An Adaptive Markov Chain Monte Carlo Method for Bayesian Finite Element Model Updating

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ABSTRACT

In this paper, an adaptive Markov Chain Monte Carlo (MCMC) approach for Bayesian finite element model updating is presented. This approach is known as the Adaptive Hamiltonian Monte Carlo (AHMC) approach. The convergence rate of the Hamiltonian/Hybrid Monte Carlo (HMC) algorithm is high due to its trajectory which is guided by the derivative of the posterior probability distribution function. This can lead towards high probability areas in a reasonable period of time. However, the HMC performance decreases when sampling from posterior functions of high dimension and when there are strong correlations between the uncertain parameters. The AHMC approach, a locally adaptive version of the HMC approach, allows efficient sampling from complex posterior distribution functions and in high dimensions. The efficiency and accuracy of the AHMC method are investigated by updating a real structure.

Keywords: Finite Element Model updating; Bayesian; Markov Chain Monte Carlo; Hybrid Monte Carlo; Adaptive.

1. Introduction

The Finite Element Model (FEM) [1, 2, 3] is one of the famous numerical methods that can be useful in numerous engineering areas such as Mechanical, Civil and Electrical Engineering. This approach derives approximate numerical models for real systems (or structures). The obtained numerical models are very accurate for simple structures or systems; however, the approximation degrades when the modelled system is sufficiently complex, and the results obtained from the Finite Element Models (FEMs) are different from those obtained from experiments. Several factors can cause the accuracy degradation of the FEMs, such as the variability of certain model parameters and the errors resulting from the modelling process. To improve the FEMs, some of the uncertain model parameters need to be adjusted to reduce the error between the measured data and the numerical model [4, 5]. The process of adjusting these uncertain parameters, to determine the most probable parameters that accurately describes the structure in the presence of the measured responses of the system, is known as "model updating" [4, 5].

They are two main strategies to perform FEM updating: the direct and indirect (iterative) updating strategies [4]. To perform a direct updating strategy, the FEM output is directly equated to the measured data. However, in the iterative strategy the differences between the measured data and the FEM output are minimized by adjusting some "flexible" uncertain variables [4]. The most used iterative algorithms are the sensitivity-based methods [4, 5] and the metaheuristics algorithms [5, 6, 7]. These methods are typically optimization problems and their objective functions are described by the error between the analytical and experimental data. The objective functions of an iterative updating strategy are then minimized by finding the optimal solution of the updated parameters of the FEM.

Other approaches used in FEM updating are to implement statistical theories [8, 9, 10]. Statistical methods (or uncertainty quantification methods) are very useful mathematical tools that can be used to update FEMs and to provide more information about the variability of the updated parameters. The use of statistical theories in model updating problems has become very attractive in the last ten years. The statistical methods that have been use in updating FEMs are divided into two main classes: the class of non-probabilistic (possibilistic) methods such as the fuzzy logic method [11, 12], which can be used to estimate the uncertainty of the model parameters where the uncertain parameters are characterized by membership functions. The second class describes the probabilistic methods where these methods treat the uncertain parameter as a random distribution with a joint probability density function (PDF) [8, 9]. The Bayesian approach [5, 10, 13, 14, 15, 16, 17], which is one of the most common probabilistic methods along with perturbation methods [9], has become very attractive in system identification and uncertainty quantification. This approach has been used numerous times for FEM updating and showed promising results for uncertainty quantification [17]. The uncertain

parameters, in the Bayesian method, are modelled as random parameters with joint probability density functions (PDFs), which is also known as the *posterior* PDF. In the case that the posterior PDF is analytically unavailable, sampling methods can be implemented to obtain numerical solutions for the posterior PDF. The most attractive sampling techniques are the Markov chain Monte Carlo (MCMC) methods. The MCMC methods have been widely used for FEM updating where the most recognized MCMC algorithm is the Metropolis–Hastings (M-H) algorithm. This algorithm has been implemented numerous times in the FEM updating problems and uncertainty quantifications [10, 13, 16, 17]. Moreover, several MCMC algorithms were used to solve Bayesian model updating problems. A Monte-Carlo inverse approach was used by Mares et al. [18] for stochastic (or Bayesian) model updating. Nichols et al. [19] applied an MCMC algorithm to sample from the posterior PDF of some nonlinear systems. The Gibbs sampling technique was implemented by Ching et al. [20] for model updating problems. Ching and Cheng [21] introduced a modified version of the M-H algorithm called the Transitional Markov Chain Monte Carlo (TMCMC) algorithm. The TMCMC algorithm was used by Muto and Beck [22] to update hysteretic structural models. Cheung and Beck [23] implemented the Hybrid Monte Carlo (HMC) method to update a structural dynamic linear system with 31 uncertain parameters. The updating process was successfully able to characterize uncertainties associated with the underlying structural system.

The HMC method had shown promising results to solve higher-dimensional complex problems. The trajectory of the HMC algorithm, which is guided by the derivative of the posterior log-density, facilitates the convergence to areas of high probability in a limited time (or limits the number of iterations) during the searching process [17, 24, 25, 26]. In the HMC algorithm, a Molecular Dynamic (MD) system is created and its total energy, or the Hamiltonian function, is used to draw samples. The total energy is evaluated through time numerically by using the leapfrog integrator (or algorithm). Unfortunately, this integrator (or even other integrators that can be used by the HMC algorithm) does not conserve the Hamiltonian function, especially when a relatively large time step is needed to speed up the convergence process or when the system size is relatively large. Boulkaibet et al. tried to solve this problem by implementing two modified versions of the HMC algorithm called the shadow hybrid Monte Carlo (SHMC) [13] and the separable shadow hybrid Monte Carlo (S2HMC) [15] algorithms for Bayesian finite element model updating. Both of these algorithms produced samples with a relatively large time step and give more accurate results than the HMC algorithm.

