easy to implement the procedure when the barriers evolve as a piecewise constant function of time. In fact we just need to set the "right" binomial mesh for the time intervals $[T_i, T_{i+1}]$ by choosing for every $i = 0, 1, ..., m$ the time step such that the barriers $L_i$ and $H_i$ match exactly two layer of nodes of the tree as for the simpler case of two constant barrier levels. Then we need to connect the meshes built for the different time intervals and this is done by using again suitable interpolations.
The procedure proposed can also be used to price contracts in which knock-out or knock-in barrier provisions are removed in some time intervals. In particular we will consider early-ending multi-step double knock-out call options, i.e. multi-step double knock-out call option in which there is no "out" condition in the last time interval.
In Section 2.8 numerical results obtained with the BIL algorithm are given and compared with the closed-form formulas provided in Guillaume ([43], 2010) in the case of 2-step double knock-out put options. The American option prices calculated with our method have no benchmark value for the comparison. We also propose two numerical experiments for pricing 16-step double knock out put options. In the European case we use as a benchmark value the price given by the Monte Carlo method in Baldi, Caramellino and Iovino ([9], 1999) with 10 millions simulations and 1000 Euler time discretization steps. As for the 2-step case no benchmark is available for the American case.

## 2. Barrier options on discontinuous payoff functions

The study of the rate of convergence for the BIL algorithm requires the knowledge of the behavior of the classical CRR binomial approximation scheme for barrier-type options (see Proposition 2.6.1). To be precise, if $n \in \mathbb{N}$ denotes the number of time steps of the tree, we need to know the asymptotic expansion of the CRR binomial approximation error that is defined as

$$\mathrm{Err}_{CRR}(n) = \mathrm{price}_{CRR}(n) - \mathrm{price}_{BS}, \tag{0.0.8}$$

where $\mathrm{price}_{CRR}(n)$ denotes the price calculated by using the CRR tree scheme.
For standard (i.e.without barriers) call options it is known that the main term in (0.0.8) is of order $\frac{1}{n}$ (see for example Diener and Diener ([31], 2004) and Chang and Palmer ([20], 2007)). For standard digital options the expression in (0.0.8) has a contribution of order $\frac{1}{\sqrt{n}}$ related to the position of the discontinuity point of the payoff function ($K$ for digital options) and for this result one can refer to Walsh and Walsh ([82], 2004) and Chang and Palmer ([20], 2007).
But what about options with barriers? The first theoretical result is due to Gobet ([42], 2001)

that gives an upper bound of the binomial approximation error (0.0.8) for a general class of continuous payoff functions with double barriers by using PDE techniques, see Section 1.2.2. He finds that the quantity in (0.0.8) is the sum of two terms of order $\frac{1}{\sqrt{n}}$ related to the distance in the tree structure between the contractual barriers and the effective ones plus a term $\mathcal{R}_n$ such that there exist a constant $C > 0$ s.t. $|\mathcal{R}_n| \leq C\frac{\log n}{n}$.
A very recent result for call options with a single barrier is the one given in Lin and Palmer ([62], 2013). Here the authors give an explicit asymptotic expansion of (0.0.8). They use a different approach that consists in writing the CRR binomial price in terms of binomial coefficients as in Reimer and Sandmann ([70], 1995) and then approximating it through the normal distribution. In agreement with the upper bound given in Gobet ([42], 2001), they find a contribution of order $\frac{1}{\sqrt{n}}$ related to the position of the contractual barrier with respect to the nodes of the tree.
The results known from the literature concerning the analysis of the quantity in (0.0.8) when dealing with barrier options always require the continuity on the payoff function (Gobet ([42], 2001), Lin and Palmer ([62], 2013)). On the other hand when the payoff is assumed to be discontinuous the analysis of the rate of convergence of the binomial algorithm is given only for vanilla options (Walsh and Walsh ([82], 2002), Chang and Palmer ([20], 2007)). This is the reason why we decided to give our contribution in order to study theoretically and numerically barrier options on discontinuous payoff functions. In particular we deal with the simplest case of digital call options (the case of digital put options being similar), that can be used to generalize the treatment to the case of generic discontinuous payoff functions with a finite number of discontinuity points.
A digital call option is an option where the payoff is equal to a fixed amount (in what follows we suppose this amount is equal to 1) if the underlying asset at maturity is greater than a predetermined level (the strike price $K$) or nothing otherwise. Practitioners that trade these products essentially predict the direction of the market without concerning in the specific the magnitude of the movements of the underlying asset price. One of the benefits with respect to standard products is that the investment and the returns are fixed, so the risk involved and the potential losses are known a priori.
Digital options can also include barrier levels. This more complex option can be used as a financial tool embedded in sophisticated products, such as accrual range notes. These notes are financial securities that are linked for example to a foreign exchange rate and then they pay a fixed interest accrual if the exchange rate remains within a specified range and nothing otherwise (see Wystup ([86], 2006) and also Hui ([48], 1996)).
We treat the option pricing problem related to these options by using lattice techniques. In order to do this we first need to find an asymptotic expansion of the binomial approximation error (0.0.8) and then we set up an algorithm such that it behaves “well”, in the sense that the worst contribution in (0.0.8) (which is of order $\frac{1}{\sqrt{n}}$) is nullified. In particular we treat the study of the error (0.0.8) in the following two cases: digital options with a single barrier and digital options with double barriers. In the first case we get a complete theoretical result that allow us to construct an efficient algorithm, in the second one we obtain a partial theoretical result and we are able to make some numerical experiments on which we can

make some conjectures. We now describe our contribution in more details.
First of all we find an explicit asymptotic expansion of the binomial approximation error for digital call options with a single barrier, see Section 3.1. It turns out that the contribution of order $\frac{1}{\sqrt{n}}$ in (0.0.8) is due both to the position of the barrier and also to the position of the strike price with respect to the nodes of the tree. For this theoretical result we follow the techniques in Chang and Palmer ([20], 2007) and Lin and Palmer ([62], 2013). We stress here that the objective in [20] and [62] is to speed up the convergence of the CRR algorithm by explicitly calculating the terms of order $\frac{1}{\sqrt{n}}$ and then subtracting them to (0.0.8) in order to get an algorithm of order $\frac{1}{n}$. Instead, we want to set directly a numerical procedure such that the contribution of order $\frac{1}{\sqrt{n}}$ is nullified. This can be done by adjusting the BIL algorithm to this specific case. In fact if the binomial mesh is constructed such that the lower barrier lies exactly on a node of the tree at maturity and the strike is located halfway between two nodes at maturity then, according to the theoretical result we obtained, we get an algorithm of order $\frac{1}{n}$. This is enhanced with the numerical experiments in Section 3.3.1.
The treatment of double barrier digital options is not straightforward. In fact no manageable binomial closed-form formulas exist in general for double barrier options and then we cannot proceed as for the single barrier case. But by using a PDE approach as described in Gobet ([42], 2001) we are able to get an upper bound of the binomial approximation error for double barrier options with a general discontinuous payoff function, but we stress here that our contribution is still partial at the moment. In Section 3.3.2 of the numerical results we propose some experiments on which we can formulate some conjectures.

# 3. Options on a model with CIR interest rate

In the last part of this thesis we study an efficient lattice method for option pricing when the underlying price process takes into account a stochastic interest rate. So we consider a generalization of the model (0.0.3): under the risk neutral probability measure, we assume that the underlying asset price $(S(t))_{t\in[0,T]}$ has the following dynamic:

$$dS(t) = r(t)S(t)dt + \sigma_S S(t)dZ_S(t), \quad S(0) = s_0 > 0, \tag{0.0.9}$$

where $r$ is the short interest rate process, $\sigma_S$ is the constant stock price volatility and $Z_S$ is a standard Brownian motion. The risk-neutralized dynamic for the interest rate is described by the CIR model, i.e.

$$dr(t) = \kappa(\theta - r(t))dt + \sigma_r\sqrt{r(t)}dZ_r(t), \quad r(0) = r_0 > 0, \tag{0.0.10}$$

where $\kappa$ is the constant reversion speed, $\theta$ is the long-term reversion target, $\sigma_r$ is the constant interest rate volatility and $Z_r$ is a standard Brownian motion. We suppose that the two Brownian noises $Z_S$ and $Z_r$ are correlated.
Starting from 1990, the introduction in financial markets of long-term options whose time to maturity is at least two years at the time of issue, such as LEAPS options (with LEAPS

standing for "Long-term Equity Anticipation Security"), involves the necessity of considering equity models with a stochastic interest rate, see for example Bakshi et al. ([7], 2000). We also observe that taking into account for stochastic interest rates is crucial for the pricing of forward starting options, i.e. options that start at a specified date in the future with an expiration date set further in the future. In fact securities with forward starting features often have long-dated maturities and are therefore much more interest rate sensitive, see Haastricht and Pelsser ([44], 2009).

Moreover, we also notice that some insurance products such as equity-linked policies with a minimum guaranteed need financial mathematics techniques in order to compute the fair premiums. These policies are products in which part of the capital spent for purchasing the policy is invested in a portfolio of equities whose performance influences the coverage of the insured. In fact at every premium payment date, the insurer can decide to continue the contract and then pay the premium again or surrender the contract and receive the maximum between the value of the fund of equities or a minimum guaranteed. Then these policies embed a Bermudan option, i.e. an option in which the buyer can exercise at a discrete set of dates before maturity. Since they are necessarily long-term contracts, it turns out to be convenient to describe the equity with a bivariate model in which the dynamic of the interest rate is stochastic, see for example Costabile et al. ([25], 2009) and Martire ([63], 2012).

The issue of pricing options with stochastic interest rate is then needed to be solved. Merton ([64], 1973) provides a closed-form formula for European options with interest rate following the Ornstein-Uhlenbeck process, but no expressions for the American counterpart are available. We also observe that a more suitable model that guarantees the positivity of interest rates is given by the CIR process, then one should consider for the interest rate the dynamic given in (0.0.10), and in this specific case no closed-form formulas are available both for the European case and the American one.

Lattice models have been studied in the literature in order to deal with a 2-dimensional diffusion process with correlated Brownian motions as in (0.0.9)-(0.0.10). Wei ([83], 1996) and then Hilliard, Schwartz and Tucker ([46], 2004) provided a bivariate tree for dealing with a stochastic interest rate. Actually, in Wei the dynamic for the short rate is given by the Vasicek model and the extension of the Wei procedure to the CIR process is described in Costabile et al. ([25], 2006). However we still call "Wei procedure" the natural extension to the CIR process.

The idea in [83] and [46] is to extend to the 2-dimensional case a technique introduced by Nelson and Ramaswamy ([65], 1990) that consists in approximating 1-dimensional diffusion processes with computationally simple binomial processes. The original contribution of Nelson and Ramaswamy is the description of an approximating binomial process for a general class of diffusions such as (0.0.1), even if the coefficients present some singularities (as the diffusion coefficient of the CIR process). They introduce the so-called "multiple jumps" that allow the discrete process to have an up jump and a down jump but not necessarily on the two adjacent nodes of the tree as for the standard CRR model, see Figure 3. These jumps are specified such that some appropriate matching conditions on the local mean and the

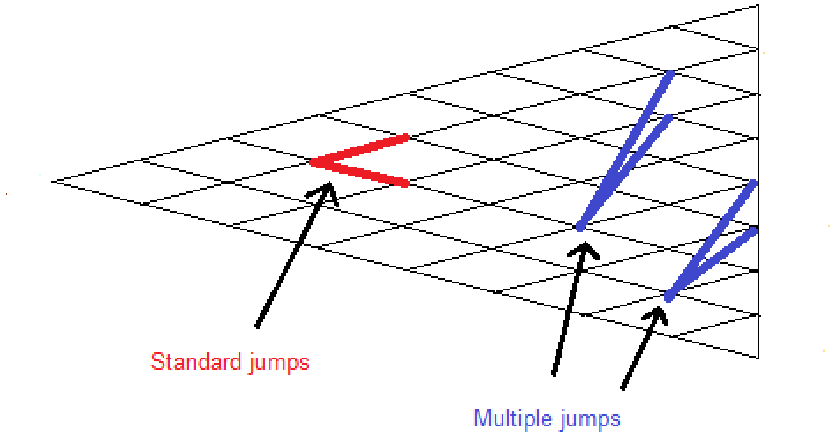


Figure 3: *Standard jumps and multiple jumps.*

local variance between the discrete and the continuous process are satisfied.
We now briefly describe the generalization of the Nelson and Ramaswamy technique as proposed in Wei. The author suggests to construct a bivariate lattice by proceeding with the following four steps:

- transform both $S$ and $r$ into unit variance processes called $\tilde{S}$ and $R$ respectively;
- define a new process $Y$ as a function of $\tilde{S}$ and $R$ that is orthogonal to $R$;
- model $R$ and $Y$ as two independent binomial processes following Nelson and Ramaswamy and then merge the two structures into a 2-dimensional tree in which each node branches into four via joint probabilities that are simply obtained by product of the individual probabilities;
- at each node of the tree convert the variables $R$ and $Y$ back to $r$ and $S$ respectively and then proceed backwardly to obtain the option prices.

The procedure in Hilliard, Schwartz and Tucker is very similar to the one in Wei: they consider different transformations allowing one to deal with independent processes (see Section 4.3.2). So, what is important is the common idea to work with uncorrelated components that allow one to define the transition probabilities of the bivariate tree by means of products.
Our algorithm (see Section 4.4) is structurally different from the previous ones in fact:

1. transformations similar to Wei and Hilliard, Scwhartz and Tucker are used but only to set up the state-space of the discrete approximation of the pair $(S, r)$;
2. the probabilistic structure for the discrete approximation of both $S$ and $r$ as individual diffusions is defined directly on the original processes (using original drifts) and in this we recover the "original" idea in Nelson and Ramaswamy;

3. the transition probabilities of the bivariate lattice take in consideration the covariance structure (instead, the Wei and Hilliard, Schwartz and Tucker methods are strongly based on the use of a bivariate diffusion with components driven by uncorrelated noises).

Numerical results show that our method is robust and moreover it is not dependent on the choice of the input parameters, and this is a significant difference with respect to the procedure provided in [83] and [46], see Section 4.6. In fact in order to obtain the convergence in law of the tree built for the pair $(S, r)$ we do not need to require the Feller condition, $2\kappa\theta \geq \sigma_r^2$, or the so-called "convergence condition", $4\kappa\theta \geq \sigma_r^2$ (see Remark 4.4.1), but we only need to assume that $\kappa > 0$ and $\theta > 0$, that turn out to be natural requirements in a financial setting.
From a theoretical point of view we get two original results: the weak convergence of the tree method and the convergence of the prices given from the algorithm to their continuous counterparts.
Let us consider first the convergence of the scheme. Using standard techniques we prove in Section 4.5 the convergence on the Skorokhod space $D([0,T];\mathbb{R}^2)$ of the càdlàg functions in $[0,T]$ with values in $\mathbb{R}^2$ of the tree method to the pair $(S, r)$ solution of the SDE (0.0.9)-(0.0.10). It means that we set $(S_i^h, r_i^h)_{i=0,\dots,n}$ the Markov chain running on the bivariate lattice and then we set $(\overline{S}_t^h, \overline{r}_t^h)_{t\in[0,T]}$ as the continuous time process defined through the piecewise constant and càdlàg interpolations in time of the chain, that is:

$$\overline{S}_t^h = S_i^h \quad \text{and} \quad \overline{r}_t^h = r_i^h, \quad \forall\, t \in [ih, (i+1)h). \tag{0.0.11}$$

We observe that we could also define $(\overline{S}_t^h, \overline{r}_t^h)_{t\in[0,T]}$ as the continuous time process obtained by linearly interpolating in time the discrete Markov chain as in (0.0.2), in fact the two approaches are equivalent (see Theorem 1.1.4). But the choice of working in the space $D([0,T];\mathbb{R}^2)$ turns out to be more convenient in the second theoretical result, i.e. the convergence of the prices.
Then we get that the family of Markov processes $(\overline{S}^h, \overline{r}^h)_h$ converges in law on the space $D([0,T];\mathbb{R}^2)$ to the diffusion process $(S, r)$ solution of the SDE (0.0.9)-(0.0.10), see Theorem 4.5.8.
Secondly, in Section 4.5.2 we discuss the convergence of European and American option prices computed with the lattice algorithm to their corresponding continuous values. The reasoning is immediate for the European prices when the payoff function is continuous and bounded. But an extension of a result proved in Amin and Khanna ([3], 1994) allows us to get the convergence of the American prices (and then also of European prices) in a more general set of conditions on the payoff function. In particular let $f(t,x) : [0,T] \times D([0,T]) \to [0,+\infty)$ denote a payoff function. Consider the following assumptions:

- **(H1)** $f$ is a continuous function (in the product topology) and for every $x, y \in D([0,T])$ such that $x_s = y_s$ for each $s \in [0,t]$ then $f(t,x) = f(t,y)$;
- **(H2)** there exists $\delta > 1$ and $h_* > 0$ s.t. $\sup_{h<h_*} \mathbb{E}\big(\sup_{t\leq T} |e^{-\int_0^t \bar{r}_s^h ds} f(t, \bar{S}^h)|^\delta\big) < \infty$.

Then under hypothesis (H1) and (H2) Amin and Khanna prove the convergence of the American prices evaluated on the tree approximation to the ones in the continuous-time model. But we can say more. In fact, suppose that $f$ fulfills the following polynomial-growth condition: there exists $C > 0$ and $\gamma > 1$ such that

$$\sup_{t\in T} |f(t,x)| \leq C(1 + \sup_{t\in T} |x_t|^\gamma). \tag{0.0.12}$$

Then hypothesis (H1) and (0.0.12) guarantee the convergence of the American prices computed with our algorithm to the corresponding continuous-time values. In fact if (0.0.12) is true, then hypothesis (H2) is verified because we prove that for every $p > 1$ there exists $h_* < 1$ such that

$$\sup_{h<h_*} \mathbb{E}\Big(\sup_{t\leq T} e^{-p\int_0^t \bar{r}_s^h ds}(\bar{S}_t^h)^p\Big) < \infty. \tag{0.0.13}$$

We remark that this is a non trivial extension of the results in Amin and Khanna ([3], 1994), see Section 4.5.2.
A further study that we don't treat in this thesis and that represents the objective of a future research is the analysis of the rate of convergence of the bivariate algorithm. We stress that in the literature results on this issue are available only for the CRR tree approximation also when the payoff function is sophisticated (as for barrier options). This study is what we actually do in Chapter 2 and Chapter 3 by using PDE techniques when the payoff function is generic and by using the normal approximation of the sum of binomial coefficients in the more specific case of call and put options. However, when the model is 2-dimensional, as the one in (0.0.9)-(0.0.10), the treatment is not straightforward and is worth to be done.

# Chapter 1

# Mathematical background

In this Chapter we present some theoretical results that we will use in the rest of the thesis. In Section 1.1 we consider the problem of the convergence of Markov chains to diffusions and we state a classical theorem that can be used for proving that the continuous time process built from the Markov chain running on a lattice scheme weakly converges to a specific diffusion process. We also introduce the Nelson and Ramaswamy technique ([65], 1990) and a theorem due to Amin and Khanna ([3], 1994) on the convergence of the American option prices computed on a discretization scheme to their continuous-time values. This part will help us to prove the convergence results on the new bivariate scheme proposed in Chapter 4. In Section 1.2 we recall some known results on the rate of convergence of binomial tree schemes for standard options and for barrier-type options. In particular we present a theorem due to Gobet ([42], 2001) that we will apply in Chapter 2 for obtaining the rate of convergence of the new binomial scheme proposed. We also state the explicit binomial error formulas due to Chang and Palmer ([20], 2007) and Lin and Palmer ([62], 2013) that will help us in the development of the contribution we give on barrier options with discontinuous payoff functions in Chapter 3.

## 1.1 Convergence of Markov chains to diffusions

In this Section we briefly describe the results about the weak convergence of Markov chains to diffusions, as described in the books of Stroock and Varadhan ([74],1979), Kushner ([57],1977), Kushner and Dupuis ([58],1992), Ethier and Kurtz ([32],1986), Billingsley ([12], 1968) and also in the notes of Pagès ([66],2001). We stress that here we use the notations adopted in Stroock and Varadhan ([74], 1979).
The basic idea is the following. Since we are concerned with computational techniques, the idea is to approximate the original stochastic process with a simpler approximating process that is a Markov chain on a finite state space. We remark that the finiteness of the approximating chain simplify the discussion in the theoretical results and it is indeed the case we are concerned with in the following chapters, but most of the theoretical results hold the same also if we remove this requirement. The approximating chain is parametrized by a

parameter $h > 0$, such that as the parameter goes to zero certain "local properties" of the approximating chain are consistent to the ones of the limit process. Under a set of broad conditions one can prove that the sequence of approximating chains converges in law to the continuous process as the approximation parameter goes to zero.
First we will present some technical results that allow us to prove the main theorem concerning the weak convergence of Markov chains to diffusions. We also briefly prove the well-known weak convergence of the CRR binomial scheme. Secondly, we will show the discrete approximation procedure due to Nelson and Ramaswamy ([65], 1990) and we will show that the sequence of Markov chains built from the discrete scheme they propose weakly converges to the original diffusion. Finally, we make some remarks on the convergence of European and American option prices by using a result due to Amin and Khanna ([3], 1994).

### 1.1.1 Weak convergence result

Let us start with a brief overview of the notion of weak convergence as presented in Billingsley ([12], 1968), that we will directly express in terms of our specific case of interest.
Let us suppose that $(S, \mathcal{S})$ is a complete and separable metric space equipped with the Borel $\sigma$-algebra. We give the following definition:

**Definition 1.1.1.** *Let us suppose that $\{\mathbb{P}^h\}_h$ is a family of probability measures on $(S, \mathcal{S})$. The family $\{\mathbb{P}^h\}_h$ is said to converge weakly to a probability measure $\mathbb{P}$ if*

$$\int f(x)\mathbb{P}^h(dx) \to \int f(x)\mathbb{P}(dx) \tag{1.1.1}$$

*for all $f(\cdot) \in C_b(S)$, where $C_b(S)$ denotes the class of all the bounded and continuous real-valued functions on $S$. Then, if $X^h$ and $X$ are random variables and $\mathbb{P}^h$ and $\mathbb{P}$ are the measures on $(S, \mathcal{S})$ induced by $X^h$ and $X$ respectively, then the weak convergence of $\{\mathbb{P}^h\}_h$ to $\mathbb{P}$ is equivalent to the convergence of $\{X^h\}_h$ to $X$ in distribution (denoted by $X^h \Rightarrow X$).*

There are a number of ways one can formulate the notion of weak convergence of probability measures, for more details see the Portmanteau Theorem 2.1 in [12].

**Remark 1.1.2.** *The random variables $X^h$ can be even defined on different probability spaces. However, one can choose a common probability space for the random variables and define a sequence $\{Y^h\}_h$ with $Y^h$ having the same law of $X^h$ for every $h$, in such a way that the convergence occurs almost surely. This is the content of the Skorokhod representation theorem (for details see for example Theorem 2.2.2 in Kushner ([57], 1937)).*

Since it is not practical to prove the weak convergence of probability measures by directly using Definition 1.1.1, the idea is to show that the two following conditions are verified:

- the *relative compactness* of the family $\{\mathbb{P}^h\}_h$, i.e. $\{\mathbb{P}^h\}_h$ admits subsequential weak limits in the space of probability measures on $(S, \mathcal{S})$;

- the *uniqueness of the subsequential weak limit*, i.e. if two different subsequences of $\{\mathbb{P}^h\}_h$ converge toward a weak limit, then the limits must be the same.

In fact, if the family $\{\mathbb{P}^h\}_h$ is relatively compact and there exists one subsequential weak limit $\mathbb{P}$, then $\{\mathbb{P}^h\}_h$ weakly converges to $\mathbb{P}$.
We remark that in complete and separable metric spaces the notion of relative compactness of measures reduces to its tightness (Prohorov's Theorem 6.1 and 6.2 in [12]), that consists, roughly speaking, in the fact that the sequence of processes $\{X^h\}_h$ (that induce on $(S, \mathcal{S})$ the family of probability measures $\{\mathbb{P}^h\}_h$) does not oscillate too widely.
In the sequel we will specify our discussion to the case in which the complete and separable metric space is $C([0,1];\mathbb{R}^d)$ of the $\mathbb{R}^d$-valued continuous functions on the finite interval $[0,1]$ equipped with the metric of the sup-norm $\rho$ that is defined as

$$\rho(x,y) = \sup_{t\in[0,1]} \parallel x(t) - y(t) \parallel, \quad \forall\, x, y \in C([0,1];\mathbb{R}^d).$$

**Remark 1.1.3.** *Without lack of generality we work in the time interval* $[0,1]$*, the case of a generic interval* $[a,b]$ *with* $-\infty \leq a < b \leq +\infty$ *being similar. In particular, in Chapter 4 we will use the results presented in this Section when the time interval is* $[0,T]$*, where* $T$ *is the maturity date of the option.*

Then we consider the problem of proving the weak convergence of a sequence of processes with trajectories in $C([0,1];\mathbb{R}^d)$ that we define in what follows to a given diffusion process $X$. Let us describe the setup we will be working with (see Stroock and Varadhan ([74], 1979)).
Given $x_0 \in \mathbb{R}^d$, let $\Pi_h(x,\cdot)$ be a transition function on $\mathbb{R}^d$. We assume that for every $h > 0$ a discrete (in time and space) Markov chain $(X^h_{ih})_i$ with associated transition probability $\Pi_h$ and deterministic starting point $x_0 \in \mathbb{R}^d$ is given. It means that for each $h > 0$ we have:

- $\mathbb{P}^h(X^h_0 = x_0) = 1$;
- $\mathbb{P}^h\Big(X^h_{(i+1)h} \in \Gamma | \mathcal{M}_{ih}\Big) = \Pi_h(X^h_{ih}, \Gamma)$, $(\mathbb{P} - \text{a.s.})\, \forall\, i \geq 0$, $\forall\, \Gamma \in \mathcal{B}_{\mathbb{R}^d}$,

where $\mathcal{M}_{ih} = \sigma\{X^h_{kh} : 0 \leq k \leq i\}$ and $\mathcal{B}_{\mathbb{R}^d}$ is the Borel $\sigma$-field of subsets of $\mathbb{R}^d$. So, the previous two conditions mean that for every $h > 0$ the process $(X^h_{ih})_i$ is a time-homogeneous Markov chain starting from $x_0$ with transition probability $\Pi_h(x,\cdot)$. We also observe that homogeneity is not really a necessary hypothesis but in practice we will be concerned with this specific case. Moreover, from the martingale formulation for discrete time processes (see [74] pages 165-166), the second condition is equivalent to say that

$$\left( f(X^h_{ih}) - \sum_{j=0}^{i-1} L_h(f(X^h_{jh})), \mathcal{M}_{ih}, \mathbb{P}^h \right)$$

is a discrete martingale for each $f \in C^\infty_0(\mathbb{R}^d)$, where $L_h$ is the operator defined as follows

$$L_h f(x) = \int (f(y) - f(x)) \Pi_h(x, dy) \tag{1.1.2}$$

and $C_0^\infty(\mathbb{R}^d)$ is the set of all the $C^\infty$-functions $f : \mathbb{R}^d \to \mathbb{R}$ having compact support. We now define for each $h$ a continuous-time process $(X_t^h)_t$ by linearly interpolating in time the discrete Markov chain $(X_{ih}^h)_i$, i.e.

$$\mathbb{P}^h\left[X_t^h = \frac{(i+1)h-t}{h}X_{ih}^h + \frac{t-ih}{h}X_{(i+1)h}^h,\ ih \le t < (i+1)h\right] = 1, \quad \forall\, i \ge 0. \tag{1.1.3}$$

Then, for every fixed $h$, $(X_t^h)_t$ is a continuous-time process with trajectories in $C([0,1];\mathbb{R}^d)$. Another possibility is to define the process $(X_t^h)_t$ by piecewise càdlàg interpolations in time, i.e.

$$\mathbb{P}^h\big[X_t^h = X_{ih}^h,\ ih \le t < (i+1)h\big] = 1, \quad \forall\, i \ge 0. \tag{1.1.4}$$

In this case $(X_t^h)_t$ has trajectories in the space $D([0,1];\mathbb{R}^d)$ of the càdlàg functions with values in $\mathbb{R}^d$. This space turns out to be a complete and separable metric space when it is equipped with the Skorokhod metric (for details see Billingsley ([12], 1968) Chapter 4).
The two approaches are equivalent, in fact the following Theorem holds:

**Theorem 1.1.4.** *The sequence of processes defined in (1.1.3) with trajectories in the space $C([0,1];\mathbb{R}^d)$ equipped with the sup-norm metric weakly converges towards a given continuous process $X$ if and only if the sequence of processes defined in (1.1.4) with trajectories in the space $D([0,1];\mathbb{R}^d)$ equipped with the Skorokhod metric weakly converges to $X$.*

**Remark 1.1.5.** *The space $C([0,1];\mathbb{R}^d)$ is a subset of $D([0,1];\mathbb{R}^d)$. Since the Skorokhod topology restricted to the space $C([0,1];\mathbb{R}^d)$ coincides with the uniform topology there, then the weak convergence of processes with trajectories in $C([0,1];\mathbb{R}^d)$ implies that the processes weakly converge as processes with trajectories in $D([0,1];\mathbb{R}^d)$. Furthermore, the weak convergence in $D([0,1];\mathbb{R}^d)$ towards a continuous process $X$ implies the weak convergence in $C([0,1];\mathbb{R}^d)$ of the linear interpolations. So the two approaches are indeed equivalent and to simplify the treatment we follow Stroock and Varadhan and we work in the space $C([0,1];\mathbb{R}^d)$. In fact, as explained in Billingsley ([12], 1968), the development of the same arguments in the space $D([0,1];\mathbb{R}^d)$ involves a characterization of the compact sets in $D([0,1];\mathbb{R}^d)$ and the study of criteria for the tightness and this requires much more technical results.*

We now want to determine conditions under which the sequence of continuous-time processes $\{X^h\}_h$ with corresponding probability measures $\{\mathbb{P}^h\}_h$ converges in distribution to a diffusion process $X$ that induces on $C([0,1];\mathbb{R}^d)$ the measure $\mathbb{P}$ and whose generator is

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}, \tag{1.1.5}$$

in which $b : \mathbb{R}^d \to \mathbb{R}^d$ and $a : \mathbb{R}^d \to \mathcal{S}(d)$, where $\mathcal{S}(d)$ denotes the set of all non-negative definite real matrices $d \times d$, are measurable functions. Moreover, we recall that if we suppose that $b$ and $\sigma$ are continuous and locally bounded functions (see Chapter 6 in Stroock and Varadhan ([74], 1979)) then the following stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0 \tag{1.1.6}$$

has a unique (in law) *weak* solution. It means that whenever two weak solutions of the SDE (1.1.6) have the same initial distribution, then they also have the same law.
Given the operator defined in (1.1.5) one can formulate the martingale problem associated to $L$ as follows:

**Definition 1.1.6.** *A solution to the martingale problem associated to $L$ (or to $b$ and $a$) starting from $x_0 \in \mathbb{R}^d$ is a probability measure $\mathbb{P}$ on the space $C([0,1];\mathbb{R}^d)$ equipped with the Borel $\sigma$-algebra that satisfies the following two conditions:*

- $\mathbb{P}(X_0 = x_0) = 1$;
- $f(X_t) - \int_0^t Lf(X_u)du$ *is a $\mathbb{P}$-martingale for all $f \in C_0^\infty(\mathbb{R}^d)$ with respect to the natural filtration $\{\mathcal{F}_t\}_t$.*

We recall here that $C_0^\infty(\mathbb{R}^d)$ is the space of all functions $f : \mathbb{R}^d \to \mathbb{R}$ having continuous derivatives of all orders and compact support.
Similarly to the discrete case, there exists a one-to-one correspondence between weak solutions of the SDE (1.1.6) and the martingale problem formulation (see Corollary 5.3.4 in Ethier and Kurtz ([32], 1986)).
In particular, we get the uniqueness in distribution of the solutions of the SDE (1.1.6) if and only if the solution to the martingale problem associated to $L$ is unique, where in the martingale problem context uniqueness means that all the solutions with identical starting points have the same law on the path-space.
We now describe the main technical result that allows us to state the weak convergence as $h \downarrow 0$ of the sequence of Markov chains $\{X^h\}_h$ defined in (1.1.3) to the diffusion process $X$ solution of the SDE (1.1.6).
We first need to define some quantities related $\{X^h\}_h$ that for every fixed $h$ are:

$$\begin{aligned}
a_{i,j}^h(x) &= \frac{1}{h}\int_{|y-x|\le 1}(y_i - x_i)(y_j - x_j)\Pi_h(x,dy), \quad \forall\, i,j,\\
b_i^h(x) &= \frac{1}{h}\int_{|y-x|\le 1}(y_i - x_i)\Pi_h(x,dy), \quad \forall\, i,\\
\Delta_\epsilon^h(x) &= \frac{1}{h}\Pi_h(x,\mathbb{R}^d \setminus B(x,\epsilon)), \quad \forall\, \epsilon > 0.
\end{aligned}$$

It is clear that $a^h(\cdot)$ and $b^h(\cdot)$ are the local second moment and the local drift of the chain respectively and that $\Delta_\epsilon^h(\cdot)$ represents a measure of the probability per unit of time of a jump of size $\epsilon$ or greater than $\epsilon$.
The main convergence result is the following (Theorem 11.2.3 in Stroock and Varadhan ([74], 1979)):

**Theorem 1.1.7.** *Let us assume that for all $R > 0$ and for all $\epsilon > 0$ the following conditions are true:*

$$\lim_{h\to 0}\sup_{|x|\le R} \| a_h(x) - a(x) \| = 0, \tag{1.1.7}$$

$$\lim_{h\to 0}\sup_{|x|\le R} |b_h(x) - b(x)| = 0, \tag{1.1.8}$$

$$\lim_{h\to 0}\sup_{|x|\le R} \Delta^h_\epsilon(x) = 0, \tag{1.1.9}$$

*where $a : \mathbb{R}^d \to \mathcal{S}(d)$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ are continuous functions. In addition, let us assume that the coefficients $a$ and $b$ have the property that for each starting point $x_0 \in \mathbb{R}^d$ the martingale problem for $a$ and $b$ has exactly one solution $\mathbb{P}$. Then $\{\mathbb{P}^h\}_h$ weakly converges to $\mathbb{P}$ on $C([0,1];\mathbb{R}^d)$ as $h \to 0$ uniformly on compact subsets of $\mathbb{R}^d$, i.e. $\{X^h\}_h$ converges in distribution to $X$.*

**Remark 1.1.8.** *We observe that conditions (1.1.7) and (1.1.8) require the convergence as $h \to 0$ of the local second moment and the local drift of the chain to the respective continuous counterparts. Moreover, condition (1.1.9) assumes that the probability $\Delta^h_\epsilon(\cdot)$ goes to zero and this is related to the fact that diffusion processes have sample paths that are continuous w.p. 1. We also remark that (1.1.7), (1.1.8), (1.1.9) are equivalent to the condition that for each $f \in C^\infty_0(\mathbb{R}^d)$*

$$\frac{1}{h}L^h f \to Lf \tag{1.1.10}$$

*uniformly on compact subsets of $\mathbb{R}^d$ (for details see Lemma 11.2.1 in ([74], 1979)).*

As explained at the beginning of this Section, in order to prove the weak convergence one needs to show that the family of probability measures $\{\mathbb{P}^h\}_h$ is relatively compact and that there exists one subsequential weak limit $\mathbb{P}$. The relative compactness is directly implied by conditions (1.1.7), (1.1.8), (1.1.9) because they are used to show that the sequence $\{X^h\}_h$ is tight (for details see Theorem 1.4.11 in [74]). Then, since the convergence condition (1.1.10) on the generators holds, it is immediate to deduce that the only possible weak subsequential limit of $\{\mathbb{P}^h\}_h$ solves the martingale problem associated to $a$ and $b$ and then it is indeed the measure $\mathbb{P}$ induced on $C([0,1];\mathbb{R}^d)$ by the diffusion process $X$. But since the martingale problem as a unique solution, then the desired conclusion follows.

### 1.1.2 Weak convergence of the CRR binomial tree

We now show that the conditions of Theorem 1.1.7 are satisfied by the classical binomial discretization scheme due to Cox, Ross and Rubinstein ([28], 1979), CRR hereafter. Let us suppose to fix the maturity time $T > 0$. The CRR model is used to approximate the stock price process $(X_t)_{t\in[0,T]}$ in the Black and Scholes model, so under the risk-neutral measure $\mathbb{P}^*$ the process $X$ is the solution of the following SDE

$$dX_t = rX_t dt + \sigma X_t dB_t, \quad X_0 = x_0 \tag{1.1.11}$$

where $r$ is the risk free rate and $\sigma$ the volatility parameter. We know that $(X_t)_{t\in[0,T]}$ can be explicitly expressed as follows

$$X_t = x_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma B_t}, \quad \forall\, t \in [0,T],$$

then we can introduce the process $(\overline{X}_t)_{t\in[0,T]}$ of the log-return defined as

$$\overline{X}_t = \log X_t, \quad \forall\, t \in [0,T],$$

and so

$$\overline{X}_t = \overline{X}_0 + \mu t + \sigma B_t, \quad \text{with} \quad \overline{X}_0 = \log x_0 \quad \text{and} \quad \mu = r - \frac{1}{2}\sigma^2.$$

Then the process $(\overline{X}_t)_{t\in[0,T]}$ follows the SDE

$$d\overline{X}_t = \mu dt + \sigma dB_t. \tag{1.1.12}$$

We set $h = T/n$, with $n \in \mathbb{N}$, that is the constant time step of the binomial tree. Let us now denote with $(\overline{X}^h_{ih})_{i=0,\dots,n}$ the discrete in time and in space process corresponding to the CRR binomial tree that is defined as

$$\overline{X}^h_{ih} = \overline{X}_0 + \sum_{j=1}^{i} \xi_j \sigma \sqrt{h}, \tag{1.1.13}$$

where $(\xi_i)_{i=0,\dots,n}$ are i.i.d. Bernouilli random variables such that

$$\begin{cases} \mathbb{P}(\xi_i = 1) = p_h = \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{h}, \\ \mathbb{P}(\xi_i = -1) = 1 - p_h = \frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{h}. \end{cases}$$

**Remark 1.1.9.** *We observe that the risk-neutral probability $p_h$ in the original work of Cox, Ross and Rubinstein ([28], 1979) is defined as*

$$p_h = \frac{e^{rh} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}. \tag{1.1.14}$$

*If we make a Taylor expansion at the first order of the expression in (1.1.14), we get*

$$p_h = \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{h} + o(\sqrt{h}) \tag{1.1.15}$$

*and this is indeed the probability we use in practice.*

First of all we observe that the coefficients $\mu$ and $\sigma$ in (1.1.12) are constant, so they are continuous and there exist a unique solution to the martingale problem associated to $\mu$ and

$a = \sigma^2$. Let us now compute the local drift, that we call $\mu_h$, and the local second moment, that we denote with $\sigma_h^2$, associated to the process (1.1.13). We get:

$$\mu_h = \frac{1}{h}[p_h \sigma\sqrt{h} + (1-p_h)(-\sigma\sqrt{h})] = \mu + \frac{o(h)}{h};$$
$$\sigma_h^2 = \frac{1}{h}[p_h (\sigma\sqrt{h})^2 + (1-p_h)(-\sigma\sqrt{h})^2] = \sigma^2 + \frac{o(h)}{h}.$$

Then it is easy to see that conditions (1.1.7) and (1.1.8) of Theorem 1.1.7, that concern the uniform convergence on the compact subsets of $\mathbb{R}$ of the local drift $\mu_h$ and the local second moment $\sigma_h^2$ of the discrete process to $\mu$ and $\sigma^2$, are verified. Moreover, since the space step of the discrete scheme is $\sigma\sqrt{h}$, then condition (1.1.9) of Theorem 1.1.7 is true as well. Then the following result holds:

**Theorem 1.1.10.** *The process $\overline{X}^h$ defined in (1.1.13) weakly converges to the diffusion process solution of (1.1.12).*

### 1.1.3 Nelson and Ramaswamy technique for diffusion approximations

We now describe the technique in Nelson and Ramaswamy ([65], 1990) used to construct a computationally simple binomial process that weakly converges to a generic 1-dimensional diffusion. As pointed out by the authors, the term "binomial process" is indeed an abuse of terminology because it does not refer to a discrete process that follows a binomial distribution (as in Cox, Ross and Rubinstein ([28], 1979)), but more generally it is used for a two-state discrete model that we briefly call *binomial tree*. We remark that a tree is defined "computationally simple" if the number of nodes in the structure grows at most linearly in the number of time intervals (i.e. the lattice structure is path independent). This is a crucial feature of the approximating process because a computationally complex tree, such as a lattice in which the number of nodes increases not linearly, is useless for purposes such as option pricing.
Let us suppose to consider in the time interval $[0, T]$ the stochastic differential equation (1.1.6) that is

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0.$$

First of all the interval $[0, T]$ is divided into $n$ subintervals of length $h = T/n$. For a fixed $h$, $(X_{i,j}^h)_{i,j}$ for $i = 0, 1, ..., n, j = 0, 1, ..., i$ is the binomial tree approximating the process $X$ defined with the following steps.
The first one is to turn the original SDE (1.1.6) into a new SDE with constant instantaneous volatility, that without lack of generality can be chosen equal to one. To this end, Nelson and Ramaswamy introduce a transformation $g(x) \in C^2(\mathbb{R}_+)$ defined on the support of $x$ as follows

$$g(x) = \int^x \frac{1}{\sigma(z)} dz. \tag{1.1.16}$$

Then they define a new process $Y$ by

$$Y_t = g(X_t), \quad \forall\, t \in [0,T]. \tag{1.1.17}$$

By Ito's formula we get

$$\begin{aligned} dY_t &= \left( \frac{\partial g(X_t)}{\partial x} b(X_t) + \frac{1}{2}\sigma^2(X_t)\frac{\partial g(X_t)}{\partial x} \right) dt + dB_t, \\ &= b^Y(X_t)dt + dB_t, \end{aligned}$$

and $Y_0 = g(x_0) =: y_0$.
The idea is to construct a binomial tree $(Y^h_{i,j})_{i,j}$ for $i = 0,1,...,n, j = 0,1,...,i$ that approximates the process $Y$ and then to convert it back through the inverse transformation of (1.1.16) in order to get the binomial tree for $X$.
They now split the discussion in two cases: the first occurs when there are no singularities in $\sigma(x)$, the second when there is a singularity at $x = 0$ such that $\sigma(0) = 0$ and $b(0) \geq 0$.

### Case 1: no singularities in $\sigma(x)$

In this case they construct a computationally simple binomial approximation for $Y$ as follows:

- $Y^h_{0,0} = y_0$;
- $Y^h_{i,j} = y_0 + (2j - i)\sqrt{h}, \forall\, j = 0,...,i$;
- starting from $Y^h_{i,j}$ at time $ih$, the process $Y^h$ may jump at time $(i+1)h$ to the two following values

$$\begin{aligned} Y^h_{i+1,j+1} &= Y^h_{i,j} + \sqrt{h}, \\ Y^h_{i+1,j} &= Y^h_{i,j} - \sqrt{h}. \end{aligned}$$

By using the inverse of (1.1.16) they now define a computationally simple tree $(X^h_{i,j})_{i,j}$ for the original process $X$ as follows:

- $X^h_{0,0} = g^{-1}(Y^h_{0,0}) = x_0$;
- $X^h_{i,j} = g^{-1}(Y^h_{i,j}), \forall\, j = 0,...,i$;
- starting from $X^h_{i,j}$ at time $ih$, the process $X^h$ may jump at time $(i+1)h$ to the two following values

$$\begin{aligned} X^h_{i+1,j+1} &= g^{-1}(Y^h_{i,j} + \sqrt{h}), \\ X^h_{i+1,j} &= g^{-1}(Y^h_{i,j} - \sqrt{h}). \end{aligned}$$

The final step is to define the transition probabilities with which an up or a down jump occurs at each node of the tree. Such probabilities are chosen such that the local drift of the discrete process for $X$ is exactly equal to the drift of the limiting diffusion (1.1.6), i.e.:

$$p_{i,j}^h = \frac{b(g^{-1}(Y_{i,j}^h))h + g^{-1}(Y_{i,j}^h) - g^{-1}(Y_{i,j}^h - \sqrt{h})}{g^{-1}(Y_{i,j}^h + \sqrt{h}) - g^{-1}(Y_{i,j}^h - \sqrt{h})}. \quad (1.1.18)$$

Since the quantity in (1.1.18) may not be a legitimate probability (it may not belong to $[0,1]$), they censor it as follows:

$$p_{i,j}^{h,*} = 0 \vee p_{i,j}^h \wedge 1. \quad (1.1.19)$$

Then the binomial scheme for the process $X$ is given by:

$$X_{i,j}^h \Longrightarrow \begin{cases} X_{i+1,j+1}^h, & p_{i,j}^{h,*}, \\ X_{i+1,j}^h, & 1 - p_{i,j}^{h,*}. \end{cases}$$

In order to prove the convergence of the discrete scheme $(X_{i,j}^h)_{i,j}$ to the diffusion $X$, they define a sequence of càdlàg Markov chains $\{X_i^h\}_{i=0,1,\dots,n}$ as in (1.1.4), i.e.:

- $X_0^h = x_0$;
- at time $ih$ the state-space for $X_i^h$ is given by

$$\mathcal{X}_i^h = \{X_{i,j}^h, j = 0, 1, ..., i\};$$

- from time $ih$ to time $(i+1)h$ the transition law on $\mathbb{R}$ is given by

$$\mathbb{P}^h(X_{i,j}^h; dx) = p_{i,j}^{h,*} \delta_{\{X_{i+1,j+1}^h\}}(dx) + (1 - p_{i,j}^{h,*}) \delta_{\{X_{i+1,j}^h\}}(dx),$$

  where $\delta_{\{a\}}$ denotes here the Dirac mass in $a \in \mathbb{R}$.

As explained in Section 1.1, in order to guarantee the convergence in distribution of $\{X^h\}_h$ to $X$, one needs to prove conditions (1.1.7), (1.1.8) and (1.1.9) of Theorem 1.1.7. We suppose here that $b$ and $\sigma$ are suitable functions that guarantee the uniqueness of the martingale problem associated to $b$ and $a = \sigma^2$.
Let us define $\mathcal{A}_* = \{(i,j) : X_{i,j}^h \leq A_*\}$. Moreover let us call the local moment of order $l$ at time $ih$ as

$$\mathcal{M}_{i,j}(l) = \mathbb{E}((X_{i+1}^h - X_i^h)^l | X_i^h = X_{i,j}^h)), \quad l = 1, 2, 4,$$

where to simplify the notations we write $\mathbb{E}$ instead of $\mathbb{E}_{\mathbb{P}_h}$.
First of all we need to prove the convergence of the local drift, i.e.:

$$\lim_{h \to 0} \sup_{(i,j) \in \mathcal{A}_*} \frac{1}{h} |\mathcal{M}_{i,j}(1) - b(X_{i,j}^h)h| = 0. \quad (1.1.20)$$

But we have that

$$\mathcal{M}_{i,j}(1) = p_{i,j}^{h,*}(g^{-1}(Y_{i,j}^h + \sqrt{h}) - g^{-1}(Y_{i,j}^h)) \\ + (1 - p_{i,j}^{h,*})(g^{-1}(Y_{i,j}^h - \sqrt{h}) - g^{-1}(Y_{i,j}^h)),$$

and by using (1.1.18) we get that $\mathcal{M}_{i,j}(1) = b(X_{i,j}^h)h$, then (1.1.20) easily follows. Then we need to prove the convergence of the local diffusion coefficient,i.e.:

$$\lim_{h\to 0} \sup_{(i,j)\in\mathcal{A}_*} \frac{1}{h}|\mathcal{M}_{i,j}(2) - \sigma^2(X_{i,j}^h)h| = 0. \tag{1.1.21}$$

We have that

$$\mathcal{M}_{i,j}(2) = p_{i,j}^{h,*}(g^{-1}(Y_{i,j}^h + \sqrt{h}) - g^{-1}(Y_{i,j}^h))^2 \\ + (1 - p_{i,j}^{h,*})(g^{-1}(Y_{i,j}^h - \sqrt{h}) - g^{-1}(Y_{i,j}^h))^2,$$

and by using a Taylor expansion at the first order we get that

$$g^{-1}(Y_{i,j}^h \pm \sqrt{h}) = g^{-1}(Y_{i,j}^h) \pm \sigma(X_{i,j}^h)\sqrt{h} + O(h), \tag{1.1.22}$$

so that

$$\mathcal{M}_{i,j}(2) = \sigma^2(X_{i,j}^h)h + O(h)$$

and then (1.1.21) easily follows.
Finally we need to prove the fast convergence to 0 of the fourth local moment, i.e.

$$\lim_{h\to 0} \sup_{(i,j)\in\mathcal{A}_*} \frac{1}{h}\mathcal{M}_{i,j}(4) = 0, \tag{1.1.23}$$

that implies condition (1.1.9) of Theorem 1.1.7. Since from the Taylor expansion (1.1.22) we essentially have that $(g^{-1}(Y_{i,j}^h - \sqrt{h}) \pm g^{-1}(Y_{i,j}^h))^4$ behaves as $h^2$, then (1.1.23) easily follows.

**Case 2:** $\sigma(0) = 0$ **and** $b(0) \geq 0$**.**

In this case Nelson and Ramaswamy define the lower limit for $Y$ by

$$\lim_{x\to 0} g(x) = y^L.$$

Moreover they slightly modify the inverse transformation of (1.1.16) as follows:

$$g^{-1}(y) = \begin{cases} x : g(x) = y, & \text{if} \quad y > y_L, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1.24}$$

Then they allow in a restricted region near the lower bound, that we call $[y^L, y^B]$, that the transformed process $Y$ jumps by a quantity greater than $\sqrt{h}$. So they construct a computationally simple binomial approximation for $Y$ as follows:

- $Y^h_{0,0} = y_0$;
- $Y^h_{i,j} = y_0 + (2j - i)\sqrt{h}, \forall\, j = 0, ..., i$;
- starting from $Y^h_{i,j}$ at time $ih$, the process $Y^h$ may jump at time $(i+1)h$ to the two following values
$$Y^h_{i+1,j_u}, \quad Y^h_{i+1,j_d},$$
with
$$j_d = \begin{cases} \text{the greatest index}\, j^* \in [0, j] : \\ g^{-1}(Y^h_{i,j}) - g^{-1}(Y^h_{i+1,j^*}) \leq b(g^{-1}(Y^h_{i,j}))h; \\ 0, \quad \text{otherwise.} \end{cases}$$
and
$$j_u = \begin{cases} \text{the smallest index}\, j^* \in [j+1, i+1] : \\ g^{-1}(Y^h_{i+1,j^*}) - g^{-1}(Y^h_{i,j}) \geq b(g^{-1}(Y^h_{i,j}))h; \\ i+1, \quad \text{otherwise.} \end{cases}$$

Then the process $Y^h$ may jump for a quantity greater than $\sqrt{h}$ and it is not constrained to necessarily reach the two adjacent nodes as in the classical CRR tree, so $j_d$ and $j_u$ are called "multiple jumps". We remark that in order to get computational simplicity, one needs to define multiple jumps just in a restricted region near the lower bound, otherwise the number of nodes might increase too fast. So, if $Y^h_{i,j} > y^B$ it is assumed that the binomial discretization for the process $Y$ behaves as in Case 1.
By using (1.1.24) they get a computationally simple tree $(X^h_{i,j})_{i,j}$ for the original process $X$:

- $X^h_{0,0} = g^{-1}(Y^h_{0,0}) = x_0$;
- $X^h_{i,j} = g^{-1}(Y^h_{i,j}), \forall\, j = 0, ..., i$;
- starting from $X^h_{i,j}$ at time $ih$, the process $X^h$ may jump at time $(i+1)h$ to the two following values
$$X^h_{i+1,j_u} = g^{-1}(Y^h_{i+1,j_u}), \quad X^h_{i+1,j_d} = g^{-1}(Y^h_{i+1,j_d}).$$

The transition probabilities are now defined as
$$p^h_{i,j} = \frac{b(g^{-1}(Y^h_{i,j}))h + g^{-1}(Y^h_{i,j}) - g^{-1}(Y^h_{i+1,j_d})}{g^{-1}(Y^h_{i+1,j_u}) - g^{-1}(Y^h_{i+1,j_d})}. \tag{1.1.25}$$

We remark that the quantities in (1.1.25) belong to $[0, 1]$ as a consequence of how $j_d$ and $j_u$ are defined. Then the binomial scheme for the process $X$ is given by:
$$X^h_{i,j} \Longrightarrow \begin{cases} X^h_{i+1,j_u}, & p^h_{i,j}, \\ X^h_{i+1,j_d}, & 1 - p^h_{i,j}. \end{cases}$$

In this case the weak convergence of the càdlàg Markov chain running on the lattice $(X^h_{i,j})_{i,j}$ (built as in Case 1) towards the limit diffusion $X$ is not straightforward. Once the martingale problem associated to $b$ and $a = \sigma^2$ has a unique solution, one possibility is to prove directly conditions (1.1.7), (1.1.8) and (1.1.9) of Theorem 1.1.7. Otherwise one can prove some additional properties on the diffusion coefficients $b$ and $\sigma$, as explained in Theorem 3 in Nelson and Ramaswamy ([65], 1990).

