On the nonlocal Cahn-Hilliard-Brinkman and Cahn-Hilliard-Hele-Shaw systems

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Abstract

The phase separation of an isothermal incompressible binary fluid in a porous medium can be described by the so-called Brinkman equation coupled with a convective Cahn-Hilliard (CH) equation. The former governs the average fluid velocity $\mathbf{u}$, while the latter rules evolution of φ, the difference of the (relative) concentrations of the two phases. The two equations are known as the Cahn-Hilliard-Brinkman (CHB) system. In particular, the Brinkman equation is a Stokes-like equation with a forcing term (Korteweg force) which is proportional to $\mu\nabla\varphi$, where μ is the chemical potential. When the viscosity vanishes, then the system becomes the Cahn-Hilliard-Hele-Shaw (CHHS) system. Both systems have been studied from the theoretical and the numerical viewpoints. However, theoretical results on the CHHS system are still rather incomplete. For instance, uniqueness of weak solutions is unknown even in 2D. Here we replace the usual CH equation with its physically more relevant nonlocal version. This choice allows us to prove more about the corresponding nonlocal CHHS system. More precisely, we first study well-posedness for the CHB system, endowed with no-slip and no-flux boundary conditions. Then, existence of a weak solution to the CHHS system is obtained as a limit of solutions to the CHB system. Stronger assumptions on the initial datum allow us to prove uniqueness for the CHHS system. Further regularity properties are obtained by assuming additional, though reasonable, assumptions on the interaction kernel. By exploiting these properties, we provide an estimate for the difference between the solution to the CHB system and the one to the CHHS system with respect to viscosity.

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1 Introduction

The phenomenon of phase separation of incompressible binary fluids in a porous medium can be modeled by means of a diffuse interface approach. Consider a mixture of two fluids occupying a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, for any time $t \in (0, T)$, $T > 0$, denote by φ the difference of the fluid (relative) concentrations and by $\boldsymbol{u}$ the (averaged) fluid velocity. Assuming that the two fluids have the same constant density, the resulting model is the so-called Cahn-Hilliard-Brinkman (CHB) system (see, e.g., [28, 30])

$$\begin{cases} \varphi_t + \nabla \cdot (\boldsymbol{u}\varphi) = \Delta\mu \\ \mu = -\Delta\varphi + F'(\varphi) \\ -\nabla \cdot (\nu\nabla\boldsymbol{u}) + \eta\boldsymbol{u} + \nabla p = \mu\nabla\varphi + \boldsymbol{h} \\ \nabla \cdot \boldsymbol{u} = 0 \end{cases} \tag{1.1}$$

in $\Omega \times (0, T)$, $T > 0$. Here $\nu > 0$ is the viscosity coefficient, $\eta > 0$ the fluid permeability and p is the fluid pressure. Other constants are supposed to be one for simplicity. The mobility is also assumed to be constant and equal to one, while F stands for a double well potential accounting for phase separation. The average velocity $\boldsymbol{u}$ obeys a modified Darcy's law proposed by H.C. Brinkman in 1947 (see [4]).

System (1.1) endowed with no-slip and no-flux boundary conditions has been analyzed from the numerical viewpoint in [6] (see also [9]). Some theoretical results can be found in [3], where well-posedness in a weak setting as well as longtime behavior of solutions (i.e., existence of the global attractor and convergence to a unique equilibrium) have been investigated. Another interesting issue is the analysis of behavior of solutions when ν goes to zero. Indeed when $\nu = 0$ system (1.1) becomes the so-called Cahn-Hilliard-Hele-Shaw (CHHS) model which is used, for instance, to describe tumor growth dynamics (see, e.g., [26] and references therein, cf. also [8]). This model presents several technical difficulties (cf. [26, 32, 33], see also [10, 9, 34] for numerical schemes). For instance, uniqueness of weak solutions is an open issue even in dimension two, as well as the existence of a global strong solution in dimension three for sufficiently general initial data (see [26]). Existence of a global weak solution to the CHHS system is obtained in [3] as limit of solutions to system (1.1) (see also [10, Thm.2.4] for an existence result). In the same paper, the difference of (strong) solutions to (1.1) and the CHHS system is estimated with respect to ν and to the initial data in dimension two. Most of the quoted papers deal with a regular potential F, that is, F is defined on the whole real line (however, see [8] for a singular potential).

In this contribution we want to analyze a nonlocal variant of (1.1) which is obtained by replacing the standard Cahn-Hilliard (CH) equation by its nonlocal version. More precisely, we consider the following nonlocal CHB system

$$\begin{cases} \varphi_t + \nabla \cdot (\boldsymbol{u}\varphi) = \Delta\mu \\ \mu = a\varphi - J * \varphi + F'(\varphi) \\ -\nabla \cdot (\nu(\varphi)\nabla \boldsymbol{u}) + \eta \boldsymbol{u} + \nabla p = \mu \nabla \varphi + \boldsymbol{h} \\ \nabla \cdot \boldsymbol{u} = 0 \end{cases} \tag{1.2}$$

in $\Omega \times (0, T)$. Here the viscosity may depend on φ, while $J : \mathbb{R}^d \to \mathbb{R}$ is a suitable interaction kernel and $a(x) = \int_\Omega J(x - y)dy$. This system is endowed with boundary and initial conditions

$$\begin{cases} \dfrac{\partial \mu}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Omega \times (0, T) \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega \times (0, T) \\ \varphi(0) = \varphi_0 & \text{in } \Omega. \end{cases} \tag{1.3}$$

We recall that the nonlocal CH equation can be justified in a more rigorous way from the physical viewpoint (cf. [19], see also [20, 21]). Also, the standard CH equation can be interpreted as an approximation of the nonlocal one. The nonlocal CH equation has been analyzed in a number of papers, under various assumptions on the potential F and on the mobility (see, e.g., [1, 7, 27, 17, 18, 24, 25, 29], cf. also [22, 23] for the numerics). In addition, a series of papers have recently been devoted to the so-called Cahn-Hilliard-Navier-Stokes (CHNS) system in its nonlocal version (cf. [5, 11, 12, 13, 14, 15, 16]). Adapting the techniques devised in [5], we can prove existence of a global weak solution to (1.2)–(1.3). Its uniqueness (for constant viscosity) also holds in dimension three. However, the main goal is the analysis of the vanishing viscosity case where the limi t problem is

$$\begin{cases} \varphi_t + \nabla \cdot (\boldsymbol{u}\varphi) = \Delta\mu \\ \eta \boldsymbol{u} + \nabla p = \mu \nabla \varphi + \boldsymbol{h} \\ \nabla \cdot \boldsymbol{u} = 0 \end{cases} \tag{1.4}$$

in $\Omega \times (0, T)$, i.e. the nonlocal CHHS system, subject to the boundary and initial conditions

$$\begin{cases} \dfrac{\partial \mu}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Omega \times (0, T) \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega \times (0, T) \\ \varphi(0) = \varphi_0 & \text{in } \Omega. \end{cases} \tag{1.5}$$

As in [3], we can prove that a solution to (1.4)–(1.5) can be obtained as a limit of solutions to (1.2)–(1.3). In addition, uniqueness holds when φ_0 is bounded (and so is φ). Here we take advantage of the fact that the nonlocal CH equation is essentially a second-order equation and not a fourth-order equation like in the standard CHHS system. Then, further reasonable assumptions on J allow us to establish some regularity properties of the solutions. These properties help us to estimate the difference, with respect to ν and the initial data, between a solution to (1.2)–(1.3) and a solution to the CHHS system.

The plan of this paper goes as follows. Notation, assumptions and statements of the main results are contained in Section 2. Results concerning existence and regularity for (1.2)–(1.3) are proven in Section 3. Existence of a weak solution to (1.4)–(1.5) is demonstrated in Section 4. Section 5 deals with uniqueness and continuous dependence on data for both problems. The final Section 6 is essentially devoted to obtain the estimate of the difference of the solutions to (1.2)–(1.3) and (1.4)–(1.5).

2 Functional setup and main results

2.1 Notation

We set $H := L^2(\Omega)$ and $V := H^1(\Omega)$. We denote by $\|\cdot\|$ and $(\cdot\,,\cdot)$ the norm and the scalar product in H, respectively, while $\langle\,\cdot\,\rangle$ stands for the duality between V' and V. For every $\varphi \in V'$ we denote by $\bar{\varphi}$ the average of φ over Ω, namely $\bar{\varphi} = |\Omega|^{-1}\langle \varphi, 1\rangle$. Then we define

$$V_2 = \Big\{ v \in H^2(\Omega) : \frac{\partial v}{\partial \boldsymbol{n}} = 0 \text{ on } \partial\Omega \Big\}.$$

The linear operator $A = -\Delta : V_2 \subset H \to H$ with dense domain is self-adjoint and non-negative. Moreover, it is strictly positive on $V_0 = \{\psi \in V \,:\, \bar{\psi} = 0\}$ and it maps V_0 isomorphically into $V_0' = \{\psi \in V' : \langle \psi, 1 \rangle = 0\}$. We will also set

$$\|\psi - \bar{\psi}\|_r = \| A^{r/2}(\psi - \bar{\psi}) \|$$

for every $r \in \mathbb{R}$. Observe that the norm $\| \cdot \|_\#$ defined as

$$\|x\|_\# := \left(\|x - \bar{x}\|_{-1}^2 + \bar{x}^2 \right)^{\frac{1}{2}},$$

is equivalent to the usual norm of V'.

