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ROBUST FEEDBACK SWITCHING CONTROL: DYNAMIC PROGRAMMING AND VISCOSITY SOLUTIONS*

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Abstract. We consider a robust switching control problem. The controller only observes the evolution of the state process, and thus uses feedback (closed-loop) switching strategies, a non-standard class of switching controls introduced in this paper. The adverse player (nature) chooses open-loop controls that represent the so-called Knightian uncertainty, i.e., misspecifications of the model. The (half) game switcher versus nature is then formulated as a two-step (robust) optimization problem. We develop the stochastic Perron's method in this framework, and prove that it produces a viscosity subsolution and supersolution to a system of HJB variational inequalities, which envelop the value function. Together with a comparison principle, this characterizes the value function of the game as the unique viscosity solution to the HJB equation, and shows as a by-product the dynamic programming principle for the robust feedback switching control problem.

Key words. model uncertainty, optimal switching, feedback strategies, stochastic games, stochastic Perron's method, viscosity solutions

AMS subject classifications. 60G40, 91A05, 49L20, 49L25

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1. Introduction. Optimal switching is a class of stochastic control problems that has attracted a lot of interest and generated important developments in applied and financial mathematics. Switching control consists in a sequence of interventions that occur at random discrete times due to switching costs, and naturally arises in investment problems with fixed transaction costs or in real options. The literature on this topic is quite large and we refer, e.g., to [33], [26], [15], [27], [3], [9], for a treatment by dynamic programming and PDE methods, to [19], [20], [13] for the connection with reflected backward stochastic differential equation methods, and to [12], [10], [17] for various applications to finance and real options in energy markets.

The standard approach to the study of a switching control problem is to give an evolution for the controlled state process, with assigned drift and diffusion coefficients. These, however, are obtained in practice through estimation procedures and are unlikely to coincide with the *real* coefficients. For this reason, in the present work we study a switching control problem *robust* to a misspecification of the model for the controlled state process. This is formalized as follows: given $s \geq 0$, $x \in \mathbb{R}^d$, and a regime $i \in \mathbb{I}_m := \{1, \ldots, m\}$, let us consider the controlled system of SDEs, for $t \geq s$,

$$\begin{cases} X_t = x + \int_s^t b(X_r, I_r, u_r)dr + \int_s^t \sigma(X_r, I_r, u_r)dW_r, \\ I_t = i\,1_{\{s \leq t < \tau_0\}} + \sum_{n\in\mathbb{N}} \iota_n 1_{\{\tau_n \leq t < \tau_{n+1}\}}. \end{cases} \tag{1.1}$$

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The piecewise constant process I denotes the regime value at any time t, whose evolution is determined by the controller through the switching control $\alpha = (\tau_n, \iota_n)_{n\in\mathbb{N}}$, while the process u, decided by nature, brings the uncertainty within the model. In the switching control problem with model uncertainty, the objective of the controller is the maximization of the following functional, over a finite time horizon $T < \infty$:

$$J(s,x,i;\alpha,u) := \mathbb{E}\bigg[\int_s^T f(X_r^{s,x,i;\alpha,u}, I_r^{s,x,i;\alpha,u}, u_r)dr + g(X_T^{s,x,i;\alpha,u}, I_T^{s,x,i;\alpha,u}) \\ - \sum_{n\in\mathbb{N}} c(X_{\tau_n}^{s,x,i;\alpha,u}, I_{\tau_n^-}^{s,x,i;\alpha,u}, I_{\tau_n}^{s,x,i;\alpha,u}) 1_{\{s\le\tau_n<T\}}\bigg],$$

playing against nature, described by u. This leads to the "robust" optimization problem

$$\sup_\alpha \Big(\inf_u J(s,x,i;\alpha,u)\Big). \tag{1.2}$$

What definition and information pattern for the switching control α and for u should we adopt? As a first attempt, if we interpret (1.2) as a game between the controller and nature, it would be reasonable to formulate it in terms of *nonanticipating strategies against controls*, as in the seminal paper by Elliott and Kalton [14]. In this case, α is a nonanticipating switching strategy, while u is an open-loop control. Then, the switcher knows the current and past choices made by the opponent (see section 4.2 below for more details on this formulation). In the context of robust optimization, the controller does not know in general the choice made by nature. He knows at most the current state of the system and its past history, that is, the evolution of X and also of I (by keeping track of his previous actions). For this reason, inspired by [1], [30] (see also [24] which considers robust controls over feedback strategies in a deterministic setting), we take α as a *feedback (also called closed-loop) switching strategy* rather than a nonanticipating strategy (namely, we present a *feedback formulation* of a switching control problem, which is quite uncommon in the literature). On the other hand, u can be an *open-loop control* (nature is aware of the all information at disposal). This leads to the formulation of *robust feedback* switching control problem where both players use controls, one in feedback form (the switcher) and the other in open-loop form (the nature), hence different from the Elliott–Kalton formulation where one player observes continuously the control (action) of the other player.

We develop the stochastic Perron's method in this framework of robust feedback switching strategy. This method was initially introduced to analyze linear problems in [4], Dynkin games in [6], and *regular* control problems in [5]. Later on, it was adapted to analyze exit time problems in [29], control problems with state constraints in [28], singular control problems in [8], stochastic differential games in [31], and stochastic control with model uncertainty in [30]. The stochastic Perron's method is similar to a *verification theorem* and avoids having to go through the dynamic programming principle first (which is not known a priori in this context) to show that the value function is a solution to the HJB equation. Actually, the dynamic programming principle is obtained as a by-product of the stochastic Perron's method and comparison principle. Unlike the classical verification theorem, the stochastic Perron's method does not require the a priori smoothness of the value function. The method is to construct viscosity (semi)solutions to the HJB equation, which envelop the value function, and rely on the comparison principle of the HJB equation to conclude that the value function is the unique viscosity solution. In order to carry out the construction, one needs to define two suitable classes of functions, denoted by $\mathcal{V}^-$ and $\mathcal{V}^+$, whose elements are known in the literature on the stochastic Perron's method as stochastic subsolutions ($\mathcal{V}^-$) and stochastic supersolutions ($\mathcal{V}^+$). The crucial property of $\mathcal{V}^-$ and $\mathcal{V}^+$ is closedness under minimization/maximization. Moreover, their members stay below/above the value function. The technical part of the proof is in showing that the supremum/infimum of the above classes give a viscosity supersolution/subsolution to the HJB equation. One of the advantages of the stochastic Perron's method is that it allows us to demonstrate that the information available to nature (whether it uses open-loop or feedback strategies) does not affect the value of the game. We do this by constructing the class $\mathcal{V}^+$ for an auxiliary problem, whose elements lie by definition above our original value function. Our results here can be thought of as a generalization of the recent work [30], in which the controller uses elementary feedback strategies. In our setting changing the value of control has a switching cost. This changes the nature of the problem as the past action of the controller needs to be stored as a state variable. The presence of this additional state variable brings about several subtle technical issues, which we resolve in this paper. For example, concatenating the feedback switching strategies needs to be done with care (not to incur an additional cost at the time of concatenation), which forces us to make appropriate changes in defining the class $\mathcal{V}^-$.

We should mention that when one can bootstrap the regularity of the viscosity solutions and show that they are classical solutions, one can still use the classical Perron method of Ishii [22]. This program is carried out by [23] for a stochastic control problem and by [7] for a robust stochastic control problem. In general, however, the PDE may not admit a smooth solution and one has to use the generalization of the Perron method, which we called the stochastic Perron's method, described above. If one attempts to use only the Perron method in [22] to construct viscosity solutions one faces a major obstacle: without additional knowledge on the properties of the value function, it does not compare with the output of the classical Perron method. In fact, this is exactly what happens in [9]. In fact, [9, section 2] shows that the system of variational inequalities has a unique viscosity solution using the classical Perron method. But when they introduce a control problem (not a game) in section 3, they still go through first proving the dynamic programming principle, to show that the value function is a viscosity solution and is therefore the unique viscosity solution they constructed in section 2.

We should emphasize that although the system of variational inequalities in [9, section 2] is quite close to the one in our paper, these authors make the connection in their section 3 with a control problem for the particular case when there is one single player using switching and regular controls. Our main result is on one hand the formulation and solution of the robust feedback switching control problem, in which the controller only observes the evolution of the state process, and thus uses feedback (closed-loop) switching strategies, a nonstandard class of switching controls introduced for the first time in this paper, and on the other hand proving *directly* that it is the unique viscosity solution to the corresponding system of dynamic programming variational inequalities.

The rest of this paper is organized as follows. In section 2, we provide a rigorous formulation of the robust feedback switching control problem. We develop in section 3 the stochastic Perron's method, and characterize the infimum (resp., supremum) of $\mathcal{V}^+$ (resp., $\mathcal{V}^-$) as the viscosity subsolution (resp., supersolution) of the HJB equation. In section 4, by using a comparison principle under a no-free-loop condition on the switching costs, we conclude that the value function is the unique viscosity solution to the HJB equation, and obtain as a by-product the dynamic programming principle. We finally compare the two formulations: robust feedback/Elliott–Kalton, in a specific example, which then gives a counterexample to uniqueness for the HJB equation. In order to keep the paper size reasonable, whenever a result has a standard proof or a similar proof can be found in the literature, we do not report all details, but we focus on the main steps providing a sketch of the proof.

2. Modeling a robust switching control problem.

2.1. Feedback switching system under model uncertainty. In this section, we consider the situation where the switcher knows just the current and past history of the state. To model this information pattern, we adopt the notion of *feedback strategies* following the definition introduced in the book [1, Chapter VIII, section 3.1] or in [30]. It is important to notice that this notion of feedback strategies differs from the notion of nonanticipating strategies à la Elliott–Kalton where the switcher-player knows the current and past choices of the control made by his/her opponent (here the nature); see also the discussion in Chapter VIII of [1] and, in particular, Lemma 3.5 which gives the connection between these two notions.

Let U be a compact metric space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space on which a d-dimensional Brownian motion $W = (W_t)_{t\geq 0}$ is defined. For any $s \geq 0$, we consider the filtration $\mathbb{F}^{W,s} = (\mathcal{F}_t^{W,s})_{t\geq s}$, which is the augmented natural filtration generated by the Brownian increments starting at s, i.e.,

$$\mathcal{F}_t^{W,s} := \sigma(W_r - W_s,\ s \leq r \leq t) \vee \mathcal{N}(\mathbb{P}, \mathcal{F}), \qquad t \geq s,$$

where $\mathcal{N}(\mathbb{P}, \mathcal{F}) := \{N \in \mathcal{F} \colon \mathbb{P}(N) = 0\}$. For each $s \geq 0$, we denote by $\mathbb{F}^s = (\mathcal{F}_t^s)_{t\geq s}$ another filtration satisfying the usual conditions, which is larger than $\mathbb{F}^{W,s}$ and keeps $(W_t - W_s)_{t\geq s}$ a Brownian motion starting at s.

We fix a finite time horizon $0 < T < \infty$. For any $s \in [0, T]$, we denote by $y(\cdot)$ or y a generic element of the space $C([s, T]; \mathbb{R}^d) \times \mathscr{L}([s, T]; \mathbb{I}_m)$, where $\mathscr{L}([s, T]; \mathbb{I}_m)$ denotes the set of càglàd paths valued in $\mathbb{I}_m$ (notice that the elements of $\mathscr{L}([s, T]; \mathbb{I}_m)$ are indeed piecewise constant paths, since $\mathbb{I}_m$ is a discrete set). We also write $y = (y^X, y^I)$ with $y^X \in C([s, T]; \mathbb{R}^d)$ and $y^I \in \mathscr{L}([s, T]; \mathbb{I}_m)$. We define the filtration $\mathbb{B}^s = (\mathcal{B}_t^s)_{s\leq t\leq T}$, where $\mathcal{B}_t^s$ is the σ-algebra generated by the canonical coordinate maps $C([s, T]; \mathbb{R}^d) \times \mathscr{L}([s, T]; \mathbb{I}_m) \to \mathbb{R}^d \times \mathbb{I}_m$, $y(\cdot) \mapsto y(r)$, $r \in [s, t]$, namely,

$$\mathcal{B}_t^s := \sigma(y(\cdot) \mapsto y(r),\ s \leq r \leq t).$$

A map $\tau \colon C([s, T]; \mathbb{R}^d) \times \mathscr{L}([s, T]; \mathbb{I}_m) \to [s, T]$ satisfying $\{\tau \leq t\} \in \mathcal{B}_t^s\ \forall\, t \in [s, T]$, is called a *stopping rule*. $\mathcal{T}^s$ denotes the family of all stopping rules starting at s. For any $s \in [0, T]$ and $\tau \in \mathcal{T}^s$, we define, as usual,

$$\mathcal{B}_{\tau^+}^s := \big\{B \in \mathcal{B}_T^s \colon \forall\, t \in [s, T],\ B \cap \{y \colon \tau(y) \leq t\} \in \mathcal{B}_{t^+}^s\big\},$$
$$\mathcal{B}_{\tau}^s := \big\{B \in \mathcal{B}_T^s \colon \forall\, t \in [s, T],\ B \cap \{y \colon \tau(y) \leq t\} \in \mathcal{B}_{t}^s\big\},$$

where $\mathcal{B}_{t^+}^s := \cap_{r>t}\mathcal{B}_r^s$, $t \in [s, T)$, and $\mathcal{B}_{T^+}^s := \mathcal{B}_T^s$. We also denote $y(T^+) := y(T)$ for any $y \in C([s, T]; \mathbb{R}^d) \times \mathscr{L}([s, T]; \mathbb{I}_m)$.

Definition 2.1 (feedback switching strategies). *Fix $s \in [0, T]$. We say that the double sequence $\alpha = (\tau_n, \iota_n)_{n\in\mathbb{N}}$ is a feedback switching control starting at s if*

- *$\tau_n \in \mathcal{T}^s$ for any $n \in \mathbb{N}$, and*

$$s \leq \tau_0 \leq \cdots \leq \tau_n \leq \cdots \leq T.$$

Moreover, $(\tau_n)_{n\in\mathbb{N}}$ satisfies the following property: $\forall\, (y_n)_{n\in\mathbb{N}} \in C([s, T]; \mathbb{R}^d) \times$

$\mathscr{L}([s,T];\mathbb{I}_m)$ with $y_n(t) = y_{n+1}(t)$, $t \in [s, \tau_n(y_n)]$, for every $n \in \mathbb{N}$, then

$$\tau_n(y_n) = T \qquad \text{for } n \text{ large enough.}$$

- *$\iota_n \colon C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m) \to \mathbb{I}_m$ is $\mathcal{B}^s_{\tau_n}$-measurable, for any $n \in \mathbb{N}$.*

$\mathcal{A}_s$ denotes the family of all feedback switching controls starting at s.

Remark 2.1. This canonical definition of the feedback switching strategy means that the stopping rules τ_n are based on the observation of the state, while the actions ι_n decided at time τ_n are based only on the knowledge of the state up to the decision time. We may alternatively call feedback switching strategy as closed-loop switching control as opposed to the notion of open-loop switching controls, where the decision times τ_n are stopping times with respect to the larger filtration $\mathbb{F}^s$, and the actions ι_n are based on a larger information given by the filtration $\mathbb{F}^s$. Consider a sequence of paths $(y_n)_{n\in\mathbb{N}}$ as in Definition 2.1. Then, the sequence $(\tau_n(y_n))_{n\in\mathbb{N}}$ is nondecreasing. Indeed, from Lemma 2.1 below we have $\tau_n(y_n) = \tau_n(y_{n+1})$. Since $\tau_n(y_{n+1}) \le \tau_{n+1}(y_{n+1})$ from the nondecreasing property of the sequence $(\tau_n)_{n\in\mathbb{N}}$, the thesis follows. See also Remark 2.3 below, where the property "$\tau_n(y_n) = T$ for n large enough" is analyzed in detail. This structure condition on the sequence (y_n) is required for ensuring well-posedness, i.e., in order to guarantee that the optimal control does not have infinitely many switches and that the SDE (2.1) of X is well-defined. This is discussed in detail below; see, in particular, Remark 2.3. □

Definition 2.2 (open-loop controls). *Fix $s \in [0,T]$. An open-loop control u starting at s, for the nature, is an $\mathbb{F}^s$-progressively measurable process $u\colon [s,T]\times\Omega \to U$. We denote by $\mathcal{U}_s$ the collection of all possible open-loop controls, given the initial deterministic time s.*

For any $(s,x,i) \in [0,T]\times\mathbb{R}^d\times\mathbb{I}_m$, $\alpha = (\tau_n,\iota_n)_{n\in\mathbb{N}} \in \mathcal{A}_s$, $u \in \mathcal{U}_s$, we can now write (1.1) on $[0,T]$ as follows:

(2.1)
$$\begin{cases} X_t = x + \int_s^t b(X_r, I_r, u_r)dr + \int_s^t \sigma(X_r, I_r, u_r)dW_r, & s \le t \le T,\\ I_t = i1_{\{s\le t<\tau_0(X_\cdot, I_{\cdot^-})\}} + \sum_{n\in\mathbb{N}} \iota_n(X_\cdot, I_{\cdot^-})1_{\{\tau_n(X_\cdot,I_{\cdot^-})\le t<\tau_{n+1}(X_\cdot,I_{\cdot^-})\}}, & s \le t < T,\\ I_T = I_{T^-} \end{cases}$$

with $I_{s^-} := I_s$. Notice that the presence of $I_{\cdot^-}$ in place of $I_\cdot$ in the arguments of τ_n, ι_n is due to the fact that the choice of (τ_n,ι_n) by the controller is based only on the information coming from the previous switching actions $(\tau_i,\iota_i)_{0\le i\le n-1}$. Moreover, the last equation $I_T = I_{T^-}$ in (2.1) means that there is no regime switching at the final time T. We impose the following assumptions on the coefficients $b\colon \mathbb{R}^d\times\mathbb{I}_m\times U \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d\times\mathbb{I}_m\times U\to\mathbb{R}^{d\times d}$ (in the following, we use the notation $\|A\|^2 = \mathrm{tr}(AA^\intercal)$ for the Hilbert–Schmidt norm of any matrix A).

(H1)
- (i) b, σ are jointly continuous on $\mathbb{R}^d\times\mathbb{I}_m\times U$.
- (ii) b, σ are uniformly Lipschitz continuous in x, i.e.,

$$|b(x,i,u) - b(x',i,u)| + \|\sigma(x,i,u) - \sigma(x',i,u)\| \le L_1|x - x'|$$

$\forall\, x, x' \in \mathbb{R}^d$, $i \in \mathbb{I}_m$, $u \in U$, for some positive constant L_1.