In this paper, another modification of the HMC algorithm is proposed. The idea is to deal with the acceptance rate (AR) degradation and to improve the results accuracy. First, the algorithm will adaptively choose the trajectory length to obtain a good Acceptance Rate (AR) without wasting computation time. This can be done by adjusting the trajectory length at every iteration to keep the AR acceptable (and controlled), and with relatively large trajectory length. Secondly, the algorithm should deal with separated regions (two high probability areas are isolated by regions of low probability). Most of the MCMC algorithms (including the HMC) have difficulties to move from one search region to another when these two regions are separated by other regions with low probability. This can be a problem when the obtained samples are only obtained from the regions with local minima. To overcome this problem, the trajectory of the AHMC algorithm is adjusted and the samples are obtained from a sequence of distributions that are more diffuse than the original posterior PDF. In this paper, an Adaptive HMC algorithm is introduced to sample the posterior PDF. This method is investigated by updating two structural examples: a three degree of freedom (DOF) linear system and an unsymmetrical H-shaped structure. The advantages and disadvantages of the AHMC technique will be discussed.

This paper is organized as following: first, the posterior distribution function of the uncertain parameters is presented in Section 2. Section 3 discusses the AHMC algorithm. Section 4 shows the implementation of Bayesian FEM updating on a 3 DOF linear system. In Section 5, the AHMC algorithm is used to update an unsymmetrical H-shaped structure with real measured data. Section 6 concludes the paper.

2. The Bayesian Theory

The Bayesian approach is governed by Bayes rule [13, 15, 17, 27]:

$$P(\boldsymbol{\theta}|\mathcal{D},\boldsymbol{\mathcal{M}}) \propto P(\mathcal{D}|\boldsymbol{\theta},\boldsymbol{\mathcal{M}})P(\boldsymbol{\theta}|\boldsymbol{\mathcal{M}}) \tag{1}$$

where $\boldsymbol{\theta} \in \boldsymbol{\Theta} \subset \mathcal{R}^d$ is the uncertain parameter vector that needs to be updated. The $\boldsymbol{\mathcal{M}}$ notation indicates the model class of the target system to be updated. Usually, the model classes are separated by the updating vector $\boldsymbol{\theta}$. $\mathcal{D}$ represents the experimental measurements (frequencies f_i^m, mode shapes ϕ_i^m, …). The $P(\boldsymbol{\theta}|\boldsymbol{\mathcal{M}})$ PDF is the prior knowledge of the uncertain parameters when the model class is known. $P(\mathcal{D}|\boldsymbol{\theta},\boldsymbol{\mathcal{M}})$ represented the likelihood function, which is obtained by the difference between the measurements and the FEM data when both $\boldsymbol{\theta}$ and $\boldsymbol{\mathcal{M}}$ are given. The term $P(\boldsymbol{\theta}|\mathcal{D},\boldsymbol{\mathcal{M}})$ is The posterior PDF which represents the probability of update parameters when both $\mathcal{D}$ and $\boldsymbol{\mathcal{M}}$ are given. Since only a single model class is considered in this paper, the notation $\boldsymbol{\mathcal{M}}$ will be ignored for the remaining equations.

The posterior $P(\boldsymbol{\theta}|\mathcal{D})$ PDF used in this paper is the same one used in [13, 15, 17]:

$$P(\boldsymbol{\theta}|\mathcal{D}) = \frac{1}{Z_s(\alpha,\beta_c)} \exp\left(-\frac{\beta_c}{2}\sum_i^{N_m}\left(\frac{f_i^m - f_i}{f_i^m}\right)^2 - \sum_i^Q \frac{\alpha_i}{2}\left\|\boldsymbol{\theta^i} - \boldsymbol{\theta_0^i}\right\|^2\right) \quad (2)$$

where $Z_s(\alpha, \beta_c)$ is a normalizing constant given by [13, 15, 17]

$$Z_s(\alpha,\beta_c) = \left(\frac{2\pi}{\beta_c}\right)^{N_m/2} \prod_{i=1}^{N_m} f_i^m \, (2\pi)^{Q/2} \prod_{i=1}^{Q} \frac{1}{\sqrt{\alpha_i}} \quad (3)$$

β_c is a constant that can be used to give more weight to the likelihood terms, f_i and f_i^m are the ith analytical natural frequency and the ith measured natural frequency. N_m is the number of measured modes used for the updating process. Q is the size of the parameter vector , $\boldsymbol{\theta}_0$ is the vector of initial values of the parameters, which are usually the mean values. α_i is the ith coefficient of the ith updating variable and these coefficients can be used to weight the prior PDF. The notation $\|*\|$ represents the Euclidean norm of the quantity $*$. In complex structures, obtaining an analytical solution from the posterior PDF is not possible. Sampling techniques however can provide a numerical solution of the PDF in Eq. (1) [13, 15, 17]. In this work, the adaptive HMC algorithm is implemented to sample from the posterior PDF.