**Remark 1.1.11.** *The advantage of the procedure proposed by Nelson and Ramaswamy is that it is constructive and, moreover, it can be applied to a generic diffusion process without restrictions on the parameters and also without requiring homogeneous coefficients (for the general case see [65]).*

**Remark 1.1.12.** *In Chapter 4 we will use the Nelson and Ramaswamy technique as in case 2 for the construction of a computationally simple binomial process for the CIR process, i.e. a process that follows the SDE*

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dB_t, \quad X_0 = x_0.$$

*We will prove directly that conditions (1.1.7), (1.1.8) and (1.1.9) of Theorem 1.1.7 are verified (see Theorem 4.5.8). Moreover, we will be able to explicitly write the region in which multiple jumps may happen (see Lemma 4.5.1). We remark that an analogous procedure can be used for the discretization of the CEV (Constant Elasticity of Variance) process, i.e. a process that follows the SDE*

$$dX_t = \kappa(\theta - X_t)dt + X_t^\gamma dB_t, \quad X_0 = x_0,$$

*with $\gamma \in [\frac{1}{2}, 1]$. By using techniques similar to the ones used in Chapter 4, it could be possible to prove also in this case the convergence conditions (1.1.7), (1.1.8) and (1.1.9) of Theorem 1.1.7.*

### 1.1.4 Convergence of the American prices

Since the seminal work of Cox, Ross and Rubinstein ([28], 1979), discrete-time models have become really popular in option valuation because they represent a useful computational tool that can approximate the continuous-time diffusion model when no simple closed-form solutions are available. From a practical point of view, it is important to obtain that the sequence of American option prices computed with respect to the discrete-time model converges to the corresponding continuous-time American option value determined from the continuous-time diffusion process.
Let us assume to work in the setup described in Section 1.1 with the time interval $[0, 1]$ replaced by the interval $[0, T]$, where $T$ denotes the maturity time of the option.
The stock price process $X$ follows the SDE (1.1.6) and we assume that for every $h > 0$, a discrete (in time and in space) Markov chain $X^h$ approximating $X$ is given. Then we denote with $\{\overline{X}^h\}_h$ the sequence of the piece-wise càdlàg interpolations in time of $X^h$ as defined in

(1.1.4). Here the interest rate is a function $r(t,x) : [0,T] \times D([0,T];\mathbb{R}^d) \to [0,+\infty)$, where $D([0,T];\mathbb{R}^d)$ denotes the space of $\mathbb{R}^d$-valued càdlàg functions on $[0,T]$, so $r$ is not assumed to be deterministic.
Let $f(t,x) : [0,T] \times D([0,T];\mathbb{R}^d) \to [0,+\infty)$ denote a payoff function, so that the American prices in the continuous-time model and in the discrete model are given by

$$\sup_{\tau\in\mathcal{G}_{0,T}} \mathbb{E}(e^{-\int_0^\tau r(s,X)ds} f(\tau,X)) \quad \text{and} \quad \sup_{\sigma\in\mathcal{G}^h_{0,T}} \mathbb{E}(e^{-\int_0^\sigma r(s,\overline{X}^h)ds} f(\sigma,\overline{X}^h))$$

respectively, where $\mathcal{G}_{0,T}$ and $\mathcal{G}^h_{0,T}$ denote the stopping times in $[0,T]$ with respect to the filtrations $\mathcal{F}_t = \sigma(X_s : s \le t)$ and $\mathcal{F}^h_t = \sigma(\overline{X}^h_s : s \le t)$ respectively. We remark that to simplify the notation we indicate with $\mathbb{E}$ the mean w.r.t. the probability measure $\mathbb{P}$ and also the mean w.r.t. $\mathbb{P}^h$.
We now consider the two following assumptions on the payoff function $f$:

**Assumption 1.1.13.** *$f$ is a continuous function (in the product topology) and for every $x,y \in D([0,T];\mathbb{R}^d)$ such that $x_s = y_s$ for each $s \in [0,t]$ then $f(t,x) = f(t,y)$.*

**Assumption 1.1.14.** *There exists $\delta > 1$ and $h_* > 0$ s.t.*

$$\sup_{h<h_*} \mathbb{E}(\sup_{t\le T} |e^{-\int_0^t r(s,\overline{X}^h_s)ds} f(t,\overline{X}^h)|^\delta < \infty).$$

Then, under Assumptions 1.1.13 and 1.1.14, Amin and Khanna ([3], 1994) prove that Theorem 1.1.7 allows to get the convergence of the American prices in the discrete-time model to the corresponding price in the continuous-time model.
Let us call $\{\rho^h\}_h$ the sequence of stopping times related to the discrete problems and $\rho$ the solution of the continuous-time stopping time problem. The proof of the convergence result in [3] consists in several steps that can be resumed as follows:

- under the hypothesis of Theorem 1.1.7 the sequence $\{X^h\}_h$ is tight in $D([0,T];\mathbb{R}^d)$. Then the sequence $\{X^h,\rho^h\}_h$, with $\rho^h \in [0,T]$ for every $h$, is tight in $D([0,T];\mathbb{R}^d) \times [0,T]$, i.e. it admits subsequential weak limits. Let us call $(\tilde{X},\tilde{\rho})$ the limit of one convergent subsequence. Obviously, $\tilde{X}$ is equal in law to $X$, so we call $(X,\tilde{\rho})$ the limit of the convergent subsequence. We remark here that we need that $\tilde{\rho}$ is a "legitimate" stopping time with respect to $X$, i.e. we need that it is indeed a stopping time with respect to the filtration generated by $X$, but we need to prove it;

- let us fix this convergent subsequence $\{X^h,\rho^h\}_h$ (we continue to use the upscript $h$ for this subsequence). For every $\epsilon > 0$, it is possible to construct a sequence $\{X^{h_k,\epsilon_k}, B^{h_k,\epsilon_k}\}_k$ such that the stopping times $\rho^{h_k}$ are adapted to the filtration generated by $B^{h_k,\epsilon_k}$ and such that $X^{h_k,\epsilon_k} \Rightarrow X$, with $X$ solution of (1.1.6), and $B^{h_k,\epsilon_k} \Rightarrow \tilde{B}$, with $\tilde{B}$ a $d$-Brownian motion;

- let us now consider the sequence $\{X^{h_k,\epsilon_k}, B^{h_k,\epsilon_k}\}_k$. It is possible to prove that every limit of a convergent subsequence of $\{X^{h_k,\epsilon_k}, B^{h_k,\epsilon_k}\}_k$ is indeed a solution of (1.1.6), so that $\{X^{h_k,\epsilon_k}, B^{h_k,\epsilon_k}\}_k$ jointly converges to $\{X, \tilde{B}\}$. Moreover, from Lemma 3.5 in [3], we get that the subsequence $\{X^{h_k,\epsilon_k}, B^{h_k,\epsilon_k}, \rho^{h_k}\}_k$ converges a.s. to $\{X, \tilde{B}, \tilde{\rho}\}$;

- finally, by using the previous results, it is possible to construct a probability space on which one can define a Brownian motion $\overline{B}$, a strong Markov process $\overline{X}$ that is a solution of (1.1.6) with $\overline{B}$ in place of $B$ and a random variable $\overline{\rho}$ such that $\{\overline{X}, \overline{\rho}\}$ has the same distribution of $\{X, \tilde{\rho}\}$ and $\overline{\rho}$ is a stopping time with respect to a filtration under which $\overline{X}$ is a strong Markov process.

Then, thanks to these results one can state the convergence of the discrete American prices to the continuous limit when the payoff function $f$ satisfies Assumptions 1.1.13 and 1.1.14. But as remarked in [3], other options can be considered, for example when $f$ is continuous and fulfills the following polynomial-growth condition: there exists $C > 0$ and $\gamma > 1$ such that

$$\sup_{t \in T} |f(t, x)| \leq C(1 + \sup_{t \in T} |x_t|^\gamma). \tag{1.1.26}$$

Provided that one proves that for every $p > 1$ there exists $h_* < 1$ such that

$$\sup_{h < h_*} \mathbb{E}(\sup_{t \leq T} e^{-p \int_0^t r(s, \overline{X}^h_s) ds} (\overline{X}^h_t)^p) < \infty, \tag{1.1.27}$$

then (1.1.26) and (1.1.27) imply Assumption 1.1.14, and if Assumption 1.1.13 and Theorem 1.1.7 hold as well then the convergence of the discrete American prices will follow.
We remark here that Amin and Khanna specify that the interest rate is assumed to be a function of $X$, with $X$ solution of the SDE (1.1.6). However one can repeat the reasoning and then obtain the same results in the case in which the interest rate $r$ is a diffusion independent of $X$ and this is the case we will be concerned with in Chapter 4. In fact we will be working with the problem of pricing European and American style options when the stock price process $X$ follows the Black and Scholes model with stochastic interest rate driven by the Cox, Ingersoll and Ross model. We propose a new bivariate tree for the pair $(X, r)$ and we prove that the continuous process $\{\overline{X}^h, \overline{r}^h\}$ built from the Markov chain running on the bivariate lattice weakly converges to $(X, r)$. We will show in details that Assumption 1.1.14 is still verified for our new binomial bivariate lattice. We remark here that this result is a non trivial extension of the one in Amin and Khanna, who prove that for the standard CRR tree Assumption 1.1.14 holds. Then we will state the convergence result for the American prices computed with our procedure.

**Remark 1.1.15.** *The previous discussion can be used to show the convergence of European-style option prices. In fact the European case is simpler than the American one because there are no stopping times. Then Theorem 1.1.7 and Assumptions 1.1.13 and 1.1.14 guarantee the convergence of the prices.*

**Remark 1.1.16.** *In the simple case in which the discrete approximation is given by the CRR binomial tree we are able to treat a more sophisticated issue: the rate of convergence of the scheme. For details see the following Section. Then this allow us to treat the rate of convergence of the new binomial scheme for pricing barrier options that we will introduce in Chapter 2.*

## 1.2 The rate of convergence of European options in the CRR tree

In the previous Section we have seen that by applying some results of convergence of Markov chains to diffusions we get that the Markov chain built from the CRR binomial tree weakly converges toward the solution of the SDE (1.1.6). Moreover, as the discretization parameter $h \downarrow 0$ (or, equivalently, as the number of time steps $n \uparrow \infty$), the price of European and American options computed with the tree model is close to the price obtained with the continuous-time model. But here we remark that there exist some other important issues on the convergence such as the ones expressed by the following questions:

- it is possible to say something about the rate of convergence of the tree scheme?
- the nature of the convergence is monotonic or oscillatory? and so the obtained results overstimate or understimate the limit?
- is the convergence of the scheme related on the position of the nodes of the tree with respect to the discontinuity points of the payoff function (as the exercise price for digital options) or to the values of the barriers (in the case it is a barrier option)?

In what follows we will consider the answers to the previous questions in two separate cases: the first is the case of standard or vanilla options and the second is the case of barrier-type options.

### 1.2.1 Rate of convergence for vanilla options

Starting from 1990, a large number of authors take into consideration the problem of deriving an asymptotic expansion of the error committed when a discrete procedure is used to approximate European option prices in the classical Black and Scholes model. So in the risk-neutral world the underlying asset price is assumed to be the solution over $[0, T]$ of the SDE (1.1.11), i.e.

$$dX_t = rX_t dt + \sigma X_t dB_t, \quad X_0 = x_0.$$

The idea is to give an explicit expression or at least the asymptotic behavior of the following quantity

$$\mathrm{Err}(n) = v^n(0, x_0) - v(0, x_0) = \mathbb{E}(f(X_T^n)) - \mathbb{E}(f(X_T)), \tag{1.2.1}$$

where $v^n(0, x_0)$ is the approximated European option price at time 0 and starting asset price $x_0$, $v(0, x_0)$ is the corresponding continuous-time value, $f$ is the payoff function and $T$ is the maturity of the option.

Talay and Tubaro ([75], 1990) prove the convergence of order $\frac{1}{n}$ for any smooth function $f$ when the continuous process $X$ is discretized with an Euler scheme (for other details see also Kloeden and Platen ([54], 1992)). This result is then generalized by Bally and Talay ([10], [11], 1996) for measurable payoffs $f$, but only in the case in which the Euler scheme uses the increments of a Brownian motion. Then, Lamberton ([59], 1999) also obtains an estimate of order $\frac{1}{n}$ when $f$ is Lipschitz continuous with bounded second derivative but under some further assumptions on the discrete model that are not satisfied by the CRR scheme.

For what concerns the convergence rate of the CRR binomial scheme, a convergence of order $\frac{1}{\sqrt{n}}$ for a general class of options is first derived in the work of Heston and Zhou ([45], 2000). They propose an adjustment approach based on smoothing the payoff function at its singular points so that it can be achieved a rate of convergence of order $\frac{1}{n}$. However, they don't provide an exact formula for the coefficients in the error expansion.

In what follows we recall the main results in the literature on the rate of convergence of the CRR binomial tree for pricing vanilla options.

For a fixed number $n \in \mathbb{N}$ of time steps, the CRR discretization scheme $(X^n_{i,j})_{i,j}$ for every $i = 0, 1, ..., n$, $j = 0, 1, ..., i$ that approximates the solution $X$ of the SDE (1.1.11) is obtained as follows:

- $X^n_{0,0} = x_0$;
- $X^n_{i,j} = x_0 e^{(2j-i)\sigma\sqrt{h}}$, with $h = \frac{T}{n}$;
- starting from $X^n_{i,j}$ at time $ih$, the process $X^n$ may jump at time $(i+1)h$ to the values

$$\begin{aligned} X^n_{i+1,j+1} &= X^n_{i,j} u, \quad \text{with probability} \quad p, \\ X^n_{i+1,j} &= X^n_{i,j} d, \quad \text{with probability} \quad 1-p, \end{aligned}$$

  where

$$u = e^{\sigma\sqrt{h}}, \quad d = u^{-1}, \quad p = \frac{e^{rh} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}.$$

We remember the notion of the rate of convergence of a binomial scheme in the following definition:

**Definition 1.2.1.** *In the n-period binomial model, if the approximation error Err(n) defined in (1.2.1) satisfies*

$$Err(n) = \frac{\gamma}{n^m} + o\left(\frac{1}{n^m}\right)$$

*as $n \uparrow \infty$, where $\gamma$ is a non zero constant, we say that the convergence of the scheme is of order $\frac{1}{n^m}$.*

The first study on the rate of convergence of binomial tree schemes (including the CRR procedure) for pricing European call options is due to Leisen and Reimer ([61], 1996). They find an upper bound of the approximation error $\mathrm{Err}(n)$ in the CRR model, in fact they prove that there exists a constant $C = C(x_0, K, r, \sigma, T) > 0$:

$$\mathrm{Err}(n) \leq \frac{C}{n}, \quad \forall\, n.$$

The result in [61] holds for a generic binomial tree, so it can be used to derive the rate of convergence of other binomial schemes such as the one in Jarrow and Rudd ([51], 1983) or the one in Tian ([77], 1993).

A more detailed study on the exact rate of convergence of the CRR tree is described in Walsh and Walsh ([82], 2002) and Walsh ([81], 2003). Since in typical financial problems the data is not smooth, they treat a more general class of payoff functions denoted with $\mathcal{K}$ and defined as follows:

**Definition 1.2.2.** *Let $\mathcal{K}$ be the class of real-valued functions $f$ on $\mathbb{R}$ which satisfy:*

(i) *$f$, $f'$, $f''$ have at most finitely many discontinuities;*

(ii) *at each $x$, $f(x) = \frac{1}{2}(f(x^+) + f(x^-))$;*

(iii) *$f$, $f'$, $f''$ are polinomially bounded: i.e. there exist $K > 0$ and $p > 0$ such that $|f(x)| + |f'(x)| + |f''(x)| \leq K(1 + |x|^p)$ for all $x$.*

We present here Theorem 4.3 in [81]. We recall that this result is obtained by using a procedure called Skorokhod embedding that consists in embedding the Markov chain (the binomial scheme) in the diffusion process $X$. By this way it is possible to closely compare the two and accurately evaluate the error. We now assert the theorem:

**Theorem 1.2.3.** *Suppose that $f \in \mathcal{K}$. Let $x_1, x_2, ..., x_k$ be the discontinuities points of $f$ and $f'$, and let $x_0$ be the initial stock price. For any real $x$, let $\tilde{x} = xe^{-rT}$. Let $n$ be an even integer. Let the time-step be $h = \frac{T}{n}$ and the space-step be $\delta = \sigma\sqrt{h}$. Then the error in the binomial tree scheme for the discounted stock price process is*

$$\begin{aligned} Err(n) &= \\ &= \frac{e^{-rT}}{n}\Bigg[\left(\frac{5}{12} + \frac{\sigma^2 T}{6} + \frac{\sigma^4 T^2}{192}\right)\mathbb{E}\{f(X_T)\} \\ &- \frac{1}{6\sigma^2 T}\mathbb{E}\{(\log(\tilde{X}_T/x_0))^2 f(X_T)\} \\ &- \frac{1}{12\sigma^4 T}\mathbb{E}\{(\log(\tilde{X}_T/x_0))^4 f(X_T)\} \end{aligned}$$

$$
\begin{aligned}
&+ \frac{2}{3}\sigma^2 T \mathbb{E}\{X_T^2 f^{''}(X_T)\} \\
&+ \sigma^2 T \sum_i \left( x_i \Delta f^{'}(x_i) - \frac{1}{2}\Delta f(x_i) \right) \\
&\left( \frac{1}{3} + 2\theta(\tilde{x}_i/x_0)(1-\theta(\tilde{x}_i/x_0)) \right) \hat{p}(\log(\tilde{x}_i/x_0)) \\
&- \frac{1}{3} \sum_{i:\log(\tilde{x}_i/x_0)\in\mathbb{N}_e^h} \log(\tilde{x}_i/x_0)\Delta f(x_i)\hat{p}(\log(\tilde{x}_i/x_0)) \\
&+ \frac{1}{6} \sum_{i:\log(\tilde{x}_i/x_0)\in\mathbb{N}_o^h} \log(\tilde{x}_i/x_0)\Delta f(x_i)\hat{p}(\log(\tilde{x}_i/x_0)) \Bigg] \\
&+ e^{-rT}\frac{\sigma\sqrt{T}}{\sqrt{n}} \sum_{i:\log(\tilde{x}_i/x_0)\notin\delta\mathbb{Z}} (2\theta(\tilde{x}_i/x_0)-1)\Delta f(x_i)\hat{p}(\log(\tilde{x}_i/x_0)) \\
&+ O\left(\frac{1}{n^{3/2}}\right),
\end{aligned}
$$

*where $\delta\mathbb{Z}$ is the set of all the multiples of $\delta$, $\mathbb{N}_e^h$ is the subset of $\delta\mathbb{Z}$ of all even multiples of $\delta$ and $\mathbb{N}_o^h$ is the subset of $\delta\mathbb{Z}$ of all odd multiples of $\delta$. We also have that*

$$
\begin{aligned}
\Delta f(x) &= f(x^+) - f(x^-), \\
\Delta f^{'}(x) &= f^{'}(x^+) - f^{'}(x^-), \\
\theta(x) &= frac\left(\frac{\log x}{2\delta}\right),
\end{aligned}
$$

*where $frac(x)$ is the fractional part of $x$, i.e. $frac(x) = x - \lfloor x \rfloor$, with $\lfloor x \rfloor$ denoting the largest integer not greater than $x$. Finally, $\hat{p}(x)$ is the density of $\log(\tilde{X}_T/x_0)$:*

$$
\hat{p}(x) = \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{(x+\frac{1}{2}\sigma^2 T)^2}{2\sigma^2 T}}.
$$

*The expectations are taken with respect to the martingale measure.*

The previous theorem essentially says that if the payoff function $f$ is continuous, then the rate of convergence of the binomial tree scheme is $\frac{1}{n}$ because $\Delta f(x_i) = 0$ for every discontinuity point $x_i$ of $f$ and so the coefficient that multiplies $\frac{1}{\sqrt{n}}$ vanishes. Instead, if the payoff is discontinuous (as for digital options) then $\mathrm{Err}(n)$ is $\frac{1}{\sqrt{n}}$. But we can say more. In fact, a possibility to eliminate the contribution of order $\frac{1}{\sqrt{n}}$ is nullifying the term $(2\theta(\tilde{x}_i/x_0) - 1)$ that multiplies $\frac{1}{\sqrt{n}}$. This happens when the discounted discontinuity points of $f$ lie in the log-scale exactly halfway between two adjacent nodes of the tree at maturity $T$.

Subsequently, Diener and Diener ([31], 2004) compute the first term of the asymptotic expansion of the price of a European call option computed with the CRR model by using some properties of the Laplace integrals. We now state their result (Theorem 4.1 in [31]):

**Theorem 1.2.4.** *In the $n$-period CRR binomial model, the binomial approximation error for a European call option with strike $K$ and maturity $T$ is equal to*

$$Err(n) = \frac{C(n)}{n} + O\left(\frac{1}{n\sqrt{n}}\right),$$

*where*

$$\begin{aligned} C(n) &= -x_0 e^{-\frac{d_{11}^2}{2}} \sqrt{\frac{2}{\pi}} \{\sigma\sqrt{T}\kappa(\kappa-1) + D_1\}, \\ d_{11} &= \frac{1}{\sigma\sqrt{T}}(\log\frac{x_0}{K} + (r + \frac{\sigma^2}{2})T), \\ D_1 &= \frac{1}{96\sigma\sqrt{T}}\left(4\left(\log\frac{x_0}{K}\right)^2 - 8rT\log\frac{x_0}{K} + 3T(4\sigma^2 - 12rT^2 - \sigma^4 T)\right), \\ \kappa &= frac\left(\frac{\log(K/x_0) - n\log d}{\log u - \log d}\right), \\ u &= e^{\sigma\sqrt{T/n}} = \frac{1}{d}. \end{aligned}$$

Then Chang and Palmer ([20], 2007) provide a slight generalization of the results of Walsh and Diener and Diener. In fact, they study the rate of convergence for the $n$-period binomial model in which the parameters $u$ and $d$ are more generic than the ones used in the CRR tree. This allows them to develop a new binomial model, called *the center binomial model*, that is of order $\frac{1}{n}$ both for call and digital options. The main theorem in [20] is the following one:

**Theorem 1.2.5.** *In the $n$-period binomial model, with*

$$u = e^{\sigma\sqrt{h}+\lambda\sigma^2 h}, d = e^{-\sigma\sqrt{h}+\lambda\sigma^2 h}, p = \frac{e^{rh} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}, \tag{1.2.2}$$

*where $\lambda$ is an arbitrary bounded function of $n$, $x_0$ is the initial stock price, $K$ is the strike price and $T$ is the maturity, the binomial approximation error $Err_{call\text{-}dig}(n)$ for pricing a European digital call option is*

$$Err_{call\text{-}dig}(n) = \frac{e^{-rT}e^{-\frac{d_{12}^2}{2}}}{\sqrt{2\pi}}\left[\frac{\Delta_n^K}{\sqrt{n}} - \frac{d_{12}(\Delta_n^K)^2}{2n} + \frac{B_n}{n}\right] + O\left(\frac{1}{n^{3/2}}\right); \tag{1.2.3}$$

*the binomial approximation error $Err_{call}(n)$ for pricing a European call option is*

$$Err_{call}(n) = \frac{x_0 e^{-\frac{d_{11}^2}{2}}}{24\sigma\sqrt{2\pi T}} \frac{A_n - 12\sigma^2 T((\Delta_n^K)^2 - 1)}{n} + O\left(\frac{1}{n^{3/2}}\right), \tag{1.2.4}$$

*where*

$$\begin{aligned}
\Delta_n^K &= 1 - 2frac\left(\frac{\log(x_0/K) - n\log d}{\log(u/d)}\right),\\
d_{11} &= \frac{\log(x_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_{12} = d_{11} - \sigma\sqrt{T},\\
B_n &= \frac{d_{11}^3 + d_{11}d_{12}^2 + 2d_{12} - 4d_{11}}{24} + \frac{(2 - d_{12}d_{22} - d_{12}^2)\sqrt{T}}{6\sigma}(r - \lambda\sigma^2)\\
&+ \frac{Td_{11}}{2\sigma^2}(r - \lambda\sigma^2)^2,\\
A_n &= -\sigma^2 T(6 + d_{11}^2 + d_{12}^2) + 4T(d_{11}^2 - d_{12}^2)(r - \lambda\sigma^2) - 12T^2(r - \lambda\sigma^2)^2.
\end{aligned}$$

**Remark 1.2.6.** *We observe that $\lambda = 0$ corresponds to the classical CRR model, so Theorems 1.2.3, 1.2.4 and 1.2.5 agree with the fact that the error in the binomial approximation for a European call option is $\frac{1}{n}$ and for a European digital call option is $\frac{1}{\sqrt{n}}$.*

We notice that the quantity $\Delta_n^K$ in Theorem 1.2.5 measures the position of the strike $K$ on the log scale in relation to two adjacent terminal stock prices. If we call $(X_{i,j})_{i,j}$, for $i = 0, ..., n$, $j = 0, ..., i$ the values of the stock price at the nodes of the $n$-step binomial tree as described at the beginning of this Section, we have that there exists an integer $j_K$ such that

$$X_{n,j_K-1} = x_0 u^{j_K-1} d^{n-j_K+1} < K \leq X_{n,j_K} = x_0 u^{j_K} d^{n-j_K}$$

and so the "effective" strike price in the binomial model is $X_{n,j_K}$. It is also possible to write in the log-scale the strike $K$ as a geometric average of $X_{n,j_K-1}$ and $X_{n,j_K}$:

$$\log K = \alpha \log X_{n,j_K} + (1-\alpha)\log X_{n,j_K-1}, \quad \text{where} \quad \alpha = \frac{1 + \Delta_n^K}{2}.$$

We remark that $-1 \leq \Delta_n^K \leq 1$ and, in particular, we have that

- $\Delta_n^K = 0 \Rightarrow \log K = \frac{1}{2}\log X_{n,j_K} + \frac{1}{2}\log X_{n,j_K-1}$;
- $\Delta_n^K = -1 \Rightarrow \log K = \log X_{n,j_K-1}$;
- $\Delta_n^K = 1 \Rightarrow \log K = \log X_{n,j_K}$.

Chang and Palmer deduce from the explicit expression of the binomial approximation error in (1.2.3) that if the parameter $\lambda$ is chosen such that the strike price is situated exactly halfway between two stock prices at maturity, then the convergence both for European call and European digital call options is $\frac{1}{n}$. In particular (see Corollary 2 in [20]), they take

$$\lambda = \frac{\log \frac{K}{x_0} - (2j_0 - 1 - n)\sigma\sqrt{h}}{n\sigma^2 h} \tag{1.2.5}$$

where $j_0 = [\tilde{\gamma}] = \min\{m \in \mathbb{N} : m \geq \tilde{\gamma}\}$ and $\tilde{\gamma} = \frac{\log \frac{K}{x_0} + n\sigma\sqrt{h}}{2\sigma\sqrt{h}}$. This choice of $\lambda$ gives the so-called *center binomial model*.

Theorems 1.2.3, 1.2.4 and 1.2.5 completely describe the problem of the study of the rate of convergence for vanilla European options. We observe that the proofs of Theorems 1.2.3 and 1.2.4 are very technical because they involve the Skorokhod embedding representation and the asymptotics of Laplace integrals respectively. The proof of Theorem 1.2.5 seems easier because it is principally based on an extension of a theorem of Uspensky ([80], 1937) on the approximation of the binomial distribution by the normal one. We remark here that the starting point in the proof of Theorem 1.2.5 is the existence of a closed-form formula expressed in terms of sums of binomial coefficients for the prices of European call options and European digital call options computed with the $n$-period binomial tree (for these formulas see for example Pliska ([68], 1997)).

### 1.2.2 Rate of convergence for barrier options

The study of the rate of convergence of the CRR binomial scheme for the computation of barrier option prices is much more complicated. However, it is an important and interesting issue, since it helps us to explain the reason why the binomial price converges very erratically to the true price with zigzag patterns (for numerical experiments see Boyle and Lau ([16], 1994)).

An interesting work, that is the only one able to deal with both single barrier options and double barrier options with generic continuous payoff functions is due to Gobet ([42], 2001). He essentially writes the error $\mathrm{Err}(n)$ defined in (1.2.1) by using the parabolic differential equation solved by the barrier option price and then he decomposes it into the sum of local errors. Let us briefly describe the setup he works with. We start by considering the case of a single barrier option with higher barrier $H$, where here $H$ denotes the barrier in the log-space. Let us suppose to work in the Black and Scholes model, so the stock price process $(X_t)_{t\in[0,T]}$ is the solution of the SDE (1.1.11). Then, the log-price $(\overline{X}_t)_{t\in[0,T]}$ satisfies the equation:

$$\overline{X}_t = \overline{X}_0 + \mu t + \sigma B_t, \quad \text{with} \quad \overline{X}_0 = \log x_0 \quad \text{and} \quad \mu = (r - \sigma^2/2).$$

Let us set $\mathcal{O} = (-\infty, H) \subset \mathbb{R}$ with $H > \overline{X}_0$. We define the stopping time $\tau$ as follows

$$\tau = \inf\{t : \overline{X}_t \notin \mathcal{O}\}.$$

The payoff of an out-type barrier option is $\mathbb{1}_{T<\tau} f(\overline{X}_T)$, where $f$ is a generic function on which we will make later some assumptions. Then, up to consider the discount factor, the function price $u(t,x)$, for every $(t,x) \in [0,T] \times \mathbb{R}$, is equal to

$$u(t,x) = \mathbb{E}_x[\mathbb{1}_{T-t<\tau} f(\overline{X}_{T-t})] = \int_{\mathcal{O}} q_{T-t}(x,y) f(y) dy, \tag{1.2.6}$$

where $q_{T-t}(x,y)$ is the transition density at time $T-t$ of the killed process $\overline{X}$ as it leaves $\mathcal{O}$ and $f$ is a measurable function such that $f$, $f^{'}$ and $f^{''}$ have at most an exponential growth. This transition density can be written as the difference of two functions, $q^1_{T-t}(x,y)$ and $q^2_{T-t}(x,y)$, that are the density functions of two normal random variables: $\mathcal{N}(x+\mu(T-t), \sigma^2(T-t))$ and $\mathcal{N}(-x+2H+\mu(T-t), \sigma^2(T-t))$ respectively. To be precise we have that:

$$\begin{aligned} q_{T-t}(x,y) &= q^1_{T-t}(x,y) - e^{\frac{2\mu(H-x)}{\sigma^2}} q^2_{T-t}(x,y), \\ q^1_{T-t}(x,y) &= \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left( -\frac{(y-x-\mu(T-t))^2}{2\sigma^2(T-t)} \right), \\ q^2_{T-t}(x,y) &= \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left( -\frac{(y+x-2H-\mu(T-t))^2}{2\sigma^2(T-t)} \right). \end{aligned}$$

We recall that $u(t,x)$ satisfies the parabolic PDE of second order with Cauchy and Dirichlet conditions, i.e.

$$\begin{cases} \partial_t u + Lu = 0, \quad (t,x) \in [0,T) \times \overline{\mathcal{O}} \\ u(t,x) = 0, \quad (t,x) \in [0,T) \times \mathcal{O}^c \\ u(T,x) = f(x), \quad x \in \mathcal{O} \end{cases} \tag{1.2.7}$$

where $L$ is the infinitesimal generator associated to the Brownian motion with constant drift and constant volatility, i.e. $Lu(x) = \mu u'(x) + \frac{1}{2}\sigma^2 u''(x)$. Then we set

$$\tau^n = \inf\{t_i : \overline{X}^n_{t_i} \notin \mathcal{O}\}$$

and

$$H_n = \inf\{\overline{X}_0 + i\sigma\sqrt{h} \geq H : i \in \{0,...,n\}\}, \tag{1.2.8}$$

i.e. $H_n$ is the first lattice point in the log-scale equal or greater than $H$.
Let us now consider the following assumptions on the payoff function $f$:

**Assumption 1.2.7.** *We assume that the payoff function $f$ satisfies:*

$$f \in C^0_b((-\infty,H],\mathbb{R}) \cap C^2_b((-\infty,K],\mathbb{R}) \cap C^2_b([K,H],\mathbb{R}) \ s.t. \ f(H) = 0,$$

*for some real $K \in (\inf_{k\in\mathcal{O}} k, \sup_{k\in\mathcal{O}} k)$.*

**Assumption 1.2.8.** *We assume that the payoff function $f$ satisfies:*

$$f \in C^2_b((-\infty,H],\mathbb{R}) \ s.t. \ f(H) \neq 0.$$

**Remark 1.2.9.** *If we are able to find the expression of the CRR binomial approximation error under Assumption 1.2.7 and also under Assumption 1.2.8, then we can treat the more general case in which the payoff function $f$ is not vanishing at the barrier $H$ and with $f'$ having a discontinuity in $K$ (as for the cases of call and put options). In fact, we can write $f$ as follows*

$$f = f_1 + f_2 =: (f - f(H)) + f(H),$$

*where $f_1 = f - f(H)$ satisfies Assumption 1.2.7 and $f_2 = f(H)$ satisfies Assumption 1.2.8.*

**Remark 1.2.10.** *The result due to Gobet ([42], 2001), that is the only one able to deal with the double barrier case, can only treat continuous payoff functions. The case of a discontinuous payoff, as the one of digital options, will be treated in Section 3.2 in which we use the same PDE technique as in [42] in order to obtain an upper bound for the binomial approximation error for double barrier digital options.*

The binomial approximation error, that we still call $\mathrm{Err}(n)$, is now defined as follows:

$$\mathrm{Err}(n) = \mathbb{E}[\mathbb{1}_{T<\tau^n} f(\overline{X}^n_T)] - \mathbb{E}[\mathbb{1}_{T<\tau} f(\overline{X}_T)].$$

The main result in [42] is Theorem 3.1, that is

**Theorem 1.2.11.** *Let $f$ be the payoff function of a barrier option with higher barrier $H$ and initial value $x_0$. Suppose that either Assumption 1.2.7 or 1.2.8 holds. Then*

$$Err(n) = \mathbb{E}_{x_0}[\partial_x u(\tau, H^-)\mathbb{1}_{\tau<T}](H - H_n) + \mathcal{R}_n, \tag{1.2.9}$$

*where the remainder term $\mathcal{R}_n$ is such that there exists a constant $C > 0 : |\mathcal{R}_n| \leq C\frac{\log n}{n}$. As a consequence, one has that for every $\alpha \in (0, 1)$*

$$Err(n) = \mathbb{E}_{x_0}[\partial_x u(\tau, H^-)\mathbb{1}_{\tau<T}](H - H_n) + o\left(\frac{1}{n}\right)^{1-\alpha},$$

*that essentially means that*

$$Err(n) = \mathbb{E}_{x_0}[\partial_x u(\tau, H^-)\mathbb{1}_{\tau<T}](H - H_n) + O\left(\frac{1}{n}\right),$$

*because $\alpha$ can be chosen arbitrarily close to zero.*

We observe that the term $H - H_n$ is $O(\frac{1}{\sqrt{n}})$ and it comes from the position of the nodes of the tree with respect to the contractual barrier. Moreover, the ratio $\frac{H-H_n}{\sigma\sqrt{h}} \in [0, 1)$ and this oscillation factor explains the "zig-zag convergence" in the prices of barrier options observed in Boyle and Lau ([16], 1994). Moreover, since the derivative $\partial_x u(t, H^-)$ is negative, it is possible to deduce that the binomial approximation overestimate the true price of the barrier option because the term $(H - H_n)\mathbb{E}_{x_0}[\partial_x u(\tau, H^-)\mathbb{1}_{\tau<T}]$ is positive. An important

contribution in Gobet ([42], 2001) is that it is possible to numerically improve the standard CRR binomial procedure by computing explicitly the main error term $\mathbb{E}_{x_0}[\partial_x u(\tau, H^-)\mathbb{1}_{\tau<T}]$. In fact, once computed, one can obtain a corrected binomial price, that we call $CBP(n)$, from the standard binomial price $BP(n)$ defined as follows:

$$CBP(n) = BP(n) + (H_n - H)\mathbb{E}_{x_0}[\partial_x u(\tau, H^-)\mathbb{1}_{\tau<T}].$$

From Theorem 1.2.11 we deduce that the error $\mathrm{Err}(n)$ for the corrected binomial price $CBP(n)$ is equal to $\mathcal{R}_n$, with $\mathcal{R}_n$ such that there exist a constant $C > 0$: $|\mathcal{R}_n| \leq C\frac{\log n}{n}$. In Proposition 4.1 in [42], Gobet provides a formula for a generic payoff $f$ that enables us to calculate the term $\mathbb{E}_{x_0}[\partial_x u(\tau, H^-)\mathbb{1}_{\tau<T}]$ and then he also provides the explicit expression for the case of put options (for details see Remark 4.3 in [42]).
The discussion above and the results are similar for the lower barrier case in which $\mathcal{O} = (L, +\infty)$, with $L$ denoting the lower barrier in the log-space, and the double barrier case in which $\mathcal{O} = (L, H)$:

**Theorem 1.2.12.** *Let $f$ be the payoff function of a barrier option with lower barrier $L$ and initial value $x_0$. Suppose that either Assumption 1.2.7 or 1.2.8 holds. Moreover, let $L_n = \sup\{\overline{X}_0 - i\sigma\sqrt{h} \leq L : i \in \{0, ..., n\}\}$ be the first lattice point less or equal to $L$. Then*

$$Err(n) = \mathbb{E}_{x_0}[\partial_x u(\tau, L^+)\mathbb{1}_{\tau<T}](L - L_n) + \mathcal{R}_n, \tag{1.2.10}$$

*where the remainder term $\mathcal{R}_n$ is such that there exists a constant $C > 0$: $|\mathcal{R}_n| \leq C\frac{logn}{n}$.*

**Theorem 1.2.13.** *Let $f$ be the payoff function of a barrier option with higher barrier $H$, lower barrier $L$ and initial value $x_0$. Suppose that either Assumption 1.2.7 or 1.2.8 holds. Let $H_n$ be defined as in (1.2.8) and $L_n$ as in Theorem 1.2.12. Then*

$$\begin{aligned} Err(n) &= \mathbb{E}_{x_0}[\partial_x u(\tau, H^-)\mathbb{1}_{\tau\leq T}\mathbb{1}_{\tau_H<\tau_L}](H - H_n) \\ &\quad + \mathbb{E}_{x_0}[\partial_x u(\tau, L^+)\mathbb{1}_{\tau\leq T}\mathbb{1}_{\tau_L<\tau_H}](L - L_n) + \mathcal{R}_n, \end{aligned} \tag{1.2.11}$$

*where $\tau_H$ is the first hitting time of $(H, +\infty)$, $\tau_L$ is the first hitting time of $(-\infty, L)$, $\tau = \tau_L \wedge \tau_H$ and the remainder $\mathcal{R}_n$ is such that there exists a constant $C > 0$: $|\mathcal{R}_n| \leq C\frac{\log n}{n}$.*

**Remark 1.2.14.** *It is possible to deduce from the proof of the above results, that the rate of convergence for vanilla options is essentially $O(\frac{1}{n})$ and this agrees with Theorem 1.2.3, Theorem 1.2.4 and Theorem 1.2.5. In fact one can repeat the proof of Gobet's results for the standard case. We recall here that the idea consists in decomposing the error $Err(n)$ into two parts: one deals with the contribution due to the lattice points near the barriers and the other is due to the lattice points "far away" from the barriers. Since the error component of order $\frac{1}{\sqrt{n}}$ comes from the first type contribution, then in the case of vanilla options this term vanishes and we essentially obtain an error term that satisfies the following equality:*

$$Err(n) = \mathcal{R}_n, \quad with \quad |\mathcal{R}_n| \leq C\frac{\log n}{n}.$$

In the case of a call option with a single barrier, Lin and Palmer ([62], 2013) obtain an explicit formulas for the coefficients of $\frac{1}{\sqrt{n}}$ and $\frac{1}{n}$ in the asymptotic expansion of the error Err$(n)$ by using similar techniques employed in Chang and Palmer ([20], 2007). They get that the coefficient that multiplies $\frac{1}{\sqrt{n}}$ is related to the position of the real barrier with respect to the nodes of the tree and this indeed agrees with Theorem 1.2.11. Let us now briefly describe the setup they work with. For example let us take into consideration the asymptotic expansion for the price of a down-and-out call option with strike $K$, lower barrier $L$ and maturity $T$. As in Theorem 1.2.5, they define the quantity $\Delta_n^K$ as

$$\Delta_n^K = 1 - 2\text{frac}\left(\frac{\log(x_0/K) - n\log d}{\log(u/d)}\right),$$

where $u = e^{\sigma\sqrt{h}} = d^{-1}$. So the quantity $\Delta_n^K$ measures the position of $K$ on the log-scale in relation to the two adjacent terminal stock prices. Corresponding to the lower barrier $L$, they introduce a "similar" quantity $\Delta_n^L$. The definition of $\Delta_n^L$ depends on whether the effective barrier (i.e. the barrier on the tree structure, that is generally different from the contractual barrier) is a terminal stock price or not. We remember that we call $(X_{i,j})_{i,j}$, for $i = 0, ..., n$, $j = 0, ..., i$ the values of the stock price at the nodes of the $n$-step binomial tree. So the value of the discretized stock price process at a generic node $(i, j)$ is given by $X_{i,j} = x_0 u^j d^{n-j}$. When the effective barrier, that we denote with $\tilde{L}$, is a terminal stock price, then there exist an integer $j_L$ such that

$$\tilde{L} = X_{n,j_L} = x_0 u^{j_L} d^{n-j_L} \leq L < \tilde{L}u = X_{n-1,j_L} = x_0 u^{j_L} d^{n-1-j_L},$$

where

$$j_L = \frac{1}{2}\lfloor 2l_L \rfloor, \quad \text{with} \quad l_L = \frac{\log(\frac{L}{x_0})}{2\sigma\sqrt{T}} + \frac{n}{2},$$

in which $\lfloor x \rfloor$ denotes the integer part of $x$, for all $x \in \mathbb{R}$, i.e. $\lfloor x \rfloor$ is the largest integer less or equal to $x$. Instead, when the effective barrier is a stock price from the penultimate period, then we have

$$\tilde{L} = X_{n-1,j_L} = x_0 u^{j_L} d^{n-1-j_L} \leq L < \tilde{L}u = X_{n,j_L+1} = x_0 u^{j_L+1} d^{n-(j_L+1)},$$

where

$$j_L = \frac{1}{2}\lfloor 2l_H \rfloor - \frac{1}{2}.$$

By resuming, the effective barrier can be written as $\tilde{L} = x_0 u^{j_L} d^{n-j_L}$ and $\tilde{j_L}$ and $j_L$ satisfy the following relation

$$j_L = \tilde{j_L} - \frac{1}{2}(1 - \epsilon_n),$$

where

$$\epsilon_n = \begin{cases} 0, & \text{if the effective barrier is not a terminal stock price} \\ 1, & \text{if the effective barrier is a terminal stock price.} \end{cases}$$

Finally they set

$$\Delta_n^L = \text{frac}(2l_H),$$

so that it is possible to write

$$\log L = (1-\Delta_n^L)\log \tilde{L} + \Delta_n^L \log(\tilde{L}u).$$

Then, the quantity $\Delta_n^L$ measures the position of the barrier $L$ on the log-scale with respect to the two adjacent stock prices, one of which is a terminal stock price and the other a price from the penultimate period. We now state the main theorem (Theorem 1.2 in [60]) for down-and-out call options:

**Theorem 1.2.15.** *In the n-period CRR binomial model, the binomial error Err(n) for European down-and-out options with lower barrier L is equal to:*
*if $L < K$ or $L = K$ and $L = K$ is not a terminal stock price*

$$Err(n) = A_1\Delta_n^L \frac{1}{\sqrt{n}} + [B_1 - D_1(\Delta_n^K)^2 - E_1(\Delta_n^L)^2]\frac{1}{n} + O\left(\frac{1}{n^{3/2}}\right),$$

*and if $L > K$ or $L = K$ and $L = K$ is a terminal stock price it is*

$$Err(n) = A_2\Delta_n^L \frac{1}{\sqrt{n}} + [B_3 - C\epsilon_n^2 - E_2(\Delta_n^L)^2]\frac{1}{n} + O\left(\frac{1}{n^{3/2}}\right).$$

*The constants are defined as follows:*

$$d_{11} = \frac{\log\frac{x_0}{K} + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_{12} = d_{11} - \sigma\sqrt{T},$$
$$d_{21} = \frac{\log\frac{L^2}{x_0K} + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_{22} = d_{21} - \sigma\sqrt{T},$$
$$d_{31} = \frac{\log\frac{x_0}{L} + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_{32} = d_{31} - \sigma\sqrt{T},$$
$$d_{41} = \frac{\log\frac{L}{x_0} + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_{42} = d_{41} - \sigma\sqrt{T},$$

$$\alpha = \frac{r - \frac{1}{2}\sigma^2}{2\sigma}, \quad \hat{\alpha} = \alpha + \frac{\sigma}{2},$$
$$\beta = \frac{\sigma^4 - 4\sigma^2 r + 12r^2}{48\sigma}, \quad \hat{\beta} = -\beta - \frac{\sigma r}{6},$$
$$\hat{g}_i = 2T(\hat{\alpha}^2 d_{i1} + \hat{\beta}\sqrt{T}) + \left(\frac{2\hat{\alpha}\sqrt{T}}{3} - \frac{d_{i1}}{12}\right)(1 - d_{i1}^2), i = 1, 2, 3, 4$$
$$g_i = 2T(\hat{\alpha}^2 d_{i2} + \hat{\beta}\sqrt{T}) + \left(\frac{2\hat{\alpha}\sqrt{T}}{3} - \frac{d_{i2}}{12}\right)(1 - d_{i2}^2), i = 1, 2, 3, 4$$

$$G_1 = \frac{x_0}{\sqrt{2\pi}} e^{-\frac{d_{11}^2}{2}}(\hat{g}_1 - g_1), \; G_2 = \frac{x_0}{\sqrt{2\pi}} \left(\frac{x_0}{L}\right)^{-1-\frac{d_{21}^2}{2}} e^{-\frac{d_{21}^2}{2}}(\hat{g}_2 - g_2),$$
$$G_3 = \frac{x_0}{\sqrt{2\pi}} e^{-\frac{d_{31}^2}{2}}(\hat{g}_3 - \frac{K}{L} g_3), \; G_4 = \frac{x_0}{\sqrt{2\pi}} e^{-\frac{d_{31}^2}{2}}(\hat{g}_4 - \frac{K}{L} g_4),$$
$$A_1 = 4\sqrt{T} h_1(d_{21}, d_{22}), \quad A_2 = 4\sqrt{T} h_1(d_{41}, d_{42}) + \frac{2x_0}{\sqrt{2\pi}} e^{-\frac{d_{31}^2}{2}} \left(1 - \frac{K}{L}\right),$$
$$A_3 = 4\sqrt{T}(h_1(-d_{41}, -d_{42}) - h_1(-d_{21}, -d_{22})) - \frac{2x_0}{\sqrt{2\pi}} e^{-\frac{d_{31}^2}{2}} \left(1 - \frac{K}{L}\right),$$
$$h_i(x, y) = \left(\frac{x_0}{L}\right)^{-\frac{2r}{\sigma^2}} \left(D\left(\frac{r + \frac{\sigma^2}{2}}{2\sigma}\right)^i \Phi(x) - \frac{x_0 K e^{-rT}}{L} \left(\frac{r - \frac{\sigma^2}{2}}{2\sigma}\right)^i \Phi(y)\right),$$
$$for \quad i = 0, 1, 2,$$
$$B_1 = G_1 - G_2 + I h_0(d_{21}, d_{22}), B_2 = B_1 - G_1,$$
$$B_3 = G_3 - G_4 + I h_0(d_{41}, d_{42}), B_4 = B_3 - G_1,$$
$$B_5 = G_2 + G_3 - G_4 + I h_0(-d_{21}, -d_{22}) - I h_0(-d_{41}, -d_{42}), B_6 = B_5 - G_1,$$
$$I = \left(\frac{4\beta + \frac{16}{3}\alpha^3}{\sigma}\right) \log\left(\frac{x_0}{L}\right) T,$$
$$C_1 = \frac{2x_0}{\sqrt{2\pi}} \left(\frac{x_0}{L}\right)^{-1-\frac{2r}{\sigma^2}} e^{-\frac{d_{21}^2}{2}} \sigma\sqrt{T}, \; C_2 = \frac{x_0}{\sqrt{2\pi}} e^{-\frac{d_{31}^2}{2}} \left(d_{31} - \frac{K}{L} d_{32}\right),$$
$$C_3 = \frac{x_0}{\sqrt{2\pi}} e^{-\frac{d_{31}^2}{2}} \left(d_{41} - \frac{K}{L} d_{42}\right), \; C = \frac{1}{2}(C_2 - C_3),$$
$$D_1 = \frac{x_0}{2\sqrt{2\pi}} e^{-\frac{d_{11}^2}{2}} \sigma\sqrt{T} - \frac{C_1}{4}, \; D_2 = \frac{C_1}{4}, D_3 = D_1 + D_2,$$
$$E_1 = 8T h_2(d_{21}, d_{22}) + C_1, E_2 = 8T h_2(d_{41}, d_{42}) + \frac{1}{2}(3C_2 + C_3),$$
$$E_3 = 8T(h_2(-d_{21}, -d_{22}) - h_2(-d_{41}, -d_{42})) - C_1 + \frac{1}{2}(3C_2 + C_3).$$

*Finally, $\Phi(x)$ is the standard normal distribution function.*

As for the proof of Theorem 1.2.5, the basic idea used to prove the statements in Theorem 1.2.15 is to write the binomial price of the down-and-out call option with a down barrier $L$ by using the binomial closed-form formula provided in Reimer and Sandmann ([70], 1996) and then a normal approximation of the sum of binomial coefficients following a generalization of a result of Uspensky ([80], 1937) provided in Lin and Palmer ([62], 2013). Similarly, it is possible to show analogous results for the down-and-in call, and the up-and-in and the up-and-out call options.
We remark that Theorem 1.2.15 shows that the convergence of the binomial prices for single barrier call options is oscillatory and the oscillation is due to the quantity $\Delta_n^L$ that is a measure of how much the contractual barrier $L$ is far from the effective barrier on the lattice. We also observe that no contribution of order $\frac{1}{\sqrt{n}}$ is due to the position of the strike $K$. In Chapter 3 we will prove that it is possible to derive a similar asymptotic expansion for the price of a digital call option with a single barrier by using arguments similar to the ones used to prove Theorem 1.2.15.

# Chapter 2

# Double and multi-step double barrier options in the Black and Scholes model

A *double barrier option* is a path dependent option because the payoff function depends on whether the stock's price path ever touches two price levels called "barriers" that are located above (*higher barrier*) and below (*lower barrier*) the current stock price. We call *step double barrier option* or *multi-barrier double touch/no touch* or simply *tunnel* a kind of option whose barriers evolve in time as piecewise constant functions.
The principal features of those contracts are the flexibility (in fact investors may set knock-out or knock-in levels they want) and the fact that it is possible to manage the risk of "sudden death" of the option by adjusting the barriers according to the investors risk-aversion.
Both barrier options and step double barrier options are traded in financial markets and in particular are embedded in a variety of structured products, in particular in range-type contracts. For instance, standard range notes are structured products that pay an above-market interest rate for each day that the underlying spot rate stays within a specified range and pay no interest for that day if the underlying asset process goes outside the range.
It is also possible to find this kind of structure in loans, in which the client pays the best interest rate below market if there are no knock-out conditions before maturity. There are also range forward contracts that allow the holders to buy/sell quantities of certain assets on a specific date at a preferential rate that is more favorable than the instant forward rate if some knock-out or knock-in conditions are satisfied. Then, there are some options called "wedding cakes" that pay coupons by considering the underlying reference rate movements. Typically, the option pays a lower coupon if the reference rate moves within the wider range and it pays nothing if it touches the barrier levels. The payoff representation is similar to a wedding cake, hence the name of this kind of options. In reality, as suggested in Guillaume ([43], 2010), there are a lot of other financial products that embed double barrier and step double barrier options and for a detailed description one can refers to the book of Wystup ([86], 2006).

Here comes the necessity to price this kind of products. We assume in what follows that the underlying process follows a geometric Brownian motion, so we work in the Black and Scholes model.
Kunitomo and Ikeda ([56], 1992) derive a closed-form formula for European double-barrier options. In particular, they give the explicit formula, expressed as an infinite series, that a geometric Brownian motion reaches the maturity date without hitting either the lower and the upper barrier. The main assumption in [56] is that the barriers are exponential functions of time (this hypothesis includes the case they are constant), so the contract is in some ways constrained by the mathematical assumption on the boundaries that on the other hand allows exact calculations. Geman and Yor ([40], 1994) derive the expression of the Laplace transform of the double-barrier option price with respect to the maturity date. Subsequently Sidenius ([73], 1998) generalizes these pricing formulas by considering the possibility of including rebates (where the rebate is the amount paid to the holder of the option if the option expires worthless) and by allowing the option to have a more complicated form (for example he considers the case in which the barriers have to be crossed several times in a certain order).
For what concerns the pricing of step double barrier options, the analytical formulas are due to Guillaume ([43], 2010) who obtains an exact expression in terms of infinite series for 2-step double barrier options, i.e. options whose lifetime is divided into two intervals to which different barrier levels are associated. The formulas in [43] are highly complicated and require the implementation of numerical algorithms for the approximation of the infinite sums. Instead, no closed-form formulas are proposed for multi-step double barrier options and in this case the author suggests the use of a Monte Carlo scheme enhanced with control variate.
Both for double barrier options and for step double barrier options, the numerical implementation of the related analytical expressions requires the approximation of the infinite sums by means of algorithms that could lead large pricing errors. Moreover these closed-form formulas are available only for European-type options and not for the American case. Then an alternative way to proceed is to use numerical lattice procedures that are able to deal with the American case and also with nonstandard payoff functions.
As observed in Boyle and Lau ([16], 1994), a naive application of the classical CRR model for barrier options may lead large pricing errors caused by the distance between the contractual barriers and the nodes of the tree structure. For this reason a possible solution is to feed the algorithm with the "right" value of the barrier as in Boyle and Lau ([16], 1994), Ritchken ([72], 1995), Cheuck and Vorst ([22], 1996), Figlewsky and Gao ([34], 1999), Gaudenzi and Lepellere ([37], 2006) and Gaudenzi and Zanette ([39], 2009). However those methods are able to deal only with a single barrier.
The first attempt for treating numerically in an efficient and accurate way the double barrier pricing issue is due to Dai and Lyuu ([29], 2010), who construct the so-called bino-trinomial tree in which the time step algorithm is chosen such that both the lower barrier $L$ and the higher barrier $H$ coincide with two layers of nodes in the binomial mesh. However numerical results show that this method is not able to deal with the "near barrier" problem, occurring

when the initial asset price is very close to one of the barriers. It means that the algorithm needs a drastically increase of the number of time steps $n \in \mathbb{N}$ in order to provide an accurate price and this could lead to require a very large value for $n$.
In Appolloni, Gaudenzi and Zanette ([5], 2013) we introduce the *Binomial Interpolated Lattice* in which the binomial mesh is constructed as in Dai and Lyuu ([29], 2010), i.e. the time step is chosen such that $L$ and $H$ match two layers of nodes in the tree, but in the pricing algorithm we introduce suitable interpolations in time and in space to solve the "near barrier" problem. The algorithm turns to be an efficient procedure for pricing double and multi-step double barrier options. Moreover, we provide a proof of the rate of convergence of the binomial approximation by using the PDE techniques described in Gobet ([42], 2001). Similarly, we provide the rate of convergence for the Dai and Lyuu algorithm and we show that, as for the Binomial Interpolated Lattice, it is equal to $o(\frac{1}{n^{1-\alpha}})$ for all $\alpha \in (0,1)$, that essentially means that it is $O(\frac{1}{n})$.
We also provide some numerical results. First we propose numerical experiments in order to compare our algorithm to the one proposed in Dai and Lyuu ([29], 2010) for pricing knock-out double barrier call options and we observe that the Binomial Interpolated Lattice performs better than the bino-trinomial procedure. Then we also compare our procedure with the PDE finite difference method implemented following Zvan, Forsyth and Vetzal ([87], 2000) both for pricing and hedging purposes. The result is that the Binomial Interpolated Lattice provides double barrier option prices and Greeks (delta, gamma, vega) with similar accuracy than the PDE method in analogue CPU times. Finally we study the performance of the Binomial Interpolated Lattice algorithm for the pricing of multi-step double barrier options.
The Chapter is organized as follows. In Section 2.1 we present the model and we define the multi-step double barrier options that we will price in Section 2.8 of the numerical results. In Section 2.2 we recall the European closed-form formulas for European double barrier knock-out put options (due to Kunitomo and Ikeda ([56], 1992)) and for European 2-step double knock-and-out put options (due to Guillaume ([43], 2010)). In Section 2.3 we describe the principal lattice techniques used to price single barrier options and in Section 2.4 we present the Dai and Lyuu algorithm that is the only one able to deal with the double barrier case. In Section 2.5 we show our new lattice technique, the Binomial Interpolated Lattice, that is described in Appolloni, Gaudenzi and Zanette ([5], 2013). Then, in Section 2.6 we discuss the rate of convergence of our scheme and that of the Dai and Lyuu one. In Section 2.7 we extend our algorithm to the case of the multi-step double barrier options and the early-ending multi-step double barrier options. Finally, Section 2.8 is devoted to the numerical results on the pricing of double barrier options (and comparisons with the Dai and Lyuu model and the finite difference approach of Zvan, Forsyth and Vetzal), 2-step double barrier options (and comparisons with the Guillaume closed-form formulas in the European case) and multi-step double barrier options (and comparisons with a Monte Carlo benchmark in the European case). No benchmark is available for 2-step and multi-step double barrier options in the American case.