Remark 2.2. From assumption (H1) it follows that b and σ satisfy a linear growth condition in x, i.e.,

$$|b(x,i,u)| + \|\sigma(x,i,u)\| \le M_1(1+|x|)$$

$\forall\, x \in \mathbb{R}^d$, $i \in \mathbb{I}_m$, $u \in U$, for some positive constant M_1. □

Remark 2.3. Fix $s \in [0,T]$ and $\alpha = (\tau_n, \iota_n)_{n\in\mathbb{N}} \in \mathcal{A}_s$. Let us consider the following properties of the nondecreasing sequence $(\tau_n)_{n\in\mathbb{N}}$:

(i) *Uniformly finite.* There exists $N \in \mathbb{N}$ such that $\forall\, y \in C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m)$,
$$\tau_n(y) = T \qquad \text{for } n \geq N.$$

(ii) *Finite along every adaptive sequence.* For every sequence $(y_n)_{n\in\mathbb{N}} \in C([s,T];\mathbb{R}^d)\times\mathscr{L}([s,T];\mathbb{I}_m)$ satisfying for every $n \in \mathbb{N}$, $y_n(t) = y_{n+1}(t)\ \forall\, t \in [s,\tau_n(y_n)]$, we have
$$\tau_n(y_n) = T \qquad \text{for } n \text{ large enough.}$$

(iii) *Finite along every path.* $\forall\, y \in C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m)$,
$$\tau_n(y) = T \qquad \text{for } n \text{ large enough.}$$

Condition (i) is the strongest, while (iii) is the weakest. In Definition 2.1 we imposed the intermediate property (ii), since it allows us to have a well-posedness result for (2.1), which is no longer guaranteed if we require only (iii). To see this latter point, we construct a counterexample. Take $s = 0$, $T = 1$, and $m = 2$ so that $\mathbb{I}_2 = \{1,2\}$. Consider the sequence $(b_n)_{n\in\mathbb{N}} \subset [0,1]$ given by
$$b_n = \sum_{j=0}^{n} \frac{1}{2^{j+2}} \qquad \forall\, n \in \mathbb{N}.$$
In particular, we have $b_0 = \frac{1}{4}$, $b_1 = \frac{1}{4} + \frac{1}{8}$, $b_2 = \frac{1}{4} + \frac{1}{8} + \frac{1}{16}, \ldots$, and in general
$$b_n = \frac{2^{n+1}-1}{2^{n+2}} \qquad \forall\, n \geq 0.$$
Notice that $(b_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence satisfying $b_n \nearrow \frac{1}{2}$ as $n \to \infty$. Now, for every $y \in C([0,1];\mathbb{R}^d) \times \mathscr{L}([0,1];\mathbb{I}_2)$ we write $y = (y^X, y^I)$ with $y^X \in C([0,1];\mathbb{R}^d)$ and $y^I \in \mathscr{L}([0,1];\mathbb{I}_2)$. Then, we define the sequence $(\tau_n)_{n\in\mathbb{N}}$ as follows:
$$\tau_n(y) = b_n 1_{\{y \in B_n\}} + 1_{\{y \in B_n^c\}} \qquad \forall\, y \in C([0,1];\mathbb{R}^d) \times \mathscr{L}([0,1];\mathbb{I}_2),\ n \in \mathbb{N},$$
where
$$\begin{aligned} B_0 &= \big\{y \in C([0,1];\mathbb{R}^d) \times \mathscr{L}([0,1];\mathbb{I}_2)\colon y^I(t) = y^I(0),\ 0 < t \leq b_0\big\}, \\ B_n &= \big\{y \in B_{n-1}\colon y^I(t) = 3 - y^I(b_{n-1}),\ b_{n-1} < t \leq b_n\big\} \qquad \forall\, n \geq 1. \end{aligned}$$
Observe that, since $y^I(t) \in \mathbb{I}_2$ then $3 - y^I(t) \in \mathbb{I}_2$; moreover, when $y^I(t) = 1$ then $3 - y^I(t) = 2$, while if $y^I(t) = 2$ then $3 - y^I(t) = 1$. We also notice that $B_n \in \mathcal{B}^0_{b_n}$, therefore $\tau_n \in \mathcal{T}^0$. Furthermore, $(\tau_n)_{n\in\mathbb{N}}$ is a nondecreasing sequence which verifies property (iii) above: this is due to the fact that every path $y \in C([0,1];\mathbb{R}^d) \times \mathscr{L}([0,1];\mathbb{I}_2)$ has only a finite number of jumps, since $\mathbb{I}_2$ is a discrete set; in other words, any y belongs to B_n^c when n is large enough (e.g., when n is strictly greater than the number of jumps of y). However, $(\tau_n)_{n\in\mathbb{N}}$ does not satisfy property (ii), as we shall prove below. We also define
$$\iota_n(y) = 3 - y^I(b_n) \qquad \forall\, y \in C([0,1];\mathbb{R}^d) \times \mathscr{L}([0,1];\mathbb{I}_2),\ n \in \mathbb{N}.$$
In other words, when $y^I(b_n) = 1$ then $\iota_n(y) = 2$, while when $y^I(b_n) = 2$ then $\iota_n(y) = 1$.

Let $\alpha = (\tau_n, \iota_n)_{n\in\mathbb{N}}$, then α satisfies Definition 2.1, but for property (ii) (see below), even if property (iii) is satisfied. Now, we solve (2.1) with $x \in \mathbb{R}^d$, $\alpha = (\tau_n, \iota_n)_{n\in\mathbb{N}}$, $u \in \mathcal{U}_{0,0}$, and $i = 1 \in \mathbb{I}_2$. Define the (deterministic) process $I\colon [0,1] \to \mathbb{I}_2$ as follows, for any $t \in [0, \frac{1}{2})$,

$$I_t = \begin{cases} 1, & 0 \le t \le b_0, \\ 2, & b_0 < t \le b_1, \\ 1, & b_1 < t \le b_2, \\ 2, & b_2 < t \le b_3, \\ \vdots \end{cases}$$

On the other hand, we do not specify I on $[\frac{1}{2}, 1]$, we only require that the limit $I_{1^-} := \lim_{t\uparrow 1} I_t$ exists and we suppose that $I_1 = I_{1^-}$. Notice that $I_{\frac{1}{2}^-}$ does not exist, therefore $I \notin \mathscr{L}([0,1];\mathbb{I}_2)$. However, the process I solves (2.1) (vice versa, every process satisfying (2.1) coincides with I on the interval $[0, \frac{1}{2})$; in particular, there does not exist a solution process with paths in $\mathscr{L}([0,1];\mathbb{I}_2)$). Moreover, under assumption (H1) we can also solve (2.1) for X. Since we did not specify the behavior of I on the entire interval $[0,1]$, we cannot have uniqueness of the solution for (2.1). Nevertheless, we notice that the sequence $(\tau_n)_{n\in\mathbb{N}}$ does not satisfy property (ii) above. Indeed, let $y_n(\cdot) := I_{\cdot\wedge b_n}$, $n \in \mathbb{N}$. Then, $y_n \in \mathscr{L}([0,1];\mathbb{I}_2)$, but $\tau_n(y_n) < \frac{1}{2}$ for any n. This shows that if we only require property (iii), then the well-posedness of (2.1) is no longer guaranteed. ∎

We now study the well-posedness of (2.1), for which we need the following two lemmas.

LEMMA 2.1. *Let $s \in [0,T]$, $\tau \in \mathcal{T}^s$, and $y_1, y_2 \in C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m)$. If $y_1(t) = y_2(t)$, $s \le t \le \tau(y^1)$, then*
 (i) *$\tau(y_1) = \tau(y_2)$,*
 (ii) *$\iota(y_1) = \iota(y_2)$ for any $\mathcal{B}^s_\tau$-measurable map $\iota\colon C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m) \to \mathbb{I}_m$.*

Proof. Let $t^* := \tau(y_1)$. We begin noting that if $B \in \mathcal{B}^s_{t^*}$ and $y_1 \in B$, then $y_2 \in B$, as well. Since τ is a stopping rule, the event $B := \{y\colon \tau(y) = t^*\}$ belongs to $\mathcal{B}^s_{t^*}$. As $y_1 \in B$, we then see that $y_2 \in B$, i.e., $\tau(y_2) = \tau(y_1)$, which gives (i). Notice that assertion (i) can be also deduced by [11, (100.1) at p. 149, Chapter IV].

Concerning (ii), let $\iota\colon C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m) \to \mathbb{I}_m$ be $\mathcal{B}^s_\tau$-measurable. By definition of ι, the event $\tilde{B} := \{y\colon \iota(y) = \iota(y_1)\}$ belongs to $\mathcal{B}^s_\tau$. Therefore, $B := \tilde{B} \cap \{\tau(y) \le t^*\} \in \mathcal{B}^s_{t^*}$. Since $y_1 \in B$, from the observation at the beginning of the proof it follows that $y_2 \in B$, which implies $y_2 \in \tilde{B}$, i.e., $\iota(y_2) = \iota(y_1)$. ∎

LEMMA 2.2. *Let $s \in [0,T]$, $\tau \in \mathcal{T}^s$, and $Y = (Y_t)_{s\le t\le T}$ be an $\mathbb{F}^s$-adapted process valued in $\mathbb{R}^d \times \mathbb{I}_m$. Suppose that every path of Y belongs to $C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m)$. Then, $\tau_Y\colon \Omega \to [s,T]$ defined as $\tau_Y(\omega) := \tau(Y_\cdot(\omega))$, $\omega \in \Omega$, is an $\mathbb{F}^s$-stopping time. Moreover, if $\iota\colon C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m) \to \mathbb{I}_m$ is $\mathcal{B}^s_\tau$-measurable then $i_Y(\omega) := \iota(Y_\cdot(\omega))$, $\omega \in \Omega$, is $\mathcal{F}^s_{\tau_Y}$-measurable.*

Proof. For any $t \in [s,T]$, we notice that the map $Y_\cdot$ is measurable from $(\Omega, \mathcal{F}^s_t)$ into $(C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m), \mathcal{B}^s_t)$. Then, $\{\omega\colon \tau_Y(\omega) \le t\} = \{\omega\colon \tau(Y_\cdot(\omega)) \le t\} = \{\omega\colon Y_\cdot(\omega) \in \tau^{-1}([s,t])\}$. Since $\tau^{-1}([s,t]) \in \mathcal{B}^s_t$, we have $\{\omega\colon Y_\cdot(\omega) \in \tau^{-1}([s,t])\} \in \mathcal{F}^s_t$, which implies that τ_Y is an $\mathbb{F}^s$-stopping time.

Let now $\iota\colon C([s,T];\mathbb{R}^d)\times\mathscr{L}([s,T];\mathbb{I}_m)\to\mathbb{I}_m$ be $\mathcal{B}^s_\tau$-measurable. We have to prove that $\{\omega\colon \iota_Y(\omega)=\underline{i}\}\in\mathcal{F}^s_{\tau_Y}$ for any $\underline{i}\in\mathbb{I}_m$, i.e., $\{\omega\colon \iota_Y(\omega)=\underline{i}\}\cap\{\omega\colon \tau_Y(\omega)\le t\}\in\mathcal{F}^s_t$ for any $\underline{i}\in\mathbb{I}_m$ and $t\in[s,T]$. Then, fix $\underline{i}\in\mathbb{I}_m$ and $t\in[s,T]$. We have

$$\begin{aligned}\big\{\omega\colon \iota_Y(\omega)=\underline{i}\big\}\cap\big\{\omega\colon \tau_Y(\omega)\le t\big\} &= \big\{\omega\colon Y_\cdot(\omega)\in\iota^{-1}(\underline{i})\big\}\cap\big\{\omega\colon Y_\cdot(\omega)\in\tau^{-1}([s,t])\big\}\\ &= \big\{\omega\colon Y_\cdot(\omega)\in\{y\colon \iota(y)=\underline{i}\}\cap\{y\colon \tau(y)\le t\}\big\}.\end{aligned}$$

Since ι is $\mathcal{B}^s_\tau$-measurable, then $\{y\colon \iota(y)=\underline{i}\}\cap\{y\colon\tau(y)\le t\}\in\mathcal{B}^s_t$. Therefore, from the observation at the beginning of the proof, we get the thesis. ∎

PROPOSITION 2.1. *Let assumption* (H1) *hold. For any* $(s,x,i)\in[0,T]\times\mathbb{R}^d\times\mathbb{I}_m$, $\alpha\in\mathcal{A}_s$, $u\in\mathcal{U}_s$, *there exists a unique (up to indistinguishability)* $\mathbb{F}^s$*-adapted process* $(X^{s,x,i;\alpha,u},I^{s,x,i;\alpha,u})=(X^{s,x,i;\alpha,u}_t,I^{s,x,i;\alpha,u}_t)_{s\le t\le T}$ *to* (2.1)*, such that every path of* $(X^{s,x,i;\alpha,u}_\cdot,I^{s,x,i;\alpha,u}_{\cdot-})$ *belongs to* $C([s,T];\mathbb{R}^d)\times\mathscr{L}([s,T];\mathbb{I}_m)$*. Moreover, for any* $q\ge 1$ *there exists a positive constant* $C_{q,T}$*, depending only on* q,T,M_1 *(independent of* s,x,i,α,u*), such that*

$$\mathbb{E}\left[\sup_{s\le t\le T}|X^{s,x,i;\alpha,u}_t|^q\right]\le C_{q,T}(1+|x|^q). \tag{2.2}$$

Remark 2.4. In Proposition 2.1 we require that every path of $(X^{s,x,i;\alpha,u}_\cdot,I^{s,x,i;\alpha,u}_{\cdot-})$ belongs to $C([s,T];\mathbb{R}^d)\times\mathscr{L}([s,T];\mathbb{I}_m)$ in order to guarantee that the maps $\tau_n(X^{s,x,i;\alpha,u}_\cdot(\omega),I^{s,x,i;\alpha,u}_{\cdot-}(\omega))$ and $\iota_n(X^{s,x,i;\alpha,u}_\cdot(\omega),I^{s,x,i;\alpha,u}_{\cdot-}(\omega))$ are well-defined for every $\omega\in\Omega$, $n\in\mathbb{N}$. ∎

Proof. Fix $(s,x,i)\in[0,T]\times\mathbb{R}^d\times\mathbb{I}_m$, $\alpha=(\tau_n,\iota_n)_{n\in\mathbb{N}}\in\mathcal{A}_s$, $u\in\mathcal{U}_s$.

Step I. *Existence.* We begin noting that, since the control α is of feedback type, we have to construct the solution $(X^{s,x,i;\alpha,u},I^{s,x,i;\alpha,u})$ and α simultaneously. To do it we proceed as follows: for any $N\in\mathbb{N}$, we solve (2.1) controlled by u and the first N switching actions $(\tau_n,\iota_n)_{0\le n\le N-1}$. This is done by induction on N. Then, noting that $(X^N,I^N)=(X^{N-1},I^{N-1})$ on the stochastic interval $[s,\tau_{N-1})$, by pasting together the various solutions we are able to construct a solution $(X^{s,x,i;\alpha,u},I^{s,x,i;\alpha,u})$ to the original (2.1) with the entire switching control α. We now report the rigorous arguments.

For any $N\in\mathbb{N}$, let $\alpha^N=(\tau^N_n,\iota^N_n)_{n\in\mathbb{N}}\in\mathcal{A}_s$ be given by

$$(\tau^N_n,\iota^N_n):=\begin{cases}(\tau_n,\iota_n), & 0\le n\le N-1,\\ (T,\iota_n), & n\ge N.\end{cases}$$

Let $N=0$ and consider (2.1) controlled by α^0 and u. Notice that I is uncontrolled, in particular, $I_t=i$, $s\le t\le T$. Then, it is well known that under assumption (H1) there exists a unique (up to indistinguishability) $\mathbb{F}^s$-adapted solution $(X^0_t,I^0_t)_{s\le t\le T}$ to this equation with $I^0_t=i$ for any $t\in[s,T]$, such that every (not only $\mathbb{P}$-a.e., simply choosing an opportune indistinguishable version) path of $(X^0_\cdot,I^0_{\cdot-})$ belongs to $C([s,T];\mathbb{R}^d)\times\mathscr{L}([s,T];\mathbb{I}_m)$.

Now, let us prove the inductive step. Let $N\in\mathbb{N}\backslash\{0\}$ and suppose that there exists an $\mathbb{F}^s$-adapted solution (X^{N-1},I^{N-1}) to (2.1) controlled by α^{N-1} and u, such that every path of $(X^{N-1}_\cdot,I^{N-1}_{\cdot-})$ belongs to $C([s,T];\mathbb{R}^d)\times\mathscr{L}([s,T];\mathbb{I}_m)$. Our aim is to solve (2.1) controlled by α^N and u. To this end, we define the process $I^N=(I^N_t)_{s\le t\le T}$ as follows:

$$\begin{cases}I^N_t=I^{N-1}_t 1_{\{s\le t<\tau_{N-1}(X^{N-1}_\cdot,I^{N-1}_{\cdot-})\}}+\iota_{N-1}(X^{N-1}_\cdot,I^{N-1}_{\cdot-})1_{\{\tau_{N-1}(X^{N-1}_\cdot,I^{N-1}_{\cdot-})\le t<T\}},\\ I^N_T=I^N_{T^-}.\end{cases}$$

From Lemma 2.2 we see that I^N is an $\mathbb{F}^s$-adapted process, with every path in $\mathscr{L}([s,T];\mathbb{I}_m)$. Then, under assumption (H1) there exists a unique (up to indistinguishability) $\mathbb{F}^s$-adapted solution $(X_t^N, I_t^N)_{s\le t\le T}$ to (2.1), such that every path of $(X_\cdot^N, I_{\cdot-}^N)$ belongs to $C([s,T];\mathbb{R}^d)\times\mathscr{L}([s,T];\mathbb{I}_m)$. Since (X^N, I^N) and (X^{N-1}, I^{N-1}) solve the same equation on $[s,\tau_{N-1}(X_\cdot^{N-1}, I_{\cdot-}^{N-1}))$, then $(X_t^N, I_t^N) = (X_t^{N-1}, I_t^{N-1})$, $t\in[s,\tau_{N-1}(X_\cdot^{N-1}, I_{\cdot-}^{N-1}))$. In particular, $(X_t^N, I_{t^-}^N) = (X_t^{N-1}, I_{t^-}^{N-1})$ for any $t\in[s,\tau_{N-1}(X_\cdot^{N-1}, I_{\cdot-}^{N-1})]$. From Lemma 2.1, it follows that

$$\big(\tau_n(X_\cdot^{N-1}, I_{\cdot-}^{N-1}), \iota_n(X_\cdot^{N-1}, I_{\cdot-}^{N-1})\big) = \big(\tau_n(X_\cdot^N, I_{\cdot-}^N), \iota_n(X_\cdot^N, I_{\cdot-}^N)\big), \qquad 0\le n\le N-1.$$

As a consequence, (X^N, I^N) solves (2.1) controlled by α^N and u.

Finally, let us define (with the convention $\tau_{-1} := s$)

$$X_t^{s,x,i;\alpha,u} := \sum_{n\in\mathbb{N}} X_t^N 1_{\{\tau_{N-1}(X_\cdot^{N-1}, I_{\cdot-}^{N-1})\le t<\tau_N(X_\cdot^N, I_{\cdot-}^N)\}}, \tag{2.3}$$

$$I_t^{s,x,i;\alpha,u} := \sum_{n\in\mathbb{N}} I_t^N 1_{\{\tau_{N-1}(X_\cdot^{N-1}, I_{\cdot-}^{N-1})\le t<\tau_N(X_\cdot^N, I_{\cdot-}^N)\}} \tag{2.4}$$

for any $s\le t<T$ and $(X_T^{s,x,i;\alpha,u}, I_T^{s,x,i;\alpha,u}) := (X_{T^-}^{s,x,i;\alpha,u}, I_{T^-}^{s,x,i;\alpha,u})$. For simplicity of notation, we denote $(X,I) := (X^{s,x,i;\alpha,u}, I^{s,x,i;\alpha,u})$. Recalling that $\tau_{N-1}(X_\cdot^{N-1}, I_{\cdot-}^{N-1}) = \tau_{N-1}(X_\cdot^N, I_{\cdot-}^N) \le \tau_N(X_\cdot^N, I_{\cdot-}^N)$, we see that the sequence $(\tau_N(X_\cdot^N, I_{\cdot-}^N))_{N\ge-1}$ is nondecreasing, so that, for any $t\in[s,T]$, there is at most one term different from zero in the series appearing in (2.3) and (2.4). Moreover, from Definition 2.1, and, more precisely, from property (ii) of Remark 2.3, we have that, for every $\omega\in\Omega$, $\tau_N(X_\cdot^N(\omega), I_{\cdot-}^N(\omega)) = T$ for N large enough. In particular, X and I are well-defined over the entire interval $[s,T]$ and they are $\mathbb{F}^s$-adapted. Furthermore, we notice that $(X_t, I_t) = (X_t^N, I_t^N)$, $t\in[s,\tau_N(X_\cdot^N, I_{\cdot-}^N))$. Then, using again property (ii) of Remark 2.3, it follows that every path of $(X_\cdot, I_{\cdot-})$ belongs to $C([s,T];\mathbb{R}^d)\times\mathscr{L}([s,T];\mathbb{I}_m)$. In addition, since $(X_t, I_{t^-}) = (X_t^N, I_{t^-}^N)$, $t\in[s,\tau_N(X_\cdot^N, I_{\cdot-}^N)]$, from Lemma 2.1 we have

$$\big(\tau_N(X_\cdot^N, I_{\cdot-}^N), \iota_N(X_\cdot^N, I_{\cdot-}^N)\big) = \big(\tau_N(X_\cdot, I_{\cdot-}), \iota_N(X_\cdot, I_{\cdot-})\big) \qquad \forall\, N\in\mathbb{N}.$$

In particular, $(X_t, I_t) = (X_t^N, I_t^N)$, $t\in[s,\tau_N(X_\cdot, I_{\cdot-}))$. This implies that (X,I) solves (2.1) on $[s,\tau_N(X_\cdot, I_{\cdot-}))$ for any $N\in\mathbb{N}$. Recalling property (ii) of Remark 2.3, we see that (X,I) solves (2.1) on $[s,T)$. Since, by definition, $(X_T, I_T) = (X_{T^-}, I_{T^-})$, it follows that (X,I) solves (2.1) on $[s,T]$.