3. Adaptive Hybrid Monte Carlo

The main idea of the AHMC algorithm is to improve the HMC trajectory by providing an adaptive trajectory length as well as a tempered trajectory. The AHMC algorithm [28, 29] is based on the original HMC algorithm (which is introduced by Duane et al. [24]). The HMC algorithm has shown encouraging results for solving higher-dimensional complex engineering problems [13, 15, 16, 17, 23, 28]. The main idea of the HMC algorithm is to combine the Molecular Dynamic (MD) trajectory and the Monte Carlo (MC) accept/rejection step. The same concepts are used for the AHMC algorithm where a new dynamical system is constructed by introducing a new auxiliary variable, called momentum, $\mathbf{p} \in R^d$. The uncertain vector $\boldsymbol{\theta}$ will be treated as the system displacement while the total energy (the Hamiltonian function) of the new dynamical system can be defined as: $H(\boldsymbol{\theta}, \mathbf{p}) = V(\boldsymbol{\theta}) + W(\mathbf{p})$. The potential energy is defined by $V(\boldsymbol{\theta}) = -\ln\big(P(\boldsymbol{\theta}|D)\big)$ while the kinetic energy of the dynamic system is given by $W(\mathbf{p}) = \mathbf{p^T M^{-1} p}/2$, where the matrix $\mathbf{M} \in R^{d\times d}$ is positive definite.

The Hamiltonian dynamics are then governed by:

$$\frac{d\boldsymbol{\theta}}{dt} = \mathbf{M^{-1}}\mathbf{p}(t), \qquad \frac{d\boldsymbol{p}}{dt} = -\nabla V\big(\boldsymbol{\theta}(t)\big) \quad (4)$$

In this paper, the joint density function $\rho(\boldsymbol{\theta}, \mathbf{p})$ follows a Boltzmann distribution [15, 28] where $\rho(\boldsymbol{\theta}, \mathbf{p}) \propto \exp(-\beta_B H(\boldsymbol{\theta}, \mathbf{p}))$, where $\beta_B = \frac{1}{T}$ and T is a constant temperature (the Boltzmann constant is neglected). It is easy to see that $\rho(\boldsymbol{\theta}, \mathbf{p})$ can be rewritten as $\rho(\boldsymbol{\theta}, \mathbf{p}) \propto \exp\left(-\frac{V(\boldsymbol{\theta})}{T}\right).\exp\left(-\frac{W(\boldsymbol{p})}{T}\right)$, or

$$\rho(\boldsymbol{\theta}, \mathbf{p}) \propto P(\boldsymbol{\theta}|D).\exp\left(-\frac{1}{T}\mathbf{p^T M^{-1} p}/2\right) \quad (5)$$

Equation (5) shows that sampling the vector $\boldsymbol{\theta}$ from the posterior PDF can also be achieved by sampling the pair of vectors $(\boldsymbol{\theta}, \mathbf{p})$ from the joint PDF $\rho(\boldsymbol{\theta}, \mathbf{p})$. The pair $(\boldsymbol{\theta}, \mathbf{p})$ is evaluated through time t by using the following leapfrog integrator [13, 15, 16, 17, 23, 28]:

$$\mathbf{p}\left(t + \frac{\delta t}{2}\right) = \mathbf{p}(t) - \frac{\delta t}{2}\nabla V\big(\boldsymbol{\theta}(t)\big) \quad (6)$$

$$\boldsymbol{\theta}(t + \delta t) = \boldsymbol{\theta}(t) + \delta t \mathbf{M^{-1}}\mathbf{p}\left(t + \frac{\delta t}{2}\right) \quad (7)$$

$$\mathbf{p}(t + \delta t) = \mathbf{p}\left(t + \frac{\delta t}{2}\right) - \frac{\delta t}{2}\nabla V\big(\boldsymbol{\theta}(t + \delta t)\big) \quad (8)$$

where δt is the time step and ∇V is the gradient which can be obtained numerically by finite difference as [15, 23]

$$\frac{\partial V}{\partial \theta_i} \cong \frac{V(\boldsymbol{\theta}+\Delta h) - V(\boldsymbol{\theta}-\Delta h)}{2h\Delta_i} \quad (9)$$

h is a scalar that dictates the size of the perturbation of $\boldsymbol{\theta}$, while $\Delta = [\Delta_1, \Delta_2, \ldots, \Delta_N]$ is the perturbation vector. After the evaluation of Eqs. (6-8), an MC accept-reject step is added to satisfy the property of Eq. (4). Thus, if the pair $(\boldsymbol{\theta}, \mathbf{p})$ is the initial vector while the pair $(\boldsymbol{\theta}^*, \mathbf{p}^*)$ is the vector after evaluating Eqs. (6-8), then the candidate $(\boldsymbol{\theta}^*, \mathbf{p}^*)$ is accepted with probability $\min\left(1, \exp\left\{-\frac{\Delta H}{T}\right\}\right)$, $\Delta H = H(\boldsymbol{\theta}^*, \mathbf{p}^*) - H(\boldsymbol{\theta}, \mathbf{p})$. Since the AHMC algorithm has the same basics as the

