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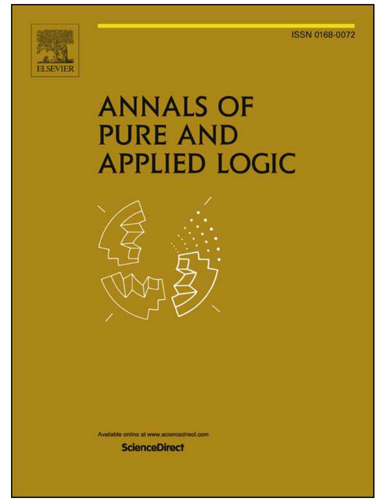

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Proof Lengths for Instances of the Paris-Harrington Principle

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Abstract

As Paris and Harrington have famously shown, Peano Arithmetic does not prove that for all numbers k, m, n there is an N which satisfies the statement $\mathrm{PH}(k, m, n, N)$: For any k-colouring of its n-element subsets the set $\{0, \ldots, N-1\}$ has a large homogeneous subset of size $\geq m$. At the same time very weak theories can establish the Σ_1-statement $\exists_N \mathrm{PH}(\overline{k}, \overline{m}, \overline{n}, N)$ for any fixed parameters k, m, n. Which theory, then, does it take to formalize *natural* proofs of these instances? It is known that $\forall_m \exists_N \mathrm{PH}(\overline{k}, m, \overline{n}, N)$ has a natural and short proof (relative to n and k) by Σ_{n-1}-induction. In contrast, we show that there is an elementary function e such that any proof of $\exists_N \mathrm{PH}(\overline{e(n)}, \overline{n+1}, \overline{n}, N)$ by Σ_{n-2}-induction is ridiculously long.

In order to establish this result on proof lengths we give a computational analysis of slow provability, a notion introduced by Sy-David Friedman, Rathjen and Weiermann. We will see that slow uniform Σ_1-reflection is related to a function that has a considerably lower growth rate than F_{ε_0} but dominates all functions F_α with $\alpha < \varepsilon_0$ in the fast-growing hierarchy.

Keywords: Peano Arithmetic, Proof Length, Paris-Harrington Principle, Finite Ramsey Theorem, Slow Consistency, Fast Growing Hierarchy
2010 MSC: 03F30, 03F20, 03F40

We recall some terminology from [1]: For a set X and a natural number n we write $[X]^n$ for the collection of subsets of X with precisely n elements. Given a function f with domain $[X]^n$, a subset Y of X is called homogeneous for f if the restriction of f to the set $[Y]^n$ is constant. A non-empty subset of $\mathbb{N}$ is called large if its cardinality is at least as big as its minimal element. Where the context suggests it we use N to denote the set $\{0, \ldots, N-1\}$. Then the Paris-Harrington Principle, or Strengthened Finite Ramsey Theorem, expresses that for all natural numbers k, m, n there is an N such that the following statement

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holds:

$$\mathrm{PH}(k,m,n,N) \quad :\equiv \quad \text{“for any function } [N]^n \to k \text{ the set } N \text{ has a large homogeneous subset with at least } m \text{ elements”}$$

Using the methods presented in [2, Section I.1(b)] it is easy to formalize the statement $\mathrm{PH}(k,m,n,N)$ in the language of first order arithmetic, as a formula that is Δ_1 in the theory $\mathbf{I\Sigma}_1$ of Σ_1-induction. The celebrated result of [1] says that the formula $\forall_{k,m,n}\exists_N \mathrm{PH}(k,m,n,N)$ is true but unprovable in Peano Arithmetic.
As is well-known, any true Σ_1-formula in the language of first-order arithmetic can be proved in a theory as weak as Robinson Arithmetic. It is thus pointless to ask whether a Σ_1-sentence is provable in a sound arithmetical theory, in contrast to the situation for Π_1-sentences (cf. Gödel's Theorems) and Π_2-sentences (provably total functions). What we can sensibly ask is whether a Σ_1-sentence has a proof with some additional property. The present paper explores this question for instances $\exists_N \mathrm{PH}(\overline{k},\overline{m},\overline{n},N)$ of the Paris-Harrington Principle. Our principal result states that, for some elementary function e, the following holds:

For sufficiently large n, no proof of the formula $\exists_N \mathrm{PH}(\overline{e(n)},\overline{n+1},\overline{n},N)$ in the theory $\mathbf{I\Sigma}_{n-2}$ can have Gödel number smaller than $F_{\varepsilon_0}(n-3)$. (1)

