On Continuous-Time White Phase Noise Channels

Luca Barletta

Institute for Advanced Study

Technische Universität München

D-85748 Garching, Germany

luca.barletta@tum.de

Gerhard Kramer

Institute for Communications Engineering

Technische Universität München

D-80333 Munich, Germany

gerhard.kramer@tum.de

Abstract—A continuous-time model for the additive white Gaussian noise (AWGN) channel in the presence of white (memoryless) phase noise is proposed and discussed. It is shown that for linear modulation the output of the baud-sampled filter matched to the shaping waveform represents a sufficient statistic. The analysis shows that the phase noise channel has the same information rate as an AWGN channel but with a penalty on the average signal-to-noise ratio, the amount of penalty depending on the phase noise statistic.

I. INTRODUCTION

A major source of impairment in radio and optical channels is multiplicative phase noise that arises due to instabilities of local oscillators used in up- and down-conversion. Estimation and compensation of phase noise is a classic problem in communication systems theory. In fiber-optic communication systems, phase noise is present due to instabilities of laser oscillators [1] or due to cross-phase modulation in multi-channel systems [2].

In the literature the phase noise process is often modeled as a discrete-time process that is output by a baud-sampled matched filter [3], [4]. This approximate model is valid as long as the variations of the phase noise process are small in one symbol period and the signal-to-noise ratio (SNR) is low or moderate [5]. In general, and especially for phase noise characterized by large linewidths, the channel impaired by phase noise should be modeled and studied in continuous-time, in order to account for the effects of filtering prior to sampling [5]–[7].

Analytical bounds on the capacity of phase noise channels are given in [8], and for high SNR in [9]. Numerical bounds on the information rate transferred through discrete-time Wiener phase noise channels are provided in [4], [10], [11], while for the continuous-time counterpart analytical lower bounds are given in [6], [7] and an analytical upper bound in [12]. Numerical results for the achievable information rates in the case of discrete-time phase noise with an arbitrary spectrum are given in [13], [14].

Few papers in the literature deal with continuous-time phase noise and, apart from trivial cases, the existence of a finite-dimensional sufficient statistic for these channels has not been investigated yet. The aim of this paper is to attempt to model the continuous-time additive white Gaussian noise (AWGN) channel in the presence of *white* phase noise, and to find a (finite-dimensional) sufficient statistic.

The motivation for studying white phase noise is that there are problems for which the phase varies more rapidly than the bandwidth of the receiving device, e.g., a "square law" detector that necessarily has limited bandwidth. In such cases, the device will "lose" energy and we wish to model this effect.

The paper is organized as follows. In Sec. II previous works on this topic are analyzed in detail. The system model is introduced and discussed in Sec. III, while a sufficient statistic is derived in Sec. IV. Results are discussed in Sec. V, and conclusions are drawn in Sec.VI.

II. PREVIOUS WORK

The output of a baud-sampled matched filter represents a sufficient statistic for the continuous-time AWGN channel with a constant phase shift given by a random parameter (see, e.g., [15, Sec. 4.4.1]).

In the context of continuous-time Wiener phase noise channels, the authors of [5] show through numerical simulations that a multi-sample receiver achieves significantly higher information rates than the baud-sampled matched filter, especially at high SNR. The dimensionality of the sufficient statistic for this channel with memory has not been investigated.

In the context of continuous-time memoryless phase noise channels, the authors of [16, Eq. (28)] model the *partially coherent* channel as

$$Y(t) = X(t)e^{j\Theta(t)} + W(t), \qquad 0 \le t \le T \qquad (1)$$

where W is a complex-valued AWGN process and Θ is a white phase noise process used to model the nonlinear effects of cross-phase modulation in multichannel fiber-optic communication systems. *White* means that $\Theta(t_1)$ and $\Theta(t_2)$ are uncorrelated random variables for any t_1, t_2 in the interval $[0, T]$ with $t_1 \ne t_2$. If, in addition, Θ is stationary and $\Theta(t)$ is distributed according to a wrapped Gaussian with zero mean and variance σ^2, then the autocorrelation function $R_{e^{j\Theta}}$ of the multiplicative disturbance $e^{j\Theta(t)}$ is

$$R_{e^{j\Theta}} = \mathsf{E}\left[e^{j\Theta(t)}e^{-j\Theta(t+\tau)}\right] = \begin{cases} 1 & \tau = 0 \\ e^{-\sigma^2} & \tau \ne 0 \end{cases} \qquad (2)$$

$$= e^{-\sigma^2} + \lim_{B \to \infty} (1 - e^{-\sigma^2}) \cdot \mathrm{sinc}(B\tau). \qquad (3)$$