## 2.1 The model

We consider a market model in the time interval $[0, T]$ where the evolution of the risky asset $(S(t))_{t\in[0,T]}$ is governed by the Black-Scholes stochastic differential equation

$$\frac{dS(t)}{S(t)} = rdt + \sigma dB(t), \quad S(0) = s_0 > 0, \tag{2.1.1}$$

where $(B(t))_{t\in[0,T]}$ is a standard Brownian motion under the risk neutral probability measure. The non-negative constant $r$ is the risk-free interest rate and $\sigma$ is the constant volatility parameter. As remarked in the introduction of this Chapter, a barrier option is a contract whose payoff depends on whether the underlying stock price path ever touches certain price levels called barriers. Once either of these barriers is breached, the status of the option is immediately determined: either the option comes into existence if the barrier is a knock-and-in type or it ceases to exist if the barrier is a knock-and-out type. In what follows we will consider the price of double barrier knock-and-out call and put options, so we recall that the payoff is given by

$$\begin{cases} \max(\theta S_T - \theta K, 0), & \text{if} \quad S_{\inf} > L \quad \text{and} \quad S_{\sup} < H \\ 0, & \text{otherwise} \end{cases} \tag{2.1.2}$$

where

- $K$ is the strike price;
- $S_{\inf} = \inf_{t\in[0,T]} S_t$ and $S_{\sup} = \sup_{t\in[0,T]} S_t$;
- $L$ and $H$ stand for the lower and the higher barrier respectively;
- $\theta = 1$ for call options and $\theta = -1$ for put options.

Let us introduce the *regular step double barrier options* as explained in Guillaume ([43], 2010). Let $\{T_0, T_1, ..., T_{n-1}, T_n\}$ be a partition of the option lifetime $[0, T]$ with

$$0 = T_0 < T_1 < ... < T_n = T.$$

A *regular n-step double barrier option* is an option in which the barriers are constant in every interval $[T_i, T_{i+1}], i = 0, ..., n-1$. Hence, at each interval $[T_i, T_{i+1}]$ a constant lower barrier $L_i$ and a constant higher barrier $H_i$ are associated. A regular $n$-step double knock-out option with payoff function $f$, has this payoff at maturity provided that the underlying asset price stays in $(L_i, H_i)$ in every interval $[T_i, T_{i+1}]$, otherwise it expires worthless or provides a contractual rebate. For example, a regular $n$-step double knock out put/call option has the following payoff:

$$\begin{cases} \max(\theta S_T - \theta K, 0), & \text{if} \quad S^i_{\inf} > L_i \quad \text{and} \quad S^i_{\sup} < H_i \quad \forall\, i = 0, 1, ..., n-1 \\ 0, & \text{otherwise} \end{cases} \tag{2.1.3}$$

where for every $i = 0, 1, ..., n-1$ we define

$$S^i_{\inf} = \inf_{t\in[T_i,T_{i+1}]} S_t \quad \text{and} \quad S^i_{\sup} = \sup_{t\in[T_i,T_{i+1}]} S_t,$$

and, as before, $\theta = 1$ for call options and $\theta = -1$ for put options.
An *early ending n-step double knock-out option* with maturity $T$ has the same payoff of a standard call or put option on the condition that the underlying asset price stays in $(L_i, H_i)$ in every interval $[T_i, T_{i+1}], i = 0, .., n-2$ (hence there are no "out" conditions on the last time interval).
We remark that it is also possible to include in all these contracts the possibility to remove the knock-out barrier provision (*partial-time step double barrier options*) and also take into account knock-in features instead of knock-out ones. We stress that, as usual, in the European case the knock-in option prices are obtained by taking the difference between the prices of the corresponding vanilla option and the knock-out option.

## 2.2 The closed-form formulas

We now present the exact formula for the price of a knock-out double barrier call as described in Kunitomo and Ikeda ([56], 1992). We recall that they work in the more general case in which the barriers are functions of the time and have an exponential form, i.e.

$$H(t) = He^{\delta_1 t}, \quad L(t) = Le^{\delta_2 t}, \tag{2.2.1}$$

with $L(t) < H(t)$, for every $t \in [0, T]$ and with $\delta_1$ and $\delta_2$ denoting the curvature of the higher and the lower barrier respectively. By choosing $\delta_1 = 0 = \delta_2$ we obtain the price when the barriers are constant. The price at time $t \in [0, T]$, that we denote with $v(t)$, of a European double barrier knock-out call option with barriers as in (2.2.1) and stock price process $(S(t))_{t\in[0,T]}$ following the SDE (2.1.1) is given by:

$$\begin{aligned}
v(t) = s_0 \sum_{n=-\infty}^{+\infty} \Bigg[ &\left(\frac{H^n}{L^n}\right)^{c^*_{1n}} \left(\frac{L}{s_0}\right)^{c_{2n}} (\Phi(d^+_{1n}) - \Phi(d^+_{2n})) \\
&- \left(\frac{L^{n+1}}{H^n s_0}\right)^{c^*_{3n}} (\Phi(d^+_{3n}) - \Phi(d^+_{4n}))\Bigg] \\
- Ke^{-r\tau} \sum_{n=-\infty}^{+\infty} \Bigg[ &\left(\frac{H^n}{L^n}\right)^{c^*_{1n}-2} \left(\frac{L}{s_0}\right)^{c_{2n}} (\Phi(d^-_{1n}) - \Phi(d^-_{2n})) \\
&- \left(\frac{L^{n+1}}{H^n s_0}\right)^{c^*_{3n}-2} (\Phi(d^-_{3n}) - \Phi(d^-_{4n}))\Bigg]
\end{aligned}$$

where $F = He^{\delta_1 T}$, $\tau = T - t$ and

$$c_{1n}^* = 2\frac{r-\delta_2-n(\delta_1-\delta_2)}{\sigma^2}+1,\ c_{2n} = 2n\frac{\delta_1-\delta_2}{\sigma^2},\ c_{3n}^* = 2\frac{r-\delta_2+n(\delta_1-\delta_2)}{\sigma^2}+1,$$
$$d_{1n}^{\pm} = \frac{\log\frac{s_0H^{2n}}{KL^{2n}}+(r\pm\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}},\ d_{2n}^{\pm} = \frac{\log\frac{s_0H^{2n}}{FL^{2n}}+(r\pm\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}},$$
$$d_{3n}^{\pm} = \frac{\log\frac{L^{2n+2}}{Ks_0H^{2n}}+(r\pm\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}},\ d_4^{\pm} = \frac{\log\frac{L^{2n+2}}{Fs_0H^{2n}}+(r\pm\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

and where $\Phi(\cdot)$ is the standard normal distribution function.
We now present the closed-form formula for the pricing at time $t = 0$ of a European 2-step double knock-out put option with constant barrier levels as described in Proposition 1 in Guillaume ([43], 2010). We recall here that $[L_i, H_i]$ $(i = 1, 2)$ is the $i$-th step of the double barrier, where $L_i$ is the lower bound and $H_i$ is the higher bound. To be precise, in the first time interval $[0, T_1]$ we consider the barriers $L_1$ and $H_1$, and in the time interval $[T_1, T_2]$ the barriers $L_2$ and $H_2$. The price at time 0, that we denote with $v_{TSD}(0)$ (with TSD standing for 2-steps double), is given by

$$v_{TSD}(0) = e^{-rT_2}K\phi_{TSD}(\mu = r-\sigma^2/2) - s_0\phi_{TSD}(\mu = r+\sigma^2/2),$$

where the function $\phi_{TSD}(\mu)$ is defined as follows:

$$\begin{aligned}\phi_{TSD}(\mu) = &\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\left(\frac{H_1^{n_1}H_2^{n_2}}{L_1^{n_1}L_2^{n_2}}\right)^{\frac{2\mu}{\sigma^2}}\Bigg[\phi_2(I_1(H_1\wedge H_2), I_2(K); \sqrt{T_1/T_2})\\
&-\Phi_2(I_1(L_1\vee L_2), I_2(K); \sqrt{T_1/T_2}) - \Phi_2(I_1(H_1\wedge H_2), I_2(L_2); \sqrt{T_1/T_2})\\
&+\Phi_2(I_1(L_1\vee L_2), I_2(L_2); \sqrt{T_1/T_2})\Bigg]\\
&-\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\left(\frac{L_1^{n_1}L_2^{n_2+1}}{s_0H_1^{n_1}H_2^{n_2}}\right)^{\frac{2\mu}{\sigma^2}}\Bigg[\phi_2(I_3(H_1\wedge H_2), I_4(K); -\sqrt{T_1/T_2})\\
&-\Phi_2(I_3(L_1\vee L_2), I_4(K); -\sqrt{T_1/T_2}) - \Phi_2(I_3(H_1\wedge H_2), I_4(L_2); -\sqrt{T_1/T_2})\\
&+\Phi_2(I_3(L_1\vee L_2), I_4(L_2); -\sqrt{T_1/T_2})\Bigg]\end{aligned}$$

$$
\begin{aligned}
&- \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \left( \frac{L_1^{n_1+1} H_2^{n_2}}{s_0 H_1^{n_1} L_2^{n_2}} \right)^{\frac{2\mu}{\sigma^2}} \Bigg[ \phi_2(I_5(H_1 \wedge H_2), I_6(K); \sqrt{T_1/T_2}) \\
&- \Phi_2(I_5(L_1 \vee L_2), I_6(K); \sqrt{T_1/T_2}) - \Phi_2(I_5(H_1 \wedge H_2), I_6(L_2); \sqrt{T_1/T_2}) \\
&+ \Phi_2(I_5(L_1 \vee L_2), I_6(L_2); \sqrt{T_1/T_2}) \Bigg] \\
&+ \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \left( \frac{L_2^{n_2+1} H_1^{n_1}}{L_1^{n_1+1} L_2^{n_2}} \right)^{\frac{2\mu}{\sigma^2}} \Bigg[ \phi_2(I_7(H_1 \wedge H_2), I_8(K); -\sqrt{T_1/T_2}) \\
&- \Phi_2(I_7(L_1 \vee L_2), I_8(K); -\sqrt{T_1/T_2}) - \Phi_2(I_7(H_1 \wedge H_2), I_8(L_2); -\sqrt{T_1/T_2}) \\
&+ \Phi_2(I_7(L_1 \vee L_2), I_8(L_2); -\sqrt{T_1/T_2}) \Bigg],
\end{aligned}
$$

where $\Phi_2(\cdot, \cdot; \theta_{12})$ is the joint cumulative distribution function of two standard normal random variables $Y_1$ and $Y_2$ with $\theta_{12}$ as correlation coefficient and where the remaining functions are defined as follows:

$$
\begin{aligned}
I_1(x) &= \frac{\log\left(\frac{x L_1^{2n_1}}{s_0 H_1^{2n_1}}\right) - \mu T_1}{\sigma\sqrt{T_1}}, & I_2(x) &= \frac{\log\left(\frac{x L_2^{2n_2} L_1^{2n_1}}{s_0 H_2^{2n_2} H_1^{2n_1}}\right) - \mu T_2}{\sigma\sqrt{T_2}}, \\
I_3(x) &= \frac{\log\left(\frac{x L_1^{2n_1}}{s_0 H_1^{2n_1}}\right) + \mu T_1}{\sigma\sqrt{T_1}}, & I_4(x) &= \frac{\log\left(\frac{x s_0 H_2^{2n_2} H_1^{2n_1}}{L_2^{2n_2+2} L_1^{2n_1}}\right) - \mu T_2}{\sigma\sqrt{T_2}}, \\
I_5(x) &= \frac{\log\left(\frac{x s_0^{2n_1} H_1^{2n_1}}{L_1^{2n_1+2}}\right) - \mu T_1}{\sigma\sqrt{T_1}}, & I_6(x) &= \frac{\log\left(\frac{x s_0 L_2^{2n_2} H_1^{2n_1}}{L_1^{2n_1+2} H_2^{2n_2}}\right) - \mu T_2}{\sigma\sqrt{T_2}}, \\
I_7(x) &= \frac{\log\left(\frac{x s_0 H_1^{2n_1}}{L_1^{2n_1+2}}\right) + \mu T_1}{\sigma\sqrt{T_1}}, & I_8(x) &= \frac{\log\left(\frac{x L_1^{2n_1+2} H_2^{2n_2}}{s_0 L_2^{2n_2+2} H_1^{2n_1}}\right) - \mu T_2}{\sigma\sqrt{T_2}}.
\end{aligned}
$$

The implementation of the previous formula, as explained in Guillaume ([43], 2010), can be done by using the Genz algorithm ([41], 2004) that computes the standard normal cumulative distribution functions with a precision for all practical purposes that is gained with the truncation of the infinite sums to $n_1, n_2 = -8, ..., 8$.

## 2.3 Lattice procedures for barrier options

In this Section we present the principal works that deal with the problem of pricing barrier options by using lattice techniques. The first tentative is described in Boyle and Lau ([16], 1994), in which the authors observe that a naive use of the CRR model for the pricing of single barrier options may cause significant convergence problems, in the sense that the method is very slow and has a persistent bias. Since a standard call option can be decomposed into a down-and-out call option and a down-and-in call option, they first analyze in both cases the convergence of the binomial prices to the Black and Scholes ones. In particular, they observe that the binomial values of the down-and-in call options always lie below the continuous-time limit and the binomial values of the down-and-out call option always lie above the continuous-time limit. Moreover, the convergence is very erratic and the oscillations of the binomial prices for the down-and-out call option are exactly a mirror image of the oscillations of the binomial prices for the down-and-in call option. This is essentially the reason why the estimates for a standard call are much more stable: the two sources of oscillations cancel each other. But the main reason why the binomial prices for barrier options are so oscillating is that the true barrier in general lies between two nodes of the tree and this induces inaccuracy in the calculations. One possibility in order to overcome this problem is to choose the number of time steps $n$ such that the contractual barrier is as close as possible to a layer of nodes. Let us suppose for example that the lower barrier $L$ lies between $m$ down jumps and $(m+1)$ down jumps of the asset price, i.e.

$$s_0 d^m > L > s_0 d^{m+1}, \quad \text{with} \quad d = e^{\sigma\sqrt{h}},$$

in which $h = T/n$ denotes as usual the time step of the binomial tree. Then if we select $n$ such that it is the largest integer smaller then

$$F(m) = \frac{m^2\sigma^2 T}{(\log(s_0/L))^2}$$

we get that the barrier is as close as possible to a layer of nodes of the tree and the method is more accurate.
Subsequently, Ritchken ([72], 1995) offers another approach for pricing single barrier options. Under a trinomial framework he constructs a tree such that the barrier exactly coincides with a layer of nodes of the tree. In particular he introduces a “stretch” parameter into the lattice which changes the time step in order to place the nodes on the barrier.
In the meantime, Derman et al. ([30], 1995) try to perform the estimates provided by the binomial method by a linear interpolation considering the two nodes above and the two below the barriers. Similar ideas have been developed in more specific contexts in Broadie and Detemple ([19], 1996), Cheuck and Vorst ([23], 1996, 1997), Tian ([79], 1999), Widdicks et al. ([84], 2002) and Chung and Shih ([24], 2007).
Figleski and Gao ([34], 1999) call the source of error caused by the barrier the “nonlinearity error”. As a solution to this problem they suggest to construct a trinomial lattice for single

barrier options in which the resolution of the tree varies in different parts of the structure. It means that they use a coarse grid for most of the tree structure and then they refine the mesh near certain critical points (the points near the barrier) where greater accuracy really matters. They call this new method the *Adaptive Mesh Model* (AMM) and they remark that the computational effort suffers only for a small increase but in the meantime the accuracy is improved.
Then Gaudenzi and Lepellere ([37], 2006) try to refine the interpolation procedure first introduced in the pricing of barrier options by Reimer and Sandmann ([70], 1995). The idea is to estimate the binomial price of a single barrier option by evaluating the price at different "computational" barriers and then by interpolating these prices on the contractual barrier. They also provide a technique in order to obtain all the interpolating data by a unique tree, so that the computational time of the new algorithm is similar to the one of the usual CRR tree.
Gaudenzi and Zanette ([39], 2009) provide an algorithm for the pricing of barrier options with discrete dividends by a binomial tree. In particular, they propose a procedure that combines the interpolation technique with the singular points approach introduced in Gaudenzi et al. ([38], 2007). The "singular points" are a set of points that are computed backwardly on the tree and that allow to construct a convex piecewise function so that one has a continuous representation of the option price function at every node of the tree. In this way they get at every node of the tree a lower bound and an upper bound of the true binomial price and so this technique permits the treatment of discrete dividends. In [39], the singular points procedure is then combined with the idea in Gaudenzi and Lepellere ([37], 2006) of constructing a tree in which all the singular points are generated by the barrier itself, and so the authors get an algorithm for the pricing of a single barrier option with discrete dividends.
All the previous methods deal with the single barrier case. The first lattice approach that is able to treat the pricing of double barrier options is due to Dai and Lyuu ([29], 2010) and it is described in details in the following Section.

## 2.4 The Dai and Lyuu procedure

In this Section we consider the description of the Dai and Lyuu procedure presented in ([29], 2010). Let $m$ be the number of time steps of the CRR binomial tree and let $\Delta\tau = \frac{T}{m}$ be the corresponding time-step. The standard CRR discrete binomial process is given by

$$S_{(i+1)\Delta\tau} = S_{i\Delta\tau} Y_{i+1}, \quad 0 \leq i \leq m-1,$$

where the random variables $Y_1, \dots, Y_m$ are independent and identically distributed with values in $\{d, u\}$. Let us denote by $p = \mathbb{P}(Y_m = u)$. The CRR tree corresponds to the choice $u = \frac{1}{d} = e^{\sigma\sqrt{\Delta\tau}}$ and

$$p = \frac{e^{r\Delta\tau} - e^{-\sigma\sqrt{\Delta\tau}}}{e^{\sigma\sqrt{\Delta\tau}} - e^{-\sigma\sqrt{\Delta\tau}}},$$

as described in (1.1.14). Let us now consider the problem of pricing a double barrier option with barriers $L$ and $H$ in which the underlying process follows the dynamics described in (2.1.1). In order to treat this problem, Dai-Lyuu in [29] introduce the following "bino-trinomial" method. After a logarithmic change of the barriers ($l = \log L$ and $h = \log H$) they first construct in the log-space a binomial CRR random walk with space step $\sigma\sqrt{\Delta T}$, where the new time step $\Delta T$ is defined as follows. Considering the CRR choice of time step $\Delta\tau = \frac{T}{m}$, the new time step is defined as

$$\Delta T = \left(\frac{h-l}{2k\sigma}\right)^2 \tag{2.4.1}$$

where

$$k = \left\lceil \frac{h-l}{2\sigma\sqrt{\Delta\tau}} \right\rceil, \tag{2.4.2}$$

with $\lceil x \rceil$ denoting the smallest integer not less than $x$, for every $x \in \mathbb{R}$. By this way, two layers of nodes in the tree coincide with the lower barrier $L$ and the higher barrier $H$ and the new number of steps is $m' = \lfloor \frac{T}{\Delta T} \rfloor$. Now, it is possible to build a binomial structure of $m'$ time steps with binomial coefficient

$$u = \frac{1}{d} = e^{\sigma\sqrt{\Delta T}} \tag{2.4.3}$$

and probability

$$p = \frac{e^{r\Delta T} - e^{-\sigma\sqrt{\Delta T}}}{e^{\sigma\sqrt{\Delta T}} - e^{-\sigma\sqrt{\Delta T}}}. \tag{2.4.4}$$

The remaining amount of time to make the whole tree span T years, that we denote with $\Delta T'$, is defined as

$$\Delta T' = T - \left(\left\lfloor \frac{T}{\Delta T} \right\rfloor - 1\right)\Delta T$$

and corresponds to the length of the first time step of the bino-trinomial tree. Finally, Dai-Lyuu construct a 1-step trinomial tree, using a moment matching procedure, starting from $s_0$ and reaching three nodes of the previous binomial CRR tree at time $\Delta T'$. Specifically, at time $\Delta T'$ they select the central node, that we call $Y_l$, such that it is the closest lattice point to the mean of the logarithmic stock price process $Y_t = \log S_t$. After choosing this point they consider two further points, one below and one above $Y_l$: $Y_{l-1}$ and $Y_{l+1}$, respectively. Then, in order to connect these three points to the starting one, they match the mean and the variance at $\Delta T'$ of the continuous process $Y_t$ with the mean and the variance of the discrete process. We recall that the mean and the variance of $Y_t$ at $\Delta T'$ are equal to

$$\begin{aligned}
\mu_{\Delta T'} &= \log s_0 + (r - \sigma^2/2)\Delta T', \\
\mathrm{Var}_{\Delta T'} &= \sigma^2 \Delta T',
\end{aligned}$$

respectively. So, the branching probabilities (that we call $p_{l-1}, p_l, p_{l+1}$) can be derived by solving the following three equations

$$\begin{aligned}
&p_{l-1}Y_{l-1} + p_l Y_l + p_{l+1}Y_{l+1} = \mu_{\Delta T'},\\
&p_{l-1}(Y_{l-1} - \mu_{\Delta T'})^2 + p_l(Y_l - \mu_{\Delta T'})^2 + p_{l+1}(Y_{l+1} - \mu_{\Delta T'})^2 = \mathrm{Var}_{\Delta T'},\\
&p_{l-1} + p_l + p_{l+1} = 1.
\end{aligned}$$

Then, the merge of the binomial tree of $m'$ steps and the 1-step trinomial tree provides the complete mesh structure of $n := m' + 1$ time steps. The pricing of European and American continuous double barrier options can be done by using the backward dynamic programming procedure on the above described bino-trinomial mesh structure. To be precise, let us denote by $v^n(t_i, S_{i,j})$ the option prices at time $t_i$ depending on the underlying asset process $S_{i,j}$. The option prices at maturity are

$$v^n(t_n, S_{n,0}) = v^n(t_n, S_{n,k}) = 0 \text{ and } \; v^n(t_n, S_{n,j}) = \max(\theta S_{n,j} - \theta K, 0),\ \forall j = 1..., k-1,$$

with $\theta = 1$ for call options and $\theta = -1$ for put options. At time steps $i = n-1, ..., 0$ the option prices are backwardly computed by means of the formulas:

1. if $n - i$ is odd:

$$\begin{aligned}
v^n(t_i, S_{i,j}) = e^{-r\Delta T}[pv^n(t_{i+1}, S_{i+1,j+1}) + (1-p)v^n(t_{i+1}, S_{i+1,j})],\\
j = 0, ..., k-1,
\end{aligned}$$

2. if $n - i$ is even:

$$\begin{aligned}
v^n(t_i, S_{i,j}) = e^{-r\Delta T}[pv^n(t_{i+1}, S_{i+1,j}) + (1-p)v^n(t_{i+1}, S_{i+1,j-1})],\\
j = 1, ..., k-1.
\end{aligned}$$

The values at the barriers $v^n(t_i, L)$, $v^n(t_i, H)$, are set equal to 0 at every step $i$ with $n - i$ even, in order to take into account the "out" feature of the barrier option. The numerical results presented in Section 2.8 show that this bino-trinomial structure is not able to treat the "near barrier" problem. In order to overcome this, we introduce a simpler binomial structure called the "Binomial Interpolated Lattice" approach, that we describe in the next Section.

## 2.5 The binomial interpolated lattice

In the following, we will use the same binomial parameters $\Delta T$, $u, d$ and $p$ of Dai-Lyuu ([29], 2010), computed as described in (2.4.1), (2.4.3) and (2.4.4) in the previous Section. But we modify the number of time steps: in fact we consider a new number of time steps $n := m' + 2$, with $m' = \lfloor \frac{T}{\Delta T} \rfloor$, in order to perform suitable interpolations in time and in space.

First of all, we construct a binomial mesh structure where all the binomial nodes are generated by the barriers. Therefore we build a tree which nodes at maturity are indeed all of type

$$S_{n,j} = Lu^{2j}, \qquad j = 0, ..., k,$$

so that $S_{n,k} = Lu^{2k} = H$ (where, as in the previous Section, $k = \lceil \frac{h-l}{2\sigma\sqrt{\Delta\tau}} \rceil$, with $\Delta\tau = \frac{T}{m}$ ). The underlying asset at a generic node $(i, j)$, $\forall\, i = 0, ..., n-1$, is

$$S_{i,j} = \begin{cases} Lu^{2j}, & j = 0, ..., k \qquad \text{if } n-i \text{ is even} \\ Lu^{2j+1}, & j = 0, ..., k-1 \quad \text{if } n-i \text{ is odd} \end{cases}$$

We now proceed to the description of the pricing algorithm in the case of double barrier knock-and-out options. We shall denote as usual by $v^n(t_i, S_{i,j})$ the option prices at time $t_i$ depending on the underlying asset $S_{i,j}$ and computed by a backward induction as described in the previous Section. The basic difference with what done in Dai and Lyuu ([29], 2010) consists in what follows.
At time steps $i = 0$ and $i = 2$ we choose four nodes (two less and two greater than $s_0$). In order to approximate the price of the double barrier option we first interpolate in time the chosen points so that we obtain four "precise" prices of the option at time 0. Then we proceed with a Lagrange four points interpolation in space, i.e. we interpolate the four prices at $s_0$. The price obtained after these interpolations in time and in space is the approximated option price at time 0 and initial underlying asset $s_0$. The procedure is illustrated in Figure 2.1.

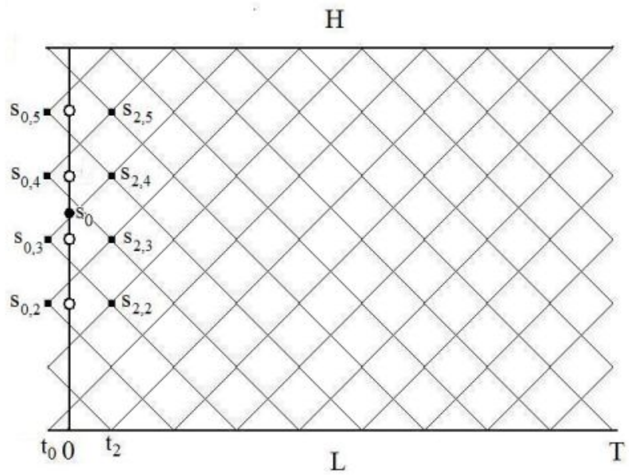


Figure 2.1: *Binomial interpolated lattice method. Double knock-out barrier option. The price at $s_0$ is obtained by a Lagrange four points interpolation in space of the prices at the empty circles, such prices are obtained by a linear interpolation in time of the prices at the nodes denoted by squares.*

**Remark 2.5.1.** *The reason why we select the points at times $t_0$ and $t_2$ is due to how the binomial mesh is constructed. In fact, at these times the choice of the four points around $s_0$ provides nodes of the same type at $t_0$ and $t_2$ ($n$ and $n-2$ have the same parity) and this makes the procedure "uniform".*

We remark that there are some cases in which we need to modify the choice of the interpolation points and this happens when $s_0$ is close to one of the barriers. Let us suppose that $s_0$ is near the lower barrier $L$. We now have two possible cases: there are no points between $s_0$ and the barrier $L$ and there is only one point between $s_0$ and the barrier $L$. In the first case we select at times $t_0$ and $t_2$ the two points above $s_0$ and the point on the barrier. So we perform three interpolations in time using the chosen points and we obtain three different prices at time 0. Then we consider the polynomial passing through these three points and we evaluate it at $s_0$ (see Figure 2.3, cases a) and b)). We remark that in case a) the mesh constructed provides at times $t_0$ and $t_2$ a node on the barrier L, while in case b) there is not a node on the barrier by construction but we can always consider it in the interpolation procedure because here we know that the price is equal to 0. In the second case we choose four points at $t_0$ and $t_2$: the two above $s_0$, the point below $s_0$ and the point on the barrier. So we linearly interpolate four times and then we evaluate at $s_0$ the polynomial passing through the four points obtained at 0. See Figure 2.2, cases c) and d). We observe again that in the case in which there is not a node on the barrier by construction we can always consider it in the interpolation procedure.

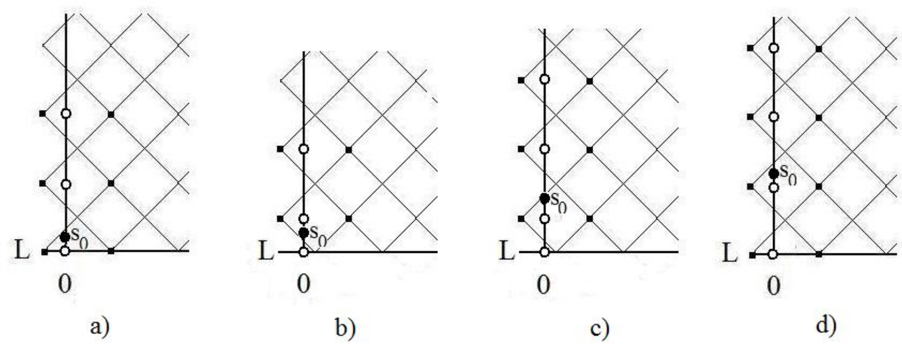


Figure 2.2: *Binomial interpolated lattice method. Double barrier knock-out options in the "near barrier case". In cases a) and b) the interpolation in space involves three nodes: at times $t_0$ and $t_2$ we select the two nodes above $s_0$ and the node on the barrier. We observe that case a) occurs when $n-2$ is even and case b) when $n-2$ is odd. In cases c) and d), instead, we select four nodes at times $t_0$ and $t_2$. Case c) occurs when $n-2$ is odd and case d) when $n-i$ is even.*

In the American case the procedure is similar with suitable differences for the values of the prices on the barriers. In particular we set $v^n(t_i, L) = \max(\theta L - \theta K, 0)$ and $v^n(t_i, H) = \max(\theta H - \theta K, 0)$ for each time step $i = 0, ..., n$ with $n-i$ even. In the backward procedure,

as usual, we need to compare the early exercise with the continuation value at each node of the tree.
The procedure previously described provides an efficient evaluation of double barrier options both in European and American case. We will show this in the Section 2.8 concerning the numerical results.

**Remark 2.5.2.** *Besides pricing, another interesting problem concerning the theory and practice of options is the hedging issue. Here we can calculate the values of Delta, Gamma and Vega by using a finite difference approximation. In particular we can use a unique tree in order to compute Delta and Gamma , i.e.:*

$$Delta = \frac{v^n(0, s_0(1+\delta)) - v^n(0, s_0(1-\delta))}{2s_0\delta},$$
$$Gamma = \frac{v^n(0, s_0(1+\delta)) - 2v^n(0, s_0) + v^n(0, s_0(1-\delta))}{s_0^2\delta^2},$$

*where $\delta$ is the relative increment. The prices $v^n(0, s_0+\delta)$ and $v^n(0, s_0-\delta)$ are computed with the interpolation procedure previously described. In the case of the Vega computation we need to implement two different trees: one with volatility parameter $\sigma$ and the other with volatility parameter $\sigma(1+\delta)$. Then the computation of Vega is given by:*

$$Vega = \frac{v^n_{\sigma(1+\delta)}(0, s_0) - v^n(0, s_0)}{\sigma\delta}.$$

## 2.6 Rate of convergence in the European case

In this Section we study the rate of convergence of the Binomial Interpolated Lattice approach. Gobet ([42], 2001) gives an asymptotic expansion of the standard CRR binomial tree error and proves that the main contribution term depends on the distance between the effective barrier and the tree overshoot of the barrier itself (see Theorem 3.3 in [42] and Section 1.2). In the following we use this result and some properties of the Lagrange polynomials providing the final interpolations to show that the approximation error of the Binomial Interpolated Lattice scheme is $o(\Delta T^{1-\alpha})$, for every $\alpha \in (0,1)$.
We recall that $v^n(t_i, S_{i,j})$ is the approximated price on the lattice at time $t_i$ and asset price $S_{i,j}$. Moreover, we call $v(t_i, S_{i,j})$ the corresponding price in the continuous-time model at time $t_i$ and asset price on the lattice $S_{i,j}$ and $v(t,s)$ the continuous price at time $t$ and underlying asset price $s$ with $(t,s) \in [0,T] \times \mathbb{R}_+$. The starting point $s_0$ is fixed.
According to what developed in Section 2.5, at time step $i = 2$ , i.e. at time $T-(n-2)\Delta T$, we choose four nodes (two less and two greater than $s_0$) and similarly we do at time step $i = 0$, i.e. at time $T - n\Delta T$. As observed at Remark 2.5.1, the mesh constructed provides

the same four nodes at times $t_0$ and $t_2$ and we set them as follows

$$S_{j-2} < S_{j-1} \leq s_0 < S_j < S_{j+1},$$

where $S_k := S_{0,k} = S_{2,k}$, for all $k \in \{j-2, j-1, j, j+1\}$. At the chosen points the algorithm gives the prices

$$v^n(t_i, S_{j-2}), v^n(t_i, S_{j-1}), v^n(t_i, S_j), v^n(t_i, S_{j+1}), \quad i = 0, 2,$$

respectively. Now, as $k = j-2, j-1, j, j+1$, we write down the expressions of the approximated option prices $v^n(0, S_k)$ obtained by linearly interpolating in time the points

$$(t_0, v^n(t_0, S_k)), (t_2, v^n(t_2, S_k)), \quad k = j-2, j-1, j, j+1.$$

This means that we set

$$v^n(0, S_k) = \overline{q}_k(0), \quad k = j-2, j-1, j, j+1,$$

where $\overline{q}_k(t)$ are the linear interpolating polynomials given by

$$\overline{q}_k(t) = a_0(t) v^n(t_0, S_k) + a_2(t) v^n(t_2, S_k),$$

with

$$a_0(t) = \frac{t - t_2}{t_0 - t_2} \quad \text{and} \quad a_2(t) = \frac{t - t_0}{t_2 - t_0}.$$

Then, in order to define the precise price at time 0 we interpolate in space through a Lagrange polynomial the points

$$(0, v^n(0, S_{j-2})), (0, v^n(0, S_{j-1})), (0, v^n(0, S_j)), (0, v^n(0, S_{j+1})).$$

This means that we set

$$v^n(0, s_0) = q(s_0),$$

where $q(x)$ is the Lagrange polynomial given by

$$\begin{aligned} q(x) &= b_{j-2}(x) v^n(0, S_{j-2}) + b_{j-1}(x) v^n(0, S_{j-1}) \\ &\quad + b_j(x) v^n(0, S_j) + b_{j+1}(x) v^n(0, S_{j+1}), \end{aligned}$$

with (for details see [85])

$$\begin{aligned} b_{j-2}(x) &= \frac{(x - S_{j-1})(x - S_j)(x - S_{j+1})}{(S_{j-2} - S_{j-1})(S_{j-2} - S_j)(S_{j-2} - S_{j+1})}, \\ b_{j-1}(x) &= \frac{(x - S_{j-2})(x - S_j)(x - S_{j+1})}{(S_{j-1} - S_{j-2})(S_{j-1} - S_j)(S_{j-1} - S_{j+1})}, \\ b_j(x) &= \frac{(x - S_{j-2})(x - S_{j-1})(x - S_{j+1})}{(S_j - S_{j-2})(S_j - S_{j-1})(S_j - S_{j+1})}, \\ b_{j+1}(x) &= \frac{(x - S_{j-2})(x - S_{j-1})(x - S_j)}{(S_{j+1} - S_{j-2})(S_{j+1} - S_{j-1})(S_{j+1} - S_j)}. \end{aligned}$$

So, by resuming, we set the approximated price at time 0 as

$$v^n(0,s_0) = \sum_{k=j-2}^{j+1} \sum_{i\in\{0,2\}} a_i(0) b_k(s_0) v^n(t_i, S_k).$$

**Proposition 2.6.1.** *The Binomial Interpolated Lattice error*

$$Err_{BIL}(n) = |v^n(0,s_0) - v(0,s_0)|$$

*resulting from the algorithm behaves as follows:*

$$Err_{BIL}(n) = o(\Delta T^{1-\alpha}),$$

*for every* $\alpha \in (0,1)$.

*Proof.* Since

$$\sum_{k\in\{j-2,j-1,j,j+1\}} b_k(s_0) = 1, \qquad \sum_{i\in\{0,2\}} a_i(0) = 1,$$

we can write

$$\begin{aligned}
\mathrm{Err}_{BIL}(n) &= \left| \sum_k \sum_i a_i(0) b_k(s_0)(v^n(t_i,S_k) - v(0,s_0)) \right| \\
&\le \left| \sum_k \sum_i a_i(0) b_k(s_0)(v^n(t_i,S_k) - v(t_i,S_k)) \right| \\
&+ \left| \sum_k \sum_i a_i(0) b_k(s_0)(v(t_i,S_k) - v(0,s_0)) \right|.
\end{aligned}$$

Let us consider first the generic term in the second sums above: by applying Taylor's formula around the point $(0, s_0)$ we can write

$$v(t_i,S_k) - v(0,s_0) = \partial_t v(0,s_0) t_i + \partial_x v(0,s_0)(S_k - s_0) + \frac{1}{2} R(i,k),$$

where

$$\begin{aligned}
R(i,k) = \partial^2_{x,x} v(0, s_0 + \theta_{i,k}(S_k - s_0))(S_k - s_0)^2 + \partial_{t,t} v(\overline{\theta}_{i,k} t_i, s_0) t_i^2 \\
+ 2\partial^2_{t,x} v(\overline{\theta}_{i,k} t_i, s_0 + \theta_{i,k}(S_k - s_0))(S_k - s_0) t_i,
\end{aligned}$$

and $\theta_{i,k}, \overline{\theta}_{i,k}$ are suitable points in $[0,1]$. By using the global estimates in Lemma 3.1 Gobet ([42], 2001), we conclude that all the partial derivatives of $v(t,s)$ are bounded around $(0,s_0)$. So, we can immediately conclude that

$$\left| \sum_k \sum_i a_i(0) b_k(s_0) R(i,k) \right| \le O(\Delta T).$$

Moreover we have

$$\sum_k \sum_i a_i(0) b_k(s_0)(\partial_t v(0,s_0) t_i + \partial_x v(0,s_0)(S_k - s_0))$$
$$= \partial_t v(0,s_0) \sum_i a_i(0) t_i + \partial_x v(0,s_0) \sum_k b_k(s_0)(S_k - s_0) = 0$$

because, by construction, $q_1(t) = \sum_i a_i(t) t_i$ is the linear polynomial which interpolates the points $(t_i, t_i)$, $i = 0, 2$, so that $q_1(t) = t$ and then $q_1(0) = 0$.
Similarly, the polynomial $q_2(x) = \sum_k b_k(x)(S_k - s_0)$ is the Lagrange polynomial which interpolates the points $(S_k, S_k - s_0)$, $k \in \{j-2, j-1, j, j+1\}$, so that $q_2(x) = x - s_0$ and again $q_2(s_0) = 0$. Therefore, we get

$$\mathrm{Err}_{BIL}(n) \leq \left| \sum_k \sum_i a_i(0) b_k(s_0)(v^n(t_i, S_k) - v(t_i, S_k)) \right| + O(\Delta T).$$

In order to deal with the generic term in the above sums we define $Y_k := \log S_k$, for all $k \in \{j-2, j-1, j, j+1\}$, $u^n(t_i, Y_k) := v^n(t_i, e^{Y_k})$ and $u(t_i, Y_k) := v(t_i, e^{Y_k})$. We now apply Theorem 3.3 in Gobet ([42], 2001). We stress that the probability $p$ of an up jump in (2.4.4) differs from the probability defined in Gobet for an $O(\sqrt{\Delta T})$, but it is easy to see that the result in [42] still remains valid in our case. Moreover, we remark that the asymptotic expansion of the standard binomial tree error, that we call $\mathrm{Err}(n)$, given in [42] (see also Theorem 1.2.11) is

$$\mathrm{Err}(n) = C_1(H - H_n) + C_2(L - L_n) + o(\sqrt{\Delta T}),$$

where $H_n$ is the first node of the tree over $H$, $L_n$ is the first node of the tree lower than $L$ and $C_1$ and $C_2$ are two positive constants. Actually, straightforward computations give that the error above can be written as follows

$$\mathrm{Err}(n) = C_1(H - H_n) + C_2(L - L_n) + \mathcal{R}_n, \tag{2.6.1}$$

where $\mathcal{R}_{\Delta T}$ is such that there exists a constant $C > 0$: $|\mathcal{R}_n| \leq C \Delta T \log(1/\Delta T)$. As a consequence, one has for every $\alpha \in (0,1)$ that

$$\mathrm{Err}(n) = C_1(H - H_n) + C_2(L - L_n) + o(\Delta T^{1-\alpha}).$$

Now, the Binomial Interpolated Lattice is constructed so that two layers of nodes coincide with the lower barrier $L$ and the higher barrier $H$, therefore the main contribution term in the error expansion (2.6.1), that is of order $O(\sqrt{\Delta T})$, vanishes. So, we get

$$v^n(t_i, S_k) - v(t_i, S_k) = u^n(t_i, Y_k) - u(t_i, Y_k) = o(\Delta T^{1-\alpha}) \quad \text{for every } i, k.$$

The statement now follows. □

**Remark 2.6.2.** *Gobet convergence result is the only one that deals with the double barrier case and, moreover, it is applicable to a generic continuous payoff function. Lin and Palmer ([62], 2013) have recently given explicit formulas for the coefficients of the asymptotic expansion of the CRR binomial price, but they treat European call options with a single barrier. In that case they obtain that the rate of convergence is $O(\Delta T)$ if the barrier lies exactly on a node of the tree. For double barrier options it may be possible to derive similar expressions for the coefficients, but it is not straightforward because no manageable closed-form formulas of the approximated price exist in terms of binomial coefficients. In fact, in the error expansion provided in [42], since $\alpha$ can be taken arbitrarily close to zero, one essentially can assert that*

$$Err(n) = C_1(H - H_n) + C_2(L - L_n) + O(\Delta T),$$

*where $C_1$ and $C_2$ are two positive constants.*

**Remark 2.6.3.** *The use of the interpolation technique is strategic in order to get that the rate of convergence of the Binomial Interpolated Lattice is $o(\Delta T^{1-\alpha})$, for $\alpha \in (0, 1)$. In fact, we do not know a priori if the initial point $s_0$ is a point of the lattice and in general it is not: our procedure allows us to numerically compute the option price for every observed starting condition $s_0 \in (L, H)$. So if we want to directly approximate $v(0, s_0)$ with $v^n(t_1, \tilde{s})$, where $(t_1, \tilde{s})$ is a point of the lattice "close" to $(0, s_0)$, then one could have $|s_0 - \tilde{s}| = O(\sqrt{\Delta T})$. Therefore, in this case one could have*

$$v(0, s_0) - v^n(t_1, \tilde{s}) \simeq O(t_1) + O(|s_0 - \tilde{s}|) = O(\sqrt{\Delta T}).$$

*Thus, it is only thanks to the interpolation rule that the error contribution of order $O(\sqrt{\Delta T})$ always vanishes.*

**Remark 2.6.4.** *Let us consider the rate of convergence of the bino-trinomial tree. As described in Section 2.4, Dai and Lyuu build the grid in the* log*-space until time $t_2$ and then they construct a 1-step trinomial tree in the remaining amount of time $\Delta T'$ using a moment matching procedure. Specifically they select at time $t_2$ three nodes that are the closest to the mean of the logarithmic process at time $\Delta T'$ and they define the three branching probabilities such that the first two moments of the logarithmic stock price process are matched (see also Dai-Lyuu ([29], 2010) for details). As in the proof of Proposition 2.6.1, we call $Y_{i,k} = \log S_{i,k}$ and $u^n(t_i, Y_k) = v^n(t_i, e^{S_k})$ for all $i, k$. Moreover we define $y_0 := \log s_0$ and we call $u(t, y) = v(t, e^y)$ for all $(t, y) \in [0, T] \times \mathbb{R}$. If we now call the chosen points at time $t_2$*

$$Y_{2,l-1}, \quad Y_{2,l} \quad \text{and} \quad Y_{2,l+1},$$

*then the Dai and Lyuu algorithm gives the prices*

$$u^n(t_2, Y_{2,l-1}), u^n(t_2, Y_{2,l}), u^n(t_2, Y_{2,l+1})$$

*and the option price of the bino-trinomial tree at time 0 is obtained by one more application of the backward induction that uses the trinomial approach, i.e.*

$$u^n_{DL}(0, y_0) = e^{-r\Delta T'} \sum_{k=-1}^{1} p_k u^n(t_2, Y_{2,l+k}).$$

*We can now proceed in the analysis of the convergence rate using arguments similar to the ones in Proposition 2.6.1. In fact, we can write*

$$\begin{aligned} Err_{DL}(n) &= |u^n_{DL}(0,y_0) - u(0,y_0)| \\ &\leq O(\Delta T') + \left| \sum_{k=-1}^{1} p_k(u^n(t_2, Y_{2,l+k}) - u(t_2, Y_{2,l+k})) \right| \\ &+ \left| \sum_{k=-1}^{1} p_k(u(t_2, Y_{2,l+k}) - u(0,y_0)) \right| \\ &\leq O(\Delta T \log(1/\Delta T)) + \left| \sum_{k=-1}^{1} p_k(u(t_2, Y_{2,l+k}) - u(0,y_0)) \right|, \end{aligned}$$

*where the estimate on the right hand side follows by applying Gobet's result. Again by using Taylor's expansion, one gets for every* $\alpha \in (0,1)$

$$Err_{DL}(n) = o(\Delta T^{1-\alpha}) + \left| \partial_x u(0,y_0) \sum_{k=-1}^{1} p_k(Y_{2,l+k} - y_0) \right|.$$

*Now, the sum in the above r.h.s. is of order* $O(\Delta T')$. *In fact we recall that the mean of the logarithmic process* $Y_t = \log S_t$ *at time* $\Delta T'$ *is* $y_0 + (r - \sigma^2/2)\Delta T'$ *and that the probabilities* $p_k$ *are calculated such that it coincides with the mean of the discrete approximating process, so we can state that* $\sum_{k=-1}^{1} p_k(Y_{2,l+k} - y_0) = O(\Delta T')$. *Therefore, we obtain*

$$Err_{DL}(n) = o(\Delta T^{1-\alpha}), \quad \forall\, \alpha \in (0,1).$$

*Then the rate of convergence of the bino-trinomial tree is an* $o(\Delta T^{1-\alpha})$ *as the for the Binomial Interpolated Lattice method.*

**Remark 2.6.5.** *There are some cases in which the procedure of the bino-trinomial tree can bring to numerical problems. In fact, if we fix the number* $n$ *of time steps, it may happen that for some values of the starting point* $s_0$ *one or two of the three nodes required at time* $\Delta T'$ *to build the 1-step trinomial tree fall out of the grid. We briefly recall that in the bino-trinomial tree procedure the central node (we call it* $Y_{2,l}$ *from Remark 2.6.4) is selected such that it is the closest to the mean of the process (i.e.* $\log s_0 + (r - \sigma^2/2)\Delta T'$*), while the nodes* $Y_{2,l+1}$ *and* $Y_{2,l-1}$ *are the two nodes adjacent to* $Y_{2,l}$ *(above and below* $Y_{2,l}$ *respectively). In the following we suppose that* $r - \sigma^2/2 \geq 0$. *Let us consider first the case in which the initial point* $\log s_0$ *is near the higher barrier* $\log H$. *We need to consider two different cases:*
*i) the mean of the process is above the barrier* $\log H$*:*

$$\log s_0 + (r - \sigma^2/2)\Delta T' \geq \log H;$$

*ii) the mean of the process is below the barrier* $\log H$*:*

$$\log s_0 + (r - \sigma^2/2)\Delta T' < \log H.$$

*The case i) is verified when*

$$He^{-(r-\sigma^2/2)\Delta T'} \leq s_0 < H, \tag{2.6.2}$$

*so for this range of values for* $s_0$ *the node* $Y_{2,l}$ *lies on the barrier (i.e.* $Y_{2,l} = \log H$*) and then the node* $Y_{2,l+1}$ *falls out of the grid. Also in case ii) it may happen the same phenomenon. It is easy to see that if*

$$He^{-(r-\sigma^2/2)\Delta T'-\sigma\sqrt{\Delta T}} \leq s_0 < H, \tag{2.6.3}$$

*then again* $Y_{2,l} = \log H$*. From (2.6.2) and (2.6.3) we deduce that for the values of* $s_0$ *such that*

$$He^{-(r-\frac{\sigma^2}{2})\Delta T'-\sigma\sqrt{\Delta T}} \leq s_0 < H, \tag{2.6.4}$$

*the bino-trinomial tree may degenerate in the above sense. As for the lower barrier, a similar discussion gives that if*

$$L < s_0 \leq Le^{-(r-\sigma^2/2)\Delta T'+\sigma\sqrt{\Delta T}}, \tag{2.6.5}$$

*then* $Y_{2,l} = \log L$*. We observe that we can write the above inequality because the exponent on the right side is greater than* $0$ *for a sufficiently large value of* $n$*. And whenever* $r - \sigma^2/2 < 0$*, one can proceed similarly and obtain the same intervals. It is clear that for* $n$ *large enough any* $s_0 \in (L, H)$ *does not satisfy both (2.6.4) and (2.6.5). Nevertheless, for fixed values of* $n$ *(2.6.4) and/or (2.6.5) may hold, so that in practice the bino-trinomial approach converges slowly than our procedure. So we conclude that asymptotically the two methods behave the same, but when* $s_0$ *is a "near barrier" point the Binomial Interpolated Lattice has the advantage of converging faster than the bino-trinomial method.*

**Remark 2.6.6.** *The proof of the rate of convergence in the case in which we take into account only three points in the space interpolation ("near to the barrier" case) can be treated similarly to the previous one ("far from the barrier" case), so we omit it. Moreover, we observe that in this specific case we always find the three points used in the interpolations as explained in Section 2.5 (see Figure 2.2).*

## 2.7 The binomial interpolated lattice for step double barrier options

In this Section we use the Binomial Interpolated Lattice algorithm introduced in Section 2.5 for pricing multi-step double barrier options. The procedure can also be applied to *early-ending* and *partial-time step double barrier options* in a straightforward way.
Let us consider for example a 2-*step double knock-out option*. We first consider the time period $[T_1, T_2]$ and we apply the Binomial Interpolated Lattice procedure described in the previous Section. It means that we compute the binomial parameters $m_2' = \lfloor \frac{T_2 - T_1}{\Delta T_2} \rfloor$, $n_2 = m_2' + 2$, $\Delta T_2$, $u_2, d_2$, $p_2$, $k_2$ in order to hit exactly the barriers $L_2$ and $H_2$. This leads to a

binomial mesh $\{S^2_{i,j}\}_{i,j}$ defined $\forall\, i = 0, ..., n_2$ as follows:

$$S^2_{i,j} = \begin{cases} L_2 u_2^{2j}, & j = 0, ..., k_2 \qquad \text{if } n_2 - i \text{ is even} \\ L_2 u_2^{2j+1}, & j = 0, ..., k_2 - 1 \quad \text{if } n_2 - i \text{ is odd} \end{cases}$$

We can then proceed using the backward procedure for $i = n_2, ..., 0$ as described in the previous Section, so that we get the option price at every node $(i, j)$, for $i = 0, ..., n_2, j = 0, ..., i$. By proceeding with suitable interpolations in time and in space we can obtain at every node $S^2_{0,j}$ at time $T_1$ the corresponding option price $v^{n_2}(T_1, S^2_{0,j})$.
We now go on similarly in time interval $[T_0, T_1]$. We compute the new binomial parameters $m_1' = \lfloor \frac{T_1 - T_0}{\Delta T_1} \rfloor$, $n_1 = m_1' + 2$, $\Delta T_1$, $u_1, d_1$, $p_1$, $k_1$ in order to hit exactly the barriers $L_1$ and $H_1$. This leads to a new binomial mesh structure $\{S^1_{i,j}\}_{i,j}$. In order to obtain the option prices on the new nodes with underlying asset $S^1_{n_1,j}$, $j = 0, ..., k_1$, we interpolate at every $S^1_{n_1,j}$, $j = 0, ..., k_1$ by a Lagrange interpolation using 4 suitable points in the set $\{(S^2_{0,j}, v^{n_2}(T_1, S^2_{0,j}))\}$, with $j = 0, ..., k_2$ if $n_2$ is even and with $j = 0, ..., k_2 - 1$ if $n_2$ is odd. In order to perform such interpolations we set $v^{n_2}(T_1, S^2_{0,j}) = 0$, for $j$ such that $S^2_{0,j} \leq L_2$ or $S^2_{0,j} \geq H_2$. Moreover, the values $v^{n_1}(T_1, S^1_{n_1,j})$ are set equal to zero if either $S^1_{n_1,j} \leq L_1$ or $S^1_{n_1,j} \geq H_1$.
Finally, we proceed backward for $i = n_1, ..., 0$ and we compute the price at $s_0$ by linear interpolations in time and a Lagrange interpolation in space as described before. We represent the mesh described above in Figure 2.3.

In the *early ending 2-step double knock-out option* we just need to add the treatment of the period $[T_2, T_3]$ where there are no "out" conditions. We start by considering the number of time steps $m_3$ and the corresponding $\Delta\tau_3$. Then we compute $k_3$ and $\Delta T_3$ in order to hit exactly the barriers $L_2$, $H_2$, i.e. $k_3 = \lceil \frac{h_2 - l_2}{2\sigma\sqrt{\Delta\tau_3}} \rceil$ and $\Delta T_3 = \left(\frac{h_2 - l_2}{2k_3\sigma}\right)^2$. The parameters $m_3', n_3, u_3, d_3, p_3$ are computed as usual. Now, starting from the nodes evaluated at time $T_2$ we can consider a tree structure $\{S^3_{i,j}\}_{i,j}$ in the time interval $[T_2, T_3]$ of $n_3$ time steps. At maturity $T_3$ we obtain the underlying assets $S_{n_3,j} = L_2 u_3^j$, $j = -n_3, ..., 2k_3 + n_3$. Then we apply the backward CRR binomial procedure starting with the maturity condition at time $T_3$. The prices at the nodes $S_{n_2,j} = L_2 u_2^j$, $j = 0, ..., k_2$ at time $T_2$ are obtained with the usual interpolations in time and space. The procedure is then the same as in the standard 2-step double barrier options (see Figure 2.4).