If we replace $\mathbf{I\Sigma}_{n-2}$ by $\mathbf{I\Sigma}_{n-3}$ (and $F_{\varepsilon_0}(n-3)$ by $F_{\varepsilon_0}(n-4)$) then we can take the constant function $e(n)=8$. It is open whether we can make e constant and keep the stronger fragment $\mathbf{I\Sigma}_{n-2}$.
Recall that F_{ε_0} is the function at stage ε_0 of the fast-growing hierarchy. Ketonen and Solovay in [3] have related it to the function that maps (k,m,n) to the smallest witness N which makes the statement $\mathrm{PH}(\overline{k},\overline{m},\overline{n},\overline{N})$ true. A classical result due to Kreisel, Wainer and Schwichtenberg [4, 5, 6] says that F_{ε_0} eventually dominates any provably total function of Peano Arithmetic. Similar to (1) we will show that the Σ_1-formula $\exists_y F_{\varepsilon_0}(\overline{n})=y$ has no short proof in the theory $\mathbf{I\Sigma}_n$.
By [2, Theorem II.1.9] the formula $\forall_m\exists_N \mathrm{PH}(\overline{k},m,\overline{n},N)$ is provable in $\mathbf{I\Sigma}_{n-1}$, for each fixed $n\geq 2$ and k. The proofs of these instances formalize perfectly natural mathematical arguments. According to [2, Section II.2(c)] they can be constructed in the meta-theory $\mathbf{I\Sigma}_1$. Since all provably total functions of $\mathbf{I\Sigma}_1$ are primitive recursive, this complements (1) by the following statement:

There is a primitive recursive function which maps (k,n) with $n\geq 2$ to a proof of the formula $\forall_m\exists_N \mathrm{PH}(\overline{k},m,\overline{n},N)$ in the theory $\mathbf{I\Sigma}_{n-1}$. (2)

Similarly, a primitive recursive construction yields proofs of $\exists_y F_{\varepsilon_0}(\overline{n})=y$ in the theories $\mathbf{I\Sigma}_{n+1}$: In view of $F_{\varepsilon_0}(x)\simeq F_{\omega_{x+1}}(x)=F_{\omega_x^{x+1}}(x)$ it suffices to prove the statements "$F_{\omega_n^{n+1}}$ is total". This is done by Π_2-induction up to ω_n^{n+1}, which is available in $\mathbf{I\Sigma}_{n+1}$ by Gentzen's classical construction (cf. [7, Theorem 4.11]).

We argue that (1) is not only a result about proof length, but also about the existence of natural proofs: Observe first that we are concerned with sequences

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p_n of proofs for a sequence of parametrized statements A_n, rather than with a single proof of a single statement. Under which conditions can such a sequence of proofs follow an intelligible uniform proof idea? It is the role of the proofs p_n to guarantee that the formulas A_n are true. On the other hand the statement "the given proof idea leads to formally correct proofs p_n of the statements A_n" should, we believe, be justified by fairly elementary means. Since elementary means cannot prove the totality of functions with a high growth rate this implies that the function mapping n to (a code of) the proof p_n cannot grow too fast. In this sense (1) shows that $\mathbf{I\Sigma}_{n-2}$-proofs of the Paris-Harrington Principle for arity n and $e(n)$ colours cannot follow a natural proof idea. The author sees no formal condition which would, on the positive side, ensure that a sequence of proofs is natural. On an informal level the construction which establishes [2, Theorem II.1.9] appears to provide natural $\mathbf{I\Sigma}_{n-1}$-proofs of the statements $\forall_m \exists_N \, \mathrm{PH}(\overline{k}, m, \overline{n}, N)$.

Let us briefly discuss connections with a line of research initiated by Harvey Friedman: Theorem 15 in [8] says that any proof of a certain Σ^0_1-statement in the theory Π^1_2-BI_0 must have at least 2_{1000} (i.e. 1000 iterated exponentials to the base 2) symbols. Obviously this goes much further than our result insofar as it involves a much stronger theory. However, there is also a more conceptual difference: Friedman's statement can, in principle, be verified explicitly (by looking at all possible proofs with less than 2_{1000} symbols) and is thus finitistically meaningful. In contrast, our statement (1) involves an unbounded existential quantifier, implicit in the phrase "sufficiently large". It is conceivable that any witness to this existential quantifier is so huge that statement (1) does not have "practical significance". On the other hand the more abstract form of (1) has the important advantage of making the statement more robust: A result like [8, Theorem 15] requires concrete numerical bounds which might depend on the formalization and are difficult to establish in full detail. To prove claim (1), on the other hand, we can rely on the more robust concept of growth rates. How exactly we arithmetize the relation "p codes a proof of the statement with Gödel number φ in the theory $\mathbf{I\Sigma}_n$" will not matter. All we require is that this relation is defined by an arithmetical formula $\mathrm{Proof}_{\mathbf{I\Sigma}_n}(p, \varphi)$ (with parameters n, p and φ) which is Δ_1 in the theory $\mathbf{I\Sigma}_1$ and $\mathbf{I\Sigma}_1$-provably equivalent to the usual formalizations of provability. Statement (1) is true for any such arithmetization; merely the concrete meaning of "sufficiently large" may change (cf. Remark 1.4 below). Another interesting comparison is with a result of Krajíček [9, Theorem 6.1]: He considers Π_2-instances of the Paris-Harrington Principle and establishes linear bounds on the number of steps in proofs in full Peano Arithmetic (rather than in restricted fragments).