The power spectral density (PSD) $S_y(f)$ of the received signal $Y(t)$ is given by

$$S_y(f) = \mathcal{F}\{R_y(\tau)\} = S_x(f) \star S_{e^{j\Theta}}(f) + S_w(f)$$
$$= e^{-\sigma^2} S_x(f) + \lim_{B \to \infty} S_x(f) \star (1 - e^{-\sigma^2}) \frac{\text{rect}(f/B)}{B}$$
$$+ S_w(f), \quad (4)$$

where $\mathcal{F}\{\cdot\}$ denotes the Fourier transform and $\star$ the convolution operator. Equation (4) shows that the PSD of the input signal $S_x(f)$ experiences a *spectral loss* [16], that is a gain equal to $e^{-\sigma^2} \leq 1$, and the remaining power proportional to $(1 - e^{-\sigma^2})$ is spread onto an infinite bandwidth. Later in this paper we will see that when using projection receivers the spectral loss is unavoidable.

III. System Model

Consider linear modulation for which we express the output of the continuous-time white phase noise channel as

$$Y(t) = X(t)e^{j\Theta(t)} + W(t)$$
$$= \sum_k A_k g(t - kT)e^{j\Theta(t)} + W(t)$$
$$= \sum_k A_k g_k(t)e^{j\Theta(t)} + W(t), \quad (5)$$

where $\{A_k\}$ is a sequence of independent and identically distributed (iid) random variables, T is the symbol period, j is the imaginary unit, $g(t)$ is the pulse shaping filter such that $\{g_k(t)\}$ is a set of orthonormal functions of $L^2(\mathbb{R})$ with the property

$$\int_{-\infty}^{\infty} g_k(t)g_l^{\star}(t)\, \mathrm{d}t = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases} \quad (6)$$

where the superscript $\star$ denotes complex conjugate, and W is a complex-valued circularly symmetric white Gaussian noise. Let

$$\{\phi_{nm}(t), n = 0, 1, \ldots; m \in \mathbb{Z}\} \quad (7)$$

be a complete orthonormal basis of $L^2(\mathbb{R})$ with $\phi_{nm}(t) = \phi_n(t - mT)$ and such that

$$\int_{-\infty}^{\infty} \phi_{nm}(t)\phi_{n'm'}^{\star}(t)\, \mathrm{d}t = \begin{cases} 1 & n = n', m = m' \\ 0 & \text{otherwise,} \end{cases} \quad (8)$$

and

$$\int_{-\infty}^{\infty} g_k(t)\phi_{nm}^{\star}(t)\, \mathrm{d}t = \begin{cases} 1 & n = 0, m = k \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

The basis functions in (7) are double indexed: the index nm denotes the n-th basis function $\phi_n(t)$ translated by mT in time, i.e., $\phi_n(t - mT)$. The choice of using a double index for basis functions is because orthogonality is also obtained by translation, as shown in (8). Property (9) states that the pulse shaping function $g(t)$ has a non-zero projection only in the first dimension, i.e., for $n = 0$. This means that $\phi_{0m}(t) = g_m(t)$.

Note that assuming property (9) comes without loss of generality, because we can always obtain all the other basis functions $\phi_n(t)$ for $n > 0$ with an orthonormalization procedure.

Example. *A complete orthonormal basis of $L^2(\mathbb{R})$ is the trigonometric system*

$$\{\phi_{nm}(t) = \phi_n(t - mT), n = 0, 1, \cdots; m \in \mathbb{Z}\}, \quad (10)$$

where

$$\phi_n(t) = \frac{1}{\sqrt{T}} \exp\left(j\frac{2\pi}{T}nt\right) \quad (11)$$

for $0 \leq t < T$ and $\phi_n(t)$ is zero elsewhere. The square pulse shaping filter

$$g_0(t) = \begin{cases} 1/\sqrt{T} & 0 \leq t < T \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

is such that (6) and (9) are satisfied.

