Stability Estimates in the Inverse Transmission Scattering Problem

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Abstract

We consider the inverse transmission scattering problem with piecewise constant refractive index. Under mild a priori assumptions on the obstacle we establish logarithmic stability estimates.

1 Introduction

In this paper we consider the scattering of acoustic time-harmonic waves in an inhomogeneous medium. More precisely we shall consider a penetrable obstacle D and we want to recover information on its location from a knowledge of Cauchy data on the boundary of a region Ω containing the obstacle D.

Given a spherical incident wave $u^i(\cdot, x_0) = \Phi(\cdot, x_0)$, where the point source x_0 is located on the boundary of a ball B of radius R, B such that $\Omega \subset B$, and Φ denotes the fundamental solution to the Helmholtz equation

$$\Phi(x, x_0) = \frac{1}{4\pi} \frac{\mathrm{e}^{ik|x-x_0|}}{|x - x_0|}, \qquad x \in \mathbb{R}^3, \quad x \neq x_0,$$

we denote by $\mathbb{G}(x, x_0) = u^i(x, x_0) + u^s(x, x_0)$ the Green's function of the equation

$$\text{div}\,(\gamma(x)\nabla\mathbb{G}(x, x_0)) + k^2 n(x)\mathbb{G}(x, x_0) = -\delta(x - x_0), \qquad \text{in } \mathbb{R}^3, \tag{1.1a}$$

where the scattered field u^s satisfies the Sommerfeld radiation condition

$$\lim_{|x|\to\infty} |x| \left(\frac{\partial u^s}{\partial r}(x) - iku^s(x) \right) = 0. \tag{1.1b}$$

Here $k > 0$ is the wave number and $r = |x|$. We shall study equation (1.1a) with piecewise constant coefficients, in particular we shall consider γ and n to be of the following form

$$\begin{cases} \gamma(x) = 1 + (a - 1)\chi_D(x) \\ n(x) = 1 + (b - 1)\chi_D(x) \\ a \geq \lambda > 0, \quad b \geq \lambda > 0, \\ (a - 1)^2 + (b - 1)^2 \geq \delta^2 > 0, \end{cases}$$

where λ and δ are given constants. We refer to [Co-Kr, Is06] for basic information on scattering problem of this type.

The unique determination of D from a knowledge of the far field data has been established by Isakov [Is90]. The purpose of the present paper is to establish a stability result. Under reasonable mild assumptions on the regularity of ∂D we show that there is a continuous dependance of D on the Cauchy data on $\partial\Omega$ with a modulus of continuity of logarithmic type. This rate of continuity appears optimal in view of the recent paper [DC-Ro] indicating the strong ill-posedness of the inverse problem.

The main ideas employed to obtain stability rely on the study of the behavior of $\mathbb{G}(x, x_0)$ when x and x_0 get close and the use of unique continuation. These ideas go bach to [Is88] where a uniqueness result for the inverse inclusion problem is proved and it has also been used in inverse scattering theory in [Is90]. In order to apply these ideas to stability some further properties on singular solutions and quantitative estimates of unique continuation are needed. We refer to [Al-DC] where similar ideas are developed for studying the stability of the inverse inclusion problem.

The stability issue in inverse scattering theory has been considered by Isakov [Is92, Is93] for the determination of a sound-soft obstacle. Hähner and Hohage [Ha-Ho] considered equation (1.1a) with $a = 1$ and $n(x)$ smooth. They showed that n depends on $\mathbb{G}(x, x_0)$, $x, x_0 \in \partial B$, with a logarithmic rate of continuity. They considered both far field data and near field data. They improve and simplify a previous result of Stefanov [St]. We finally mention a result obtained by Potthast [Po] for impenetrable obstacles which is also based on the use of singular solutions.

The plan of the paper is the following. In Section 2 we give the a priori assumptions we need and we state the stability theorem. In Section 3 the proof of the stability theorem is given based on some auxiliary results whose proofs are collected in Section 4 and Section 5. In particular, in Section 4 we establish some results on singular solutions of equation (1.1a) and in Section 5 we study quantitative estimates of unique continuation.

2 The Main Result

In this section we state the stability theorem. Before doing this we shall give some definitions we need and introduce the a priori assumptions on the regularity of the obstacle. For any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and any $r > 0$ we denote by $B_r(x)$ the open ball in $\mathbb{R}^3$ of radius r centered in the point x, $B_r(0) = B_r$ and for $x' = (x_1, x_2) \in \mathbb{R}^2$ we denote by $B'_r(x')$ the open ball in $\mathbb{R}^2$ of radius r centered in the point x'. In places, we shall denote a point $x \in \mathbb{R}^3$ by $x = (x', x_3)$ where $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$.

Definition 2.1. *Let Ω be a bounded domain in $\mathbb{R}^3$. Given α, $0 < \alpha \leq 1$, we shall say that a portion S of $\partial\Omega$ is of class $C^{1,\alpha}$ with constants r_0, $L > 0$ if for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P = 0$ and*

$$\Omega \cap B_{r_0} = \{x \in B_{r_0} \,:\, x_3 > \varphi(x')\},$$

where φ is a $C^{1,\alpha}$ function on $B'_{r_0} \subset \mathbb{R}^2$ satisfying $\varphi(0) = |\nabla\varphi(0)| = 0$ and $\|\varphi\|_{C^{1,\alpha}(B'_{r_0})} \leq L r_0$.

Definition 2.2. *We shall say that a portion S of $\partial\Omega$ is of Lipschitz class with constants r_0, $L > 0$ if for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P = 0$ and*

$$\Omega \cap B_{r_0} = \{x \in B_{r_0} \, : \, x_3 > \varphi(x')\},$$

where φ is a Lipschitz continuous function on $B'_{r_0} \subset \mathbb{R}^2$ satisfying $\varphi(0) = 0$ and $\|\varphi\|_{C^{0,1}(B'_{r_0})} \leq Lr_0$.

Remark 2.1. *We use the convention to scale all norms in such a way that they are dimensionally equivalent to their argument. For instance, for any $\psi \in C^{1,\alpha}(B'_{r_0})$ we set*

$$\|\psi\|_{C^{1,\alpha}(B'_{r_0})} = \|\psi\|_{L^\infty(B'_{r_0})} + r_0\|\nabla\psi\|_{L^\infty(B'_{r_0})} + r_0^{1+\alpha}|\nabla\psi|_{\alpha,B'_{r_0}}.$$

Assumptions on the obstacle D

For given numbers r_0, $L > 0$, $0 < \alpha < 1$, we shall assume there exists a bounded domain Ω such that the obstacle D satisfies the following conditions:

(2.2a) $\quad D \subset \Omega;$

(2.2b) $\quad \Omega \setminus \overline{D}$ is connected;

(2.2c) $\quad \partial D$ is of class $C^{1,\alpha}$ with constants r_0, L.

In the sequel we shall refer to numbers r_0, L, α, R, a, b and k as the a priori data.

The inverse problem we are concerned with is the determination of the obstacle D from the knowledge of the Cauchy data of the singular solutions $\mathbb{G}(\cdot, x_0)$ on $\partial\Omega$ for all points source x_0 located on ∂B.