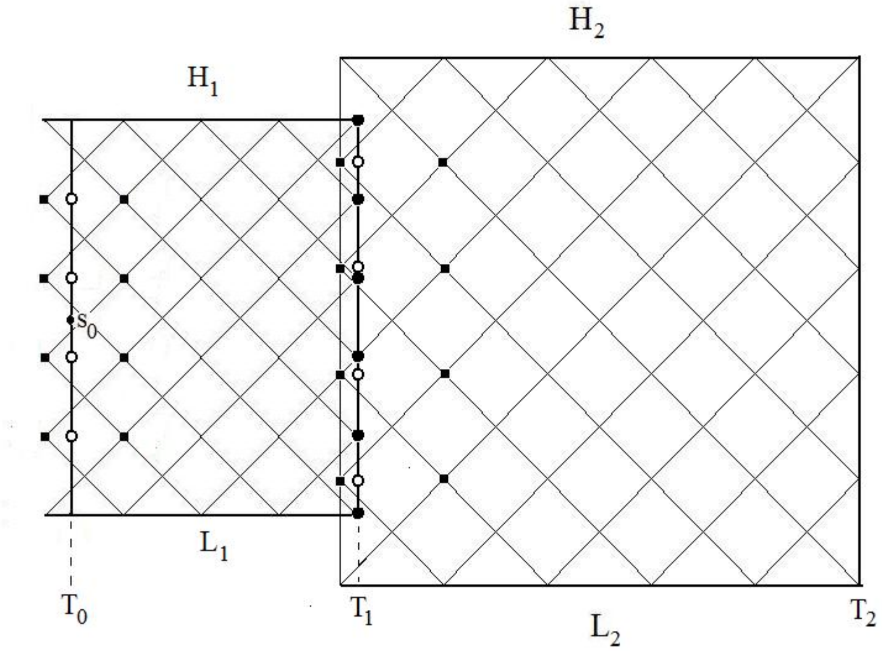


Figure 2.3: *Binomial interpolated lattice method. 2-step double knock-out options. The prices at time $T_1$ are obtained by a Lagrange space interpolation of the prices at the nodes denoted by empty circles, such prices are obtained by a linear interpolation in time of the prices at the adjacent nodes denoted by squares. Similarly we obtain the prices at time $T_0$.*

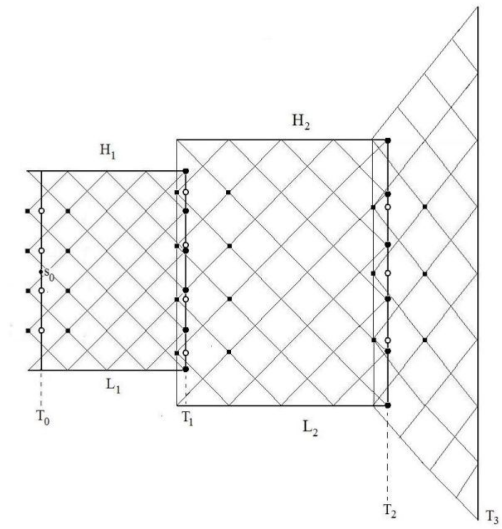


Figure 2.4: *Binomial interpolated lattice method. Early ending knock-out options. The prices at time $T_2$ are obtained by a Lagrange space interpolation of the prices at the nodes denoted by empty circles, such prices are obtained by a linear interpolation in time of the prices at the adjacent nodes denoted by squares. Similarly we obtain the prices at time $T_0$ and $T_1$.*

In the *n-step double barrier options* case we just apply the procedure described above for 2-step double barrier options recursively.
The treatment of *partial-time step double barrier options* is straightforward, in fact one can easily remove the knock-out barrier provision in some time intervals. Moreover, we remark that it is possible to take into account knock-in features instead of knock-out ones.

**Remark 2.7.1.** *The above technique can also be applied to the case in which the volatility and the interest rate are piecewise constant functions of time. In the sequel we will treat only the case of piecewise constant volatility because when the interest rate is piecewise constant the treatment is straightforward. Let us consider first the case in which the volatility varies at every time interval* $[T_i, T_{i+1}]$ *in which the barriers change, i.e.* $\sigma = \sigma_i$ *in* $[T_i, T_{i+1}]$*, for* $i = 0, ..., n-1$*. Then for each interval* $[T_i, T_{i+1}]$ *we need to construct the mesh matching the barriers by using the volatility parameter* $\sigma_i$ *and suitable interpolations in time and in space as explained before. In the general case in which the volatility varies during the "volatility time intervals"* $[\tilde{T}_j, \tilde{T}_{j+1}]$*, for* $j = 0, ..., n_\sigma - 1$ *(i.e. the volatility varies at some instants that are different from the ones at which the barriers change), we need to construct a new mesh, not only at the generic barrier time* $T_i$ *but also at the volatility time* $\tilde{T}_i$*. So, in this general case we construct* $n + n_\sigma$ *different mesh structures.*

## 2.8 Numerical results

We provide some numerical results of the algorithms presented in the Sections 2.5 and 2.7 in the case of double barrier options, 2-step double barrier options and multi-step double barrier options. All the computations presented in the tables have been performed in double precision on a PC with a processor Intel Core i5 at 1.7 Ghz.

### 2.8.1 Double barrier options: comparisons with the Day-Lyuu method

We first consider the comparisons with the Day-Lyuu (DL) method using the numerical experiments proposed in Day-Lyuu ([29], 2010) for pricing knock-out double-barrier call options. We also add further comparisons with different parameters. The volatility of the stock price is $\sigma = 0.25$, the interest rate is $r = 0.1$, the time to maturity is $T = 1$, the strike price is $K = 100$ and the two barriers are $L = 90$ and $H = 140$. We consider three possible values for the initial stock price: $s_0 = 95, 90.05, 139.95$.
We observe that in the tables below the number of time steps is $m$ and it refers to the original CRR model, as defined in Section 2.4. So for each $m$ the Dai-Lyuu and the Binomial Interpolated Lattice (BIL) compute a new number of time steps $n := m' + 1$ in DL and $n := m' + 2$ in BIL) that are of the same order as $m$.
We use as benchmark value the price computed with the closed formula provided by Kunitomo and Ikeda ([56], 1992) that is recalled in Section 2.2. We observe that in the cases in

which $s_0$ is a point near to the barrier, the Binomial Interpolated Lattice performs better than the bino-trinomial procedure. The results are given in Table 2.1.

| | $s_0 = 95$ | | | $s_0 = 90.05$ | | |
|---|---|---|---|---|---|---|
| m | DL | KI | BIL | DL | KI | BIL |
| 100 | 1.423589 | | 1.422126 | 0.313771 | | 0.016004 |
| 200 | 1.440976 | | 1.440586 | 0.279768 | | 0.016235 |
| 400 | 1.449705 | 1.458435 | 1.450008 | 0.226927 | 0.016268 | 0.016267 |
| 800 | 1.453358 | | 1.453441 | 0.149755 | | 0.016263 |
| 1600 | 1.456403 | | 1.456350 | 0.084767 | | 0.016254 |
| 3200 | 1.457183 | | 1.457179 | 0.059701 | | 0.016259 |

| | $s_0 = 139.95$ | | |
|---|---|---|---|
| m | DL | KI | BIL |
| 100 | 0.196833 | | 0.006562 |
| 200 | 0.176563 | | 0.006688 |
| 400 | 0.144278 | 0.006656 | 0.006673 |
| 800 | 0.093742 | | 0.006661 |
| 1600 | 0.051903 | | 0.006650 |
| 3200 | 0.035926 | | 0.006653 |

Table 2.1: *Knock-out double barrier call option prices with* $T = 1$, $r = 0.1$, $\sigma = 0.25$, $K = 100$, $L = 90$, $H = 140$ *and* $s_0$ *varying.*

In Table 2.2 we give the price of a knock-out double barrier call option in other "near to the barrier" cases. Here $\sigma = 0.25$, $r = 0.1$, $T = 1$ and $K = 100$. We now vary $s_0$, $L$ and $H$ so that we can consider both the case in which $s_0$ is close to $L$ and the case in which $s_0$ is close to $H$. In the first table we choose $s_0 = 95$, $L = 94.9$ and $H = 140$, so we have that the starting point $s_0$ is near to the lower barrier $L$. Instead, in the second one $s_0$ is near to the higher barrier $H$ and we choose $s_0 = 139.9$, $L = 95$ ad $H = 140$.

| | $s_0 = 95, L = 94.9, H = 140$ | | | $s_0 = 139.9, L = 95, H = 140$ | | |
|---|---|---|---|---|---|---|
| m | DL | KI | BIL | DL | KI | BIL |
| 100 | 0.274716 | | 0.024774 | 0.211312 | | 0.011449 |
| 200 | 0.227618 | | 0.025281 | 0.085969 | | 0.011148 |
| 400 | 0.109875 | 0.025305 | 0.025182 | 0.083269 | 0.011238 | 0.011188 |
| 800 | 0.128411 | | 0.025305 | 0.087552 | | 0.011253 |
| 1600 | 0.062742 | | 0.025274 | 0.060058 | | 0.011243 |
| 3200 | 0.053451 | | 0.025286 | 0.048296 | | 0.011239 |

Table 2.2: *Knock-out double barrier call option prices with* $T = 1$, $r = 0.1$, $\sigma = 0.25$, $K = 100$. *The values of* $s_0$, $L$ *and* $H$ *vary.*

We remark that in the "near barrier" cases presented in Table 2.1 and Table 2.2 for each value of $m$ the starting point $s_0$ belongs to the critical intervals (2.6.4) and (2.6.5) defined in Section 2.6. So, thanks to the sufficient condition given in Remark 2.6.5, we explain why the Binomial Interpolated Lattice method performs better than the bino-trinomial tree. In fact if $s_0$ belongs to the interval $(L, Le^{-(r-\sigma^2/2)\Delta T'+\sigma\sqrt{\Delta T}}]$ or to the interval $[He^{-(r-\frac{\sigma^2}{2})\Delta T'-\sigma\sqrt{\Delta T}}, H)$ the Dai and Lyuu procedure may degenerate. As pointed out in Remark 2.6.5, it is clear that if we choose $m$ such that

$$s_0 > Le^{-(r-\sigma^2/2)\Delta T'+\sigma\sqrt{\Delta T}}, \tag{2.8.1}$$

(in the case in which $s_0$ is near to the lower barrier $L$), or if we choose $m$ such that

$$s_0 < He^{-(r-\sigma^2/2)\Delta T' - \sigma\sqrt{\Delta T}}, \tag{2.8.2}$$

(when $s_0$ is near to the higher barrier $H$), then the bino-trinomial tree may not degenerate. But the values of $m$ we need to consider in order to satisfy these conditions are very large. In fact in the case of Table 2.1 with $s_0 = 90.05$, we have to choose $m \geq 201840$. Instead, in the case in which $s_0 = 139.95$ we need to choose $m \geq 489104$ in order to satisfy condition (2.8.2). Similarly, in Table 2.2, we need to choose $m \geq 55987$ if $s_0 = 95$ and $m \geq 122107$ if $s_0 = 139.9$ to satisfy conditions (2.8.1) and (2.8.2) respectively.
In Table 2.3 we show two more examples of pricing a knock-out double barrier call option with $\sigma = 0.25$, $r = 0.1$, $T = 1$, $K = 100$, $L = 90$ and $H = 140$ in the "near to the barrier" case. We vary the starting point and we choose $s_0 = 92$ and $s_0 = 138$. The numerical results show that the BIL method converges faster than the bino-trinomial tree also in the case in which $s_0$ is chosen not so much close to the barriers. We also observe that when $s_0 = 92$ we need $m \geq 104$ to satisfy (2.8.1) and when $s_0 = 138$ we have to choose $m \geq 289$ to verify condition (2.8.2). Moreover, also if $m$ is such that (2.8.1) or (2.8.2) is satisfied, the Binomial Interpolated Lattice converges faster than the bino-trinomial tree.

| | $s_0 = 92$ | | | $s_0 = 138$ | | |
|---|---|---|---|---|---|---|
| m | DL | KI | BIL | DL | KI | BIL |
| 100 | 0.753689 | | 0.611674 | 0.365983 | | 0.265373 |
| 200 | 0.675166 | | 0.620375 | 0.338188 | | 0.270340 |
| 400 | 0.667178 | 0.626377 | 0.623008 | 0.316626 | 0.271825 | 0.270702 |
| 800 | 0.624209 | | 0.624220 | 0.281821 | | 0.270875 |
| 1600 | 0.625457 | | 0.625476 | 0.271367 | | 0.271286 |
| 3200 | 0.625861 | | 0.625833 | 0.271534 | | 0.271548 |

Table 2.3: *Knock-out double barrier call option prices with* $T = 1$, $r = 0.1$, $\sigma = 0.25$, $K = 100$, $L = 90$, $H = 140$ *and* $s_0$ *varying.*

### 2.8.2 Double barrier options: comparisons with a finite difference method

In order to test the efficiency of the Binomial Interpolated Lattice (BIL) approach, both for pricing and hedging purposes, we compare it with the PDE finite difference method (FD) implemented following Zvan, Forsyth and Vetzal ([87], 2000). In particolar we use a fully implicit scheme with uniformly spaced grid. We denote by $m_s$ the number of space steps and as before $m$ the number of time steps. We use the same parameters used in the previous Section changing the number of time steps. We consider four different test cases: A, B, C, D as resumed in Table 2.4. Each case is associated to a different standard number of steps of the corresponding tree. The number of steps, in the four different cases, are chosen in such a way that the considered algorithms (BIL and FD) have similar times of computation. In Table 2.5, Table 2.6, Table 2.7 we provide the numerical pricing and hedging (delta, gamma, vega) results. The numerical experiments show that the BIL method is reliable and accurate both for the computation of the option prices and the Greeks. Compared with the standard finite difference scheme the present methodology delivers double barrier option prices and Greeks with similar accuracy in analogue CPU times.

| | A | | B | | C | | D | |
|---|---|---|---|---|---|---|---|---|
| | $m/m_s$ | *CPU time* | $m/m_s$ | *CPU time* | $m/m_s$ | *CPU time* | $m/m_s$ | *CPU time* |
| BIL | 500 | 0.0034 | 1500 | 0.0104 | 3500 | 0.0324 | 12000 | 0.1352 |
| FD | 200/50 | 0.0033 | 400/100 | 0.0108 | 800/200 | 0.0352 | 1600/400 | 0.1321 |

Table 2.4: Different cases : Time/Space Steps and CPU times

| | $s_0 = 95$ | | | $s_0 = 90.05$ | | |
|---|---|---|---|---|---|---|
| Case | FD | KI | BIL | FD | KI | BIL |
| A | 1.466963 | | 1.451349 | 0.016074 | | 0.016266 |
| | 0.255061 | | 0.252350 | 0.327332 | | 0.324974 |
| | -0.016582 | | -0.016331 | -0.012978 | | -0.013982 |
| | -21.207882 | | -21.183731 | -0.240449 | | -0.247137 |
| B | 1.462677 | 1.458435 | 1.456207 | 0.016181 | 0.016268 | 0.016270 |
| | 0.254362 | 0.253605 | 0.253204 | 0.326101 | 0.325082 | 0.325072 |
| | -0.016559 | -0.016527 | -0.016471 | -0.012305 | -0.011651 | -0.013065 |
| | -21.254516 | -21.295842 | -21.269938 | -0.244398 | -0.247919 | -0.247805 |
| C | 1.460531 | | 1.457437 | 0.016233 | | 0.016261 |
| | 0.253992 | | 0.253438 | 0.325562 | | 0.324922 |
| | -0.016545 | | -0.016512 | -0.011949 | | -0.012265 |
| | -21.275779 | | -21.280831 | -0.246337 | | -0.247777 |
| D | 1.459458 | | 1.458075 | 0.016258 | | 0.016265 |
| | 0.253802 | | 0.253544 | 0.325314 | | 0.325010 |
| | -0.016538 | | -0.016521 | -0.011766 | | -0.011962 |
| | -21.285916 | | -21.291278 | -0.247297 | | -0.247871 |

| | $s_0 = 139.95$ | | |
|---|---|---|---|
| Case | FD | KI | BIL |
| A | 0.006787 | | 0.006668 |
| | -0.134492 | | -0.133415 |
| | 0.002774 | | 0.002059 |
| | -0.073870 | | -0.072093 |
| B | 0.006720 | 0.006656 | 0.006660 |
| | -0.133826 | -0.133216 | -0.133257 |
| | 0.002912 | 0.003028 | 0.002510 |
| | -0.073067 | -0.072327 | -0.072300 |
| C | 0.006687 | | 0.006654 |
| | -0.133513 | | -0.133142 |
| | 0.002978 | | 0.002816 |
| | -0.072663 | | -0.072247 |
| D | 0.006670 | | 0.006656 |
| | -0.133362 | | -0.133184 |
| | 0.003010 | | 0.002921 |
| | -0.072460 | | -0.072307 |

Table 2.5: *Knock-out double barrier call option prices, delta, gamma and vega with* $T = 1$, $r = 0.1$, $\sigma = 0.25$, $K = 100$, $L = 90$, $H = 140$ *and* $s_0$ *varying.*

| | $s_0 = 95, L = 94.9, H = 140$ | | | $s_0 = 139.9, L = 95, H = 140$ | | |
|---|---|---|---|---|---|---|
| Case | FD | KI | BIL | FD | KI | BIL |
| A | 0.025237 | | 0.025213 | 0.011480 | | 0.011267 |
| | 0.255861 | | 0.251614 | -0.114002 | | -0.112749 |
| | -0.009910 | | -0.010320 | 0.002304 | | 0.001506 |
| | -0.466251 | | -0.473302 | -0.162609 | | -0.160230 |
| B | 0.025295 | 0.025305 | 0.025271 | 0.011352 | 0.011238 | 0.011222 |
| | 0.254125 | 0.252623 | 0.252228 | -0.113219 | -0.112503 | -0.112332 |
| | -0.009245 | -0.008694 | -0.009636 | 0.002439 | 0.002535 | 0.002181 |
| | -0.471336 | -0.475184 | -0.474126 | -0.161217 | -0.160094 | -0.159848 |
| C | 0.025322 | | 0.025305 | 0.011288 | | 0.011239 |
| | 0.253347 | | 0.252571 | -0.112853 | | -0.112496 |
| | -0.008900 | | -0.009638 | 0.002503 | | 0.002180 |
| | -0.473829 | | -0.474661 | -0.160517 | | -0.160008 |
| D | 0.025335 | | 0.025304 | 0.011255 | | 0.011237 |
| | 0.252982 | | 0.252581 | -0.112676 | | -0.112487 |
| | -0.008725 | | -0.009160 | 0.002535 | | 0.002362 |
| | -0.475062 | | -0.475082 | -0.160145 | | -0.160086 |

Table 2.6: *Knock-out double barrier call option prices, delta, gamma and vega with* $T = 1$, $r = 0.1$, $\sigma = 0.25$, $K = 100$. *The values of* $s_0$, $L$ *and* $H$ *vary.*

| | $s_0 = 92$ | | | $s_0 = 138$ | | |
|---|---|---|---|---|---|---|
| Case | FD | KI | BIL | FD | KI | BIL |
| A | 0.629982 | | 0.623382 | 0.274247 | | 0.270743 |
| | 0.301449 | | 0.297709 | -0.139743 | | -0.137431 |
| | -0.014131 | | -0.013982 | 0.002456 | | 0.002059 |
| | -9.339698 | | -9.330881 | -3.009659 | | -2.975722 |
| B | 0.628163 | 0.626377 | 0.625416 | 0.273028 | 0.271825 | 0.271323 |
| | 0.300709 | 0.299884 | 0.299436 | -0.139163 | -0.138561 | -0.138313 |
| | -0.014107 | -0.014074 | -0.014015 | 0.002448 | 0.002439 | 0.002424 |
| | -9.360235 | -9.378668 | -9.367329 | -3.005481 | -3.001289 | -2.994041 |
| C | 0.627254 | | 0.625943 | 0.272420 | | 0.271599 |
| | 0.300305 | | 0.299669 | -0.138864 | | -0.138447 |
| | -0.014092 | | -0.014054 | 0.002444 | | 0.002432 |
| | -9.369625 | | -9.372070 | -3.003413 | | -2.997151 |
| D | 0.626801 | | 0.626215 | 0.272116 | | 0.271744 |
| | 0.300095 | | 0.299804 | -0.138712 | | -0.138519 |
| | -0.014085 | | -0.014068 | 0.002442 | | 0.002437 |
| | -9.374109 | | -9.376531 | -3.002383 | | -3.000212 |

Table 2.7: *Knock-out double barrier call option prices, delta, gamma and vega with* $T = 1$, $r = 0.1$, $\sigma = 0.25$, $K = 100$, $L = 90$, $H = 140$ *and* $s_0$ *varying.*

### 2.8.3 2-step double barrier options

Let us now consider the numerical experiments proposed in Guillame ([43], 2010) for pricing 2-step double barrier knock-out put options. In the European case we compare our method (BIL-EU) with the benchmark value given by the closed formula (GUI) provided in Guillaume ([43], 2010). No benchmark is available in the American case (BIL-AM). The volatility of the stock price is $\sigma = 0.3$, the interest rate is $r = 0.03$, the current stock price is $s_0 = 100$ and the strike price varies: $K = 90, 100, 110$. In Table 2.8 we report the values of 2-step double knock-out put options with double barrier with parameters: $T_1 = 0.25$, $T_2 = T = 0.5$, $L_1 = 70$, $H_1 = 130$, $L_2 = 75$ and $H_2 = 125$.

| | $K = 90$ | | | $K = 100$ | | | $K = 110$ | | |
|---|---|---|---|---|---|---|---|---|---|
| m | BIL-EU | GUI | BIL-AM | BIL-EU | GUI | BIL-AM | BIL-EU | GUI | BIL-AM |
| 100 | 0.806457 | | 3.549508 | 3.152257 | | 7.694393 | 7.089792 | | 13.516271 |
| 200 | 0.819128 | | 3.541653 | 3.186487 | | 7.694650 | 7.196594 | | 13.574689 |
| 400 | 0.815156 | 0.821806 | 3.545095 | 3.187184 | 3.194080 | 7.703400 | 7.186754 | 7.186905 | 13.576866 |
| 800 | 0.820657 | | 3.555169 | 3.191445 | | 7.712047 | 7.182939 | | 13.580487 |
| 1600 | 0.821165 | | 3.555965 | 3.192845 | | 7.713536 | 7.184522 | | 13.581967 |
| 3200 | 0.821348 | | 3.556271 | 3.192542 | | 7.713324 | 7.184678 | | 13.582068 |

Table 2.8: *2-step double knock-out put option prices with* $s_0 = 100$, $r = 0.03$, $\sigma = 0.3$, $T_1 = 0.25$, $T_2 = T = 0.5$, $L_1 = 70$, $H_1 = 130$, $L_2 = 75$, $H_2 = 125$ *and* $K$ *varying.*

In Table 2.9 we consider an early-ending 2-step double knock-out call with $K = 120$, $s_0 = 100$, $r = 0.03$ and double barrier parameters $T_1 = 0.125$, $T_2 = 0.25$, $T_3 = T = 0.5$, $L_1 = 75$, $H_1 = 125$, $L_2 = 70$, $H_2 = 130$. The volatility varies: $\sigma = 0.15, 0.3$.

| | $\sigma = 0.15$ | | | $\sigma = 0.3$ | | |
|---|---|---|---|---|---|---|
| m | BIL-EU | GUI | BIL-AM | BIL-EU | GUI | BIL-AM |
| 100 | 0.263380 | | 9.962265 | 1.575926 | | 9.728607 |
| 200 | 0.265933 | | 9.962439 | 1.613757 | | 9.741826 |
| 400 | 0.271830 | 0.2755 | 9.962422 | 1.605417 | 1.6165 | 9.745550 |
| 800 | 0.273739 | | 9.962533 | 1.608395 | | 9.750609 |
| 1600 | 0.275081 | | 9.962435 | 1.612841 | | 9.755035 |
| 3200 | 0.275201 | | 9.962400 | 1.615489 | | 9.758268 |

Table 2.9: *Early-ending 2-step double knock-out call option prices with* $s_0 = 100$, $r = 0.03$, $K = 120$, $T_1 = 0.125$, $T_2 = 0.25$, $T_3 = T = 0.5$, $L_1 = 75$, $H_1 = 125$, $L_2 = 70$, $H_2 = 130$ *and* $\sigma$ *varying.*

The numerical results show that the method is accurate also in the 2-step double knock-out option case.

### 2.8.4 Multi step double barrier options

In Table 2.9, we propose the results obtained with our method for a 16-steps knock out double barrier put option. The volatility of the stock price is $\sigma = 0.3$, the interest rate is $r = 0.03$, the current stock price is $s_0 = 100$, the strike price is $K = 110$, the time to maturity is $T = 2$ and the barrier parameters are: $T_i = i \cdot 0.125$, $L_i = 70 - i$, $H_i = 130 + i$, for all $i = 0, ..., 15$. In the European case we use as benchmark value the Monte Carlo method provided in Baldi, Caramellino and Iovino ([9], 1999) with 10 millions simulations and 1000 Euler time discretization steps (with the 99% confidence interval in parenthesis). We observe that the computation times are very fast. For example, for $m = 12800$ and $m = 25600$ they are respectively 0.028221 and 0.072933 seconds.

| m | BIL-EU | MC | BIL-AM |
|---|---|---|---|
| 100 | 6.585345 | | 17.570376 |
| 200 | 6.399257 | | 17.598079 |
| 400 | 6.288664 | 6.197331 | 17.623151 |
| 800 | 6.243341 | [6.187387-6.207276] | 17.627720 |
| 1600 | 6.233008 | | 17.629993 |
| 3200 | 6.208183 | | 17.633762 |
| 6400 | 6.203391 | | 17.634432 |
| 12800 | 6.194374 | | 17.635186 |
| 25600 | 6.191878 | | 17.635486 |

Table 2.10: *16-step double knock-out put option prices with* $\sigma = 0.3$*,* $r = 0.03$*,* $s_0 = 100$*,* $K = 110$*,* $T = 2$*. The barrier parameters are* $T_i = i \cdot 0.125$*,* $L_i = 70 - i$*,* $H_i = 130 + i, i = 0, ..., 15$.

Finally, we test a case with different volatilities in every observation period (see Remark 2.7.1). In particular, we consider the following volatilities:

$$\sigma_i = 0.2 + 0.02 \cdot i, \quad i = 0, ..., 15.$$

| m | BIL-EU | MC | BIL-AM |
|---|---|---|---|
| 100 | 4.632436 | | 18.890209 |
| 200 | 4.565130 | | 18.924618 |
| 400 | 4.474004 | 4.423045 | 18.943538 |
| 800 | 4.466596 | [4.414167-4.431922] | 18.949408 |
| 1600 | 4.446335 | | 18.944733 |
| 3200 | 4.430396 | | 18.944736 |
| 6400 | 4.426262 | | 18.944366 |
| 12800 | 4.420738 | | 18.944321 |
| 25600 | 4.418433 | | 18.944005 |

Table 2.11: *16-step double knock-out put option prices with* $r = 0.03$*,* $s_0 = 100$*,* $K = 110$*,* $T = 2$*. The barrier parameters are* $T_i = i \cdot 0.125$*,* $L_i = 70 - i$*,* $H_i = 130 + i$*,* $\sigma_i = 0.2 + 0.02 \cdot i, i = 0, ..., 15$.

# Chapter 3

# Digital barrier options in the Black and Scholes model

In this Chapter we consider the problem of pricing digital barrier options when the stock price process $(S_t)_{t\in[0,T]}$ follows the Black and Scholes model, see (2.1.1).
In Section 3.1 we find an explicit asymptotic expansion of the binomial approximation error $\mathrm{Err}(n)$ (see Definition (1.2.1)) for a digital call option with lower barrier $L$. We observe that the case of a higher barrier $H$ is similar, so we omit it. This is an original result principally based on the techniques used in Chang and Palmer ([20], 2007) (who find an explicit expression of the error $\mathrm{Err}(n)$ for vanilla digital call options, that is without barriers, for details see Theorem 1.2.5) and in Lin and Palmer ([62], 2013) (who find an expression for $\mathrm{Err}(n)$ for call options with a single barrier, for details see Theorem 1.2.15).
The idea is to find a closed-form formula in terms of binomial coefficients of the price of a digital call option with barrier $L$ and then use the approximation of the binomial distribution by the normal one in order to find explicit coefficients in the asymptotic expansion. The expression of $\mathrm{Err}(n)$ suggests us how to set the barrier $L$ and the strike $K$ in the binomial algorithm such that the rate of convergence is $\frac{1}{n}$. In particular, here we adapt the Binomial Interpolated Lattice presented in Section 2.5 so that $L$ and $K$ are placed in the binomial mesh in a "right" way. In fact if the barrier $L$ lies on a layer of nodes of the tree and the strike $K$ is located halfway between two nodes at maturity $T$ (i.e. it is a node from the penultimate node before maturity), then we get a procedure of order $\frac{1}{n}$. The numerical results are presented in Section 3.3.1.
We stress that since there are no manageable closed-form formulas for the binomial price of digital call options with double barriers, we are not able to replicate the discussion when there is both a lower barrier $L$ and a higher barrier $H$. The only work that is able to treat the study of the rate of convergence of double barrier options is due to Gobet ([42], 2001). Gobet uses a completely different approach, based on PDE techniques, that allows one to treat the double barrier case for a generic class of continuous payoff functions (for details see Section 1.2.2). We notice that he finds that the contribution of type $O(\frac{1}{\sqrt{n}})$ in $\mathrm{Err}(n)$ explicitly depends on the distance between the first lattice point above the higher barrier

$H$ and $H$ itself and also on the distance between the lower barrier $L$ and the first lattice point below $L$. This result suggests that if we construct a binomial algorithm such that the contractual barriers are exactly two nodes of the tree then the rate of convergence of the algorithm is improved.
Then, in Section 3.2 we study the rate of convergence for digital options with double barriers by using PDE techniques and we find an upper bound for the binomial approximation error $\mathrm{Err}(n)$. Our initial goal was to find an expression in which apart from a contribution of order $\frac{1}{\sqrt{n}}$ due to the position of the barriers, as in the case of continuous payoff functions studied in Gobet ([42], 2001), there was also an additional term of type $O(\frac{1}{\sqrt{n}})$ that explicitly depends on the position of the strike. Currently we are not able to get the desired result because we obtain that the error is bounded from above by two terms that are related on the position of the barriers plus a term $\tilde{\mathcal{R}}_n$, with $|\tilde{\mathcal{R}}_n| \leq C\frac{1}{\sqrt{n}}$, for a constant $C > 0$. But a careful analysis of the proof suggests that the dependence on the position of $K$ is "hidden" in $\tilde{\mathcal{R}}_n$. We stress that the advantage of this result is that it can be easily extended to a more general payoff function with a finite number of discontinuity points, but without lack of generality we will treat in details only the case of digital payoffs. Finally, in Section 3.3.2 we present some numerical results in order to efficiently price double barrier digital options.

## 3.1 Digital call options with single barrier

### 3.1.1 Black and Scholes prices

In this Section we recall the continuous prices at time $t = 0$ of European single barrier digital call options when the underlying process follows the SDE (2.1.1), i.e.

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t, \quad S_0 = s_0 > 0.$$

As usual we will use the following notations: $s_0$ is the stock price at time $t = 0$, $K$ is the strike price, $r$ is the continuously compounded interest rate, $\sigma$ is the volatility, $T$ is the time to maturity and $L$ is the lower barrier.
We stress here that we will consider the case of down options, the case of up options being similar.
We first recall the prices of standard call options with barrier $L$ that are due to Merton ([64], 1973), who first derived the analytical formula for the down-and-out call option, and Reiner and Rubinstein ([71], 1991), that provided the formulas in details for single barriers call and put options of the all the types, i.e. down-and-out, down-and-in, up-and-in and up-and-out. Then, we derive the respective European continuous prices for the digital case. These formulas are also resumed in the work of Cheng ([21], 2003).

**Remark 3.1.1.** *The price of a digital option is essentially the derivative with respect to the strike $K$ of the price of the respective call option. If we denote with $C_{call}(K)$ the price of a* *single barrier call option as a function of K, i.e.*

$$C_{call}(K) = \mathbb{E}((S_T - K)_+ \mathbb{1}_{S_{\inf} > L}), \quad \textit{with} \quad S_{\inf} = \inf_{t\in[0,T]} S_t,$$

*we get that the price as a function of K of the respective digital call option, that we call* $C_{digital-call}(K)$*, can be deduced by the following relations:*

$$\begin{aligned}\frac{d}{dK} C_{call}(K) &= \mathbb{E}\left(\frac{d}{dK}(S_T - K)_+ \mathbb{1}_{S_{\inf}>L}\right) \\ &= \mathbb{E}(-\mathbb{1}_{S_T \geq K} \mathbb{1}_{S_{\inf}>L}) = -C_{digital\text{-}call}(K).\end{aligned}$$

We stress that we need the Black and Scholes prices of digital call options with lower barrier $L$ of the two following types:

1. a down-and-in call option with $L < K$;
2. a down-and-out call option with $L > K$.

We remark that we consider these two cases because the corresponding binomial formulas for the prices are manageable and permits a simple treatment. Then we will see that by using the binomial formulas for the vanilla digital call option, it is possible to find the asymptotic expansion of the error also for a down-and-out digital call option with $L < K$ and a down-and-in digital call option with $L > K$.

### Prices of down-and-in call and digital call options with $L < K$

The Black and Scholes price of a down-and-in call option with barrier $L < K$, that we denote with $C_{di}^{BS}(s_0, K, T, L)$, is given by:

$$C_{di}^{BS}(s_0, K, T, L) = s_0 \left(\frac{s_0}{L}\right)^{-1-\frac{2r}{\sigma^2}} \Phi(d_{21}) - Ke^{-rT} \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}} \Phi(d_{22}), \tag{3.1.1}$$

with

$$d_{21} = \frac{\log \frac{L^2}{s_0 K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_{22} = \frac{\log \frac{L^2}{s_0 K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Then, the price of a digital down-and-in call option with barrier $L < K$, that we call $C_{di,digital}^{BS}(s_0, K, T, L)$, is given by

$$C_{di,digital}^{BS}(s_0, K, T, L) = e^{-rT} \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}} \Phi(d_{22}). \tag{3.1.2}$$

**Remark 3.1.2.** *In order to obtain the expression (3.1.2) we use Remark 3.1.1, so we have that*

$$
\begin{aligned}
\frac{d}{dK}P_{call}(K) &= s_0\left(\frac{s_0}{L}\right)^{-1-\frac{2r}{\sigma^2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_{21}^2}{2}}\left(-\frac{1}{\sigma\sqrt{T}K}\right)+ \\
&- e^{-rT}\left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}}\left[\Phi(d_{22})+K\frac{1}{\sqrt{2\pi}}e^{-\frac{d_{22}^2}{2}}\left(-\frac{1}{\sigma\sqrt{T}K}\right)\right] \\
&= -e^{-rT}\left(\frac{s_0}{H}\right)^{1-\frac{2r}{\sigma^2}}\Phi(d_{22}) + remainder
\end{aligned}
$$

*with*

$$
\begin{aligned}
remainder = s_0\left(\frac{s_0}{L}\right)^{-1-\frac{2r}{\sigma^2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_{21}^2}{2}}\left(-\frac{1}{\sigma\sqrt{T}K}\right)+ \\
- e^{-rT}K\left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}}\frac{1}{\sqrt{2\pi}}\left(-\frac{1}{\sigma\sqrt{T}K}\right).
\end{aligned}
$$

*But*

$$
remainder = 0.
$$

*In fact we have that*

$$
\begin{aligned}
remainder &= \frac{1}{\sqrt{2\pi}}\left(-\frac{1}{\sigma\sqrt{T}K}\right)\left(\frac{s_0}{L}\right)^{-\frac{2r}{\sigma^2}}\left[s_0\frac{L}{s_0}e^{-\frac{d_{21}^2}{2}} - e^{-rT}\frac{s_0}{L}Ke^{-\frac{d_{22}^2}{2}}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left(-\frac{1}{\sigma\sqrt{T}K}\right)\left(\frac{s_0}{L}\right)^{-\frac{2r}{\sigma^2}}\left[\frac{L^2e^{-\frac{d_{21}^2}{2}} - e^{-rT}s_0Ke^{-\frac{d_{22}^2}{2}}}{L}\right] = 0,
\end{aligned}
$$

*where the last equality comes from the fact that*

$$
L^2e^{-\frac{d_{21}^2}{2}} - e^{-rT}s_0Ke^{-\frac{d_{22}^2}{2}} = 0.
$$

*So indeed the Black and Scholes price of a digital down-and-in call option is given in (3.1.2).*

### Prices of down-and-out call and digital call options with $L > K$

The price of a down-and-out call option with barrier $L > K$, that we call $C_{do}^{BS}(s_0, K, T, L)$, is

$$
\begin{aligned}
C_{do}^{BS}(s_0, K, T, L) &= s_0\Phi(d_{31}) - Ke^{-rT}\Phi(d_{32}) \\
&- \left[s_0\left(\frac{s_0}{L}\right)^{-1-\frac{2r}{\sigma^2}}\Phi(d_{41}) - Ke^{-rT}\left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}}\Phi(d_{42})\right],
\end{aligned}
$$

with

$$d_{31} = \frac{\log \frac{s_0}{L} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_{32} = d_{31} - \sigma\sqrt{T},$$
$$d_{41} = \frac{\log \frac{L}{s_0} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_{42} = d_{41} - \sigma\sqrt{T}.$$

Then, the price of a down-and-out digital call option with barrier $L > K$, that we call $C_{do-digital}(s_0, K, T, L)$ is:

$$C^{BS}_{do-digital}(s_0, K, T, L) = e^{-rT}\left[\Phi(d_{32}) - \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}} \Phi(d_{42})\right]. \tag{3.1.3}$$

**Remark 3.1.3.** *We get the Black and Scholes price (3.1.3) by using again Remark 3.1.1.*

### 3.1.2 Binomial prices

We now present the binomial formulas for a down-and-in digital call with $L < K$ and a down-and-out digital call with $L > K$. We will principally follow the results in Reimer and Sandmann ([70], 1995) and Lin and Palmer ([60], 2013).

#### Notations

Let us first introduce two quantities as defined in Lin and Palmer ([60], 2013), that we call $\Delta_n^K$ and $\Delta_n^L$, that will have a crucial role in what follows. We recall that, as usual, in the $n$-period CRR binomial model it is assumed that the stock price at any time arises by a factor $u = e^{\sigma\sqrt{h}}$ or falls by a factor $d = u^{-1}$ at the next period, where $h = T/n$. We also recall that the probability of an up jump is equal to $p = \frac{e^{rh}-d}{u-d}$ and that the down jump occurs with probability $1 - p$.
The quantity $\Delta_n^K$ is set as

$$\Delta_n^K = 1 - 2\text{frac}\left(\frac{\log(s_0/K)}{2\sigma\sqrt{h}} - \frac{n}{2}\right), \tag{3.1.4}$$

where for every real number $x$, the fractional part of $x$ is defined as $\text{frac}(x) = x - \lfloor x \rfloor$, with $\lfloor x \rfloor$ indicating the largest integer preceding $x$.
We observe that $\Delta_n^K$ is a measure of the position of $K$ in the log-scale in relation to two adjacent terminal stock prices (for details see Section 1.2.2). The nodes $(S_{i,j})_{i,j}$ of the binomial approximation scheme are given by

$$S_{i,j} = s_0 u^j d^{i-j}, \quad \forall\, i = 0, ..., n,\ j = 0, ..., i,$$

and in particular the $n+1$ nodes at maturity $T$ are equal to

$$S_{n,j} = s_0 u^j d^{n-j}, \quad \forall\, j = 0, ..., n.$$

So if we define $j_K$ the integer such that

$$S_{n,j_K-1} = s_0 u^{j_K-1} d^{n-j_K+1} < K \leq S_{n,j_K} = s_0 u^{j_K} d^{n-j_K},$$

then it is possible to write (for details see Lin and Palmer ([62], 2013))

$$\log K = \alpha \log S_{n,j_K} + (1-\alpha) \log S_{n,j_K-1}, \quad \text{with} \quad \alpha = \frac{1+\Delta_n^K}{2}.$$

We observe that $-1 \leq \Delta_n^K \leq 1$, then in the log-scale: if $\Delta_n^K = -1$ the strike $K$ is the node $S_{n,j_k-1}$ at maturity, if $\Delta_n^K = 0$ then $K$ lies halfway between the two nodes $S_{n,j_k-1}$ and $S_{n,j_k}$ at maturity (i.e. it is a node from the first period before maturity), and if $\Delta_n^K = 1$ then $K$ is the node $S_{n,j_k}$ at maturity.
We now describe a *similar* quantity corresponding to the lower barrier $L$ that we call $\Delta_n^L$. First, we call $\tilde{L}$ the effective barrier on the tree structure, that is generally different from the contractual barrier $L$. Then we need to consider two possible cases:

1. $\tilde{L}$ is a terminal stock price;
2. $\tilde{L}$ is a stock price from the penultimate period, i.e. the first period before maturity.

Let us call $j_L$ the number of up jumps required to reach the effective barrier $\tilde{L}$. Moreover, let us define the real number

$$l_L = \frac{\log \frac{L}{s_0}}{2\sigma\sqrt{h}} + \frac{n}{2}.$$

So, when $\tilde{L}$ is a terminal stock price, i.e.

$$\tilde{L} = S_{n,j_L} = s_0 u^{j_L} d^{n-j_L} \leq L < s_0 u^{j_L} d^{n-1-j_L} = S_{n-1,j_L} = \tilde{L}u, \tag{3.1.5}$$

we have that

$$j_L = \frac{1}{2}\lfloor 2l_L \rfloor;$$

instead, when the effective barrier $\tilde{L}$ is a stock price from a penultimate period, i.e.

$$\tilde{L} = S_{n-1,j_L} = s_0 u^{j_L} d^{n-1-j_L} \leq L < s_0 u^{j_L+1} d^{n-1-j_L} = S_{n,j_L+1} = \tilde{L}u, \tag{3.1.6}$$

it happens that

$$j_L = \frac{1}{2}\lfloor 2l_L \rfloor - \frac{1}{2}.$$

Now, if we define

$$\tilde{j_L} = \frac{1}{2}\lfloor 2l_L \rfloor$$

then

$$j_L = \tilde{j}_L - \frac{1}{2}(1-\epsilon_n),$$

where

$$\epsilon_n = \begin{cases} 0, & \text{if the effective barrier is not a terminal stock price,} \\ 1, & \text{if the effective barrier is a terminal stock price,} \end{cases}$$

and, moreover, the effective barrier $\tilde{L}$ can be written as

$$\tilde{L} = s_0 u^{\tilde{j}_L} d^{n-\tilde{j}_L}.$$

So, we define

$$\Delta_n^L = \text{frac}(2l_L)$$

and then in the log-space we have

$$\log L = (1-\Delta_n^L)\log \tilde{L} + \Delta_n^L \log(\tilde{L}u). \tag{3.1.7}$$

Then $\Delta_n^L \in [0,1]$ measures in the log-scale the position of $L$ in relation to two adjacent stock prices, one of which is a node at maturity and the other is a node of the first time before maturity. In the special cases in which $\Delta_n^L = 0$ and $\Delta_n^L = 1$ we get that the effective barrier $\tilde{L}$ lies exactly on a node of the tree (see (3.1.5) and (3.1.6) to understand if the node is a node at maturity or a node of a period before maturity).

**Price of down-and-in digital call options with $L < K$**

We now introduce some notations as in Reimer and Sandmann ([70], 1995).
We first recall that the effective barrier is

$$\tilde{L} = S_{n,\tilde{j}_L} = s_0 u^{\tilde{j}_L} d^{n-\tilde{j}_L}$$

and that the effective strike price is

$$S_{n,j_K} = s_0 u^{j_k} d^{n-j_K},$$

where $j_K$ is the first integer such that $K \leq S_{n,j_K}$, $\tilde{j}_L = \frac{1}{2}\lfloor 2l_L \rfloor$ and, as usual, $u = e^{\sigma\sqrt{h}} = d^{-1}$, with $h = \frac{T}{n}$.
We denote with $(S_{jh}^n)_{j=0,1,\dots,n}$ the discrete approximation of the process $S$. Let us suppose to fix $j$, with $j \in \{0,\dots,n\}$. We define $\pi_d(n,j,\tilde{j}_L)$ as the price at time 0 of a security which pays on unit at time $T$ if the asset price at the time step $n$ is equal to $S_{n,j} = s_0 u^j d^{n-j}$ and if there exists a pair $(i,l)$ with $i \in \{0,\dots,n\}$ and $l \in \{0,\dots i\}$, such that $S_{i,l} = s_0 u^l d^{i-l} \leq \tilde{L}$, and otherwise nothing, i.e.

$$\begin{aligned} \pi_d(n,j,\tilde{j}_L) &= e^{-rT}\mathbb{E}[\mathbb{1}_{S_T^n = S_{n,j}} \cdot \mathbb{1}_{\exists\, i\leq n, \exists\, l \leq i: S_{i,l} = s_0 u^l d^{i-l} \leq \tilde{L}}] \\ &= e^{-rT}\mathbb{P}(S_T^n = S_{n,j}; \exists\, i \leq n, \exists\, l \leq i : S_{i,l} \leq \tilde{L}). \end{aligned} \tag{3.1.8}$$

In order to calculate (3.1.8), we first need to count the number of paths $Z_d(n, j, \tilde{j}_L)$ in the binomial tree which reach the terminal stock price $S_{n,j}$ after touching or passing through the effective barrier $\tilde{L}$. The reflection principle (see Feller ([33]), 1968)) yields the number $Z_d(n, j, \tilde{j}_L)$ that for every $j = 0, ..., n$ is equal to

$$Z_d(n, j, \tilde{j}_L) = \begin{cases} \binom{n}{j}, & \text{if} \quad j \leq \tilde{j}_L, \\ \binom{n}{2\tilde{j}_L - j}, & \text{if} \quad \tilde{j}_L < j \leq 2\tilde{j}_L, \\ 0, & \text{if} \quad j > 2\tilde{j}_L. \end{cases} \tag{3.1.9}$$

We briefly recall the proof of (3.1.9).
In the case in which $j \leq \tilde{j}_L$, then $S_{n,j} = s_0 u^j d^{n-j} \leq \tilde{L} = s_0 u^{\tilde{j}_L} d^{n-\tilde{j}_L}$, so all the paths from $s_0$ to $S_{n,j}$ must contact $\tilde{L}$, so that $Z_d(n, j, \tilde{j}_L) = \binom{n}{j}$.
When $\tilde{j}_L < j \leq 2\tilde{j}_L$, then $S_{n,j} = s_0 u^j d^{n-j} > \tilde{L} = s_0 u^{\tilde{j}_L} d^{n-\tilde{j}_L}$. For the reflection principle the number of paths from $s_0$ to $S_{n,j}$ that touch $\tilde{L}$ are equal to the number of paths from $s_0$ to $s_0 u^{\tilde{j}_L - (j - \tilde{j}_L)} d^{n - \tilde{j}_L - (n - j - n + \tilde{j}_L)} = s_0 u^{2\tilde{j}_L - j} d^{n - (2\tilde{j}_L - j)}$, so that $Z_d(n, j, \tilde{j}_L) = \binom{n}{2\tilde{j}_L - j}$.
Finally let us consider the case $j > 2\tilde{j}_L$. First to get from $s_0$ to the barrier $\tilde{L} = s_0 u^{\tilde{j}_L} d^{n-\tilde{j}_L} = s_0 d^{n-2\tilde{j}_L}$ we need $n - 2\tilde{j}_L$ down steps, secondly from $\tilde{L} = s_0 u^{\tilde{j}_L} d^{n-\tilde{j}_L} = s_0 u^{2\tilde{j}_L - n}$ to $S_{n,j} = s_0 u^j d^{n-j} = s_0 u^{2j-n}$ we need $2j - n - (2\tilde{j}_L - n) = 2(j - \tilde{j}_L)$ up steps. So the total number of steps required is at least $n - 2\tilde{j}_L + 2(j - \tilde{j}_L) = n + 2(j - 2\tilde{j}_L) > n$, thus no path hits $\tilde{L}$ before reaching $S_{n,j}$ and so $Z_d(n, j, \tilde{j}_L) = 0$.
Then, by using (3.1.9) we get that

$$\pi_d(n, j, \tilde{j}_L) = \begin{cases} e^{-rT}\binom{n}{j} p^j (1-p)^{n-j}, & \text{if} \quad j \leq \tilde{j}_L, \\ e^{-rT}\binom{n}{2\tilde{j}_L - j} p^j (1-p)^{n-j}, & \text{if} \quad \tilde{j}_L < j \leq 2\tilde{j}_L, \\ 0, & \text{if} \quad j > 2\tilde{j}_L. \end{cases} \tag{3.1.10}$$

We can now prove the following result:

**Proposition 3.1.4.** *The binomial price $C_{di-digital}(s_0, K, T, L, n)$ of a down-and-in digital call option with barrier $L < K < s_0$ is equal to*

$$C_{di-digital}(s_0, K, T, L, n) = e^{-rT} \sum_{i=j_K}^{2\tilde{j}_L} \binom{n}{2\tilde{j}_L - i} p^i (1-p)^{n-i}. \tag{3.1.11}$$

*Proof.* Let us denote with $G(S_{n,j})$ the payoff of a digital call option at node $S_{n,j}$, i.e.

$$G(S_{n,j}) = \begin{cases} 1, & \text{if} \quad S_{n,j} \geq K, \\ 0, & \text{if} \quad S_{n,j} < K. \end{cases} \tag{3.1.12}$$

So the price at time 0 of a down-and-in digital call option is equal to

$$
\begin{aligned}
C_{di-digital}(s_0,K,T,L,n) &= e^{-rT}\mathbb{E}[G(S_T^n)\cdot \mathbb{1}_{\exists\, i\le n, \exists\, l\le i: S_{i,l}=s_0u^ld^{i-l}\le \tilde{L}}] \\
&= e^{-rT}\sum_{j=0}^{n}\mathbb{E}[G(S_T^n)\mathbb{1}_{S_T^n=S_{n,j}}\mathbb{1}_{\exists\, i\le n, \exists\, l\le i: S_{i,l}=s_0u^ld^{i-l}\le \tilde{L}}] = \sum_{j=j_K}^{n}\pi_d(n,j,\tilde{j}_L) \\
&= e^{-rT}\sum_{j=j_K}^{n}\left[\binom{n}{j}p^j(1-p)^{n-j}\mathbb{1}_{j\le\tilde{j}_L} + \binom{n}{2\tilde{j}_L-j}p^j(1-p)^{n-j}\mathbb{1}_{\tilde{j}_L<j\le 2\tilde{j}_L}\right] \\
&= e^{-rT}\sum_{j=j_K+1}^{2\tilde{j}_L}\binom{n}{2\tilde{j}_L-j}p^j(1-p)^{n-j}, \qquad (3.1.13)
\end{aligned}
$$

where the last equality comes from the fact that here we suppose $L<K$ (i.e. $\tilde{j}_L<j_K$) and as a consequence one has that the contribution due to the first sum vanishes. So the proof is complete. □

### Price of a down-and-out digital call option with $L>K$

We derive the price of a down-and-out digital call option with $L>K$, that we call $C_{do-digital}(s_0,K,T,L,n)$, as the difference between the binomial price of the vanilla digital call option and the binomial price of the down-and-in digital call option with $L>K$.
Let us start from the binomial price of the vanilla digital call option, that we denote with $C_{digital}(s_0,K,T,n)$. Let us call with $\pi(n,j)$ the price at time 0 of a security which pays one unit at time $T$ if the asset price is equal to $S_{n,j}=s_0u^jd^{n-j}$ and otherwise nothing, i.e.

$$
\pi(n,j) = e^{-rT}\mathbb{E}[\mathbb{1}_{S_T^n=S_{n,j}}] = e^{-rT}\binom{n}{j}p^j(1-p)^{n-j}.
$$

We denote as before with $G(S_{n,j})$ the payoff of a digital call option at node $S_{n,j}$, see (3.1.12). So the price at time 0 of a vanilla digital call option is equal to

$$
\begin{aligned}
C_{digital}(s_0,K,T,n) &= e^{-rT}\mathbb{E}[G(S_T^n)] = e^{-rT}\sum_{j=0}^{n}\mathbb{E}[G(S_T^n)\mathbb{1}_{S_T^n=S_{n,j}}] \\
&= \sum_{j=j_K}^{n}\pi(n,j) = e^{-rT}\left[\sum_{j=j_K}^{n}\binom{n}{j}p^j(1-p)^{n-j}\right].
\end{aligned}
\qquad (3.1.14)
$$

Let us now consider the price of a down-and-in digital call option with $L>K$. From the proof of Proposition 3.1.4 we know that the binomial price of a down-and-in digital call option with $L > K$ (i.e. $\tilde{j}_L > j_K$) can be written as

$$C_{di-digital}(s_0, K, T, L, n)$$
$$= e^{-rT}\left[\sum_{j=j_K}^{\tilde{j}_L}\binom{n}{j}p^j(1-p)^{n-j} + \sum_{j=\tilde{j}_L+1}^{2\tilde{j}_L}\binom{n}{2\tilde{j}_L - j}p^j(1-p)^{n-j}\right]. \tag{3.1.15}$$

We can now state the following result:

**Proposition 3.1.5.** *The binomial price $C_{do-digital}(s_0, K, T, L, n)$ of a down-and-out digital call option with barrier $L > K$ is equal to*

$$C_{do-digital}(s_0, K, T, L, n) = e^{-rT}\left[\sum_{i=\tilde{j}_L+1}^{n}\binom{n}{i}p^i(1-p)^{n-i} - \sum_{i=\tilde{j}_L+1}^{2\tilde{j}_L}\binom{n}{2\tilde{j}_L - i}p^i(1-p)^{n-i}\right].$$

*Proof.* The price $C_{do-digital}(s_0, K, T, L, n)$ of a down-and-in digital call option with $L > K$ is then obtained by subtracting the price $C_{digital}(s_0, K, T, L)$ given in (3.1.14) to the price $C_{di-digital}(s_0, K, T, L, n)$ given in (3.1.15).