The phase noise process Θ is defined as a stationary random process such that

$$\mathsf{E}\left[\left\langle g_k(t)e^{j\Theta(t)}, \phi_{nm}(t) \right\rangle\right] \triangleq \mu_\Theta \cdot \begin{cases} 1 & n = 0, m = k \\ 0 & \text{otherwise,} \end{cases} \quad (13)$$

$$\mathsf{E}\left[\left\langle g_k(t)e^{j\Theta(t)}, \phi_{nm}(t) \right\rangle \cdot \left\langle g_{k'}(s)e^{-j\Theta(s)}, \phi_{n'm'}(s) \right\rangle\right]$$
$$\triangleq |\mu_\Theta|^2 \cdot \begin{cases} 1 & n = n' = 0, m = k, m' = k', \\ 0 & \text{otherwise,} \end{cases} \quad (14)$$

where $\mu_\Theta = \mathsf{E}\left[e^{j\Theta(t)}\right]$. Note that (13) and (14) give

$$\mathsf{Var}\left[\left\langle g_k(t)e^{j\Theta(t)}, \phi_{nm}(t) \right\rangle\right] = 0 \quad (15)$$

for any choice of k, n, and m. In other words, we can view the projections $\left\langle g_k(t)e^{j\Theta(t)}, \phi_{nm}(t) \right\rangle$ as constants.

Defining the phase noise process through (13)-(15) is reasonable if we interpret the model (5) as the limit of successive refinements of a related discrete-time model. For the discrete-time model, consider the finite interval $[-S, S]$ and time instants $t_i = iS/l$ for $i = -l, \ldots, l - 1$. The extension to the whole real axis is straightforward by letting $S \to \infty$. Since the continuous-time AWGN $W(t)$ is defined only when using projection receivers, also for the phase noise process it is desirable to preserve the statistical properties of its projections onto the basis functions, when passing from the discrete-time to the continuous-time model. Mathematically speaking, it is useful to define

$$\left\langle g_k(t)e^{j\Theta(t)}, \phi_{nm}(t) \right\rangle \triangleq \lim_{l \to \infty} \frac{1}{2l} \sum_{i=-l}^{l-1} g_k(t_i)e^{j\Theta(t_i)}\phi_{nm}^{\star}(t_i), \quad (16)$$

where the equality represents some kind of convergence. The condition and the type of convergence is given by the following Lemma.

Lemma. *If $|g_k(t)\phi_{nm}(t)|$ is uniformly bounded in t, then the right-hand side of (16), that is the projection of the phase noise process $g_k(t)e^{j\Theta(t)}$ onto the basis function $\phi_{nm}(t)$ of $L^2([-S,S])$, converges (almost surely) to the expected value of the projection:*

$$\left\langle g_k(t)e^{j\Theta(t)}, \phi_{nm}(t) \right\rangle \stackrel{a.s.}{=} \mu_\Theta \int_{-S}^{S} g_k(t)\phi_{nm}^\star(t)\,\mathrm{d}t, \quad (17)$$

for any value of k, n, and m.

Proof: We show the convergence result for the real part of the projection; the procedure for the imaginary part is analogous. Consider the independent random variables

$$Z_{i+1} = \Re\{g_k(t_i)\phi_{nm}^\star(t_i)e^{j\Theta(t_i)}\}, \qquad i = 0, \dots, l-1, \quad (18)$$

where $\Re\{x\}$ is the real part of complex number x, then Kolmogorov's criterion for the strong law of large numbers [17, Th. 1-3, pag. 238] for $l \to \infty$ reads as

$$
\begin{aligned}
\sum_{i=1}^{\infty} \frac{\mathsf{Var}\,[Z_i]}{i^2} &\leq \sum_{i=1}^{\infty} \frac{\mathsf{E}\left[\Re\{g_k(t_{i-1})\phi_{nm}^\star(t_{i-1})e^{j\Theta(t_{i-1})}\}^2\right]}{i^2} \\
&\leq \sum_{i=1}^{\infty} \frac{|g_k(t_{i-1})\phi_{nm}(t_{i-1})|^2}{i^2} \\
&\leq \frac{K\pi^2}{6} < \infty,
\end{aligned}
\quad (19)
$$

where (19) follows by the boundedness condition $|g_k(t)\phi_{nm}(t)|^2 \leq K$ for any $t \in [-S,S]$, with $K < \infty$. The finiteness of (19) implies the strong law of large numbers for independent and non-identically distributed random variables:

$$
\begin{aligned}
\frac{1}{l}\sum_{i=1}^{l} Z_i &\stackrel{a.s.}{\longrightarrow} \lim_{l\to\infty} \frac{1}{l}\sum_{i=1}^{l} \mathsf{E}\,[Z_i] \\
&= \Re\left\{\mu_\Theta \int_{0}^{S} g_k(t)\phi_{nm}^\star(t)\,\mathrm{d}t\right\}
\end{aligned}
\quad (20)
$$

where the last equality follows by the definitions of μ_Θ and of the Riemann integral.