For two possible obstacles D_1, D_2 satisfying (2.2) we shall denote by $\mathbb{G}_i$, $i = 1, 2$, the corresponding solutions to (1.1a) satisfying the Sommerfeld radiation condition (1.1b).

Theorem 2.2. *Let D_1, D_2 be two obstacles satisfying* (2.2)*. If, given $\varepsilon > 0$, we have*

$$\sup_{x\in\partial B}\left(\left\|\frac{\partial\mathbb{G}_1(\cdot,x)}{\partial\nu} - \frac{\partial\mathbb{G}_2(\cdot,x)}{\partial\nu}\right\|_{L^2(\partial\Omega)} + \|\mathbb{G}_1(\cdot,x) - \mathbb{G}_2(\cdot,x)\|_{L^2(\partial\Omega)}\right) \leq \varepsilon, \tag{2.3}$$

then

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq \omega(\varepsilon),$$

where ω is an increasing function on $[0, +\infty)$, which satisfies

$$\omega(t) \leq C|\log t|^{-\eta}, \qquad \textit{for every} \quad 0 < t < 1$$

and C, η, $C > 0$, $0 < \eta \leq 1$, are constants only depending on the a priori data.

Remark 2.3. *We stress the fact that we don't need any assumption on k.*

3 Proof of the Stability Theorem

We denote by $\mathcal{G}$ the connected component of $\Omega \setminus (D_1 \cup D_2)$ such that $\partial\Omega \subset \overline{\mathcal{G}}$ and $\Omega_D = \Omega \setminus \mathcal{G}$.

Theorem 2.2 evaluates how close the two inclusions are in term of the Hausdorff distance $d_{\mathcal{H}}$. We recall a definition of this metric.

$$d_{\mathcal{H}}(D_1, D_2) = \max\left\{\sup_{x\in D_1} dist(x, D_2), \sup_{x\in D_2} dist(x, D_1)\right\}.$$

In order to deal with the Hausdorff distance we introduce a simplified variation of it which we call modified distance.

Definition 3.1. *We shall call modified distance between D_1 and D_2 the number*

$$d_\mu(D_1, D_2) = \max\left\{\sup_{x\in\partial D_1\cap\partial\Omega_D} \operatorname{dist}(x, D_2), \sup_{x\in\partial D_2\cap\partial\Omega_D} \operatorname{dist}(x, D_1)\right\}. \tag{3.4}$$

We wish to remark here that such modified distance does not satisfy the axioms of a metric and in general does not dominate the Hausdorff distance (see [Al-Be-Ro-Ve, §3] for related arguments).

Proposition 3.1. *Let D_1, D_2 be two obstacles satisfying* (2.2)*. Then*

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \le c d_\mu(D_1, D_2), \tag{3.5}$$

where c depends only on the a priori assumptions.

Proof. See [Al-DC, Proposition 3.1] □

With no loss of generality, we can assume that there exists a point O of $\partial D_1 \cap \partial\Omega_D$, where the maximum in the Definition 3.1 is attained, that is

$$d_\mu = d_\mu(D_1, D_2) = \operatorname{dist}(O, D_2). \tag{3.6}$$

We remark that $\mathbb{G}$ is solution to

$$\operatorname{div}(\gamma(x)\nabla\mathbb{G}(x,y)) + k^2 n(x)\mathbb{G}(x,y) = -\delta(x,y).$$

We shall denote by $\mathbb{G}_1$ and $\mathbb{G}_2$ Green's functions when $D = D_1$ and D_2 respectively and γ_i, n_i, $i = 1, 2$, the corresponding coefficients.

Integrating by parts we have

$$\begin{aligned}
&(a-1)\left\{\int_{D_1} \nabla\mathbb{G}_1(\cdot,y)\cdot\nabla\mathbb{G}_2(\cdot,w) - \int_{D_2} \nabla\mathbb{G}_1(\cdot,y)\cdot\nabla\mathbb{G}_2(\cdot,w)\right\}\\
&+k^2(b-1)\left\{\int_{D_1} \mathbb{G}_2(\cdot,w)\mathbb{G}_1(\cdot,y) - \int_{D_2} \mathbb{G}_1(\cdot,y)\mathbb{G}_2(\cdot,w)\right\}\\
=& \int_{\partial\Omega}\left(\frac{\partial\mathbb{G}_1(\cdot,y)}{\partial\nu}\mathbb{G}_2(\cdot,w) - \mathbb{G}_1(\cdot,y)\frac{\partial\mathbb{G}_2(\cdot,w)}{\partial\nu}\right)\\
=& \int_{\partial\Omega}\frac{\partial\mathbb{G}_1(\cdot,y)}{\partial\nu}(\mathbb{G}_2(\cdot,w) - \mathbb{G}_1(\cdot,w))\\
&+\int_{\partial\Omega}\mathbb{G}_1(\cdot,y)\left(\frac{\partial\mathbb{G}_1(\cdot,w)}{\partial\nu} - \frac{\partial\mathbb{G}_2(\cdot,w)}{\partial\nu}\right) \qquad \forall\, y, w \in \mathcal{C}B.
\end{aligned} \tag{3.7}$$

Let us define for $y, w \in \mathcal{C}B$

$$S_1(y,w) = (a-1)\int_{D_1} \nabla\mathbb{G}_1(\cdot,y)\cdot\nabla\mathbb{G}_2(\cdot,w) + k^2(b-1)\int_{D_1} \mathbb{G}_1(\cdot,y)\mathbb{G}_2(\cdot,w),$$

$$S_2(y,w) = (a-1)\int_{D_2} \nabla\mathbb{G}_1(\cdot,y)\cdot\nabla\mathbb{G}_2(\cdot,w) + k^2(b-1)\int_{D_2} \mathbb{G}_1(\cdot,y)\mathbb{G}_2(\cdot,w),$$

$$f(y,w) = S_1(y,w) - S_2(y,w).$$

Thus (3.7) can be rewritten as

$$\begin{aligned} f(y,w) \quad &= \quad \int_{\partial\Omega} \frac{\partial\mathbb{G}_1(\cdot,y)}{\partial\nu}(\mathbb{G}_2(\cdot,w)-\mathbb{G}_1(\cdot,w)) \\ &+ \int_{\partial\Omega} \mathbb{G}_1(\cdot,y)\left(\frac{\partial\mathbb{G}_1(\cdot,w)}{\partial\nu} - \frac{\partial\mathbb{G}_2(\cdot,w)}{\partial\nu}\right) \qquad \forall y,w\in\mathcal{C}B. \end{aligned} \tag{3.8}$$

Let us fix $P \in \partial D$. We can assume $P \equiv 0$. We denote by $\nu(P)$ the outer unit normal vector to Ω_D in P and we rotate the coordinate system in such a way that $\nu(P) = (0,0,-1)$.

Let us denote by $\chi^+(x)$ the characteristic function of the half-space and by $\mathbb{G}_+$ the Green's function of $\mathrm{div}((1+(a-1)\chi^+)\nabla) + k^2(1+(b-1)\chi^+)$.