Besides, let $\mathcal{V}$ be the space of divergence-free test functions defined by

$$\mathcal{V} = \{\boldsymbol{v} \in C_0^\infty(\Omega, \mathbb{R}^d) : \nabla \cdot \boldsymbol{v} = 0\}.$$

We shall use the following canonical spaces (see, e.g., [31, Chapter I])

$$\boldsymbol{H} = \overline{\mathcal{V}}^{H^d} \qquad \text{and} \qquad \boldsymbol{V} = \{\boldsymbol{v} \in V^d : \nabla \cdot \boldsymbol{v} = 0\}.$$

Recall that $\boldsymbol{v} \in \boldsymbol{V}$ yields $\boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}$, while $\boldsymbol{v} \in \boldsymbol{H}$ is such that $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$. We will still use $(\cdot\,,\cdot)$ and $\langle \cdot \rangle$ to denote the scalar product in $\mathbf{H}$ and the duality between $\boldsymbol{V}'$ and $\boldsymbol{V}$, respectively.

Finally, c will indicate a generic nonnegative constant depending on Ω, J, F, and $\boldsymbol{h}$ at most. Instead, N will stand for a generic positive constant which has further dependence on T and/or on some norm of φ_0. The value of c and N may vary even within the same line.

2.2 Assumptions

Following [1] and [5] (cf. also [3]) we introduce the following assumptions.

(H0) $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is open, bounded and connected with a smooth boundary.

(H1) $J \in W^{1,1}(\mathbb{R}^d)$ satisfies

$$J(x) = J(-x), \qquad a(x) := \int_\Omega J(x-y)\, dy \geq 0, \text{ a.e. } x \in \Omega.$$

(H2) $F \in C^{2,1}_{loc}(\mathbb{R})$ and there exists $c_0 > 0$ such that

$$F''(s) + a(x) \geq c_0, \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega.$$

(H3) There exist $c_1 > 0$, $c_2 > 0$ and $q > 0$ if $d = 2$, $q \geq \frac{1}{2}$ if $d = 3$ such that

$$F''(s) + a(x) \geq c_9|s|^{2q} - c_{10}, \qquad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega.$$

(H4) There exist $c_3 > 0$ and $p \in (1, 2]$ such that
$$|F'(s)|^p \leq c_4(|F(s)| + 1), \quad \forall s \in \mathbb{R}.$$

(H5) $\eta \in L^\infty(\Omega)$ and
$$\eta(x) \geq 0, \quad \text{a.e. } x \in \Omega.$$

(H6) ν is locally Lipschitz on $\mathbb{R}$ and there exist $\nu_0, \nu_1 > 0$ such that
$$\nu_0 \leq \nu(s) \leq \nu_1, \quad \forall s \in \mathbb{R}.$$

(H7) $\boldsymbol{h} \in L^2(0, T; \mathbf{V}')$.

Remark 2.1 *Assumption (H2) implies that the potential F is a quadratic perturbation of a strictly convex function. Indeed F can be represented as*
$$F(s) = G(s) - \frac{a^*}{2} s^2 \tag{2.1}$$
with $G \in C^{2,1}(\mathbb{R})$ strictly convex, since $G'' \geq c_0$ in Ω. Here $a^* = \|a\|_{L^\infty(\Omega)}$ and observe that $a \in L^\infty(\Omega)$ derives from (H1).

Remark 2.2 *Since F is bounded from below, it is easy to see that (H4) implies that F has polynomial growth of order p', where $p' \in [2, \infty)$ is the conjugate index to p. Namely there exist $c_4 > 0$ and $c_5 \geq 0$ such that*
$$|F(s)| \leq c_4|s|^{p'} + c_5, \qquad \forall s \in \mathbb{R}.$$
Besides, it can be shown that (H3) implies the existence of $c_6, c_7 > 0$ such that
$$F(s) \geq c_6|s|^{2+2q} - c_5, \qquad \forall s \in \mathbb{R}.$$

Remark 2.3 *The usual double well potential $F(s) = \frac{1}{4}(s^2 - 1)^2$ satisfies all the hypotheses on F.*

Remark 2.4 *One easily realizes that (H4) implies*
$$|F'(s)| \leq c(|F(s)| + 1), \quad \forall s \in \mathbb{R};$$
furthermore (H3) implies that
$$|F(s)| \leq F(s) + 2\max\{0, c_2\}, \quad \forall s \in \mathbb{R}.$$

Remark 2.5 *Note that (H5) allows, in particular, $\eta = 0$. Thus the so-called Cahn-Hilliard-Stokes system is also included (see [30]).*

Remark 2.6 *The convective nonlocal CH equation can formally be rewritten as follows*
$$\varphi_t = \nabla \cdot \big((F''(\varphi) + a)\nabla\varphi\big) + \nabla \cdot \big(\nabla a\varphi - \boldsymbol{u}\varphi\big) - \nabla J * \varphi$$
from which the crucial role of (H2) is evident, namely, we are dealing with a convection-diffusion integrodifferential equation.

2.3 Statement of the main results

Let us introduce the definition of weak solution to (1.2)–(1.3).

Definition 2.1 *Let $T > 0$ be given and let $\varphi_0 \in H$ be such that $F(\varphi_0) \in L^1(\Omega)$. A pair $(\varphi, \boldsymbol{u})$ is a weak solution to (1.2)–(1.3) on $[0, T]$ if*

$$\begin{aligned}
&\varphi \in C([0,T]; H) \cap L^2(0,T;V)\\
&\varphi_t \in L^2(0,T;V')\\
&\mu = a\varphi - J * \varphi + F'(\varphi) \in L^2(0,T;V)\\
&\boldsymbol{u} \in L^2(0,T;\, \boldsymbol{V})
\end{aligned}$$

and it satisfies

$$\langle \varphi_t,\, \psi\rangle + (\nabla\mu,\, \nabla\psi) = (\boldsymbol{u}\varphi,\, \nabla\psi), \quad \forall \psi \in V, \quad a.e.\ in\ (0,T), \tag{2.2}$$

$$(\nu(\varphi)\nabla\boldsymbol{u},\, \nabla\boldsymbol{v}) + (\eta\boldsymbol{u},\, \boldsymbol{v}) = (\mu\nabla\varphi,\, \boldsymbol{v}) + \langle \boldsymbol{h},\, \boldsymbol{v}\rangle, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \quad a.e.\ in\ (0,T), \tag{2.3}$$

$$\varphi(0) = \varphi_0, \quad a.e.\ in\ \Omega. \tag{2.4}$$

Remark 2.7 *Observe that if we choose $\psi = 1$ in (2.2) we obtain*

$$\frac{d}{dt}\bar{\varphi} = 0.$$

Thus the total mass of any weak solution is conserved.

Global existence of a weak solution is given by

Theorem 2.2 *Let $\varphi_0 \in H$ be such that $F(\varphi_0) \in L^1(\Omega)$ and suppose that (H0)-(H7) are satisfied. Then there exists a weak solution $(\varphi, \boldsymbol{u})$ to (1.2)–(1.3). Furthermore, $F(\varphi)$ is in $L^\infty(0,T;L^1(\Omega))$ and setting*

$$\mathcal{E}(\varphi(t)) = \frac{1}{4}\int_\Omega\int_\Omega J(x-y)(\varphi(x,t) - \varphi(y,t))^2\, dx\, dy + \int_\Omega F(\varphi(x,t))\, dx. \tag{2.5}$$

the following energy equality holds for almost every $t \in (0,T)$

$$\frac{d}{dt}\mathcal{E}(\varphi(t)) + \|\nabla\mu\|^2 + \|\sqrt{\nu(\varphi)}\nabla\boldsymbol{u}\|^2 + \|\sqrt{\eta}\boldsymbol{u}\|^2 = \langle \boldsymbol{h}, \boldsymbol{u}\rangle. \tag{2.6}$$

Furthermore, we have

Corollary 2.1 *Let (H0)-(H6) hold. If $\boldsymbol{h} \in L^\infty(0,T;\, \boldsymbol{V}')$ for some $T > 0$. Then, any weak solution $(\varphi, \boldsymbol{u})$ to (1.2)–(1.3) is such that*

$$\varphi \in L^4(0,T;L^4(\Omega)), \qquad \boldsymbol{u} \in L^\infty(0,T;V).$$

Weak solutions can be regular provided φ_0 is bounded. Indeed we have

Proposition 2.1 *Let the assumptions of Theorem 2.2 hold. If $\varphi_0 \in L^\infty(\Omega)$ then, any solution $(\varphi, \boldsymbol{u})$ to problem (1.2) on $[0,T]$ corresponding to φ_0 satisfies*

$$\varphi, \mu \in L^\infty(\Omega \times (0,T)).$$

In particular, we have

$$\|\varphi\|_{L^\infty(\Omega\times(0,T))} \le M, \qquad \|\mu\|_{L^\infty(\Omega\times(0,T))} \le M,$$

for some $M > 0$, independent of ν and T.