Step II. *Uniqueness.* Let (X^1, I^1) and (X^2, I^2) be two solutions of (2.1). Set $\underline{\tau}_0 := \tau_0(X_\cdot^1, I_{\cdot-}^1)\wedge\tau_0(X_\cdot^2, I_{\cdot-}^2)$. Notice that (X^1, I^1) and (X^2, I^2) solve the same equation on $[0,\underline{\tau}_0)$. Therefore (X^1, I^1) and (X^2, I^2) are equal (up to indistinguishability) on $[0,\underline{\tau}_0)$. Consider $\omega\in\Omega$ such that $\underline{\tau}_0(\omega) = \tau_0(X_\cdot^1(\omega), I_{\cdot-}^1(\omega))$. Since $(X_t^1(\omega), I_{t^-}^1(\omega)) = (X_t^2(\omega), I_{t^-}^2(\omega))$, $t\in[s,\underline{\tau}_0(\omega)] = [s,\tau_0(X_\cdot^1(\omega), I_{\cdot-}^1(\omega))]$, from Lemma 2.1 it follows that $\tau_0(X_\cdot^1(\omega), I_{\cdot-}^1(\omega)) = \tau_0(X_\cdot^2(\omega), I_{\cdot-}^2(\omega))$. When $\underline{\tau}_0(\omega) = \tau_0(X_\cdot^2(\omega), I_{\cdot-}^2(\omega))$, a similar argument shows that we still have $\tau_0(X_\cdot^1(\omega), I_{\cdot-}^1(\omega)) = \tau_0(X_\cdot^2(\omega), I_{\cdot-}^2(\omega))$. From the arbitrariness of ω, we conclude that $\underline{\tau}_0 = \tau_0(X_\cdot^1, I_{\cdot-}^1) = \tau_0(X_\cdot^2, I_{\cdot-}^2)$. Using again Lemma 2.1, we also deduce $\iota_0(X_\cdot^1, I_{\cdot-}^1) = \iota_0(X_\cdot^2, I_{\cdot-}^2)$. By induction on n, we can prove that

$$\begin{aligned}\big(\tau_n(X^1_\cdot, I^1_{\cdot-}), \iota_n(X^1_\cdot, I^1_{\cdot-})\big) &= \big(\tau_n(X^2_\cdot, I^2_{\cdot-}), \iota_n(X^2_\cdot, I^2_{\cdot-})\big)\\ &\qquad \forall\, n \in \mathbb{N},\\ (X^1_t, I^1_t) &= (X^2_t, I^2_t)\\ &\qquad \forall\, t \in [s, \tau_n(X^1_\cdot, I^1_{\cdot-})),\ n \in \mathbb{N}.\end{aligned}$$

From Definition 2.1 and, more precisely, from property (ii) of Remark 2.3, we have that, for any $\omega \in \Omega$, $\tau_n(X^1_\cdot(\omega), I^1_{\cdot-}(\omega)) = T$ for n large enough. As a consequence, (X^1, I^1) and (X^2, I^2) are equal (up to indistinguishability) on $[s, T)$. Since $(X^1_T, I^1_T) = (X^1_{T^-}, I^1_{T^-})$ and $(X^2_T, I^2_T) = (X^2_{T^-}, I^2_{T^-})$, we conclude that (X^1, I^1) and (X^2, I^2) are equal (up to indistinguishability) on $[s, T]$.

Step III. *Estimate* (2.2). Under (H1), estimate (2.2) is well known; see, e.g., [27, Theorem 1.3.15]. ∎

Remark 2.5. Notice that $\mathbb{F}^s$ is the filtration generated by the noise and $\mathbb{B}^s$ is the filtration generated by the state variable X. Since we have strong existence the latter is a subset of the former but not vice versa since the volatility is allowed to degenerate. α is the control of the switcher (the maximizer of our problem) and it is of feedback type. That is the switcher is only allowed to make a decision by observing the state variable. He is not allowed to observe the noise or the actions of the nature, which uses open-loop control, i.e., its control is adapted to $\mathbb{F}^s$. ∎

2.2. The value function. The value function associated with the robust switching control problem is defined as follows:

$$V(s,x,i) := \sup_{\alpha \in \mathcal{A}_s} \inf_{u \in \mathcal{U}_s} J(s,x,i;\alpha,u) \qquad \forall\, (s,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_m, \tag{2.5}$$

with

$$\begin{aligned} J(s,x,i;\alpha,u) := \mathbb{E}\bigg[\int_s^T & f(X^{s,x,i;\alpha,u}_r, I^{s,x,i;\alpha,u}_r, u_r)dr + g(X^{s,x,i;\alpha,u}_T, I^{s,x,i;\alpha,u}_T)\\ &- \sum_{n \in \mathbb{N}} c(X^{s,x,i;\alpha,u}_{\tau_n}, I^{s,x,i;\alpha,u}_{\tau_n^-}, I^{s,x,i;\alpha,u}_{\tau_n}) 1_{\{s \le \tau_n < T\}}\bigg], \end{aligned} \tag{2.6}$$

where τ^n stands for $\tau^n(X^{s,x,i;\alpha,u}_\cdot, I^{s,x,i;\alpha,u}_{\cdot-})$.

Remark 2.6. This definition of game value function with the outside player (switcher) using feedback strategies (i.e., closed-loop controls) and the inside player (nature) using open-loop controls is the same as the one used in [1, Definition 3.6, Chapter VIII], and called there the B-feedback value. It is also pointed out that the B-feedback value is smaller than the upper value of a game where the outside player uses nonanticipating strategies à la Elliott–Kalton; see also our section 4.2. ∎

We impose the following conditions on the functions $g\colon \mathbb{R}^d \times \mathbb{I}_m \to \mathbb{R}$, $f\colon \mathbb{R}^d \times \mathbb{I}_m \times U \to \mathbb{R}$, and $c\colon \mathbb{R}^d \times \mathbb{I}_m \times \mathbb{I}_m \to \mathbb{R}$.

(H2)

(i) g, f, c are jointly continuous on their domains.

(ii) c is nonnegative.

(iii) g, f, c satisfy a polynomial growth condition in x, i.e.,

$$|g(x,i)| + |f(x,i,u)| + |c(x,i,j)| \le M_2(1 + |x|^p)$$

$\forall\, x \in \mathbb{R}^d$, $i, j \in \mathbb{I}_m$, $u \in U$, for some positive constants M_2 and $p \ge 1$.

(iv) g satisfies

$$g(x,i) \geq \max_{j\neq i} \big[g(x,j) - c(x,i,j)\big]$$

for any $x \in \mathbb{R}^d$ and $i \in \mathbb{I}_m$.

Remark 2.7. Notice that V satisfies the polynomial growth condition:

$$|V(s,x,i)| \leq C(1+|x|^p) \qquad \forall\,(s,x,i) \in [0,T]\times\mathbb{R}^d\times\mathbb{I}_m \tag{2.7}$$

for some positive constant C, depending only on T, M_1, M_2, and with the same p as in assumption (H2)(iii). Indeed, since c is nonnegative, we find

$$V(s,x,i) \leq \sup_{\alpha\in\mathcal{A}_s}\inf_{u\in\mathcal{U}_s}\mathbb{E}\bigg[\int_s^T f(X_r^{s,x,i;\alpha,u}, I_r^{s,x,i;\alpha,u}, u_r)dr + g(X_T^{s,x,i;\alpha,u})\bigg]. \tag{2.8}$$

On the other hand, let $\alpha^* = (\tau_n^*, \iota_n^*)_{n\in\mathbb{N}} \in \mathcal{A}_s$ be given by $(\tau_n^*, \iota_n^*) = (T, \underline{i})\ \forall\, n\in\mathbb{N}$ for some fixed $\underline{i} \in \mathbb{I}_m$. Then

$$\begin{aligned} V(s,x,i) &\geq \inf_{u\in\mathcal{U}_s} J(s,x,i;\alpha^*,u) \\ &= \inf_{u\in\mathcal{U}_s}\mathbb{E}\bigg[\int_s^T f(X_r^{s,x,i;\alpha^*,u}, I_r^{s,x,i;\alpha^*,u}, u_r)dr + g(X_T^{s,x,i;\alpha^*,u})\bigg]. \end{aligned} \tag{2.9}$$

From (2.8) and (2.9), we obtain

$$|V(s,x,i)| \leq \sup_{\alpha\in\mathcal{A}_s}\sup_{u\in\mathcal{U}_s}\mathbb{E}\bigg[\int_s^T |f(X_r^{s,x,i;\alpha,u}, I_r^{s,x,i;\alpha,u}, u_r)|dr + |g(X_T^{s,x,i;\alpha,u})|\bigg].$$

Now, from estimate (2.2) and the polynomial growth condition of f and g in (H2)(iii), we see that estimate (2.7) holds. As a consequence, in (2.5) we could take the supremum only over $\alpha = (\tau_n, \iota_n)_{n\in\mathbb{N}} \in \mathcal{A}_s$ satisfying (τ^n stands for $\tau^n(X_\cdot^{s,x,i;\alpha,u}, I_{\cdot^-}^{s,x,i;\alpha,u})$)

$$\inf_{u\in\mathcal{U}_s}\mathbb{E}\bigg[-\sum_{n\in\mathbb{N}} c(X_{\tau_n}^{s,x,i;\alpha,u}, I_{\tau_n^-}^{s,x,i;\alpha,u}, I_{\tau_n}^{s,x,i;\alpha,u})1_{\{s\leq\tau_n<T\}}\bigg] > -\infty. \qquad \square$$

Our aim is to prove that V is the unique viscosity solution to the dynamic programming equation associated to the robust switching control problem, which turns out to be a system of variational inequalities of HJB type of the following form:

$$\begin{cases} \min\Big\{-\dfrac{\partial V}{\partial t}(s,x,i) - \inf_{u\in U}\big[\mathcal{L}^{i,u}V(s,x,i) + f(x,i,u)\big], \\ \qquad V(s,x,i) - \max_{j\neq i}\big[V(s,x,j) - c(x,i,j)\big]\Big\} = 0, \quad (s,x,i)\in[0,T)\times\mathbb{R}^d\times\mathbb{I}_m, \\ V(T,x,i) = g(x,i), \quad (x,i)\in\mathbb{R}^d\times\mathbb{I}_m, \end{cases} \tag{2.10}$$

where

$$\mathcal{L}^{i,u}V(s,x,i) = b(x,i,u).D_xV(s,x,i) + \frac{1}{2}\mathrm{tr}\big[\sigma\sigma^\intercal(x,i,u)D_x^2V(s,x,i)\big].$$

We need the definition of (discontinuous) viscosity solution to (2.10), that we now provide. To this end, given a locally bounded function $v\colon [0,T)\times\mathbb{R}^d\times\mathbb{I}_m \to \mathbb{R}$, we define its lower semicontinuous (lsc for short) envelope $v_*\colon [0,T]\times\mathbb{R}^d\times\mathbb{I}_m\to\mathbb{R}$, and upper semicontinuous (usc for short) envelope $v^*\colon [0,T]\times\mathbb{R}^d\times\mathbb{I}_m\to\mathbb{R}$, by

$$v_*(s,x,i)=\liminf_{\substack{(s',x')\to(s,x)\\(s',x')\in[0,T)\times\mathbb{R}^d}} v(s',x',i)\qquad\text{and}\qquad v^*(s,x,i)=\limsup_{\substack{(s',x')\to(s,x)\\(s',x')\in[0,T)\times\mathbb{R}^d}} v(s',x',i),$$

$\forall\,(s,x,i)\in[0,T]\times\mathbb{R}^d\times\mathbb{I}_m$.

Definition 2.3 (viscosity solution to (2.10)).

(i) *An lsc (resp., usc) function v on $[0,T]\times\mathbb{R}^d\times\mathbb{I}_m$ is called a viscosity supersolution (resp., subsolution) to* (2.10) *if*

$$v(T,x,i)\geq (resp.,\ \leq)\ g(x,i)$$

for any $(x,i)\in\mathbb{R}^d\times\mathbb{I}_m$, and

$$\min\Big\{-\frac{\partial\varphi}{\partial t}(s,x)-\inf_{u\in U}\big[\mathcal{L}^{i,u}\varphi(s,x)+f(x,i,u)\big],\\ v(s,x,i)-\max_{j\neq i}\big[v(s,x,j)-c(x,i,j)\big]\Big\}\geq (resp.,\ \leq)\ 0$$

for any $(s,x,i)\in[0,T)\times\mathbb{R}^d\times\mathbb{I}_m$ and any $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^d)$ such that

$$v(s,x,i)-\varphi(s,x)=\min_{(s',x')\in[0,T]\times\mathbb{R}^d}\big[v(s',x',i)-\varphi(s',x')\big]$$

$$\Big\{resp.,\quad v(s,x,i)-\varphi(s,x)=\max_{(s',x')\in[0,T]\times\mathbb{R}^d}\big[v(s',x',i)-\varphi(s',x')\big]\Big\}.$$

(ii) *A locally bounded function v on $[0,T)\times\mathbb{R}^d\times\mathbb{I}_m$ is called a viscosity solution to* (2.10) *if v_* is a viscosity supersolution and v^* is a viscosity subsolution to* (2.10).

3. Stochastic Perron's method.

Our aim is to prove that V is a viscosity solution to the dynamic programming equation (2.10) and satisfies the dynamic programming principle. To derive these results, we exploit the *stochastic Perron's method*, which allows us to obtain the viscosity properties of V without relying on the dynamic programming principle, but by means of the comparison theorem for viscosity solutions to (2.10) (the dynamic programming principle will be obtained as a by-product of this procedure).

3.1. An auxiliary robust switching problem.

We begin with the formulation of an auxiliary robust switching control problem where nature adopts closed-loop controls (also called feedback strategies) in place of open-loop controls. Using the comparison principle for (2.10), we shall see that the corresponding value function, denoted by $\overline{V}$, coincides with V. In other words, the information available to nature does not affect the value of the game. This is not the only motivation for the introduction of this auxiliary robust control problem. Indeed, in the implementation of the stochastic Perron's method we encountered the following difficulty: given two different controls u_1 and u_2, for nature, we have to concatenate them at some stopping rule $\tau=\tau(X_\cdot,I_{\cdot-})$. If u^1 and u^2 are open-loop controls, the control $u^1\otimes_\tau u^2$ resulting from the concatenation of u^1 and u^2 at the stopping rule τ, given by

$$(u^1\otimes_\tau u^2)(t,\omega,y)=u^1(t,\omega)1_{\{s\leq t\leq\tau(y)\}}+u^2(t,\omega)1_{\{\tau(y)<t\leq T\}},$$

is no longer of open-loop type, since it also depends on y. On the other hand, if u^1 and u^2 are closed-loop controls, then $u^1 \otimes_\tau u^2$ is still a closed-loop control. For this technical reason, to study the original control problem with corresponding value function V, we also need to consider another robust switching control problem, in which nature adopts closed-loop controls. In particular, inspired by [31] and [30], it turns out that it is more convenient, and it is enough, to consider only piecewise constant closed-loop controls, i.e., the *elementary feedback strategies* that we now define.

Definition 3.1 (elementary feedback strategies). *Fix $s \in [0,T]$. We say that u is an elementary feedback strategy starting at s if*

- *$\tau_k \in \mathcal{T}^s$ for any $k = 1, \dots, n$, and*

$$s =: \tau_0 \leq \cdots \leq \tau_k \leq \cdots \leq \tau_n = T;$$

- *$\xi_k \colon C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m) \to U$ is $\mathcal{B}^s_{\tau^+_{k-1}}$-measurable for any $k = 1, \dots, n$.*

The control $u \colon [s,T] \times C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m) \to U$ is given by

$$u(t,y) := \xi_1(y) 1_{\{t=s\}} + \sum_{k=1}^n \xi_k(y) 1_{\{\tau_{k-1}(y) < t \leq \tau_k(y)\}}.$$

$\mathcal{U}^E_s$ denotes the family of all elementary feedback strategies (also called elementary closed-loop controls) starting at s.

Remark 3.1. We notice that Definition 3.1 is inspired by Definition 2.2 in [31] (see also Definition 2.1 in [30]), the only difference being that ξ_k is $\mathcal{B}^s_{\tau^+_{k-1}}$-measurable instead of $\mathcal{B}^s_{\tau_{k-1}}$-measurable. This implies that the map $\xi_k = \xi_k(y)$ depends on y through the values $\{y(t),\, s \leq t \leq \tau_{k-1}(y)\} \cup \{y(\tau_{k-1}(y)^+)\}$, so that ξ_k can also depend on $y(\tau_{k-1}(y)^+)$. Recalling that in our setting y denotes a generic path of $(X_t, I_{t^-})_{s \leq t \leq T}$, this means that ξ_k depends on $(X_t, I_t)_{s \leq t \leq \tau_{k-1}(X_\cdot, I_{\cdot^-})}$ rather than on $(X_t, I_{t^-})_{s \leq t \leq \tau_k(X_\cdot, I_{\cdot^-})}$. Therefore, nature reacts to the switcher using all the information at disposal at time $\tau_{k-1} = \tau_{k-1}(X_\cdot, I_{\cdot^-})$, including $I_{\tau_{k-1}}$ (in particular, if τ_{k-1} coincides with a switching action, nature is aware of the action that the switcher has just performed). We point out that elementary feedback strategies are different from strategies in the sense of Elliott–Kalton where strategies are used by the outside player (i.e., the switcher here) and not by the inside player (the nature here). Actually, the set of elementary feedback strategies (closed-loop controls) is obviously a subset of open-loop controls since they correspond to controls which are piecewise constant on one hand, and with actions decided based only on the knowledge of the state, hence with less information than the one generated by $\mathbb{F}^s$. In other words, we have $\mathcal{U}^E_s \subset \mathcal{U}_s$: for any feedback control $u \in \mathcal{U}^E_s$ we can construct an open-loop control $(v_t)_{s \leq t \leq T} \triangleq (u(t, X_\cdot^{s,x,\alpha,u}))_{s \leq t \leq T} \in \mathcal{U}_s$ which shows the inclusion above. ☐

We have the following well-posedness result for (2.1) when u is an elementary feedback strategy (so that u_r stands for $u(r, X_\cdot, I_{\cdot^-})$), where the only difference with Proposition 2.1 is that now the solution is adapted to the smaller filtration $\mathbb{F}^{W,s}$, since $\mathbb{F}^s$ plays no role when $u \in \mathcal{U}^E_s$.

Proposition 3.1. *Let assumption* (H1) *hold. For any $(s,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_m$, $\alpha \in \mathcal{A}_s$, $u \in \mathcal{U}^E_s$, there exists a unique (up to indistinguishability) $\mathbb{F}^{W,s}$-adapted process $(X^{s,x,i;\alpha,u}, I^{s,x,i;\alpha,u}) = (X_t^{s,x,i;\alpha,u}, I_t^{s,x,i;\alpha,u})_{s \leq t \leq T}$ to* (2.1)*, such that every path of $(X_\cdot^{s,x,i;\alpha,u}, I_{\cdot^-}^{s,x,i;\alpha,u})$ belongs to $C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m)$. Moreover, for any*

$q \geq 1$ *there exists a positive constant* $C_{q,T}$*, depending only on* q, T, M_1 *(independent of* s, x, i, α, u*), such that*

$$\mathbb{E}\left[\sup_{s\leq t\leq T} |X_t^{s,x,i;\alpha,u}|^q\right] \leq C_{q,T}(1+|x|^q). \tag{3.1}$$

Proof. The proof can be done along the lines of the proof of Proposition 2.1. We simply note that in Proposition 2.1 we used the following result: if $u \in \mathcal{U}_s$ and $I = (I_t)_{s\leq t\leq \tau}$ is known up to a certain $\mathbb{F}^s$-stopping time τ, then there exists a unique (up to indistinguishability) $\mathbb{F}^s$-adapted solution $X = (X_t)_{s\leq t\leq\tau}$ to the equation

$$X_t = x + \int_s^t b(X_r, I_r, u_r)dr + \int_s^t \sigma(X_r, I_r, u_r)dW_r, \qquad s \leq t \leq \tau, \tag{3.2}$$

such that every path of X belongs to $C([s,T];\mathbb{R}^d)$. The validity of this result is well known under (H1). On the other hand, it is not immediately clear when $u \in \mathcal{U}_s^E$ is an elementary feedback strategy. However, the result is still valid and follows from [31, Proposition 2.4]; see also [30, Theorem 2.2]. Moreover, when $u \in \mathcal{U}_s^E$ it turns out that the process X is adapted to the smaller filtration $\mathbb{F}^{W,s}$. Finally, under assumption (H1), estimate (3.1) is well known; see, e.g., [27, Theorem 1.3.15]. ∎

We can finally introduce the value function for the robust switching control problem where nature adopts the elementary feedback strategies:

$$\overline{V}(s,x,i)$$
$$:= \sup_{\alpha\in\mathcal{A}_s}\inf_{u\in\mathcal{U}_s^E} \mathbb{E}\left[\int_s^T f(X_t, I_t, u'_t)dt + g(X_T, I_T) - \sum_{n\in\mathbb{N}} c(X_{\tau'_n}, I_{(\tau'_n)^-}, I_{\tau'_n})1_{\{s\leq\tau'_n<T\}}\right]$$

for every $(s,x,i) \in [0,T]\times\mathbb{R}^d\times\mathbb{I}_m$, with the shorthand $X = X^{s,x,i;\alpha,u}$, $I = I^{s,x,i;\alpha,u}$, $\tau'_n = \tau_n(X_\cdot, I_{\cdot^-})$, and $u'_t = u(t, X_\cdot, I_{\cdot^-})$. This auxiliary formulation of the robust switching problem where both players use feedback strategies (or closed-loop controls) is the same as the one used in [31]. Notice that $u' \in \mathcal{U}_s$ and we have

$$\overline{V}(s,x,i) := \sup_{\alpha\in\mathcal{A}_s}\inf_{u\in\mathcal{U}_s^E} J(s,x,i;\alpha,u') \qquad \forall\,(s,x,i)\in[0,T]\times\mathbb{R}^d\times\mathbb{I}_m.$$

In particular, $V(s,x,i) \leq \overline{V}(s,x,i)$ for any $(s,x,i) \in [0,T]\times\mathbb{R}^d\times\mathbb{I}_m$. Moreover, proceeding as in Remark 2.7, we can show that $\overline{V}$ satisfies a polynomial growth condition in x: $|\overline{V}(s,x,i)| \leq C(1+|x|^p) < \infty$ for some positive constant C, depending only on T, M_1, M_2, and with the same p as in assumption (H2)(iii).