To conclude this introduction, let us summarize the different sections of the paper: In Section 1 we show how the analysis of reflection leads to lower bounds on proof sizes. Given a theory $\mathbf{T}$, the uniform reflection principle for the formula $\exists_y \varphi(x, y)$ expresses that "for all p and n there is an N such that if p is a $\mathbf{T}$-proof of $\exists_y \varphi(\overline{n}, y)$ then $\varphi(\overline{n}, \overline{N})$ is true". If we have a bound on the provably

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total functions of reflection then we know that the witness N cannot be too much bigger than the code of the proof p. Vice versa p cannot be too small if $\exists_y \varphi(\overline{n}, y)$ has only large witnesses. We suppose that this line of argumentation is known (it occurs e.g. in [10]), but the author knows of no article that would develop it in general form.
The method just described applies to sequences of proofs in a single theory $\mathbf{T}$, while statement (1) is concerned with a sequence of proofs that may contain axioms from increasingly strong theories. This discrepancy is resolved in Section 2: We consider a notion of "slow proof" in Peano Arithmetic, deduced from the slow consistency statement introduced by Sy-David Friedman, Rathjen and Weiermann in [11]. The idea is to penalize complex induction axioms by a drastic increase in proof size. This generates an interplay between proof length and the use of induction. At the same time it makes the construction of proofs more difficult, thus weakening the reflection and consistency statement. We can then apply the method of Section 1 to show that any *slow* **PA**-proof of $\exists_N \mathrm{PH}(\overline{e(n+2)}, \overline{n+3}, \overline{n+2}, N)$ must be long. Claim (1) will easily follow.
The results of Section 2 rely on certain bounds on the provably total functions of slow reflection. The proof of these bounds follows in Section 3. There we relate slow uniform Σ_1-reflection to a "slow variant" $F^{\diamond}_{\varepsilon_0}$ of the function F_{ε_0}. We will see that each function F_α with $\alpha < \varepsilon_0$ is dominated by $F^{\diamond}_{\varepsilon_0}$ while $F^{\diamond}_{\varepsilon_0}$ itself grows much slower than F_{ε_0}. This computational analysis of slow reflection is complemented by the results of [12], where we investigate the consistency strength (Π_1-consequences) of slow reflection. Further results on slow provability can be found in [13].

1. Bounding Proof Sizes via Reflection Principles

In this section we show how bounds on the provably total functions of uniform Σ_1-reflection lead to lower bounds on the sizes of proofs. To formulate the reflection principle we will need a Σ_1-formula $\mathrm{True}_{\Sigma_1}(\varphi)$ that defines truth for Σ_1-formulas (in the large sense, i.e. the formula may start with several existential quantifiers). The theory $\mathbf{I\Sigma_1}$ should be able to prove Tarski's truth conditions (as guaranteed by [2, Theorem I.1.75]). With respect to the proof predicate we must develop the theory in some generality:

Definition 1.1. A proof predicate is a Π_1-formula $\mathrm{Proof}(p, \varphi)$ in the language of first-order arithmetic, with only the variables p and φ free. Given a proof predicate we have the associated Σ_1-reflection principle

$$\mathrm{RFN}_{\Sigma_1} :\equiv \forall_\varphi(\text{“}\varphi \text{ is a closed } \Sigma_1\text{-formula”} \land \exists_p \mathrm{Proof}(p, \varphi) \to \mathrm{True}_{\Sigma_1}(\varphi)).$$

For a natural number p and a formula φ with Gödel number $\ulcorner \varphi \urcorner$ we say that "p is a proof of φ" if the formula $\mathrm{Proof}(\overline{p}, \overline{\ulcorner \varphi \urcorner})$ is true in the standard model.