If the viscosity ν is constant then we have a continuous dependence estimate

Proposition 2.2 *Let hypotheses (H0)-(H5) hold. Suppose that ν is a positive constant and $\boldsymbol{h} \in L^\infty(0,T;\boldsymbol{V}')$. Consider two weak solutions to (1.2)–(1.3), namely $(\varphi_1, \boldsymbol{u}_1)$ and $(\varphi_2, \boldsymbol{u}_2)$, corresponding to the initial data $\varphi_{1,0}$ and $\varphi_{2,0}$, respectively. Here $\varphi_{i,0} \in L^2(\Omega)$ and $F(\varphi_{i,0}) \in L^1(\Omega)$, $i = 1,2$. Then there exists $N = N(T) > 0$ such that, for any $t \in [0,T]$,*

$$\|\varphi_1(t) - \varphi_2(t)\|_\#^2 + \int_0^t \|\boldsymbol{u}_1(y) - \boldsymbol{u}_2(y)\|_{\boldsymbol{V}}^2 dy \le N\big(\|\varphi_{1,0} - \varphi_{2,0}\|_\#^2 + |\bar{\varphi}_{1,0} - \bar{\varphi}_{2,0}|\big). \tag{2.7}$$

In particular, (1.2)–(1.3) has a unique weak solution.

The limit $\nu \to 0$. As a second step in our analysis we study the limit of (1.2)–(1.3) with constant viscosity ν, as ν tends to 0. We recall that the resulting limit system is (1.4)–(1.5) whose weak formulation is given by the following definition.

Definition 2.3 *Let $T > 0$ be given and let $\varphi_0 \in L^\infty(\Omega)$.*

A pair $(\varphi, \boldsymbol{u})$ is a weak solution to (1.4)–(1.5) on $(0,T)$ if

$$\begin{aligned}
&\varphi \in L^\infty(\Omega \times (0,T)) \cap L^2(0,T;V)\\
&\varphi_t \in L^2(0,T;V')\\
&\mu = a\varphi - J * \varphi + F'(\varphi) \in L^2(0,T;V)\\
&\boldsymbol{u} \in L^2(0,T;\boldsymbol{H})
\end{aligned}$$

and it satisfies

$$\langle \varphi_t, \psi\rangle + (\nabla\mu, \nabla\psi) = (u\varphi, \nabla\psi), \quad \forall \psi \in V, \quad a.e.\ in\ (0,T), \tag{2.8}$$

$$(\eta\boldsymbol{u}, \boldsymbol{v}) = (\mu\nabla\varphi, \boldsymbol{v}) + (\boldsymbol{h}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{H}, \quad a.e.\ in\ (0,T), \tag{2.9}$$

$$\varphi(0) = \varphi_0, \quad a.e.\ in\ \Omega.$$

To analyze (1.4)–(1.5) we replace assumption (H5) with the stronger

(H8) $\eta \in L^\infty(\Omega)$ and there exists $\eta_0 > 0$ such that

$$\eta(x) \geq \eta_0, \quad \text{a.e. } x \in \Omega.$$

Furthermore, for the sake of simplicity, we let $\boldsymbol{h} = \mathbf{0}$. Then we have the following existence theorem

Theorem 2.4 *Let (H0)-(H4), (H8) hold and let $\varphi_0 \in L^\infty(\Omega)$. Then, for any given $T > 0$, if $\{\nu_k\}$ is a sequence of positive constants converging to 0, the weak solution to (1.2)–(1.3) with $\nu = \nu_k$ converges, up to a subsequence, to a weak solution $(\varphi, \boldsymbol{u})$ to (1.4)–(1.5). More precisely, we have*

$$\begin{aligned} \varphi_k \to \varphi \quad & \textit{strongly in } L^2(0,T;H) \\ \boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \quad & \textit{weakly in } L^2(0,T;\boldsymbol{H}) \end{aligned}$$

Furthermore, the following energy equality holds for almost any $t \in (0,T)$:

$$\frac{d}{dt}\mathcal{E}(\varphi(t)) + \|\nabla\mu\|^2 + \|\sqrt{\eta}\boldsymbol{u}\|^2 = 0, \tag{2.10}$$

where $\mathcal{E}$ is defined by (2.5).

Next corollary is related to further regularity in the case where η is constant.

Corollary 2.2 *Let the assumptions of Theorem 2.4 hold and η be a positive constant, then $\boldsymbol{u} \in L^\infty(0,T;[L^p(\Omega)]^d)$ for each $p \geq 1$.*

This fact allows us to prove uniqueness of the (weak) solution to (1.4)–(1.5) for constant parameter η. More precisely, we have

Proposition 2.3 *Let the assumptions of Corollary 2.2 hold. Consider two weak solutions to (1.4)–(1.5), namely $(\varphi_1, \boldsymbol{u}_1)$, $(\varphi_2, \boldsymbol{u}_2)$ corresponding to bounded initial data $\varphi_{1,0}$, $\varphi_{2,0}$, respectively. Then there exists $N = N(T) > 0$ such that, for every $t \in [0,T]$,*

$$\|\varphi_1 - \varphi_2\|_\#^2 + \int_0^t \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{H}}^2 \leq N\big(\|\varphi_{1,0} - \varphi_{2,0}\|_\#^2 + |\bar{\varphi}_{1,0} - \bar{\varphi}_{2,0}|\big).$$

In particular, there exists a unique bounded weak solution to (1.4)–(1.5).

In case J is more regular, we gain regularity also for the velocity field $\boldsymbol{u}$. For the sake of completeness, we first recall the definition of admissible kernel (see [2, Definition 1]).

Definition 2.5 *A kernel* $J \in W^{1,1}_{loc}(\mathbb{R}^d)$*,* $d = 2, 3$ *is admissible if the following conditions are satisfied:*

- $J \in C^3(\mathbb{R}^d \setminus \{0\})$*;*
- J *is radially symmetric, i.e.,* $J(x) = \tilde{J}(|x|)$ *with* $\tilde{J}$ *non-increasing;*
- $\tilde{J}''(r)$ *and* $\tilde{J}'(r)/r$ *are monotone on* $(0, r_0)$ *for some* $r_0 > 0$*;*
- $|D^3 J(x)| \le C|x|^{-(d+1)}$ *for some* $C > 0$*.*

Then we state the following regularity result

Proposition 2.4 *Let the assumptions of Theorem 2.4 hold,* η *be constant and* J *be admissible or* $J \in W^{2,1}$*. Then*

$$\boldsymbol{u} \in L^2(0, T; \boldsymbol{V}).$$

Thanks to the above regularity result we can obtain an estimate of the difference between a solution to (1.2)–(1.3) and a solution to (1.4)–(1.5). Indeed we have

Theorem 2.6 *Let (H0), (H2)-(H4), (H8) hold. Suppose* ν*,* η *constant,* $\mathbf{h} = \mathbf{0}$*, and* J *either be admissible or* $J \in W^{2,1}(\mathbb{R}^2)$*. Take* $\varphi_0^\nu, \varphi_0 \in L^\infty(\Omega)$ *and*

$$R := \sup_{\nu > 0}\{\|\varphi_0^\nu\|_{L^\infty}, \|\varphi_0\|_{L^\infty}\} < \infty.$$

Let $(\varphi_\nu, \boldsymbol{u}_\nu)$ *be the unique weak solution to* (1.2)–(1.3) *with initial datum* φ_0^ν*, and* $(\varphi, \boldsymbol{u})$ *the unique solution to* (1.4)–(1.5) *with initial datum* φ_0*. Then, for any given* $T > 0$*, there exists* $C_{R,T} > 0$ *such that*

$$\|\varphi_\nu(t) - \varphi(t)\|_\#^2 + \int_0^t \|\boldsymbol{u}_\nu(y) - \boldsymbol{u}(y)\|^2 \, dy \le \big(\|\varphi_0^\nu - \varphi_0\|_\#^2 + |\bar{\varphi}_0^\nu - \bar{\varphi}_0|\big) e^{C_{R,T}} + C_{R,T}\nu,$$

for each $t \in [0, T]$*. In particular, if* $\varphi_0^\nu = \varphi_0$*, then* $\varphi_\nu \to \varphi$ *in* $L^\infty(0, T; V')$ *and in* $L^2(0, T; H)$ *as* $\nu \to 0$*.*

3 Existence and regularity for the CHB system

The first part of this section is devoted to prove Theorem 2.2. Then, in the second part, the proofs of Corollary 2.1 and Proposition 2.1 are given.

Proof of Theorem 2.2

The proof will be carried out by means of a Faedo–Galerkin approximation scheme, following closely [5]. We first prove existence of a solution when $\varphi_0 \in V_2$ and $\boldsymbol{h} \in C([0,T];\mathbf{H})$; then, by a density argument, we will recover the same result for any initial datum $\varphi_0 \in H$ with $F(\varphi_0) \in L^1(\Omega)$ and any $\boldsymbol{h} \in L^2([0,T];\mathbf{V}')$.