Proving the convergence in the time interval $[-S,0]$ is analogous, and the two parts together prove the thesis. ∎

Result (17) is in accordance with the definitions (13)-(15). The condition of boundedness of the product of the basis functions required by the Lemma is mild and can be easily met, therefore we assume it is met.

Note that the proof of the Lemma requires a strong condition: independence among phase noise samples $\Theta(t_i)$. But this independence condition is not unrealistic, as phase noise samples can be uncorrelated and Gaussian distributed, and thus independent. This can be the case, e.g., with phase noise generated by cross-phase modulation caused by neighboring channels in multichannel fiber-optic communication systems [2], [16].

IV. SUFFICIENT STATISTIC

The average mutual information, per unit time, between the input and the output of the continuous-time channel in (5) is defined as [18, pag. 370]

$$I(X;Y) \triangleq \lim_{M\to\infty} \frac{1}{2M+1} \lim_{N\to\infty} I(X^{N,M}; Y^{N,M}), \quad (21)$$

where

$$X^{N,M} = (X_0^M, X_1^M, \dots, X_N^M), \quad (22)$$

and

$$X_n^M = (X_{n,-M}, X_{n,-M+1}, \dots, X_{nM}), \quad (23)$$

where

$$
\begin{aligned}
X_{nm} &= \langle X(t), \phi_{nm}(t)\rangle \quad &(24) \\
&= \sum_k A_k \left\langle g_k(t), \phi_{nm}(t)\right\rangle = \begin{cases} A_m & n = 0, \\ 0 & n > 0. \end{cases} \quad &(25)
\end{aligned}
$$

In other words, $X^{N,M}$ is a vector that collects the $(N+1)\cdot(2M+1)$ projections of process X onto the basis functions $\phi_{nm}(t)$ for $n = 0, 1, \dots, N$ and $m = -M, -M+1, \dots, M$. An analogous notation holds for process Y.

The projection of Y onto $\phi_{nm}(t)$ is

$$
\begin{aligned}
Y_{nm} &= \langle Y(t), \phi_{nm}(t)\rangle \quad &(26) \\
&= \left\langle X(t)e^{j\Theta(t)} + W(t), \phi_{nm}(t)\right\rangle \quad &(27) \\
&= \sum_k A_k \left\langle g_k(t)e^{j\Theta(t)}, \phi_{nm}(t)\right\rangle + W_{nm} \quad &(28) \\
&\stackrel{a.s.}{=} \mu_\Theta \sum_k A_k \int_{-\infty}^{\infty} g_k(t)\phi_{nm}^\star(t)\,\mathrm{d}t + W_{nm} \quad &(29)
\end{aligned}
$$

where the W_{nm}'s are iid circularly-symmetric Gaussian variates, and the last step follows by the Lemma in the previous section. By using (9) in (29) we have

$$Y_{nm} \stackrel{a.s.}{=} \begin{cases} \mu_\Theta A_m + W_{nm} & n = 0, \\ W_{nm} & n > 0. \end{cases} \quad (30)$$

Since Y_{nm} does not carry information about the input A_m for $n > 0$, and the sequences of random variables converge almost surely to their limits, we have

$$
\begin{aligned}
I(X;Y) &= \lim_{M\to\infty} \frac{1}{2M+1} I(X_0^M; Y_0^M) \quad &(31) \\
&= \lim_{M\to\infty} \frac{1}{2M+1} \sum_{m=-M}^{M} I(X_{0m}; Y_{0m}) \quad &(32) \\
&= I(X_{00}; Y_{00}) = I(A_0; Y_{00}) \quad &(33)
\end{aligned}
$$

where (32) follows because the channel is memoryless by (30), and (33) follows by stationarity and by (25). The result (33) shows that a sufficient statistic for inferring A_k given Y is

$$Y_{0k} = \langle Y(t), \phi_{0k}(t)\rangle = \langle Y(t), g_k(t)\rangle \quad (34)$$

that is the output of the sampled filter that is matched to $g(t)$.

V. Discussion

By virtue of (30), at the output of the sampled matched filter the continuous-time white phase noise channel is equivalent to a discrete-time AWGN channel

$$Y_{0k} = \mu_\Theta A_k + W_{0k} \tag{35}$$

so the effect of the spectral loss μ_Θ is to reduce the SNR. In the limit of $\mu_\Theta \to 0$, reliable communication over this channel is not possible.