HMC algorithm, all previous equations and properties will be used by the AHMC algorithm. However, the AHMC algorithm has certain modifications that improves the sampling performance. To avoid the non-ergodicity problem and to ensure a good performance of the AHMC algorithm, the evaluation of the leapfrog algorithm needs to be done L steps during each iteration. This will increase the algorithm trajectory and ensure large steps. The value of L can be uniformly chosen from the interval $\{1, L_{max}\}$. Moreover, since the time step used by the leapfrog integrator is bounded ($\delta t_{min} < \delta t < \delta t_{max}$), the time step will be careful adjusted after a number of iterations. Adjusting the time step after a fixed number of iterations avoids a large rejection rate when the time step is large [28, 29]. Also, the adaptation of the time step avoids using a small trajectory length and therefore more iteration are needed to converge. To adapt the time step, a random value of the δt is chosen from the interval$[\delta t_{min}\ \delta t_{max}]$. Then the Eqs. (6-8) will be evaluated for N_b iterations. The N_b samples obtained will be used to calculate the acceptance rate $\bar{\alpha}^b$and used to decide if the time step has to increased or decreased as following:

$$\delta t^{i+1} = \begin{cases} \delta t^i + \gamma^i \delta t^i, & \bar{\alpha}^b < \bar{\alpha} \\ \delta t^i - \gamma^i \delta t^i, & \bar{\alpha}^b \geq \bar{\alpha} \end{cases} \tag{10}$$

where γ^i is a random variable selected from the interval $[0.01\ 0.05]$ and $\bar{\alpha}$ is the target acceptance rate. The value of the target acceptance rate can be selected high to ensure more different samples are involved in computing the mean values of the uncertain parameters. Then, after each N_b samples (or iterations) the time step will be adapted by increasing or decreasing the time step by a small value (between 1% to 5%). This strategy will ensure that the time step does not produce small trajectory moves and is not relatively large (significant numbers of iterations are not wasted). The second modification proposed in this algorithm is to sample from distributions that are more diffuse than the original posterior PDF [30]. This strategy can facilitate the movement between high probability areas separated by regions of low probability. This can be done by increasing the temperature T which will eventually give more diffuse distribution (the posterior PDF is when $T = 1$) [30]. In this paper, the temperature is changed at each iteration by a small value according to: $T(i+1) = \bar{\alpha}^T T(i)$, where $\bar{\alpha}^T > 1$. The AHMC algorithm is than summarised as:

1) A value $\boldsymbol{\theta}_0$ is used to initiate the algorithm
2) Initiate $\mathbf{p}_0$ such that $\mathbf{p}_0 \sim N(0, \boldsymbol{M})$
3) Run the following steps N_b times:
    a. Initiate the leapfrog integrator by the previously accepted pair $(\boldsymbol{\theta}, \mathbf{p})$ and run the algorithm for L time steps to obtain $(\boldsymbol{\theta}^*, \mathbf{p}^*)$.
    b. Update the FEM and use the obtained analytical frequencies to compute $H(\boldsymbol{\theta}^*, \mathbf{p}^*)$.
    c. Accept $(\boldsymbol{\theta}^*, \mathbf{p}^*)$ with probability $\min\left(1, \exp\left\{-\frac{\Delta H}{T(i)}\right\}\right)$.
    d. Change the temperature according to: $T(i+1) = \bar{\alpha}^T T(i)$
4) Use the N_b samples to obtain the acceptance rate $\bar{\alpha}^b$
5) Adjust the time step according to Eq. (10)
6) Repeat steps (3-5) to for N_s samples.

In the next sections, the performance of the AHMC algorithm is discussed when two different systems are updated.

4. Application 1: A 3 DOF Linear System

To investigate the AHMC algorithm, a simulation of a 3 DOF mass-spring linear system is considered. The system is shown in Fig. 1. The deterministic parameters of this system are: $m_1 = 2\text{kg}, m_2 = 1\text{kg}, m_1 = 3\text{kg}$. The nominal mean values of the uncertain parameters are: $k_1 = 10$ N/m, $k_2 = 6$ N/m, $k_2 = k_3$ and $k_4 = 10$ N/m, and these values are used to obtain the following natural frequencies of interest of this structure: $\omega_1 = 1.12\text{Hz},\ \omega_2 = 3.5\text{Hz}$ and $\omega_3 = 4.1\text{Hz}$. However, the initial values of the uncertain parameters are: $k_1 = 12$ N/m, $k_2 = 4$ N/m, $k_2 = k_3$ and $k_4 = 7$ N/m. Thus, the parameters to be updated are: k_1, k_2 and k_4, and can be represented by a vector of $d = 3$ variables $\boldsymbol{\theta} = \{\theta_1, \theta_2, \theta_3\}$.

4.1.1 Updating the Stiffness Parameters

In this subsection, the 3 DOF linear system is updated by adjusting a vector of 3 parameters $\boldsymbol{\theta} = \{\theta_1, \theta_2, \theta_3\}$ using the Bayesian approach, while the AHMC algorithm is used to sample from the posterior PDF. The number of samples is set to $N_s = 5000$, the coefficients α_i in Eq. (2) were set equal to $\frac{1}{\sigma_i^2}$, where σ_i^2 is the variance of θ_i. Since only the stiffness parameters are updated, the σ_i $\{i = 1,2,3\}$ have equal values and are set to 500. The constant β_c, the weight of the likelihood term, in Eq. (2) was set equal to 1. The updating parameters θ_i were bounded with a maximum value of 14 N/m and a minimum value of 2 N/m.