We consider the families $\{\psi_j\}_{j\in\mathbb{N}} \subset V_2$ and $\{\boldsymbol{v}_j\}_{j\in\mathbb{N}} \subset \mathbf{V}$ respectively eigenvectors of $A+I : V_2 \to H$ and of the Stokes operator, which are both self-adjoint, positive and linear. Let us define the n-dimensional subspaces $\Psi_n := \langle \psi_1, ..., \psi_n\rangle$ and $\mathcal{W}_n := \langle \mathbf{w}_1, ..., \mathbf{w}_n\rangle$ with the related orthogonal projectors on this subspace $P_n := P_{\Psi_n}$ and $\tilde{P}_n := P_{\mathcal{W}_n}$. We then look for three functions of the following form:

$$\varphi_n(t) = \sum_{k=1}^{n} b_k^{(n)}(t)\psi_k, \qquad \mu_n(t) = \sum_{k=1}^{n} c_k^{(n)}(t)\psi_k, \qquad \boldsymbol{u}_n(t) = \sum_{k=1}^{n} d_k^{(n)}(t)\mathbf{w}_k$$

that solve the following discretized problem

$$(\varphi_n', \psi) + (\nabla\rho_n, \nabla\psi) = (\boldsymbol{u}_n\varphi_n, \nabla\psi) + (\nabla J * \varphi_n, \nabla\psi) \tag{3.1}$$

$$(\nu(\varphi_n)\nabla\boldsymbol{u}_n,\ \nabla\mathbf{w}) + (\eta\boldsymbol{u}_n,\ \mathbf{w}) + (\varphi_n\nabla\mu_n,\ \mathbf{w}) = \langle\boldsymbol{h},\ \mathbf{w}\rangle \tag{3.2}$$

$$\rho_n := a(\cdot)\varphi_n + F'(\varphi_n), \tag{3.3}$$

$$\mu_n = P_n(\rho_n - J * \varphi_n), \tag{3.4}$$

$$\varphi_n(0) = \varphi_{0n}, \tag{3.5}$$

for every $\psi \in \Psi_n$, every $\mathbf{w} \in \mathcal{W}_n$ and where $\varphi_{0n} := P_n\varphi_0$.

By using the definition of φ_n, μ_n and $\boldsymbol{u}_n$, problem (3.1)–(3.5) becomes equivalent to a Cauchy problem for a system of ordinary differential equations in the n unknowns $b_i^{(n)}$. Thanks to (H2), the Cauchy-Lipschitz theorem yields that there exists a unique solution $b^{(n)} \in C^1([0,T_n^*];\mathbb{R}^n)$ for some maximal time $T_n^* \in (0,+\infty]$.

Let us show that $T_n^* = +\infty$, for all $n \geq 1$. Indeed, using $\psi = \mu_n$ as test function in (3.1) and $\mathbf{w} = \boldsymbol{u}_n$ in (3.2) we get the following identity:

$$(\varphi_n', \mu_n) + (\nabla\rho_n, \nabla\mu_n) + \|\sqrt{\nu(\varphi_n)}\nabla\boldsymbol{u}_n\|^2 + \|\sqrt{\eta}\boldsymbol{u}_n\|^2 = (\nabla J * \varphi_n, \nabla\mu_n) + \langle\boldsymbol{h}, \boldsymbol{u}_n\rangle. \tag{3.6}$$

Let us first notice that

$$(\varphi_n', \mu_n) = \frac{d}{dt}\Big(\frac{1}{4}\int_\Omega\int_\Omega J(x-y)(\varphi_n(x) - \varphi_n(y))^2 + \int_\Omega F(\varphi_n)\Big), \tag{3.7}$$

$$(\nabla\mu_n, \nabla P_n(J * \varphi_n)) \leq \frac{1}{4}\|\nabla\mu_n\|^2 + \|\varphi_n\|^2\|J\|^2_{W^{1,1}}, \tag{3.8}$$

$$(\nabla J * \varphi_n, \nabla\mu_n) \leq \frac{1}{4}\|\nabla\mu_n\|^2 + \|\varphi_n\|^2\|J\|^2_{W^{1,1}}. \tag{3.9}$$

By means of (H3), we can deduce the existence of a positive constant α such that

$$\begin{aligned}&\frac{1}{2}\int_\Omega\int_\Omega J(x-y)(\varphi_n(x)-\varphi_n(y))^2\,dx\,dy+2\int_\Omega F(\varphi_n)\\&=\|a\varphi_n\|^2+2\int_\Omega F(\varphi_n)-(\varphi_n,J*\varphi_n)\geq\alpha\Big(\|\varphi_n\|^2+\int_\Omega F(\varphi_n)\Big)-c.\end{aligned}\tag{3.10}$$

By using (H6) and Poincaré's inequality, it is easy to show that there exists $\beta>0$ such that

$$\beta\|\boldsymbol{u}_n\|_{\mathbf{V}}^2\leq\|\sqrt{\nu}\nabla\boldsymbol{u}_n\|^2,\tag{3.11}$$

and, on account of (H7,) we have

$$\langle\boldsymbol{h},\boldsymbol{u}_n\rangle\leq c\|\boldsymbol{h}\|_{\mathbf{V}'}^2+\frac{\beta}{2}\|\boldsymbol{u}_n\|_{\mathbf{V}}^2.\tag{3.12}$$

Let us now exploit (3.7) in (3.6) and integrate it with respect to time between 0 and $t\in(0,T_n^*)$. Taking (3.8)–(3.12) into account, we find

$$\begin{aligned}&\alpha\Big(\|\varphi_n\|^2+\int_\Omega F(\varphi_n)\Big)+\int_0^t\Big(\frac{\beta}{2}\|\boldsymbol{u}_n(\tau)\|_{\mathbf{V}}^2+\|\sqrt{\eta}\boldsymbol{u}_n(\tau)\|^2+\|\nabla\mu_n(\tau)\|^2\Big)\,d\tau\\&\leq M+K\int_0^t\Big(\|\varphi_n(\tau)\|^2+\int_\Omega F(\varphi_n(\tau))\Big)\,d\tau,\end{aligned}\tag{3.13}$$

which holds for all $t\in[0,T_n^*)$, where

$$M=c\Big(1+\|\varphi_0\|^2+\int_\Omega F(\varphi_0)+\|\boldsymbol{h}\|_{L^2(0,T;\mathbf{V}')}^2\Big),$$

and $K=2\|J\|_{W^{1,1}}^2$. Here, we have used the fact that that φ_0 and $\varphi_{0,n}$ are supposed to belong to V_2. We point out that M and K do not depend on n.

Thus, inequality (3.13) entails that $T_n^*=+\infty$, for all $n\geq1$. As a consequence, (3.1)–(3.5) has a unique global-in-time solution. Furthermore, we obtain the following estimates, holding for any given $0<T<+\infty$:

$$\|\varphi_n\|_{L^\infty(0,T;H)}\leq N\tag{3.14}$$

$$\|\nabla\mu_n\|_{L^2(0,T;H)}\leq N\tag{3.15}$$

$$\|F(\varphi_n)\|_{L^\infty(0,T;L^1(\Omega))}\leq N\tag{3.16}$$

$$\|\boldsymbol{u}_n\|_{L^2(0,T;\mathbf{V})}\leq\frac{N}{\sqrt{\nu_0}}\tag{3.17}$$

where N is independent of n. Observe that, in light of (H3), (3.16) implies

$$\|\varphi_n\|_{L^\infty(0,T;L^{2+2q}(\Omega))} \leq N \tag{3.18}$$

Thanks to (H2), recalling (3.4), we get

$$\begin{aligned}\frac{c_0}{4}\|\nabla\varphi_n\|^2 + \frac{1}{c_0}\|\nabla\mu_n\|^2 &\geq (a\nabla\varphi_n + \varphi_n\nabla a + F''(\varphi_n)\nabla\varphi_n - \nabla J * \varphi_n, \nabla\varphi_n)\\ &\geq c_0\|\nabla\varphi_n\|^2 - 2\|\nabla J\|_{L^1}\|\nabla\varphi_n\|\|\varphi_n\|\\ &\geq \frac{c_0}{2}\|\nabla\varphi_n\|^2 - c\|\varphi_n\|^2,\end{aligned}$$

thus (3.14) and (3.15) yield

$$\|\varphi_n\|_{L^2(0,T;V)} \leq N. \tag{3.19}$$

The next step is to deduce a (uniform) bound for μ_n in $L^2(0,T;V)$. Thanks to Remark 2.4 and to the identity

$$(P_n(-J * \varphi_n + a\varphi_n), 1) = (-J * \varphi_n + a\varphi_n, 1) = 0$$

we get

$$\left|\int_\Omega \mu_n\right| = \left|(F'(\varphi_n), 1)\right| \leq \int_\Omega |F'(\varphi_n)| \leq c\int_\Omega F(\varphi_n) + c \leq N. \tag{3.20}$$