We conclude that the presence of rapidly varying continuous-time phase noise affects not only the information encoded in the *phase* of $X(t)$, but also the information encoded in its *amplitude*. We further conclude that even a noncoherent receiver is affected by white phase noise and suffers from an SNR loss. This is true also for fiber-optic communication systems based on *direct detection*, e.g., when the front-end is a photodiode that performs photon counting, the numbers of photons being proportional to the energy the photodiode harvests. The output of the photodiode is usually modeled as the modulus-square of the field at the input, so in principle the phase noise would not be an issue. But real photodiodes are bandlimited and should be modeled as being preceded by a bandpass filter that reduces the energy collected by the receiver, and the presence of these filters will cause SNR loss.

VI. Conclusion

We have discussed the modeling of the continuous-time AWGN channel in the presence of white (memoryless) phase noise. We have found that, under mild conditions, the sufficient statistic for this channel is the output of a baud-sampled matched filter, and that the channel is equivalent to a discrete-time AWGN channel with an SNR penalty.

References

[1] G. Foschini, L. Greenstein, and G. Vannucci, "Noncoherent detection of coherent lightwave signals corrupted by phase noise," *IEEE Trans. Commun.*, vol. 36, no. 3, pp. 306–314, 1988.

[2] R. Essiambre, G. Kramer, P. Winzer, G. Foschini, and B. Goebel, "Capacity limits of optical fiber networks," *J. Lightwave Technol.*, vol. 28, no. 4, pp. 662–701, 2010.

[3] A. Spalvieri and L. Barletta, "Pilot-aided carrier recovery in the presence of phase noise," *IEEE Trans. Commun.*, vol. 59, no. 7, pp. 1966–1974, 2011.

[4] L. Barletta, M. Magarini, and A. Spalvieri, "The information rate transferred through the discrete-time Wiener's phase noise channel," *J. Lightwave Technol.*, vol. 30, no. 10, pp. 1480–1486, 2012.

[5] H. Ghozlan and G. Kramer, "Multi-sample receivers increase information rates for Wiener phase noise channels," in *Global Telecom. Conf. (GLOBECOM)*, Dec. 2013, pp. 1919–1924.

[6] ——, "On Wiener phase noise channels at high signal-to-noise ratio," in *IEEE Int. Symp. Inf. Theory (ISIT)*, 2013, pp. 2279–2283.

[7] ——, "Phase modulation for discrete-time Wiener phase noise channels with oversampling at high SNR," in *IEEE Int. Symp. Inf. Theory (ISIT)*, 2014.

[8] M. Katz and S. Shamai, "On the capacity-achieving distribution of the discrete-time noncoherent and partially coherent AWGN channels," *IEEE Trans. Inf. Theory*, vol. 50, no. 10, pp. 2257–2270, 2004.

[9] A. Lapidoth, "On phase noise channels at high SNR," in *IEEE Inf. Theory Workshop*, 2002, pp. 1–4.

[10] L. Barletta, M. Magarini, and A. Spalvieri, "Estimate of information rates of discrete-time first-order Markov phase noise channels," *IEEE Photon. Technol. Lett.*, vol. 23, no. 21, pp. 1582–1584, 2011.

[11] A. Barbieri and G. Colavolpe, "On the information rate and repeat-accumulate code design for phase noise channels," *IEEE Trans. Commun.*, vol. 59, no. 12, pp. 3223–3228, 2011.

[12] L. Barletta and G. Kramer, "Signal-to-noise ratio penalties for phase noise channels," in *Intern. Conf. on Cognitive Radio Oriented Wirel. Networks (CROWNCOM 2014)*, Jun. 2014.

[13] L. Barletta, M. Magarini, and A. Spalvieri, "Tight upper and lower bounds to the information rate of the phase noise channel," in *IEEE Int. Symp. Inf. Theory (ISIT)*, 2013, pp. 2284–2288.

[14] L. Barletta, M. Magarini, S. Pecorino, and A. Spalvieri, "Upper and lower bounds to the information rate transferred through first-order Markov channels with free-running continuous state," *IEEE Trans. Inf. Theory*, 2014, forthcoming.

[15] H. Van Trees, *Detection, Estimation, and Modulation Theory: Part I*. Wiley, 1968.

[16] B. Goebel, R. Essiambre, G. Kramer, P. Winzer, and N. Hanik, "Calculation of mutual information for partially coherent Gaussian channels with applications to fiber optics," *IEEE Trans. Inf. Theory*, vol. 57, no. 9, pp. 5720–5736, 2011.

[17] W. Feller, *An Introduction to Probability Theory and its Applications, 2nd ed., 2nd volume*. Wiley, 1971.

[18] R. Gallager, *Information Theory and Reliable Commmunication*. Wiley, 1968.