Proposition 3.2. *Let $D \subset \Omega$ be a bounded open set whose boundary is of class $C^{1,\alpha}$ with constants r_0, L. Then there exist constants c_1, c_2 depending on a, α, k and L such that*

$$|\nabla_x\mathbb{G}(x,y)| \le c_1|x-y|^{-2}, \tag{3.9}$$

$$|\nabla_x\mathbb{G}_+(x,y)| \le c_2|x-y|^{-2} \tag{3.10}$$

for every $x, y \in \mathbb{R}^3$.

Proof. (3.9) and (3.10) can be obtained following [Al-DC, Proposition 3.1]. In [Al-DC] the key point is the piecewise regularity of the transmission problem. For a proof of that we refer to [DB-El-Fr] and [Li-Vo]. □

We shall state now two propositions that describe the behavior of $f(y)$ and $S_1(y)$ when we move the singularity y toward the boundary of the inclusion. We postpone their proofs in the last Section 5.

Proposition 3.3. *Let D_1, D_2 two obstacles verifying* (2.2) *and let $y = h\nu(O)$, with O defined in* (3.6)*. If, given $\varepsilon > 0$ we have*

$$\sup_{x\in\partial B}\left(\left\|\frac{\partial\mathbb{G}_1(\cdot,x)}{\partial\nu} - \frac{\partial\mathbb{G}_2(\cdot,x)}{\partial\nu}\right\|_{L^2(\partial\Omega)} + \|\mathbb{G}_1(\cdot,x)-\mathbb{G}_2(\cdot,x)\|_{L^2(\partial\Omega)}\right) \le \varepsilon,$$

then for every h, $0 < h < \bar{c}r_0$, with $\bar{c}\in(0,1)$ depending on L,

$$|f(y,y)| \le c\frac{\varepsilon^{Bh^F}}{h^A},$$

where $0 < A < 1$ and $c, B, F > 0$ are constants that depend only on the a priori data.

Proposition 3.4. *Let D_1, D_2 two obstacles verifying* (2.2) *and let $y = h\nu(O)$, with O defined in* (3.6). *Then for every h, $0 < h < \min\{\overline{r}_2, d_\mu\}$*

$$|S_1(y,y)| \geq c_1 h^{-2} - c_2(d_\mu - h)^{-2} + c_3 \tag{3.11}$$

where c_1, c_2, c_3 and $\overline{r}_2$ are positive constants only depending on the a priori data.

Proof of Theorem 2.2. Let $O \in \partial D_1$ as defined (3.6), that is

$$d_\mu(D_1, D_2) = \operatorname{dist}(O, D_2) = d_\mu.$$

Then, for $y = h\nu(O)$, with $0 < h < h_1$, where $h_1 = \min\{d_\mu, \overline{c}r_0, \overline{r}_2/2\}$, using (3.9), we have

$$|S_2(y,y)| \leq c \int_{D_2} \frac{1}{(d_\mu - h)} \frac{1}{(d_\mu - h)} dx = c \frac{1}{(d_\mu - h)^2} |D_2|. \tag{3.12}$$

Using Proposition 3.3, we have

$$|S_1(y,y)| - |S_2(y,y)| \leq |S_1(y,y) - S_2(y,y)| = |f(y,y)| \leq c \frac{\varepsilon^{Bh^F}}{h^A}.$$

On the other hand, by Proposition 3.4 and (3.12), there exists $h_0 > 0$, only depending on the a priori data, such that for h, $0 < h < h_0$

$$|S_1(y,y)| - |S_2(y,y)| \geq c_1 h^{-2} - c_4 (d_\mu - h)^{-2}.$$

Thus we have

$$c_1 h^{-2} - c_4 (d_\mu - h)^{-2} \leq \frac{\varepsilon^{Bh^F}}{h^A}.$$

Let $h = h(\varepsilon)$ where $h(\varepsilon) = \min\{|\ln \varepsilon|^{-\frac{1}{2F}}, d_\mu\}$, for $0 < \varepsilon \leq \varepsilon_1$, with $\varepsilon_1 \in (0,1)$ such that $\exp(-B|\ln \varepsilon_1|^{1/2}) = 1/2$. If $d_\mu \leq |\ln \varepsilon|^{-\frac{1}{2F}}$ the theorem follows using Proposition 3.1. In the other case we have

$$c_4 (d_\mu - h)^{-2} \quad \geq \quad c_3 h^{-2} - \frac{\varepsilon^{Bh^F}}{h^A} \geq c_5 h^{-2} \big(1 - \varepsilon^{Bh^F} h^{\widetilde{A}}\big),$$

where $\widetilde{A} = 2 - A$, $\widetilde{A} > 0$. Since

$$\varepsilon^{Bh(\varepsilon)^F} h(\varepsilon)^{\widetilde{A}} \leq \varepsilon^{B|\ln \varepsilon|^{-1/2}} \leq \exp\big(-B|\ln \varepsilon|^{1/2}\big),$$

for any ε, $0 < \varepsilon < \varepsilon_1$,

$$(d_\mu - h(\varepsilon))^{-2} \geq c_6 h(\varepsilon)^{-2},$$

that is, solving for d_μ, and recalling that, in this case, $h(\varepsilon) = |\ln \varepsilon|^{-\frac{1}{2F}}$

$$d_\mu \leq c_7 |\ln \varepsilon|^{-\frac{\delta}{2}}$$

where $\delta = 1/(2F)$. When $\varepsilon \geq \varepsilon_1$, then

$$d_\mu \leq \operatorname{diam}\Omega$$

and, in particular when $\varepsilon_1 \leq \varepsilon < 1$

$$d_\mu \leq \operatorname{diam}\Omega \frac{|\ln \varepsilon|^{-\frac{1}{2F}}}{|\ln \varepsilon_1|^{-\frac{1}{2F}}}.$$

Finally, using Proposition 3.1, the theorem follows. □

4 Remarks on Singular Solutions

Proposition 4.1. *Let $D \subset \mathbb{R}^3$ be an open set with $C^{1,\alpha}$ boundary with constants r_0, L, let P be a point in ∂D and let us denote with $\nu(P)$ the outer normal vector to D in P. There exist positive constants c_3, c_4 depending on a, k, α and L such that*

$$|\mathbb{G}(x,y) - \mathbb{G}_+(x,y)| \le \frac{c_3}{r_0^\alpha}|x-y|^{-1+\alpha}, \tag{4.13}$$

$$|\nabla_x \mathbb{G}(x,y) - \nabla_x \mathbb{G}_+(x,y)| \le \frac{c_4}{r_0^{\alpha^2}}|x-y|^{-2+\alpha^2}, \tag{4.14}$$

for every $x \in D \cap B_r(P)$ and $y = h\nu(P)$, with $0 < r < (\min\{\frac{1}{2}(8L)^{-1/\alpha}, \frac{1}{2}\})r_0 = \overline{r}_0$, $0 < h < (\min\{\frac{1}{2}(8L)^{-1/\alpha}, \frac{1}{2}\})\frac{r_0}{2}$.