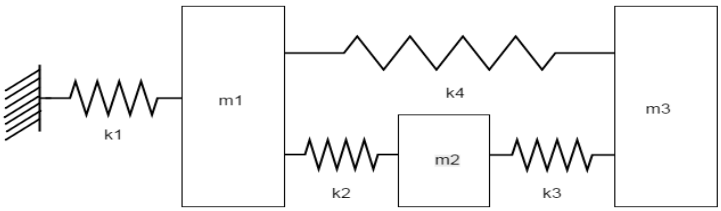

Figure 1: The 3 DOF mass-spring linear system

The initial vector of $\boldsymbol{\theta}$ is set to $\boldsymbol{\theta_0} = \{12, 4, 7\}$. The initial time step used in the HMC algorithm is $\delta t^0 = 1 \times 10^{-3}$s while the time step is bounded with a minimum value of $\delta t^0 = 0.001 \times 10^{-3}$s and a maximum value of $\delta t^0 = 15 \times 10^{-3}$s. L is uniformly distributed on the interval $\{1, 30\}$, the target acceptance rate is $\bar{\alpha} = 0.95$ (95%), the initial value of the temperature is $T(1) = 1$, and $\bar{\alpha}^T$ is set to 1.0012. $N_b$is set to 50, and the results of the updating are given in Tables 1 and 2.

Figure 2 shows the scatter plots for the three uncertain parameters using the AHMC algorithm. The uncertain parameters plotted in Fig. 2 are normalised by dividing their values by the initial value θ_i^0. Also, an error ellipse (or confidence ellipse) for the obtained samples is shown in the same figure. The error ellipse represents a contour that allows the visualization of the confidence interval, given as the region that contains 95% of the parameter samples. The plot shows that the AHMC algorithm has found the high probability areas after a few iterations.

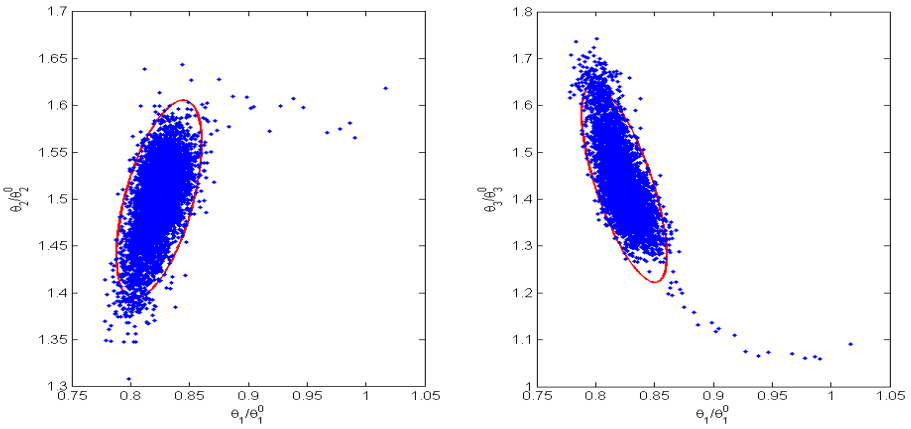

Figure 2: The scatter plots with the error ellipses using the AHMC method

The updated values of the uncertain parameters are presented in Table 1 along with their initial values, nominal values and the coefficient of variation (c.o.v) values. The c.o.v values are obtained by dividing the obtained standard deviation by the updated vector $\boldsymbol{\theta}$, and can be used to describe the errors in the updated parameters. Table 1 shows that the c.o.v values are small for the AHMC algorithm (less than 6%) which indicates that the AHMC algorithm has efficiently estimated the uncertain parameters. This can be seen from the updated parameters which are nearly identical to the nominal values.

Table 1: The updated vector of the stiffness parameters using AHMC technique

Stiffness Parameters (N/m)					
	Initial	Nominal values	Error (%)	AHMC algorithm (μ_i)	$\frac{\sigma_i}{\mu_i}$ (%) c.o.v
θ_1	12.00	10.00	20	9.89	1.77
θ_2	4.00	6.00	50	5.99	2.82
θ_3	7.00	10.00	30	10.05	5.79

Table 2 shows the initial and updated natural frequencies, together with the absolute errors obtained by $\frac{|f_i^m - f_i|}{f_i^m}$, the total average error (TAE) as a percentage, where $\text{TAE} = \frac{1}{N_m}\sum_{i=1}^{N_m}\frac{|f_i^m - f_i|}{f_i^m}$ $(N_m = 3)$, and the coefficient of variation which is obtained by dividing the standard deviation of the natural frequencies by the updated values. The results obtained by the AHMC algorithm are, on average, better than the initial natural frequencies. The initial error for the first frequency was 0.16%, and when the AHMC algorithm is applied to update the FEM the error was reduced to 0.05%. The same comment can be made for the other frequencies. In general, using the AHMC algorithm to update the system reduces the total average error from 9.13% to only 0.04%. Also, the coefficient of variation obtained by the AHMC algorithm indicates that the error for all modes is very small.

Table 2: Frequencies and Errors when AHMC techniques used to update stiffness parameters.