The Poincaré inequality implies

$$\left\|\mu_n - \frac{1}{|\Omega|}\int_\Omega \mu_n\right\| \leq c\|\nabla\mu_n\|, \tag{3.21}$$

and from (3.15) and (3.20) we deduce that

$$\|\mu_n\|_{L^2(0,T;V)} \leq N. \tag{3.22}$$

Observe now that, calling $\tilde\rho_n = P_n\rho_n$,

$$\|\tilde\rho_n\|_V^2 = \|\mu_n + P_n(J * \varphi_n)\|_V^2 \leq 2\|\mu_n\|_V^2 + 2(\|J\|_{L^1}^2 + \|\nabla J\|_{L^1}^2)\|\varphi_n\|^2,$$

so that from (3.22) we immediately get

$$\|\tilde\rho_n\|_{L^2(0,T;V)} \leq N. \tag{3.23}$$

Furthermore, recalling (3.3) and invoking (H4), we obtain

$$\|\rho_n\|_{L^p} \leq ca^*\|\varphi_n\| + \|F'(\varphi_n)\|_{L^p} \leq cN + c\Big(\int_\Omega |F(\varphi_n)|\Big)^{1/p} \leq N,$$

which yields the bound

$$\|\rho_n\|_{L^\infty(0,T;L^p(\Omega))} \leq N. \tag{3.24}$$

We finally provide an estimate for the sequence φ_n'. We take a generic test function $\psi \in V$ and we write it as $\psi = \psi_1 + \psi_2$, where $\psi_1 = P_n\psi \in \Psi_n$ and $\psi_2 = \psi - \psi_1 \in \Psi_n^\perp$. It is easy to see that

$$|(\nabla\rho_n, \nabla\psi_1)| \leq \|\nabla\tilde{\rho}_n\|\|\nabla\psi_1\| \leq \|\nabla\tilde{\rho}_n\|\|\nabla\psi\|_V, \tag{3.25}$$

and

$$|(\boldsymbol{u}_n\varphi_n, \nabla\psi_1)| \leq \|\boldsymbol{u}_n\|_{[L^{\frac{2+2q}{q}}]^d}\|\nabla\psi_1\|\|\varphi_n\|_{L^{2+2q}} \leq N\|\boldsymbol{u}_n\|_{\mathbf{V}}\|\psi\|_V, \qquad d=2, \tag{3.26}$$

$$|(\boldsymbol{u}_n\varphi_n, \nabla\psi_1)| \leq \|\boldsymbol{u}_n\|_{[L^6]^d}\|\nabla\psi_1\|\|\varphi_n\|_{L^3} \leq N\|\boldsymbol{u}_n\|_{\mathbf{V}}\|\psi\|_V, \qquad d=3. \tag{3.27}$$

By using Young's lemma we infer

$$\left|\int_\Omega \nabla J * \varphi_n \nabla\psi_1\right| \leq \|\psi\|_V\|\nabla J\|_{L^1}\|\varphi_n\| \leq N\|\nabla J\|_{L^1}\|\psi\|_V. \tag{3.28}$$

From (3.1), owing to (3.25)-(3.28), we have that

$$|(\varphi_n', \psi)| \leq N(1 + \|\nabla\rho_n\| + \|\boldsymbol{u}_n\|_{\mathbf{V}})\|\psi\|_V, \tag{3.29}$$

which gives

$$\|\varphi_n'\|_{L^2(0,T;V')} \leq N, \tag{3.30}$$

owing to (3.17) and (3.23). Collecting estimates (3.14), (3.19), (3.22)–(3.24), (3.30), we find

$$\begin{aligned}
&\varphi \in L^\infty(0,T;L^{2+2q}(\Omega)) \cap L^2(0,T;V) \cap H^1(0,T;V'),\\
&\mu \in L^2(0,T;V),\\
&\tilde{\rho} \in L^2(0,T;V),\\
&\rho \in L^\infty(0,T;L^p(\Omega)),\\
&\boldsymbol{u} \in L^2(0,T;\mathbf{V}),
\end{aligned}$$

such that, up to a subsequence,

$$\varphi_n \rightharpoonup \varphi \quad \text{weakly* in } L^\infty(0,T;H), \tag{3.31}$$

$$\varphi_n \rightharpoonup \varphi \quad \text{weakly in } L^2(0,T;V), \tag{3.32}$$

$$\varphi_n \to \varphi \quad \text{strongly in } L^\gamma(0,T;H) \text{ and a.e. in } \Omega\times(0,T), \tag{3.33}$$

$$\mu_n \rightharpoonup \mu \quad \text{weakly in } L^2(0,T;V), \tag{3.34}$$

$$\tilde{\rho}_n \rightharpoonup \tilde{\rho} \quad \text{weakly in } L^2(0,T;V), \tag{3.35}$$

$$\rho_n \rightharpoonup \rho \quad \text{weakly* in } L^\infty(0,T;L^p(\Omega)), \tag{3.36}$$

$$\varphi_n' \rightharpoonup \varphi_t \quad \text{weakly in } L^2(0,T;V'), \tag{3.37}$$

$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \quad \text{weakly in } L^2(0,T;\mathbf{V}). \tag{3.38}$$

Here $\gamma = 2 + 2q$ if $d = 2$, $\gamma = \min\{2 + 2q, 4\}$ if $d = 3$. We now pass to the limit in (3.1)–(3.5) in order to prove that $(\varphi, \boldsymbol{u})$ is a weak solution to CHB system according to Definition 2.1. First of all, from the pointwise convergence (3.33) we have $\rho_n \to a\varphi + F'(\varphi)$ almost everywhere in $\Omega \times (0, T)$, therefore from (3.36) we have $\rho = a\varphi + F'(\varphi)$. Now, for every $\phi \in \Psi_j$, every $j \leq n$ with j fixed and for every $\chi \in C_0^\infty(0, T)$, we have that

$$\int_0^T (\rho_n, \phi)\chi(t) = \int_0^T (\tilde{\rho}_n, \phi)\chi(t).$$

Passing to the limit in this equation, using (3.35) and (3.36), and on account of the density of $\{\Psi_j\}_{j\geq 1}$ in H, we get $\tilde{\rho}(\cdot, \varphi) = \rho(\cdot, \varphi) = a\varphi + F'(\varphi)$. Moreover, since $\mu_n = P_n(\rho_n - J * \varphi_n)$, then, for every $\phi \in \Psi_j$, every $k \leq j$ with j fixed and for every $\chi \in C_0^\infty(0, T)$, there holds

$$\int_0^T (\mu_n(t), \phi)\chi(t)dt = \int_0^T (\rho_n - J * \varphi_n, \phi)\chi(t)dt.$$

By passing to the limit in the above identity, and using the convergences (3.33), (3.34) and (3.36), we eventually get

$$\mu = a\varphi - J * \varphi + F'(\varphi) = \rho - J * \varphi.$$

It still remains to pass to the limit in (3.1) and (3.2) in order to recover (2.2), (2.3) and initial condition (2.4). This can be obtained in a standard way, so we refer the reader to [5, Proof of Theorem 1] where all the technicalities are detailed. In order to conclude to proof, let us now assume that $\varphi_0 \in H$ with $F(\varphi_0) \in L^1(\Omega)$ and $\mathbf{h} \in L^2(0, T; \mathbf{V}')$. In this case, we first choose an approximating sequence of initial data $\varphi_{0n} \in V_2$ such that $\varphi_{0n} \to \varphi_0$ in H, and a sequence $\mathbf{h}_n \in C(0, T; \mathbf{H})$ in such a way that $\mathbf{h}_n \to \mathbf{h}$ in $L^2(0, T; \mathbf{V}')$. Then, arguing as in [5, Proof of Theorem 1] the existence of a solution to (1.2)–(1.3) is obtained by passing to the limit $n \to \infty$. In particular, on account of (3.10)-(3.12), we find that $F(\varphi) \in L^\infty(0, T; L^1(\Omega))$.

We are left to prove the energy identity (2.6). Let us take $\psi = \mu(t)$ in equation (2.2). This yields

$$\langle \varphi_t, \mu \rangle + \|\sqrt{\nu}\nabla \boldsymbol{u}\|^2 + \|\sqrt{\eta}\boldsymbol{u}\|^2 + \|\nabla\mu\|^2 = \langle \boldsymbol{h}, \boldsymbol{u} \rangle. \tag{3.39}$$

By arguing as in [5, proof of Corollary 2], one can prove the identity

$$\langle \varphi_t, \mu \rangle = \langle \varphi_t, a\varphi + F'(\varphi) - J * \varphi \rangle = \frac{d}{dt}\mathcal{E}(\varphi(t))$$

which holds for almost every $t > 0$. Thus (2.6) follows directly from (3.39). □

Proof of Corollary 2.1

We recall that a standard application of the Gagliardo-Nirenberg inequality gives

$$\|\varphi\|_{L^4} \leq \|\varphi\|^{1/2}\|\nabla\varphi\|^{1/2}, \qquad d = 2,$$
$$\|\varphi\|_{L^4} \leq \|\varphi\|_{L^3}^{1/2}\|\nabla\varphi\|^{1/2}, \qquad d = 3.$$

On account of Theorem 2.2, we have $\varphi \in L^2(0,T;V)$. Moreover, owing to (H3), we have $\varphi \in L^\infty(0,T;L^2(\Omega))$ if $d = 2$ and $\varphi \in L^\infty(0,T;L^3(\Omega))$ if $d = 3$. Then we easily deduce

$$\int_0^T \|\varphi\|_{L^4}^4 \leq N \int_0^T \|\varphi\|_V^2 \leq N.$$

In order to prove the estimate for $\boldsymbol{u}$, let us first recall the following identity (see [13, Proof of Thm. 2])

$$(\mu\nabla\varphi, \boldsymbol{u}) = (\nabla J * \varphi, \varphi\boldsymbol{u}) - (\frac{1}{2}\nabla a\varphi^2, \boldsymbol{u}). \tag{3.40}$$

Thanks to (3.40), equation (2.3) with $\boldsymbol{v} = \boldsymbol{u}$ can be rewritten as follows

$$\|\sqrt{\nu}\nabla\boldsymbol{u}\|^2 + \|\sqrt{\eta}\boldsymbol{u}\|^2 = (\nabla J * \varphi,\, \varphi\boldsymbol{u}) - \frac{1}{2}(\nabla a\varphi^2,\, \boldsymbol{u}) + \langle \boldsymbol{h},\, \boldsymbol{u}\rangle. \tag{3.41}$$