3.2. Concatenation of feedback strategies. In the present section, we need to introduce the concept of feedback control starting at a certain stopping rule τ and to define the notion of concatenation at τ of two feedback controls, which will be crucial in the development of the stochastic Perron's method.

Definition 3.2 (feedback switching strategies starting strictly later than τ). *Fix* s *in* $[0,T]$ *and* $\tau \in \mathcal{T}^s$*. We say that the double sequence* $\alpha = (\tau_n, \iota_n)_{n\in\mathbb{N}}$ *is a feedback switching strategy starting strictly later than* τ *if* $\alpha \in \mathcal{A}_s$*, with* $\tau \leq \tau_0$ *and* $\tau < \tau_0$ *on the set* $\{\tau < T\}$*.* $\mathcal{A}_{s,\tau^+}$ *denotes the family of all feedback switching strategies for the controller, given the initial deterministic time* s *and starting strictly later than* τ*. When* $\tau \equiv s$*, we simply write* $\mathcal{A}_{s^+}$ *instead of* $\mathcal{A}_{s,s^+}$*.*

Following [31, Definition 2.7], and recalling Remark 3.1, we now define the elementary feedback strategies starting at some stopping rule τ.

Definition 3.3 (elementary feedback strategies starting at τ). *Fix $s \in [0,T]$ and $\tau \in \mathcal{T}^s$. We say that u is an elementary feedback strategy starting at τ if*

- *$\tau_k \in \mathcal{T}^s$ for any $k = 1,\dots,n$, and*

$$\tau =: \tau_0 \leq \cdots \leq \tau_k \leq \cdots \leq \tau_n = T;$$

- *$\xi_k \colon C([s,T];\mathbb{R}^d)\times\mathscr{L}([s,T];\mathbb{I}_m) \to U$ is $\mathcal{B}^s_{\tau^+_{k-1}}$-measurable, for any $k = 1,\dots,n$.*

The elementary feedback strategy

$$u\colon \big\{(t,y) \in [s,T] \times \big(C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m)\big)\colon \tau(y) \leq t \leq T\big\} \longrightarrow U$$

is given by

$$u(t,y) := \xi_1(y)1_{\{t=\tau(y)\}} + \sum_{k=1}^n \xi_k(y)1_{\{\tau_{k-1}(y)<t\leq\tau_k(y)\}}.$$

$\mathcal{U}^E_{s,\tau}$ denotes the family of all elementary feedback strategies given the initial deterministic time s and starting at τ.

Notice that, when $\tau = s$ in Definition 3.3, the set $\mathcal{U}^E_s$ is just $\mathcal{U}^E_s$.

Remark 3.2. Definition 3.2 is inspired by [31, Definition 2.7] with, in addition, the condition "$\tau < \tau_0$ on the set $\{\tau < T\}$," which justifies the presence of the adverb *strictly* in the name. Indeed, our aim is to define the set $\mathcal{A}_{s,\tau^+}$ in such a way that when we concatenate two feedback switching strategies $\alpha \in \mathcal{A}_s$ and $\tilde{\alpha} \in \mathcal{A}_{s,\tau^+}$ at a stopping rule $\tau \in \mathcal{T}^s$ (see Proposition 3.2 below) then $\alpha \otimes_\tau \tilde{\alpha}$ coincides with α at time τ (this property plays an important role in what follows, e.g., in the proof of Theorem 3.1). On the other hand, when we concatenate two elementary feedback strategies $u \in \mathcal{U}^E_s$ and $\tilde{u} \in \mathcal{U}^E_{s,\tau}$, then $u \otimes_\tau \tilde{u}$ coincides with u at time τ, simply adopting the same definition for $\mathcal{U}^E_{s,\tau}$ as in [31] combined with Remark 3.1. ∎

As in [31, Lemma 2.8 and Proposition 2.9], we have the two following results, whose simple proof is only sketched for Lemma 3.1 and omitted for Proposition 3.2.

Lemma 3.1. *Fix $s \in [0,T]$, $\tau \in \mathcal{T}^s$, $\alpha^1 = (\tau^1_n,\iota^1_n)_{n\in\mathbb{N}}, \alpha^2 = (\tau^2_n,\iota^2_n)_{n\in\mathbb{N}} \in \mathcal{A}_{s,\tau^+}$, $u^1,u^2 \in \mathcal{U}^E_{s,\tau}$, and $B \in \mathcal{B}^s_{\tau^+}$.*

- *The double sequence $\alpha = (\tau_n,\iota_n)_{n\in\mathbb{N}}$ given by*

  $$\big(\tau_n(y),\iota_n(y)\big) = \big(\tau^1_n(y),\iota^1_n(y)\big)1_{\{y\in B\}} + \big(\tau^2_n(y),\iota^2_n(y)\big)1_{\{y\in B^c\}}$$

  is in $\mathcal{A}_{s,\tau^+}$.
- *The map*

  $$u\colon \big\{(t,y) \in [s,T] \times \big(C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m)\big)\colon \tau(y) \leq t \leq T\big\} \longrightarrow U$$

  given by

  $$u(t,y) = u^1(t,y)1_{\{y\in B\}} + u^2(t,y)1_{\{y\in B^c\}}$$

  is in $\mathcal{U}^E_{s,\tau}$.

Proof. We only prove the first item, where we focus on the two main points. In particular, the proof that $\tau_n \in \mathcal{T}^s$ and $\iota_n \in \mathcal{B}^s_{\tau_n}$ is based on the observation that

$B \in \mathcal{B}^s_{\tau^+} \subset \mathcal{B}^s_{\tau^1_n}, \mathcal{B}^s_{\tau^2_n}$ for any $n \in \mathbb{N}$, which is a consequence of the property $\tau < \tau^1_0, \tau^2_0$ on the set $\{\tau < T\}$. The other nontrivial part is the proof that α satisfies property (ii) of Remark 2.3. To prove it, consider $(y_n)_{n\in\mathbb{N}} \in C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m)$, with $y_n(t) = y_{n+1}(t)$, $t \in [s, \tau_n(y_n)]$. Since $\tau_0 \le \tau_n$ for any $n \in \mathbb{N}$, we have

$$y_0(t) = y_n(t) \qquad \forall\, t \in [s, \tau_0(y_0)],\ n \in \mathbb{N}.$$

As $\tau < \tau_0$ on the set $\{\tau < T\}$, it follows that

$$y_0(t^+) = y_n(t^+) \qquad \forall\, t \in [s, \tau(y_0)],\ n \in \mathbb{N}. \tag{3.3}$$

In particular $y_0(\tau(y_0)^+) = y_n(\tau(y_0)^+)$. Moreover, from Lemma 2.1 we get $\tau(y_0) = \tau(y_n)$, so that $y_0(\tau(y_0)^+) = y_n(\tau(y_n)^+)$. Therefore, $y_0 \in B$ if and only if $y_n \in B$, for any $n \in \mathbb{N}$. In conclusion, property (ii) of Remark 2.3 for $(\tau_n)_{n\in\mathbb{N}}$ follows from the definitions of $(\tau^1_n)_{n\in\mathbb{N}}$ and $(\tau^2_n)_{n\in\mathbb{N}}$. ∎

PROPOSITION 3.2 (concatenation). *Fix $s \in [0,T]$, $\tau, \rho \in \mathcal{T}^s$ with $\tau \le \rho \le T$, $\tilde\alpha = (\tilde\tau_n, \tilde\iota_n)_{n\in\mathbb{N}} \in \mathcal{A}_{s,\rho^+}$, $\tilde u \in \mathcal{U}^E_{s,\rho}$. Then*

- *for each $\alpha = (\tau_n, \iota_n)_{n\in\mathbb{N}} \in \mathcal{A}_s$ (resp., $\alpha = (\tau_n, \iota_n)_{n\in\mathbb{N}} \in \mathcal{A}_{s,\tau^+}$), the double sequence $\alpha \otimes_\rho \tilde\alpha = (\tau_n^{\otimes_\rho}, \iota_n^{\otimes_\rho})_{n\in\mathbb{N}}$ given by*
$$\big(\tau_n^{\otimes_\rho}(y), \iota_n^{\otimes_\rho}(y)\big) = \big(\tau_n(y), \iota_n(y)\big)1_{\{\tau_n(y)\le\rho(y)\}} + \big(\tilde\tau_n(y), \tilde\iota_n(y)\big)1_{\{\tau_n(y)>\rho(y)\}}$$
is in $\mathcal{A}_s$ (resp., $\mathcal{A}_{s,\tau^+}$);
- *for each $u \in \mathcal{U}^E_{s,\tau}$, the map*
$$u \otimes_\rho \tilde u \colon \big\{(t,y) \in [s,T] \times \big(C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m)\big) \colon \tau(y) \le t \le T\big\} \longrightarrow U$$
given by
$$(u \otimes_\rho \tilde u)(t,y) = u(t,y)1_{\{\tau(y)\le t\le\rho(y)\}} + \tilde u(t,y)1_{\{\rho(y)<t\le T\}}$$
is in $\mathcal{U}^E_{s,\tau}$.

3.3. Definitions of $\mathcal{V}^-$, $\mathcal{V}^+$ and their properties. We can now provide the definitions of the classes of functions $\mathcal{V}^-$ and $\mathcal{V}^+$, which are the cornerstones of the stochastic Perron's method. Their elements are known in the literature on the stochastic Perron's method as stochastic subsolutions ($\mathcal{V}^-$) and stochastic supersolutions ($\mathcal{V}^+$); see, e.g., [5].

DEFINITION 3.4. *$\mathcal{V}^-$ is the set of functions $v\colon [0,T] \times \mathbb{R}^d \times \mathbb{I}_m \to \mathbb{R}$ which have the following properties:*

- *v is continuous and satisfies the terminal condition $v(T,x,i) \le g(x,i)$, $(x,i) \in \mathbb{R}^d \times \mathbb{I}_m$, together with the polynomial growth condition*
$$\sup_{(s,x,i)\in[0,T]\times\mathbb{R}^d\times\mathbb{I}_m} \frac{|v(s,x,i)|}{1+|x|^q} < \infty$$
for some $q \ge 1$.
- *For any $s \in [0,T]$ and $\tau, \rho \in \mathcal{T}^s$ with $\tau \le \rho \le T$, there exists $\tilde\alpha = (\tilde\tau_n, \tilde\iota_n)_{n\in\mathbb{N}} \in \mathcal{A}_{s,\tau^+}$ (possibly depending on s, τ, ρ) such that, for any $\alpha = (\tau_n, \iota_n)_{n\in\mathbb{N}} \in \mathcal{A}_s$, $u \in \mathcal{U}_s$, and $(x,i) \in \mathbb{R}^d \times \mathbb{I}_m$, we have*
$$v(\tau', X_{\tau'}, I_{\tau'}) \le \mathbb{E}\bigg[\int_{\tau'}^{\rho'} f(X_t, I_t, u_t)dt + v(\rho', X_{\rho'}, I_{\rho'}) - \sum_{n\in\mathbb{N}} c(X_{\tilde\tau'_n}, I_{(\tilde\tau'_n)^-}, I_{\tilde\tau'_n}) 1_{\{\tau'\le\tilde\tau'_n<\rho'\}} \bigg| \mathcal{F}^s_{\tau'}\bigg] \qquad \mathbb{P}\text{-a.s.}$$

with the shorthand $X = X^{s,x,i;\alpha\otimes_\tau\tilde{\alpha},u}$, $I = I^{s,x,i;\alpha\otimes_\tau\tilde{\alpha},u}$, $\tau' = \tau(X_\cdot, I_{\cdot-})$, $\rho' = \rho(X_\cdot, I_{\cdot-})$, *and* $\tilde{\tau}'_n = \tilde{\tau}_n(X_\cdot, I_{\cdot-})$.

Definition 3.5. $\mathcal{V}^+$ *is the set of functions* $v\colon [0,T]\times\mathbb{R}^d\times\mathbb{I}_m \to \mathbb{R}$ *which have the following properties:*

- v *is continuous and satisfies the terminal condition* $v(T,x,i) \geq g(x,i)$, $(x,i) \in \mathbb{R}^d\times\mathbb{I}_m$, *together with the polynomial growth condition*

$$\sup_{(s,x,i)\in[0,T]\times\mathbb{R}^d\times\mathbb{I}_m} \frac{|v(s,x,i)|}{1+|x|^q} < \infty$$

for some $q \geq 1$.

- *For any* $s \in [0,T]$, $\tau \in \mathcal{T}^s$, *and* $\alpha = (\tau_n,\iota_n)_{n\in\mathbb{N}} \in \mathcal{A}_s$, *there exists* $\tilde{u} \in \mathcal{U}^E_{s,\tau}$ *(possibly depending on* s,τ,α*) such that, for any* $u \in \mathcal{U}^E_s$, $(x,i)\in\mathbb{R}^d\times\mathbb{I}_m$, *and* $\rho\in\mathcal{T}^s$ *with* $\tau\leq\rho\leq T$, *we have*

$$v(\tau',X_{\tau'},I_{\tau'}) \geq \mathbb{E}\bigg[\int_{\tau'}^{\rho'} f(X_t,I_t,\tilde{u}_t)dt + v(\rho',X_{\rho'},I_{\rho'}) - \sum_{n\in\mathbb{N}} c(X_{\tau'_n}, I_{(\tau'_n)^-}, I_{\tau'_n})1_{\{\tau'\leq\tau'_n<\rho'\}}\bigg|\mathcal{F}^s_{\tau'}\bigg] \qquad \mathbb{P}\text{-a.s.}$$

with the shorthand $X = X^{s,x,i;\alpha,u\otimes_\tau\tilde{u}}$, $I = I^{s,x,i;\alpha,u\otimes_\tau\tilde{u}}$, $\tau' = \tau(X_\cdot, I_{\cdot-})$, $\rho' = \rho(X_\cdot, I_{\cdot-})$, $\tau'_n = \tau_n(X_\cdot, I_{\cdot-})$, *and* $\tilde{u}_t = \tilde{u}(t, X_\cdot, I_{\cdot-})$.

Remark 3.3. The definitions of $\mathcal{V}^-$ and $\mathcal{V}^+$ are inspired by [31, Definitions 3.1–3.2–3.3], but for the fact that in Definition 3.4 above we fix ρ before $\tilde{\alpha}$, so that $\tilde{\alpha}$ can depend on ρ. This greater freedom in the choice of $\tilde{\alpha}$ turns out to be fundamental in the implementation of the stochastic Perron's method, Theorem 3.1, and it is due to the condition "$\tau < \tau_0$ on the set $\{\tau < T\}$" in the definition of $\mathcal{A}_{s,\tau^+}$, already discussed in Remark 3.2. Indeed, using the set $\mathcal{A}_{s,\tau^+}$, the existence of an "optimal" feedback switching strategy $\tilde{\alpha} = (\tilde{\tau}_n,\tilde{\iota}_n)_{n\in\mathbb{N}} \in \mathcal{A}_{s,\tau^+}$, which works for every $\rho\in\mathcal{T}^s$ with $\tau\leq\rho\leq T$, is not guaranteed. For example, it could happen that every optimal feedback switching strategy which works $\forall\,\rho$ has to satisfy $\tilde{\tau}_0 = \tau$, therefore it cannot belong to $\mathcal{A}_{s,\tau^+}$. To avoid this problem, first we fix ρ, then we choose an optimal $\tilde{\alpha}\in\mathcal{A}_{s,\tau^+}$. Another possibility would be to look for an "ε-optimal" $\tilde{\alpha}\in\mathcal{A}_{s,\tau^+}$ which works for every ρ. □

We first notice that, as stated below, the two sets $\mathcal{V}^-$ and $\mathcal{V}^+$ are not empty, moreover, every $v\in\mathcal{V}^-$ (resp., $v\in\mathcal{V}^+$) satisfies the subdynamic (resp., superdynamic) programming principle, also known as suboptimality (resp., superoptimality) principle; see [32].

Lemma 3.2. *Let assumptions* (H1) *and* (H2) *hold.*

(i) $\mathcal{V}^- \neq \emptyset$ *and* $\mathcal{V}^+ \neq \emptyset$.

(ii) *Every* $v\in\mathcal{V}^-$ *satisfies the subdynamic programming principle: for any* $(s,x,i)\in[0,T]\times\mathbb{R}^d\times\mathbb{I}_m$ *and* $\rho\in\mathcal{T}^s$,

$$v(s,x,i) \leq \sup_{\alpha\in\mathcal{A}_{s^+}} \inf_{u\in\mathcal{U}_s} \mathbb{E}\bigg[\int_s^{\rho'} f(X_t,I_t,u_t)dt + v(\rho',X_{\rho'},I_{\rho'}) - \sum_{n\in\mathbb{N}} c(X_{\tau'_n}, I_{(\tau'_n)^-}, I_{\tau'_n})1_{\{s\leq\tau'_n<\rho'\}}\bigg] \tag{3.4}$$

with the shorthand $X = X^{s,x,i;\alpha,u}$, $I = I^{s,x,i;\alpha,u}$, $\rho' = \rho(X_\cdot, I_{\cdot -})$, *and* $\tau_n' = \tau_n(X_\cdot, I_{\cdot -})$.

(iii) *Every* $v \in \mathcal{V}^+$ *satisfies the superdynamic programming principle: for any* $(s,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_m$ *and* $\rho \in \mathcal{T}^s$,

$$v(s,x,i) \geq \sup_{\alpha \in \mathcal{A}_{s^+}} \inf_{u \in \mathcal{U}_s^E} \mathbb{E}\Big[\int_s^{\rho'} f(X_t, I_t, u_t)dt + v(\rho', X_{\rho'}, I_{\rho'}) - \sum_{n \in \mathbb{N}} c(X_{\tau_n'}, I_{(\tau_n')^-}, I_{\tau_n'}) 1_{\{s \leq \tau_n' < \rho'\}} \Big] \tag{3.5}$$

with the shorthand $X = X^{s,x,i;\alpha,u}$, $I = I^{s,x,i;\alpha,u}$, $\rho' = \rho(X_\cdot, I_{\cdot -})$, $\tau_n' = \tau_n(X_\cdot, I_{\cdot -})$, *and* $u_t = u(t, X_\cdot, I_{\cdot -})$.

Proof. We begin proving that $\mathcal{V}^- \neq \emptyset$. Let us consider the function $v\colon [0,T] \times \mathbb{R}^d \times \mathbb{I}_m \to \mathbb{R}$ given by

$$v(s,x,i) := -Ce^{\lambda(T-s)}(1 + |x|^q) \qquad \forall\, (s,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_m, \tag{3.6}$$

where $q = \max\{4, p\}$, with p as in assumption (H2)(iii), and C, λ are positive constants to be determined later. Set $h(x) = |x|^q$. Notice that $h \in C^2(\mathbb{R}^d)$ and there exists a positive constant M_h (depending only on q) such that $|D_x h(x)| \leq M_h |x|^{q-1}$ and $D_x^2 h(x) \leq M_h |x|^{q-2}$ $\forall\, x \in \mathbb{R}^d$.

From the polynomial growth condition of g in assumption (H2)(iii), we see that $v(T,x,i) \leq g(x,i)$ if we choose C large enough.