Proof. Let us fix $r_1 = \min\{\frac{1}{2}(8L)^{-1/\alpha}r_0, \frac{r_0}{2}\}$. In the ball $B_{r_0}(P)$ the boundary of D can be represented as the graph of a $C^{1,\alpha}$ function φ. Let us introduce now the following change of variable that transform in $B_{r_0}(P)$ ∂D in the x'-axis. For every $r > 0$, let $Q_r(P)$ be the cube centered at P, with sides of length $2r$ and parallel to the coordinates axes. We have that the ball $B_r(P)$ is inscribed into $Q_r(P)$. We define

$$\begin{aligned} \Psi : Q_{2r_1}(P) &\to Q_{2r_1}(P) \\ \begin{pmatrix} x' \\ x_n \end{pmatrix} &\to \begin{pmatrix} \xi' = x' \\ \xi_n = x_n - \varphi(x')\theta(\frac{|x'|}{r_1})\theta(\frac{x_n}{r_1}) \end{pmatrix}, \end{aligned}$$

where $\theta \in C^\infty(\mathbb{R})$ be such that $0 \le \theta \le 1$, $\theta(t) = 1$, for $|t| < 1$, $\theta(t) = 0$, for $|t| > 2$ and $|\frac{d\theta}{dt}| \le 2$. Since the $C^{1,\alpha}$ regularity of φ, it is possible to verify that the following inequalities hold:

$$c^{-1}|x_1 - x_2| \le |\Psi(x_1) - \Psi(x_2)| \le c|x_1 - x_2|, \tag{4.15a}$$

$$|\Psi(x) - x| \le \frac{c}{r_0^\alpha}|x|^{1+\alpha} \qquad \forall\, x \in \mathbb{R}^3, \tag{4.15b}$$

$$|D\Psi(x) - I| \le \frac{c}{r_0^\alpha}|x|^\alpha \qquad \forall\, x \in \mathbb{R}^3 \tag{4.15c}$$

where $c \ge 1$ depends on L and α only. Ψ is a $C^{1,\alpha}$ diffeomorphism from $\mathbb{R}^3$ into itself. Let us define the cylinder C_{r_1} as $C_{r_1} = \{x \in \mathbb{R}^3 : |x'| < r_1, |x_n| < r_1\}$. For $x, y \in C_{r_1}$, we shall denote

$$\widetilde{\mathbb{G}}(x,y) = \mathbb{G}(\Psi^{-1}(x), \Psi^{-1}(y)). \tag{4.16}$$

$\widetilde{\mathbb{G}}(x,y)$ is solution of

$$\begin{aligned} \operatorname{div}((1 + (a-1)\chi^+)B\nabla\widetilde{\mathbb{G}}(x,y)) \\ + k^2\zeta(1 - (b-1)\chi_+(x))B\widetilde{\mathbb{G}}(x,y) = -\delta(x-y), \end{aligned} \tag{4.17}$$

where $B = \frac{JJ^T}{\det J}$, with $J = \frac{\partial\xi}{\partial x}(\Psi^{-1}(\xi))$, is of class C^α, $B(0) = I$ and $\zeta = \det J$. Since $\mathbb{G}_+$ is solution to

$$\begin{aligned} \operatorname{div}((1 + (a-1)\chi^+)\mathbb{G}_+(x,y)) + \\ k^2(1 - (b-1)\chi_+(x))\mathbb{G}_+(x,y) = -\delta(x,y), \end{aligned} \tag{4.18}$$

subtracting (4.18) to (4.17) we obtain that $\widetilde{R}(x,y) = \widetilde{\mathbb{G}}(x,y) - \mathbb{G}_+(x,y)$ is solution to

$$\begin{aligned}(4.19)\quad \operatorname{div}((1+(a-1)\chi^+)\tilde{R}(x,y)) \\ + k^2(1+(b-1)\chi_+)\tilde{R}(x,y) \\ = \operatorname{div}((1+(a-1)\chi^+)[B(x)-I]\nabla\widetilde{\mathbb{G}}(x,y)) \\ + k^2(1-\zeta)(1+(b-1)\chi_+)\tilde{\mathbb{G}}(x,y).\end{aligned}$$

Let $\tilde{L}$, depending on the a priori data, be such that $\overline{\Omega} \subset B_{\tilde{L}}(0)$, then using the fundamental solution $\mathbb{G}_+$ we get

$$\begin{aligned}-\tilde{R}(x,y) = \int_{B_{\tilde{L}}(0)} (1+(a-1)\chi^+)[B(z)-I]\nabla_x\tilde{\mathbb{G}}(z,y)\cdot\nabla_x\mathbb{G}_+(z,x)dz \\ + \int_{\partial B_{\tilde{L}}(0)} [B(z)-I]\left[\tilde{R}(x,z)\frac{\partial\mathbb{G}_+}{\partial\nu}(z,y) + \mathbb{G}_+(z,y)\frac{\partial\tilde{R}}{\partial\nu}(x,z)\right]d\sigma(z) \\ + k^2(1-\zeta)\int_{B_{\tilde{L}}(0)} (1+(b-1)\chi_+)\tilde{\mathbb{G}}(z,x)\mathbb{G}_+(z,y)dz+ \\ k^2(1-\zeta)\int_{\partial B_{\tilde{L}}(0)} (1+(a-1)\chi_+)\left[\tilde{R}(x,z)\frac{\partial\mathbb{G}_+}{\partial\nu}(z,y) + \mathbb{G}_+(z,y)\frac{\partial\tilde{R}}{\partial\nu}(x,z)\right]d\sigma(z)\end{aligned}$$

Integrals over $\partial B_{\tilde{L}}(0)$ are bounded by a constant. Let us split

$$B_{\tilde{L}}(0) = (B_{\tilde{L}}(0) \setminus C_{r_1}) \cup (B_{\tilde{L}}(0) \cap C_{r_1}).$$

For $|x|, |y| \le r_1/2$, in $B_{\tilde{L}}(0) \setminus C_{r_1}$ we are away from the singularity thus the integrals over $B_{\tilde{L}}(0) \setminus C_{r_1}$ are bounded. Let us evaluate integrals over $B_{\tilde{L}}(0) \cap C_{r_1}$. We have

$$\begin{aligned}\left|\int_{B_{\tilde{L}}(0)\cap C_{r_1}} (1+(a-1)\chi^+)[B(z)-I]\nabla_x\tilde{\mathbb{G}}(z,y)\cdot\nabla_x\mathbb{G}_+(z,x)dz\right| \\ \le c\int_{B_{\tilde{L}}(0)\cap C_{r_1}} |z|^\alpha|z-y|^{-2}|z-x|^{-2}dz = I\end{aligned}$$

where c depends on L, α, a and n. We can split $I = I_1 + I_2$ where

$$I_1 = \int_{\{|z|<4h\}\cap C_{r_1}} |z|^\alpha|x-z|^{-2}|y-z|^{-2}dz,$$

$$I_2 = \int_{\{|z|>4h\}\cap C_{r_1}} |z|^\alpha|x-z|^{-2}|y-z|^{-2}dz.$$

Now

$$\begin{aligned}
I_1 &\leq \int_{|w|<4} h^\alpha |w|^\alpha h^{-2}\Big|\frac{x}{h}-w\Big|^{-2} h^{-2}\Big|\frac{y}{h}-w\Big|^{-2} h^3 dw \\
&= h^{\alpha-1}\int_{|w|<4} |w|^\alpha \Big|\frac{x}{h}-w\Big|^{-2}\Big|\frac{y}{h}-w\Big|^{-2} dw \\
&\leq h^{\alpha-1}F(\xi,\eta),
\end{aligned}$$