Modes	Nominal Frequency (Hz)	Initial Frequency (Hz)	Error (%)	Frequency AHMC Method (Hz)	c.o.v values (%)	Error (%)
1	1.120	1.122	0.16	1.119	0.43	0.05
2	3.500	2.932	16.23	3.498	0.48	0.07
3	4.100	3.649	11.01	4.100	0.45	0.01
Total Average Error	_____	_____	**9.13**	_____	_____	**0.04**

Figure 3 displays the total average error versus the iteration number for the first 5000 iterations. Figure 3 is obtained by computing the mean values of samples at every iteration as $\widehat{\boldsymbol{\theta}} = E(\boldsymbol{\theta}) \cong \frac{1}{N_s}\sum_{j=1}^{i}\boldsymbol{\theta}^i$, where i represents the current iteration. Then, the obtained mean value is used to obtain the new natural frequencies from the FEM and the absolute total average error is computed by $\text{TAE(i)} = \frac{1}{N_m}\sum_{j=1}^{N_m}\frac{|f_j^m - f_j|}{f_j^m}$. The result plotted in Fig. 3 shows that the AHMC algorithm converges reasonably fast and within the first 300 iterations

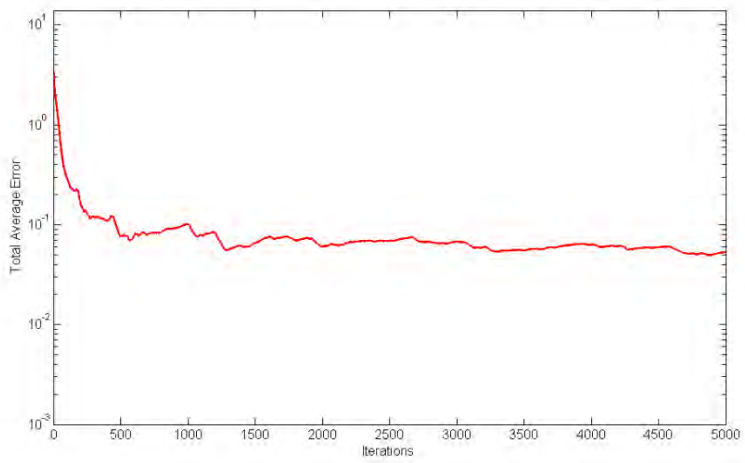

Figure 3: The total average error using the AHMC algorithm

In the next section, an unsymmetric H-shaped aluminium structure with real measured data is used to test the AHMC algorithm and the results obtained will be compared with those obtained previously with the HMC algorithm.

5. The Unsymmetric H-shaped Structure

In this section, an unsymmetric H-shaped aluminium [5, 17] structure with real measured data is updated using the AHMC algorithm. This structure is shown in Fig. 4. The structure was modelled by assembling 12 beam elements where each element was modelled as an Euler-Bernoulli beam. The position specified by the double arrow in the middle beam (see Fig. 4) showed the location where the structure was excited. An electromagnetic shaker was used to excite the structure, and an accelerometer was used to measure the response. A set of 15 frequency-response functions were measured. More details about the structure can be found in [5].

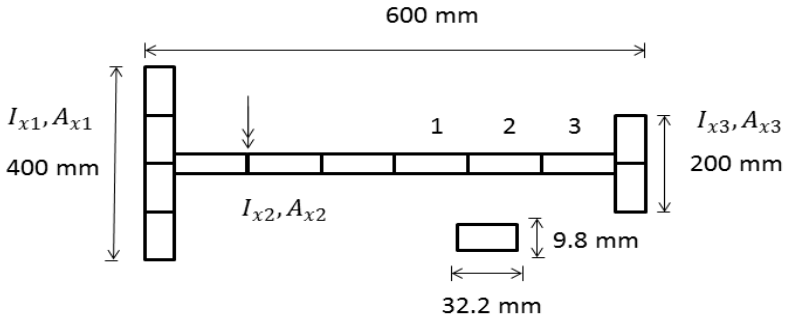

Figure 4: The H-shaped aluminium structure

The measured natural frequencies are: 53.9 Hz, 117.3 Hz, 208.4 Hz, 254.0 Hz and 445.0 Hz. In this example, the uncertain parameters are the moments of inertia and the section areas of the three beams (as shown in Fig. 4). The AHMC algorithm will be used to obtain the updated vector $\boldsymbol{\theta} = \{I_{x1}, I_{x2}, I_{x3}, A_{x1}, A_{x2}, A_{x3}\}$.

5.1 The Unsymmetric H-shaped Structure Simulation

The parameters of the unsymmetric H-shaped aluminium structure are given as follows: the Young's modulus is set at 7.2×10^{10}N/m^2, the density is set to 2785 kg/m^3. The same simulation sets and boundaries used in [13, 14, 15, 16, 17] are used for the AHMC algorithm. To help to keep the uncertain parameters physically realistic, the updated parameter vector is bounded by maximum and minimum values equal to: $[3.73 \times 10^{-8}, 3.73 \times 10^{-8}, 3.73 \times 10^{-8}, 4.16 \times 10^{-4}, 4.16 \times 10^{-4}, 4.16 \times 10^{-4}]$ and $[1.73 \times 10^{-8}, 1.73 \times 10^{-8}, 1.73 \times 10^{-8}, 2.16 \times 10^{-4}, 2.16 \times 10^{-4}, 2.16 \times 10^{-4}]$, respectively.