Now, we choose λ opportunely. Fix $s \in [0,T]$ and $\tau, \rho \in \mathcal{T}^s$ with $\tau \leq \rho \leq T$. We choose $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{\iota}_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s,\tau^+}$ as follows: for any $n \in \mathbb{N}$, $\tilde{\tau}_n \equiv T$ and $\tilde{\iota}_n \equiv \underline{i}$ for some fixed $\underline{i} \in \mathbb{I}_m$. Let $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}} \in \mathcal{A}_s$, $u \in \mathcal{U}_s$, and $(x,i) \in \mathbb{R}^d \times \mathbb{I}_m$. Set $X = X^{s,x,i;\alpha \otimes_\tau \tilde{\alpha},u}$, $I = I^{s,x,i;\alpha \otimes_\tau \tilde{\alpha},u}$, $\tau' = \tau(X,I)$, and $\rho' = \rho(X,I)$. Then, noting that $v(r, X_r, I_r)$ is constant with respect to I_r, and applying Itô's formula to $\int_{\tau'}^r f(X_t, I_t, u_t)dt + v(r, X_r, I_r)$ between τ' and ρ', we obtain

$$\begin{aligned}
&\int_{\tau'}^{\rho'} f(X_t, I_t, u_t)dt + v(\rho', X_{\rho'}, I_{\rho'}) \\
&\quad = \int_{\tau'}^{\rho'} f(X_t, I_t, u_t)dt + v(\tau', X_{\tau'}, I_{\tau'}) - C \int_{\tau'}^{\rho'} e^{\lambda(T-t)} D_x h(X_t).b(X_t, I_t, u_t)dt \\
&\qquad - C \int_{\tau'}^{\rho'} e^{\lambda(T-t)} (D_x h(X_t))^\intercal \sigma(X_t, I_t, u_t)dW_t + \lambda C \int_{\tau'}^{\rho'} e^{\lambda(T-t)}(1 + h(X_t))dt \\
&\qquad - \frac{1}{2} C \int_{\tau'}^{\rho'} e^{\lambda(T-t)} \mathrm{tr}\big[\sigma\sigma^\intercal(X_t, I_t, u_t) D_x^2 h(X_t)\big] dt.
\end{aligned} \tag{3.7}$$

Consider the $\mathbb{F}^s$-local martingale $M_r = \int_s^r 1_{[\tau',T]}(t) e^{\lambda(T-t)} (D_x h(X_t))^\intercal \sigma(X_t, I_t, u_t)dW_t$, $r \in [s,T]$. In order to prove that M is a true martingale, we show that $\mathbb{E}[\sup_{s \leq r \leq T} |M_r|] < \infty$. From Burkholder–Davis–Gundy's inequality, we see that it is enough to prove $\mathbb{E}[\sqrt{\langle M \rangle_T}] < \infty$, namely,

$$\mathbb{E}\left[\sqrt{\int_{\tau'}^T e^{2\lambda(T-t)} |D_x h(X_t)|^2 \|\sigma(X_t, I_t, u_t)\|^2 dt} \right] < \infty.$$

This latter inequality holds since $|D_x h(x)| \le M_h |x|^{q-1}$, $\|\sigma(x,i,u)\| \le M_1(1+|x|)$ (see Remark 2.2), and X satisfies estimate (2.2). From the martingale property of M and Doob's optional sampling theorem, we have in particular

$$\mathbb{E}\bigg[\int_{\tau'}^{\rho'} e^{\lambda(T-t)}(D_x h(X_t))^\intercal \sigma(X_t,I_t,u_t)dW_t\bigg|\mathcal{F}^s_{\tau'}\bigg] = \mathbb{E}\big[M_{\rho'}\big|\mathcal{F}^s_{\tau'}\big] = 0.$$

Therefore, taking the conditional expectation with respect to $\mathcal{F}^s_{\tau'}$ in (3.7), using the linear growth conditions of b, σ, f, and the estimates on $D_x h(x)$ and $D_x^2 h(x)$, we find

$$\begin{aligned}
&\mathbb{E}\bigg[\int_{\tau'}^{\rho'} f(X_t,I_t,u_t)dt + v(\rho',X_{\rho'},I_{\rho'})\bigg|\mathcal{F}^s_{\tau'}\bigg] \\
&\ge v(\tau',X_{\tau'},I_{\tau'}) + \mathbb{E}\bigg[-M_2\int_{\tau'}^{\rho'}(1+|X_t|^p)dt \\
&\qquad - CM_hM_1\int_{\tau'}^{\rho'} e^{\lambda(T-t)}|X_t|^{q-1}(1+|X_t|)dt \\
&\qquad + \lambda C\int_{\tau'}^{\rho'} e^{\lambda(T-t)}(1+|X_t|^q)dt \\
&\qquad - \frac{1}{2}CM_hM_1^2\int_{\tau'}^{\rho'} e^{\lambda(T-t)}|X_t|^{q-2}(1+|X_t|)^2dt\bigg|\mathcal{F}^s_{\tau'}\bigg].
\end{aligned}$$

We see that there exists a positive constant $\bar{C}$ (depending only on C, M_h, M_1, M_2) such that

$$\begin{aligned}
&\mathbb{E}\bigg[\int_{\tau'}^{\rho'} f(X_t,I_t,u_t)dt + v(\rho',X_{\rho'},I_{\rho'})\bigg|\mathcal{F}^s_{\tau'}\bigg] \\
&\ge v(\tau',X_{\tau'},I_{\tau'}) \\
&\quad + (\lambda C - \bar{C})\,\mathbb{E}\bigg[\int_{\tau'}^{\rho'} e^{\lambda(T-t)}(1+|X_t|^q)dt\bigg|\mathcal{F}^s_{\tau'}\bigg].
\end{aligned}$$

Now, we choose $\lambda \ge 0$ such that $\lambda C - \bar{C} \ge 0$. Then, we have

$$\mathbb{E}\bigg[\int_{\tau'}^{\rho'} f(X_t,I_t,u_t)dt + v(\rho',X_{\rho'},I_{\rho'})\bigg|\mathcal{F}^s_{\tau'}\bigg] \ge v(\tau',X_{\tau'},I_{\tau'}).$$

From the definition of $\tilde{\alpha}$, we see that $\sum_{n\in\mathbb{N}} c(X_{\tilde{\tau}'_n}, I_{(\tilde{\tau}'_n)^-}, I_{\tilde{\tau}'_n})1_{\{\tau'\le\tilde{\tau}'_n<\rho'\}} = 0$. Therefore, it follows that $v \in \mathcal{V}^-$. In a similar way we can prove that $-v \in \mathcal{V}^+$, so that $\mathcal{V}^+ \neq \emptyset$.

Concerning (ii), let $v \in \mathcal{V}^-$ and fix $s \in [0,T]$, $\tau, \rho \in \mathcal{T}^s$, with $s \equiv \tau \le \rho \le T$. From the second item of the definition of $\mathcal{V}^-$, there exists $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{\iota}_n)_{n\in\mathbb{N}} \in \mathcal{A}_{s^+}$ such that, for any $u \in \mathcal{U}_s$ and $(x,i) \in \mathbb{R}^d \times \mathbb{I}_m$ (we choose $\alpha = (\tau_n, \iota_n)_{n\in\mathbb{N}} \in \mathcal{A}_s$ with $\tau_n \equiv T$ and $\iota_n \equiv i$ for any $n \in \mathbb{N}$; with this choice we have $(X^{s,x,i;\alpha\otimes_s\tilde{\alpha},u}, I^{s,x,i;\alpha\otimes_s\tilde{\alpha},u}) = (X^{s,x,i;\tilde{\alpha},u}, I^{s,x,i;\tilde{\alpha},u})$; in particular, $I_s^{s,x,i;\alpha\otimes_s\tilde{\alpha},u} = i$), we find

$$\begin{aligned}
v(s,x,i) \le \mathbb{E}\bigg[&\int_s^{\rho'} f(X_t,I_t,u_t)dt + v(\rho',X_{\rho'},I_{\rho'}) \\
&- \sum_{n\in\mathbb{N}} c(X_{\tilde{\tau}'_n}, I_{(\tilde{\tau}'_n)^-}, I_{\tilde{\tau}'_n})1_{\{s\le\tilde{\tau}'_n<\rho'\}}\bigg|\mathcal{F}^s_s\bigg] \qquad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{3.8}$$

with the shorthand $X = X^{s,x,i;\tilde{\alpha},u}$, $I = I^{s,x,i;\tilde{\alpha},u}$, $\rho' = \rho(X_\cdot, I_{\cdot-})$, and $\tilde{\tau}'_n = \tilde{\tau}_n(X_\cdot, I_{\cdot-})$. Taking the expectation in (3.8) and the infimum with respect to $u \in \mathcal{U}_s$, we get

$$\begin{aligned} v(s,x,i) &\leq \inf_{u\in\mathcal{U}_s} \mathbb{E}\bigg[\int_s^{\rho'} f(X_t, I_t, u_t)dt + v(\rho', X_{\rho'}, I_{\rho'}) \\ &\qquad - \sum_{n\in\mathbb{N}} c(X_{\tilde{\tau}'_n}, I_{(\tilde{\tau}'_n)^-}, I_{\tilde{\tau}'_n}) 1_{\{s\leq\tilde{\tau}'_n<\rho'\}}\bigg] \\ &\leq \sup_{\alpha\in\mathcal{A}_{s^+}} \inf_{u\in\mathcal{U}_s} \mathbb{E}\bigg[\int_s^{\rho'} f(X_t, I_t, u_t)dt + v(\rho', X_{\rho'}, I_{\rho'}) \\ &\qquad - \sum_{n\in\mathbb{N}} c(X_{\tau'_n}, I_{(\tau'_n)^-}, I_{\tau'_n}) 1_{\{s\leq\tau'_n<\rho'\}}\bigg]. \end{aligned}$$

In a similar way we can prove statement (iii). ∎

As stated below, every $v \in \mathcal{V}^-$ is less than every $v \in \mathcal{V}^+$, while the value functions V and $\overline{V}$ are squeezed between them.

LEMMA 3.3. *Let assumptions* (H1) *and* (H2) *hold.*

(i) $\sup_{v\in\mathcal{V}^-} v =: v^- \leq V \leq \overline{V} \leq v^+ := \inf_{v\in\mathcal{V}^+} v$.

(ii) v^- *is lsc and satisfies the polynomial growth condition*

$$\sup_{(s,x,i)\in[0,T]\times\mathbb{R}^d\times\mathbb{I}_m} \frac{|v^-(s,x,i)|}{1+|x|^q} < \infty \tag{3.9}$$

for some $q \geq 1$.

(iii) v^+ *is usc and satisfies the polynomial growth condition*

$$\sup_{(s,x,i)\in[0,T]\times\mathbb{R}^d\times\mathbb{I}_m} \frac{|v^+(s,x,i)|}{1+|x|^q} < \infty$$

for some $q \geq 1$.

Proof. Concerning (i), to obtain the inequality $v \leq \forall\, v \in \mathcal{V}^-$ (resp., $\overline{V} \leq v \,\forall\, v \in \mathcal{V}^+$) we take $\rho \equiv T$ in the subdynamic programming principle (3.4) (resp., superdynamic programming principle (3.5)) and we use the inequality $v(T,x,i) \leq g(x,i)$ (resp., $v(T,x,i) \geq g(x,i)) \,\forall\, (x,i) \in \mathbb{R}^d \times \mathbb{I}_m$. Regarding (ii), we notice that v^- is lsc since it is the supremum of a family of lsc (actually, continuous) functions. Moreover, let $\underline{v} \in \mathcal{V}^-$ and $\bar{v} \in \mathcal{V}^+$. From (i) it follows that $\underline{v} \leq v^- \leq \bar{v}$, and from the polynomial growth condition of $\underline{v}, \bar{v}$ we see that v^- satisfies the polynomial growth condition (3.9). Statement (iii) can be proved in a similar way. ∎

We can now state our main result.

THEOREM 3.1 (stochastic Perron's method). *Let assumptions* (H1) *and* (H2) *hold. Then,* v^- *is a viscosity supersolution to* (2.10) *and* v^+ *is a viscosity subsolution to* (2.10).

In order to prove Theorem 3.1, we need the following two lemmas. In particular, Lemma 3.4 states that $\mathcal{V}^-$ (resp., $\mathcal{V}^+$) is stable by supremum (resp., infimum), which gives the existence of a monotone approximating sequence for v^- (resp., v^+) in Lemma 3.5.

LEMMA 3.4. *Let assumptions* (H1) *and* (H2) *hold.*
(i) *If* $v^1, v^2 \in \mathcal{V}^-$ *then* $v := v^1 \vee v^2 \in \mathcal{V}^-$.
(ii) *If* $v^1, v^2 \in \mathcal{V}^+$ *then* $v := v^1 \wedge v^2 \in \mathcal{V}^+$.

Proof. Let us prove (i). As the first item in Definition 3.4 clearly holds, we prove that v satisfies the second item. To this end, fix $s \in [0,T]$ and $\tau, \rho \in \mathcal{T}^s$ with $\tau \leq \rho \leq T$. Let $\tilde{\alpha}^1 = (\tilde{\tau}_n^1, \tilde{\iota}_n^1)_{n\in\mathbb{N}}, \tilde{\alpha}^2 = (\tilde{\tau}_n^2, \tilde{\iota}_n^2)_{n\in\mathbb{N}} \in \mathcal{A}_{s,\tau^+}$ be the two feedback switching controls, starting strictly later than τ, corresponding to v^1 and v^2. Now, consider the set $B := \{(v^1 - v^2)(\tau(y), y(\tau(y)^+)) \geq 0\} \in \mathcal{B}^s_{\tau^+}$ and define the double sequence $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{\iota}_n)_{n\in\mathbb{N}}$ as follows:

$$\big(\tilde{\tau}_n(y), \tilde{\iota}_n(y)\big) := \big(\tilde{\tau}_n^1(y), \tilde{\iota}_n^1(y)\big)1_{\{y\in B\}} + \big(\tilde{\tau}_n^2(y), \tilde{\iota}_n^2(y)\big)1_{\{y\in B^c\}}$$

for any $y \in C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m)$, $n \in \mathbb{N}$. From Lemma 3.1 it follows that $\tilde{\alpha} \in \mathcal{A}_{s,\tau^+}$. Now, we prove that $\tilde{\alpha}$ satisfies the condition in the second item of Definition 3.4. Take $\alpha = (\tau_n, \iota_n)_{n\in\mathbb{N}} \in \mathcal{A}_s$, $u \in \mathcal{U}_s$, and $(x,i) \in \mathbb{R}^d \times \mathbb{I}_m$. We adopt the shorthand

$$\begin{aligned} X &= X^{s,x,i;\alpha\otimes_\tau\tilde{\alpha},u}, & X^1 &= X^{s,x,i;\alpha\otimes_\tau\tilde{\alpha}^1,u}, & X^2 &= X^{s,x,i;\alpha\otimes_\tau\tilde{\alpha}^2,u}, \\ I &= I^{s,x,i;\alpha\otimes_\tau\tilde{\alpha},u}, & I^1 &= I^{s,x,i;\alpha\otimes_\tau\tilde{\alpha}^1,u}, & I^2 &= I^{s,x,i;\alpha\otimes_\tau\tilde{\alpha}^2,u}. \end{aligned}$$

We also denote $\tau' = \tau(X_\cdot, I_{\cdot^-})$, $\rho' = \rho(X_\cdot, I_{\cdot^-})$, $\rho^{1,'} = \rho(X^1_\cdot, I^1_{\cdot^-})$, $\rho^{2,'} = \rho(X^2_\cdot, I^2_{\cdot^-})$, $\tilde{\tau}_n' = \tilde{\tau}_n(X_\cdot, I_{\cdot^-})$, $\tilde{\tau}_n^{1,'} = \tilde{\tau}_n^1(X^1_\cdot, I^1_{\cdot^-})$, and $\tilde{\tau}_n^{2,'} = \tilde{\tau}_n^2(X^2_\cdot, I^2_{\cdot^-})$. Notice that $(X_t, I_{t^-}) = (X^1_t, I^1_{t^-}) = (X^2_t, I^2_{t^-})$, $t \in [s,\tau']$. Therefore, from Lemma 2.1 we see that $\tau' = \tau(X^1_\cdot, I^1_{\cdot^-}) = \tau(X^2_\cdot, I^2_{\cdot^-})$. Moreover, for any $t \in [\tau', T]$,

$$(X_t, I_t) = (X^1_t, I^1_t)1_{\{(v^1-v^2)(\tau',X_{\tau'},I_{\tau'})\geq 0\}} + (X^2_t, I^2_t)1_{\{(v^1-v^2)(\tau',X_{\tau'},I_{\tau'})<0\}}.$$

As a consequence,

$$\begin{aligned} \rho' &= \rho^{1,'}1_{\{(v^1-v^2)(\tau',X_{\tau'},I_{\tau'})\geq 0\}} + \rho^{2,'}1_{\{(v^1-v^2)(\tau',X_{\tau'},I_{\tau'})<0\}}, \\ \tilde{\tau}_n' &= \tilde{\tau}_n^{1,'}1_{\{(v^1-v^2)(\tau',X_{\tau'},I_{\tau'})\geq 0\}} + \tilde{\tau}_n^{2,'}1_{\{(v^1-v^2)(\tau',X_{\tau'},I_{\tau'})<0\}}. \end{aligned}$$

Therefore, from the previous identities and the properties of v^1, we obtain

$$\begin{aligned} &v^1(\tau', X_{\tau'}, I_{\tau'})1_{\{(v^1-v^2)(\tau',X_{\tau'},I_{\tau'})\geq 0\}} \\ &\quad = v^1(\tau', X^1_{\tau'}, I^1_{\tau'})1_{\{(v^1-v^2)(\tau',X_{\tau'},I_{\tau'})\geq 0\}} \\ &\quad \leq \mathbb{E}\bigg[\bigg(\int_{\tau'}^{\rho^{1,'}} f(X^1_t, I^1_t, u_t)dt + v^1(\rho^{1,'}, X^1_{\rho^{1,'}}, I^1_{\rho^{1,'}}) \\ &\qquad\qquad - \sum_{n\in\mathbb{N}} c(X^1_{\tilde{\tau}_n^{1,'}}, I^1_{(\tilde{\tau}_n^{1,'})^-}, I^1_{\tilde{\tau}_n^{1,'}})1_{\{\tau'\leq\tilde{\tau}_n^{1,'}<\rho^{1,'}\}}\bigg)1_{\{(v^1-v^2)(\tau',X_{\tau'},I_{\tau'})\geq 0\}}\bigg|\mathcal{F}^s_{\tau'}\bigg] \\ &\quad \leq \mathbb{E}\bigg[\bigg(\int_{\tau'}^{\rho'} f(X_t, I_t, u_t)dt + v(\rho', X_{\rho'}, I_{\rho'}) \\ &\qquad\qquad - \sum_{n\in\mathbb{N}} c(X_{\tilde{\tau}_n'}, I_{(\tilde{\tau}_n')^-}, I_{\tilde{\tau}_n'})1_{\{\tau'\leq\tilde{\tau}_n'<\rho'\}}\bigg)1_{\{(v^1-v^2)(\tau',X_{\tau'},I_{\tau'})\geq 0\}}\bigg|\mathcal{F}^s_{\tau'}\bigg]. \end{aligned}$$

Concerning v^2, proceeding similarly we get

$$\begin{aligned}&v^2(\tau', X_{\tau'}, I_{\tau'})1_{\{(v^1-v^2)(\tau',X_{\tau'},I_{\tau'})<0\}}\\&\quad\le \mathbb{E}\bigg[\bigg(\int_{\tau'}^{\rho'} f(X_t, I_t, u_t)dt + v(\rho', X_{\rho'}, I_{\rho'})\\&\qquad\quad - \sum_{n\in\mathbb{N}} c(X_{\tilde{\tau}'_n}, I_{(\tilde{\tau}'_n)^-}, I_{\tilde{\tau}'_n})1_{\{\tau'\le\tilde{\tau}'_n<\rho'\}}\bigg)1_{\{(v^1-v^2)(\tau',X_{\tau'},I_{\tau'})<0\}}\bigg|\mathcal{F}^s_{\tau'}\bigg].\end{aligned}$$

In conclusion, we find

$$\begin{aligned}&v(\tau', X_{\tau'}, I_{\tau'})\\&\quad= v^1(\tau', X_{\tau'}, I_{\tau'})1_{\{(v^1-v^2)(\tau',X_{\tau'},I_{\tau'})\ge0\}} + v^2(\tau', X_{\tau'}, I_{\tau'})1_{\{(v^1-v^2)(\tau',X_{\tau'},I_{\tau'})<0\}}\\&\quad\le \mathbb{E}\bigg[\int_{\tau'}^{\rho'} f(X_t, I_t, u_t)dt + v(\rho', X_{\rho'}, I_{\rho'}) - \sum_{n\in\mathbb{N}} c(X_{\tilde{\tau}'_n}, I_{(\tilde{\tau}'_n)^-}, I_{\tilde{\tau}'_n})1_{\{\tau'\le\tilde{\tau}'_n<\rho'\}}\bigg|\mathcal{F}^s_{\tau'}\bigg],\end{aligned}$$

which shows that $v \in \mathcal{V}^-$.