where $h=|x-y|$ and

$$F(\xi,\eta)=4^\alpha\int_{|w|<4}|\xi-w|^{-2}|\eta-w|^{-2}dw$$

and $\xi=x/h$ and $\eta=y/h$. From standard bounds (see, for instance, [Mi, Ch. 2, § 11]), it is not difficult to see that

$$F(\xi,\eta)\leq \text{const.}<\infty,$$

for all $\xi,\eta\in\mathbb{R}^3$, $|\xi-\eta|=1$. Thus

$$I_1\leq c|x-y|^{\alpha-1}.$$

Let us consider now I_2. Since $|y|=-y_n\leq|x-y|=h$, we can deduce $|z|\leq\frac{4}{3}|y-z|$ and $|z|\leq 2|x-z|$ and thus obtain that

$$I_2\leq c\int_{|z|>4h}|z|^{\alpha+1-n+1-n}dz\leq ch^{\alpha-1}.$$

Then we conclude

$$|\widetilde{R}(x,y)|\leq c|x-y|^{-1+\alpha}, \tag{4.20}$$

for every $|x|,|y|\leq r_1/2$, where c depends on L, α, k and a only.

We observe that if $x\in\Psi^{-1}(B^+_{r_1/2}(0))$ and $y=e_3y_3$, with $y_3\in(-r_1/2,0)$ then

$$c^{-1}|x|\leq|\Psi(x)|\leq|\Psi(x)-y|\leq c|x-y|. \tag{4.21}$$

From (4.20) and (4.21) we can conclude

$$|\widetilde{R}(x,y)|\leq c|x-y|^{-1+\alpha}. \tag{4.22}$$

Now, since

$$\begin{aligned}
&\mathbb{G}(x,y)-\mathbb{G}_+(x,y)\\
=&\mathbb{G}(x,y)-\mathbb{G}_+(x,y)+\mathbb{G}_+(\Psi(x),\Psi(y))-\mathbb{G}_+(\Psi(x),\Psi(y))\\
=&\widetilde{R}(\Psi(x),\Psi(y))+\mathbb{G}_+(\Psi(x),y)-\mathbb{G}_+(x,y),
\end{aligned}$$

using Theorem 4.1 of [Li-Vo], the properties of Ψ and (4.22) we obtain

$$\begin{aligned}
&|\mathbb{G}(x,y)-\mathbb{G}_+(x,y)|\\
\leq\;& \frac{c}{r_0^\alpha}|x-y|^{\alpha-1}+\frac{c}{r_0^\alpha}\|\nabla\mathbb{G}_+(\cdot,y)\|_{L^\infty(Q_{r_1})}|x-\Psi(x)|\\
\leq\;& \frac{c}{r_0^\alpha}|x-y|^{\alpha-1}+\frac{c'}{r_0^\alpha}|x-y|^{1+\alpha}h^{-2}\\
\leq\;& \frac{c''}{r_0^\alpha}|x-y|^{\alpha-1},
\end{aligned}$$

where c'' depends on k, α and L only.

We estimate now the first derivative of R. To estimate the first derivative of $\widetilde{R}$ let us consider a cube $Q \subset B^+_{r_1/4}(x)$ of side $cr_1/4$, with $0 < c < 1$, such that $x \in \partial Q$. The following interpolation inequality holds:

$$\|\nabla \widetilde{R}(\cdot, y)\|_{L^\infty(Q)} \leq c\|\widetilde{R}(\cdot, y)\|^{1-\delta}_{L^\infty(Q)} |\nabla \widetilde{R}(\cdot, y)|^{\delta}_{\alpha,Q},$$

where $\delta = \frac{1}{1+\alpha}$, c depends on L only and

$$|\nabla \widetilde{R}|_{\alpha,Q} = \sup_{x,x' \in Q, x \neq x'} \frac{|\nabla \widetilde{R}(x, y) - \nabla \widetilde{R}(x', y)|}{|x - x'|^\alpha}.$$

Since, from the piecewise Hölder continuity of $\nabla \mathbb{G}$ and of $\nabla \mathbb{G}_+$, we have that

$$|\nabla \widetilde{R}(x, y)|_{\alpha,Q} \leq |\nabla \widetilde{\mathbb{G}}(x, y)|_{\alpha,Q} + |\nabla \mathbb{G}_+(x, y)|_{\alpha,Q} \leq ch^{-\alpha-2},$$

where c depends on L only, thus we conclude

$$|\nabla_x \widetilde{R}(x, y)| \leq \frac{c}{r_0^\eta} h^{(\alpha-1)(1-\delta)} h^{(-\alpha-2)\delta} = \frac{c}{r_0^\eta} h^{-2+\eta},$$

where $\eta = \frac{\alpha^2}{1+\alpha}$. Thus

$$|\nabla_x \widetilde{R}(x, y)| \leq \frac{c}{r_0^\eta} |x - y|^{\eta-2}, \tag{4.23}$$

where $\eta = \frac{\alpha^2}{1+\alpha}$ and c depends on L only. Concerning $\mathbb{G}_+$ we have

$$\begin{aligned}
&|\nabla_x \mathbb{G}_+(\Psi(x), y) - \nabla_x \mathbb{G}_+(x, y)| \\
= \;& |D\Psi(x)^T \nabla \mathbb{G}_+(\cdot, y)_{|\Psi(x)} - \nabla_x \mathbb{G}_+(x, y)| \\
\leq \;& |(D\Psi(x)^T - I)\nabla \mathbb{G}_+(\cdot, y)_{|\Psi(x)}| + |\nabla \mathbb{G}_+(\cdot, y)_{|\Psi(x)} - \nabla_x \mathbb{G}_+(x, y)| \\
\leq \;& \frac{c}{r_0^\alpha} \|\nabla \mathbb{G}_+(\cdot, y)\|_{L^\infty(Q_{r_1})} |x - \Psi(x)| + |\nabla \mathbb{G}_+(\cdot, y)|_{\alpha,Q} |\Psi(x) - x|^\alpha \\
\leq \;& \frac{c'}{r_0^\alpha} h^{1+\alpha} h^{-2} + \frac{c}{r_0^{\alpha^2}} h^{-\alpha-2} h^{(1+\alpha)\alpha} \\
\leq \;& \frac{c}{r_0^{\alpha^2}} h^{-2+\alpha^2},
\end{aligned}$$

where c depends on k, α and L only. □

Let us denote by $\mathbb{G}^0_+$ the Green's function of the operator $\mathrm{div}((1 + (a - 1)\chi_+)\nabla)$.