Observe now that

$$\begin{aligned}
&(\nabla J * \varphi,\, \varphi\boldsymbol{u}) - \frac{1}{2}(\nabla a\varphi^2,\, \boldsymbol{u}) \\
&\leq \left(\frac{1}{2}\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1}\right)\|\varphi\|\|\varphi\|_{L^{2+2q}}\|\boldsymbol{u}\|_{L^{\frac{2+2q}{q}}}, \qquad d = 2,\\
&(\nabla J * \varphi,\, \varphi\boldsymbol{u}) - \frac{1}{2}(\nabla a\varphi^2,\, \boldsymbol{u}) \\
&\leq \left(\frac{1}{2}\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1}\right)\|\varphi\|\|\varphi\|_{L^3}\|\boldsymbol{u}\|_{L^6}, \qquad d = 3,
\end{aligned}$$

and, as $\varphi \in L^\infty(0,T;L^{2+2q}(\Omega))$ when $d = 2$, we obtain

$$(\nabla J * \varphi,\, \varphi\boldsymbol{u}) - \frac{1}{2}(\nabla a\varphi^2,\, \boldsymbol{u}) \leq N\|\boldsymbol{u}\|_{\mathbf{V}}.$$

On the other hand we get (cf. (H6))

$$\|\sqrt{\nu}\nabla\boldsymbol{u}\|^2 \geq \nu_0\|\nabla\boldsymbol{u}\|^2 \geq c\|\boldsymbol{u}\|_{\mathbf{V}}^2.$$

Hence, by (H8) and (3.41), we end up with

$$c\|\boldsymbol{u}\|_{\mathbf{V}}^2 \leq N\|\boldsymbol{u}\|_{\mathbf{V}}$$

which yields $\boldsymbol{u} \in L^\infty(0,T;\mathbf{V})$. □

Proof of Proposition 2.1

In order to prove that $\varphi \in L^\infty(\Omega \times (0,T))$ we can use a Moser-Alikakos type argument (see [15, Proof of Thm. 3] for the details). The boundedness of μ follows from its definition by comparison. □

4 Existence and regularity for CHHS system

Proof of Theorem 2.4

Let $(\varphi_k, \boldsymbol{u}_k)$ be the solution of problem (1.2) with $\nu = \nu_k$, thus satisfying (2.5). Therefore, for every $k \geq 1$ we have

$$\mathcal{E}(\varphi_k(t)) + \int_0^t \big(\|\nabla\mu_k\|^2 + \|\sqrt{\nu}\nabla\boldsymbol{u}\|^2 + \|\sqrt{\eta}\boldsymbol{u}\|^2\big) = \mathcal{E}(\varphi_0)$$

and thanks to (3.10) it is possible to deduce (3.14)–(3.16) and

$$\|\boldsymbol{u}_k\|_{L^2(0,T;\mathbf{V})} \leq \frac{N}{\sqrt{\nu}_k} \tag{4.1}$$

$$\|\boldsymbol{u}_k\|_{L^2(0,T;\mathbf{H})} \leq N. \tag{4.2}$$

Furthermore, by arguing as in the proof of Theorem 2.2, it is possible to recover (3.19) and (3.22). Then from Proposition 2.1 we deduce the following bound

$$\|\varphi_k\|_{L^\infty(\Omega\times(0,T))} \leq N. \tag{4.3}$$

Also, we observe that

$$(\nabla\mu_k,\, \nabla\psi) \leq \|\nabla\mu\|\|\nabla\psi\| \tag{4.4}$$

and (see (4.3))

$$(\boldsymbol{u}_k\varphi_k, \nabla\psi) \leq \|\boldsymbol{u}_k\|_{\mathbf{H}}\|\varphi\|_{L^\infty}\|\nabla\psi\|. \tag{4.5}$$

By exploiting (4.4)–(4.5) in (2.2) we deduce (3.30) by comparison. We recall that N does not depend neither on k nor on ν_k. Summing up, we deduce the existence of

$$\begin{aligned}
&\varphi \in L^\infty(\Omega \times (0,T)) \cap L^2(0,T;V) \cap H^1(0,T;V'),\\
&\mu \in L^2(0,T;V),\\
&\boldsymbol{u} \in L^2(0,T;\mathbf{H}),
\end{aligned}$$

such that, up to a subsequence,

$$\varphi_k \rightharpoonup \varphi \quad \text{weakly* in } L^\infty(\Omega \times (0,T)), \tag{4.6}$$

$$\varphi_k \rightharpoonup \varphi \quad \text{weakly in } L^2(0,T;V), \tag{4.7}$$

$$\varphi_k \to \varphi \quad \text{strongly in } L^2(0,T;L^\beta(\Omega)) \text{ and a.e. in } \Omega \times (0,T), \tag{4.8}$$

$$\mu_k \rightharpoonup \mu \quad \text{weakly in } L^2(0,T;V), \tag{4.9}$$

$$\varphi_k' \rightharpoonup \varphi_t \quad \text{weakly in } L^2(0,T;V'), \tag{4.10}$$

$$\boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \quad \text{weakly in } L^2(0,T;\mathbf{H}). \tag{4.11}$$

Here β is such that $\frac{1}{2} = \frac{1}{d+\varepsilon} + \frac{1}{\beta}$ for some $\varepsilon > 0$.

It is now possible to pass to the limit as $k \to \infty$ in the weak formulation of (1.2)–(1.3). We will do that restricting ourselves to the case $\psi \in W^{1,d+\varepsilon}(\Omega) \subset V$ in (2.2) and then recovering the fact that (2.8) holds for every $\psi \in V$ by a density argument. Some attention is needed when passing to the limit in the viscous term of the Brinkman equation; as a matter of fact we have

$$\nu_k(\nabla \boldsymbol{u}_k,\, \nabla \boldsymbol{v}) \le \nu_k \|\nabla \boldsymbol{u}_k\| \|\nabla \boldsymbol{v}\| \le \sqrt{\nu_k} N \|\nabla \boldsymbol{v}\|$$

which tends to 0 as $\nu_k \to 0$. The convective term can be treated as follows:

$$\int_t^{t+r} (\boldsymbol{u}_k \varphi_k - \boldsymbol{u}\varphi,\, \nabla\psi) = \int_t^{t+r} (\boldsymbol{u}_k(\varphi_k - \varphi),\, \nabla\psi) + \int_t^{t+r} ((\boldsymbol{u}_k - \boldsymbol{u})\varphi,\, \nabla\psi)$$

where $r \ge 0$ is arbitrary. Here the second term vanishes thanks to the boundedness of φ and (4.11). The first one goes to 0 thanks to (4.2), (4.8) and the fact that

$$\int_t^{t+r} (\boldsymbol{u}_k(\varphi_k - \varphi),\, \nabla\psi) \le \|\varphi - \varphi_k\|_{L^2(0,T;L^\beta)} \|\boldsymbol{u}_k\|_{L^2(0,T;\mathbf{H})} \|\nabla\psi\|_{L^{d+\varepsilon}}.$$

Finally, we can pass to the limit into the the Korteweg force since, for every $r \ge 0$, we have

$$\int_t^{t+r} (\nabla\mu_k \varphi_k - \nabla\mu\varphi,\, \boldsymbol{v}) = \int_t^{t+r} (\nabla\mu_k(\varphi_k - \varphi),\, \boldsymbol{v}) + \int_t^{t+r} (\nabla(\mu_k - \mu)\varphi,\, \boldsymbol{v})$$

and the second term goes to 0 thanks to the boundedness of φ and (4.9), while the first one vanishes thanks to (3.22) and (4.8) and the inequality

$$\int_t^{t+r} (\nabla\mu_k(\varphi_k - \varphi),\, \boldsymbol{v}) \le \|\boldsymbol{v}\|_{\mathbf{V}} \|\mu_k\|_{L^2(0,T;V)} \|\varphi - \varphi_k\|_{L^2(0,T;L^3)}.$$

It is easy to see that (2.9) makes sense also for every $\boldsymbol{v} \in \mathbf{H}$. Furthermore, thanks to (4.5) we can deduce that (2.8) holds also for every $\psi \in V$ by a density argument. Thus, we showed that there is a subsequence of $(\varphi_k, \boldsymbol{u}_k)$ converging to a $(\varphi, \boldsymbol{u})$ which is a weak solution to (1.4)–(1.5). □

4.1 Proof of Corollary 2.2

As shown in (3.40), we can rewrite (2.9) as

$$\eta(\boldsymbol{u}, \boldsymbol{v}) = (\nabla J * \varphi, \varphi \boldsymbol{v}) - (\frac{1}{2}\nabla a \varphi^2, \boldsymbol{v}), \qquad \text{a.e. in } [0,T], \forall \boldsymbol{v} \in \mathbf{H}. \tag{4.12}$$

On account of Lemma 2.1 in [26] we can deduce

$$\|\boldsymbol{u}\|_{[L^p]^d} \le c(\|(\nabla J * \varphi)\varphi\|_{L^p} + \|\nabla a \varphi^2\|_{L^p}).$$

Furthermore, from Theorem 2.4 we have $\varphi \in L^\infty(0,T;\Omega)$, which, thanks to (H1), leads to $\boldsymbol{u} \in L^\infty(0,T;L^p(\Omega))$ for each $p \ge 1$.