□

### 3.1.3 Binomial error for digital call options with a single barrier

We now give the explicit coefficients of $\frac{1}{\sqrt{n}}$ and $\frac{1}{n}$ in the asymptotic expansion of the binomial error for the price of digital call options with barrier $L$. The idea is to use the closed-form formulas of the binomial prices given in Proposition 3.1.4 and Proposition 3.1.5 and then approximate them by using Lemma 4.1 in Lin and Palmer ([60], 2013) that is:

**Lemma 3.1.6.** *Let $p_n$ be given by*

$$p_n = \frac{1}{2} + \frac{\alpha}{\sqrt{n}} + \frac{\beta}{n^{3/2}} + O\left(\frac{1}{n^{5/2}}\right),$$

*for some coefficients $\alpha$ and $\beta$ and let $j_n$ be*

$$j_n = \frac{1-b_n}{2} + \gamma\sqrt{n} + \frac{n}{2},$$

*where $b_n$ is bounded. Then*

$$\sum_{k=j_n}^{n}\binom{n}{k}p_n^k(1-p_n)^{n-k} = \Phi(\overline{d}) + \frac{e^{-\frac{\overline{d}^2}{2}}}{\sqrt{2\pi}}\left(\frac{b_n}{\sqrt{n}} + \frac{g - \overline{d}b_n^2/2}{n}\right) + O\left(\frac{1}{n^{3/2}}\right),$$

*where*

$$\overline{d} = 2(\alpha - \gamma) \quad \text{and} \quad g = 2(\beta + \alpha^2 \overline{d}) + (2\alpha/3 - \overline{d}/12)(1 - \overline{d}^2).$$

The above Lemma is a generalization due to Lin and Palmer ([62], 2013) of a result of Uspensky ([80], 1937) on the approximation of the binomial distribution by the normal one. The principal idea is to rewrite the binomial formulas in a form such that Lemma 3.1.6 can be applied.

### Down-and-in and down-and-out digital call options with $L < K$

**Theorem 3.1.7.** *In the n-period CRR binomial model, the binomial error Err(n) for the prices of European digital call options with barrier $L < K$ is:*

- *for a down-and-in digital call option:*

$$Err(n) = e^{-rT}\Bigg[(\tilde{A}_1 \Delta_n^K + \tilde{A}_2 \Delta_n^L)\frac{1}{\sqrt{n}} + (\tilde{B}_1 + \tilde{B}_2(\Delta_n^K)^2 + \tilde{B}_3 \Delta_n^K \Delta_n^L + \tilde{B}_4(\Delta_n^L)^2)\frac{1}{n}\Bigg] + O\left(\frac{1}{n^{3/2}}\right);$$

- *for a down-and-out digital call option:*

$$Err(n) = e^{-rT}\Bigg[(\tilde{C}_1 \Delta_n^K + \tilde{C}_2 \Delta_n^L)\frac{1}{\sqrt{n}} + (\tilde{D}_1 + \tilde{D}_2(\Delta_n^K)^2 + \tilde{D}_3 \Delta_n^K \Delta_n^L + \tilde{D}_4(\Delta_n^L)^2)\frac{1}{n}\Bigg] + O\left(\frac{1}{n^{3/2}}\right),$$

*with*

$$\tilde{A}_1 = \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}} \frac{e^{-\frac{d_{22}^2}{2}}}{\sqrt{2\pi}},$$
$$\tilde{A}_2 = -2\tilde{A}_1 - 4\alpha\sqrt{T}\Phi(d_{22})\left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}},$$
$$\tilde{B}_1 = \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}} \left[g_2 \frac{e^{-\frac{d_{22}^2}{2}}}{\sqrt{2\pi}} - I\Phi(d_{22})\right],$$
$$\tilde{B}_2 = \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}} \left(-\frac{d_{22}}{2}\right) \frac{e^{-\frac{d_{22}^2}{2}}}{\sqrt{2\pi}},$$
$$\tilde{B}_3 = \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}} \frac{e^{-\frac{d_{22}^2}{2}}}{\sqrt{2\pi}}[2d_{22} - 4\alpha\sqrt{T}],$$
$$\tilde{B}_4 = \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}} \left[\frac{e^{-\frac{d_{22}^2}{2}}}{\sqrt{2\pi}}(-2d_{22} + 8\alpha\sqrt{T}) + 8\alpha^2 T\Phi(d_{22})\right],$$
$$c_1 = \frac{e^{-\frac{d_{12}^2}{2}}}{\sqrt{2\pi}},\ c_2 = -\frac{d_{12}}{2}\frac{e^{-\frac{d_{12}^2}{2}}}{\sqrt{2\pi}},$$
$$\tilde{c} = \frac{d_{11}^3 + d_{11}d_{12}^2 + 2d_{12} - 4d_{11}}{24} + \frac{(2 - d_{11}d_{12} - d_{11}^2)\sqrt{T}}{6\sigma}r + \frac{Td_{11}}{2\sigma^2}r^2,$$
$$c_3 = \tilde{c}\frac{e^{-\frac{d_{12}^2}{2}}}{\sqrt{2\pi}},$$
$$\tilde{C}_1 = c_1 - \tilde{A}_1,\ \tilde{C}_2 = -\tilde{A}_2,$$
$$\tilde{D}_1 = c_2 - \tilde{B}_1,\ \tilde{D}_2 = c_3 - \tilde{B}_2,$$
$$\tilde{D}_3 = -\tilde{B}_3,\ \tilde{D}_4 = -\tilde{B}_4.$$

*The list of the other constants can be found in the statement of Theorem 1.2.15.*

*Proof.* Let us consider first the asymptotic expansion for the binomial price of the down-and-in digital call option given in Proposition 3.1.4, that is

$$C_{di-digital}(s_0, K, T, L, n) = e^{-rT} \sum_{i=j_K}^{2\tilde{j}_L} \binom{n}{2\tilde{j}_L - i} p^i (1-p)^{n-i}. \tag{3.1.16}$$

From equation (5.1) in Lin and Palmer ([62],2013) we can write (3.1.16) as follows

$$C_{di-digital}(s_0, K, T, L, n) = e^{-rT}\left(\frac{1-p}{p}\right)^{n-2\tilde{j}_L} \sum_{i=0}^{2\tilde{j}_L - j_K} \binom{n}{i} p^{n-i}(1-p)^i. \tag{3.1.17}$$

From equation (5.7) in [60] we know that

$$\left(\frac{1-p}{p}\right)^{n-2\tilde{j}_L} = a_1 + a_2\frac{1}{\sqrt{n}} + a_3\frac{1}{n} + O\left(\frac{1}{n^{3/2}}\right), \tag{3.1.18}$$

with

$$a_1 = \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}}, \; a_2 = -4\alpha\sqrt{T}\Delta_n^L\left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}},$$
$$a_3 = (-I + 8\alpha^2 T(\Delta_n^L)^2)\left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}};$$

moreover, from equation (5.3) in [62] we have that

$$\sum_{i=0}^{2\tilde{j}_L - j_K} \binom{n}{i} p^{n-i}(1-p)^i = b_1 + b_2\frac{1}{\sqrt{n}} + b_3\frac{1}{n} + O\left(\frac{1}{n^{3/2}}\right), \tag{3.1.19}$$

with

$$b_1 = \Phi(d_{22}), \; b_2 = \frac{e^{-\frac{d_{22}^2}{2}}}{\sqrt{2\pi}}(\Delta_n^K - 2\Delta_n^L),$$
$$b_3 = \frac{e^{-\frac{d_{22}^2}{2}}}{\sqrt{2\pi}}\left(g_2 - \frac{d_{22}}{2}(-\Delta_n^K + 2\Delta_n^L)^2\right).$$

We stress that the asymptotic expansion (3.1.19) is obtained by applying Lemma 3.1.6 to

$$\sum_{i=2\tilde{j}_L - j_K + 1}^{n} \binom{n}{i} p^{n-i}(1-p)^i,$$

where $2\tilde{j}_L - j_K + 1 = \frac{1}{2}(1 + \Delta_n^K - 2\Delta_n^L) + (-\hat{\alpha}\sqrt{T} + d_{21}/2)\sqrt{n} + \frac{n}{2}$.
In order to obtain an asymptotic expansion for (3.1.16) we combine together (3.1.18) and

(3.1.19), i.e.

$$\begin{aligned}
&C_{di-digital}(s_0, K, T, L, n) \\
&= e^{-rT}\left[a_1b_1 + (a_1b_2 + a_2b_1)\frac{1}{\sqrt{n}} + (a_1b_3 + a_2b_2 + a_3b_1)\frac{1}{n}\right] + O\left(\frac{1}{n^{3/2}}\right) \\
&= C^{BS}_{di-digital}(s_0, K, T, L) + e^{-rT}\left[(\tilde{A}_1\Delta_n^K + \tilde{A}_2\Delta_n^L)\frac{1}{\sqrt{n}}\right. \\
&\left. + (\tilde{B}_1 + \tilde{B}_2(\Delta_n^K)^2 + \tilde{B}_3\Delta_n^K\Delta_n^L + \tilde{B}_4(\Delta_n^L)^2)\frac{1}{n}\right] + O\left(\frac{1}{n^{3/2}}\right),
\end{aligned}$$

with

$$\begin{aligned}
\tilde{A}_1 &= \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}} \frac{e^{-\frac{d_{22}^2}{2}}}{\sqrt{2\pi}}, \\
\tilde{A}_2 &= -2\tilde{A}_1 - 4\alpha\sqrt{T}\Phi(d_{22})\left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}}, \\
\tilde{B}_1 &= \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}} \left[g_2\frac{e^{-\frac{d_{22}^2}{2}}}{\sqrt{2\pi}} - I\Phi(d_{22})\right], \\
\tilde{B}_2 &= \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}} \left(-\frac{d_{22}}{2}\right)\frac{e^{-\frac{d_{22}^2}{2}}}{\sqrt{2\pi}}, \\
\tilde{B}_3 &= \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}} \frac{e^{-\frac{d_{22}^2}{2}}}{\sqrt{2\pi}}[2d_{22} - 4\alpha\sqrt{T}], \\
\tilde{B}_4 &= \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}} \left[\frac{e^{-\frac{d_{22}^2}{2}}}{\sqrt{2\pi}}(-2d_{22} + 8\alpha\sqrt{T}) + 8\alpha^2T\Phi(d_{22})\right].
\end{aligned}$$

Let us now consider the asymptotic expansion for the down-and-out digital call option. We observe that it can be derived from the asymptotic expansion for the down-and-in option and that for the corresponding vanilla option. From Chang and Palmer ([20], 2007), we know that the asymptotic expansion for the vanilla option is the following:

$$C_{digital}(s_0, K, T, L, n) = C^{BS}_{digital}(s_0, K, T, L) + e^{-rT}\left[c_1\Delta_n^K \frac{1}{\sqrt{n}} + (c_2 + c_3(\Delta_n^K)^2)\frac{1}{n}\right] + O\left(\frac{1}{n^{3/2}}\right),$$

with

$$\begin{aligned}
c_1 &= \frac{e^{-\frac{d_{12}^2}{2}}}{\sqrt{2\pi}},\ c_2 = -\frac{d_{12}}{2}\frac{e^{-\frac{d_{12}^2}{2}}}{\sqrt{2\pi}},\\
\tilde{c} &= \frac{d_{11}^3 + d_{11}d_{12}^2 + 2d_{12} - 4d_{11}}{24} + \frac{(2 - d_{11}d_{12} - d_{11}^2)\sqrt{T}}{6\sigma}r + \frac{Td_{11}}{2\sigma^2}r^2,\\
c_3 &= \tilde{c}\frac{e^{-\frac{d_{12}^2}{2}}}{\sqrt{2\pi}}.
\end{aligned}$$

So, the difference between the asymptotic expansion for the vanilla option and the one for the down-and-in option, gives us the expansion for the down-and-out type option, i.e.:

$$\begin{aligned}
&C_{do-digital}(s_0, K, T, L, n) = C^{BS}_{do-digital}(s_0, K, T, L)\\
&+ e^{-rT}\left[(\tilde{C}_1\Delta_n^K + \tilde{C}_2\Delta_n^L)\frac{1}{\sqrt{n}} + (\tilde{D}_1 + \tilde{D}_2(\Delta_n^K)^2 + \tilde{D}_3\Delta_n^K\Delta_n^L + \tilde{D}_4(\Delta_n^L)^2)\frac{1}{n}\right]\\
&+ O\left(\frac{1}{n^{3/2}}\right),
\end{aligned}$$

with

$$\begin{aligned}
&\tilde{C}_1 = c_1 - \tilde{A}_1,\ \tilde{C}_2 = -\tilde{A}_2,\\
&\tilde{D}_1 = c_2 - \tilde{B}_1,\ \tilde{D}_2 = c_3 - \tilde{B}_2,\\
&\tilde{D}_3 = -\tilde{B}_3,\ \tilde{D}_4 = -\tilde{B}_4
\end{aligned}$$

and the proof is complete. □

**Remark 3.1.8.** *Theorem 3.1.7 shows that the contribution of the type $\frac{1}{\sqrt{n}}$ in the asymptotic expansion is due to the position of both the barrier and the strike price with respect to the nodes of the tree. In order to obtain an algorithm of order $\frac{1}{n}$, we need to set $\Delta_n^K = 0$ and $\Delta_n^L = 0$. It means that in the* log-scale *the strike $K$ must be positioned halfway between two nodes at maturity (i.e. it is a node of the penultimate period before maturity) and the barrier $L$ must lie on a layer of nodes of the tree. So, we can adapt the algorithm described in Chapter 2 for the pricing of double barrier options to this specific case, as explained in Section 3.3.1 of the numerical results, and then get a rate of convergence of order $\frac{1}{n}$.*

**Down-and-in and down-and-out digital call options with $L > K$**

**Theorem 3.1.9.** *In the n-period CRR binomial model, the binomial error Err(n) for the prices of European digital call options with barrier $L > K$ is:*

- *for a down-and-out digital call option:*

$$Err(n) = e^{-rT}\left[(\tilde{E}_1 + \tilde{E}_2\Delta_n^L)\frac{1}{\sqrt{n}} + (\tilde{F}_1 + \tilde{F}_2\Delta_n^L + \tilde{F}_3(\Delta_n^L)^2\frac{1}{n}\right] + O\left(\frac{1}{n^{3/2}}\right);$$

- *for a down-and-in digital call option:*

$$Err(n) = e^{-rT}\left[(\tilde{G}_1 + \tilde{G}_2\Delta_n^K + \tilde{G}_3\Delta_n^L)\frac{1}{\sqrt{n}} + (\tilde{H}_1 + \tilde{H}_2(\Delta_n^K)^2 + \tilde{H}_3\Delta_n^L + \tilde{H}_4(\Delta_n^L)^2)\frac{1}{n}\right] + O\left(\frac{1}{n^{3/2}}\right),$$

*with*

$$\tilde{E}_1 = -\epsilon_n\frac{e^{-\frac{d_{32}^2}{2}}}{\sqrt{2\pi}} + \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}}\frac{e^{-\frac{d_{42}^2}{2}}}{\sqrt{2\pi}},$$

$$\tilde{E}_2 = \frac{e^{-\frac{d_{32}^2}{2}}}{\sqrt{2\pi}} + \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}}\left(\frac{e^{-\frac{d_{42}^2}{2}}}{\sqrt{2\pi}} + 4\alpha\sqrt{\Phi}(d_{42})\right),$$

$$\tilde{F}_1 = \frac{e^{-\frac{d_{32}^2}{2}}}{\sqrt{2\pi}}\left(g_3 - \frac{d_{32}}{2}\epsilon_n^2\right) + \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}}\frac{e^{-\frac{d_{42}^2}{2}}}{\sqrt{2\pi}}\left(\frac{d_{42}}{2}\epsilon_n^2 - g_4\right) + \Phi(d_{42})I\left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}},$$

$$
\begin{aligned}
\tilde{F}_2 &= \frac{e^{-\frac{d_{32}^2}{2}}}{\sqrt{2\pi}} d_{32}\epsilon_n + \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}} \left(\frac{e^{-\frac{d_{42}^2}{2}}}{\sqrt{2\pi}}\epsilon_n d_{42} - 4\epsilon_n \alpha\sqrt{T}\right),\\
\tilde{F}_3 &= -\frac{d_{32}}{2}\frac{e^{-\frac{d_{32}^2}{2}}}{\sqrt{2\pi}} + \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}} \frac{e^{-\frac{d_{42}^2}{2}}}{\sqrt{2\pi}}\left(\frac{d_{42}}{2} - 4\alpha\sqrt{T}\right)\\
&- \left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}} \Phi(d_{42})8\alpha^2 T,\\
\tilde{G}_1 &= -\tilde{E}_1,\ \tilde{G}_2 = c_1, \tilde{G}_3 = -\tilde{E}_2,\\
\tilde{H}_1 &= c_2 - \tilde{F}_1,\ \tilde{H}_2 = c_3,\ \tilde{H}_3 = -\tilde{F}_2,\ \tilde{H}_4 = -\tilde{F}_3.
\end{aligned}
$$

*The other constants can be found in the statement of Theorem 1.2.15.*

*Proof.* By proceeding similarly to the proof of Theorem 3.1.7, we need here to find an asymptotic expansion of the binomial price for a down-and-out digital call found in Proposition 3.1.5. In order to do this we apply (5.7), (5.11) and (5.15) in Lin and Palmer ([62], 2013). The down-and-in case is then obtained by considering the difference of the asymptotic expansion for the vanilla digital call and the down-and-out digital call. The proof is straightforward, so we omit it. □

**Remark 3.1.10.** *As noticed in Lin and Palmer ([60], 2013), the term $\Delta_n^K$ does not appear in the error expansion for the down-and-out option, and the intuitive reason is that in this case $L > K$ and since the option stays alive if the stock price is above $L$, and therefore above $K$, the position of $K$ has no influence. The situation is different in the down-and-in case in which the position of the strike $K$ appears in the coefficient of $\frac{1}{\sqrt{n}}$. In fact in this case the option is activated if the stock price process is below $L$, then since $L > K$ the position of the strike $K$ has influence in the error expansion.*

**Remark 3.1.11.** *From the statements in Theorem* 3.1.9 *we observe that in the error expansion $Err(n)$ for down-and-in and down-and-out digital call options with $L > K$ it is not possible to vanish the contribution of order $\frac{1}{\sqrt{n}}$ by setting $\Delta_n^L = 0$ and $\Delta_n^K = 0$. In fact there is another constant term of order $\frac{1}{\sqrt{n}}$ that can't be nullified. A possibility in order to get an algorithm of order $\frac{1}{n}$ is to set $L$ and $K$ such that $\Delta_n^L = 0 = \Delta_n^K$ and then explicitly calculate the constant coefficient that multiplies $\frac{1}{\sqrt{n}}$ in order to subtract it to the binomial approximated price.*

Theorem 3.1.7 and Theorem 3.1.9 suggest us how to set the barrier $L$ and the strike $K$ in the binomial tree scheme. This theoretical result is enhanced by numerical examples presented in Section 3.3.1. As observed at the beginning of the Chapter, we are not able to extend these results for double barrier options since no manageable binomial closed-form formulas exist. The idea in order to treat this case is to use a different approach that we describe in the following Section.

## 3.2 Double barrier options on discontinuous payoffs

In this Section we study an upper bound for the binomial approximation error in the case of double barrier options on a discontinuous payoff by using the PDE techniques in Gobet ([42], 2001). We stress here that we consider the setup described in Section 1.2.2 that we briefly recall. We suppose as usual that the stock price process $(S_t)_{t\in[0,T]}$ follows the SDE (2.1.1), hence the log-price process that we call $(\overline{S}_t)_{t\in[0,T]}$ satisfies the equation:

$$\overline{S}_t = \overline{S}_0 + \mu t + \sigma B(t), \quad \text{with} \quad \overline{S}_0 = \log s_0 \quad \text{and} \quad \mu = r - \frac{1}{2}\sigma^2.$$

In what follows we consider a European knock-out option with a single lower barrier $L$ on a generic discontinuous payoff function. We remark that in this Section $L$ denotes the barrier in the log-space. So we define $\mathcal{O} = (L, +\infty) \subset \mathbb{R}$ and the stopping time

$$\tau_L = \inf\{t > 0 : \overline{S}_t \notin \mathcal{O}\}. \tag{3.2.1}$$

Then the payoff is equal to

$$\mathbb{1}_{T<\tau_L} f(\overline{S}_T)$$

with $f$ having only one discontinuity point and such that $f$, $f^{'}$ and $f^{''}$ have at most an exponential growth. In particular, in what follows, we consider the case in which the strike $K$ is the only discontinuity point, i.e. $f$ is the payoff of a digital option. We also remark that in this Section $K$ denotes the strike in the log-space and that we assume in what follows that $L < K$.

**Remark 3.2.1.** *Gobet ([42], 2001) considers a general class of continuous payoff functions (for details see Section 1.2.2). A generalization of his result should consider all the discontinuous payoffs with at most a finite number of discontinuity points and such that $f$, $f^{'}$ and $f^{''}$ have at most an exponential growth. But there is no lack of generality if we assume that $f$ has only one discontinuity point equal to $K$, since the discussion can be easily extended to the more general case.*

We consider a binomial approximation $(\overline{S}^n_{t_i})_{i=0,...,n}$, where $t_i = ih$ and $h = T/n$, as described in equation (1.1.13). Moreover, we set

$$\tau^n_L = \inf\{t_i : \overline{S}^n_{t_i} \notin \mathcal{O}\} \tag{3.2.2}$$

and

$$L_n = \sup\{\overline{S}_0 + i\sigma\sqrt{h} \le L : i \in \{0, ..., n\}\}, \tag{3.2.3}$$

i.e. $L_n$ is the first lattice point below the barrier $L$. Moreover, we recall that

$$u(t, x) = \mathbb{E}[\mathbb{1}_{T-t<\tau_L} f(\overline{S}_{T-t})] = \int_{\mathcal{O}} q_{T-t}(x, y) f(y) dy, \tag{3.2.4}$$

where $q_{T-t}(x,y)$ is the transition density at time $T-t$ of the killed process $\overline{S}$ as it leaves $\mathcal{O}$, defined as follows:

$$q_{T-t}(x,y) = q^1_{T-t}(x,y) - e^{\frac{2\mu(L-x)}{\sigma^2}} q^2_{T-t}(x,y),$$

where

$$q^1_{T-t}(x,y) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left(-\frac{(y-x-\mu(T-t))^2}{2\sigma^2(T-t)}\right),$$

$$q^2_{T-t}(x,y) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left(-\frac{(y+x-2L-\mu(T-t))^2}{2\sigma^2(T-t)}\right).$$

We also remember that $u(t,x)$ satisfies the following parabolic PDE of the second order with Cauchy and Dirichlet conditions, i.e.

$$\begin{cases} \partial_t u + Lu = 0, & (t,x) \in [0,T) \times \overline{\mathcal{O}} \\ u(t,x) = 0, & (t,x) \in [0,T) \times \mathcal{O}^c \\ u(T,x) = f(x), & x \in \mathcal{O} \end{cases}$$

where $L$ is the infinitesimal generator: $Lu(x) = \mu u'(x) + \frac{1}{2}\sigma^2 u''(x)$. Our purpose is to find an upper bound for the binomial approximation error that is

$$\text{Err}(n) = \mathbb{E}(\mathbb{1}_{T<\tau^n_L} f(\overline{S}^n_T)) - \mathbb{E}(\mathbb{1}_{T<\tau_L} f(\overline{S}_T)).$$

By using a telescoping sum and rearranging the indicator functions we can write

$$\begin{aligned} \text{Err}(n) &= \sum_{i=0}^{n-1} \mathbb{E}[\mathbb{1}_{t_{i+1}<\tau^n_L} u(t_{i+1}, \overline{S}^n_{t_{i+1}}) - \mathbb{1}_{t_i<\tau^n_L} u(t_i, \overline{S}^n_{t_i})] \\ &= \sum_{i=0}^{n-1} \mathbb{E}[\mathbb{1}_{t_i<\tau^n_L}(u(t_{i+1}, \overline{S}^n_{t_{i+1}}) - u(t_i, \overline{S}^n_{t_i}))] \\ &= \sum_{i=0}^{n-1} A_i + \sum_{i=0}^{n-1} B_i, \end{aligned}$$

where

$$A_i = \mathbb{E}[1_{t_i<\tau^n_L} 1_{\overline{S}^n_{t_i}=L_n+\sigma\sqrt{h}}(u(t_{i+1}, \overline{S}^n_{t_{i+1}}) - u(t_i, \overline{S}^n_{t_i}))], \tag{3.2.5}$$

$$B_i = \mathbb{E}[1_{t_i<\tau^n_L} 1_{\overline{S}^n_{t_i}>L_n+\sigma\sqrt{h}}(u(t_{i+1}, \overline{S}^n_{t_{i+1}}) - u(t_i, \overline{S}^n_{t_i}))]. \tag{3.2.6}$$

Then, the idea is to decompose the analysis of the error $\text{Err}(n)$ by considering the two sums above: $\sum_{i=0}^{n-1} A_i$ refers to the lattice points near the lower barrier $L$ and $\sum_{i=0}^{n-1} B_i$ deals with all the lattice points "far" from the barrier $L$.
Let us consider the two following hypothesis on the payoff function $f$:

**Assumption 3.2.2.** *The payoff function $f$ satisfies:*

$$f \in C_b^2([L,K],\mathbb{R}) \cap C_b^2([K,+\infty),\mathbb{R}) \quad \text{and} \quad f(L)=0.$$

**Assumption 3.2.3.** *The payoff function $f$ satisfies:*

$$f \in C_b^2([L,+\infty),\mathbb{R}) \ s.t. \ f(L) \neq 0.$$

As observed in Remark 1.2.9, if we prove the result for $f$ that satisfies Assumption 3.2.2 and also for $f$ that satisfies Assumption 3.2.3, then this is sufficient for the treatment of the general case in which $f$ has a discontinuity in $K$ and is not vanishing in $L$, as for the digital case. Gobet ([42], 2013) proves the statement with $f$ under Assumption 3.2.3 and this result is given in Theorem 1.2.12. We recall here that he proves under Assumption 3.2.3 that

$$\mathrm{Err}(n) = (L-L_n)\mathbb{E}[\partial_x u(\tau_L, L^+)\mathbb{1}_{\tau_L \leq T}] + \mathcal{R}_n,$$

where $\mathcal{R}_n$ is such that there exists a constant $C>0$: $|\mathcal{R}_n| \leq C\frac{\log n}{n}$ and where the term $\mathbb{E}[\partial_x u(\tau_L, L^+)\mathbb{1}_{\tau_L \leq T}]$ is a finite value. In particular, looking at the proof of Theorem 3.1 in Gobet ([42], 2001), we notice that the contribution in $\mathrm{Err}(n)$ of order $\frac{1}{\sqrt{n}}$ (that is the main term in the error expansion above) derives from the terms $A_i$ defined in (3.2.5), that are related to the lattice points near the barrier $L$. Instead, the terms $B_i$ defined in (3.2.6) have a negligible contribution that is resumed by the term $\mathcal{R}_n$. Our purpose is to find an estimate for the error when $f$ satisfies Assumption 3.2.2, so that we also have an estimate of the error when $f$ is the payoff of a digital option. In order to prove our result we need some preliminary lemmas on the boundary and global estimates of the function $u$ solution of the PDE.

### 3.2.1 Preliminary results

The following two lemmas extend Lemma 3.1 in Gobet ([42], 2001) to the case in which Assumption 3.2.2 holds. The proofs are postponed in Appendix A.

**Lemma 3.2.4.** *Let $f$ be a function such that*

$$f \in C_b^2([L,K],\mathbb{R}) \cap C_b^2([K,+\infty),\mathbb{R}) \quad \text{and} \quad f(L)=0.$$

*Then there exist two positive constants $C$ and $c$ such that the following boundary estimates hold:*

$$\sup_{(t,x)\in[0,T)\times(L,\frac{K+L}{2}]} |\partial_t u(t,x)| + |\partial_x u(t,x)| + |\partial^2_{x,x} u(t,x)| \leq C,$$

$$\sup_{(t,x)\in[0,T)\times(L,\frac{K+L}{2}]} |\partial^2_{t,x} u(t,x)| \leq \frac{C}{\sqrt{T-t}},$$

*with $u$ defined in (3.2.4).*

**Lemma 3.2.5.** *Let $f$ be a function such that*

$$f \in C_b^2([L,K],\mathbb{R}) \cap C_b^2([K,+\infty),\mathbb{R}) \quad \text{and} \quad f(L)=0.$$

*Then there exist two positive constants $C$ and $c$ such that the following global estimates hold:*

$$|\partial^2_{x,t}u(t,x)| + |\partial^3_{x,x,x}u(t,x)| \leq \frac{C}{\sqrt{T-t}}\left(1+\frac{1}{(T-t)}e^{-c\frac{(x-K)^2}{T-t}}\right),$$
$$|\partial^2_{t,t}u(t,x)| + |\partial^3_{x,x,t}u(t,x)| + |\partial^4_{x,x,x,x}u(t,x)|$$
$$\leq \frac{C}{(T-t)}\left(1+\frac{1}{(T-t)}e^{-c\frac{(x-K)^2}{T-t}}\right),$$

*with $u$ defined in (3.2.4).*

**Remark 3.2.6.** *In Lemma 3.1 Gobet ([42], 2001) finds boundary and global estimates when the payoff function $f$ is such that*

$$f \in C^0([L,+\infty),\mathbb{R}) \cap C_b^2([L,K],\mathbb{R}) \cap C_b^2([K,+\infty),\mathbb{R}) \quad \text{and} \quad f(L)=0.$$

*We notice that the boundary estimates he obtains under the continuity assumption of $f$ are exactly the same as the estimates in Lemma 3.2.4. Instead, the global estimates he gets are the following:*

$$|\partial^2_{x,t}u(t,x)| + |\partial^3_{x,x,x}u(t,x)| \leq \frac{C}{\sqrt{T-t}}\left(1+\frac{1}{\sqrt{T-t}}e^{-c\frac{(x-K)^2}{T-t}}\right),$$
$$|\partial^2_{t,t}u(t,x)| + |\partial^3_{x,x,t}u(t,x)| + |\partial^4_{x,x,x,x}u(t,x)|$$
$$\leq \frac{C}{(T-t)}\left(1+\frac{1}{\sqrt{T-t}}e^{-c\frac{(x-K)^2}{T-t}}\right).$$

*We observe that since the global estimates are related on the strike $K$, they are better than the ones we get in Lemma 3.2.5.*

We recall here Lemma 3.1 in Gobet ([42], 2001) that we will use in the following Section:

**Lemma 3.2.7.** *For $c > 0$, one has*

$$\mathbb{E}\left[\exp\left(-c\frac{(\overline{S}_{t_i}-\alpha)^2}{\epsilon}\right)\right] \leq C\left(\frac{1}{\sqrt{i}}+\sqrt{\frac{\epsilon}{\epsilon+t_i}}\right),$$

*for a positive constant $C = C(c,\mu,\sigma,T)$, uniform in $\epsilon, t_i = ih, n, \overline{S}_0$ and $\alpha$.*

### 3.2.2 Upper bound for the error

We state here our result:

**Theorem 3.2.8.** *Let $f$ satisfy Assumption 3.2.2. The error $Err(n)$ in the binomial lattice approximation satisfies:*

$$Err(n) = (L - L_n)\mathbb{E}[\partial_x u(\tau_L, L^+)\mathbb{1}_{\tau_L<T}] + \tilde{\mathcal{R}}_n,$$

*with $\tilde{\mathcal{R}}_n$ such that there exists a constant $C > 0 : |\tilde{\mathcal{R}}_n| \leq \frac{C}{\sqrt{n}}$, $L_n$ defined in (3.2.3) and $\tau_L$ defined in (3.2.1).*

*Proof.* We stress here that for the analysis of the contribution given by the terms $A_i$ we can exactly repeat the arguments used in the proof of Theorem 3.1 in Gobet ([42], 2001), since the boundary estimates found in Lemma 3.2.4 are the same as the boundary estimates given in Lemma 3.1 in Gobet ([42], 2001) under the hypothesis of continuity of the payoff.
The main difference is given by the study of the terms $B_i$: in fact under Assumption 3.2.2 we get worst global estimates than the ones in [42] (see Remark 3.2.6). In fact the global estimates “reflect” the behavior around the discontinuity point that explicitly appears in the exponential term, so it is reasonable that the estimates obtained for a payoff with a discontinuity point are worst than the global estimates under the assumption of continuity of the payoff function.

**Contribution of the terms $A_i$**

We have that

$$A_i = \mathbb{E}[1_{t_i<\tau_L^n} 1_{\overline{S}^n_{t_i}=L_n+\sigma\sqrt{h}}(u(t_{i+1}, \overline{S}^n_{t_{i+1}}) - u(t_i, \overline{S}^n_{t_i}))],$$

so if we define $\hat{A}_i = \mathbb{E}[A_i|\overline{S}^n_{t_i}]$, then

$$A_i = \mathbb{E}[1_{t_i<\tau_L^n} 1_{\overline{S}^n_{t_i}=L_n+\sigma\sqrt{h}}\hat{A}_i].$$

Let us now consider $\hat{A}_i$. We recall here that $p_h$ is the Taylor expansion at the first order as defined in (1.1.15). By using the law of $\overline{S}^n_{t_{i+1}}$ given $\overline{S}^n_{t_i}$ and the fact that $u(t_{i+1}, L_n) = u(t_{i+1}, L)$, then a Taylor expansion around the point $(t_{i+1}, L)$ gives

$$\begin{aligned}
\hat{A}_i &= p_h u(t_{i+1}, L_n + 2\sigma\sqrt{h}) + (1 - p_h)u(t_{i+1}, L_n) - u(t_i, L_n + \sigma\sqrt{h}) \\
&= (1 - p_h)[u(t_{i+1}, L) - u(t_i, L_n + \sigma\sqrt{h})] \\
&+ p_h[u(t_{i+1}, L_n + 2\sigma\sqrt{h}) - u(t_i, L_n + \sigma\sqrt{h})] \\
&= (1 - p_h)[\partial_x u(t_{i+1}, L^+)(L - \sigma\sqrt{h} - L_n) + O(h)] \\
&+ p_h[\partial_x u(t_{i+1}, L^+)(\sigma\sqrt{h}) + O(h)],
\end{aligned}$$

where we remark that the remainder terms of the Taylor expansion are $O(h)$ uniformly in $i$ because of the boundary estimates given in Lemma 3.2.4. Then, by using that $2p_h - 1 = C \cdot O(h)$ we obtain

$$\hat{A}_i = (1 - p_h)\partial_x u(t_{i+1}, L^+)(L - L_n) + O(h),$$

so that

$$A_i = [\partial_x u(t_{i+1}, L^+)(L - L_n) + O(h)]\mathbb{P}(t_{i+1} = \tau_L^n),$$

because

$$(1 - p_h)\mathbb{P}[t_i < \tau_L^n; \overline{S}^n_{t_{i+1}} = L_n + \sigma\sqrt{h}] = \mathbb{P}(t_{i+1} = \tau_L^n).$$

Then we obtain that

$$\sum_{i=0}^{n-1} A_i = (L - L_n)\mathbb{E}[\partial_x u(\tau_L^n, L^+)\mathbb{1}_{\tau_L^n \le T}] + O(h),$$

but $\mathbb{E}[\partial_x u(\tau_L^n, L^+)\mathbb{1}_{\tau_L^n \le T}]$ tends to $\mathbb{E}[\partial_x u(\tau_L, L^+)\mathbb{1}_{\tau_L \le T}]$ as $n \uparrow \infty$, and so we get that

$$\sum_{i=0}^{n-1} A_i = (L - L_n)\mathbb{E}[\partial_x u(\tau_L, L^+)\mathbb{1}_{\tau_L \le T}] + O(h). \tag{3.2.7}$$

### Contribution of the terms $B_i$

We have that

$$B_i = \mathbb{E}[1_{t_i < \tau_L^n} 1_{\overline{S}^n_{t_i} > L_n + \sigma\sqrt{h}}(u(t_{i+1}, \overline{S}^n_{t_{i+1}}) - u(t_i, \overline{S}^n_{t_i}))],$$

so if we define $\hat{B}_i = \mathbb{E}[B_i | \overline{S}^n_{t_i}]$, then

$$B_i = \mathbb{E}[1_{t_i < \tau_L^n} 1_{\overline{S}^n_{t_i} > L_n + \sigma\sqrt{h}} \hat{B}_i].$$

As for the previous case of the terms $\hat{A}_i$, we can make a Taylor expansion of the terms $\hat{B}_i$ and obtain:

$$\begin{aligned}\hat{B}_i =& h^2\left(\mu\partial^2_{t,x}u(t_i,\overline{S}^n_{t_i})+\mu\frac{\sigma^2}{6}\partial^3_{x,x,x}u(t_i,\overline{S}^n_{t_i})\right)+\\ &+p_h\sigma^2h^2\int_{[0,1]^2}ds_1ds_2s_1\partial^3_{t,x,x}u(t_i,\overline{S}^n_{t_i}+s_1s_2\sigma\sqrt{h})+\\ &+(1-p_h)\sigma^2h^2\int_{[0,1]^2}ds_1ds_2s_1\partial^3_{t,x,x}u(t_i,\overline{S}^n_{t_i}-s_1s_2\sigma\sqrt{h})+\\ &+p_hh^2\int_{[0,1]^2}ds_1ds_2s_1\partial^2_{t,t}u(t_i+s_1s_2h,\overline{S}^n_{t_i}+\sigma\sqrt{h})+\\ &+(1-p_h)h^2\int_{[0,1]^2}ds_1ds_2s_1\partial^2_{t,t}u(t_i+s_1s_2h,\overline{S}^n_{t_i}-\sigma\sqrt{h})+\\ &+p_h\sigma^4h^2\int_{[0,1]^2}ds_1ds_2ds_3ds_4s_1^3s_2^2s_3\partial^4_{x,x,x,x}u(t_i,\overline{S}^n_{t_i}+s_1s_2s_3s_4\sigma\sqrt{h})+\\ &+(1-p_h)\sigma^4h^2\int_{[0,1]^2}ds_1ds_2ds_3ds_4s_1^3s_2^2s_3\partial^4_{x,x,x,x}u(t_i,\overline{S}^n_{t_i}-s_1s_2s_3s_4\sigma\sqrt{h}),\end{aligned}$$

(for details see also (3.19) and (3.20) in Gobet ([42], 2001)). By using the global estimates in Lemma 3.2.5 and the expression of $\hat{B}_i$, we get that $\sum_{i=0}^{n-1} B_i$ can be transformed in few sums that are bounded by the following two expressions:

$$S_1(n)=\frac{1}{n^2}\sum_{i=0}^{n-1}\frac{C}{T-t_i}; \tag{3.2.8}$$

$$S_2(n)=\frac{1}{n^2}\sum_{i=0}^{n-1}\frac{C}{(T-t_i)^2}\mathbb{E}\left[\exp\left(-c\frac{(\overline{S}^n_{t_i}-k_n)^2}{T-t_i}\right)\right], \tag{3.2.9}$$

with $|K-k_n|\leq\sigma\sqrt{h}$. Let us consider first the sum (3.2.8). We have that

$$S_1(n)=\frac{1}{n^2}\sum_{i=0}^{n-1}\frac{C}{T-t_i}=\frac{1}{n^2}\sum_{i=0}^{n-1}\frac{C}{\frac{1}{n}(n-i)}=\frac{1}{n}\sum_{i=0}^{n-1}\frac{C}{n-i}=O\left(\frac{\log n}{n}\right).$$

Let us now consider the sum (3.2.9). By considering that the exponential function is bounded when $t_i$ is far enough from maturity $T$ and by using Lemma 3.2.7 we get

$$S_2(n)\leq\frac{1}{n^2}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-1}\frac{C}{(T-t_i)^2}+\frac{1}{n^2}\sum_{i=\lfloor\frac{n}{2}\rfloor}^{n-1}\frac{C}{(T-t_i)^2}\left(\frac{1}{\sqrt{i}}+\sqrt{\frac{T-t_i}{T}}\right).$$

We now study the three terms above separately:

$$\frac{1}{n^2}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-1}\frac{C}{(T-t_i)^2}=\frac{1}{n^2}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-1}\frac{C}{(n-i)^2}=O\left(\frac{1}{n}\right);$$
$$\frac{1}{n^2}\sum_{i=\lfloor\frac{n}{2}\rfloor}^{n-1}\frac{C}{(T-t_i)^2}\frac{1}{\sqrt{i}}=\sum_{i=\lfloor\frac{n}{2}\rfloor}^{n-1}\frac{C}{(n-i)^2\sqrt{i}}=O\left(\frac{1}{\sqrt{n}}\right);$$
$$\frac{1}{n^2}\sum_{i=\lfloor\frac{n}{2}\rfloor}^{n-1}\frac{C}{(T-t_i)^2}\sqrt{\frac{T-t_i}{T}}=\frac{1}{\sqrt{n}}\sum_{i=\lfloor\frac{n}{2}\rfloor}^{n-1}\frac{C}{(n-i)^{3/2}}=O\left(\frac{1}{\sqrt{n}}\right).$$

So we get that

$$\sum_{i=0}^{n-1}B_i\le\tilde{\mathcal{R}}_n,$$

with $\tilde{\mathcal{R}}_n$ such that there exists a constant $C>0:|\tilde{\mathcal{R}}_n|\le\frac{C}{\sqrt{n}}$. The statement now follows. □

**Remark 3.2.9.** *The error term (3.2.7) due to the* $\sum_{i=0}^{n}A_i$ *shows that a part of the contribution of type* $\frac{1}{\sqrt{n}}$ *in the binomial error is due to the position of the barrier with respect to the nodes of the tree. In fact, since* $\mathbb{E}[\partial_x u(\tau_L,L^+)\mathbb{1}_{\tau_L\le T}]$ *is finite, the leading term in (3.2.7) is* $L-L_n$ *that is proportional to* $\frac{1}{\sqrt{n}}$ *because*

$$L-L_n=\sigma\sqrt{h}\left(1-frac\left(\frac{\log s_0-L}{\sigma\sqrt{h}}\right)\right).$$

**Remark 3.2.10.** *Our goal was originally to find in the asymptotic expansion of the binomial approximation error a term in the coefficient of* $\frac{1}{\sqrt{n}}$ *explicitly dependent on the position of the strike* $K$*. In fact, this result is suggested in the more specific case of digital call options from Theorem 3.1.7 and Theorem 3.1.9. However, we find that the contribution* $O(\frac{1}{\sqrt{n}})$ *depends on two factors: on the position of the barrier* $L$ *(and this comes from the analysis of the terms* $A_i$*) and also on the nodes of the tree "far from the barrier" (this comes from the analysis of the terms* $B_i$*). We stress that the contribution of the terms* $B_i$ *is essentially a consequence of the global estimates of Lemma 3.2.5 that in the case of a payoff function with a discontinuity point in* $K$ *are worst than the ones obtained for continuous payoffs. We also remark that the exponential function in the global estimates reflects the behavior around the discontinuity point* $K$ *and this allows us to say that there exist a contribution of* $\frac{1}{\sqrt{n}}$ *in the binomial error due to* $K$*, also if we are not able to write an explicit dependence.*

Now that the result is proved both for $f$ satisfying Assumption 3.2.2 (Theorem 3.2.8) and for $f$ satisfying Assumption 3.2.3 (see Appendix B in Gobet ([42], 2001)), we can say that the result in Theorem 3.2.8 can also be applied to functions $f\in C_b^2$ with at most one discontinuity point in $K$ and not vanishing at $L$.

The advantage of this procedure is that Theorem 3.2.8 can be easily extended to the case of double barrier options. In fact, by proceeding similarly it is possible to prove the following theorem:

**Theorem 3.2.11.** *Let $L$ denote the lower barrier and $H$ the higher barrier. We assume that the payoff function $f$ is such that $f \in C_b^2([L,K],\mathbb{R}) \cap C_b^2([K,H],\mathbb{R})$ and such that $f(L) = 0 = f(H)$. The error Err$(n)$ in the binomial lattice approximation satisfies:*

$$\begin{aligned} Err(n) &= (L - L_n)\mathbb{E}[\partial_x u(\tau_L, L^+)\mathbb{1}_{\tau \le T}\mathbb{1}_{\tau_L < \tau_H}] \\ &\quad + (H - H_n)\mathbb{E}[\partial_x u(\tau_H, H^-)\mathbb{1}_{\tau \le T}\mathbb{1}_{\tau_H < \tau_L}] + \tilde{\mathcal{R}}_n, \end{aligned}$$

*with $\tilde{\mathcal{R}}_n$ such that there exists a constant $C > 0 : |\tilde{\mathcal{R}}_n| \le \frac{C}{\sqrt{n}}$, $L_n$ defined in (3.2.3), $\tau_L$ defined in (3.2.1), $\tau = \tau_H \wedge \tau_L$, $\tau_H = \inf\{t > 0 : \overline{S}_t \le H\}$ and $H_n = \inf\{\overline{S}_0 + i\sigma\sqrt{h} \ge H : i \in \{0, ..., n\}\}$.*

*Proof.* The proof is similar to the one of Theorem 3.2.8, so we omit it. □

We observe that Gobet proves that for $f \in C_b^2([L,H],\mathbb{R})$ and such that $f(L) \ne 0 \ne f(H)$ the binomial approximation error can be written as follows

$$\begin{aligned} \mathrm{Err}(n) &= (L - L_n)\mathbb{E}[\partial_x u(\tau_L, L^+)\mathbb{1}_{\tau \le T}\mathbb{1}_{\tau_L < \tau_H}] \\ &\quad + (H - H_n)\mathbb{E}[\partial_x u(\tau_H, H^-)\mathbb{1}_{\tau \le T}\mathbb{1}_{\tau_H < \tau_L}] + \mathcal{R}_n, \end{aligned} \tag{3.2.10}$$

with $\mathcal{R}_n$ such that there exists a constant $C > 0 : |\mathcal{R}_n| \le C\frac{\log n}{n}$ (see Appendix B in ([42], 2001)). Then, if we combine the result in Theorem 3.2.11 with the result in (3.2.10) we can state that:

**Theorem 3.2.12.** *Let $L$ denote the lower barrier and $H$ the higher barrier. We assume that the payoff function $f$ is such that*

$$f \in C_b^2([L,K],\mathbb{R}) \cap C_b^2([K,H],\mathbb{R}).$$

*The error Err$(n)$ in the binomial lattice approximation satisfies:*

$$\begin{aligned} Err(n) &= (L - L_n)\mathbb{E}[\partial_x u(\tau_L, L^+)\mathbb{1}_{\tau \le T}\mathbb{1}_{\tau_L < \tau_H}] \\ &\quad + (H - H_n)\mathbb{E}[\partial_x u(\tau_H, H^-)\mathbb{1}_{\tau \le T}\mathbb{1}_{\tau_H < \tau_L}] + \tilde{\mathcal{R}}_n, \end{aligned}$$

*with $\tilde{\mathcal{R}}_n$ such that there exists a constant $C > 0 : |\tilde{\mathcal{R}}_n| \le \frac{C}{\sqrt{n}}$, $L_n$ defined in (3.2.3), $\tau_L$ defined in (3.2.1), $\tau = \tau_H \wedge \tau_L$, $\tau_H = \inf\{t > 0 : \overline{S}_t \le H\}$ and $H_n = \inf\{\overline{S}_0 + i\sigma\sqrt{h} \ge H : i \in \{0, ..., n\}\}$.*

*Proof.* The proof is a direct consequence of Remark 1.2.9. □

**Remark 3.2.13.** *We stress again that another advantage of the PDE technique is that Theorem 3.2.11, and then also Theorem 3.2.12, can be extended to the more general case in which the payoff function has more than one discontinuity point.*

## 3.3 Numerical results

### 3.3.1 Single barrier digital options

Theorem 3.1.7 and Theorem 3.1.9 on the asymptotic expansion of the binomial approximation error, suggest that an algorithm of order $\frac{1}{n}$ can be obtained if the lower barrier $L$ lies exactly on a node of the tree and if the strike $K$ is positioned halfway between two nodes at maturity.
In Chapter 2, we introduced a new binomial algorithm, called the Binomial Interpolated Lattice (for details see Section 2.5), that is able to treat efficiently the pricing of options with double barrier $L$ and $H$. The idea is to construct a binomial mesh such that both $L$ and $H$ are set on a layer of nodes of the tree.
Then, here we can adapt the Binomial Interpolated Lattice algorithm such that the lower barrier $L$ is a node of the tree (in particular we set it as a node at maturity) and the strike $K$ is a node of the first period before maturity. In fact we just need to modify the choice of $k$ defined in (2.4.2) and the time step $\Delta T$ defined in (2.4.1) such that the previous two conditions are satisfied. As in the Binomial Interpolated Lattice algorithm the number of time steps of the binomial procedure is set as $n = \lfloor \frac{T}{\Delta T} \rfloor + 2$. Then we provide the price at time 0 by a backward induction and by proceeding through interpolations in time and in space involving some specified prices at times $t_0 = 0$ and $t_2 = 2\Delta T$ (see Section 2.5 for details). Let us denote with $m$ the number of time steps of the CRR binomial approximation and with $\Delta\tau = T/m$. If we denote with $\tilde{k} = \log K$ and with $l = \log L$, then we define $k$ as follows

$$k = \left\lceil \frac{\tilde{k} - l}{2\sigma\sqrt{\Delta\tau}} \right\rceil + \frac{1}{2},$$

so that the new time step $\Delta T$ is

$$\Delta T = \left( \frac{\tilde{k} - l}{2\sigma k} \right)^2.$$

We can then construct a binomial mesh such that $L$ and $K$ are set as suggested by Theorems 3.1.7 and 3.1.9. In the sequel we will call this procedure adapted to the pricing of single barrier digital options "adjusted BIL" algorithm, where BIL stands for Binomial Interpolated Lattice.
We stress here that we will consider the out-type barrier options as in Chapter 2, the case of the in-type barrier options being similar.

**Remark 3.3.1.** *Since we are considering out-type barrier options, in the BIL algorithm we just need to construct a binomial mesh between the barriers L and H because at the barriers the price of the option is set equal to* 0. *In the adjusted BIL algorithm it is not enough to construct a mesh between L and K, so we need to extend it above K, but this is straightforward.*

We present here some numerical results in order to compare the prices for single barrier digital options obtained with the standard CRR algorithm and those obtained with the adjusted BIL algorithm.

### Down-and-out digital call option with $L < K$

We consider the problem of pricing a down-and-out digital call option with lower barrier $L = 60$, strike $K = 100$ and initial stock value equal to $s_0 = 150$. The other parameters are: $r = 0.1$, $\sigma = 0.25$ and $T = 1$. In Figure 3.1 we plot the prices obtained by using the CRR binomial approximation and the true price that is calculated by using the Black and Scholes formula, i.e.

$$C^{BS}_{do-digital} = e^{-rT}\left[\Phi(d_{12}) - \Phi(d_{22})\left(\frac{s_0}{L}\right)^{1-\frac{2r}{\sigma^2}}\right]. \tag{3.3.1}$$

We observe that the binomial price oscillates widely around the true price and this is due both on the position of $L$ and the position of $K$ with respect to the nodes of the tree.

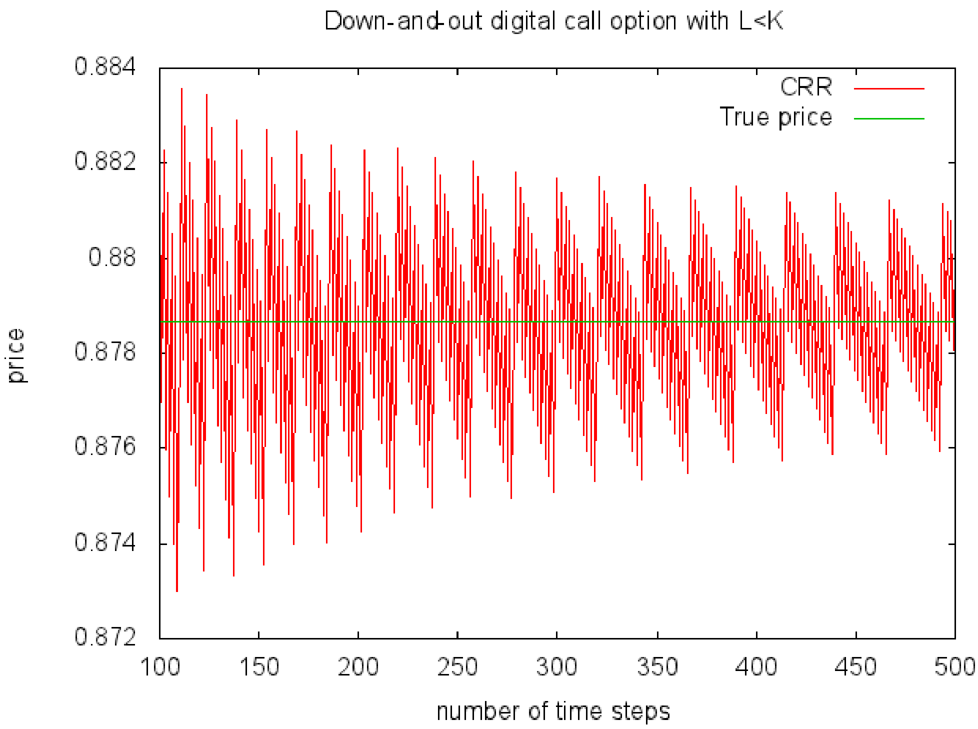


Figure 3.1: *CRR binomial approximation. Knock-out digital call option with* $L = 60, K = 100, s_0 = 150, r = 0.1, \sigma = 0.25$ *and* $T = 1$.

In Figure 3.2 we plot the prices obtained by using the adjusted BIL algorithm in order to match solely the lower barrier $L$, so that we can observe a behavior of order $\frac{1}{\sqrt{n}}$ due to the position of the strike K. We stress that we construct the binomial mesh as described in Remark 3.3.1, but with the trick of actually matching the barrier $L$ and another level different from $K$. We also remark that in the $x$-axis we report the number $m$ of time steps corresponding to the CRR binomial approximation. In fact we recall that in the adjusted BIL algorithm we define a new number of time steps $n$ different from $m$ but having the same order of magnitude. We notice that the period of the oscillations is greater than the one in the CRR binomial approximations, but there are still high peaks around the true price due to position of K.

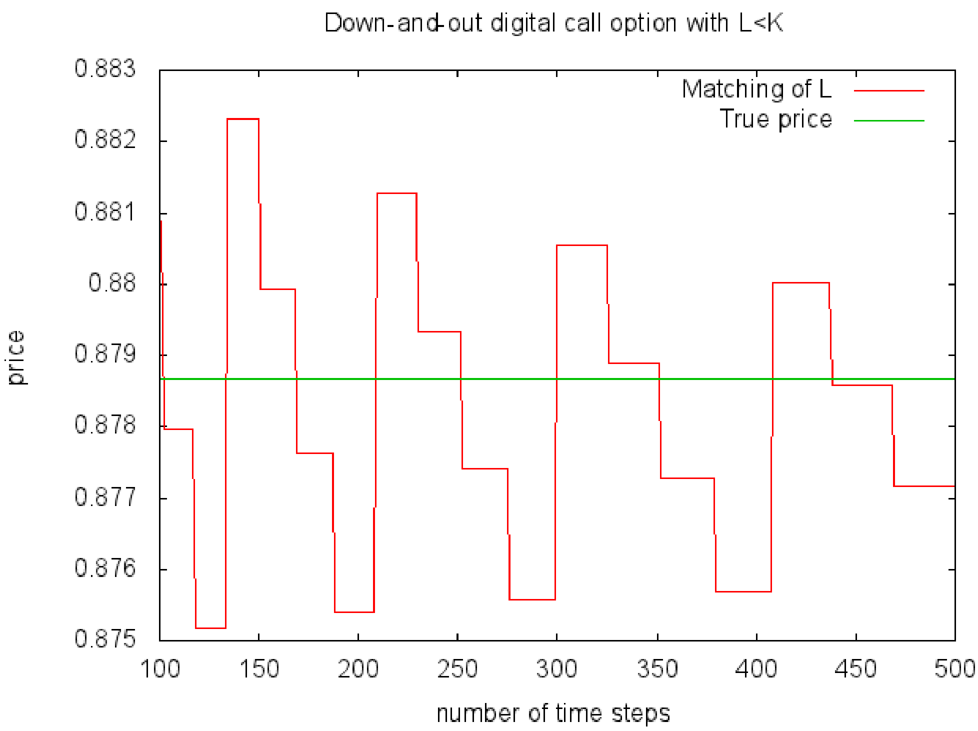


Figure 3.2: *Matching of the lower barrier L. Knock-out digital call option with* $L = 60, K = 100, s_0 = 150, r = 0.1, \sigma = 0.25$ *and* $T = 1$.