A similar argument allows us to prove the stability with respect to infimum of $\mathcal{V}^+$ in (ii). In particular, fix $s \in [0,T]$, $\tau \in \mathcal{T}^s$, and $\alpha = (\tau_n, \iota_n)_{n\in\mathbb{N}} \in \mathcal{A}_s$. Let $\tilde{u}^1, \tilde{u}^2 \in \mathcal{U}^E_{s,\tau}$ be the two elementary feedback strategies, for the nature, starting at τ and corresponding to v^1 and v^2. Let $B := \{(v^1 - v^2)(\tau(y), y(\tau(y)^+)) \le 0\} \in \mathcal{B}^s_{\tau+}$. Then, from Lemma 3.1 we see that the map

$$\tilde{u}(t, y) := \tilde{u}^1(t, y)1_{\{y\in B\}} + \tilde{u}^2(t, y)1_{\{y\in B^c\}}$$

is an elementary feedback strategy starting at τ, which allows us to prove that $v \in \mathcal{V}^+$. ∎

LEMMA 3.5. *Let assumptions* (H1) *and* (H2) *hold.*

(i) *There exists a nondecreasing sequence* $(v_n)_{n\in\mathbb{N}} \subset \mathcal{V}^-$ *such that* $v_n \nearrow v^-$.

(ii) *There exists a nonincreasing sequence* $(v_n)_{n\in\mathbb{N}} \subset \mathcal{V}^+$ *such that* $v_n \searrow v^+$.

Proof. From [4, Proposition 4.1] we can find a sequence $(\tilde{v}_n)_{n\in\mathbb{N}} \subset \mathcal{V}^-$ satisfying $v^- = \sup_{n\in\mathbb{N}} \tilde{v}_n$. Set $v_n := \tilde{v}_0 \vee \cdots \vee \tilde{v}_n$, $n \in \mathbb{N}$. Then $v_n \nearrow v^-$ as $n \to \infty$, and from Lemma 3.4 we see $(v_n)_{n\in\mathbb{N}} \subset \mathcal{V}^-$. In a similar way we can prove statement (ii). ∎

We are now in a position to prove Theorem 3.1. First, we just state here, in the spirit of Lemma 2.4 in [6], the following technical result, which will be used several times in the proof of Theorem 3.1.

LEMMA 3.6. *Let* $\mathcal{C} \subset [0,T]\times\mathbb{R}^d$ *be a compact set and consider a continuous function* $F : \mathbb{R}^m\times\mathcal{C} \to \mathbb{R}$, *which is nondecreasing in each of its first* m *components. If there exists* $\delta > 0$ *such that* $\inf_{(t,x)\in\mathcal{C}} F(v^-(t, x, \cdot), t, x) > \delta$ (*resp.,* $\sup_{(t,x)\in\mathcal{C}} F(v^+(t, x, \cdot), t, x) < -\delta$), *then*

$$\inf_{(t,x)\in\mathcal{C}} F(v(t, x, \cdot), t, x) > \delta$$

$$\left(\textit{resp.} \sup_{(t,x)\in\mathcal{C}} F(v(t, x, \cdot), t, x) < -\delta\right)$$

for some $v \in \mathcal{V}^-$ (*resp.,* $v \in \mathcal{V}^+$).

Proof. Notice that, from the strict inequality $\inf_{(t,x)\in\mathcal{C}} F(v^-(t, x, \cdot), t, x) > \delta$ we can find $\varepsilon > 0$ such that $F(v^-(t, x, \cdot), t, x) > \delta + \varepsilon$, for any $(t, x) \in \mathcal{C}$. Recall from

Lemma 3.5 that there exists a nondecreasing sequence $(v_n)_{n\in\mathbb{N}} \subset \mathcal{V}^-$ such that $v_n \nearrow v^-$. Let

$$A_n := \big\{(t,x) \in \mathcal{C}\colon F(v_n(t,x,\cdot),t,x) \le \delta + \varepsilon/2\big\}.$$

Notice that A_n is closed, $A_{n+1} \subset A_n$, and $\cap_{n=0}^\infty A_n = \emptyset$. Since $A_n \subset \mathcal{C}$, using the compactness we see that there exists an n_0 such that $A_{n_0} = \emptyset$, namely, $F(v_{n_0}(t,x,\cdot),t,x) > \delta + \varepsilon$ for any $(t,x) \in \mathcal{C}$. In particular, $\inf_{(t,x)\in\mathcal{C}} F(v_{n_0}(t,x,\cdot),t,x) > \delta$. We then take $v := v_{n_0}$. In a similar way we can prove the statement for v^+. ∎

Proof of Theorem 3.1. *Step* I. *v^- is a viscosity supersolution to the HJB equation* (2.10).

Step I(i). *Interior viscosity supersolution property.* Let $(t_0,x_0) \in [0,T) \times \mathbb{R}^d$, $i \in \mathbb{I}_m$, and consider a test function $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^d)$ such that $v^-(\cdot,\cdot,i) - \varphi(\cdot,\cdot)$ attains a strict global minimum equal to zero at (t_0,x_0). Reasoning by contradiction, we assume that

$$\min\bigg\{ -\frac{\partial\varphi}{\partial t}(t_0,x_0) - \inf_{u\in U}\big[\mathcal{L}^{i,u}\varphi(t_0,x_0) + f(x_0,i,u)\big], \\ v^-(t_0,x_0,i) - \max_{j\neq i}\big[v^-(t_0,x_0,j) - c(x_0,i,j)\big]\bigg\} < 0.$$

We distinguish two cases.

Case a. $-\frac{\partial\varphi}{\partial t}(t_0,x_0) - \inf_{u\in U}[\mathcal{L}^{i,u}\varphi(t_0,x_0) + f(x_0,i,u)] < 0$. Then, there exists $\varepsilon \in (0,T-t_0)$ such that

$$-\frac{\partial\varphi}{\partial t}(t_0,x_0) - \inf_{u\in U}\big[\mathcal{L}^{i,u}\varphi(t_0,x_0) + f(x_0,i,u)\big] < -\varepsilon.$$

From the continuity of b,σ,f, together with the compactness of U, we see that we can choose a smaller $\varepsilon \in (0,T-t_0)$ such that

$$-\frac{\partial\varphi}{\partial t}(t,x) - \inf_{u\in U}\big[\mathcal{L}^{i,u}\varphi(t,x) + f(x,i,u)\big] < -\varepsilon \qquad \forall\,(t,x) \in B(t_0,x_0,\varepsilon),$$

where

$$B(t_0,x_0,\varepsilon) = \big\{(t,x) \in [0,T]\times\mathbb{R}^d\colon \max\{|t-t_0|,|x-x_0|\} < \varepsilon\big\}. \tag{3.10}$$

Since $v^-(\cdot,\cdot,i) - \varphi(\cdot,\cdot)$ is lsc and strictly positive on the compact set $\mathcal{C} := \overline{B(t_0,x_0,\varepsilon)}\setminus B(t_0,x_0,\varepsilon/2)$, there exists $\delta > 0$ such that $\inf_{(t,x)\in\mathcal{C}}(v^-(t,x,i) - \varphi(t,x)) > \delta$. Denoting $F(p,t,x) := p - \varphi(t,x)$, it follows from Lemma 3.6 that there exists $v \in \mathcal{V}^-$ such that $\varphi(t,x) + \delta < v(t,x,i)$ on $\mathcal{C}$. Now, define

$$v^\delta(t,x,i) = \begin{cases} (\varphi(t,x)+\delta) \vee v(t,x,i) & \text{on } \overline{B(t_0,x_0,\varepsilon)}, \\ v(t,x,i) & \text{outside } \overline{B(t_0,x_0,\varepsilon)}. \end{cases}$$

Moreover, $v^\delta(t,x,j) = v(t,x,j)$ for any $(t,x,j) \in [0,T]\times\mathbb{R}^d\times\mathbb{I}_m$ with $j \neq i$. Our aim is to prove that $v^\delta \in \mathcal{V}^-$, which would give a contradiction, since $v^\delta(t_0,x_0,i) > v^-(t_0,x_0,i)$. Clearly, v^δ satisfies the first item in Definition 3.4, therefore, it remains to prove the second item. To this end, fix $s \in [0,T]$ and $\tau,\rho \in \mathcal{T}^s$ with $\tau \le \rho \le T$. Let $\tilde\alpha^0 = (\tilde\tau_n^0,\tilde\iota_n^0)_{n\in\mathbb{N}}$ be given by

$$(\tilde\tau_n^0,\tilde\iota_n^0) = (T,i) \quad \forall\, n \in \mathbb{N}.$$

Notice that $\tilde\alpha^0 \in \mathcal{A}_{s,\tau^+}$. Introduce now the stopping rule

$$\rho_1\colon C([s,T];\mathbb{R}^d)\times\mathscr{L}([s,T];\mathbb{I}_m)\to[s,T],\quad \tau\le\rho_1\le T,$$

$$\rho_1(y)=\inf\big\{t\in[\tau(y),T]\colon (t,y^X(t))\notin B(t_0,x_0,\varepsilon/2)\big\}\wedge T. \tag{3.11}$$

We denote by $\tilde\alpha^1=(\tilde\tau_n^1,\tilde\iota_n^1)_{n\in\mathbb{N}}\in\mathcal{A}_{s,(\rho_1\wedge\rho)^+}$ the feedback switching strategy in Definition 3.4, corresponding to $s,\rho_1\wedge\rho,\rho$, for v. Then, we define $\tilde\alpha^2=\tilde\alpha^0\otimes_{\rho_1\wedge\rho}\tilde\alpha^1$, which belongs to $\mathcal{A}_{s,\tau^+}$ thanks to Proposition 3.2. Moreover, let $\tilde\alpha^3=(\tilde\tau_n^3,\tilde\iota_n^3)_{n\in\mathbb{N}}\in\mathcal{A}_{s,\tau^+}$ be the feedback switching strategy corresponding to s,τ,ρ for v. Then, we define $\tilde\alpha=(\tilde\tau_n,\tilde\iota_n)_{n\in\mathbb{N}}$ by (for any $y\in C([s,T];\mathbb{R}^d)\times\mathscr{L}([s,T];\mathbb{I}_m)$ we write $y=(y^X,y^I)$ with $y^X\in C([s,T];\mathbb{R}^d)$ and $y^I\in\mathscr{L}([s,T];\mathbb{I}_m)$)

$$\begin{aligned}
&(\tilde\tau_n(y),\tilde\iota_n(y))\\
&\quad=(\tilde\tau_n^2(y),\tilde\iota_n^2(y))1_{\{(\tau(y),y^X(\tau(y)))\in B(t_0,x_0,\varepsilon),\,(v-\varphi)(\tau(y),y(\tau(y)^+))<\delta,\,y^I(\tau(y)^+)=i\}}\\
&\qquad+(\tilde\tau_n^3(y),\tilde\iota_n^3(y))1_{\{(\tau(y),y^X(\tau(y)))\in B(t_0,x_0,\varepsilon),\,(v-\varphi)(\tau(y),y(\tau(y)^+))<\delta,\,y^I(\tau(y)^+)=i\}^c}.
\end{aligned}$$

From Lemma 3.1 it follows that $\tilde\alpha\in\mathcal{A}_{s,\tau^+}$. Moreover, the feedback switching strategy $\tilde\alpha$ satisfies the condition in the second item of Definition 3.4 for v^δ. To see this, fix $\alpha=(\tau_n,\iota_n)_{n\in\mathbb{N}}\in\mathcal{A}_s$, $u\in\mathcal{U}_s$, and $(x,i)\in\mathbb{R}^d\times\mathbb{I}_m$. We adopt the shorthand

$$\begin{aligned}
(X,I)&=(X^{s,x,i;\alpha\otimes_\tau\tilde\alpha,u},I^{s,x,i;\alpha\otimes_\tau\tilde\alpha,u}),\\
(X^1,I^1)&=(X^{s,x,i;\alpha\otimes_\tau\tilde\alpha^2,u},I^{s,x,i;\alpha\otimes_\tau\tilde\alpha^2,u}),\\
(X^2,I^2)&=(X^{s,x,i;\alpha\otimes_\tau\tilde\alpha^3,u},I^{s,x,i;\alpha\otimes_\tau\tilde\alpha^3,u}).
\end{aligned}$$

We also denote $\tau'=\tau(X_\cdot,I_{\cdot-})$, $\rho_1'=\rho_1(X_\cdot,I_{\cdot-})$, and $\rho'=\rho(X_\cdot,I_{\cdot-})$. Notice that

$$\begin{aligned}
(X,I)=&(X^1,I^1)1_{\{(\tau',X_{\tau'})\in B(t_0,x_0,\varepsilon),\,(v-\varphi)(\tau',X_{\tau'},I_{\tau'})<\delta,\,I_{\tau'}=i\}}\\
&+(X^2,I^2)1_{\{(\tau',X_{\tau'})\in B(t_0,x_0,\varepsilon),\,(v-\varphi)(\tau',X_{\tau'},I_{\tau'})<\delta,\,I_{\tau'}=i\}^c}.
\end{aligned}$$

In particular, it is useful to decompose $v^\delta(\tau',X_{\tau'},I_{\tau'})$ as follows:

$$\begin{aligned}
v^\delta(\tau',X_{\tau'},I_{\tau'})=&\big(\varphi(\tau',X^1_{\tau'})+\delta\big)1_{\{(\tau',X_{\tau'})\in B(t_0,x_0,\varepsilon),\,(v-\varphi)(\tau',X_{\tau'},I_{\tau'})<\delta,\,I_{\tau'}=i\}}\\
&+v(\tau',X^2_{\tau'},I^2_{\tau'})1_{\{(\tau',X_{\tau'})\in B(t_0,x_0,\varepsilon),\,(v-\varphi)(\tau',X_{\tau'},I_{\tau'})<\delta,\,I_{\tau'}=i\}^c}.
\end{aligned} \tag{3.12}$$

We now consider the two terms on the right-hand side of (3.12) individually. Regarding the first term, we apply Itô's formula to φ between τ' and $\rho_1'\wedge\rho'$, observing that $I_t^1=i$ for any $t\in[\tau',\rho_1'\wedge\rho']$; afterwards, we use the property in the second item of Definition 3.4 for v with corresponding feedback switching strategy $\tilde\alpha^1$. Finally, concerning the other term in (3.12), the result follows from the properties of v and the definition of $\tilde\alpha^3$.

Case b. $v^-(t_0,x_0,i)<\max_{j\neq i}[v^-(t_0,x_0,j)-c(x_0,i,j)]$ and $-\frac{\partial\varphi}{\partial t}(t_0,x_0)-\inf_{u\in U}[\mathcal{L}^{i,u}\varphi(t_0,x_0)+f(x_0,i,u)]\ge 0$. Since v^- is lsc and c is continuous, there exists $\varepsilon\in(0,T-t_0)$ such that

$$v^-(t_0,x_0,i)+\varepsilon<\inf_{(t,x)\in\overline{B(t_0,x_0,\varepsilon)}}\max_{j\neq i}[v^-(t,x,j)-c(x,i,j)].$$

Set $F(p,t,x)=\max_{j\neq i}[p_j-c(x,i,j)]$, for any $(p,t,x)\in\mathbb{R}^m\times\overline{B(t_0,x_0,\varepsilon)}$. Then, from Lemma 3.6 it follows that there exists $v\in\mathcal{V}^-$ such that $F(v(t,x,\cdot),t,x)>$

$v^-(t_0,x_0,i)+\varepsilon \geq v(t_0,x_0,i)+\varepsilon$ for any $(t,x) \in \overline{B(t_0,x_0,\varepsilon)}$. We also suppose that the function v given by Lemma 3.6 satisfies $v^-(t_0,x_0,i)-v(t_0,x_0,i) < \varepsilon/2$. Since v is continuous on $\overline{B(t_0,x_0,\varepsilon)}$, we can find $\delta > 0$ such that

$$\sup_{(t',x')\in\overline{B(t_0,x_0,\delta)}} v(t',x',i)+\varepsilon < \inf_{(t,x)\in\overline{B(t_0,x_0,\varepsilon)}} \max_{j\neq i}\big[v(t,x,j)-c(x,i,j)\big]. \tag{3.13}$$

Let $M > 0$ be an upper bound for the continuous function $|f(x,i,u)|$ on the compact set $\overline{B(t_0,x_0,\varepsilon)} \times \mathbb{I}_m \times U$. We suppose that $\delta \leq \varepsilon/(4M)$. Now, define (we adopt the notation $\|(t,x)\| = \max\{|t|,|x|\}$)

$$v^\delta(t,x,i) = \begin{cases} v(t,x,i)+\frac{\varepsilon}{2\delta}(\delta-\|(t-t_0,x-x_0)\|) & \text{on } \overline{B(t_0,x_0,\delta)}, \\ v(t,x,i) & \text{outside } \overline{B(t_0,x_0,\delta)}. \end{cases}$$

Moreover, $v^\delta(t,x,j) = v(t,x,j)$ for any $(t,x,j) \in [0,T]\times\mathbb{R}^d\times\mathbb{I}_m$ with $j \neq i$. As $v^\delta(t_0,x_0,i) > v^-(t_0,x_0,i)$, we get a contradiction if we prove that $v^\delta \in \mathcal{V}^-$. In order to do so, fix $s \in [0,T]$ and $\tau,\rho \in \mathcal{T}^s$ with $\tau \leq \rho \leq T$. We have to determine $\tilde{\alpha} = (\tilde{\tau}_n,\tilde{\iota}_n)_{n\in\mathbb{N}} \in \mathcal{A}_{s,\tau^+}$ which works for v^δ. To this end, define $\rho_1 \in \mathcal{T}^s$ as follows:

$$\rho_1(y) = \inf\big\{t \in [\tau(y),T]\colon (t,y^X(t)) \notin B(t_0,x_0,\delta)\big\} \wedge T.$$

Let $\tilde{\alpha}^0 = (\tilde{\tau}_n^0,\tilde{\iota}_n^0)_{n\in\mathbb{N}}$ be given by $(\tilde{\tau}_n^0,\tilde{\iota}_n^0) = (T,i)$ for any $n \geq 1$, and

$$\begin{aligned}\tilde{\tau}_0^0(y) &= \rho_1(y)1_{\{(\tau(y),y^X(\tau(y)))\in B(t_0,x_0,\delta)\}} + T\,1_{\{(\tau(y),y^X(\tau(y)))\notin B(t_0,x_0,\delta)\}},\\ \tilde{\iota}_0^0(y) &= \min\big\{j\neq i\colon v(\tilde{\tau}_0^0(y),y^X(\tilde{\tau}_0^0(y)),j)-c(y^X(\tilde{\tau}_0^0(y)),i,j) = m(y)\big\},\end{aligned}$$

where $m\colon C([s,T];\mathbb{R}^d)\times\mathscr{L}([s,T];\mathbb{I}_m) \to \mathbb{R}$ is defined as

$$m(y) = \max_{j\neq i}\big[v(\tilde{\tau}_0^0(y),y^X(\tilde{\tau}_0^0(y)),j)-c(y^X(\tilde{\tau}_0^0(y)),i,j)\big].$$

Notice that m is $\mathcal{B}^s_{\tilde{\tau}_0^0}$-measurable, so that $\tilde{\iota}_0^0$ is $\mathcal{B}^s_{\tilde{\tau}_0^0}$-measurable. Moreover, $\tau < \tilde{\tau}_0^0$ on the set $\{\tau < T\}$. In particular, $\tilde{\alpha}^0 \in \mathcal{A}^s_{\tau^+}$. Now, consider the feedback switching strategy $\tilde{\alpha}^1 = (\tilde{\tau}_n^1,\tilde{\iota}_n^1)_{n\in\mathbb{N}} \in \mathcal{A}_{s,(\tilde{\tau}_0^0\wedge\rho)^+}$ in Definition 3.4, corresponding to $s,\tilde{\tau}_0^0\wedge\rho,\rho$, for v. We define $\tilde{\alpha}^2 = \tilde{\alpha}^0 \otimes_{\tilde{\tau}_0^0\wedge\rho} \tilde{\alpha}^1$, which belongs to $\mathcal{A}_{s,\tau^+}$ thanks to Proposition 3.2. Consider also the feedback switching strategy $\tilde{\alpha}^3 = (\tilde{\tau}_n^3,\tilde{\iota}_n^3)_{n\in\mathbb{N}} \in \mathcal{A}_{s,\tau^+}$, corresponding to s,τ,ρ, for v. Then, let $\tilde{\alpha} = (\tilde{\tau}_n,\tilde{\iota}_n)_{n\in\mathbb{N}}$ be given by

$$\begin{aligned}(\tilde{\tau}_n(y),\tilde{\iota}_n(y)) &= (\tilde{\tau}_n^2(y),\tilde{\iota}_n^2(y))1_{\{(\tau(y),y^X(\tau(y)))\in B(t_0,x_0,\delta),\, y^I(\tau(y)^+)=i\}}\\ &\quad + (\tilde{\tau}_n^3(y),\tilde{\iota}_n^3(y))1_{\{(\tau(y),y^X(\tau(y)))\in B(t_0,x_0,\delta),\, y^I(\tau(y)^+)=i\}^c}.\end{aligned}$$