4.2 Proof of Proposition 2.4

As η is constant we can take advantage of Lemma 2.1 in [26] and, rewriting the Korteweg force as in (4.12) we can write

$$\|\boldsymbol{u}\|_{\mathbf{V}} \le c(\|\varphi \nabla J * \varphi\|_V + \frac{1}{2}\|\nabla a \varphi^2\|_V). \tag{4.13}$$

As $\varphi \in L^\infty(0,T;\Omega)$, from (H1) we can easily deduce that

$$\|\varphi \nabla J * \varphi\| \le c\|\varphi\|^2_{L^\infty}, \qquad \|\nabla a \varphi^2\| \le c\|\varphi\|^2_{L^\infty}. \tag{4.14}$$

Besides, we have

$$\|\nabla(\varphi \nabla J * \varphi)\| \le \|(\nabla J * \varphi) \otimes \nabla \varphi\| + \|\varphi \nabla^2 J * \varphi\| \le c(\|\varphi\|_{L^\infty}\|\nabla \varphi\| + \|\varphi\|^2_{L^\infty}) \tag{4.15}$$

and

$$\|\nabla(\nabla a \varphi^2)\| \le \|\nabla^2 a \varphi^2\| + 2\|\varphi \nabla a \otimes \nabla \varphi\| \le c(\|\varphi\|_{L^\infty}\|\nabla \varphi\| + \|\varphi\|^2_{L^\infty}). \tag{4.16}$$

Therefore, collecting (4.13)-(4.16) we finally conclude the proof of the proposition.

5 Continuous dependence and uniqueness

Proof of Proposition 2.2

Let $(\varphi_1, \mathbf{u}_1)$ and $(\varphi_2, \mathbf{u}_2)$ be two weak solutions to the system (1.2)–(1.3) corresponding to $\varphi_{1,0}$ and $\varphi_{2,0}$, respectively. Here $N > 0$ will denote a generic constant

depending on T and $\|\varphi_{i,0}\|$, $i = 1, 2$.
Setting $\varphi = \varphi_1 - \varphi_2$, $\tilde{\mu} = \mu(\varphi_1) - \mu(\varphi_2)$ and $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, we have

$$\langle \varphi_t, \psi \rangle + (\nabla \tilde{\mu}, \nabla \psi) = (\mathbf{u}\varphi_1, \nabla \psi) + (\mathbf{u}_2 \varphi, \nabla \psi), \quad \forall \psi \in V, \quad \text{a.e. in } (0,T), \tag{5.1}$$
$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + (\eta \boldsymbol{u}, \boldsymbol{v}) = (\tilde{\mu} \nabla \varphi_1, \boldsymbol{v}) + (\mu_2 \nabla \varphi, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathbf{V}, \quad \text{a.e. in } (0,T), \tag{5.2}$$
$$\varphi(0) = \varphi_{1,0} - \varphi_{2,0}, \quad \text{a.e. in } \Omega. \tag{5.3}$$

Choosing $\psi = 1$ we readily obtain that $\bar{\varphi}(t) = \varphi(0)$ for all $t \in [0,T]$. On account of this, let us take $\psi = (-\Delta)^{-1}(\varphi - \bar{\varphi})$ in (5.1) and find

$$\frac{1}{2}\frac{d}{dt}\|\varphi - \bar{\varphi}\|_{-1}^2 + (\tilde{\mu}, \varphi - \bar{\varphi}) = I_1 + I_2, \tag{5.4}$$

where

$$I_1 = (\mathbf{u}\varphi_1, \nabla(-\Delta)^{-1}(\varphi - \bar{\varphi})), \qquad I_2 = (\mathbf{u}_2\varphi, \nabla(-\Delta)^{-1}(\varphi - \bar{\varphi})).$$

Furthermore, taking $\boldsymbol{v} = \boldsymbol{u}$ in (5.2), we get

$$\nu\|\nabla \boldsymbol{u}\|^2 + \|\sqrt{\eta}\boldsymbol{u}\|^2 = (\tilde{\mu}\nabla\varphi_1, \boldsymbol{u}) + (\mu_2 \nabla \varphi, \boldsymbol{u}).$$

After standard computations in light of (3.40), we obtain

$$(\tilde{\mu}\nabla\varphi_1, \boldsymbol{u}) + (\mu_2\nabla\varphi, \boldsymbol{u}) = (\nabla J * \varphi_1, \, \varphi\boldsymbol{u}) + (\nabla J * \varphi, \, \varphi_2\boldsymbol{u}) - \frac{1}{2}(\nabla a(\varphi_1 + \varphi_2), \, \varphi\boldsymbol{u}).$$

If $d = 2$, since $\varphi_i \in L^\infty(0,T; L^{2+2q}(\Omega))$, $i = 1, 2$, then we obtain

$$\begin{aligned}
&(\tilde{\mu}\nabla\varphi_1, \boldsymbol{u}) + (\mu_2\nabla\varphi, \boldsymbol{u}) \\
&\le \max\left\{\frac{1}{2}\|\nabla a\|_{L^\infty}, \, \|\nabla J\|_{L^1}\right\} \|\boldsymbol{u}\|_{[L^{\frac{2+2q}{q}}]^d}(\|\varphi_1\|_{L^{2+2q}} + \|\varphi_2\|_{L^{2+2q}})\|\varphi\| \\
&\le N\|\boldsymbol{u}\|_{\mathbf{V}}\|\varphi\|.
\end{aligned} \tag{5.5}$$

Analogously, if $d = 3$, recalling that $\varphi_i \in L^\infty(0,T; L^3(\Omega))$, $i = 1, 2$, we deduce

$$\begin{aligned}
(\tilde{\mu}\nabla\varphi_1, \boldsymbol{u}) + (\mu_2\nabla\varphi, \boldsymbol{u}) &\le \max\{\|\nabla a\|_{L^\infty}, \, \|\nabla J\|_{L^1}\} \|\boldsymbol{u}\|_{[L^6]^d}(\|\varphi_1\|_{L^3} + \|\varphi_2\|_{L^3})\|\varphi\| \\
&\le N\|\boldsymbol{u}\|_{\mathbf{V}}\|\varphi\|.
\end{aligned}$$

Observe now that

$$\nu\|\nabla\boldsymbol{u}\|_{\mathbf{V}}^2 + \|\sqrt{\eta}\boldsymbol{u}\|^2 \ge c\|\boldsymbol{u}\|_{\mathbf{V}}^2$$

gives

$$\|\boldsymbol{u}\|_{\mathbf{V}} \le N\|\varphi\|. \tag{5.6}$$

Let us now estimate the terms in the differential equality (5.4). In order to estimate $(\tilde{\mu}, \varphi - \bar{\varphi})$ we argue as in [7, proof of Proposition 2.1] to deduce

$$(a\varphi + F'(\varphi_1) - F'(\varphi_2), \varphi - \bar{\varphi}) \geq \frac{7c_0}{8}\|\varphi\|^2 - c\bar{\varphi}^2 - N|\bar{\varphi}| \tag{5.7}$$

and

$$(J * \varphi,\, \varphi - \bar{\varphi}) \leq \frac{c_0}{8}\|\varphi\|^2 + c\|\varphi - \bar{\varphi}\|_{\#}^2. \tag{5.8}$$

On the other hand, we have

$$(\mathbf{u}\varphi_1, \nabla(-\Delta)^{-1}(\varphi - \bar{\varphi})) \leq \|\varphi_1\|_{L^{2+2q}}\|\boldsymbol{u}\|_{[L^{\frac{2+2q}{q}}]^d}\|\varphi - \bar{\varphi}\|_{\#}, \qquad d = 2,$$
$$(\mathbf{u}\varphi_1, \nabla(-\Delta)^{-1}(\varphi - \bar{\varphi})) \leq \|\varphi_1\|_{L^3}\|\boldsymbol{u}\|_{[L^6]^d}\|\varphi - \bar{\varphi}\|_{\#}, \qquad d = 3.$$

implying

$$I_1 \leq N\|\boldsymbol{u}\|_{\mathbf{V}}\|\varphi - \bar{\varphi}\|_{\#}. \tag{5.9}$$

Concerning I_2, suppose $d = 2$ first and observe that

$$(\mathbf{u}_2\varphi, \nabla(-\Delta)^{-1}(\varphi - \bar{\varphi})) \leq \frac{c_0}{16}\|\varphi\|^2 + c\|\boldsymbol{u}_2\|_{[L^4]^d}^2\|\nabla(-\Delta)^{-1}(\varphi - \bar{\varphi})\|_{L^4}^2$$

and

$$\begin{aligned}\|\nabla(-\Delta)^{-1}(\varphi - \bar{\varphi})\|_{L^4}^2 &\leq c\|\nabla(-\Delta)^{-1}(\varphi - \bar{\varphi})\|\|\nabla(-\Delta)^{-1}(\varphi - \bar{\varphi})\|_V \\ &\leq c\|\varphi - \bar{\varphi}\|\|\varphi - \bar{\varphi}\|_{\#}.\end{aligned}$$