In Figure 3.3 we plot the prices obtained by applying the adjusted Binomial Interpolated Lattice as explained at the beginning of this Section. In this figure there are no oscillations and the convergence is $O(1/n)$: the oscillations due $L$ and $K$ disappear and the convergence is monotone.

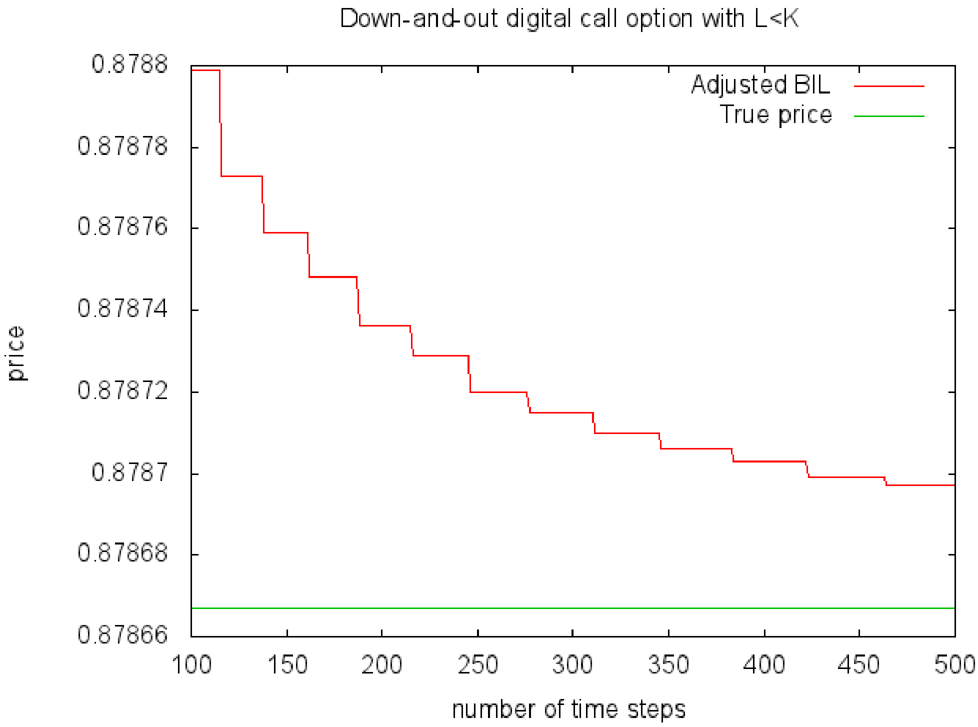


Figure 3.3: *Adjusted binomial interpolated lattice. Knock-out digital call option with* $L = 60$, $K = 100$, $s_0 = 150$, $r = 0.1$, $\sigma = 0.25$ *and* $T = 1$.

In Figure 3.4 we plot together the prices obtained with the CRR binomial approximation in Figure 3.1, the adjusted BIL algorithm in order to match solely the lower barrier $L$ as in Figure 3.2 and the adjusted BIL algorithm for the matching of $L$ and $K$ as in Figure 3.3. We also plot the true price obtained by using (3.3.1). The improvement obtained by setting $L$ and $K$ as suggested in Theorem 3.1.7 is evident.

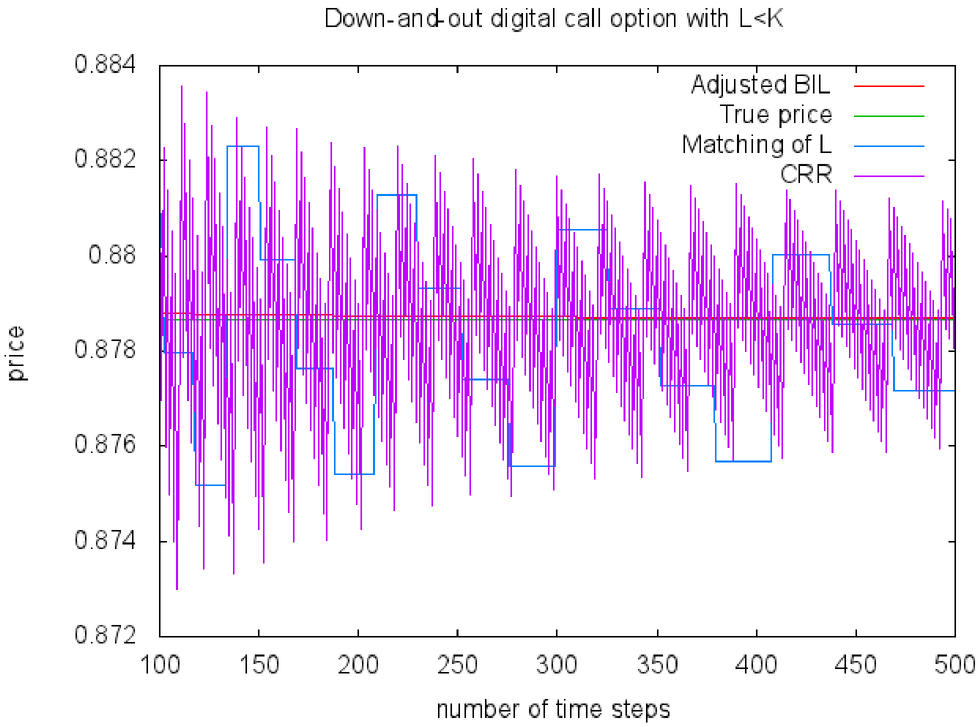


Figure 3.4: *Knock-out digital call option with* $L = 60$*,* $K = 100$*,* $s_0 = 150$*,* $r = 0.1$*,* $\sigma = 0.25$ *and* $T = 1$*.*

In Table 3.1 we report the prices of the down-and-out digital call option with lower barrier $L$ obtained with the CRR algorithm and the adjusted BIL algorithm. In the first column we write the number $m$ of time steps of the CRR binomial approximation. The true price is calculated by using (3.3.1).

| | | $L < K < s_0$ | |
|---|---|---|---|
| m | CRR | True | adjusted BIL |
| 100 | 0.883147 | | 0.878791 |
| 200 | 0.879006 | | 0.878732 |
| 400 | 0.880340 | 0.878667 | 0.878700 |
| 800 | 0.876786 | | 0.878684 |
| 1600 | 0.878863 | | 0.878676 |
| 3200 | 0.877873 | | 0.878671 |

Table 3.1: *Knock-out digital call option prices with* $L = 60$*,* $K = 100$*,* $s_0 = 150$*,* $r = 0.1$*,* $\sigma = 0.25$ *and* $T = 1$*.*

### Down-and-out digital call option with $L > K$

We consider the price of a down-and-out digital call option with strike $K = 60$, lower barrier $L = 100$ and initial stock price $s_0 = 150$. The other parameters are: $r = 0.1$, $\sigma = 0.25$ and $T = 1$. In Figure 3.5 we plot the CRR binomial prices and in Figure 3.4 the adjusted

Binomial Interpolated Lattice prices. We remark that in this case the oscillations of the CRR prices are fewer than the case $L < K$ and this is due to the fact that the position of $K$ has no influence in the error expansion since the option stays alive when the stock price is above $L$ and therefore above $K$. So, the term $O(\frac{1}{\sqrt{n}})$ is only due on the position of $L$, as remarked in Theorem 3.1.9. In Figure 3.6 we plot the prices obtained with the adjusted BIL algorithm and we observe that here the convergence is monotone since we construct the tree such that the lower barrier $L$ lies exactly on a layer of nodes. The true price is obtained by using the Black and Scholes formula (3.1.3).

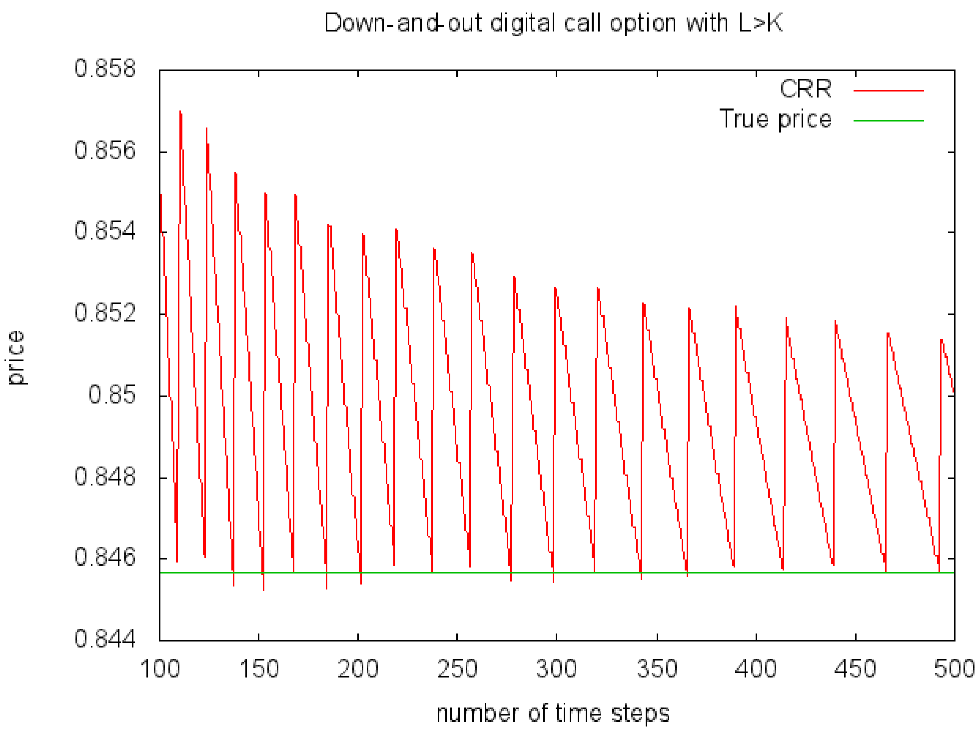


Figure 3.5: *CRR binomial approximation. Knock-out digital call option with* $K = 60$, $L = 100$, $s_0 = 150$, $r = 0.1$, $\sigma = 0.25$ *and* $T = 1$.

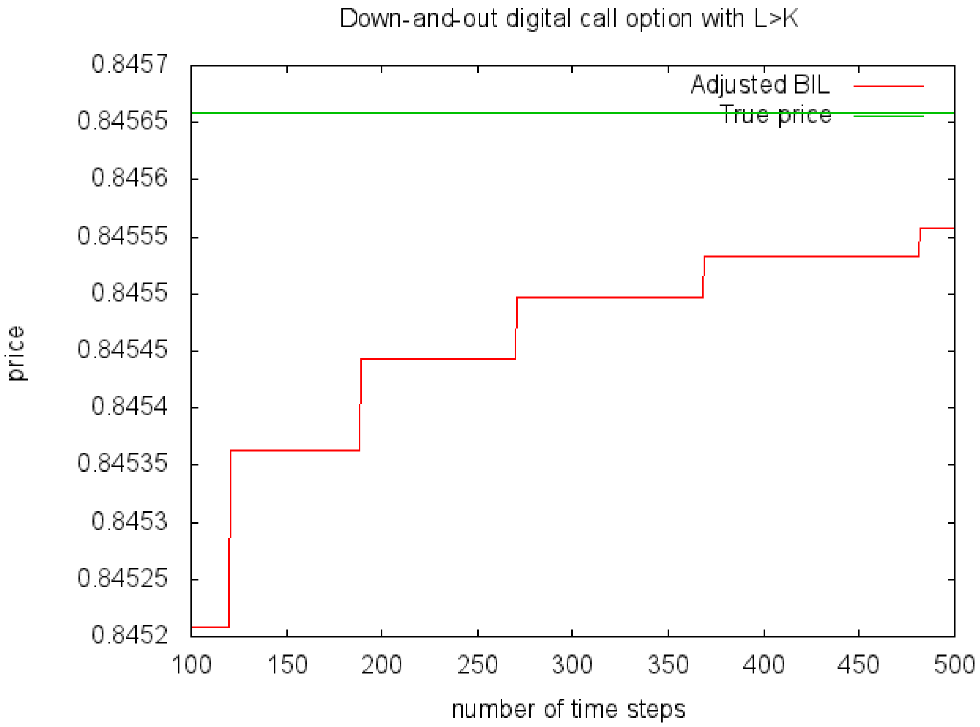


Figure 3.6: *Adjusted binomial interpolated lattice. Knock-out digital call option with* $K = 60$*,* $L = 100$*,* $s_0 = 150$*,* $r = 0.1$*,* $\sigma = 0.25$ *and* $T = 1$*.*

In Table 3.2 we report the prices of the down-and-out digital call option with lower barrier $L > K$ obtained with the CRR algorithm and the adjusted BIL algorithm. As usual, $m$ denotes the number of time steps of the CRR binomial approximation. The true price is calculated by using (3.1.3).

| | | $K < L < s_0$ | |
|---|---|---|---|
| m | CRR | True | adjusted BIL |
| 100 | 0.855913 | | 0.844983 |
| 200 | 0.846415 | | 0.845304 |
| 400 | 0.849497 | | 0.845484 |
| 800 | 0.846188 | 0.845659 | 0.845571 |
| 1600 | 0.846107 | | 0.845615 |
| 3200 | 0.846252 | | 0.845637 |

Table 3.2: *Knock-out digital call option prices with* $K = 60$*,* $L = 100$*,* $s_0 = 150$*,* $r = 0.1$*,* $\sigma = 0.25$ *and* $T = 1$*.*

### 3.3.2 Double barrier digital options

We now present some numerical results on the pricing of double barrier digital call options. From Theorem 3.2.12 we know that the binomial approximation is of order $\frac{1}{\sqrt{n}}$ and this is due both on the position of the barriers (the lower barrier $L$ and the higher barrier $H$) and the position of the strike $K$. We consider for example a double barrier digital call option with lower barrier $L = 60$, $K = 100$, $s_0 = 150$ and $H = 180$. The other parameters are: $r = 0.1$, $\sigma = 0.25$ and $T = 1$. In Figure 3.7 we plot the prices obtained by using the CRR binomial approximation and the true price obtained with the Ikeda and Kunitomo formula ([56], 1992) that gives for the previous parameters a price equal to 0.372300. We observe that the binomial price oscillates with high frequency around the true price and then the convergence is $O(\frac{1}{\sqrt{n}})$.

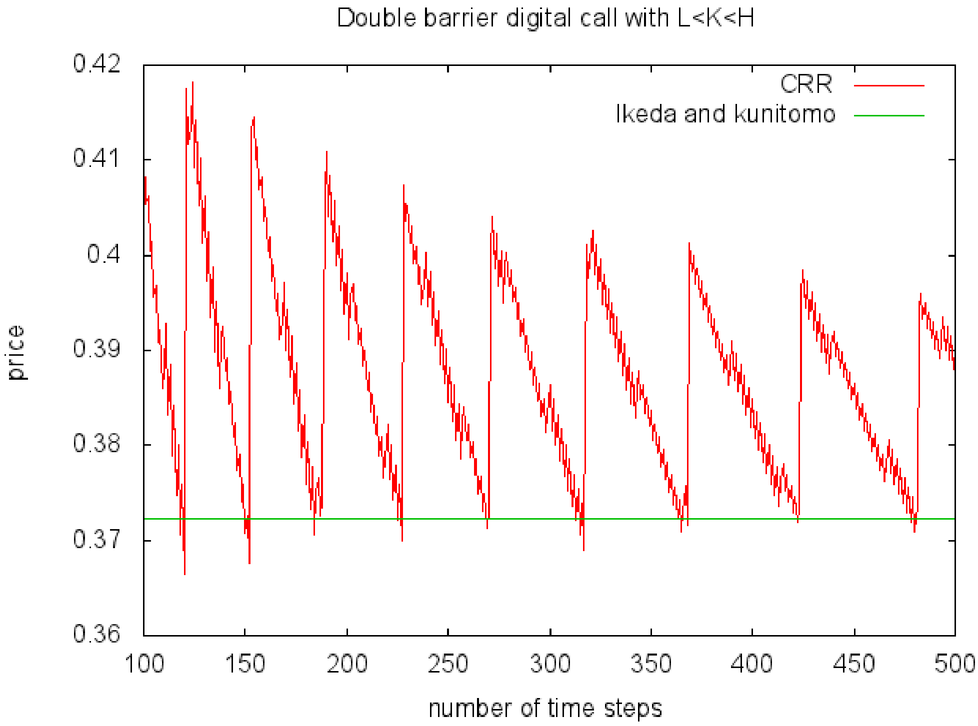


Figure 3.7: *CRR binomial approximation. Knock-out double barrier digital call option with* $K = 100$, $L = 60$, $H = 180$, $s_0 = 150$, $r = 0.1$, $\sigma = 0.25$ *and* $T = 1$.

We now apply the Binomial Interpolated Lattice (BIL) described in Section 2.5 to price a double barrier digital call option with the same parameters as in Figure 3.7. Then we are able to match exactly the barriers $L$ and $H$, so the only contribution of $O(\frac{1}{\sqrt{n}})$ is due to the position of the strike $K$. In Figure 3.8 we plot the prices obtained with the Binomial Interpolated Lattice and the true price calculated with the Ikeda and Kunitomo formula. The result we obtain is interesting: in fact part of the oscillatory behavior disappears (the one due to the barriers $L$ and $H$), but the convergence is still of the type $O(\frac{1}{\sqrt{n}})$. We can explain this effect by looking at the statement of Theorem 3.2.12: the binomial error $\mathrm{Err}(n)$ is bounded from the above by a quantity $\tilde{\mathcal{R}}_n$ that is indeed a $O(\frac{1}{\sqrt{n}})$ and that is related to the position of the strike $K$. However it seems that the oscillations due to $K$ are not so large.

In fact the contribution of order $\frac{1}{\sqrt{n}}$ due to the position of the strike has a lower magnitude than the one caused by the position of the barriers with respect to the nodes of the tree.

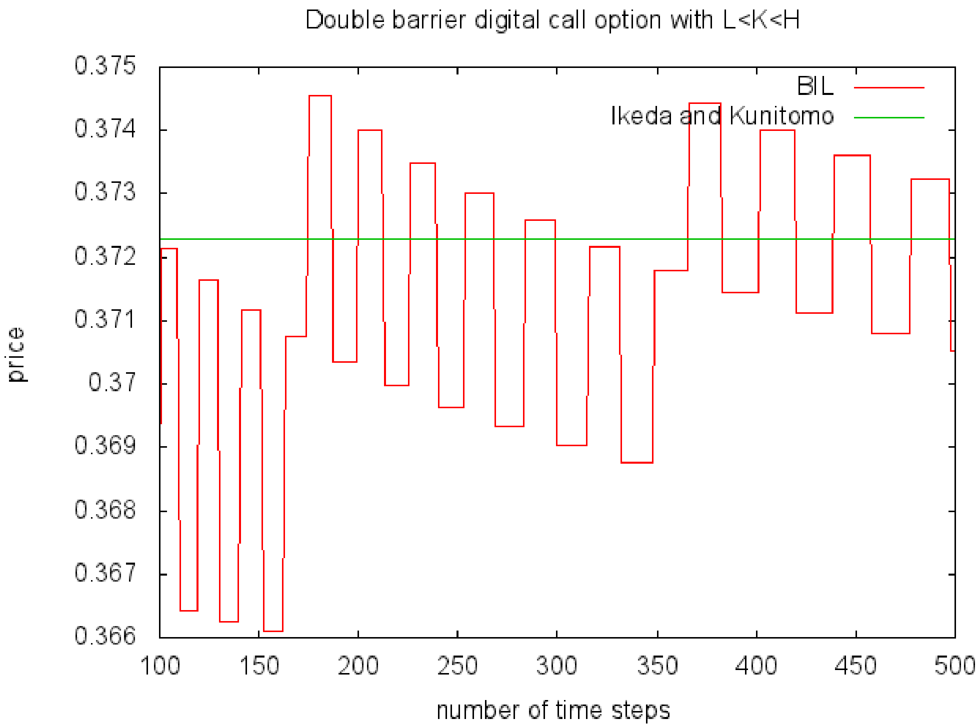


Figure 3.8: *Binomial interpolated lattice. Knock-out double barrier digital call option with* $K = 100,\ L = 60,\ H = 180,\ s_0 = 150,\ r = 0.1,\ \sigma = 0.25$ *and* $T = 1$.

In Figure 3.9 we plot the prices obtained with the adjusted Binomial Interpolated Lattice as described in Section 3.3.1 and the true price calculated with the Ikeda and Kunitomo formula. We observe that here the contributions of order $O(\frac{1}{\sqrt{n}})$ due to the position of the lower barrier $L$ and the strike $K$ vanish, but in the error approximation there is still a component of order $O(\frac{1}{\sqrt{n}})$ due to the position of the higher barrier $H$.

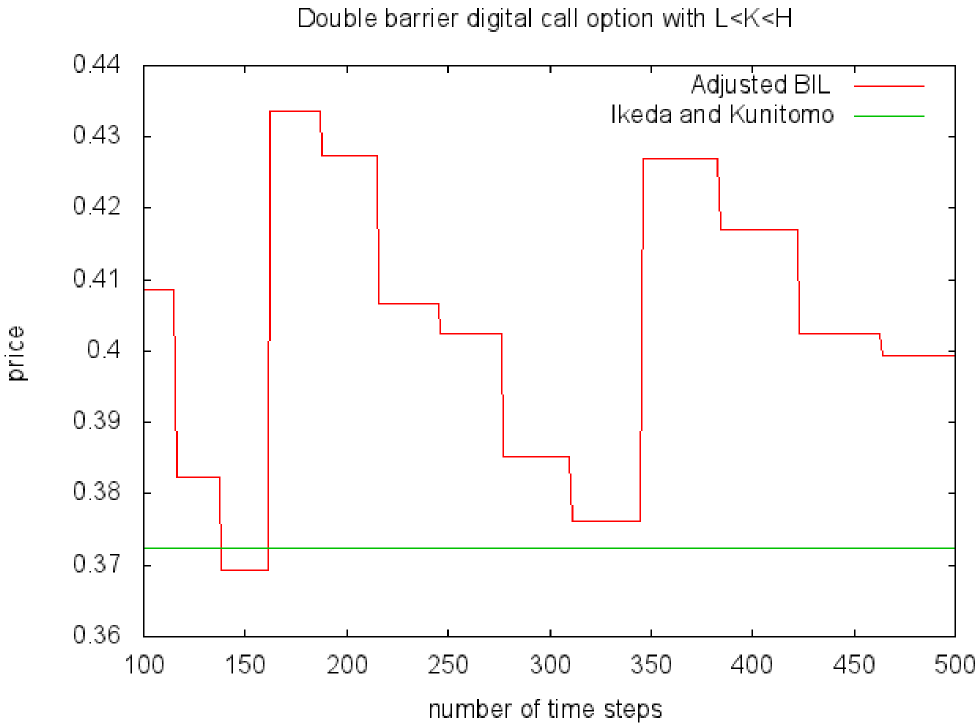


Figure 3.9: *Adjusted binomial interpolated lattice. Knock-out double barrier digital call option with* $K = 100$, $L = 60$, $H = 180$, $s_0 = 150$, $r = 0.1$, $\sigma = 0.25$ *and* $T = 1$.

From Theorem 3.2.11 we know that the contribution of order $O(\frac{1}{\sqrt{n}})$ related to the position of the higher barrier $H$ is given by

$$(H - H_n)\mathbb{E}[(\partial_x u(\tau_H, H^-)\mathbb{1}_{\tau \leq T}\mathbb{1}_{\tau_H < \tau_L}].$$

Once computed the expectation above, following the idea in Section 4 in Gobet ([42], 2001), we can easily improve the adjusted Binomial Interpolated Lattice (which gives the price $aBIL(n)$) to obtain a corrected adjusted Binomial Interpolated Lattice price ($caBIL(n)$) by the following way:

$$caBIL(n) = aBIL(n) + (H_n - H)\mathbb{E}[(\partial_x u(\tau_H, H^-)\mathbb{1}_{\tau \leq T}\mathbb{1}_{\tau_H < \tau_L}].$$

From Proposition 4.1 in [42] we can directly compute the expectation we need, in particular we get that

$$\begin{aligned}&\mathbb{E}[(\partial_x u(\tau_H, H^-)\mathbb{1}_{\tau \leq T}\mathbb{1}_{\tau_H < \tau_L}]\\ &\approx CC_1\sigma\sqrt{T}[-\sigma\sqrt{T}(e^{-\frac{B_2^2}{2}} - e^{-\frac{B_1^2}{2}}) + \mu T\sqrt{2\pi}(\Phi(B_2) - \Phi(B_1))],\end{aligned} \tag{3.3.2}$$

with

$$C = \frac{2}{\sigma^3 T\sqrt{2\pi T}} e^{-\frac{\mu^2 T}{2\sigma^2}}, \quad \mu = r - \frac{1}{2}\sigma^2, C_1 = e^{-\frac{1}{2\sigma^2 T}[4\mu T(\log s_0 - \log H)] - \mu^2 T^2},$$

$$B_1 = \frac{\log K - 2\log H + \log s_0 - \mu T}{\sigma\sqrt{T}}, \; B_2 = \frac{-\log H + \log s_0 - \mu T}{\sigma\sqrt{T}}.$$

As usual, $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$.

**Remark 3.3.2.** *From Proposition 4.1 in [42] we know that the expectation above is indeed a finite sum of terms of type (3.3.2). In practice, to keep only the first term of the sum, that is the one in (3.3.2), leads to a good accuracy as we will see from the numerical results. We also remark that the correction term in (3.3.2) is specific for the case of digital call options. In fact the expectation above depends on the payoff function, so actually the procedure based on the correction term is not really practical.*

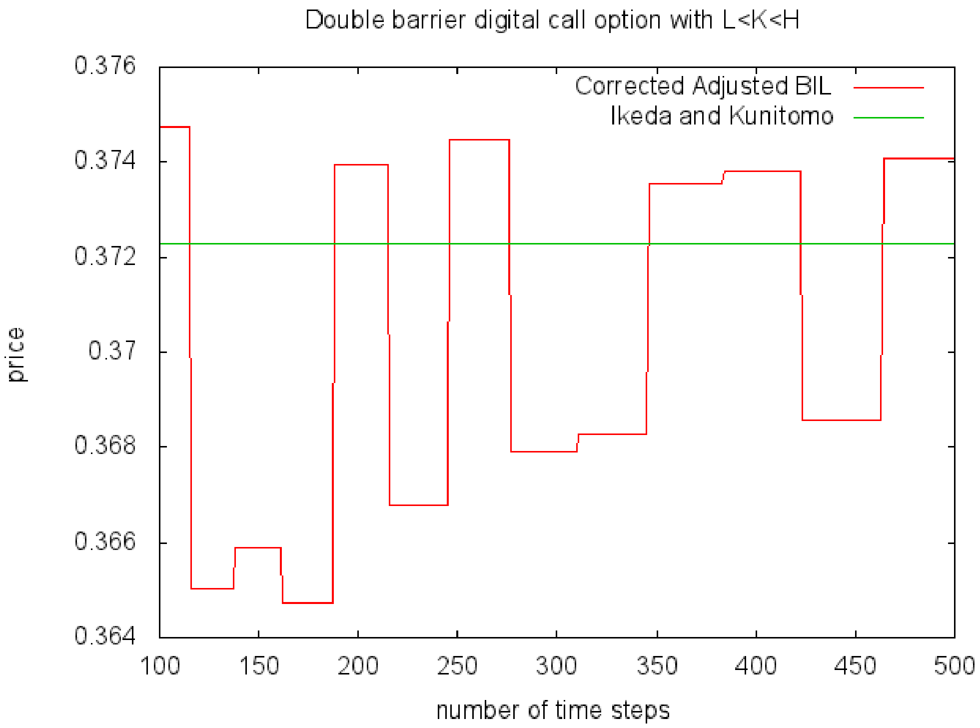


Figure 3.10: *Adjusted binomial interpolated lattice and correction for the higher barrier. Knock-out double barrier digital call option with $K = 100$, $L = 60$, $H = 180$, $s_0 = 150$, $r = 0.1$, $\sigma = 0.25$ and $T = 1$.*

In Figure 3.10 we plot the prices obtained with the corrected adjusted Binomial Interpolated Lattice, that is the adjusted Binomial Interpolated Lattice implemented with the correction

term for the higher barrier $H$ given in (3.3.2). The true price is given by the Ikeda and Kunitomo formula.
In Figure 3.11 we plot together the prices obtained with the CRR binomial approximation (in which the barriers $L$ and $H$ and the strike $K$ do not coincide with a node of the tree), the BIL procedure (in which we set the barriers $L$ and $H$ such that they are on a layer of nodes of the tree), the adjusted BIL algorithm (in which the lower barrier $L$ is a node of the tree and the strike $K$ is a node of the penultimate period), the corrected adjusted BIL algorithm (in which $L$ is a node of the tree, $K$ is a node from the penultimate period and where we also add the correction term that is proportional to (3.3.2)) and the Ikeda and Kunitomo price. We observe that the BIL algorithm and the corrected adjusted BIL algorithm perform better than the other procedures. In fact the first one cancels the contribution of order $O(\frac{1}{\sqrt{n}})$ due to the position of the barriers $L$ and $H$, and in the second one all the contributions of order $O(\frac{1}{\sqrt{n}})$ (due to $L, H$ and $K$) vanish. We still remark that the correction term calculated by using (3.3.2) is indeed an approximation: from Proposition 4.1 in [42] we get that actually it is equal to a sum of $(2M+1)^2$ term with $M > 0$. However in practice to keep only one term ($M = 0$) is enough to get good results.

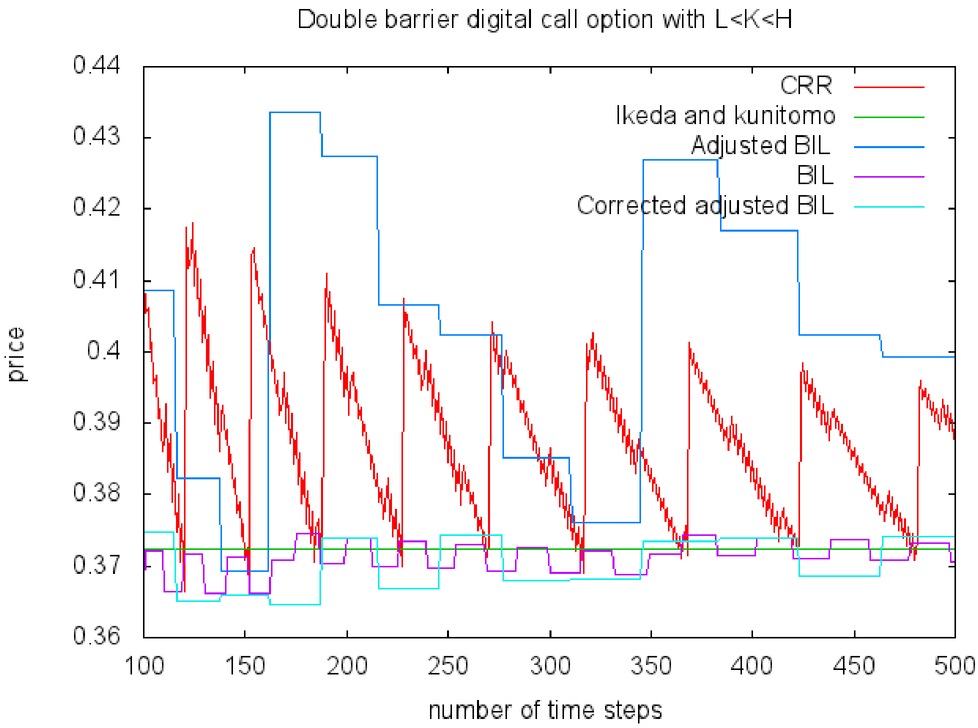


Figure 3.11: *Comparisons. Knock-out double barrier digital call option with* $K = 100$, $L = 60$, $H = 180$, $s_0 = 150$, $r = 0.1$, $\sigma = 0.25$ *and* $T = 1$.

In Table 3.3 we report the prices of the double barrier digital call option obtained with the CRR algorithm, the BIL algorithm and the corrected adjusted BIL algorithm (caBIL). We observe that as $m$ (and then the true number of time steps $n$ of the BIL algorithm) increases the prices computed with the BIL algorithm and the caBIL algorithm are much more closer to the true price calculated with the Ikeda and Kunitomo formula than the ones obtained with the CRR tree. This is due to the fact that first of all with the BIL algorithm the contribution of order $\frac{1}{\sqrt{n}}$ due to the barriers vanishes. Our conjecture is that the only term of order $\frac{1}{\sqrt{n}}$ is due to the distance $|\tilde{K} - K|$ between the contractual strike $K$ and the node $\tilde{K}$ of the tree at a period before maturity that is the closest to $K$, and as $n$ increases this distance becomes smaller and smaller. We also obtain a good approximation by using the caBIL algorithm: in this case we eliminate the contribution of order $\frac{1}{\sqrt{n}}$ due to $L$ and $K$ and by using the correction term we are also able to remove part of the error due to the position of $H$.

| | | $L < K < H$ | | |
|---|---|---|---|---|
| m | CRR | BIL | caBIL | KI |
| 100 | 0.411097 | 0.366569 | 0.374740 | |
| 200 | 0.398082 | 0.374004 | 0.373933 | |
| 400 | 0.384829 | 0.371442 | 0.373827 | |
| 800 | 0.377426 | 0.371153 | 0.373804 | 0.372300 |
| 1600 | 0.383279 | 0.373327 | 0.370515 | |
| 3200 | 0.378424 | 0.371445 | 0.373219 | |

Table 3.3: *Knock-out double barrier digital call options prices with* $K = 100$, $L = 60$, $H = 180$, $s_0 = 150$, $r = 0.1$, $\sigma = 0.25$ *and* $T = 1$.

# Chapter 4

# Option pricing with CIR interest rate

The scenarios that have been prevailing on financial markets in the last decades suggest that equity models need to take into account for stochastic interest rates to capture the reality. For example, let us think to long-lived contracts such as equity-linked life insurance policies with an asset value guaranteed introduced by Brennan and Schwartz ([18], 1976). This kind of product is not really an insurance policy at all, but it is indeed an investment program in which the insurance company invests part of the premium in bonds that guarantee a minimum benefit to the policyholder and another part in an investment portfolio whose value at expiration is uncertain. Then, it is actually a financial product whose pricing issue involves the necessity to allow for stochastic interest rates because of its long life time.
Moreover, starting from 1990, practitioners introduced a kind of options whose time-to-maturity is at least 2 or 3 years at the time of issue. Those options are known as LEAPS, an acronym for Long-term Equity AnticiPation Security, and actually they are available on approximately 2500 equities and 20 indexes. Although a closed-form formula due to Merton ([64], 1973) exists for a European option with stochastic interest rate that follows an Ornestein-Uhlenbeck process, its American counterpart is unknown. Here comes the necessity of deriving an alternative way for the valuation of American-style options with stochastic interest rate by using numerical methods. Then the starting continuous model is a two-dimensional diffusion in which the equity value is a geometric Brownian motion with drift driven by a square root process (see Section 4.2).

Although Boyle ([14], 1988) and then Boyle et al.([15], 1989) propose lattice models for the pricing of options with two state variables, those methods are specific for the case in which the two variables have a bivariate lognormal distribution.
The first two attempts to solve the problem in which one of the two state variables is a mean-reverting process, such as the CIR process, are due to Kishimoto ([53], 1989) and Hull and White ([50], 1990). In few words, Kishimoto combine the Ho-Lee term structure model ([47], 1986) with the CRR binomial tree ([28], 1979), while Hull and White propose a finite difference approach (for details see Section 4.1). In both cases the numerical procedure is complex and for this reason first Wei ([83], 1996) and then Hilliard, Schwartz and Tucker ([46], 2004) develop simpler lattice procedures based on a slight modification of the Nelson

and Ramaswamy idea ([65], 1990) that consists in approximating one dimensional diffusions with *simple binomial processes* (for details see Section 4.3.1 and Section 4.3.2). Nevertheless, numerical experiments show that the methods proposed by Wei and Hilliard, Scwhartz and Tucker are not stable and robust from a practical point of view whenever the parameter volatility of the interest rate increases.
We describe here the procedure introduced in Appolloni, Caramellino and Zanette ([4], 2013), that is a new lattice approach which permits very efficient and precise numerical results without any restriction on the parameters. Moreover, we provide a theoretical proof of the convergence of the method by using standard techniques of convergence of Markov chains to diffusions. In particular we discuss both the convergence in law of the new bivariate tree model to the original two-dimensional diffusion and the convergence of the American prices obtained from the algorithm to the true ones. We remark that in [4] we first construct two separate binomial lattices, one for the underlying asset price and the other for the stochastic interest rate process, by recovering the original idea in Nelson and Ramaswamy ([65], 1990). Then we introduce a probabilistic structure on the bivariate lattice. Since the two original processes are driven by two Brownian noises that are supposed to be correlated, we define the transition probability by using the correlation structure in the two-dimensional model. But other techniques could be used for the discretization of the original diffusion. For example, in Martire ([63], 2012) the author proposes a similar probabilistic structure for the two-dimensional approximating process but the tree for the stochastic interest rate is different, being based on an idea suggested in Costabile and Massabò ([26], 2010). Roughly speaking, they construct a lattice for $r$ by using the standard approximation of the Brownian motion with the binomial random walk but, since the lattice structure does not recombine, they provide a method in order to get a recombining binomial simple lattice.

The Chapter is organized as follows. In Section 4.1 we provide a brief review of the existing literature on the pricing of American-style options with two state variables and the first attempts of approximating a stochastic interest rate by means of a tree structure. In Section 4.2 we describe the bivariate continuous model for the joint evolution of the equity and the interest rate processes. Sections 4.3.1 and 4.3.2 are devoted to a detailed description of the Wei and Hilliard, Schwartz and Tucker algorithms respectively. Then, Section 4.4 refers to the description of the new lattice algorithm proposed and Section 4.5 is dedicated to the main convergence theoretical results with some preliminary lemmas. We first provide the weak convergence of the Markov chain associated to the new bivariate algorithm (subsection 4.5.1). Then we prove the convergence of European and American put option prices obtained with the lattice proposed to their continuous counterparts (subsection 4.5.2). Finally, Section 4.6 refers to the numerical comparisons for European and American option pricing problems between our method and the procedures of Wei and Hilliard, Schwartz and Tucker.

## 4.1 Pricing American options on two state variables or with stochastic interest rate

In this Section we briefly recall the main papers that deal with the pricing of American-style options with two or more state variables.
The first two attempts are due to Boyle (1988, [14]) and Boyle at al. (1989, [15]). Boyle in [14] develops a procedure for the valuation of options when the two underlying state variables $S_1$ and $S_2$ have a bivariate lognormal distribution. The basic idea is first to modify the CRR lattice binomial approach in the case of a single state variable by using a trinomial scheme and then generalize the procedure when two underlying assets are involved. So, in the case of a single state variable, the approximating discrete process is a three-jump process involving up, horizontal and down movements, instead of the classical two-jump process (that just involves up and down movements) and the usual matching conditions on the first two moments are used to define the transition probabilities. Instead, in the case of two state variables whose joint density is a bivariate lognormal distribution, the author finds that a five-point discrete process is the most suitable in order to obtain an efficient algorithm. It means that given the pair $(S_1, S_2)$ that represents the current value, the process may jump to one of the following values

$$(S_1u_1, S_2u_2), (S_1u_1, S_2d_2), (S_1d_1, S_2u_2), (S_1d_1, S_2d_2), (S_1, S_2),$$

where $u_1$ and $u_2$ are the multiplicative factors related to an up jump of the assets $S_1$ and $S_2$ respectively. These five points correspond in the three-dimensional space to a lattice structure resembling an inverted pyramid. As for the one-dimensional case, the transition probabilities are the solution of a linear system obtained by equating the mean and the variance of the discrete distribution to the mean and the variance of the continuous counterpart. But it is not guaranteed that the system produces positive probabilities, for this reason Boyle defines $u_1$ and $u_2$ as follows

$$u_1 = e^{\lambda\sigma_1\sqrt{h}} \quad \text{and} \quad u_2 = e^{\lambda\sigma_2\sqrt{h}},$$

where $\sigma_1$ and $\sigma_2$ are the constant volatility parameters of $S_1$ and $S_2$ respectively, and $\lambda$ is a constant greater than 1. This is indeed a structural difference with the classical CRR approach that instead requires the jump sizes be of the type $e^{\sigma\sqrt{h}}$. By using different values for $\lambda$, a range of values for $u_1$ and $u_2$ is obtained, and there is an interval within this range that produces acceptable values for all the probabilities. The author proposes different numerical examples by comparing the results obtained with the lattice procedure with closed-form expressions for the price of some European-style options.
Subsequently Boyle et al. ([15], 1989) extend the Boyle lattice procedure for the valuation of contingent claims involving several underlying assets using a generalized lattice framework. They follow the CRR approach for the definition of the jump sizes and then they choose the probabilities in order to match the second-order Taylor expansion of the characteristic functions of the discrete and the continuous-time processes. They are able to obtain closed-form

solutions for such probabilities but here no conditions are given to ensure their positivity, so they need to check it at each application. They also present some numerical experiments involving the values of European options with three underlying assets.

However, both the procedures of Boyle and Boyle et al. require that the interest rate is constant, precluding it from being one of the state variables. The first two works that allow the interest rate to be stochastic are due to Kishimoto (1989, [53]) and Hull and White (1990, [50]).

Kishimoto essentially combines the classical CRR binomial lattice with the Ho and Lee term structure model (1986, [47]) in order to work with the two sources of uncertainty, the equity value and the interest rate. Ho and Lee in [47] study the problem of pricing interest rate contingent claims by modeling the term structure movements and by relating the movements to the assets' prices. In particular, they describe the term structure in terms of a binomial lattice and then they explain how the price of a discount bond is derived. The main idea in Kishimoto ([53], 1989) is to extend Ho and Lee procedure for the pricing of assets whose risk is not solely determined by interest rate movements, but instead is influenced also by another risky asset component. The author proposes to resolve the two uncertainties one by one: he divides each time step into two subperiods in which the two uncertainties resolve separately. So, during the first subperiod, the interest rate uncertainty resolves and at the end of the first subperiod the discount function takes one of two possible shapes: if an *upstate of the term structure* is attained, then the discount function shifts up; if a *downstate of the term structure is attained*, then the discount function shifts down. During the second subperiod, the discount function is supposed to remain unchanged and the asset price movements are modeled. In this case, the movements are decomposed in two more components. The first captures the co-movements of the asset price with the discount function during the first subperiod, so it is called the *interest-dependent component*. The second one, instead, called the *asset-specific component*, is a residual noise and follows a binomial process. The problem with this kind of approach is that the programming code can be extremely complex .
After few years, Hull and White ([50], 1990) propose a finite-difference approach to value any derivative security dependent on a single or more state variables. In particular, they deal with the *explicit* finite difference method that relates the value of the derivative security at a generic time $t$ to three alternative values at time $t + \Delta t$ giving conditions such that the convergence of the approximating value to the correct solution is ensured. Brennan and Schwartz ([18], 1978) showed that this method is equivalent to a trinomial lattice approach. Generally speaking, the explicit difference method consists in considering the PDE satisfied by the price function and approximating it with a finite difference equation by using suitable approximations of the partial derivatives involved. When there are two state variables with correlation $\rho$, the first step is to transform them in order to get constant instantaneous standard deviations. Then, the variables are transformed again to eliminate the correlation so that it is possible to model the uncorrelated variables by using a two-dimensional lattice with nine branches emanating from each node. The joint transition probabilities are then obtained by the product of the individual probabilities that are solutions of two finite difference

equations. We remark that this procedure can be applied also when one of the variables is a mean-reverting process, such as for example the CIR process. In this specific case the method is modified in order to ensure the convergence to the exact solution of the PDE. To be precise, instead of insisting the movement from a node $(i, j)$ for some $i, j$ to one of the nodes $(i+1, j-1)$, $(i+1, j)$ and $(i+1, j+1)$ in time $\Delta t$, the method allows a movement from $(i, j)$ to one of the nodes $(i+1, k-1)$, $(i+1, k)$ and $(i+1, k+1)$, where $k$ is not necessary equal to $j$. The authors provide some numerical results, but the implementation of the procedure is expensive from a computational point of view because the approach requires a large number of nodes.
Hull ([49], 1992, pages 601-603) outlines also a two-state binomial model, i.e. a three-dimensional lattice in which each node branches into four, that handles two correlated geometric Brownian motions. As before, the procedure consists in removing the correlation and combining the two uncorrelated variables by using two separate binomial trees. The two trees can then be combined together into a single bivariate tree in which the transition probabilities are defined by means of products. This method is less expensive than the one in [50], but here the disadvantage is again that it can be applied only for modeling two geometric Brownian motions.

## 4.2 The bivariate continuous model

We are concerned in a geometric Brownian motion describing the evolution of the equity value with drift driven by a square root process. So, we consider, under the risk-neutral probability measure, the following dynamics for the equity value

$$\frac{dS(t)}{S(t)} = r(t)dt + \sigma_S dZ_S(t), \quad S(0) = s_0 > 0, \tag{4.2.1}$$

where $r$ is the short interest rate process, $\sigma_S$ is the constant stock price volatility and $Z_S$ is a standard Brownian motion. The risk-neutralized process for the short rate is described as in the Cox, Ingersoll and Ross (CIR) model (see [27] for more details), that is

$$dr(t) = \kappa(\theta - r(t))dt + \sigma_r \sqrt{r(t)} dZ_r(t), \quad r(0) = r_0 > 0, \tag{4.2.2}$$

where $\theta$ is the long term reversion target, $\kappa$ is a constant representing the reversion speed, $\sigma_r > 0$ is the constant interest rate volatility and $Z_r$ is a standard Brownian motion. The Brownian noises $Z_S$ and $Z_r$ are supposed to be correlated and we let $\rho$ denote the correlation:

$$d < Z_S, Z_r > (t) = \rho dt.$$

In what follows, we call the model described in equation (4.2.1)-(4.2.2) the BS-CIR model. We remark that the interest rate behavior implied by the structure defined in (4.2.2) has the following properties:

1. negative interest rates are precluded;

2. if the interest rate reaches zero, it can subsequently become positive.

We now recall the well-known *Feller condition*: if $r_0 > 0$ and if $2\kappa\theta \geq \sigma_r^2$ then a.s. the process $r$ never hits 0.
The main trouble with all the models briefly presented in the previous Section is that when they can be applied to a two-dimensional diffusion such as (4.2.1)-(4.2.2), they are all computationally expensive. Instead, the manageable procedures are built for two-state models that don't allow for a stochastic interest rate.
The first attempt to derive a closed-form formula for the price of European put and call options under stochastic interest rate is due to Merton (1973, [64]) that uses the same technique as Black and Scholes (1973, [13]) of valuing the price of a derivative security by creating a replicating portfolio. In particular Merton (1973, [64]), Rabinovitch (1989, [69]) and Amin and Jarrow (1992, [2]) derive accurate approximating formulas (quasi-closed form) under a Gaussian interest rate but these results can also be adapted to the case in which the interest rate process follows the dynamic in (4.2.2) (for that generalization see Kim (2002, [52])).
On the other hand, the pricing of American-style options whose underlying satisfies (4.2.1)-(4.2.2) is much more complicated because it involves an optimal stopping time problem. Several approaches have been suggested in the literature and they can be classified into three main types: finite-difference methods, lattice methods and various analytical methods. Among these, lattice techniques are very simple to implement and have the advantage of permitting early exercise in order to treat the American case by using the same tree structure built for the pricing of European-style options and this is the reason why they are so appealing.
The next Section is devoted to the description of the first two algorithms that are able to combine numerical simplicity with the treatment of the dynamics (4.2.1)-(4.2.2): the Wei procedure and the Hilliard, Schwartz and Tucker procedure.

## 4.3 Existing lattice methods in the BS-CIR model

### 4.3.1 The Wei procedure

In this Section we describe the procedure proposed in Wei ([83], 1996), that consists in generalizing the Nelson and Ramaswamy technique in order to price American-style options on a model with a CIR stochastic interest rate. Actually, in Wei the dynamic for the short rate is given by the Vasicek model and the extension of the Wei procedure to the CIR process is described in Costabile et al.([25], 2006). However, in what follows, we still call "Wei procedure" the natural extension to the CIR process.
As explained in Section 1.1.3, Nelson and Ramaswamy show that one dimensional diffusion processes can be approximated with a computationally simple binomial process by transforming the original process into a diffusion with unit variance. Starting from this idea, Wei suggests to build a bivariate lattice following four simple steps:

- transform both $S$ and $r$ into unit variance processes that we call $\tilde{S}$ and $R$ respectively;
- define a new process $Y$ as a function of $\tilde{S}$ and $R$ that is orthogonal to $R$;
- model $R$ and $Y$ as two independent binomial processes following the Nelson and Ramaswamy technique and then merge the two structures into a bivariate tree in which each node branches into four via joint probabilities;
- at each node of the tree convert the variables $R$ and $Y$ back to $r$ and $S$ respectively and then proceed backwardly to obtain the option prices.