Proposition 4.2. *Let $\mathbb{G}_+$ and $\mathbb{G}^0_+$ as above, then there exist positive constants c_5, c_6 depending on the a priori data such that for every $x, y \in \mathbb{R}^3$ we have*

$$|\mathbb{G}_+(x, y) - \mathbb{G}^0_+(x, y)| \leq c_5 |x - y| \tag{4.24}$$

$$|\nabla_x \mathbb{G}_+(x, y) - \nabla_x \mathbb{G}^0_+(x, y)| \leq c_6 |x - y|^{-1} \tag{4.25}$$

Proof. Defining $R(x,y) = \mathbb{G}_+(x,y) - \mathbb{G}_+^0(x,y)$, we have that

$$\mathrm{div}((1+(b-1)\chi_+)\nabla R(x,y)) = -k^2(1+((b-1)\chi_+)\mathbb{G}_+(x,y). \tag{4.26}$$

Thus

$$-R(x,y) = k^2 \int_\Omega (1+(b-1)\chi_+)\mathbb{G}_+(z,y)\mathbb{G}_+^0(x,z)dz.$$

Hence for [Li-St-We] we have

$$|R(x,y)| \leq C \int_\Omega |x-z|^{-1}|y-z|^{-1}dz.$$

Let decompose $\Omega = B_{\frac{|x-y|}{3}}(x) \cup B_{\frac{|x-y|}{3}}(y) \cup \mathcal{G}$.
For $z \in B_{\frac{|x-y|}{3}}(x)$ we have that

$$\begin{aligned} |y-z| &\geq |y|-|z| \geq |y|-|z-y|-|x| \\ &\geq |x-y| - \frac{|x-y|}{3} = \frac{2}{3}|x-y|. \end{aligned}$$

Thus

$$\int_{B_{\frac{|x-y|}{3}}(x)} |x-z|^{-1}|y-z|^{-1}dz \leq \frac{2}{3}|x-y|^{-1}\int_0^{\frac{|x-y|}{3}} \rho\, d\rho \leq c|x-y|^2.$$

Similarly it can be evaluated the integral over $B_{\frac{|x-y|}{3}}(y)$.
Let us consider now the integral over $\mathcal{G}$. For $z \in \mathcal{G}$ we have that $|z-y| \geq \frac{|x-z|}{3}$, then we obtain

$$\begin{aligned} &\int_{\mathcal{G}} |x-z|^{-1}|y-z|^{-1}dz \leq c\int_{\mathcal{G}} |x-z|^{-1}|x-z|^{-1}dz \\ \leq\; & c\int_{\Omega\setminus B_{\frac{|x-y|}{3}}(x)} |x-z|^{-1}|x-z|^{-1}dz \\ \leq\; & c\int_{\frac{|x-y|}{3}}^{2\tilde{L}} \rho d\rho \leq c_1|x-y|^{-2} + c_2. \end{aligned}$$

Let us prove now (4.25). We use the interpolation inequality

$$\|\nabla R(\cdot)\|_{L^\infty(Q)} \leq \|R(\cdot)\|^{1-\delta}_{L^\infty(Q)} |\nabla R(\cdot,y)|^\delta_{\alpha,Q}.$$

As in Proposition 4.1, since

$$|\nabla R(\cdot,y)|_{\alpha,Q} \leq h^{-\alpha-2},$$

we obtain

$$|\nabla R(x,y)| \leq ch^{-2+\eta} \leq ch^{-1}.$$

□

5 Proof of Proposition 3.3 and 3.4

Proof of Proposition 3.3. Let us consider $f(y,\overline{w})$, where $\overline{w}$ is a fixed point in $\overline{\mathcal{C}B}$. Since f, as a function of y, is a radiating solution of

$$\mathcal{L}_y f = \Delta_y f + k^2 f = 0 \qquad \text{in } \mathcal{C}\Omega_D,$$

then by [Co-Kr, Theorem 2.14], for $y \in \overline{\mathcal{C}B}$ we have

$$f(y,\overline{w}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m h_n^{(1)}(k|y|) Y_n^m(\hat{y}),$$

where $\hat{y} = y/|y|$, Y_n^m is a spherical harmonic of order n and $h_n^{(1)}$ is a spherical Hankel function of the first kind of order n. Let us consider y such that $R < R_1 < |y| < R_2$. For an integer N, using Schwarz inequality and the asymptotic behavior of Hankel function (see [Co-Kr, (2.38) pg. 28]) we have

$$\begin{aligned}
&\left[\sum_{n=0}^{N} \sum_{m=-n}^{n} a_n^m h_n^{(1)}(k|y|) Y_n^m(\hat{y})\right]^2 \\
\leq\ & \sum_{n=0}^{N} \left|\frac{h_n^{(1)}(k|y|)}{h_n^{(1)}(kR)}\right|^2 \sum_{n=0}^{N} \sum_{m=-n}^{n} |a_n^m|^2 |h_n^{(1)}(kR)|^2 |Y_n^m(\hat{y})|^2. \\
\leq\ & c\sum_{n=0}^{N} \sum_{m=-n}^{n} |a_n^m|^2 |h_n^{(1)}(kR)|^2 |Y_n^m(\hat{y})|^2,
\end{aligned}$$

for some constant c depending on R, R_1 and R_2. Thus, taking the limit as $N \to +\infty$, we can conclude that

$$|f(y,\overline{w})|^2 \leq c|f(\cdot,\overline{w})_{|\partial B}|^2, \quad \forall\, y \in B_{R_2} \setminus \overline{B}_{R_1},$$

where c depends on R, R_1 and R_2. Analogous considerations can be carried on fixing y and varying w. Thus, we can conclude that for all $(y,w) \in \left[B_{R_2} \setminus \overline{B}_{R_1}\right]^2$

$$|f(y,w)| \leq |f_{|\partial B \times \partial B}| \leq c\varepsilon.$$

For $y \in \mathcal{G}^h$, where $\mathcal{G}^h = \{x \in \mathcal{G} \,:\, \mathrm{dist}(x,\Omega_D) \geq h\}$,

$$|S_1(y,\overline{w})| \leq c\int_{D_1} |x-y|^{-2} \leq ch^{-2},$$

where $c = c(L,R)$. Similarly $|S_2(y,\overline{w})| \leq ch^{-2}$. Then we conclude that

$$|f(y,\overline{w})| \leq ch^{-2} \qquad \text{in } \mathcal{G}^h. \tag{5.27}$$

At this stage we shall make use iteratively of the three spheres inequality (see [La, Ku]). Let u be a solution of $\mathcal{L}u = 0$ in $\mathcal{G}$, let $x \in \mathcal{G}$. There exist r_1, r, r_2, $0 < r_1 < r < r_2 < R$ and $\tau \in (0,1)$ such that

$$\|u\|_{L^\infty(B_r(x))} \leq c\|u\|^{\tau}_{L^\infty(B_{r_1}(x))} \|u\|^{1-\tau}_{L^\infty(B_{r_2}(x))}, \tag{5.28}$$

where c and τ depend on R, r/r_2, r_1/r_2 and L. Applying (5.28) to $u(\cdot) = f(\cdot, \overline{w})$, with $x = \overline{x} \in B_{4R} \setminus \overline{B}_{3R}$, $r_1 = r_0/2$, $r = 3r_0/2$ and $r_2 = 2r_0$ we obtain

$$\|f\|_{L^\infty(B_{3r_0/2}(\overline{x}))} \le c\|f\|^{\tau}_{L^\infty(B_{r_0/2}(\overline{x}))}\|f\|^{1-\tau}_{L^\infty(B_{2r_0}(\overline{x}))},$$