The following observation is easy but crucial:

Lemma 1.2. *Let* $\mathrm{Proof}(p, \varphi)$ *be a proof predicate, and let* $\mathbf{T}$ *be a sound extension of* $\mathbf{I\Sigma_1}$ *that proves the* Σ_1*-reflection principle associated with* $\mathrm{Proof}(p, \varphi)$.

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For any Σ_1-formula $\psi(x,y)$ there is a $\mathbf{T}$-provably total function $g:\mathbb{N}^2\to\mathbb{N}$ such that $\psi(\overline{n},\overline{g(p,n)})$ is true whenever p is a proof of $\exists_y\psi(\overline{n},y)$.

Note that, since $\mathbf{T}$ must be sound, the lemma can only be applied to proof predicates which are themselves sound for Σ_1-formulas.

Proof. Since the theory $\mathbf{T}$ extends $\mathbf{I\Sigma}_1$ it is strong enough to handle Feferman's dot notation, and it proves the "It's snowing"-Lemma (see [2, Corollary I.1.76]). Combining this with the reflection principle for $\mathrm{Proof}(p,x)$ we obtain

$$\mathbf{T}\vdash\forall_x(\exists_p\,\mathrm{Proof}(p,\ulcorner\exists_y\psi(\dot{x},y)\urcorner)\to\exists_y\psi(x,y)).$$

Prefixing quantifiers transforms this into

$$\mathbf{T}\vdash\forall_{p,x}\exists_y(\mathrm{Proof}(p,\ulcorner\exists_y\psi(\dot{x},y)\urcorner)\to\psi(x,y)).$$

We remark that it is only mildly non-constructive to prefix the existential quantifier in the consequent: A computation of the witness y will use the proof p but rather not the computational content of the statement $\mathrm{Proof}(p,\ulcorner\exists_y\psi(\dot{x},y)\urcorner)$. In any case the formula $\mathrm{Proof}(p,\ulcorner\exists_y\psi(\dot{x},y)\urcorner)\to\psi(x,y)$ is Σ_1 in $\mathbf{I\Sigma}_1$. As we have seen the theory $\mathbf{T}$ shows that this formula defines a left-total relation. To obtain a single-valued function we apply a standard minimization argument. Note that we cannot simply pick the minimal value for y since this would yield a function with a Δ_2-graph; instead we simultaneously minimize over y and the witness to the existential quantifier implicit in $\mathrm{Proof}(p,\ulcorner\exists_y\psi(\dot{x},y)\urcorner)\to\psi(x,y)$. This results in a Σ_1-formula $\chi(p,x,y)$ such that we have

$$\mathbf{T}\vdash\forall_{p,x,y}(\chi(p,x,y)\to(\mathrm{Proof}(p,\ulcorner\exists_y\psi(\dot{x},y)\urcorner)\to\psi(x,y)))$$

and $\mathbf{T}\vdash\forall_{x,p}\exists!_y\chi(p,x,y)$. Since $\mathbf{T}$ is sound the formula $\chi(p,x,y)$ does indeed define a $\mathbf{T}$-provably total function $g:\mathbb{N}^2\to\mathbb{N}$, which satisfies $\mathbb{N}\vDash\chi(\overline{p},\overline{n},\overline{g(p,n)})$ for all natural numbers p and n. By the above we also have

$$\mathbb{N}\vDash\mathrm{Proof}(\overline{p},\overline{\ulcorner\exists_y\psi(\overline{n},y)\urcorner})\to\psi(\overline{n},\overline{g(p,n)})\qquad\text{for all } p,n\in\mathbb{N}.$$

Lifting the implication to the meta-language gives the desired claim. ☐

We can deduce the promised lower bound on proof sizes:

Proposition 1.3. *Let* $\mathrm{Proof}(p,\varphi)$ *be a proof predicate, and let* $\mathbf{T}$ *be a sound extension of* $\mathbf{I\Sigma}_1$ *that proves the* Σ_1*-reflection principle for* $\mathrm{Proof}(p,\varphi)$. *Consider a* Σ_1*-formula* $\psi(x,y)$ *and define a function* $F_\psi:\mathbb{N}\to\mathbb{N}\cup\{\infty\}$ *by setting*

$$F_\psi(n):=\begin{cases}m & \text{if } m \text{ is the least number for which } \psi(\overline{n},\overline{m}) \text{ is true,}\\ \infty & \text{if } \exists_y\psi(\overline{n},y) \text{ is false.}\end{cases}$$

Let $f:\mathbb{N}\to\mathbb{N}$ be a function with $f(n)\geq n$ and such that, whenever g is $\mathbf{T}$-provably total, the function $g\circ f$ is eventually dominated by F_ψ (considering ∞ as bigger than any natural number). Then there is a bound N such that we have

$$p>f(n)\qquad\text{whenever } p \text{ is a proof of } \exists_y\psi(\overline{n},y) \text{ with } n\geq N.$$

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To avoid misunderstanding, we stress that the notion of proof in the last line of the proposition is induced by the proof predicate in the first line, via Definition 1.1.