Thus, on account of Corollary 2.1, we get

$$\begin{aligned}(\mathbf{u}_2\varphi, \nabla(-\Delta)^{-1}(\varphi - \bar{\varphi})) &\leq \frac{c_0}{8}\|\varphi\|^2 + \|\boldsymbol{u}_2\|_{\mathbf{V}}^4\|\varphi - \bar{\varphi}\|_{\#}^2 + c\bar{\varphi}^2 \\ &\leq \frac{c_0}{8}\|\varphi\|^2 + N\|\varphi - \bar{\varphi}\|_{\#}^2,\end{aligned}$$

so that

$$I_2 \leq \frac{c_0}{8}\|\varphi\|^2 + N\|\varphi - \bar{\varphi}\|_{\#}^2. \tag{5.10}$$

Inequality (5.10) can also be proved in the case $d = 3$ by considering

$$(\mathbf{u}_2\varphi, \nabla(-\Delta)^{-1}(\varphi - \bar{\varphi})) \leq \frac{c_0}{16}\|\varphi\|^2 + \|\boldsymbol{u}_2\|_{[L^6]^d}^2\|\nabla(-\Delta)^{-1}(\varphi - \bar{\varphi})\|_{L^3}^2,$$

and observing that

$$\begin{aligned}\|\nabla(-\Delta)^{-1}(\varphi - \bar{\varphi})\|_{L^3}^2 &\leq c\|\nabla(-\Delta)^{-1}(\varphi - \bar{\varphi})\|\|\nabla(-\Delta)^{-1}(\varphi - \bar{\varphi})\|_V \\ &\leq c\|\varphi - \bar{\varphi}\|\|\varphi - \bar{\varphi}\|_{\#}.\end{aligned}$$

Collecting (5.7)–(5.10), we deduce from (5.4) the differential inequality

$$\frac{1}{2}\frac{d}{dt}\|\varphi-\bar{\varphi}\|_{-1}^2+\frac{c_0}{4}\|\varphi\|^2 \leq N\|\boldsymbol{u}\|_{\mathbf{V}}\|\varphi-\bar{\varphi}\|_{\#}+N\|\varphi-\bar{\varphi}\|_{\#}^2+c\bar{\varphi}^2-N|\bar{\varphi}|. \quad (5.11)$$

Taking (5.6) into account, we deduce

$$\frac{1}{2}\frac{d}{dt}\|\varphi-\bar{\varphi}\|_{\#}^2+\frac{c_0}{8}\|\varphi\|^2 \leq N\|\varphi-\bar{\varphi}\|_{\#}^2+N|\bar{\varphi}| \quad (5.12)$$

and Gronwall's lemma yields

$$\|\varphi_1(t)-\varphi_2(t)\|_{\#}^2 \leq N\big(\|\varphi_{1,0}-\varphi_{2,0}\|_{\#}^2+|\bar{\varphi}_{1,0}-\bar{\varphi}_{2,0}|\big).$$

The estimate for $\boldsymbol{u}$ follows from (5.6) by integrating (5.12) on $[0,t]$, $t\in(0,T]$. □

Proof of Proposition 2.3

We argue in the same way as in the Proof of Proposition 2.2. However, in this case we take advantage of the inequality

$$(\eta\boldsymbol{u},\,\boldsymbol{u}) \geq \eta_0\|\boldsymbol{u}\|^2. \quad (5.13)$$

Moreover, we observe that (5.5) can be replaced by

$$\begin{aligned}(\tilde{\mu}\nabla\varphi_1,\,\boldsymbol{u})+(\mu_2\nabla\varphi,\,\boldsymbol{u}) &= (\nabla J*\varphi_1,\,\varphi\boldsymbol{u})+(\nabla J*\varphi,\,\varphi_2\boldsymbol{u})-(\nabla a(\varphi_1+\varphi_2),\,\varphi\boldsymbol{u})\\ &\leq \max\{\|\nabla a\|_{L^\infty},\,\|\nabla J\|_{L^1}\}\|\boldsymbol{u}\|\big(\|\varphi_1\|_{L^\infty}+\|\varphi_2\|_{L^\infty}\big)\|\varphi\|. \quad (5.14)\end{aligned}$$

Leveraging on the fact that φ_1 and φ_2 are bounded, we obtain

$$\|\boldsymbol{u}\| \leq N\|\varphi\|.$$

Consider now (5.4). Instead of controlling I_1 as in (5.9), we obtain

$$I_1=(\boldsymbol{u}\varphi_1,\,\nabla(-\Delta)^{-1}(\varphi-\bar{\varphi})) \leq N\|\boldsymbol{u}\|\|\varphi-\bar{\varphi}\|_{\#} \quad (5.15)$$

Also, exploiting the estimates for $\boldsymbol{u}$ and arguing as in the proof of Proposition 2.2, thanks to Corollary 2.2 we have

$$\begin{aligned}I_2 &= (\mathbf{u}_2\varphi,\nabla(-\Delta)^{-1}(\varphi-\bar{\varphi}))\\ &\leq \frac{c_0}{8}\|\varphi\|^2+\|\boldsymbol{u}_2\|_{L^{2d}}^4\|\varphi-\bar{\varphi}\|_{\#}^2+c\bar{\varphi}^2 \leq \frac{c_0}{8}\|\varphi\|^2+N\|\varphi-\bar{\varphi}\|_{\#}^2.\end{aligned}$$

Thus we can still prove inequality (5.11) and the proof can be completed arguing as above. □

6 Convergence of solutions as $\nu \to 0$

In this section we prove Theorem 2.6.

6.1 Proof of Theorem 2.6

We first define $\psi = \varphi_\nu - \varphi$, $\tilde{\mu} = \mu(\varphi_\nu) - \mu(\varphi)$ and $\boldsymbol{v} = \boldsymbol{u}_\nu - \boldsymbol{u}$. Let us now take $\boldsymbol{v}$ in the weak formulation of the equation for $\boldsymbol{v}$. Adding $-\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})$ to both sides of the resulting identity, we get

$$\nu\|\nabla \boldsymbol{v}\|^2 + \|\sqrt{\eta}\boldsymbol{v}\|^2 = (\tilde{\mu}\nabla\varphi_\nu, \boldsymbol{v}) + (\mu\nabla\psi, \boldsymbol{v}) - \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})$$

Since

$$-\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) \leq \nu\|\nabla \boldsymbol{u}\|^2 + \nu\|\nabla \boldsymbol{v}\|^2$$

we obtain

$$\eta\|\boldsymbol{v}\|^2 \leq |(\tilde{\mu}\nabla\varphi_\nu, \boldsymbol{v}) + (\mu\nabla\psi, \boldsymbol{v})| + \nu\|\nabla \boldsymbol{u}\|^2.$$

Reasoning as in (5.14) we find

$$\begin{aligned}|(\tilde{\mu}\nabla\varphi_\nu,\, \boldsymbol{v}) + (\mu\nabla\psi,\, \boldsymbol{v})| &\leq \max\left(\|\nabla a\|_{L^\infty},\, \|\nabla J\|_{L^1}\right)\|\boldsymbol{v}\|\big(\|\varphi_\nu\|_{L^\infty} + \|\varphi\|_{L^\infty}\big)\|\psi\| \\ &\leq C\|\boldsymbol{v}\|\|\psi\|,\end{aligned}$$

hence

$$\eta\|\boldsymbol{v}\|^2 \leq C\|\boldsymbol{v}\|\|\psi\| + \nu\|\nabla \boldsymbol{u}\|^2.$$

Note that this implies

$$\|\boldsymbol{v}\| \leq \frac{C}{\eta}\|\psi\| + \frac{\sqrt{\nu}}{\sqrt{\eta}}\|\nabla \boldsymbol{u}\|. \tag{6.1}$$

On the other hand, we have

$$\frac{1}{2}\frac{d}{dt}\|\psi - \bar{\psi}\|_{-1}^2 + (\tilde{\mu}, \psi - \bar{\psi}) = I_1 + I_2,$$

where

$$I_1 = (\mathbf{v}\varphi_\nu, \nabla(-\Delta)^{-1}(\psi - \bar{\psi})), \qquad I_2 = (\mathbf{u}\psi, \nabla(-\Delta)^{-1}(\psi - \bar{\psi})).$$

Now, by arguing as in proof of Proposition 2.3 and exploiting boundedness of $\boldsymbol{u}$ we deduce

$$\frac{1}{2}\frac{d}{dt}\|\psi - \bar{\psi}\|_{-1}^2 + \frac{c_0}{4}\|\psi\|^2 \leq N\|\boldsymbol{v}\|\|\psi - \bar{\psi}\|_\# + N\|\psi - \bar{\psi}\|_\#^2 + c\bar{\psi}^2 + N|\bar{\psi}|.$$

Thus, taking (6.1) into account, we end up with

$$\frac{1}{2}\frac{d}{dt}\|\psi-\bar{\psi}\|_{\#}^{2}+\frac{c_0}{8}\|\psi\|^{2}\leq N\|\psi-\bar{\psi}\|_{\#}^{2}+N|\bar{\psi}|+N\nu\|\nabla \boldsymbol{u}\|^{2}. \tag{6.2}$$

An application of the Gronwall lemma on $[0,T]$, on account of Proposition 2.4 provides

$$\|\varphi_\nu(t)-\varphi(t)\|_{\#}^{2}\leq \big(\|\varphi_0^\nu-\varphi_0\|_{\#}^{2}+|\bar{\varphi}_0^\nu-\bar{\varphi}_0|\big)e^{C_T}+C_T\nu.$$

Now a further integration of (6.2), and (6.1) complete the proof. □

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