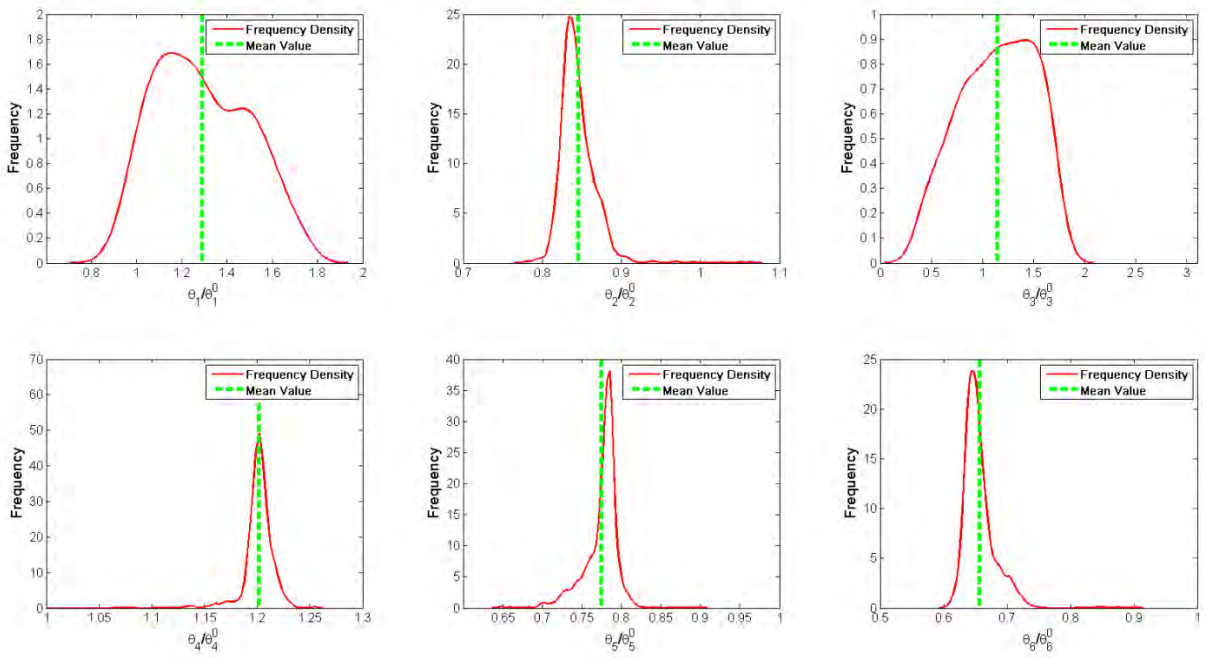

Figure 5: The Kernel smoothing density estimation of updating model parameters using the AHMC method

The likelihood weight β_c (see Eq. (2)) is set equal to 10. The coefficients α_i are set equal to $\frac{1}{\sigma_i^2}$ where $\boldsymbol{\sigma} = [5 \times 10^{-8}, 5 \times 10^{-8}, 5 \times 10^{-8}, 5 \times 10^{-4}, 5 \times 10^{-4}, 5 \times 10^{-4}]$. The number of samples N_s is set to 1000, the initial time step used in the HMC algorithm is $\delta t^0 = 4.5 \times 10^{-3}$s while the time step is bounded with a minimum value of $\delta t^0 = 0.001 \times 10^{-3}$s and a maximum value of $\delta t^0 = 7 \times 10^{-3}$s. L is uniformly distributed on the interval $\{1, 30\}$, the target

acceptance rate is $\bar{\alpha} = 0.95$ (95%), the initial value of the temperature is $T(1) = 1$, and $\bar{\alpha}^T$ is set to 1.0081. $N_b$is set to 20. The results obtained using the AHMC algorithm, namely the updated parameters and the updated frequencies, are presented in Tables 3 and 4, respectively.

Figure 5 shows the Kernel smoothing density estimation of the updating parameters along with the updated values of the uncertain parameters. The θ_i refers to the sequential numbering of the updating parameters while the normalisation constants θ_i^0 are the initial values of the updated parameters. The obtained results show that the AHMC algorithm identified the high probability region. Moreover, the shapes of the density functions are not Gaussian.

The updated parameter values, the initial values of these parameters and their c.o.v values are given in Table 3. The total acceptance rate for the AHMC algorithm is 96.3% which is very good (only 3.7% of the computational time was wasted). The AHMC algorithm successfully updated the uncertain parameters (the updated values are different to the initial $\boldsymbol{\theta}_0$). The coefficients of variation obtained by the AHMC algorithm shows that the estimation for the middle beam parameters are better than those of the left and the right beams parameters (the c.o.v of the middle beam is smaller than that for the other two beams). This is obvious since structure was excited in the middle beam, and more information on the middle beam was used in the updating process.

Table 3: Initial and updated parameters using the AHMC algorithm

	$\boldsymbol{\theta_0}$ vector Initial	$\boldsymbol{\theta}$ vector, AHMC Method	$\frac{\sigma_i}{\theta_i}$ (%)
I_{x1}	2.73×10^{-8}	3.51×10^{-8}	15.56
I_{x2}	2.73×10^{-8}	2.30×10^{-8}	2.80
I_{x3}	2.73×10^{-8}	3.12×10^{-8}	14.50
A_{x1}	3.16×10^{-4}	3.79×10^{-4}	1.35
A_{x2}	3.16×10^{-4}	2.45×10^{-4}	2.83
A_{x3}	3.16×10^{-4}	2.28×10^{-4}	3.98

The correlation between all updated parameters, when the AHMC algorithm is used to update the structure, is shown in Fig. 6. The uncertain parameters are weakly correlated except for the pair (I_{x2}, A_{x2}) which are highly correlated (the correlation is equal to 0.73).