Proof. Let $g : \mathbb{N}^2 \to \mathbb{N}$ be the function provided by Lemma 1.2. We can make g monotone in both arguments: First define $g_0 : \mathbb{N}^2 \to \mathbb{N}$ by the primitive recursion

$$\begin{aligned} g_0(p, 0) &:= g(p, 0), \\ g_0(p, n+1) &:= \max\{g(p, n+1), g_0(p, n)\}. \end{aligned}$$

This yields $g_0(p, n) \geq g(p, n)$ for all numbers p and n, as well as $g_0(p, n) \leq g_0(p, n')$ whenever we have $n \leq n'$. Now define $g_1 : \mathbb{N}^2 \to \mathbb{N}$ by setting

$$\begin{aligned} g_1(0, n) &:= g_0(0, n), \\ g_1(p+1, n) &:= \max\{g_0(p+1, n), g_1(p, n)\}. \end{aligned}$$

It is obvious that we have $g_1(p, n) \geq g_0(p, n) \geq g(p, n)$ for all numbers p and n, and that $p \leq p'$ implies $g_1(p, n) \leq g_1(p', n)$. By induction on p one can also show that $g_1(p, n) \leq g_1(p, n')$ holds whenever we have $n \leq n'$. Lemma 1.2 implies that we have

$$F_\psi(n) \leq g_1(p, n) \qquad \text{whenever } p \text{ is a proof of } \exists_y \psi(\overline{n}, y).$$

Since the theory $\mathbf{T}$ extends $\mathbf{I\Sigma}_1$ its provably total functions are closed under primitive recursion, by [2, Theorem I.1.54]. Thus g_1 is still $\mathbf{T}$-provably total. We define another $\mathbf{T}$-provably total function $g^\triangle : \mathbb{N} \to \mathbb{N}$, diagonalizing over g_1, as

$$g^\triangle(p) := g_1(p, p) + 1.$$

By assumption there is a bound N such that we have

$$(g^\triangle \circ f)(n) \leq F_\psi(n) \qquad \text{for all } n \geq N.$$

Let us show that the same bound N satisfies the claim of the proposition: Consider an arbitrary $n \geq N$ and assume that p is a proof of the formula $\exists_y \psi(\overline{n}, y)$. Aiming at a contradiction we assume $p \leq f(n)$. Then we have

$$F_\psi(n) \leq g_1(p, n) \leq g_1(f(n), f(n)) < (g^\triangle \circ f)(n) \leq F_\psi(n),$$

which is indeed absurd. $\square$

It is a nice property of the proposition that the bounds it establishes are invariant under basic transformations of proofs:

Remark 1.4. If f satisfies the conditions of the proposition and h is $\mathbf{T}$-provably total (e.g. primitive recursive) with $h(p) \geq p$ then $h \circ f$ satisfies these conditions as well. Thus proofs of $\exists_y \psi(\overline{n}, y)$ will even be bigger than $h(f(n))$ for all n above some (possibly increased) bound.

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This is useful because it allows us to preprocess proofs: Consider a modified notion proof′ and a sequence of formulas φ_n, not necessarily of the form $\varphi(\overline{n})$ and not necessarily in the syntactic class Σ_1. Assume that there is a Σ_1-formula $\psi(x,y)$ and a primitive recursive function h which transforms any proof′ of φ_n into (an upper bound for) a proof of $\exists_y \psi(\overline{n},y)$. Possibly increasing h we can assume that h is monotone and satisfies $h(p) \geq p$. Using the proposition we may be able to show that $p > h(f(n))$ holds whenever p is a proof of $\exists_y \psi(\overline{n},y)$, with n sufficiently large. We want to deduce $q > f(n)$ where q is a proof′ of φ_n. Indeed, $q \leq f(n)$ would imply $h(q) \leq h(f(n))$. This would mean that there exists a proof of $\exists_y \psi(\overline{n},y)$ below $h(f(n))$, which we have seen to be false. The proof of Lemma 2.6 contains a detailed application of this argument.