For every $\overline{y} \in \mathcal{G}^h$, we denote by γ a simple arc in $\mathcal{G}$ joining $\overline{x}$ to $\overline{y}$. Let us define $\{x_i\}$, $i = 1, \dots, s$ as follows $x_1 = \overline{x}$, $x_{i+1} = \gamma(t_i)$, where $t_i = \max\{t : |\gamma(t) - x_i| = r_0\}$ if $|x_i - \overline{y}| > r_0$, otherwise let $i = s$ and stop the process. By construction, the balls $B_{r_0/2}(x_i)$ are pairwise disjoint, $|x_{i+1} - x_i| = r_0$ for $i = 1, \dots, s-1$, $|x_s - \overline{y}| \le r_0$. There exists β such that $s \le \beta$. An iterated application of the three spheres inequality (5.28) for f (see for instance [Al-Be-Ro-Ve, pg. 780], [Al-DB, Appendix E]) gives that for any r, $0 < r < r_0$

$$\|f\|_{L^\infty(B_{r/2}(\overline{y}))} \le c\|f\|^{\tau^s}_{L^\infty(B_{r/2}(\overline{x}))}\|f\|^{1-\tau^s}_{L^\infty(\mathcal{G})}. \tag{5.29}$$

We can estimate the right hand side of (5.29) by (5.27) and obtain for any r, $0 < r < r_0$

$$\|f\|_{L^\infty(B_{r/2}(\overline{y}))} \le c(h^{-2})^{1-\tau^s}\varepsilon^{\tau^s} \le ch^{-A}\varepsilon^{\tilde{\beta}}, \tag{5.30}$$

where $\tilde{\beta} = \tau^\beta$ and $A = 2(1 - \tilde{\beta})$. Let $O \in \partial D_1$ as defined in (3.6), that is

$$d(O, D_2) = d_\mu(D_1, D_2).$$

There exists a $C^{1,\alpha}$ neighborhood U of O in $\partial\Omega_D$ with constants r_0 and L. Thus there exists a non-tangential vector field $\widetilde{\nu}$, defined on U such that the truncated cone

$$C(O, \widetilde{\nu}(O), \theta, r_0) = \left\{x \in \mathbb{R}^3 : \frac{(x - O) \cdot \widetilde{\nu}(O)}{|x - O|} > \cos\theta,\ |x - O| < r_0\right\} \tag{5.31}$$

satisfies

$$C(O, \widetilde{\nu}(O), \theta, r_0) \subset \mathcal{G},$$

where $\theta = \arctan(1/\overline{L})$. Let us define

$$\lambda_1 = \min\left\{\frac{r_0}{1 + \sin\theta}, \frac{r_0}{3\sin\theta}\right\}, \qquad \theta_1 = \arcsin\left(\frac{\sin\theta}{4}\right),$$

$$G_1 = O + \lambda_1\nu, \qquad \rho_1 = \lambda_1 \sin\theta_1.$$

We have that $B_{\rho_1}(G_1) \subset C(O, \widetilde{\nu}(O), \theta_1, r_0)$, $B_{4\rho_1}(G_1) \subset C(O, \widetilde{\nu}(O), \theta, r_0)$. Let $\overline{G} = G_1$, since $\rho_1 \le r_0/2$, we can use (5.30) in the ball $B_{\rho_1}(\overline{G})$ and we can approach $O \in \partial D_1$ by constructing a sequence of balls contained in the cone $C(O, \widetilde{\nu}(O), \theta_1, r_0)$. We define, for $k \ge 2$

$$G_k = O + \lambda_k\nu, \qquad \lambda_k = \chi\lambda_{k-1}, \qquad \rho_k = \chi\rho_{k-1}, \qquad \text{with } \chi = \frac{1 - \sin\theta_1}{1 + \sin\theta_1}.$$

Hence $\rho_k = \chi^{k-1}\rho_1$, $\lambda_k = \chi^{k-1}\lambda_1$ and

$$B_{\rho_{k+1}}(G_{k+1}) \subset B_{\rho_{3k}}(G_k) \subset B_{\rho_{4k}}(G_k) \subset C(O, \nu, \theta, r_0).$$

Denoting $d(k) = |G_k - O| - \rho_k = \lambda_k - \rho_k$, we have $d(k) = \chi^{k-1} d(1)$, with $d(1) = \lambda_1(1 - \sin\theta)$. For any r, $0 < r \le d(1)$, let $k(r)$ be the smallest integer such that $d(k) \le r$, that is

$$\frac{\left|\log \frac{r}{d(1)}\right|}{|\log \chi|} \le k(r) - 1 \le \frac{\left|\log \frac{r}{d(1)}\right|}{|\log \chi|} + 1.$$

By an iterated application of the three spheres inequality over the chain of balls $B_{\rho_1}(G_1), \dots, B_{\rho_{k(r)}}(G_{k(r)})$, we have

(5.32)
$$\|f(\cdot, \overline{w})\|_{L^\infty(B_{\rho_{k(r)}}(G_{k(r)}))} \le ch^{-A(1-\tau^{k(r)-1})} \varepsilon^{\tilde{\beta}\tau^{k(r)-1}} \le ch^{-A} \varepsilon^{\tilde{\beta}\tau^{k(r)-1}},$$

for $0 < r < cr_0$, where c, $0 < c < 1$, depends on L. Let us consider now $f(y, w)$ as a function of w. First we observe that

$$\mathcal{L}_w f = 0 \quad \text{in } \mathcal{C}\Omega_D, \qquad \text{for all } y \in \mathcal{C}\Omega_D.$$

For $y, w \in \mathcal{G}^h$, $y \neq w$, using (3.9)

$$|S_1(y, w)| \le c \int_{D_1} |x - y|^{-2} |x - w|^{-2} dx \le ch^{-4}.$$

Similarly for S_2. Therefore

$$|f(y, w)| \le ch^{-4} \quad \text{with } y, w \in \mathcal{G}^h.$$

For $w \in B_{4R} \setminus B_{3R}$ and $y \in \mathcal{G}^h$, using (5.32), we have

$$|f(y, w)| \le ch^{-A} \varepsilon^{\tilde{\beta}\tau^{k(r)-1}}.$$

Proceeding as before, let us fix $y \in \mathcal{G}$ such that $\mathrm{dist}(y, \Omega_D) = h$ and $\tilde{w} \in B_{4R} \setminus B_{3R}$ such that $\mathrm{dist}(\tilde{w}, \partial B_R) = R/2$. Taking $r = R/2$, $r_1 = 3r$, $r_2 = 4r$, $w_1 = O + \lambda_1 \nu$ and using iteratively the three spheres inequality, we have

$$\|f(y, w)\|_{L^\infty(B_{R/2}(w_1)} \le \|f(y, w)\|^{\tau^s}_{L^\infty(B_{R/2}(\tilde{w})} \|f(y, w)\|^{1-\tau^s}_{L^\infty(\mathcal{G})},$$

where τ and s are as above. Therefore

$$\begin{aligned} \|f(y, w)\|_{L^\infty(B_{R/2}(w_1)} &\le c(h^{-4})^{1-\tau^s} h^{-A\tau^s} (\varepsilon^{\beta\tau^{k(h)-1}})^{\tau^s} \\ &\le c(h^{-4})^{1-\gamma} h^{-A\tau^s} (\varepsilon^{\beta\tau^{k(h)-1}})^{\gamma} \le ch^{-A'} (\varepsilon^{\beta\tau^{k(h)-1}})^{\gamma}, \end{aligned}$$

where $\gamma = \tau^\beta$, with β as above, so $0 < \gamma < 1$ and $A' = A\tau^s - 4 + \gamma$. Once again, let us apply the three spheres inequality over a chain of balls contained in a cone with vertex in O, choosing $y = w = h\nu(O)$ we obtain