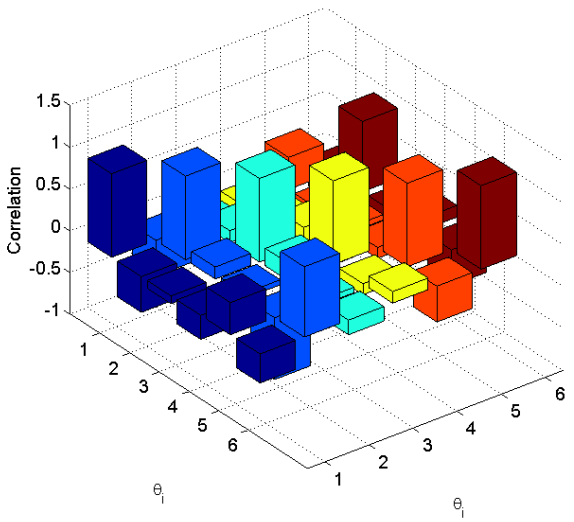

Figure 6: The correlation between uncertain parameters

The unsymmetric H-shaped aluminium structure has been updated many times by different (deterministic and statistic) methods [5]. The Nelder Mead (NM) Simplex method gave a total average error equal 2.14% [5] while the Genetic Algorithm (GA) [5] reduced the TAE to 1.1%. On the other hand, the Response-Surface (RS) method [5] produced a higher TAE than the GA algorithm (the total average error was equal to 1.84%). The Particle Swarm Optimization (PSO) algorithm [5] gave better results and the error was reduced to 0.4%. Moreover, three MCMC algorithms were applied to this structure, the M-H, SS and HMC algorithms [17], and the results were 3.01%, 2.98% and 0.73%, respectively. Another two modified versions of the HMC algorithm (SHMC and S2HMC algorithms) were also applied to update this structure and the results were 0.66% and 0.58%, respectively.

Table 7 presents the updated frequencies obtained by the AHMC algorithm. The initial error for the first measured natural frequency was 4.63%. When the AHMC algorithm was used to update the structure the error was reduced to 0.85%. In general, the resulted obtained by the AHMC algorithm are far better than the initial FEM. The AHMC algorithm reduced the total error to 0.48% which is good compared to the results obtained in previous works (better than all algorithms except for the PSO algorithm).

Figure 7 shows the variation in the total average error through time (iterations). The strategy used in the first example to plot Figure 3 is used again to plot Figure 7. The plotted results show that AHMC algorithm converges fast and within the first 100 iterations (100 samples will be enough for the structure to be updated).

Table 4: Natural frequencies and errors when AHMC algorithm is used to update the structure

Mode	Measured Frequency (Hz)	Initial Frequency (Hz)	Error (%)	Frequencies AHMC Method (Hz)	Error (%)
1	53.90	51.40	4.63	53.44 (0.87%)	0.85
2	117.30	116.61	0.59	118.96 (0.96%)	1.42
3	208.40	201.27	3.42	208.38 (1.04%)	0.01
4	254.00	247.42	2.59	254.30 (1.40%)	0.12
5	445.00	390.33	12.28	445.08 (1.21%)	0.02
Total average error	_____	_____	**4.70**	_____	**0.48**

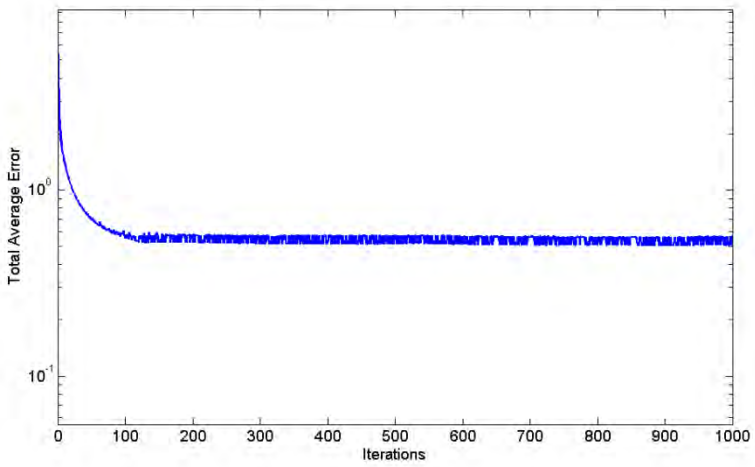

Figure 7: The total average error for the AHMC algorithm

6. Conclusion

In this paper, an adaptive MCMC algorithm, the Adaptive Hybrid Monte Carlo algorithm, is proposed to solve the Bayesian FEM updating formulation. In this method, the time step was adaptively selected to improve the trajectory length of the algorithm. The method was tested by updating two structural systems: a simulated 3 DOF linear system and an unsymmetric H-shaped aluminium structure. In the first simulation the AHMC technique gave good results and reduced the error to less than 0.05%. In the second example, the AHMC method also gave good results and reduced the total error to 0.48%. The results obtained by the AHMC algorithms are better than those obtained by HMC, SHMC and S2HMC algorithms in previous works. The SHMC and S2HMC methods use relatively larger time steps (which means large trajectory moves in the search space), whereas the AHMC algorithm adjusts its time step and temperature at each iteration and this gave the algorithm a significant advantage to escape local solutions. Further work will consider the differences between the above method and other adaptive MCMC algorithms. Also, the AHMC algorithm will be used to update more complex structures.

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