To conclude this section we illustrate what a simple application of the proposition can yield. Adopting the notation from [3] we have

$$\sigma(n,k) = \min\{N \mid \mathrm{PH}(\overline{k}, \overline{n+1}, \overline{n}, \overline{N}) \text{ is true}\},$$

i.e. the number $\sigma(n,k)$ is the smallest witness for the Paris-Harrington Principle with arity n and k colours. We know from [1, Theorem 3.2] that the function $n \mapsto \sigma(n,n)$ eventually dominates any provably total function of Peano Arithmetic. The following result on proof sizes is considerably weaker than (1), insofar as it speaks about fixed fragments of Peano Arithmetic.

Corollary 1.5. *For any number k the (total) function*

$$n \mapsto \text{“the smallest Gödel number of a proof of the } \Sigma_1\text{-formula } \exists_N \mathrm{PH}(\overline{n}, \overline{n}+1, \overline{n}, N) \text{ by } \Sigma_k\text{-induction”}$$

eventually dominates any provably total function of Peano Arithmetic.

Proof. Let f be an arbitrary $\mathbf{PA}$-provably total function. Assume that $f(n) \geq n$ holds for all n, possibly after replacing f by the function $n \mapsto \max\{f(n), n\}$. We apply Proposition 1.3 to the usual proof predicate $\mathrm{Proof}_{\mathbf{I\Sigma}_k}(p,\varphi)$ for the theory of Σ_k-induction (or rather to a Π_1-formula that is equivalent to $\mathrm{Proof}_{\mathbf{I\Sigma}_k}(p,\varphi)$ over $\mathbf{I\Sigma}_1$), to the theory $\mathbf{T} = \mathbf{PA}$, to the formula $\psi(x,y) \equiv \mathrm{PH}(x, x+1, x, y)$, and to the function f. Then $n \mapsto \sigma(n,n)$ is the function F_ψ of Proposition 1.3. The assumptions of the proposition are satisfied: It is well known that Peano Arithmetic proves uniform Σ_1-reflection for the theory $\mathbf{I\Sigma}_k$ (see e.g. [2, Corollary I.4.34]). For any $\mathbf{PA}$-provably total function g the composition $g \circ f$ is $\mathbf{PA}$-provably total as well, and thus indeed dominated by $n \mapsto \sigma(n,n)$. The result of Proposition 1.3 is nothing but the claim of the corollary. □

The bound of the corollary is reasonably accurate, in the sense that the function computing the minimal proofs is not much faster than the provably total functions of Peano Arithmetic: Recall that $\mathrm{PH}(k,m,n,N)$ is Δ_1 in $\mathbf{I\Sigma}_1$. Thus not only $\sigma(n,n)$ itself but the witnesses to all unbounded quantifiers of the Σ_1-formula $\exists_N \mathrm{PH}(\overline{n}, \overline{n}+1, \overline{n}, N)$ are bounded by a primitive recursive function in n and $\sigma(n,n)$. Furthermore, the Σ_1-completeness theorem is established by

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a primitive recursive construction of proofs. Thus there is a primitive recursive function $h : \mathbb{N}^2 \to \mathbb{N}$ such that $h(n, \sigma(n, n))$ is the Gödel number of a proof of $\exists_N \mathrm{PH}(\overline{n}, \overline{n}+1, \overline{n}, N)$ in the theory $\mathbf{I\Sigma}_0$.

2. No Short Proofs for Instances of the Paris-Harrington Principle

In this section we refine Corollary 1.5 by varying k alongside with n. On first sight it may seem astonishing that Proposition 1.3, which only deals with one proof predicate at a time, can be used to this effect. We will see, however, that a single proof predicate can inform us about proofs in various theories: The slow **PA**-proofs that we will introduce penalize the use of complex induction axioms by a drastic increase in proof length, thus creating an interplay between proof length and the amount of induction used in the proof.
Before we can define the notion of a slow proof we need some preliminaries on ordinal notations and the fast-growing hierarchy of functions. Ordinal notations are required for the ordinals below ε_0, the smallest fixed point of the function $\alpha \mapsto \omega^\alpha$. As usual they will be based on the Cantor normal form

$$\alpha = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k \qquad \text{with } k \in \mathbb{N},\, n_i \in \mathbb{N}\backslash\{0\} \text{ and } \alpha_1 > \cdots > \alpha_k.$$