We now explain in details how the procedure works. The first step is to transform the processes $S$ and $r$ into unit variance processes and this is done by introducing two transformations as in (1.1.16) that are

$$\tilde{S} = (\log S)/\sigma_S \quad \text{and} \quad R = 2\sqrt{r}/\sigma_r$$

respectively. The dynamics of $\tilde{S}$ and $R$ may be easily derived by applying Ito's Lemma. Hence,

$$\begin{aligned} d\tilde{S}(t) &= \mu_{\tilde{S}}(R(t))dt + dZ_S(t), \quad \tilde{S}(0) = (\log s_0)/\sigma_S, \\ dR(t) &= \mu_R(R(t))dt + dZ_r(t), \quad R(0) = 2\sqrt{r_0}/\sigma_r, \end{aligned}$$

where

$$\mu_{\tilde{S}}(R) = \frac{\sigma_r^2 R^2/4 - \sigma_S^2/2}{\sigma_S} \quad \text{and} \quad \mu_R(R) = \frac{\kappa(4\theta - R^2\sigma_r^2) - \sigma_r^2}{2R\sigma_r^2}. \tag{4.3.1}$$

The second step is to define a new process $Y$, function of both $\tilde{S}$ and $R$, that is orthogonal to the unit variance process $R$, so one needs to consider the transformation

$$Y = \frac{\tilde{S} - \rho R}{\sqrt{1-\rho^2}}. \tag{4.3.2}$$

Then the dynamic of the diffusion process $Y$ is

$$dY(t) = \mu_Y(R(t))dt + dZ_Y(t), \quad Y(0) = Y_0 = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{\log s_0}{\sigma_S} - \frac{2\rho\sqrt{r_0}}{\sigma_r}\right),$$

where

$$\mu_Y(R) = \frac{\mu_{\tilde{S}}(R) - \rho\mu_R(R)}{\sqrt{1-\rho^2}}$$

and $Z_Y$ is the standard Brownian motion given by

$$Z_Y = \frac{Z_S - \rho Z_r}{\sqrt{1-\rho^2}}.$$

We remark that $Z_Y$ is orthogonal to $Z_r$ so that $Y$ and $R$ have null covariance. The third step is the crucial one. In fact, we first need to construct the discrete approximations of the processes $R$ and $Y$ by using two independent binomial trees. Then, we merge the two structures into a bivariate lattice with each node branching into four via joint transition probabilities that are simply obtained by the product of the individual probabilities. So, we define $h = T/n$ and then we build the two binomial structures with $n$ steps of length $h$. We label $(0,0)$ the starting node where the $R$-process has value $R(0)$. After $i$ time steps $(i = 0,\dots,n)$, $R$ may be located at one of the nodes $(i,k)$ $(k = 0,\dots,i)$ corresponding to the values

$$R_{i,k} = R_0 + (2k - i)\sqrt{h}. \tag{4.3.3}$$

Analogously, for the discrete process approximating $Y$, we label $(0,0)$ the starting node where the $Y$-process has value $Y(0)$. After $i$ time steps $(i = 0,...,n)$, $Y$ may be located at one of the nodes $(i,j)$ $(j = 0,\dots,i)$ corresponding to the values

$$Y_{i,j} = Y_0 + (2j - i)\sqrt{h}. \tag{4.3.4}$$

Then the transition probabilities have to be specified for both processes to ensure the matching of the local drift and the local variance between the discrete approximations and the respective continuous counterparts. To this end, one has to take into account that in some regions of the tree it may happen that multiple jumps are needed to satisfy properly the matching conditions. Hence, starting from $R_{i,k}$ at time $ih$, the process $R$ may jump at time $(i+1)h$ to the value $R_{i+1,k_d}$ or $R_{i+1,k_u}$, with $k_d$ and $k_u$ defined as

$$k_d = \begin{cases} 0 & \text{if } R_{i,k} + (\mu_R)_{i,k}h < R_{i+1,0} \\ i & \text{if } R_{i,k} + (\mu_R)_{i,k}h > R_{i+1,i+1} \\ \text{the largest index } k^* \in [0,i] \text{ s.t.} & \\ \qquad R_{i,k} + (\mu_R)_{i,k}h \ge R_{i+1,k^*} & \text{otherwise} \end{cases}$$

and

$$k_u = k_d + 1,$$

in which $(\mu_R)_{i,k} = \mu_R(R_{i,k})$. It is clear that in the case $R_{i,k} = 0$ the drift $\mu_R$ cannot be evaluated. So, in practice one chooses the number $n$ of the monitoring instants in such a way that the lattice for $R$ never hits 0. We briefly recall here the construction of the discrete approximation for $R$ as described in Remark 1 in [25]: setting

$$k_0 = \mathrm{int}\Big(\frac{R_0}{\sqrt{h}}\Big) \quad \text{and} \quad \gamma = \min\big(|R_0 - k_0\sqrt{h}|, |R_0 - (k_0+1)\sqrt{h}|\big)$$

(that is, $\gamma$ is the the minimum, in absolute value, of the lattice points for $R$), the number of steps $n$ giving $h = T/n$ is chosen such that $\gamma$ is not too small $(\gamma \ge 10^{-6})$. Hereafter, $\mathrm{int}(x)$ denotes the integer part of $x \in \mathbb{R}$, that is, for $x \ge 0$ then $\mathrm{int}(x)$ is the largest integer not exceeding $x$ and for $x < 0$ we set $\mathrm{int}(x) = -\mathrm{int}(-x)$. One easily gets that

$$k_d = k + \mathrm{int}\Big(\frac{(\mu_R)_{i,k}\sqrt{h} + 1}{2}\Big).$$

Now, the probability that the $R$-process, located at the node $R_{i,k}$, reaches $R_{i+1,k_u}$, is well defined by setting

$$p_{i,k} = 0 \vee \frac{(\mu_R)_{i,k}h + R_{i,k} - R_{i+1,k_d}}{R_{i+1,k_u} - R_{i+1,k_d}} \wedge 1.$$

Obviously, the probability to reach $R_{i+1,k_d}$ is $1 - p_{i,k}$. Concerning the $Y$-process, one has to take into account also the behavior of $R$ because $\mu_Y = \mu_Y(R)$ . So, let us assume that the pair is located at $(R_{i,k}, Y_{i,j})$ at the time-step $i$. Then, for the jumps in the $Y$-direction, one sets

$$j_d = \begin{cases} 0 & \text{if } Y_{i,j} + (\mu_Y)_{i,k}h < Y_{i+1,0} \\ i & \text{if } Y_{i,j} + (\mu_Y)_{i,k}h > Y_{i+1,i+1} \\ \text{the largest index } j^* \in [0,i] \text{ s.t.} & \\ \qquad Y_{i,j} + (\mu_Y)_{i,k}h \geq Y_{i+1,j^*} & \text{otherwise} \end{cases}$$

and $j_u = j_d + 1$, in which $(\mu_Y)_{i,k} = \mu_Y(R_{i,k})$. Again, one has

$$j_d = j + \mathrm{int}\Big(\frac{(\mu_Y)_{i,k}\sqrt{h} + 1}{2}\Big).$$

Furthermore, the transition probability that the $Y$-process jumps to $Y_{i+1,j_u}$ is defined by setting

$$\hat{p}_{i,j,k} = 0 \vee \frac{(\mu_Y)_{i,k}h + Y_{i,j} - Y_{i+1,j_d}}{Y_{i+1,j_u} - Y_{i+1,j_d}} \wedge 1$$

and $1 - \hat{p}_{i,j,k}$ is the probability of the down-jump. We are now ready to describe the discrete approximation scheme for the joint evolution of the processes $R$ and $Y$ by considering a bivariate tree obtained by merging the two univariate binomial trees. At each time step $i$ ($i = 0, \dots, n$), the tree has $(i+1)^2$ nodes that we label $(i, j, k)$ corresponding to the values $R_{i,k}$ and $Y_{i,j}$ ($k, j = 0, \dots, i$). Starting from the node $(i, j, k)$, in consideration of possible multiple jumps and taking into account the tree structure, the process may reach one of the following four nodes:

$$\begin{aligned} &(i+1, j_u, k_u), && \text{with probability } q_{i,j_u,k_u}, \\ &(i+1, j_u, k_d), && \text{with probability } q_{i,j_u,k_d}, \\ &(i+1, j_d, k_u), && \text{with probability } q_{i,j_d,k_u}, \\ &(i+1, j_d, k_d), && \text{with probability } q_{i,j_d,k_d}, \end{aligned}$$

where $j_u, j_d, k_u, k_d$ are related to multiple jumps on the tree in the $Y$ and $R$ direction respectively, and $q_{i,j_u,k_u}, q_{i,j_u,k_d}, q_{i,j_d,k_u}, q_{i,j_d,k_d}$ are the associated transition probabilities. Such probabilities can be computed, due to the orthogonality of the noises driving the two processes, as follows

$$\begin{aligned} q_{i,j_u,k_u} &= \hat{p}_{i,j,k} p_{i,k}, & q_{i,j_d,k_u} &= (1 - \hat{p}_{i,j,k}) p_{i,k}, \\ q_{i,j_u,k_d} &= \hat{p}_{i,j,k}(1 - p_{i,k}), & q_{i,j_d,k_d} &= (1 - \hat{p}_{i,j,k})(1 - p_{i,k}). \end{aligned} \tag{4.3.5}$$

The last step is to work backwards along the lattice and compute the American option value. Since the lattice is built for the transformed processes $R$ and $Y$, we need to convert them back into $r$ and $S$ at each node using the inverse transformations of (1.1.16) that in this specific case are

$$r = \begin{cases} \frac{R^2\sigma_r^2}{4} & \text{if} \quad R > 0 \\ 0 & \text{otherwise} \end{cases}$$
$$S = \exp[\sigma_S(\sqrt{1-\rho^2}Y + \rho R)],$$

respectively. Therefore, each node $(i, j, k)$ of the bivariate tree for $(r, S)$ corresponds to the values

$$r_{i,k} = \begin{cases} \frac{R_{i,k}^2\sigma_r^2}{4} & \text{if} \quad R_{i,k} > 0, \\ 0 & \text{otherwise}, \end{cases} \tag{4.3.6}$$

and

$$S_{i,j,k} = \exp[\sigma_S(\sqrt{1-\rho^2}Y_{i,j} + \rho R_{i,k})]. \tag{4.3.7}$$

The resulting successor values are easily identified with

$$(r_{i+1,k_u}, S_{i+1,j_u,k_u}), (r_{i+1,k_d}, S_{i+1,j_u,k_d}), (r_{i+1,k_u}, S_{i+1,j_u,k_u}), (r_{i+1,k_d}, S_{i+1,j_d,k_d})$$

and the transition probabilities are kept those defined in (4.3.5). Finally, if for example we consider an American put option with maturity $T$ and strike $K$, its price at time 0 is computed by the following backward dynamic programming equations:

$$\begin{cases} v_{n,j,k} = (K - S_{n,j,k})_+ \\ v_{i,j,k} = \max\Bigg( (K - S_{i,j,k})_+, e^{-r_{i,k}h}\Big[q_{i,j_u,k_u}v_{i+1,j_u,k_u} + q_{i,j_u,k_d}v_{i+1,j_u,k_d} + \\ \qquad\qquad + q_{i,j_d,k_u}v_{i+1,j_d,k_u} + q_{i,j_d,k_d}v_{i+1,j_d,k_d}\Big]\Bigg), \end{cases} \tag{4.3.8}$$

where $v_{i,j,k}$, $i = 0, ..., n$ and $j, k = 0, ..., i$, provides the American option price at every node $(i, j, k)$ of the tree structure.

### 4.3.2 The Hilliard,Schwartz and Tucker procedure

We briefly describe here the procedure in Hilliard, Schwartz and Tucker ([46], 2004) that represents another way to extend the Nelson and Ramaswamy technique for the valuation of American-style options with two underlying state variables. The pricing idea is similar to the one in Wei ([83], 1996) except that for the transformations used to convert the two stochastic processes to give constant volatilities and for the transformations employed to

ensure that the once-transformed processes are orthogonal.
In fact the first step is to introduce the following two transformations

$$\tilde{S} = (\log S)/\sigma_S \quad \text{and} \quad R = 2\sqrt{r}$$

that convert $S$ into the unit variance process $\tilde{S}$ and $r$ into the constant variance process $R$. In fact, from Ito's lemma we get

$$\begin{aligned} d\tilde{S}(t) &= \mu_{\tilde{S}}(R(t))dt + dZ_S(t), \quad \tilde{S}(0) = (\log s_0)/\sigma_S, \\ dR(t) &= \mu_R(R(t))dt + \sigma_r dZ_r(t), \quad R(0) = 2\sqrt{r_0}, \end{aligned}$$

where

$$\mu_{\tilde{S}}(R) = \frac{R^2/4 - \sigma_S^2/2}{\sigma_S} \quad \text{and} \quad \mu_R(R) = \frac{\kappa(4\theta - R^2)2 - 2\sigma_r^2}{4R}.$$

Then, the second step consists in transforming the once-transformed processes $\tilde{S}$ and $R$ into two new processes $X_1$ and $X_2$ by defining

$$X_1 = \sigma_r\tilde{S} + R \quad \text{and} \quad X_2 = \sigma_r\tilde{S} - R.$$

It is easy to verify that $X_1$ and $X_2$ have zero covariance. Now, similarly to Wei, the idea is modeling $X_1$ and $X_2$ as two separate binomial processes in which multiple jumps may occur to ensure the matching of the mean and to legitimate probabilities. Then the two binomial trees are combined together into a bivariate tree via four joint probabilities simply obtained by products. The final step is to transform at each node $X_1$ and $X_2$ back to $S$ and $r$ using the following transformations

$$S = \exp\left(\sigma_S \frac{X_1 + X_2}{2\sigma_r}\right) \quad \text{and} \quad r = \frac{(X_1 - X_2)^2}{8}.$$

The usual backward procedure is then applied in order to compute the American option price of the derivative security.

## 4.4 The new bivariate algorithm

In this Section we introduce the new lattice procedure presented in Appolloni, Caramellino and Zanette ([4], 2013) for the description of the approximating evolution of the pair $(S, r)$. We recall that the drift coefficients appearing in (4.2.1) and (4.2.2) are given respectively by

$$\mu_S(S, r) = rS \quad \text{and} \quad \mu_r(r) = \kappa(\theta - r).$$

Consider first the interest rate process $r$. Following a remark of Tian (see ([78], 1994) at page 100, or also ([76], 1992)), the right implementation of the Nelson and Ramaswamy algorithm presented in Section 1.1.3 for the CIR process consists in setting the probabilistic structure directly on $r$, and not through $R$ (for more details see Remark 4.4.1). Thus, we first construct a tree for $R$ as in Wei by means of (4.3.3) and we transform it into the computationally simple lattice for $r$ with the map in (4.3.6), i.e. at time step $i = 0, ..., n$, for $k = 0, ..., i$, we set

$$r_{i,k} = \begin{cases} \frac{R_{i,k}^2 \sigma_r^2}{4} & \text{if} \quad R_{i,k} > 0 \\ 0 & \text{otherwise,} \end{cases}$$

$R_{i,k}$ being defined in (4.3.3). Secondly, the transition probabilities are specified directly on $r$: the matching of the local drift between the discrete and the continuous model is set by using the original interest rate CIR process $r$. According to Nelson and Ramaswamy ([65], 1990), at the node $(i, k)$, we set $(\mu_r)_{i,k} = \mu_r(r_{i,k})$ and we define

$$\begin{aligned} k_d(i,k) &= \max\{k^* \,:\, 0 \le k^* \le k \text{ and } r_{i,k} + (\mu_r)_{i,k} h \ge r_{i+1,k^*}\}, \\ k_u(i,k) &= \min\{k^* \,:\, k+1 \le k^* \le i+1 \text{ and } r_{i,k} + (\mu_r)_{i,k} h \le r_{i+1,k^*}\} \end{aligned}$$

with the understanding $k_d(i, k) = 0$ if $\{k^* : 0 \le k^* \le k \text{ and } r_{i,k} + (\mu_r)_{i,k} h \ge r_{i+1,k^*}\} = \emptyset$ and $k_u(i, k) = i + 1$ if $\{k^* : k + 1 \le k^* \le i + 1 \text{ and } r_{i,k} + (\mu_r)_{i,k} h \le r_{i+1,k^*}\} = \emptyset$. The transition probabilities are now defined as follows: starting from $(i, k)$, the probability that the process jumps to $(i + 1, k_u(i, k))$ is set as

$$p_{i,k} = 0 \vee \frac{(\mu_r)_{i,k} h + r_{i,k} - r_{i+1,k_d(i,k)}}{r_{i+1,k_u(i,k)} - r_{i+1,k_d(i,k)}} \wedge 1. \tag{4.4.1}$$

And of course, the jump to $(i + 1, k_d(i, k))$ happens with probability $1 - p_{i,k}$.
Concerning the lattice for $S$, we proceed as follows. We first consider a computationally simple tree-structure $(U_{i,j})_{i,j}$ defined as

$$U_{i,j} = U_0 + (2j - i)\sqrt{h}, \quad U_0 = \frac{1}{\sigma_S} \log s_0, \quad i = 0, \dots, n, \ j = 0, \dots, i. \tag{4.4.2}$$

Then, we apply the (new) transformation

$$S = e^{\sigma_S U} \tag{4.4.3}$$

and we get a tree $(S_{i,j})_{i,j}$ for $S$: $S_{i,j} = e^{\sigma_S U_{i,j}}$. By combining the two lattices, we obtain a bivariate tree

$$(S_{i,j}, r_{i,k}), \quad i = 0, \dots, n, \ j, k = 0, \dots, i.$$

It is worth to say that the transformation (4.4.3) does not seem to be natural in order to describe the evolution of the pair. Nevertheless, this is not important. In fact, at this stage, we only need to set up the state-space of the Markov chain that we want to approximate the continuous-time process $(S, r)$, and in fact as $h \to 0$ one really gets that the discrete state-space converges to $\mathbb{R}_+^2$. What is important now is to define the transition probabilities in order to link the tree with the diffusion pair $(S, r)$.

We first set the probabilistic behavior of the jumps for $S$ at the time-step $i+1$ given the position $(S_{i,j}, r_{i,k})$ by matching w.r.t. the drift $\mu_S$ of $S$. So, we set $(\mu_S)_{i,j,k} = \mu_S(S_{i,j}, r_{i,k})$ and we define

$$j_d(i,j,k) = \max\{j^* : 0 \le j^* \le j \text{ and } S_{i,j} + (\mu_S)_{i,j,k}h \ge S_{i+1,j^*}\}$$
$$j_u(i,j,k) = \min\{j^* : j+1 \le j^* \le i+1 \text{ and } S_{i,j} + (\mu_S)_{i,j,k}h \le S_{i+1,j^*}\}$$

with the usual understanding $j_d(i,j,k) = 0$ if $\{j^* : 0 \le j^* \le j \text{ and } S_{i,j} + (\mu_S)_{i,j,k}h \ge S_{i+1,j^*}\} = \emptyset$ and $j_u(i,j,k) = i+1$ if $\{j^* : j+1 \le j^* \le i+1 \text{ and } S_{i,j}+(\mu_S)_{i,j,k}h \le S_{i+1,j^*}\} = \emptyset$. Now, starting from $(i,j,k)$, the probability of an up-jump of the tree for $S$ is set as

$$\hat{p}_{i,j,k} = 0 \vee \frac{(\mu_S)_{i,j,k}h + S_{i,j} - S_{i+1,j_d(i,j,k)}}{S_{i+1,j_u(i,j,k)} - S_{i+1,j_d(i,j,k)}} \wedge 1. \tag{4.4.4}$$

The down-jump obviously may happen with probability $1 - \hat{p}_{i,j,k}$.
We must finally set the covariance structure for the joint evolution of the processes $S$ and $r$ on the bivariate tree. Starting from the node $(i,j,k)$, by considering possible multiple jumps and by taking into account the tree structure, the process may reach one of the four nodes: $(i+1, j_u, k_u)$ with probability $q_{i,j_u,k_u}$, $(i+1, j_u, k_d)$ with probability $q_{i,j_u,k_d}$, $(i+1, j_d, k_u)$ with probability $q_{i,j_d,k_u}$, $(i+1, j_d, k_d)$ with probability $q_{i,j_d,k_d}$. The transition probabilities $q_{i,j_u,k_u}, q_{i,j_u,k_d}, q_{i,j_d,k_u}, q_{i,j_d,k_d}$ are computed by matching (at the first order in $h$) the conditional mean and the conditional covariance between the continuous and the discrete processes of $S$ and $r$. The matching conditions lead to solving the following system:

$$\begin{cases} q_{i,j_u,k_u} + q_{i,j_u,k_d} = \hat{p}_{i,j,k} \\ q_{i,j_u,k_u} + q_{i,j_d,k_u} = p_{i,k} \\ q_{i,j_u,k_u} + q_{i,j_d,k_u} + q_{i,j_u,k_d} + q_{i,j_d,k_d} = 1 \\ m_{i,j_u,k_u}q_{i,j_u,k_u} + m_{i,j_u,k_d}q_{i,j_u,k_d} + m_{i,j_d,k_u}q_{i,j_d,k_u} + m_{i,j_d,k_d}q_{i,j_d,k_d} = \rho\sigma_r\sqrt{r_{i,k}}\sigma_S S_{i,j}h \end{cases} \tag{4.4.5}$$

where $\hat{p}_{i,j,k}$ and $p_{i,k}$ are given in (4.4.1) and (4.4.4) respectively and

$$\begin{aligned} m_{i,j_u,k_u} &= (S_{i+1,j_u} - S_{i,j})(r_{i+1,k_u} - r_{i,k}), \quad m_{i,j_u,k_d} = (S_{i+1,j_u} - S_{i,j})(r_{i+1,k_d} - r_{i,k}), \\ m_{i,j_d,k_u} &= (S_{i+1,j_d} - S_{i,j})(r_{i+1,k_u} - r_{i,k}), \quad m_{i,j_d,k_d} = (S_{i+1,j_d} - S_{i,j})(r_{i+1,k_d} - r_{i,k}). \end{aligned} \tag{4.4.6}$$

Now, the backward induction procedure (4.3.8) can be applied, by using the probabilities that solve the system (4.4.5). This completely concludes the description of the approximating process for the pair $(S, r)$.

**Remark 4.4.1.** *Our procedure uses directly both the evolution of the process $r$ and the correlation between $S$ and $r$: those are the structural differences with what done in Wei ([83], 1996) and Hilliard, Schwartz and Tucker ([46], 2004). Therefore, we first remark that the transformed process $R$ is exploited only to construct the state-space of the Markov chain approximating $r$ and not to build up the transition probabilities. Indeed, their methods are strongly based on the use of a bivariate diffusion process whose components are driven* *by uncorrelated noises ($R$ and $Y$ for Wei, $X_1$ and $X_2$ for Hilliard, Schwartz and Tucker), leading to the definition of the transition probabilities by means of a product. And this can be done only by handling the process $R$. Moreover, the use of the process $r$ for the definition of the probabilities makes the main substantial difference between our method and the previous ones. In fact, we use the drift coefficient*

$$\mu_r(r) = \kappa(\theta - r)$$

*in the definition of the probabilities $p_{i,k}$, instead both Wei and Hilliard, Schwartz and Tucker employ the drift*

$$\mu_R(R) = \frac{1}{2\sigma_r^2}\left(\frac{4\kappa\theta - \sigma_r^2}{R} - \kappa\sigma_r^2 R\right).$$

*The presence of $R$ in the denominator is the source of numerical troubles when $R \downarrow 0$ (see Section 4.6 for more details). In fact, as discussed in Tian ([76], 1992) and Tian ([78], 1994), the convergence in law for the tree built for the process $R$ holds under the "convergence condition" $4\kappa\theta \geq \sigma_r^2$, that represents a limit for the parameters values. In fact when $4\kappa\theta \geq \sigma_r^2$, looking at the expression of $\mu_R(R)$, we get that $R$ has an infinite positive drift when $R \downarrow 0$ and this make the boundary zero inaccessible. Instead when $4\kappa\theta < \sigma_r^2$, then $R$ has an infinite negative drift coefficient when $R \downarrow 0$ that "pushes" the process under 0. In this case it happens that $R = 0$ is an absorbing barrier for the process $R$ and this is inconsistent with the fact that $r = 0$ is a reflecting barrier of the process $r$. As a result, the convergence in the Wei procedure and the Hilliard, Schwartz and Tucker one is not always guaranteed. On the contrary, we prove that our lattice procedure converges for every values of the parameters, because we directly work on the process $r$. As we will see in Section 4.5, in order to obtain the weak convergence we do not need to require the Feller condition ($2\kappa\theta \geq \sigma_r^2$) or the so-called "convergence condition" ($4\kappa\theta \geq \sigma_r^2$), but we only suppose that $\kappa > 0$ and $\theta > 0$, that are natural requirements in a financial setting.*

# 4.5 The convergence

## 4.5.1 Weak convergence of the associated approximating Markov chain

In this Section we use standard techniques to prove the theoretical convergence on the set $D([0,T];\mathbb{R}^2)$ of the tree method to the pair $(S,r)$ solution of the SDE (4.2.1)-(4.2.2). We recall here that $D([0,T];\mathbb{R}^2)$ is the set of all the càdlàg functions on $[0,T]$ with values in $\mathbb{R}^2$. The preliminary results to the main convergence theorem concern the analysis of

1. the up and down jumps of the tree for the process $r$;
2. the up and down jumps of the tree for the process $S$;
3. the probability transition matrix of the bivariate tree for the pair $(S,r)$.

### Up and down jumps for the spot rate

Let us start from the first theoretical result concerning the jumps of the tree built for the stochastic rate. To this end, we briefly recall the lattice structure for $r$: for a fixed node $(i,k)$, $i=0,1,\dots,n$, $k=0,1,\dots,i+1$, one has

$$r_{i,k}=\frac{R_{i,k}^2\sigma_r^2}{4}\mathbb{1}_{R_{i,k}>0} \quad \text{and} \quad R_{i,k}=R_0+(2k-i)\sqrt{h}, \tag{4.5.1}$$

with $R_0=\frac{2}{\sigma_r}\sqrt{r_0}$. We also remember that $k_d(i,k)=\max K_d(i,k)$ and $k_u(i,k)=\min K_u(i,k)$, where

$$K_d(i,k)=\{k^*\in\{0,...,i-1\} \,:\, 0\le k^*\le k \text{ and } r_{i,k}+(\mu_r)_{i,k}h\ge r_{i+1,k^*}\},$$
$$K_u(i,k)=\{k^*\in\{1,...,i\} \,:\, k+1\le k^*\le i+1 \text{ and } r_{i,k}+(\mu_r)_{i,k}h\le r_{i+1,k^*}\}$$

with the understanding $k_d(i,k)=0$ if $K_d(i,k)=\emptyset$ and $k_u(i,k)=i+1$ if $K_u(i,k)=\emptyset$. The probability that the process jumps to $(i+1,k_u(i,k))$ is set as

$$p_{i,k}=0\vee\frac{(\mu_r)_{i,k}h+r_{i,k}-r_{i+1,k_d(i,k)}}{r_{i+1,k_u(i,k)}-r_{i+1,k_d(i,k)}}\wedge 1$$

and of course, the jump to $(i+1,k_d(i,k))$ happens with probability $1-p_{i,k}$. The behavior of the up and down jumps is given in the following

**Lemma 4.5.1.** *Let $\theta_*<\theta$ and $\theta^*>\theta$ be such that*

$$0<\theta_*<\frac{(\theta\wedge r_0)}{2} \quad \textit{and} \quad \theta^*>2(\theta\vee r_0).$$

*Then there exists a positive constant $h_1=h_1(\theta_*,\theta^*,\kappa,\theta,\sigma_r)<1$ such that for every $h<h_1$ the following statements hold.*

*(i) If $0\le r_{i,k}<\theta_*\sqrt{h}$ then $k_d(i,k)=k$ and $k_u(i,k)\in\{k+1,\dots,i+1\}$. Moreover, there exists a positive constant $C_*>0$ such that*

$$|r_{i+1,k_d(i,k)}-r_{i,k}|<C_*h^{3/4} \quad \textit{and} \quad |r_{i+1,k_u(i,k)}-r_{i,k}|<C_*h^{3/4}. \tag{4.5.2}$$

*(ii) If $\theta_*\sqrt{h}\le r_{i,k}\le\theta^*/\sqrt{h}$ then $k_d(i,k)=k$ and $k_u(i,k)=k+1$. Moreover, one has*

$$r_{i+1,k_d(i,k)}-r_{i,k}=-\sigma_r\sqrt{r_{i,k}h}+\frac{\sigma_r^2}{4}h \quad \textit{and} \quad r_{i+1,k_u(i,k)}-r_{i,k}=\sigma_r\sqrt{r_{i,k}h}+\frac{\sigma_r^2}{4}h\,. \tag{4.5.3}$$

*Proof.* We start by considering $h_1\in(0,1)$ and in what follows, we will “calibrate” the value of $h_1$.

$(i)$ Let us first notice that since $r_{i,k}<\theta$ one has $(\mu_r)_{i,k}>0$ and then for every $k^*\le k$ one has $r_{i+1,k^*}\le r_{i,k}\le r_{i,k}+(\mu_r)_{i,k}h$, which gives $k_d(i,k)=k$. Let us prove (4.5.2) for $k_d(i,k)$.

If $r_{i,k} = 0$ then $r_{i+1,k} = 0$ as well, and (4.5.2) trivially holds. If instead $0 < r_{i,k} < \theta_* \sqrt{h}$, then one can have both $r_{i+1,k} = 0$ and $r_{i+1,k} > 0$. In the first case, it must be

$$0 \geq R_{i+1,k} = R_{i,k} - \sqrt{h},$$

so that one actually has $0 < R_{i,k} \leq \sqrt{h}$ and then $r_{i,k} \leq \sigma_r^2 h/4$. Therefore, $|r_{i+1,k_d} - r_{i,k}| = r_{i,k} \leq \sigma_r^2 h/4$, and (4.5.2) holds. Consider now the second case, that is $0 < r_{i,k} < \theta_* \sqrt{h}$ and $r_{i+1,k} > 0$. Then it must be

$$r_{i+1,k} = \frac{\sigma_r^2}{4} R_{i+1,k}^2 = \frac{\sigma_r^2}{4} (R_{i,k} - \sqrt{h})^2 = r_{i,k} - \sigma_r \sqrt{r_{i,k} h} + \frac{\sigma_r^2}{4} h$$

so that

$$|r_{i+1,k_d(i,k)} - r_{i,k}| = \Big| - \sigma_r \sqrt{r_{i,k} h} + \frac{\sigma_r^2}{4} h \Big| \leq \sigma_r \sqrt{r_{i,k} h} + \frac{\sigma_r^2}{4} h < \Big( \sigma_r \sqrt{\theta_*} + \frac{\sigma_r^2}{4} \Big) h^{3/4}$$

and (4.5.2) again holds.
Let us now discuss the up-jump. We notice that

$$r_{i+1,i+1} - r_{i,k} - (\mu_r)_{i,k} h \geq r_0 - \theta_* \sqrt{h} - \kappa\theta h \geq r_0 - \theta_* - \kappa\theta h.$$

So, by taking $h_1 < (r_0 - \theta_*)/(\kappa\theta)$ we get $K_u(i,k) \neq \emptyset$, and we can proceed by looking for the smallest integer $k^* \geq k+1$ such that $k^* \leq i+1$ and the following statements hold:

$$r_{i+1,k^*} \geq r_{i,k} + (\mu_r)_{i,k} h \quad \text{and} \quad r_{i+1,k^*-1} < r_{i,k} + (\mu_r)_{i,k} h. \tag{4.5.4}$$

Notice that in particular the first condition gives $r_{i+1,k^*} > 0$. Assume first the case $r_{i+1,k^*-1} = 0$, that is $R_{i+1,k^*} > 0$ and $R_{i+1,k^*-1} \leq 0$. Then,

$$0 < R_{i+1,k^*} = R_{i+1,k^*-1} + 2\sqrt{h} \leq 2\sqrt{h},$$

so that $r_{i+1,k^*} \leq \sigma_r^2 h$. Since $r_{i+1,k^*} \geq r_{i,k}$, one gets $|r_{i+1,k^*} - r_{i,k}| \leq r_{i+1,k^*} \leq \sigma_r^2 h$, and this proves (4.5.2). So, it remains to study the case $r_{i+1,k^*-1} > 0$. Here, we have

$$\begin{aligned} r_{i+1,k^*} &= \frac{\sigma_r^2}{4} R_{i+1,k^*}^2 = \frac{\sigma_r^2}{4} \big( R_{i+1,k^*-1} + 2\sqrt{h} \big)^2 \\ &= r_{i+1,k^*-1} + 2\sigma_r \sqrt{r_{i+1,k^*-1} h} + \sigma_r^2 h. \end{aligned}$$

Now, by using (4.5.4) we get

$$\begin{aligned} 0 \leq r_{i+1,k^*} - r_{i,k} &= r_{i+1,k^*-1} - r_{i,k} + 2\sigma_r \sqrt{r_{i+1,k^*-1} h} + \sigma_r^2 h \\ &< (\mu_r)_{i,k} h + 2\sigma_r \sqrt{(r_{i,k} + (\mu_r)_{i,k} h) h} + \sigma_r^2 h \end{aligned}$$

and by recalling that $r_{i,k} \leq \theta_* \sqrt{h}$, we get (4.5.2) also in this last case.

$(ii)$ Here, we split our reasonings in two different cases: $(ii.a)$ $\theta_* \sqrt{h} \leq r_{i,k} \leq \theta$ and $(ii.b)$ $\theta \leq r_{i,k} \leq \theta^*/\sqrt{h}$.

Assume $(ii.a)$. As for the down-jump, here we have $(\mu_r)_{i,k} > 0$. Since for every $k^* \leq k$ one has $r_{i+1,k^*} \leq r_{i,k}$, we can immediately conclude that $k_d(i,k) = k$. Concerning the up-jump, we first notice that for $k^* \geq k+1$ one has $R_{i+1,k^*} = R_{i,k} + (2(k^*-k)-1)\sqrt{h} > 0$, so that using the square relation with $R$ we can write

$$r_{i+1,k^*} - r_{i,k} = (2(k^*-k)-1)\sigma_r\sqrt{r_{i,k}h} + \frac{(2(k^*-k)-1)^2}{4}\sigma_r^2 h.$$

Therefore, we must look for the smallest integer $k^* \geq k+1$ such that the above r.h.s. is larger or equal to $(\mu_r)_{i,k}h$, and this reduces to require that

$$2(k^*-k)-1 \geq 2\frac{\sqrt{r_{i,k}+(\mu_r)_{i,k}h} - \sqrt{r_{i,k}}}{\sigma_r\sqrt{h}}.$$

So, we prove that we can choose $h_1$ such that for every $h < h_1$ the above inequality holds for $k^* = k+1$, that is

$$1 \geq 2\frac{\sqrt{r_{i,k}+(\mu_r)_{i,k}h} - \sqrt{r_{i,k}}}{\sigma_r\sqrt{h}}.$$

In fact, by recalling that $0 \leq (\mu_r)_{i,k} = \kappa(\theta - r_{i,k}) \leq \kappa\theta$ and $r_{i,k} \geq \theta_*\sqrt{h}$, we can write

$$\begin{aligned} 2\frac{\sqrt{r_{i,k}+(\mu_r)_{i,k}h} - \sqrt{r_{i,k}}}{\sigma_r\sqrt{h}} &= \frac{2(\mu_r)_{i,k}\sqrt{h}}{\sigma_r(\sqrt{r_{i,k}+(\mu_r)_{i,k}h} + \sqrt{r_{i,k}})} \\ &\leq \frac{\kappa\theta\sqrt{h}}{\sigma_r\sqrt{r_{i,k}}} \leq \frac{\kappa\theta}{\sigma_r\sqrt{\theta_*}}\, h^{1/4} \end{aligned}$$

and the statement actually holds for $h < h_1 < (\frac{\sigma_r\sqrt{\theta_*}}{\kappa\theta})^4$.

$(ii.b)$ This case can be treated similarly to the previous one, so we omit the proof.

□

**Remark 4.5.2.** *Lemma 4.5.1 essentially states that when the approximating process for r is located in a restricted region near the lower bound zero, then it is "pushed up" away from the lower bound and therefore it is not absorbed by zero. Instead, when the discrete process is far enough from the lower bound, it behaves as the classical CRR tree that moves up or down by a quantity equal to* $\sqrt{h}$*. This indeed agrees with the Nelson and Ramaswamy method, that allows for "multiple jumps" only in restricted region where singularities may lead to numerical problems. In fact they explain that allowing multiple jumps everywhere in the tree may affect computational simplicity by increasing the number of nodes at a rapid rate and in fact we overcome this problem by forcing the jumps to remain inside the tree structure.*

**Remark 4.5.3.** *As a consequence of the proof of Lemma 4.5.1, for $h < h_1$ we get that $r_{i+1,k_u(i,k)} - r_{i+1,k_d(i,k)} > 0$, $(\mu_r)_{i,k}h + r_{i,k} - r_{i+1,k_d(i,k)} \geq 0$ and $r_{i+1,k_u(i,k)} - r_{i,k} - (\mu_r)_{i,k}h \geq 0$. Therefore, we can actually drop the $0\vee$ and $\wedge 1$ appearing in (4.4.1) and for $h < h_1$ we can directly write*

$$p_{i,k} = \frac{(\mu_r)_{i,k}h + r_{i,k} - r_{i+1,k_d(i,k)}}{r_{i+1,k_u(i,k)} - r_{i+1,k_d(i,k)}}. \tag{4.5.5}$$

*We also notice that whenever $r_{i,k} = 0$ one has $r_{i+1,k_d(i,k)} = 0$ as well and we have seen that $(\mu_r)_{ik}h \leq r_{i+1,k_u} = r_{i+1,k_u-1} + 2\sigma_r\sqrt{r_{i+1,k_u-1}h} + \sigma_r^2 h$. So, since $r_{i+1,k_u-1} < r_{i,k} + (\mu_r)_{i,k}h$, we get*

$$\kappa\theta h \leq r_{i+1,k_u} \leq (\sqrt{\kappa\theta} + \sigma_r)^2 h$$

*and by substituting in (4.4.1) we obtain*

$$p_{i,k} = \frac{\kappa\theta h}{r_{i+1,k_u(i,k)}} \in \Big[\frac{1}{(1+\sigma_r/\sqrt{\kappa\theta})^2}, 1\Big]. \tag{4.5.6}$$

*So, we state a lower bound for the up-jump probability, which is close to 1 when the volatility parameter $\sigma_r$ is is close to 0.*

### Up and down jumps for the stock price

Now the second step is the analysis of the jumps for the discrete process approximating the stock price $S$. It turns out that the behavior of the up and down jumps for the process $S$ is much easier to study. We recall that for a fixed node $(i,j)$, $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, i+1$, the lattice structure on $S$ is given by

$$S_{i,j} = e^{\sigma_S U_{i,j}} \quad \text{and} \quad U_{i,j} = U_0 + (2j - i)\sqrt{h}, \tag{4.5.7}$$

with $U_0 = \frac{1}{\sigma_S}\log s_0$. Here, the up and down jumps depend also on the position of the process $r$: for a fixed $k = 1, \ldots, i+1$, $j_d(i,j,k) = \max J_d(i,j,k)$ and $j_u(i,j,k) = \min J_d(i,j,k)$, where

$$J_d(i,j,k) = \{j^* \in \{0, ..., i-1\} \,:\, 0 \leq j^* \leq j \text{ and } S_{i,j} + (\mu_S)_{i,j,k}h \geq S_{i+1,j^*}\},$$
$$J_u(i,j,k) = \{j^* \in \{1, ..., i\} \,:\, j+1 \leq j^* \leq i+1 \text{ and } S_{i,j} + (\mu_S)_{i,j,k}h \leq S_{i+1,j^*}\}$$

again with the understanding $j_d(i,j,k) = 0$ if $J_d(i,j,k) = \emptyset$ and $j_u(i,j,k) = i+1$ if $J_u(i,j,k) = \emptyset$. The probability that the process jumps to $(i+1, j_u(i,j,k))$ is set as

$$\hat{p}_{i,j,k} = 0 \vee \frac{(\mu_S)_{i,j,k}h + S_{i,j} - S_{i+1,j_d(i,j,k)}}{S_{i+1,j_u(i,j,k)} - S_{i+1,j_d(i,j,k)}} \wedge 1 \tag{4.5.8}$$

and of course, the jump to $(i+1, j_d(i,j,k))$ happens with probability $1 - \hat{p}_{i,j,k}$.
The behavior of the up and down jumps is given in the following

**Lemma 4.5.4.** *Let $r_* > 0$ be fixed. Then there exists $h_2 = h_2(s_0, r_*, \sigma_S) < 1$ such that for every $h < h_2$ and $(i, k)$ such that $r_{i,k} \in [0, r_*]$ one has*

$$j_d(i,j,k) = j \quad \textit{and} \quad j_u(i,j,k) = j + 1.$$

*As a consequence, for $h < h_2$ and for every $(i, k)$ such that $r_{i,k} \in [0, r_*]$ one has*

$$S_{i+1,j_u(i,j,k)} - S_{i,j} = S_{i,j}(e^{\sigma_S\sqrt{h}} - 1) \quad \textit{and} \quad S_{i+1,j_d(i,j,k)} - S_{i,j} = S_{i,j}(e^{-\sigma_S\sqrt{h}} - 1). \tag{4.5.9}$$

*Proof.* First of all, notice that for $j^* \le j$ one has $U_{i+1,j^*} \le U_{i,j}$, so that $S_{i+1,j^*} - S_{i,j} \le 0 \le (\mu_S)_{i,j,k}h$. This gives $j_d(i,j,k) = j$, for every $i, j, k$ and $h$, so that

$$S_{i+1,j_d(i,j,k)} - S_{i,j} = S_{i+1,j} - S_{i,j} = S_{i,j}(e^{-\sigma_S\sqrt{h}} - 1).$$

Concerning the up-jump, it is sufficient to prove that for every $h$ sufficiently small one has

$$S_{i+1,j+1} - S_{i,j} \ge (\mu_S)_{i,j,k}h,$$

that is equivalent to

$$S_{i,j}(e^{\sigma_S\sqrt{h}} - 1 - r_{i,k}h) \ge 0.$$

Since $e^x - 1 \ge x$ for $x > 0$, for $r_{i,k} \le r_*$ we can write

$$S_{i,j}(e^{\sigma_S\sqrt{h}} - 1 - r_{i,k}h) \ge S_{i,j}(\sigma_S\sqrt{h} - r_*h) = S_{i,j}\sqrt{h}(\sigma_S - r_*\sqrt{h}),$$

and the last term is positive for $h < h_2 < (\frac{\sigma_S}{r_*})^2$. Finally,

$$S_{i+1,j_u(i,j,k)} - S_{i,j} = S_{i+1,j+1} - S_{i,j} = S_{i,j}(e^{\sigma_S\sqrt{h}} - 1)$$

and this completes the proof. □

**Remark 4.5.5.** *We notice that Lemma 4.5.4 says that the discrete process $S$ does not need "multiple jumps" in order to preserve the matching conditions on the local mean and the local variance, but it is a tree whose jump sizes are everywhere equal to $\sqrt{h}$.*

**Remark 4.5.6.** *We can state a remark similar to Remark 4.5.3: in Lemma 4.5.4 we actually proved that for $h < h_2$ we have $S_{i+1,j_u(i,j,k)} - S_{i+1,j_d(i,j,k)} > 0$, $(\mu_S)_{i,j,k}h + S_{i,j} - S_{i+1,j_d(i,j,k)} \ge 0$ and $S_{i+1,j_u(i,j,k)} - (\mu_S)_{i,j,k}h - S_{i,k} \ge 0$. Therefore, as $h < h_2$, the up-jump probability in (4.4.4) can be rewritten as*

$$\hat{p}_{i,j,k} = \frac{(\mu_S)_{i,j,k}h + S_{i,j} - S_{i+1,j_d(i,j,k)}}{S_{i+1,j_u(i,j,k)} - S_{i+1,j_d(i,j,k)}}. \tag{4.5.10}$$

### The bivariate transition matrix

We are now ready to set up and discuss the transition matrix of the bivariate Markov chain. We recall that, starting from the node $(i,j,k)$, we have used the following notations: $q_{i,j_u,k_u}$, $q_{i,j_u,k_d}$, $q_{i,j_d,k_u}$ and $q_{i,j_d,k_d}$ stand for the probability to reach $(i+1,j_u,k_u)$, $(i+1,j_u,k_d)$, $(i+1,j_d,k_u)$ and $(i+1,j_d,k_d)$ respectively.

**Proposition 4.5.7.** *Let $r_*>0$ and $S_*>0$ be fixed and set $A_*=\{(i,j,k) : r_{i,k}\leq r_*, S_{i,j}\leq S_*\}$. Let $\theta_*$ be as in Lemma 4.5.1 and $(i,j,k)\in A_*$. We set:*

*i) if $(i,j,k)\in A_*$ and $r_{i,k}\leq \theta_*\sqrt{h}$ then*

$$\begin{aligned} q_{i,j_u,k_u} &= \hat{p}_{i,j_u,k_u}p_{i,k_u}, & q_{i,j_u,k_d} &= \hat{p}_{i,j_u,k_u}(1-p_{i,k_u}),\\ q_{i,j_d,k_u} &= (1-\hat{p}_{i,j_u,k_u})p_{i,k_u}, & q_{i,j_d,k_d} &= (1-\hat{p}_{i,j_u,k_u})(1-p_{i,k_u}); \end{aligned}$$

*ii) if $(i,j,k)\in A_*$ and $r_{i,k}\geq \theta_*\sqrt{h}$ then $q_{i,j_u,k_u}$, $q_{i,j_u,k_d}$, $q_{i,j_d,k_u}$ and $q_{i,j_d,k_d}$ are set as the solutions of the linear system 4.4.5.*

*Then there exists $h_3<1$ and a positive constant $C$ such that for every $h<h_3$ and $(i,j,k)\in A_*$ the above probabilities are actually well defined.*

*Proof.* We fix the node $(S_{i,j},r_{i,k})$, with $(i,j,k)\in A_*$.
*i)* If $r_{i,k}=0$, the defined transition probabilities solve system (4.4.5). And if $r_{i,k}$ is small enough then the transition probabilities are supposed to be close to the ones in 0. So, what is said in *i)* is that for $r_{i,k}$ positive but small, we just consider the behavior in 0.
*ii)* We assume here that $r_{i,k}\geq\theta_*\sqrt{h}$. If $M$ denote the $4\times 4$ matrix underlying the linear system (4.4.5), then straightforward computations give

$$\det M = -(S_{i+1,j_u(i,j,k)}-S_{i+1,j_d(i,j,k)})(r_{i+1,k_u(i,k)}-r_{i+1,k_d(i,k)})$$

which is non null because, by construction, both factors are positive. So, a unique solution $x\in\mathbb{R}^4$ really exists for every $h$. One has only to check that this actually gives a probability distribution, and this reduces to check that all entries of $x$ are non negative. We set the solution as follows:

$$\begin{aligned} x_1 &= \hat{p}_{i,j_u,k_u}p_{i,k_u}(1+g_{i,j,k}(\rho)), & x_2 &= \hat{p}_{i,j_u,k_u}(1-p_{i,k_u})(1-g_{i,j,k}(\rho)),\\ x_3 &= (1-\hat{p}_{i,j_u,k_u})p_{i,k_u}(1-g_{i,j,k}(\rho)), & x_4 &= (1-\hat{p}_{i,j_u,k_u})(1-p_{i,k_u})(1+g_{i,j,k}(\rho)) \end{aligned}$$

(just to be clear, we have implicity given to the four transition nodes the following ordering: $(i+1,j_u,k_u)$, $(i+1,j_u,k_d)$, $(i+1,j_d,k_u)$, $(i+1,j_d,k_d)$). It is clear that $x$ solves the first three equations. As for the fourth one, we take $h<h_1\wedge h_2$, $h_1$ and $h_2$ given in Lemma 4.5.1 and Lemma 4.5.4 respectively, so that we can use both (4.5.5) and (4.5.10). And easy

computations give

$$\begin{aligned}
g_{i,j,k}(\rho) =& \frac{\rho\sigma_r\sigma_S\sqrt{r_{i,k}}\,S_{i,j}h - (\mu_r)_{i,k}(\mu_S)_{i,j,k}h^2}{(\Delta_u S_{i,j,k}\hat{p}_{i,j,k} - \Delta_d S_{i,j,k}(1-\hat{p}_{i,j,k}))(\Delta_u r_{i,k}p_{i,k} - \Delta_d r_{i,k}(1-p_{i,k}))}\\
=& \frac{\sigma_r\sigma_S\sqrt{r_{i,k}}\,S_{i,j}h}{(\Delta_u S_{i,j,k}\hat{p}_{i,j,k} - \Delta_d S_{i,j,k}(1-\hat{p}_{i,j,k}))(\Delta_u r_{i,k}p_{i,k} - \Delta_d r_{i,k}(1-p_{i,k}))}\times\\
&\times\Big(\rho - \frac{(\mu_r)_{i,k}(\mu_S)_{i,j,k}h^2}{\sigma_r\sigma_S\sqrt{r_{i,k}}\,S_{i,j}h}\Big),
\end{aligned}$$

in which we have set $\Delta_u S_{i,j,k} = S_{i+1,j_u(i,j,k)} - S_{i,j}$, $\Delta_d S_{i,j,k} = S_{i+1,j_d(i,j,k)} - S_{i,j}$, $\Delta_u r_{i,k} = r_{i+1,k_u(i,k)} - r_{i,k}$ and $\Delta_d r_{i,k} = r_{i+1,k_d(i,k)} - r_{i,k}$. So, we only need to show that for small values of $h$ one gets

$$\sup_{(i,j,k)\in A_* \text{ and } r_{i,k}\geq\theta_*\sqrt{h}} |g(\rho)| < 1.$$

We write

$$g_{i,j,k}(\rho) = \frac{1}{\alpha_{i,j,k}}\Big(\rho - \frac{(\mu_r)_{i,k}(\mu_S)_{i,j,k}h^2}{\sigma_r\sigma_S\sqrt{r_{i,k}}\,S_{i,j}h}\Big).$$

with

$$\alpha_{i,j,k} = \frac{(\Delta_u S_{i,j,k}\hat{p}_{i,j,k} - \Delta_d S_{i,j,k}(1-\hat{p}_{i,j,k}))(\Delta_u r_{i,k}p_{i,k} - \Delta_d r_{i,k}(1-p_{i,k}))}{\sigma_r\sigma_S\sqrt{r_{i,k}}\,S_{i,j}h}$$

We use (4.5.3), (4.5.9) and we write, for $h$ small,

$$\begin{aligned}
\Delta_u S_{i,j,k} &= S_{i,j}e^{\sigma_S\sqrt{h}}(1-e^{-\sigma_S\sqrt{h}}) \geq S_{i,j}(1-e^{-\sigma_S\sqrt{h}}) = -\Delta_d S_{i,j,k},\\
\Delta_u r_{i,k} &= \sigma_r\sqrt{r_{i,k}h} + \frac{\sigma_r^2}{4}h \geq \sigma_r\sqrt{r_{i,k}h} - \frac{\sigma_r^2}{4}h = -\Delta_d r_{i,k},
\end{aligned}$$

so that

$$\alpha_{i,j,k} \geq \frac{S_{i,j}(1-e^{-\sigma_S\sqrt{h}})(\sigma_r\sqrt{r_{i,k}h} - \frac{\sigma_r^2}{4}h)}{\sigma_r\sigma_S\sqrt{r_{i,k}}S_{i,j}h} = \frac{(1-e^{-\sigma_S\sqrt{h}})(\sigma_r\sqrt{r_{i,k}h} - \frac{\sigma_r^2}{4}h)}{\sigma_r\sigma_S\sqrt{r_{i,k}}h}.$$

By recalling that $1-e^{-\sigma_S\sqrt{h}} \geq \sigma_S\sqrt{h} - \frac{\sigma_S^2}{2}he^{\sigma_S}$, we get

$$\alpha_{i,j,k} \geq \left(1 - \frac{\sigma_S e^{\sigma_S}}{2}\sqrt{h}\right)\left(1 - \frac{\sigma_r}{4}\sqrt{\frac{h}{r_{i,k}}}\right) \geq \left(1 - \frac{\sigma_S e^{\sigma_S}}{2}h^{\frac{1}{2}}\right)\left(1 - \frac{\sigma_r}{4\sqrt{\theta_*}}h^{\frac{1}{4}}\right)$$

in which we have used that $r_{i,k}\geq\theta_*\sqrt{h}$. Setting $c_* = \frac{\sigma_r}{4\sqrt{\theta_*}} \wedge \frac{\sigma_S e^{\sigma_S}}{2}$, we get

$$\alpha_{i,j,k} \geq (1 - c_* h^{\frac{1}{4}})^2.$$

Therefore, we obtain

$$\begin{aligned}|g_{i,j,k}(\rho)| &\le \frac{1}{(1-c_*h^{\frac14})^2}\left|\rho-\frac{(\mu_r)_{i,k}(\mu_S)_{i,j,k}h^2}{\sigma_r\sigma_S\sqrt{r_{i,k}}S_{i,j}h}\right|\\ &\le \frac{1}{(1-c_*h^{\frac14})^2}\left(|\rho|+\frac{(\mu_r)_{i,k}(\mu_S)_{i,j,k}h^2}{\sigma_r\sigma_S\sqrt{r_{i,k}}S_{i,j}h}\right)\\ &\le \frac{1}{(1-c_*h^{\frac14})^2}\left(|\rho|+\frac{\kappa(\theta+r_*)}{\sigma_r\sqrt{\theta_*}}h^{\frac14}+\frac{r_*}{\sigma_S}h^{\frac12}\right)\end{aligned}$$

and since $|\rho|<1$, the last r.h.s. can be set less than 1 for $h$ small enough. □

### The Markov chain and the weak convergence

We can now state the main result. In order to do this, we set $(S_i^h, r_i^h)_{i=0,1,\dots,n}$ the Markov chain running on the bivariate lattice structure:

- $S_0^h = s_0$ and $r_0^h = r_0$;
- at time $ih$, the state-space for the pair $(S_i^h, r_i^h)$ is given by $\{(S_{i,j}, r_{i,k}) \,:\, j,k = 0,1,\dots,i\}$;
- from time $ih$ to time $(i+1)h$ the transition law on $\mathbb{R}^2$ is given by

$$\begin{aligned}\Pi_h(r_{i,k},S_{i,j};dx) =& q_{i,j_u,k_u}\delta_{\{(S_{i+1,j_u},r_{i+1,k_u})\}}(dx)+q_{i,j_u,k_d}\delta_{\{(S_{i+1,j_u},r_{i+1,k_d})\}}(dx)+\\ &+q_{i,j_d,k_u}\delta_{\{(S_{i+1,j_d},r_{i+1,k_u})\}}(dx)+q_{i,j_d,k_d}\delta_{\{(S_{i+1,j_d},r_{i+1,k_d})\}}(dx),\end{aligned}$$

  where $\delta_{\{a\}}$ denotes here the Dirac mass in $a\in\mathbb{R}^2$ and the above probabilities are given in Proposition 4.5.7.

We set now $(\bar S_t^h, \bar r_t^h)_{t\in[0,T]}$ as the continuous-time process defined through the piecewise constant and càdlàg interpolation in time of the chain: for $t\in[ih,(i+1)h)$,

$$\bar S_t^h = S_i^h \quad\text{and}\quad \bar r_t^h = r_i^h.$$

We now state the convergence in law of the Markov chain $(r_i^h, S_i^h)_{i=0,1,\dots,n}$ to the diffusion process of our interest.

**Theorem 4.5.8.** *As $h\to 0$, the family of Markov processes $(\bar S^h, \bar r^h)_h$ converges in law on the space $D([0,T];\mathbb{R}^2)$ to the diffusion process $(S,r)$ solution to the equations (4.2.1)-(4.2.2).*

*Proof.* The proof is standard and is based on the convergence results of Nelson and Ramaswamy ([65], 1990), Ethier and Kurtz ([32], 1986), Kushner and Dupuis ([58], 1992) or

Stroock and Varadhan ([74], 1979) described in Section 1.1. To simplify the notations, let us set

$$\begin{aligned}
\mathcal{M}^S_{i,j,k}(l) &= \mathbb{E}\big((S^h_{i+1}-S^h_i)^l \,|\, (S^h_i, r^h_i) = (S_{i,j}, r_{i,k})\big), \quad l = 1,2,4,\\
\mathcal{M}^r_{i,j,k}(l) &= \mathbb{E}\big((r^h_{i+1}-r^h_i)^l \,|\, (S^h_i, r^h_i) = (S_{i,j}, r_{i,k})\big), \quad l = 1,2,4,\\
\mathcal{M}^{S,r}_{i,j,k} &= \mathbb{E}\big((S^h_{i+1}-S^h_i)(r^h_{i+1}-r^h_i) \,|\, (S^h_i, r^h_i) = (S_{i,j}, r_{i,k})\big).
\end{aligned}$$

We stress that here the mean is indicated simply with $\mathbb{E}$ also if we should write $\mathbb{E}_{\mathbb{P}_h}$ that is related to the probability measure $\mathbb{P}_h$ of the probability space in which the process $(\bar{S}^h_t, \bar{r}^h_t)$ is defined. But we do not care these notations for simplicity. Moreover, it is clear that $\mathcal{M}^S_{i,j,k}(l)$ is the local moment of order $l$ at time $ih$ related to $S$, as well as $\mathcal{M}^r_{i,k}(l)$ is similar but related to the component $r$, and $\mathcal{M}^{S,r}_{i,j,k}$ is the local cross-moment of the two components at the generic time step $i$.
So, the proof of the theorem relies in checking that for $r_* > 0$ and $S_* > 0$ fixed, setting $A_* = \{(i,j,k) \,:\, r_{i,k} \le r_*, S_{i,j} \le S_*\}$, then the following properties $i)$, $ii)$ and $iii)$ hold:

$i)$ (convergence of the local drift)

$$\lim_{h\to 0} \sup_{(i,j,k)\in A_*} \frac{1}{h} \left|\mathcal{M}^S_{i,j,k}(1) - (\mu_S)_{i,j,k} h\right| = 0,$$
$$\lim_{h\to 0} \sup_{(i,j,k)\in A_*} \frac{1}{h} \left|\mathcal{M}^r_{i,j,k}(1) - (\mu_r)_{i,k} h\right| = 0;$$

$ii)$ (convergence of the local diffusion coefficient)

$$\lim_{h\to 0} \sup_{(i,j,k)\in A_*} \frac{1}{h} \left|\mathcal{M}^S_{i,j,k}(2) - \sigma_S^2 S_{i,j}^2 h\right| = 0,$$
$$\lim_{h\to 0} \sup_{(i,j,k)\in A_*} \frac{1}{h} \left|\mathcal{M}^r_{i,j,k}(2) - \sigma_r^2 r_{i,k} h\right| = 0$$
$$\lim_{h\to 0} \sup_{(i,j,k)\in A_*} \frac{1}{h} \left|\mathcal{M}^{S,r}_{i,j,k} - \rho\sigma_r\sigma_S S_{i,j}\sqrt{r_{i,k}} h\right| = 0;$$

$iii)$ (fast convergence to 0 of the fourth order local moments)

$$\lim_{h\to 0} \sup_{(i,j,k)\in A_*} \frac{1}{h} \mathcal{M}^S_{i,j,k}(4) = 0,$$
$$\lim_{h\to 0} \sup_{(i,j,k)\in A_*} \frac{1}{h} \mathcal{M}^r_{i,j,k}(4) = 0.$$

So, we consider $h$ such that $h < \min(h_1, h_2, h_3)$, with $h_1$, $h_2$ and $h_3$ given in Lemma 4.5.1, Lemma 4.5.4 and Proposition 4.5.7 respectively, so that we can use both Remark 4.5.3 and Remark 4.5.6.