From Lemma 3.1 it follows that $\tilde{\alpha} \in \mathcal{A}_{s,\tau^+}$. Moreover, $\tilde{\alpha}$ is the feedback switching strategy which satisfies the condition in the second item of Definition 3.4 for v^δ. To see this, fix $\alpha = (\tau_n,\iota_n)_{n\in\mathbb{N}} \in \mathcal{A}_s$, $u \in \mathcal{U}_s$, and $(x,i) \in \mathbb{R}^d\times\mathbb{I}_m$. We adopt the shorthand introduced in Case *a*. Consider the event $A := \{(\tau',X_{\tau'}) \in B(t_0,x_0,\delta),\, I_{\tau'} = i\}$. On A^c the result follows from the properties of v and the definition of $\tilde{\alpha}^3$. On the other hand, on A we have

$$\begin{aligned}v^\delta(\tau',X_{\tau'},I_{\tau'})1_A &= v^\delta(\tau',X^1_{\tau'},i)1_A\\ &= \Big[v(\tau',X^1_{\tau'},i)+\frac{\varepsilon}{2\delta}\big(\delta-\|(\tau'-t_0,X^1_{\tau'}-x_0)\|\big)\Big]1_A\\ &\leq \Big[v(\tau',X^1_{\tau'},i)+\frac{\varepsilon}{2}\Big]1_A.\end{aligned}$$

Using (3.13) and taking the conditional expectation with respect to $\mathcal{F}^s_{\tau'}$, we obtain (denoting $\tilde\tau_0^{0,'} = \tilde\tau_0^0(X_\cdot, I_{\cdot-})$)

$$v^\delta(\tau', X_{\tau'}, I_{\tau'})1_A \le \mathbb{E}\Big[v\big(\tilde\tau_0^{0,'}\wedge\rho', X^1_{\tilde\tau_0^{0,'}\wedge\rho'}, I^1_{\tilde\tau_0^{0,'}\wedge\rho'}\big) - c\big(X^1_{\tilde\tau_0^{0,'}\wedge\rho'}, i, I^1_{\tilde\tau_0^{0,'}\wedge\rho'}\big) - \frac{\varepsilon}{2}\Big|\mathcal{F}^s_{\tau'}\Big]1_A.$$

Observe that $\tilde\tau_0^{0,'} \le \rho'$ on A. Therefore, the above inequality can be written as

$$v^\delta(\tau', X_{\tau'}, I_{\tau'})1_A \le \mathbb{E}\Big[v\big(\tilde\tau_0^{0,'}, X^1_{\tilde\tau_0^{0,'}}, I^1_{\tilde\tau_0^{0,'}}\big) - c\big(X^1_{\tilde\tau_0^{0,'}}, i, I^1_{\tilde\tau_0^{0,'}}\big) - \frac{\varepsilon}{2}\Big|\mathcal{F}^s_{\tau'}\Big]1_A.$$

Adding and subtracting $\int_{\tau'}^{\tilde\tau_0^{0,'}} f(X^1_t, I^1_t, u_t)dt$, noting that $(\tilde\tau_0^{0,'} - \tau')1_A \le 2\delta$ and $2\delta M - \varepsilon/2 \le 0$, we find

$$\begin{aligned} &v^\delta(\tau', X_{\tau'}, I_{\tau'})1_A \\ &\quad\le \mathbb{E}\bigg[\int_{\tau'}^{\tilde\tau_0^{0,'}} f(X^1_t, I^1_t, u_t)dt + v\big(\tilde\tau_0^{0,'}, X^1_{\tilde\tau_0^{0,'}}, I^1_{\tilde\tau_0^{0,'}}\big) - c\big(X^1_{\tilde\tau_0^{0,'}}, i, I^1_{\tilde\tau_0^{0,'}}\big)\bigg|\mathcal{F}^s_{\tau'}\bigg]1_A. \end{aligned}$$

Finally, using that v satisfies the second item of Definition 3.4, with corresponding feedback switching strategy $\tilde\alpha^1$, and from the inequality $v \le v^\delta$, we deduce that $v^\delta \in \mathcal{V}^-$.

Step I(ii). *Terminal condition.* Reasoning by contradiction, we assume that there exist $x_0 \in \mathbb{R}^d$ and $i \in \mathbb{I}_m$ such that

$$v^-(T, x_0, i) < g(x_0, i).$$

Since g is continuous, there exists $\varepsilon > 0$ such that $v^-(T, x_0, i) \le g(x, i) - \varepsilon$ whenever $|x - x_0| \le \varepsilon$. Consider the compact set

$$\mathcal{C} := \big(\overline{B(T, x_0, \varepsilon)}\backslash B(T, x_0, \varepsilon/2)\big) \cap \big([0,T]\times\mathbb{R}^d\big),$$

where $B(T, x_0, \varepsilon) = \{(t,x) \in [0,T]\times\mathbb{R}^d\colon \max\{|t - t_0|, |x - x_0|\} < \varepsilon\}$. Since v^- is lsc, it is bounded from below on $\mathcal{C}$. Therefore, we can find $\eta > 0$ small enough (possibly depending on ε) such that

$$v^-(T, x_0, i) - \frac{\varepsilon^2}{4\eta} < -\varepsilon + \inf_{(t,x)\in\mathcal{C}} v^-(t, x, i).$$

From Lemma 3.6 with $F(p, t, x) = p$ for any $(p, t, x) \in \mathbb{R}\times\mathcal{C}$, we can find $v \in \mathcal{V}^-$ such that

$$v^-(T, x_0, i) - \frac{\varepsilon^2}{4\eta} < -\varepsilon + \inf_{(t,x)\in\mathcal{C}} v(t, x, i). \tag{3.14}$$

For $k > 0$ define

$$\varphi^{\eta,\varepsilon,k}(t,x) = v^-(T, x_0, i) - \frac{|x - x_0|^2}{\eta} - k(T - t).$$

Since b, σ, f are continuous, we can choose k large enough such that

$$-\frac{\partial\varphi^{\eta,\varepsilon,k}}{\partial t}(t,x) - \inf_{u\in U}\big[\mathcal{L}^{i,u}\varphi^{\eta,\varepsilon,k}(t,x) + f(x,i,u)\big] < 0, \qquad \forall\,(t,x) \in \overline{B(T, x_0, \varepsilon)}.$$

From (3.14) it follows that $\varphi^{\eta,\varepsilon,k}(t,x) < -\varepsilon + v(t,x,i)$ on $\mathcal{C}$. Moreover

$$\varphi^{\eta,\varepsilon,k}(T,x) \leq v^-(T,x_0,i) \leq g(x,i) - \varepsilon \qquad \text{whenever } |x - x_0| \leq \varepsilon.$$

Now, for $\delta \in (0,\varepsilon)$ define

$$v^\delta(t,x,i) = \begin{cases} (\varphi^{\eta,\varepsilon,k}(t,x) + \delta) \vee v(t,x,i) & \text{on } \overline{B(t_0,x_0,\varepsilon)}, \\ v(t,x,i) & \text{outside } \overline{B(t_0,x_0,\varepsilon)}. \end{cases}$$

Moreover, $v^\delta(t,x,j) = v(t,x,j)$ for any $(t,x,j) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_m$ with $j \neq i$. As $v^\delta(T,x_0,i) > v^-(T,x_0,i)$, we get a contradiction if we are able to prove that $v^\delta \in \mathcal{V}^-$. In particular, for any $s \in [0,T]$ and $\tau, \rho \in \mathcal{T}^s$ with $\tau \leq \rho \leq T$, we have to find $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{\iota}_n)_{n\in\mathbb{N}} \in \mathcal{A}_{s,\tau^+}$ which works for v^δ. Consider the feedback switching strategy $\tilde{\alpha}$ defined in Step I(i), Case a, with ρ_1 the exit time from $B(T,x_0,\varepsilon/2)$. Then, proceeding as in Case a of Step I(i), we can prove that $\tilde{\alpha}$ satisfies the condition in the second item of Definition 3.4 for v^δ.

Step II. *v^+ is a viscosity subsolution to the HJB equation* (2.10).

Step II(i). *Interior viscosity subsolution property.* Let $(t_0,x_0) \in [0,T) \times \mathbb{R}^d$, $i \in \mathbb{I}_m$, and consider a test function $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$ such that $v^+(\cdot,\cdot,i) - \varphi(\cdot,\cdot)$ attains a strict global maximum equal to zero at (t_0,x_0). Reasoning by contradiction, we assume that

$$\min\bigg\{ -\frac{\partial\varphi}{\partial t}(t_0,x_0) - \inf_{u\in U}\big[\mathcal{L}^{i,u}\varphi(t_0,x_0) + f(x_0,i,u)\big], \\ v^+(t_0,x_0,i) - \max_{j\neq i}\big[v^+(t_0,x_0,j) - c(x_0,i,j)\big]\bigg\} > 0.$$

Then, there exists $\varepsilon > 0$ and $\underline{u} \in U$ such that

$$-\frac{\partial\varphi}{\partial t}(t_0,x_0) - \mathcal{L}^{i,\underline{u}}\varphi(t_0,x_0) - f(x_0,i,\underline{u}) > \varepsilon.$$

From the continuity of b, σ, f, it follows that we can find a smaller $\varepsilon > 0$ such that

$$-\frac{\partial\varphi}{\partial t}(t,x) - \mathcal{L}^{i,\underline{u}}\varphi(t,x) - f(x,i,\underline{u}) > \varepsilon \qquad \forall\,(t,x) \in B(t_0,x_0,\varepsilon),$$

where $B(t_0,x_0,\varepsilon)$ is given by (3.10). As $v^+(\cdot,\cdot,i) - \varphi(\cdot,\cdot)$ is usc and strictly negative on the compact set $\mathcal{C} := \overline{B(t_0,x_0,\varepsilon)} \backslash B(t_0,x_0,\varepsilon/2)$, we see that there exists $\delta > 0$ such that $\sup_{(t,x)\in\mathcal{C}}(v^+(t,x,i) - \varphi(t,x)) < -\delta$. Denoting $F(p,t,x) := p - \varphi(t,x)$, it follows from Lemma 3.6 that there exists $v \in \mathcal{V}^+$ such that $\varphi(t,x) - \delta > v(t,x,i)$ on $\mathcal{C}$. Now, define

$$v^\delta(t,x,i) = \begin{cases} (\varphi(t,x) - \delta) \wedge v(t,x,i) & \text{on } \overline{B(t_0,x_0,\varepsilon)}, \\ v(t,x,i) & \text{outside } \overline{B(t_0,x_0,\varepsilon)}. \end{cases}$$

Moreover, $v^\delta(t,x,j) = v(t,x,j)$ for any $(t,x,j) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_m$ with $j \neq i$. As $v^\delta(t_0,x_0,i) < v^+(t_0,x_0,i)$, we find a contradiction if we are able to prove that $v^\delta \in \mathcal{V}^+$. To this end, fix $s \in [0,T]$, $\tau \in \mathcal{T}^s$, and $\alpha = (\tau_n,\iota_n)_{n\in\mathbb{N}} \in \mathcal{A}_s$. We have to construct an elementary feedback strategy $\tilde{u} \in \mathcal{U}^E_{s,\tau}$ which works for v^δ. Consider the stopping rule $\rho_1 \in \mathcal{T}^s$ given by (3.11), and let $\tilde{u}^1 \in \mathcal{U}^E_{s,\rho_1}$ be the elementary feedback strategy for v, corresponding to s, ρ_1, α. Then, we define $\tilde{u}^2 = \underline{u} \otimes_{\rho_1} \tilde{u}^1$, which belongs to $\mathcal{U}^E_{s,\tau}$, thanks to Proposition 3.2. Now, let $\tilde{u}^3 \in \mathcal{U}^E_{s,\tau}$ be the elementary feedback strategy for v, corresponding to s, τ, α. Then, we define

$$\tilde{u}(t,y) = \tilde{u}^2(t,y) 1_{\{(\tau(y), y^X(\tau(y))) \in B(t_0,x_0,\varepsilon),\, (v-\varphi)(\tau(y), y(\tau(y)^+)) > -\delta,\, y^I(\tau(y)^+) = i\}} + \tilde{u}^3(t,y) 1_{\{(\tau(y), y^X(\tau(y))) \in B(t_0,x_0,\varepsilon), (v-\varphi)(\tau(y), y(\tau(y)^+)) > -\delta\,,\, y^I(\tau(y)^+) = i\}^c}.$$

From Lemma 3.1 we see that $\tilde{u} \in \mathcal{U}^E_{s,\tau}$. Moreover, $\tilde{u}$ is the elementary feedback strategy for the second item of Definition 3.5 for v^δ. Indeed, fix $u \in \mathcal{U}^E_s$, $(x,i) \in \mathbb{R}^d \times \mathbb{I}_m$, and $\rho \in \mathcal{T}^s$ with $\tau \le \rho \le T$. We adopt the shorthand

$$\begin{aligned}
(X, I) &= (X^{s,x,i;\alpha,u\otimes_\tau \tilde{u}}, I^{s,x,i;\alpha,u\otimes_\tau \tilde{u}}),\\
(X^1, I^1) &= (X^{s,x,i;\alpha,u\otimes_\tau \tilde{u}^2}, I^{s,x,i;\alpha,u\otimes_\tau \tilde{u}^2}),\\
(X^2, I^2) &= (X^{s,x,i;\alpha,u\otimes_\tau \tilde{u}^3}, I^{s,x,i;\alpha,u\otimes_\tau \tilde{u}^3}).
\end{aligned}$$

We also denote $\tau' = \tau(X_\cdot, I_{\cdot-})$, $\rho_1' = \rho_1(X_\cdot, I_{\cdot-})$, and $\rho' = \rho(X_\cdot, I_{\cdot-})$. Notice that

$$(X, I) = (X^1, I^1) 1_{\{(\tau', X_{\tau'}) \in B(t_0,x_0,\varepsilon),\, (v-\varphi)(\tau', X_{\tau'}, I_{\tau'}) > -\delta,\, I_{\tau'} = i\}} + (X^2, I^2) 1_{\{(\tau', X_{\tau'}) \in B(t_0,x_0,\varepsilon),\, (v-\varphi)(\tau', X_{\tau'}, I_{\tau'}) > -\delta,\, I_{\tau'} = i\}^c}.$$

Moreover, write $v^\delta(\tau', X_{\tau'}, I_{\tau'})$ as follows:

$$v^\delta(\tau', X_{\tau'}, I_{\tau'}) = \big(\varphi(\tau', X^1_{\tau'}) - \delta\big) 1_{\{(\tau', X_{\tau'}) \in B(t_0,x_0,\varepsilon),\, (v-\varphi)(\tau', X_{\tau'}, I_{\tau'}) > -\delta,\, I_{\tau'} = i\}} + v(\tau', X^2_{\tau'}, I^2_{\tau'}) 1_{\{(\tau', X_{\tau'}) \in B(t_0,x_0,\varepsilon),\, (v-\varphi)(\tau', X_{\tau'}, I_{\tau'}) > -\delta,\, I_{\tau'} = i\}^c}.$$

Then, applying Itô's formula to φ and using the properties of v, we see that $v^\delta \in \mathcal{V}^+$.

Step II(ii). *Terminal condition.* Reasoning by contradiction, we assume that there exist $x_0 \in \mathbb{R}^d$ and $i \in \mathbb{I}_m$ such that

$$v^+(T, x_0, i) > g(x_0, i).$$

Since g is continuous, there exists $\varepsilon > 0$ such that $v^+(T, x_0, i) \ge g(x, i) + \varepsilon$ whenever $|x - x_0| \le \varepsilon$. Consider the compact set

$$\mathcal{C} := \big(\overline{B(T, x_0, \varepsilon)} \backslash B(T, x_0, \varepsilon/2)\big) \cap \big([0,T] \times \mathbb{R}^d\big).$$

As v^+ is usc, it is bounded from above on $\mathcal{C}$. Therefore, we can find $\eta > 0$ small enough (possibly depending on ε) such that

$$v^+(T, x_0, i) + \frac{\varepsilon^2}{4\eta} > \varepsilon + \sup_{(t,x)\in\mathcal{C}} v^+(t, x, i).$$

From Lemma 3.6 with $F(p,t,x) = p$ for any $(p,t,x) \in \mathbb{R} \times \mathcal{C}$, we can find $v \in \mathcal{V}^+$ such that

$$v^+(T, x_0, i) + \frac{\varepsilon^2}{4\eta} > \varepsilon + \sup_{(t,x)\in\mathcal{C}} v(t, x, i). \tag{3.15}$$

For $k > 0$ define

$$\varphi^{\eta,\varepsilon,k}(t,x) = v^+(T, x_0, i) + \frac{|x - x_0|^2}{\eta} + k(T - t).$$

Since b, σ, f are continuous, we can choose k large enough and $\underline{u} \in U$ such that

$$-\frac{\partial \varphi^{\eta,\varepsilon,k}}{\partial t}(t,x) - \mathcal{L}^{i,\underline{u}}\varphi^{\eta,\varepsilon,k}(t,x) - f(x,i,\underline{u}) > 0 \qquad \forall\,(t,x) \in \overline{B(T,x_0,\varepsilon)}.$$

From (3.15) it follows that $\varphi^{\eta,\varepsilon,k}(t,x) > \varepsilon + v(t,x,i)$ on $\mathcal{C}$. Moreover,

$$\varphi^{\eta,\varepsilon,k}(T,x) \geq v^+(T,x_0,i) \geq g(x,i) + \varepsilon \qquad \text{whenever } |x - x_0| \leq \varepsilon.$$

Now, for $\delta \in (0,\varepsilon)$ define

$$v^\delta(t,x,i) = \begin{cases} (\varphi^{\eta,\varepsilon,k}(t,x) - \delta) \wedge v(t,x,i) & \text{on } \overline{B(t_0,x_0,\varepsilon)}, \\ v(t,x,i) & \text{outside } \overline{B(t_0,x_0,\varepsilon)}. \end{cases}$$

Moreover, $v^\delta(t,x,j) = v(t,x,j)$ for any $(t,x,j) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_m$ with $j \neq i$. As $v^\delta(T,x_0,i) < v^+(T,x_0,i)$, we get a contradiction if we prove that $v^\delta \in \mathcal{V}^+$. In particular, for any $s \in [0,T]$, $\tau \in \mathcal{T}^s$, and $\alpha = (\tau_n, \iota_n)_{n\in\mathbb{N}} \in \mathcal{A}_s$, we have to find $\tilde{u} \in \mathcal{U}^E_{s,\tau}$ for the second item of Definition 3.5 for v^δ. Let $\tilde{u} \in \mathcal{U}^E_{s,\tau}$ be the elementary feedback strategy defined in Step II(i), with ρ_1 the exit time from $B(T,x_0,\varepsilon/2)$. Then, we can prove, as in Step II(i), that $\tilde{u}$ satisfies the condition in the second item of Definition 3.5 for v^δ. ∎

4. Dynamic programming and viscosity properties of V.

In the present section, by means of the comparison principle for (2.10), we prove that V satisfies the dynamic programming principle and is a viscosity solution to (2.10), which therefore turns out to be the dynamic programming equation of the robust switching control problem.

4.1. Comparison principle and viscosity characterization.

We need to make an additional assumption on the switching costs in order to get the comparison principle.

(H3)

The switching cost function c satisfies the *no free loop property*: for any sequence of indices $i_1, \ldots, i_k \in \mathbb{I}_m$, with $k \in \mathbb{N}\backslash\{0,1,2\}$, $i_1 = i_k$, and $\mathrm{card}\{i_1, \ldots, i_k\} = k - 1$, we have

$$c(x,i_1,i_2) + c(x,i_2,i_3) + \cdots + c(x,i_{k-1},i_k) + c(x,i_k,i_1) > 0 \qquad \forall\, x \in \mathbb{R}^d.$$

We also assume that $c(x,i,i) = 0$, for any $x \in \mathbb{R}^d$ and $i \in \mathbb{I}_m$.