Crucially, $\alpha < \varepsilon_0$ implies $\alpha_1 < \alpha$ so that the Cantor normal form inductively yields finite term notations. Basic ordinal arithmetic can be translated into syntactic operations on these terms. The operations are sufficiently elementary to make ordinal arithmetic available in the theory $\mathbf{I\Sigma}_1$, after arithmetization of the finite term syntax. In fact, Sommer in [14, Sections 2 and 3] shows that theories much weaker than $\mathbf{I\Sigma}_1$ suffice if one encodes the terms efficiently. In this paper we are not interested in very weak theories, but it is nevertheless convenient to adopt the encoding of Sommer: This allows us to use his Δ_0-definition of the functions in the fast-growing hierarchy.
We remark that the ordinal arithmetic of [14] includes fundamental sequences: The fundamental sequence $(\{\alpha\}(n))_{n\in\mathbb{N}}$ of a limit ordinal α is a strictly increasing sequence of ordinals with supremum α. Precisely, any limit ordinal α can uniquely be written as $\alpha = \beta + \omega^\gamma \cdot (k+1)$ where $\gamma > 0$ is the smallest exponent of the Cantor normal form of α, and β contains the larger summands. We then have

$$\begin{aligned} \{\beta + \omega^\gamma \cdot (k+1)\}(n) &= \beta + \omega^\gamma \cdot k + \omega^\delta \cdot (n+1) && \text{if } \gamma = \delta + 1, \\ \{\beta + \omega^\gamma \cdot (k+1)\}(n) &= \beta + \omega^\gamma \cdot k + \omega^{\{\gamma\}(n)} && \text{if } \gamma \text{ is a limit.} \end{aligned}$$

For zero and successor ordinals one sets $\{0\}(n) := 0$ and $\{\beta+1\}(0) := \beta$.
Next, consider the "stack of ω's"-function defined by the recursion

$$\omega_0^\alpha = \alpha, \qquad \omega_{n+1}^\alpha = \omega^{\omega_n^\alpha}.$$

As usual, ω_n abbreviates ω_n^1. This function is not part of the ordinal arithmetic encoded by Sommer (although it is, of course, part of his meta-theory). Since

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Sommer does encode the function $\alpha \mapsto \omega^\alpha$ it is immediate to make the function $(n, \alpha) \mapsto \omega_n^\alpha$ (operating on the codes) available in $\mathbf{I\Sigma}_1$. However, we will need more, namely a Δ_0-formula defining the graph and explicit bounds on the values of this function. Write $\ulcorner\alpha\urcorner$ for the term notation of α, represented as a list with digits from $\{1, \ldots, 4\}$ as in [14]. Then ω_n^α is represented by the following concatenation of lists:

$$\ulcorner\omega_n^\alpha\urcorner = \langle\underbrace{4, \ldots, 4}_{n \text{ characters } 4}\rangle^\frown\ulcorner\alpha\urcorner^\frown\langle\underbrace{3, 1, \ldots, 3, 1}_{n \text{ alternations}}\rangle$$

Indeed, with each character 4 we move to the exponent of the leftmost summand of the Cantor normal form, while 3 instructs us to leave the exponent and look at the corresponding coefficient, which in the present case is always 1 (represented by the base two notation of 1, which happens to be the list $\langle 1\rangle$ itself). Now to verify the relation $\omega_n^\alpha = \beta$ we only have to compare digits in the sequence representations of α and β, and this can be cast into a Δ_0-formula (see [14, Section 2.2]). Using [14, Proposition 2.1], which relates the code of a list of digits to its length, we can also establish the following inequality between the codes of α and ω_n^α:

$$\mathbf{I\Sigma}_1 \vdash \forall_{n,\alpha}\, \omega_n^\alpha \leq 4^{3n+1} \cdot (\alpha + 1). \tag{3}$$

Let us remark that we do not extend the ordinal notation system by a symbol for ε_0, in order to keep it closed under the usual operations of ordinal arithmetic. By a harmless abuse of notation we will sometimes refer to the "fundamental sequence" of ε_0, which we define as $\{\varepsilon_0\}(n) := \omega_{n+1}$.
Using fundamental sequences we can define the fast-growing hierarchy of functions indexed by ordinals below and including ε_0. The definition varies slightly within the literature; our version differs from the classic [5, 6] and coincides e.g. with [14]:

$$F_0(x) := x + 1,$$
$$F_{\alpha+1}(x) := F_\alpha^{x+1}(x),$$
$$F_\lambda(x) := F_{\{\lambda\}(x)}(x) \quad \text{for } \lambda \text{ a limit ordinal.}$$