*Proof of i).* By using (4.5.5) and (4.5.10), we immediately get that

$$\mathcal{M}^S_{i,j,k}(1) = (\mu_S)_{i,j,k}h \quad \text{and} \quad \mathcal{M}^r_{i,j,k}(1) = (\mu_r)_{i,k}h,$$

and $i$) holds.
*Proof of ii).* As for the cross-moment, this is immediate when $r_{i,k} \geq \theta_*\sqrt{h}$: the transition probabilities solve system (4.4.5), whose last equation is actually $\mathcal{M}^{S,r}_{i,j,k} = \rho\sigma_r\sigma_S S_{i,j}\sqrt{r_{i,k}}h$. Consider now the case $r_{i,k} < \theta_*\sqrt{h}$: here the transition probabilities are of the product-type, so that $\mathcal{M}^{S,r}_{i,j,k} = (\mu_r)_{i,k}(\mu_S)_{i,j,k}h^2$. Therefore,

$$\begin{aligned}|\mathcal{M}^{S,r}_{i,j,k} - \rho\sigma_r\sigma_S S_{i,j}\sqrt{r_{i,k}}h| &\leq (\mu_r)_{i,k}(\mu_S)_{i,j,k}h^2 + \rho\sigma_r\sigma_S S_{i,j}\sqrt{r_{i,k}}h\\ &\leq \kappa(\theta + r_*)r_*S_*h^2 + \rho\sigma_r\sigma_S S_*\sqrt{\theta_*}h^{5/4}\\ &\leq Ch^{5/4}.\end{aligned}$$

As for the second order moment, again by using (4.5.5) and (4.5.10), it follows that

$$\begin{aligned}\mathcal{M}^S_{i,j,k}(2) &= (S_{i+1,j_u} + S_{i+1,j_d} - 2S_{i,j})(\mu_S)_{i,j,k}h + (S_{i+1,j_u} - S_{i,j})(S_{i,j} - S_{i+1,j_d})\\ \mathcal{M}^r_{i,k}(2) &= (r_{i+1,k_u} + r_{i+1,k_d} - 2r_{i,k})(\mu_r)_{i,k}h + (r_{i+1,k_u} - r_{i,k})(r_{i,k} - r_{i+1,k_d}).\end{aligned}$$

So, by using (4.5.9), we get

$$\mathcal{M}^S_{i,j,k}(2) = S^2_{i,j}\Big(2r_{i,k}\big(\cosh(\sigma_S\sqrt{h}) - 1\big)h + e^{-\sigma_S\sqrt{h}}\big(e^{\sigma_S\sqrt{h}} - 1\big)^2\Big).$$

Therefore, for $(i,j,k) \in A_*$ we get

$$|\mathcal{M}^S_{i,j,k}(2) - \sigma_S^2 S^2_{i,j}h| \leq 2S_*^2 r_*(\cosh(\sigma_S\sqrt{h}) - 1\big)h + S_*^2\Big|e^{-\sigma_S\sqrt{h}}\big(e^{\sigma_S\sqrt{h}} - 1\big)^2 - \sigma_S^2 h\Big|$$

and the statement holds. Concerning the second moment on $r$, we first study the case $r_{i,k} \leq \theta_*\sqrt{h}$, $\theta_*$ as in Lemma 4.5.1. Then by using (4.5.2) we have

$$|\mathcal{M}^r_{i,k}(2) - \sigma_r^2 r_{i,k}h| \leq 2C_*\kappa\theta h^{3/4+1} + C_*^2 h^{3/2} \leq Ch^{3/2}.$$

Consider now the case $\theta_*\sqrt{h} \leq r_{i,k} \leq r_*$. Then, for $h \leq (\theta^*/r_*)^2$ then $r_{i,k} \leq \theta^*/\sqrt{h}$, $\theta^*$ as in Lemma 4.5.1. So, we use (4.5.3) and we obtain

$$|\mathcal{M}^r_{i,k}(2) - \sigma_r^2 r_{i,k}h| \leq \kappa(\theta + r_*)\frac{\sigma_r^2}{4}\,h^2 + \frac{\sigma_r^4}{16}h^2$$

and the statements again holds.
*Proof of iii).* First, straightforward computations give

$$\begin{aligned}\mathcal{M}^S_{i,j,k}(4) =&\big((\mu_S)_{i,j,k}h + S_{i,j} - S_{i+1,j_d}\big)(S_{i+1,j_u} + S_{i+1,j_d} - 2S_{i,j})\times\\ &\times\Big((S_{i+1,j_u} - S_{i,j})^2 + (S_{i+1,j_d} - S_{i,j})^2\Big) + (S_{i+1,j_d} - S_{i,j})^4\\ \mathcal{M}^r_{i,k}(4) =&\big((\mu_r)_{i,k}h + r_{i,k} - r_{i+1,k_d}\big)(r_{i+1,k_u} + r_{i+1,k_d} - 2r_{i,k})\times\\ &\times\Big((r_{i+1,k_u} - r_{i,k})^2 + (r_{i+1,k_d} - r_{i,k})^2\Big) + (r_{i+1,k_d} - r_{i,k})^4.\end{aligned}$$

As for the first quantity, we immediately get

$$\begin{aligned}\mathcal{M}^S_{i,j,k}(4) \leq & 4S_*^4(r_*h+1-e^{-\sigma_S\sqrt{h}})\big(\cosh(\sigma_S\sqrt{h})-1\big)2(e^{\sigma_S\sqrt{h}}-1)^2+\\ &+S_*^4(e^{-\sigma_S\sqrt{h}}-1)^4 \leq Ch^2\end{aligned}$$

Concerning the 4th moment for $r$, first notice that if $r_{i,k} \leq \theta_*\sqrt{h}$ then (4.5.2) gives $\mathcal{M}^r_{i,k}(4) \leq C_*h^3$. If instead $\theta_*\sqrt{h} \leq r_{i,k} \leq r_*$ then from (4.5.3) one gets

$$\begin{aligned}\mathcal{M}^r_{i,k}(4) \leq & \Big(\kappa(\theta+r_*)h+\sigma_r\sqrt{r_*h}+\frac{\sigma_r^2}{4}h\Big)\Big(\sigma_r\sqrt{r_*h}+\frac{\sigma_r^2}{4}h\Big)^2\sigma_r^2h+\\ &+\Big(\sigma_r^2\sqrt{r_*h}+\frac{\sigma_r^2}{4}h\Big)^4\\ &\leq Ch^2.\end{aligned}$$

So, the proof is completed. □

**Remark 4.5.9.** *We still remark that in order to obtain the weak convergence we only request that $\kappa > 0$ and $\theta > 0$ and we do not need the Feller condition ($2\kappa\theta \geq \sigma_r^2$) or the "convergence condition" ($4\kappa\theta \geq \sigma_r^2$) to hold. This indeed justifies in theory the fact that the proposed lattice approach permits accurate and efficient option pricing numerical results without any restriction on the model parameters.*

## 4.5.2 The convergence of the prices

Theorem 4.5.8 of the previous Section can be used in order to discuss the convergence of European and American prices computed with the lattice algorithm to their corresponding continuous values. In the European case the reasoning is immediate. In fact, since the binomial process weakly converges to the bivariate diffusion, if the option payoff is a continuous and bounded function, the convergence result is an immediate consequence of the continuous mapping theorem. In the American case, since the option valuation problem involves a control, the corresponding result is not so simple. However, thanks to the results proved in Amin and Khanna ([3], 1994) and Lamberton and Pagès ([60], 1990), it is possible to obtain under a fairly general set of conditions on the payoff function that also for the American prices the convergence holds. In particular, we essentially can weaken the boundedness condition on the payoff function by requiring a property of uniform integrability. To be precise, let $f(t,x) : [0,T] \times D([0,T]) \to [0,+\infty)$ denote a payoff function, so that the price in the continuous-time model and in the discrete model are given by

$$\sup_{\tau\in\mathcal{G}_{0,T}} \mathbb{E}\big(e^{-\int_0^\tau r_s ds}f(\tau,S)\big) \quad \text{and} \quad \sup_{\sigma\in\mathcal{G}^h_{0,T}} \mathbb{E}\big(e^{-\int_0^\sigma \bar{r}^h_s ds}f(\sigma,\bar{S}^h)\big)$$

respectively, where $\mathcal{G}_{0,T}$ and $\mathcal{G}^h_{0,T}$ denote the stopping times in $[0,T]$ w.r.t. the filtration $\mathcal{F}_t = \sigma((S_s,r_s) : s \leq t)$ and $\mathcal{F}^h_t = \sigma((\bar{S}^h_s,\bar{r}^h_s) : s \leq t)$ respectively. As remarked before, for the sake of simplicity, we use the symbol $\mathbb{E}$ to indicate the mean with respect to the probability measure $\mathbb{P}$ in which the process $(S, r)$ is defined and also the mean with respect to the probability measure $\mathbb{P}_h$ related to the process $(\bar{S}^h, \bar{r}^h)$. Let us consider the two following assumptions:

(H1) $f$ is a continuous function (in the product topology) and for every $x, y \in D([0,T])$ such that $x_s = y_s$ for each $s \in [0,t]$ then $f(t,x) = f(t,y)$;

(H2) there exists $\delta > 1$ and $h_* > 0$ s.t. $\sup_{h<h_*} \mathbb{E}\big(\sup_{t\leq T} |e^{-\int_0^t \bar{r}^h_s ds} f(t, \bar{S}^h)|^\delta\big) < \infty$.

Then, under the hypothesis (H1) and (H2), Theorem 4.5.8 allows one to get the convergence of the American price in the discrete-time model to the corresponding price in the continuous-time model. In fact one can repeat the arguments used in Amin and Khanna ([3], 1994) to treat our context. Then American put options can be numerically evaluated with the algorithm proposed in Section 4.4. But other options can be considered, for example when $f$ is continuous and fulfills the following polynomial-growth condition: there exist $C > 0$ and $\gamma > 1$ such that

$$\sup_{t\leq T} |f(t,x)| \leq C\big(1 + \sup_{t\leq T} |x_t|^\gamma\big). \tag{4.5.11}$$

In fact, if we prove that for every $p > 1$ there exists $h_* < 1$ such that

$$\sup_{h<h_*} \mathbb{E}\Big(\sup_{t\leq T} e^{-p\int_0^t \bar{r}^h_s ds} (\bar{S}^h_t)^p\Big) < \infty, \tag{4.5.12}$$

then (4.5.11) and (4.5.12) imply the assumption (H2), and if (H1) holds as well then the convergence of the binomial American price will follow. This is a development of the arguments used in Amin and Khanna ([3], 1994).
We now state the following result:

**Proposition 4.5.10.** *For every $p > 1$ there exist $h_* < 1$, a positive constant $C_{p,T}$ depending on $p, T$ and a universal constant $c > 0$ such that*

$$\sup_{h<h_*} \mathbb{E}\Big(\sup_{t\leq T} e^{-p\int_0^t \bar{r}^h_s ds} (\bar{S}^h_t)^p\Big) \leq C_{p,T}\, s_0^p \exp\Big(c\,\frac{p(p-1)}{2}\,\sigma_S^2\, T\Big).$$

**Proof.** We use here the notation $\mathbb{E}_i$ to denote the conditional expectation given the $\sigma$-algebra $\mathcal{F}^h_i = \sigma\big((S^h_k, r^h_k) \,:\, k \leq i\big)$, $i \leq n$. Moreover, we notice that

$$\sup_{t\leq T} e^{-p\int_0^t \bar{r}^h_s ds} (\bar{S}^h_t)^p = \sup_{i\leq n} e^{-p\sum_{k=0}^{i-1} r^h_k h} (S^h_i)^p,$$

with the understanding $\sum_{k=0}^{-1}(\cdot) = 0$, so we work with the quantity in the r.h.s. above.
For $i = 0, 1, \dots, n$, we set $\tilde{S}^h_i = \exp(-\sum_{k=0}^{i-1} r^h_k h) S^h_i$, so that

$$\tilde{S}^h_{i+1} = \tilde{S}^h_i \times e^{-r^h_i h} \xi^h_{i+1} \quad \text{with} \quad \xi^h_{i+1} = \frac{S^h_{i+1}}{S^h_i}.$$

We recall that, by construction, the conditional law of $S^h_{i+1}$ given $\mathcal{F}^h_i$ is the following:

- on the set $\{1+r_i^h h \le e^{\sigma_S\sqrt{h}}\}$, $S_{i+1}^h$ can assume the values $S_i^h e^{\sigma_S\sqrt{h}}$ and $S_i^h e^{-\sigma_S\sqrt{h}}$ with probability

$$\hat{p}_i^h = \frac{1+r_i^h h - e^{-\sigma_S\sqrt{h}}}{e^{\sigma_S\sqrt{h}} - e^{-\sigma_S\sqrt{h}}} \quad \text{and} \quad 1-\hat{p}_i^h = \frac{e^{\sigma_S\sqrt{h}} - 1 - r_i^h h}{e^{\sigma_S\sqrt{h}} - e^{-\sigma_S\sqrt{h}}}$$

  respectively; we also recall that, on this set, $\mathbb{E}_i(S_{i+1}^h) = S_i^h(1+r_i^h h)$;

- on the set $\{1+r_i^h h > e^{\sigma_S\sqrt{h}}\}$, $S_{i+1}^h$ assumes the (unique) value $S_i^h e^{\sigma_S\sqrt{h}}$.

As a consequence, we get

$$e^{-r_i^h h}\mathbb{E}_i(\xi_{i+1}^h) = e^{-r_i^h h}(1+r_i^h h)\mathbb{1}_{\{1+r_i^h h\le e^{\sigma_S\sqrt{h}}\}} + e^{-r_i^h h}e^{\sigma_S\sqrt{h}}\mathbb{1}_{\{1+r_i^h h> e^{\sigma_S\sqrt{h}}\}} \le 1$$

and

$$\Big(\xi_{i+1}^h - \mathbb{E}_i(\xi_{i+1}^h)\Big)\mathbb{1}_{\{1+r_i^h h> e^{\sigma_S\sqrt{h}}\}} = 0$$

Therefore, we can write

$$\begin{aligned}\tilde{S}_{i+1}^h &= \tilde{S}_i^h e^{-r_i^h h}\mathbb{E}_i(\xi_{i+1}^h) + \tilde{S}_i^h e^{-r_i^h h}\big(\xi_{i+1}^h - \mathbb{E}_i(\xi_{i+1}^h)\big)\\ &\le \tilde{S}_i^h + \tilde{S}_i^h e^{-r_i^h h}\big(\xi_{i+1}^h - (1+r_i^h h)\big)\mathbb{1}_{\{1+r_i^h h\le e^{\sigma_S\sqrt{h}}\}}.\end{aligned}$$

So, by setting

$$Y_{i+1}^h = e^{-r_i^h h}\big(\xi_{i+1}^h - (1+r_i^h h)\big)\mathbb{1}_{\{1+r_i^h h\le e^{\sigma_S\sqrt{h}}\}}$$

we get

$$\tilde{S}_{i+1}^h \le \tilde{S}_i^h + \tilde{S}_i^h Y_{i+1}^h = \tilde{S}_i^h(1+Y_{i+1}^h).$$

This first gives that $1+Y_{i+1}^h \ge 0$ for every $i$ and secondly, $\tilde{S}_i^h \le s_0 M_i^h$ for every $i = 0,1,...,n$, where

$$M_0^h = 1 \text{ and for } i = 1,\dots,n,\ M_i^h = \prod_{k=1}^i (1+Y_k^h).$$

Since $\sup_{i\le n}(\tilde{S}_i^h)^p \le s_0^p \sup_{i\le n}(M_i^h)^p$, we proceed by proving that $\sup_{i\le n}(M_i^h)^p$ is integrable and by studying an upper estimate for its expectation which is independent of the choice of $h$.

We first notice that $\mathbb{E}_i(Y_{i+1}^h) = 0$, so that $(M_i^h)_{i\le n}$ is a martingale. Let us prove that it is bounded in $L^p$ - this will allow us to apply the Doob inequality.

On the set $\{1+r_i^h h \le e^{\sigma_S\sqrt{h}}\}$, we can find $h_1$ such that for every $h<h_1$ we have $|\xi_{i+1}^h - 1| \le 2\sigma_S\sqrt{h}$ and $r_i^h h \le 2\sigma_S\sqrt{h}$. This gives that

$$|Y_{i+1}^h| \le 4\sigma_S\sqrt{h}$$

as $h < h_1$. By using the Taylor expansion of the function $x \mapsto (1+x)^p$ for $|x| \leq 4\sigma_S\sqrt{h}$, there exists a positive (non random) constant $c_{1,p}$ such that

$$(1+Y^h_{i+1})^p \leq 1 + pY^h_{i+1} + \frac{p(p-1)}{2}\,(Y^h_{i+1})^2 + c_{1,p}h\sqrt{h},$$

so that

$$\mathbb{E}_i\big((1+Y^h_{i+1})^p\big) \leq 1 + \frac{p(p-1)}{2}\,\mathbb{E}_i\big((Y^h_{i+1})^2\big) + c_{1,p}h\sqrt{h}.$$

Now, straightforward computations give

$$\mathbb{E}_i\big((Y^h_{i+1})^2\big) = e^{-2r^h_i h}(e^{\sigma_S\sqrt{h}} - 1 - r^h_i h)(1 + r^h_i h - e^{-\sigma_S\sqrt{h}})\mathbb{1}_{\{1+r^h_i h \leq e^{\sigma_S\sqrt{h}}\}}$$

and for $h$ small enough we have

$$\begin{aligned}\mathbb{E}_i\big((Y^h_{i+1})^2\big) &\leq \big(\sigma_S^2 h - (r^h_i h)^2 + c_2 h\sqrt{h}\big)\mathbb{1}_{\{1+r^h_i h \leq e^{\sigma_S\sqrt{h}}\}}\\ &\leq 5\sigma_S^2 h + c_2 h\sqrt{h}\end{aligned}$$

where $c_2$ denotes a suitable positive constant. So, there exist $h_* > 0$ and $c_p > 0$ such that for every $h < h_*$

$$\mathbb{E}_i\big((1+Y^h_{i+1})^p\big) \leq 1 + 5\frac{p(p-1)}{2}\,\sigma_S^2\,h + c_p h\sqrt{h}.$$

Now we are done, because

$$\mathbb{E}\big((M^h_{i+1})^p\big) = \mathbb{E}\Big((M^h_i)^p\mathbb{E}_i\big((1+Y^h_{i+1})^p\big)\Big) \leq \mathbb{E}\big((M^h_i)^p\big)\Big(1 + 5\frac{p(p-1)}{2}\,\sigma_S^2\,h + c_p h\sqrt{h}\Big)$$

and by iteration,

$$\begin{aligned}\mathbb{E}\big((M^h_{i+1})^p\big) &\leq \Big(1 + 5\frac{p(p-1)}{2}\,\sigma_S^2\,h + c_p h\sqrt{h}\Big)^{i+1}\\ &\leq \exp\Big(5\frac{p(p-1)}{2}\,\sigma_S^2\,h(i+1) + c_p h\sqrt{h}(i+1)\Big).\end{aligned}$$

In particular,

$$\sup_{i\leq n-1}\mathbb{E}\big((M^h_{i+1})^p\big) \leq \exp\Big(5\frac{p(p-1)}{2}\,\sigma_S^2\,T + c_p T\sqrt{h}\Big)$$

and this holds for every $h < h_*$. We can apply the Doob inequality and we get

$$\begin{aligned}\mathbb{E}\big(\sup_{i\leq n-1}(M^h_{i+1})^p\big) &\leq \Big(\frac{p}{p-1}\Big)^p \exp\Big(1 + 5\frac{p(p-1)}{2}\,\sigma_S^2\,T + c_p T\sqrt{h}\Big)\\ &\leq C_{p,T}\exp\Big(5\frac{p(p-1)}{2}\,\sigma_S^2 T\Big)\end{aligned}$$

and we stress the the above r.h.s. does not depend on $h$. The statement now follows.

## 4.6 Numerical results

In this Section we compare the performance of our lattice algorithm (called ACZ) with the procedures of Wei (WEI) and Hilliard, Schwartz and Tucker (HST) for the computation of European and American options in the BS-CIR model.
In the European and American option contracts we are dealing with, we consider a set of parameters already used in Hilliard, Schwartz and Tucker [46]: $s_0 = 100$, $\sigma_S = 0.25$, $r_0 = 0.06$, $\theta = 0.1$, $\kappa = 0.5$, $\rho = -0.25$. In order to study the numerical robustness of the algorithms, we choose different values for $\sigma_r$: we set $\sigma_r = 0.08, 0.35, 0.5, 1, 3$. Let us remark that for $\sigma_r = 0.35, 0.5, 1, 3$, the Feller condition $2\kappa\theta \geq \sigma_r^2$ is not satisfied at all. The parameters of the option contracts are the following: the strike is $K = 100$ and the maturity $T$ is varying, since we set $T = 1, 2$ years. Finally, the number of time steps $n$ varies: $n = 50, 100, 150, 200, 300$.
Tables 4.1 and 4.2 report European put option prices for $T = 1$ and $T = 2$ respectively. The benchmark value is obtained using a Monte Carlo with very large number of Monte Carlo simulation (10 million iterations) using the accurate Alfonsi ([1], 2010) discretization scheme for the CIR process with $M = 300$ discretization time steps (this method provides a Monte Carlo weak second-order scheme for the CIR process, without any restriction on its parameters). We also provide the results for American put option prices, as reported in Table 4.3 and Table 4.4 (no benchmarks are available in this case).
The numerical results show that our method provides very reliable and stable outcomes. Even if no odd results appear as $\sigma_S$ varies, on the contrary both WEI and HST fail when $\sigma_r$ increases. As already observed by Tian ([76], 1992) and Tian ([78], 1994), this follows from the procedure set up to approximate the CIR process and can be explained by looking at the behavior of the drift $\mu_R$ (see (4.3.1)) associated to the transformed process $R$, which is the one used to define the transition probabilities (see Remark 4.4.1). Indeed, one can write

$$\mu_R(R) = \frac{1}{2\sigma_r^2}\Big(\frac{4\kappa\theta - \sigma_r^2}{R} - \kappa\sigma_r^2 R\Big).$$

We recall that the convergence is achieved when $4\kappa\theta - \sigma_r^2 \geq 0$. As $\sigma_r = 0.5, 1, 3$, one gets $\sigma_r^2 - 4\kappa\theta > 0$, so that when $R \downarrow 0$ one has $\mu_R(R) \downarrow -\infty$ and $p_{i,k} \to 0$. This gives that, in some sense, the process $r$ tends to be absorbed in 0 (see Remark 4.4.1). As $\sigma_r = 0.35$ the Feller condition is not satisfied but the convergence condition holds, instead as $\sigma_r = 0.08$ both conditions are true.
We remark that the numerical results show that for $\sigma_r = 0.08$ the three methods are competitive, instead some differences start to arise when $\sigma_r = 0.35$. In fact, for this parameter value, also if the "convergence condition" holds, the WEI procedure is oscillating, instead the HST ad the ACZ algorithms are robust. Finally, for $\sigma_r = 0.50, 1, 3$, both WEI and HST present some troubles, while the ACZ procedure is still accurate and efficient.

| | $n$ | WEI | HST | ACZ | MC Benchmark |
|---|---|---|---|---|---|
| $\sigma_r$ =0.08 | 50 | 6.599590 | 6.579999 | 6.547556 | |
| | 100 | 6.596199 | 6.586245 | 6.569844 | (6.580864) |
| | 150 | 6.594964 | 6.588312 | 6.577486 | 6.586622 |
| | 200 | 6.594367 | 6.589255 | 6.581016 | (6.592380) |
| | 300 | 6.593705 | 6.590247 | 6.584744 | |
| $\sigma_r$ =0.35 | 50 | 6.525565 | 6.501327 | 6.456607 | |
| | 100 | 8.098863 | 6.485796 | 6.480764 | (6.487199) |
| | 150 | 6.506420 | 6.488178 | 6.488831 | 6.492901 |
| | 200 | 6.518722 | 6.453131 | 6.492840 | (6.498603) |
| | 300 | 6.506538 | 6.502675 | 6.496892 | |
| $\sigma_r$ =0.50 | 50 | 6.522023 | 6.495957 | 6.515875 | |
| | 100 | 6.606752 | nan | 6.543780 | (6.544551) |
| | 150 | 6.708564 | 6.430882 | 6.551733 | 6.550315 |
| | 200 | 6.588212 | 6.308088 | 6.556571 | (6.556078) |
| | 300 | 6.492437 | 6.280427 | 6.560239 | |
| $\sigma_r$ =1.00 | 50 | 7.397553 | 3.640714 | 7.054686 | |
| | 100 | 4.786837 | nan | 7.109525 | (7.153304) |
| | 150 | 8.046617 | 4.094663 | 7.123767 | 7.159471 |
| | 200 | 4.846171 | 4.082429 | 7.127271 | (7.165637) |
| | 300 | 7.340537 | 4.135390 | 7.126012 | |
| $\sigma_r$ =3.00 | 50 | 9.582836 | 0.042281 | 8.491568 | |
| | 100 | 3.026463 | 0.074981 | 8.636541 | (8.756826) |
| | 150 | 15.966217 | 0.085990 | 8.671644 | 8.763625 |
| | 200 | 9.103675 | 0.081284 | 8.648793 | (8.770423) |
| | 300 | 1.604996 | 0.084082 | 8.681638 | |

Table 4.1: *European put options with* $T = 1$*,* $s_0 = 100$*,* $K = 100$*,* $\sigma_S = 0.25$*,* $r_0 = 0.06$*,* $\theta = 0.1$*,* $\kappa = 0.5$*,* $\rho = -0.25$*,* $\sigma_r$ *varying.*

| | $n$ | WEI | HST | ACZ | MC Benchmark |
|---|---|---|---|---|---|
| $\sigma_r$ =0.08 | 50 | 7.113252 | 7.075978 | 7.044844 | |
| | 100 | 7.109863 | 7.090808 | 7.075287 | (7.090164) |
| | 150 | 7.108682 | 7.095834 | 7.085454 | 7.096171 |
| | 200 | 7.108057 | 7.098328 | 7.090541 | (7.102178) |
| | 300 | 7.107381 | 7.100858 | 7.095698 | |
| $\sigma_r$ =0.35 | 50 | 19.700288 | 6.894164 | 7.150781 | |
| | 100 | 7.392442 | 7.187941 | 7.184543 | (7.180155) |
| | 150 | 7.190269 | 7.138343 | 7.196207 | 7.186305 |
| | 200 | 8.938681 | 7.089212 | 7.201783 | (7.192455) |
| | 300 | 7.205803 | 7.103088 | 7.207558 | |
| $\sigma_r$ =0.50 | 50 | 7.590023 | nan | 7.532236 | |
| | 100 | 7.384775 | 7.151360 | 7.573197 | (7.575268) |
| | 150 | 8.099945 | 7.157489 | 7.590692 | 7.581702 |
| | 200 | 7.866820 | nan | 7.598707 | (7.588135) |
| | 300 | 8.627843 | 7.029057 | 7.600614 | |
| $\sigma_r$ =1.00 | 50 | 10.511291 | nan | 9.029287 | |
| | 100 | 10.377910 | 1.513991 | 9.113234 | (9.312621) |
| | 150 | 11.204261 | 1.603589 | 9.204665 | 9.319903 |
| | 200 | 3.979039 | nan | 9.192707 | (9.327185) |
| | 300 | 12.541620 | 1.839346 | 9.206984 | |
| $\sigma_r$ =3.00 | 50 | 0.003130 | 0.000216 | 11.801614 | |
| | 100 | 13.515823 | 0.000344 | 11.656249 | (12.046144) |
| | 150 | 12.799415 | 0.000594 | 11.820232 | 12.054222 |
| | 200 | 1.864718 | 0.000832 | 11.873238 | (12.062299) |
| | 300 | 24.377573 | 0.001110 | 11.919532 | |

Table 4.2: *European put options with* $T = 2$, $s_0 = 100$, $K = 100$, $\sigma_S = 0.25$, $r_0 = 0.06$, $\theta = 0.1$, $\kappa = 0.5$, $\rho = -0.25$, $\sigma_r$ *varying.*

| | $n$ | WEI | HST | ACZ |
|---|---|---|---|---|
| $\sigma_r$ =0.08 | 50 | 7.459623 | 7.422693 | 7.438208 |
| | 100 | 7.456394 | 7.437777 | 7.445392 |
| | 150 | 7.455096 | 7.442827 | 7.447790 |
| | 200 | 7.454461 | 7.445104 | 7.448865 |
| | 300 | 7.453750 | 7.447485 | 7.449971 |
| $\sigma_r$ =0.35 | 50 | 6.025684 | 7.539697 | 7.536313 |
| | 100 | 5.953750 | 7.544594 | 7.545435 |
| | 150 | 7.547122 | 7.547791 | 7.548400 |
| | 200 | 6.179678 | 7.554119 | 7.549846 |
| | 300 | 7.551032 | 7.550363 | 7.551299 |
| $\sigma_r$ =0.50 | 50 | 7.647938 | 7.596269 | 7.650304 |
| | 100 | 7.698584 | nan | 7.662723 |
| | 150 | 7.751229 | 7.586111 | 7.665955 |
| | 200 | 7.684290 | 7.472087 | 7.668003 |
| | 300 | 7.620723 | 7.627011 | 7.669464 |
| $\sigma_r$ =1.00 | 50 | 8.187062 | 6.511775 | 8.084019 |
| | 100 | 6.750600 | nan | 8.106474 |
| | 150 | 8.743109 | 6.506826 | 8.113036 |
| | 200 | 6.822003 | 6.501217 | 8.115025 |
| | 300 | 8.164598 | 6.534435 | 8.116760 |
| $\sigma_r$ =3.00 | 50 | 9.699021 | 2.902701 | 8.953802 |
| | 100 | 5.821256 | 3.165914 | 9.009359 |
| | 150 | 16.069688 | 3.405395 | 9.028517 |
| | 200 | 9.212406 | 3.307452 | 9.025299 |
| | 300 | 4.740153 | 3.049339 | 9.037878 |

Table 4.3: *American put options with* $T = 1$, $s_0 = 100$, $K = 100$, $\sigma_S = 0.25$, $r_0 = 0.06$, $\theta = 0.1$, $\kappa = 0.5$, $\rho = -0.25$, $\sigma_r$ *varying.*

| | $n$ | WEI | HST | ACZ |
|---|---|---|---|---|
| $\sigma_r$ =0.08 | 50 | 9.161152 | 9.087704 | 9.149096 |
| | 100 | 9.162292 | 9.125087 | 9.155979 |
| | 150 | 9.162352 | 9.137484 | 9.158033 |
| | 200 | 9.162318 | 9.143667 | 9.159035 |
| | 300 | 9.162238 | 9.149867 | 9.160028 |
| $\sigma_r$ =0.35 | 50 | 4.459093 | 9.284505 | 9.499658 |
| | 100 | 4.956463 | 9.479192 | 9.507529 |
| | 150 | 9.494545 | 9.442510 | 9.510407 |
| | 200 | 4.826695 | 9.480988 | 9.511623 |
| | 300 | 9.505248 | 9.494247 | 9.512845 |
| $\sigma_r$ =0.50 | 50 | 9.814438 | nan | 9.821675 |
| | 100 | 9.717844 | 9.555458 | 9.833291 |
| | 150 | 10.113582 | 9.570541 | 9.838509 |
| | 200 | 9.986565 | nan | 9.841491 |
| | 300 | 10.358344 | 9.576959 | 9.842231 |
| $\sigma_r$ =1.00 | 50 | 11.421349 | 6.586697 | 10.790323 |
| | 100 | 11.365461 | 6.934987 | 10.834057 |
| | 150 | 11.826422 | 6.392858 | 10.868361 |
| | 200 | 7.790298 | nan | 10.863552 |
| | 300 | 13.160854 | 6.897618 | 10.871842 |
| $\sigma_r$ =3.00 | 50 | 2.837551 | 3.291170 | 12.488520 |
| | 100 | 13.585301 | 2.903662 | 12.458445 |
| | 150 | 12.865822 | 2.672643 | 12.516032 |
| | 200 | 6.335779 | 3.166955 | 12.539396 |
| | 300 | 24.420598 | 3.405717 | 12.564801 |

Table 4.4: *American put options with* $T = 2$, $s_0 = 100$, $K = 10$, $\sigma_S = 0.25$, $r_0 = 0.06$, $\theta = 0.1$, $\kappa = 0.5$, $\rho = -0.25$, $\sigma_r$ *varying.*

# Appendix A

# Proof of Lemma 3.2.4 and Lemma 3.2.5

## A.1 Proof of Lemma 3.2.4

First of all we observe that the function $u(t,x)$ satisfies the PDE in (1.2.7), i.e. $\partial_t u + Lu = 0$, with $L = \mu\partial_x u + \frac{1}{2}\sigma^2\partial_{xx}u$. So, in order to find the estimates for $|\partial_t u(t,x)|$ we need to find estimates for $|\partial_x u(t,x)|$ and $|\partial_{xx}u(t,x)|$. Similarly, we obtain an estimate for $|\partial_{t,x}u(t,x)|$ from the estimates of $|\partial_{x,x}u(t,x)|$ and $|\partial_{x,x,x}u(t,x)|$. Then we will just consider the problem of finding a boundary estimate for $|\partial_x u(t,x)|$, $|\partial_{x,x}u(t,x)|$ and $|\partial_{x,x,x}u(t,x)|$. Moreover, we recall that the transition function $q_s(x,y)$ satisfies for all $(s,x,y) \in (0,T] \times \overline{\mathcal{O}} \times \overline{\mathcal{O}}$ the following global estimates

$$\left|\frac{\partial^{\alpha+\beta+\gamma} q_s(x,y)}{\partial x^\alpha \partial y^\beta \partial s^\gamma}\right| \leq \frac{C}{s^{\frac{\alpha+\beta+2\gamma+1}{2}}} \exp\left(-c\frac{(y-x)^2}{s}\right), \tag{A.1.1}$$

(for details see Friedman ([35], 1964)). We recall that here $\overline{\mathcal{O}} = [L,+\infty)$.
We are assuming that the payoff function $f$ is such that

$$f \in C_b^2([L,K];\mathbb{R}) \cap C_b^2([K,+\infty),\mathbb{R}) \quad \text{and} \quad f(L) = 0.$$

- Let us first consider $|\partial_x u(t,x)|$. We have that

$$\begin{aligned}\partial_x u(t,x) = &\int_L^{+\infty} -\partial_y q^1_{T-t}(x,y) f(y)dy \\ &+ \frac{2\mu}{\sigma^2} e^{\frac{2\mu(L-x)}{\sigma^2}} \int_L^{+\infty} q^2_{T-t}(x,y) f(y)dy \\ &- e^{\frac{2\mu(L-x)}{\sigma^2}} \int_L^{+\infty} \partial_y q^2_{T-t}(x,y) f(y)dy\end{aligned}$$

$$
\begin{aligned}
&= [-q^1_{T-t}(x,y)f(y)]^K_L + [-q^1_{T-t}(x,y)f(y)]^{+\infty}_K \\
&+ \int_L^{+\infty} q^1_{T-t}(x,y)f'(y)dy + \frac{2\mu}{\sigma^2}e^{\frac{2\mu(L-x)}{\sigma^2}} \int_L^{+\infty} q^2_{T-t}(x,y)f(y)dy \\
&- e^{\frac{2\mu(L-x)}{\sigma^2}}\{[q^2_{T-t}(x,y)f(y)]^K_L \\
&+ [q^2_{T-t}(x,y)f(y)]^{+\infty}_K - \int_L^{+\infty} q^2_{T-t}(x,y)f'(y)dy\} \\
&= \Delta f(K)[q^1_{T-t}(x,K) + e^{\frac{2\mu(L-x)}{\sigma^2}} q^2_{T-t}(x,K)] \\
&+ \int_L^{+\infty} q^1_{T-t}(x,y)f'(y)dy \\
&+ \frac{2\mu}{\sigma^2}e^{\frac{2\mu(L-x)}{\sigma^2}} \int_L^{+\infty} q^2_{T-t}(x,y)f(y)dy \\
&+ e^{\frac{2\mu(L-x)}{\sigma^2}} \int_L^{+\infty} q^2_{T-t}(x,y)f'(y)dy,
\end{aligned}
$$

where $\Delta f(K) = f(K^+) - f(K^-)$. We observe that all the integrals in the last equality are bounded, so we just need to look at the term

$$\Delta f(K)[q^1_{T-t}(x,K) + e^{\frac{2\mu(L-x)}{\sigma^2}} q^2_{T-t}(x,K)].$$

For the estimates (A.1.1) with $\alpha = \beta = \gamma = 0$ we get

$$|q^1_{T-t}(x,K)|, |q^2_{T-t}(x,K)| \leq \frac{C}{\sqrt{T-t}} e^{-c\frac{(x-K)^2}{T-t}} \leq C,$$

where the last inequality is a consequence of the fact that we are considering the boundary estimate, i.e. $x \in (L, \frac{K+L}{2}]$, so we can conclude that

$$\sup_{(t,x)\in[0,T)\times(L,\frac{K+L}{2}]} |\partial_x u(t,x)| \leq C.$$

- Let us now consider $|\partial^2_{x,x}u(t,x)|$. We have that

$$
\begin{aligned}
\partial^2_{x,x}u(t,x) &= \int_L^{+\infty} \partial^2_{y,y} q^1_{T-t}(x,y)f(y)dy \\
&- e^{\frac{2\mu(L-x)}{\sigma^2}} \left( \left( -\frac{2\mu}{\sigma^2} \right)^2 \int_L^{+\infty} q^2_{T-t}(x,y)f(y)dy \right. \\
&\left. - \frac{4\mu}{\sigma^2} \int_L^{+\infty} \partial_y q^2_{T-t}(x,y)f(y)dy + \int_L^{+\infty} \partial^2_{y,y} q^2_{T-t}(x,y)f(y)dy \right).
\end{aligned}
$$

So the "new" terms we need to consider are

$$\int_L^{+\infty} \partial^2_{y,y} q^1_{T-t}(x,y)f(y)dy - e^{\frac{2\mu(L-x)}{\sigma^2}} \int_L^{+\infty} \partial^2_{y,y} q^2_{T-t}(x,y)f(y)dy.$$

From a double integration by parts we have that

$$\begin{aligned}
&\int_L^{+\infty} \partial^2_{y,y} q^1_{T-t}(x,y)f(y)dy - e^{\frac{2\mu(L-x)}{\sigma^2}} \int_L^{+\infty} \partial^2_{y,y} q^2_{T-t}(x,y)f(y)dy \\
&= [\partial_y q^1_{T-t}(x,y)f(y)]_L^K + [\partial_y q^1_{T-t}(x,y)f(y)]_K^{+\infty} - [q^1_{T-t}(x,y)f'(y)]_L^K \\
&- [q^1_{T-t}(x,y)f'(y)]_K^{+\infty} + \int_L^{+\infty} q^1_{T-t}(x,y)f''(y)dy \\
&- e^{\frac{2\mu(L-x)}{\sigma^2}} \{[\partial_y q^2_{T-t}(x,y)f(y)]_L^K + [\partial_y q^2_{T-t}(x,y)f(y)]_K^{+\infty} \\
&- [q^2_{T-t}(x,y)f'(y)]_L^K - [q^2_{T-t}(x,y)f'(y)]_K^{+\infty} \\
&+ \int_L^{+\infty} q^2_{T-t}(x,y)f''(y)dy\} \\
&= -\partial_y q(x,K)\Delta f(K) + q(x,K)\Delta f'(K) - q(x,L)f'(L^+) \\
&+ \int_L^{+\infty} q_{T-t}(x,y)f''(y)dy \\
&= -\partial_y q(x,K)\Delta f(K) + q(x,K)\Delta f'(K) + \int_L^{+\infty} q_{T-t}(x,y)f''(y)dy,
\end{aligned}$$

where the last equality comes from the fact that $q(x,L) = 0$. We recall that $\Delta f'(K) = f'(K^+) - f'(K^-)$.
Now we have that $\int_L^{+\infty} q_{T-t}(x,y)f''(y)dy$ is bounded and if we consider the two terms $-\partial_y q(x,K)\Delta f(K)$ and $q(x,K)\Delta f'(K)$ the one that gives a worst contribution to the estimate is $-\partial_y q(x,K)\Delta f(K)$. So, by using estimates (A.1.1) with $\alpha = \gamma = 0$ and $\beta = 1$ we obtain

$$|\partial_y q(x,K)\Delta f(K)| \leq \frac{C}{(T-t)} e^{-c\frac{(K-x)^2}{T-t}} \leq C$$

because $x \in (L, \frac{K+L}{2}]$.
So we have that

$$\sup_{(t,x)\in[0,T)\times(L,\frac{K+L}{2}]} |\partial^2_{x,x} u(t,x)| \leq C$$

and we can also say that

$$\sup_{(t,x)\in[0,T)\times(L,\frac{K+L}{2}]} |\partial_t u(t,x)| + |\partial_x u(t,x)| + |\partial^2_{x,x} u(t,x)| \leq C.$$

- Now let us consider $|\partial^3_{x,x,x} u(t,x)|$. We have that

$$\begin{aligned}\partial^3_{x,x,x} u(t,x) = &\int_L^{+\infty} -\partial^3_{y,y,y} q^1_{T-t}(x,y) f(y) dy \\ &- \left( \left(-\frac{2\mu}{\sigma^2}\right)^3 e^{\frac{2\mu(L-x)}{\sigma^2}} \int_L^{+\infty} q^2_{T-t}(x,y) f(y) dy \right. \\ &+ \left(-\frac{2\mu}{\sigma^2}\right)^2 e^{\frac{2\mu(L-x)}{\sigma^2}} \int_L^{+\infty} \partial_y q^2_{T-t}(x,y) f(y) dy \\ &\left(-\frac{2\mu}{\sigma^2}\right) e^{\frac{2\mu(L-x)}{\sigma^2}} \int_L^{+\infty} \partial^2_{y,y} q^2_{T-t}(x,y) f(y) dy \\ &\left. + e^{\frac{2\mu(L-x)}{\sigma^2}} \int_L^{+\infty} \partial^3_{y,y,y} q^2_{T-t}(x,y) f(y) dy \right).\end{aligned}$$

Here the terms that give a "new" contribution to the estimate are

$$\begin{aligned}&\int_L^{+\infty} -\partial^3_{y,y,y} q^1_{T-t}(x,y) f(y) dy - e^{\frac{2\mu(L-x)}{\sigma^2}} \int_L^{+\infty} \partial^3_{y,y,y} q^2_{T-t}(x,y) f(y) dy \\ &= -[\partial^2_{y,y} q^1_{T-t}(x,y) f(y)]_L^K - [\partial^2_{y,y} q^1_{T-t}(x,y) f(y)]_K^{+\infty} + [\partial_y q^1_{T-t}(x,y) f'(y)]_L^K \\ &+ [\partial_y q^1_{T-t}(x,y) f'(y)]_K^{+\infty} - \int_L^{+\infty} \partial_y q^1_{T-t}(x,y) f''(y) dy \\ &- e^{\frac{2\mu(L-x)}{\sigma^2}} \left( [\partial^2_{y,y} q^2_{T-t}(x,y) f(y)]_L^K \right. \\ &+ [\partial^2_{y,y} q^2_{T-t}(x,y) f(y)]_K^{+\infty} - [\partial_y q^2_{T-t}(x,y) f'(y)]_L^K \\ &\left. - [\partial_y q^2_{T-t}(x,y) f'(y)]_K^{+\infty} + \int_L^{+\infty} \partial_y q^2_{T-t}(x,y) f''(y) dy \right) \\ &= \Delta f(K)(\partial^2_{y,y} q^1_{T-t}(x,K) + e^{\frac{2\mu(L-x)}{\sigma^2}} \partial^2_{y,y} q^2_{T-t}(x,K)) + \Delta f'(K)(-\partial_y q^1_{T-t}(x,K) \\ &- e^{\frac{2\mu(L-x)}{\sigma^2}} \partial_y q^2_{T-t}(x,K)) + f'(L^+)(\partial_y q^1_{T-t}(x,L) + e^{\frac{2\mu(L-x)}{\sigma^2}} \partial_y q^2_{T-t}(x,L)) \\ &+ \int_L^{+\infty} (-\partial_y q^1_{T-t}(x,y) - e^{\frac{2\mu(L-x)}{\sigma^2}} \partial_y q^2_{T-t}(x,y)) f''(y) dy.\end{aligned}$$

Now let us consider each single term we found. Between $\Delta f(K)(\partial^2_{y,y} q^1_{T-t}(x,K) + e^{\frac{2\mu(L-x)}{\sigma^2}} \partial^2_{y,y} q^2_{T-t}(x,K))$ and $\Delta f'(K)(-\partial_y q^1_{T-t}(x,K) - e^{\frac{2\mu(L-x)}{\sigma^2}} \partial_y q^2_{T-t}(x,K))$ the term that gives a worst contribution is the first one, but from estimates (A.1.1) and the fact that $x \in (L, \frac{K+L}{2}]$ we can write

$$|\Delta f(K)(\partial^2_{y,y} q^1_{T-t}(x,K) + e^{\frac{2\mu(L-x)}{\sigma^2}} \partial^2_{y,y} q^2_{T-t}(x,K))| \leq \frac{C}{(T-t)^{3/2}} e^{-c\frac{(x-K)^2}{T-t}} \leq C.$$

Then we have that

$$\begin{aligned}
&|f'(L^+)(\partial_y q^1_{T-t}(x,L) + e^{\frac{2\mu(L-x)}{\sigma^2}} \partial_y q^2_{T-t}(x,L))| \\
&\qquad = |f'(L^+)\frac{2\mu}{\sigma^2}\frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-\frac{(L-x-\mu(T-t))^2}{2\sigma^2(T-t)}}| \\
&\qquad \leq \frac{C}{\sqrt{T-t}} e^{-c\frac{(x-L)^2}{T-t}} \leq \frac{C}{\sqrt{T-t}}
\end{aligned}$$

where the last inequality comes from the fact that $x \in (L, \frac{K+L}{2}]$. Finally we have that

$$\left| \int_L^{+\infty} (-\partial_y q^1_{T-t}(x,y) - e^{\frac{2\mu(L-x)}{\sigma^2}} \partial_y q^2_{T-t}(x,y)) f''(y) dy \right| \leq \frac{C}{\sqrt{T-t}}$$

by using estimates (A.1.1) and the integral of a gaussian density. Moreover, we also need to check the following term

$$\begin{aligned}
\left| \int_L^{+\infty} \partial^2_{y,y} q^2_{T-t}(x,y) f(y) dy \right| &= \Big| \Delta f(K) \partial_y q^2_{T-t}(x,K) + \Delta f'(K) q^2_{T-t}(x,K) \\
&- q^2_{T-t}(x,L) f'(L^+) + \int_L^{+\infty} q^2_{T-t}(x,y) f''(y) dy \Big| \leq \frac{C}{\sqrt{T-t}},
\end{aligned}$$

where the last inequality comes from

$$\left| \Delta f(K) \partial_y q^2_{T-t}(x,K) + \Delta f'(K) q^2_{T-t}(x,K) + \int_L^{+\infty} q^2_{T-t}(x,y) f''(y) dy \right| \leq C$$

and

$$|q^2_{T-t}(x,L) f'(L^+)| \leq \frac{C}{\sqrt{T-t}}.$$

So we can conclude that

$$\sup_{(t,x)\in[0,T)\times(L,\frac{K+L}{2}]} |\partial^2_{t,x} u(t,x)| \leq \frac{C}{\sqrt{T-t}}$$

and the proof is complete.

## A.2 Proof of Lemma 3.2.5

As in the proof of Lemma 3.2.4, since $u(t,x)$ solves the PDE equation (1.2.7), we just need to consider the global estimates of $|\partial^3_{x,x,x}u(t,x)|$ and $|\partial^4_{x,x,x}u(t,x)|$. As for Lemma 3.2.4, we are assuming that the payoff function $f$ is such that

$$f \in C^2_b([L,K];\mathbb{R}) \cap C^2_b([K,+\infty),\mathbb{R}) \quad \text{and} \quad f(L)=0.$$

- Let us start with $|\partial^3_{x,x,x}u(t,x)|$. As for the boundary estimates we just need to check

$$\begin{aligned}
&\int_L^{+\infty} -\partial^3_{y,y,y}q^1_{T-t}(x,y)f(y)dy - e^{\frac{2\mu(L-x)}{\sigma^2}}\int_L^{+\infty}\partial^3_{y,y,y}q^2_{T-t}(x,y)f(y)dy\\
&= \Delta f(K)(\partial^2_{y,y}q^1_{T-t}(x,K) + e^{\frac{2\mu(L-x)}{\sigma^2}}\partial^2_{y,y}q^2_{T-t}(x,K)) + \Delta f'(K)(-\partial_y q^1_{T-t}(x,K)\\
&- e^{\frac{2\mu(L-x)}{\sigma^2}}\partial_y q^2_{T-t}(x,K)) + f'(L^+)(\partial_y q^1_{T-t}(x,L) + e^{\frac{2\mu(L-x)}{\sigma^2}}\partial_y q^2_{T-t}(x,L))\\
&+ \int_L^{+\infty}(-\partial_y q^1_{T-t}(x,y) - e^{\frac{2\mu(L-x)}{\sigma^2}}\partial_y q^2_{T-t}(x,y))f''(y)dy.
\end{aligned}$$

Then

$$\begin{aligned}
&|\Delta f(K)(\partial^2_{y,y}q^1_{T-t}(x,K) + e^{\frac{2\mu(L-x)}{\sigma^2}}\partial^2_{y,y}q^2_{T-t}(x,K))\\
&+ \Delta f'(K)(-\partial_y q^1_{T-t}(x,K) - e^{\frac{2\mu(L-x)}{\sigma^2}}\partial_y q^2_{T-t}(x,K))| \le \frac{C}{(T-t)^{3/2}}e^{-c\frac{(x-K)^2}{T-t}}
\end{aligned}$$

and

$$\begin{aligned}
&|f'(L^+)(\partial_y q^1_{T-t}(x,L) + e^{\frac{2\mu(L-x)}{\sigma^2}}\partial_y q^2_{T-t}(x,L))|\\
&= \left| f'(L^+)\frac{2\mu}{\sigma^2}\frac{1}{\sqrt{2\pi\sigma^2(T-t)}}e^{-\frac{(L-x-\mu(T-t))^2}{2\sigma^2(T-t)}}\right|\\
&\le \frac{C}{\sqrt{T-t}}e^{-c\frac{(x-L)^2}{T-t}} \le C
\end{aligned}$$

and finally

$$\left|\int_L^{+\infty}(-\partial_y q^1_{T-t}(x,y) - e^{\frac{2\mu(L-x)}{\sigma^2}}\partial_y q^2_{T-t}(x,y))f''(y)dy\right| \le \frac{C}{\sqrt{T-t}}$$

so we can conclude saying that

$$|\partial^3_{x,x,x}u(t,x)| \le \frac{C}{\sqrt{T-t}}\left(1 + \frac{1}{(T-t)}e^{-c\frac{(x-K)^2}{T-t}}\right)$$

and then

$$|\partial^2_{x,t}u(t,x)| + |\partial^3_{x,x,x}u(t,x)| \le \frac{C}{\sqrt{T-t}}\left(1 + \frac{1}{(T-t)}e^{-c\frac{(x-K)^2}{T-t}}\right).$$

- Let us now consider $|\partial^4_{x,x,x,x}v(t,x)|$. We have that

$$\begin{aligned}
\partial^4_{x,x,x,x}u(t,x) &= \int_L^{+\infty} \partial^4_{y,y,y,y}q^1_{T-t}(x,y)f(y)dy \\
&- \left(\left(-\frac{2\mu}{\sigma^2}\right)^4 e^{\frac{2\mu(L-x)}{\sigma 2}} \int_L^{+\infty} q^2_{T-t}(x,y)f(y)dy \right.\\
&+ 4\left(-\frac{2\mu}{\sigma^2}\right) e^{\frac{2\mu(L-x)}{\sigma 2}} \int_L^{+\infty} \partial^3_{y,y,y}q^2_{T-t}(x,y)f(y)dy \\
&+ 6\left(-\frac{2\mu}{\sigma^2}\right)^2 e^{\frac{2\mu(L-x)}{\sigma 2}} \int_L^{+\infty} \partial^2_{y,y}q^2_{T-t}(x,y)f(y)dy \\
&+ 4\left(-\frac{2\mu}{\sigma^2}\right)^3 e^{\frac{2\mu(L-x)}{\sigma 2}} \int_L^{+\infty} \partial_{y}q^2_{T-t}(x,y)f(y)dy \\
&\left.+ e^{\frac{2\mu(L-x)}{\sigma 2}} \int_L^{+\infty} \partial^4_{y,y,y,y}q^2_{T-t}(x,y)f(y)dy \right).
\end{aligned}$$

So the "new" terms we need to consider are

$$\begin{aligned}
&\int_L^{+\infty} \partial^4_{y,y,y,y}q^1_{T-t}(x,y)f(y)dy - e^{\frac{2\mu(L-x)}{\sigma 2}} \int_L^{+\infty} \partial^4_{y,y,y,y}q^2_{T-t}(x,y)f(y)dy \\
&= -\Delta f(K)\partial^3_{y,y,y}q_{T-t}(x,K) + \Delta f'(K)\partial^2_{y,y}q_{T-t}(x,K) - f'(L^+)\partial^2_{y,y}q_{T-t}(x,L) \\
&+ \int_L^{+\infty} \partial^2_{y,y}q_{T-t}(x,y)f(y)dy.
\end{aligned}$$

From estimates (A.1.1) with $\alpha = 0 = \gamma$ and $\beta = 3$ we get

$$|-\Delta f(K)\partial^3_{y,y,y}q_{T-t}(x,K) + \Delta f'(K)\partial^2_{y,y}q_{T-t}(x,K)| \le \frac{C}{(T-t)^2}e^{-c\frac{(x-K)^2}{T-t}}$$

and from estimates (A.1.1) with $\alpha = 0 = \gamma$ and $\beta = 2$ we have

$$|f'(L^+)\partial^2_{y,y}q_{T-t}(x,L)| \le \frac{C}{(T-t)^{3/2}}e^{-c\frac{(x-L)^2}{T-t}} \le C.$$

By using estimates (A.1.1) with $\alpha = 0 = \gamma$ and $\beta = 2$ and the integral of a gaussian density we obtain that

$$\left| \int_L^{+\infty} \partial^2_{y,y} q_{T-t}(x,y) f(y) dy \right| \leq \frac{C}{(T-t)}.$$

Then we finally get

$$|\partial^4_{x,x,x,x} u(t,x)| \leq \frac{C}{T-t} \left( 1 + \frac{1}{T-t} e^{-c\frac{(x-K)^2}{T-t}} \right).$$

The proof is then complete because as a consequence we have that

$$\begin{aligned} &|\partial^2_{t,t} u(t,x)| + |\partial^3_{x,x,t} u(t,x)| + |\partial^4_{x,x,x,x} u(t,x)| \\ &\leq \frac{C}{(T-t)} \left( 1 + \frac{1}{(T-t)} e^{-c\frac{(x-K)^2}{T-t}} \right). \end{aligned}$$

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