(5.33)
$$|f(y, y)| \le ch^{-A'} (\varepsilon^{\beta\tau^{k(h)-1}})^{\gamma\tau^{k(h)-1}}.$$

We observe that, for $0 < h < cr_0$, where $0 < c < 1$ depends on L, $k(h) \le c|\log h| = -c\log h$, so we can write

$$\tau^{k(h)} = \mathrm{e}^{-c\log h \log \tau} = h^{-c\log\tau} = h^{c|\log\tau|} = h^F,$$

with $F = c|\log\tau|$. Therefore

$$\begin{aligned}|f(y,y)| &\leq h^{-A'}\varepsilon^{B\tau^{k(h)}} = \mathrm{e}^{-A'\log h}\mathrm{e}^{B\tau^{k(h)}\log\varepsilon}\\ &= \mathrm{e}^{-A'\log h + B'h^F\log\varepsilon}\end{aligned}$$

Then in (5.33) we obtain

$$|f(y,y)| \leq \mathrm{e}^{-A'\log h + B'h^F\log\varepsilon} = \frac{\varepsilon^{B'h^F}}{h^{A'}}.$$

□

Proof of Proposition 3.4. Let us define $\overline{r}_2 = \min\{\overline{r}_0, r_2\}$, where $\overline{r}_0$ is the one of Proposition 4.1 and r_2 will be fixed later. For every x, y such that $|x-y| < r$, with $0 < r < \overline{r}_2$, the following asymptotic formula holds (cf. Proposition 4.1)

$$|\mathbb{G}_1(x,y) - \mathbb{G}_+(x,y)| \leq c|x-y|^{-1+\alpha}.$$

We now distinguish two situations:

1) $x \in B_r \cap (D_1 \cap D_2)$;

2) $x \in B_r \cap (D_1 \setminus D_2)$.

If case 1) occurs then the asymptotic formula (4.14) holds also for $\mathbb{G}_2$ since the hypothesis of Proposition 4.1 are met. From [Al, Lemma 3.1] there exists r_2, depending on the a priori data, such that

$$\nabla\mathbb{G}_1(x,y)\cdot\nabla\mathbb{G}_2(x,y) \geq c|x-y|^{-2}. \tag{5.34}$$

Let us consider case 2). In $B_r \cap (D_1 \setminus D_2)$ we consider a smaller ball $B_\rho(0)$ with radius ρ where $0 < \rho < \min\{d_\mu, r_2\}$. Since the definition of d_μ we have $B_\rho \cap D_2 = \emptyset$. If x and y are in B_ρ and denoting by $\mathcal{L} = \Delta + k^2$ we have

$$\mathcal{L}\left(\mathbb{G}_2(x,y) - \Phi(x,y)\right) = 0 \qquad \text{in } B_\rho$$

where Φ is the fundamental solution of the Helmholtz equation, with the boundary condition

$$[\mathbb{G}_2(x,y) - \Phi(x,y)]_{|\partial B_\rho} \leq c\rho^{-1}.$$

Thus by maximum principle

$$|\mathbb{G}_2(x,y) - \Phi(x,y)| \leq c_1\rho^{-1} \qquad \forall\, x, y \in B_\rho$$

and by interior gradient bound

$$|\nabla\mathbb{G}_2(x,y) - \nabla\Phi(x,y)| \leq c_2\rho^{-2} \qquad \forall\, x \in B_{\rho/2}, \forall\, y \in B_\rho.$$

Thus using Lemma 3.1 of [Al], in $B_{\rho/2}(O)$ we obtain the formula formula

$$\nabla\mathbb{G}_1(x,y)\cdot\nabla\mathbb{G}_2(x,y) \geq c|x-y|^{-2} - c_4\rho^{-2}. \tag{5.35}$$

Let us consider $h < \overline{r}_2/2$ and $0 < r < \overline{r}_2$. Then we have

$$
\begin{aligned}
&\left|\int_{D_1} \nabla\mathbb{G}_1(x,y)\cdot\nabla\mathbb{G}_2(x,y)dx\right| \\
= &\left|\int\limits_{D_1\cap B_r(O)} \nabla\mathbb{G}_1(x,y)\cdot\nabla\mathbb{G}_2(x,y) + \int\limits_{D_1\setminus B_r(O)} \nabla\mathbb{G}_1(x,y)\cdot\nabla\mathbb{G}_2(x,y)\right| \\
\geq &\left|\int\limits_{D_1\cap B_r(O)} \nabla\mathbb{G}_1(x,y)\cdot\nabla\mathbb{G}_2(x,y)\right| - \left|\int\limits_{D_1\setminus B_r(O)} \nabla\mathbb{G}_1(x,y)\cdot\nabla\mathbb{G}_2(x,y)\right|
\end{aligned}
$$

The first integral can be estimated as follows

$$
\begin{aligned}
&\left|\int_{D_1\cap B_r(O)} \nabla\mathbb{G}_1(x,y)\cdot\nabla\mathbb{G}_2(x,y)dx\right| \\
= &\left|\int_{(D_1\cap D_2)\cap B_r(O)} \nabla\mathbb{G}_1(x,y)\cdot\nabla\mathbb{G}_2(x,y)dx\right. \\
&\left.+\int_{(D_1\setminus D_2)\cap B_r(O)} \nabla\mathbb{G}_1(x,y)\cdot\nabla\mathbb{G}_2(x,y)dx\right| \\
\geq &\left|\int\limits_{[(D_1\cap D_2)\cap B_\rho(O)]\cup[(D_1\setminus D_2)\cap B_\rho]} \nabla\mathbb{G}_1(x,y)\cdot\nabla\mathbb{G}_2(x,y)dx\right| \\
&-\left|\int\limits_{[(D_1\setminus D_2)\cap B_r(O)]\setminus B_\rho} \nabla\mathbb{G}_1(x,y)\cdot\nabla\mathbb{G}_2(x,y)dx\right|
\end{aligned}
$$

In conclusion, choosing $\rho = h$ and using (5.34), (5.35) and (3.9) we obtain

$$
\begin{aligned}
|S_1(y)| \geq c_1 &\int\limits_{[(D_1\cap D_2)\cap B_\rho(O)]\cup[(D_1\setminus D_2)\cap B_\rho]} |x-y|^{-2}dx \\
- c_2 &\int\limits_{[(D_1\setminus D_2)\cap B_r(O)]\setminus B_\rho} |x-y|^{-1}|x-y|^{-1}dx - c_3 \int\limits_{D_1\setminus B_r(O)} |x-y|^{-1}|x-y|^{-1}dx \\
&\geq c_4 h^{-2} - c_5 d_\mu^{-2} - c_6.
\end{aligned}
$$

□

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