THEOREM 4.1 (comparison principle). *Let assumptions* (H1), (H2), *and* (H3) *hold and consider a viscosity subsolution $\check{v}$ (resp., supersolution $\hat{v}$) to* (2.10). *Suppose that*

$$\sup_{(t,x,i)\in[0,T]\times\mathbb{R}^d\times\mathbb{I}_m} \frac{|\check{v}(t,x,i)| + |\hat{v}(t,x,i)|}{1 + |x|^q} < \infty$$

for some $q \geq 1$. Then, we have $\check{v}(t,x,i) \leq \hat{v}(t,x,i)$ for any $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_m$.

Remark 4.1. The proof can be done along the lines of Proposition 3.1 in [18], apart from minor changes due to the presence of the infimum over U in (2.10), which are dealt with by the uniform Lipschitz condition in (H1)(ii). More precisely, it is proved, as usual, proceeding by contradiction and then using the doubling variable technique. We simply note here that (2.10) requires a particular step. Indeed, along the sequence of maximum points $(t_n, x_n)_n$ coming through the doubling of variables, we require

$$\check{v}(t_n, x_n, i) > \max_{j\neq i}\big[\check{v}(t_n, x_n, j) - c(x_n, i, j)\big], \tag{4.1}$$

so that, from the viscosity subsolution property of $\check{v}$, we can derive an inequality for the PDE part of (2.10) (concerning $\hat{v}$, the viscosity supersolution property implies already the nonnegativity of both terms in (2.10)). Condition (4.1) is obtained from a "no-loop" argument presented in [21, Theorem 3.1] (see also [2, Lemma A.2] and [18, Proposition 3.1]), which is based on the no-free-loop-property in (H3). ∎

COROLLARY 4.1. *Under assumptions* (H1), (H2), *and* (H3), *we have* $v^- = V = \overline{V} = v^+$. *In particular,* V *(as* $v^-, \overline{V}, v^+$*) is continuous. Moreover,* V *is the unique viscosity solution to* (2.10) *satisfying a polynomial growth condition. Furthermore,* V *satisfies the dynamic programming principle: for any* $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ *and* $\rho \in \mathcal{T}^s$,

$$\begin{aligned} V(s,x,i) &= \sup_{\alpha\in\mathcal{A}_{s^+}} \inf_{u\in\mathcal{U}_s} \mathbb{E}\bigg[\int_s^{\rho'} f(X_t, I_t, u_t)dt + V(\rho', X_{\rho'}, I_{\rho'}) \\ &\qquad - \sum_{n\in\mathbb{N}} c(X_{\tau_n'}, I_{(\tau_n')^-}, I_{\tau_n'})1_{\{s\le\tau_n'<\rho'\}}\bigg] \\ &= \sup_{\alpha\in\mathcal{A}_{s^+}} \inf_{u\in\mathcal{U}_s^E} \mathbb{E}\bigg[\int_s^{\rho'} f(X_t, I_t, u_t')dt + V(\rho', X_{\rho'}, I_{\rho'}) \\ &\qquad - \sum_{n\in\mathbb{N}} c(X_{\tau_n'}, I_{(\tau_n')^-}, I_{\tau_n'})1_{\{s\le\tau_n'<\rho'\}}\bigg] \end{aligned}$$

with the shorthand $X = X^{s,x,i;\alpha,u}$, $I = I^{s,x,i;\alpha,u}$, $\rho' = \rho(X_\cdot, I_{\cdot-})$, $\tau_n' = \tau_n(X_\cdot, I_{\cdot-})$, *and* $u_t' = u(t, X_\cdot, I_{\cdot-})$.

Proof. The equality $v^- = V = \overline{V} = v^+$ follows from the comparison principle Theorem 4.1. Since v^- is lsc and v^+ is usc, we see that V is continuous. Moreover, from Remark 2.7 and Theorems 3.1 and 4.1 it follows that V is the unique viscosity solution to (2.10) satisfying a polynomial growth condition. Finally, let us prove the dynamic programming principle for V. We begin noting that v^- and v^+ satisfy, respectively, the sub- and super-dynamic programming principles: for any $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ and $\rho \in \mathcal{T}^s$,

$$\begin{aligned} v^-(s,x,i) \le \sup_{\alpha\in\mathcal{A}_{s^+}} \inf_{u\in\mathcal{U}_s} \mathbb{E}\bigg[&\int_s^{\rho'} f(X_t, I_t, u_t)dt + v^-(\rho', X_{\rho'}, I_{\rho'}) \\ &- \sum_{n\in\mathbb{N}} c(X_{\tau_n'}, I_{(\tau_n')^-}, I_{\tau_n'})1_{\{s\le\tau_n'<\rho'\}}\bigg] \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} v^+(s,x,i) \ge \sup_{\alpha\in\mathcal{A}_{s^+}} \inf_{u\in\mathcal{U}_s^E} \mathbb{E}\bigg[&\int_s^{\rho'} f(X_t, I_t, u_t')dt + v^+(\rho', X_{\rho'}, I_{\rho'}) \\ &- \sum_{n\in\mathbb{N}} c(X_{\tau_n'}, I_{(\tau_n')^-}, I_{\tau_n'})1_{\{s\le\tau_n'<\rho'\}}\bigg] \end{aligned} \tag{4.3}$$

with the shorthand $X = X^{s,x,i;\alpha,u}$, $I = I^{s,x,i;\alpha,u}$, $\rho' = \rho(X_\cdot, I_{\cdot-})$, $\tau'_n = \tau_n(X_\cdot, I_{\cdot-})$, and $u'_t = u(t, X_\cdot, I_{\cdot-})$. As a matter of fact, let $(v_n)_{n\in\mathbb{N}} \subset \mathcal{V}^-$ be the sequence in Lemma 3.5(i). From Lemma 3.2 we know that each v_n satisfies the subdynamic programming principle: for any $(s,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_m$ and $\rho \in \mathcal{T}^s$,

$$v_n(s,x,i) \leq \sup_{\alpha\in\mathcal{A}_{s^+}} \inf_{u\in\mathcal{U}_s} \mathbb{E}\bigg[\int_s^{\rho'} f(X_t, I_t, u_t)dt + v_n(\rho', X_{\rho'}, I_{\rho'}) - \sum_{n\in\mathbb{N}} c(X_{\tau'_n}, I_{(\tau'_n)^-}, I_{\tau'_n}) 1_{\{s\leq\tau'_n<\rho'\}}\bigg].$$

Since $v_n \leq v^-$, we get

$$v_n(s,x,i) \leq \sup_{\alpha\in\mathcal{A}_{s^+}} \inf_{u\in\mathcal{U}_s} \mathbb{E}\bigg[\int_s^{\rho'} f(X_t, I_t, u_t)dt + v^-(\rho', X_{\rho'}, I_{\rho'}) - \sum_{n\in\mathbb{N}} c(X_{\tau'_n}, I_{(\tau'_n)^-}, I_{\tau'_n}) 1_{\{s\leq\tau'_n<\rho'\}}\bigg]. \tag{4.4}$$

Letting $n \to \infty$ in (4.4), we finally obtain the subdynamic programming principle (4.2) for v^-. In a similar way we can prove (4.3). Combining (4.2) and (4.3) with the equalities $v^- = V = v^+$, gives us the dynamic programming principle for V. ☐

4.2. Elliott–Kalton formulation. We now describe the Elliott–Kalton formulation of the robust switching control problem, and we present in the next paragraph an example which shows that this is, in general, a different control problem than the robust feedback switching control problem studied here. As a by-product of this example, we will find a counterexample to uniqueness for (2.10). Let us begin by introducing the concept of nonanticipating strategy for the switcher. First, we define a standard switching control, not necessarily of feedback form.

DEFINITION 4.1 (switching controls). *Fix $s \in [0,T]$. We say that the double sequence $\alpha = (\tau_n, \iota_n)_{n\in\mathbb{N}}$ is a switching control starting at s if*

- *τ_n is an $\mathbb{F}^s$-stopping time, for any $n \in \mathbb{N}$, and*

$$s \leq \tau_0 \leq \cdots \leq \tau_n \leq \cdots \leq T.$$

Moreover, $(\tau_n)_{n\in\mathbb{N}}$ satisfies the following property: for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\tau_n(\omega) = T \qquad \text{for } n \text{ large enough;}$$

- *$\iota_n \colon \Omega \to \mathbb{I}_m$ is $\mathcal{F}^s_{\tau_n}$-measurable for any $n \in \mathbb{N}$.*

$\overline{\mathcal{A}}_s$ denotes the family of all switching controls starting at s.

When using switching controls as defined above, the well-posedness of (2.1) becomes easier. In particular, we have the following result, whose standard proof is omitted.

PROPOSITION 4.1. *Let assumption* (H1) *hold. For any $(s,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_m$, $\alpha \in \overline{\mathcal{A}}_s$, $u \in \mathcal{U}_s$, there exists a unique (up to indistinguishability) $\mathbb{F}^s$-adapted process $(X^{s,x,i;\alpha,u}, I^{s,i;\alpha}) = (X_t^{s,x,i;\alpha,u}, I_t^{s,i;\alpha})_{s\leq t\leq T}$ to* (2.1). *Moreover, estimate* (2.2) *holds.*

We can now introduce the concept of nonanticipating strategy for the switcher.

DEFINITION 4.2 (nonanticipating strategies). *Fix $s \in [0,T]$. We say that the map*

$$\beta\colon \mathcal{U}_s \longrightarrow \overline{\mathcal{A}}_s,$$
$$u \longmapsto \beta[u] = \big(\tau_n[u], \iota_n[u]\big)_{n\in\mathbb{N}},$$

is a nonanticipating strategy starting at s if

$$\mathbb{P}\big[(\tau_n[u^1], \iota_n[u^1])1_{\{\tau_n[u^1]\le t\}} = (\tau_n[u^2], \iota_n[u^2])1_{\{\tau_n[u^2]\le t\}}\ \forall\, n \in \mathbb{N}\big] = 1$$

whenever $\mathbb{P}(u^1_r = u^2_r, \forall\, r \in [s,t]) = 1$ for any $t \in [s,T]$ and $u^1, u^2 \in \mathcal{U}_s$. Δ_s denotes the family of all nonanticipating strategies starting at s.

We can now define the corresponding value function

$$\hat V(s,x,i) := \sup_{\beta\in\Delta_s} \inf_{u\in\mathcal{U}_s} J(s,x,i;\beta[u],u),$$

$\forall\,(s,x,i) \in [0,T]\times\mathbb{R}^d\times\mathbb{I}_m$. Notice that

$$V(s,x,i) \le \hat V(s,x,i) \qquad \forall\,(s,x,i) \in [0,T]\times\mathbb{R}^d\times\mathbb{I}_m. \tag{4.5}$$

Under assumptions (H1) and (H2), we expect that $\hat V$ (as V) is a viscosity solution to (2.10). Therefore, when (H3) holds, by comparison, we have $V = \hat V$. However, if (H3) is not assumed, the above inequality (4.5) might be strict at some $(s,x,i) \in [0,T]\times\mathbb{R}^d\times\mathbb{I}_m$. The following example illustrates this latter point.

Example. Fix $d = 1$, $m = 2$ so that $\mathbb{I}_2 = \{1,2\}$, and take $U = \mathbb{I}_2$. Moreover, set $b(x,i,u) = -|i-u|$ and $\sigma \equiv 0$. Notice that $b \in \{-1,0\}$. Since assumption (H1) is satisfied, from Proposition 4.1 it follows that, for any $(s,x,i) \in [0,T]\times\mathbb{R}\times\mathbb{I}_2$, $\alpha \in \overline{\mathcal{A}}_s$, $u \in \mathcal{U}_s$, there exists a unique solution $(X^{s,x,i;\alpha,u}, I^{s,i;\alpha}) = (X^{s,x,i;\alpha,u}_t, I^{s,i;\alpha}_t)_{s\le t\le T}$ to (2.1).

Set $g(x,i) = x$, $f \equiv 0$, and $c \equiv 0$. Our aim is now to determine the explicit form of $\hat V$ and V. To this end, it is convenient to give the following definition.

DEFINITION 4.3 (step controls). *Fix $s \in [0,T]$. We say that u is a step control starting at s if there exists $n \in \mathbb{N}\backslash\{0\}$ such that*

- *$s =: t_0 \le \cdots \le t_k \le \cdots \le t_n := T$,*
- *$\xi_k\colon \Omega \to U$ is $\mathcal{F}^s_{t_k}$-measurable, for any $k = 0,\dots,n-1$.*

The control $u\colon [s,T]\times\Omega \to U$ is given by $u_t := \sum_{k=0}^{n-1} \xi_k 1_{\{t_k\le t<t_{k+1}\}}$. $\mathcal{U}^S_s$ denotes the family of all step controls starting at s.

Let us now determine the form of the function $\hat V$. Since the terminal payoff g is strictly increasing and the drift b is nonpositive, the aim of the switcher is to keep the system still. Having this in mind, we define, for every $\varepsilon > 0$, the strategy $\beta^\varepsilon \in \Delta_s$ with $\beta^\varepsilon[u] = (\tau^\varepsilon_n[u], \iota^\varepsilon_n[u])_{n\in\mathbb{N}}$ $\forall\, u \in \mathcal{U}_s$, as follows:

(i) For any $u_t = \sum_{k=0}^{n-1} \xi_k 1_{\{t_k\le t<t_{k+1}\}}$ in $\mathcal{U}^S_s$, we set

$$\big(\tau^\varepsilon_k[u], \iota^\varepsilon_k[u]\big) := (t_k, \xi_k) \qquad \forall\, k = 0,\dots,n-1.$$

With this choice, $X^{s,x,i;\beta^\varepsilon[u],u}_t = x$ for any $t \in [s,T]$ and $J(s,x,i;\beta^\varepsilon[u],u) = x$.

(ii) For any $u \in \mathcal{U}_s\backslash\mathcal{U}^S_s$, it follows from the approximation result in [25, Lemma 3.2.6], that there exists $u^\varepsilon \in \mathcal{U}^S_s$ such that $\mathbb{E}[\int_s^T |u_t - u^\varepsilon_t|dt] \le \varepsilon$. Then we define $\beta^\varepsilon[u] := \beta^\varepsilon[u^\varepsilon]$, where $\beta^\varepsilon[u^\varepsilon]$ has already been defined in item (i), since $u^\varepsilon \in \mathcal{U}_s^S$. Therefore

$$\begin{aligned} J(s,x,i;\beta^\varepsilon[u],u) = \mathbb{E}\big[X_T^{s,x,i;\beta^\varepsilon[u],u}\big] &= x - \mathbb{E}\bigg[\int_s^T \big|I_T^{s,x,i;\beta^\varepsilon[u],u} - u_t\big|dt\bigg] \\ &= x - \mathbb{E}\bigg[\int_s^T \big|I_T^{s,x,i;\beta^\varepsilon[u^\varepsilon],u} - u_t\big|dt\bigg] \\ &= x - \mathbb{E}\bigg[\int_s^T \big|u_t^\varepsilon - u_t\big|dt\bigg] \geq x - \varepsilon. \end{aligned}$$

In conclusion, we find, for every $\varepsilon > 0$,

$$J(s,x,i;\beta^\varepsilon[u],u) \geq x - \varepsilon \qquad \forall\, u \in \mathcal{U}_s,$$

which implies $\inf_{u\in\mathcal{U}_s} J(s,x,i;\beta^\varepsilon[u],u) \geq x-\varepsilon$, and then $\hat{V}(s,x,i) \geq x-\varepsilon$. From the arbitrariness of ε, we obtain $\hat{V}(s,x,i) \geq x$. On the other hand, since $J(s,x,i;\beta[u],u) = \mathbb{E}[X_T^{s,x,i;\beta[u],u}] = x - \mathbb{E}[\int_s^T |I_t^{s,x,i;\beta[u],u} - u_t|dt] \leq x$, we deduce that

$$\hat{V}(s,x,i) = g(x,i) = x \qquad \forall\,(s,x,i) \in [0,T]\times\mathbb{R}\times\mathbb{I}_2.$$

As a consequence of this result, we also have

$$\hat{V}(s,x,i) = \sup_{\beta\in\Delta_s}\inf_{u\in\mathcal{U}_s^S} J(s,x,i;\beta[u],u),\ \forall (s,x,i) \in [0,T]\times\mathbb{R}^d\times\mathbb{I}_m.$$

Let us now find the expression for V. Fix $(s,x,i) \in [0,T]\times\mathbb{R}\times\mathbb{I}_2$ and $\alpha = (\tau_n,\iota_n)_{n\in\mathbb{N}} \in \mathcal{A}_s$. The aim of nature is to minimize the quantity $J(s,x,i;\alpha,u)$ over $\mathcal{U}_s$, which means to maximize the drift b, i.e., to keep it at the value -1. This can be done as follows. Define $u \in \mathcal{U}_s$, depending on α, by

$$u_t := (3-i)\,1_{\{s\leq t\leq\tau_0\}} + \sum_{n\in\mathbb{N}}(3-\iota_n)1_{\{\tau_n<t\leq\tau_{n+1}\}}, \qquad \forall\, t \in [s,T].$$

Observe that, since $i,\iota_n \in \mathbb{I}_2$ then $3-i, 3-\iota_n \in \mathbb{I}_2$; moreover, when $i=1$ then $3-i=2$, while if $i=2$ then $3-i=1$. Notice that, for $\mathbb{P}$-a.e. $\omega\in\Omega$ we have $I_t^{s,i;\alpha}(\omega) = 3-u_t(\omega)\ \forall\, t\in[s,T]$ with $t\neq\tau_n(\omega)$, $n\in\mathbb{N}$. Therefore, $\mathbb{P}$-a.s.,

$$b\big(X_t^{s,x,i;\alpha,u}, I_t^{s,i;\alpha}, u_t\big) = -\big|I_t^{s,i;\alpha} - u_t\big| = -1,$$

$\forall\, t\in[s,T]$, with $t\neq\tau_n$, $n\in\mathbb{N}$. It follows that, $\mathbb{P}$-a.s. we have $X_T^{s,x,i;\alpha,u} = x-(T-s)$. In other words, we obtain

$$V(s,x,i) = x-(T-s) \qquad \forall\,(s,x,i)\in[0,T]\times\mathbb{R}\times\mathbb{I}_2.$$

In conclusion, $V < \hat{V}$ on $[0,T)\times\mathbb{R}\times\mathbb{I}_2$. We finally observe that both V and $\hat{V}$ are classical solutions to (2.10), so that the comparison does not hold. This is due to the fact that while assumptions (H1) and (H2) hold, the no-free-loop property in (H3) is not satisfied.

Remark 4.2. In the example above, because of the assumption that the switching costs are always zero ($c \equiv 0$), it would be more natural, at least intuitively, to formulate the robust switching control problem as a classical two-player zero-sum stochastic differential game as in [16]. In this latter setting, we recall from [16, Theorem 2.6] that the lower value function V^{FS} (see [16, Definition 1.4]) is the unique viscosity solution to the lower Bellman–Isaacs equation:

$$(4.6)\quad \begin{cases} -\dfrac{\partial w}{\partial t}(s,x) - \max_{i\in\mathbb{I}_2}\min_{u\in\mathbb{I}_2}\left[\mathcal{L}^{i,u}w(s,x)\right] = 0, & (s,x)\in[0,T)\times\mathbb{R}^d,\\ w(T,x) = x, \quad x\in\mathbb{R}^d, \end{cases}$$

where $\mathcal{L}^{i,u}w(s,x) = -|i-u|D_x w(s,x)$. On the other hand, the upper value function U^{FS} (see [16, Definition 1.4]) is the unique viscosity solution to the upper Bellman–Isaacs equation:

$$(4.7)\quad \begin{cases} -\dfrac{\partial w}{\partial t}(s,x) - \min_{u\in\mathbb{I}_2}\max_{i\in\mathbb{I}_2}\left[\mathcal{L}^{i,u}w(s,x)\right] = 0, & (s,x)\in[0,T)\times\mathbb{R}^d,\\ w(T,x) = x, \quad x\in\mathbb{R}^d. \end{cases}$$

By direct calculation, we see that V satisfies (4.6), so that it coincides with the lower value function V^{FS} (this is expected from the results of [16] and [30], since V is the sup/inf over feedback strategies/open-loop controls), while $\hat{V}$ satisfies (4.7), and therefore it coincides with the upper value function U^{FS} (this is also not surprising, since $\hat{V}$ is the sup/inf over strategies/open-loop controls). Notice that in the present framework the Isaacs condition does not hold:

$$\max_{i\in\mathbb{I}_2}\min_{u\in\mathbb{I}_2}[-|i-u|p] \;\neq\; \min_{u\in\mathbb{I}_2}\max_{i\in\mathbb{I}_2}[-|i-u|p] \qquad \forall\, p\in\mathbb{R}. \qquad \square$$

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