Here and in the following an exponent to a function symbol denotes the number of times the function is to be iterated. Given an arithmetization of ordinal arithmetic it is easy to define the graph of $(\alpha, x, i) \mapsto F_\alpha^i(x)$ by a Σ_1-formula in the language of first-order arithmetic: As described in [15, Section 4.1] one can compute $F_\alpha^i(x)$ by simplifying expressions of the form $F_{\alpha_1}^{i_1}(F_{\alpha_2}^{i_2}(\cdots(F_{\alpha_k}^{i_k}(z))\cdots))$, so one only needs to state the existence of such a computation sequence. What is remarkable is that the size of an (improved) computation sequence can be bounded by a polynomial in the value of $F_\alpha^i(x)$. This is worked out in [15, Appendix A] (see also the less detailed [14, Section 5.2]) and leads to a Δ_0-formula $F_\alpha^i(x) = y$ with free variables x, y, α, i which defines the functions F_α for $\alpha < \varepsilon_0$, as well as their iterations. By [14, Theorem 5.3] the defining equations of the fast-growing hierarchy are provable in $\mathbf{I\Sigma}_1$ (under the assumption

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that the involved computations terminate, which is of course unprovable in $\mathbf{I\Sigma}_1$). As Sommer only encodes the hierarchy below ε_0 we should show separately that the formula

$$F_{\varepsilon_0}(x) = y \quad :\equiv \quad \exists_\alpha(\alpha = \omega_{x+1} \wedge F_\alpha(x) = y)$$

is Δ_0 in $\mathbf{I\Sigma}_1$: The only task is to bound the existentially quantified α. By [11, Lemma 2.3, Proposition 2.12] the inequalities

$$F_{\omega_{x+1}}(x) \geq F_\omega(x) \geq F_2(x) = 2^{x+1} \cdot (x+1) - 1 \geq 2^{x+1} \qquad \text{for } x \geq 1$$

are provable in $\mathbf{I\Sigma}_1$. Combining this with (3) we obtain

$$\mathbf{I\Sigma}_1 \vdash x \geq 1 \rightarrow (F_{\varepsilon_0}(x) = y \leftrightarrow \exists_{\alpha \leq y^6 \cdot 4 \cdot (\ulcorner 1 \urcorner + 1)}(\alpha = \omega_{x+1} \wedge F_\alpha(x) = y)), \quad (4)$$

where $\ulcorner 1 \urcorner$ denotes the code of the ordinal 1.
Writing $\langle \cdot, \cdot \rangle$ for the Cantor pairing function with projections $\pi_1(\cdot), \pi_1(\cdot)$ we can now define slow proofs in Peano Arithmetic. The idea is to penalize the use of complex induction axioms by a drastic increase in proof length, and thus to create an interplay between proof size and the amount of induction used in the proof.

Definition 2.1 (cf. [11])**.** A pair $\langle q, N \rangle$ is a slow $\mathbf{PA}$-proof of a formula φ if there is a number n such that we have $N = F_{\varepsilon_0}(n)$ and such that q codes a (usual) proof of φ in the theory $\mathbf{I\Sigma}_{n+1}$. This notion is defined by the formula

$$\mathrm{Proof}^\diamond_{\mathbf{PA}}(p, \varphi) :\equiv \exists_x(\mathrm{Proof}_{\mathbf{I\Sigma}_{x+1}}(\pi_1(p), \varphi) \wedge F_{\varepsilon_0}(x) = \pi_2(p)),$$

which is Δ_1 in $\mathbf{I\Sigma}_1$ since by [14, Proposition 5.4] the second conjunct implies the bound $x \leq \pi_2(p)$.

For a formula $F(x) = y$ let us abbreviate $\exists_y F(x) = y$ by $F(x)\!\downarrow$. Also, we write $\mathrm{Pr}_{\mathbf{I\Sigma}_x}(\varphi)$ for the formula $\exists_p \mathrm{Proof}_{\mathbf{I\Sigma}_x}(p, \varphi)$. It is easy to see that the slow provability predicate

$$\mathrm{Pr}^\diamond_{\mathbf{PA}}(\varphi) :\equiv \exists_p \mathrm{Proof}^\diamond_{\mathbf{PA}}(p, \varphi)$$

satisfies the equivalence

$$\mathbf{I\Sigma}_1 \vdash \mathrm{Pr}^\diamond_{\mathbf{PA}}(\varphi) \leftrightarrow \exists_x(\mathrm{Pr}_{\mathbf{I\Sigma}_{x+1}}(\varphi) \wedge F_{\varepsilon_0}(x)\!\downarrow).$$

The slow uniform Σ_1-reflection principle

$$\mathrm{RFN}^\diamond_{\Sigma_1}(\mathbf{PA}) :\equiv \forall_\varphi(\text{“}\varphi \text{ is a closed } \Sigma_1\text{-formula”} \wedge \mathrm{Pr}^\diamond_{\mathbf{PA}}(\varphi) \rightarrow \mathrm{True}_{\Sigma_1}(\varphi))$$

and the slow consistency statement

$$\mathrm{Con}^\diamond(\mathbf{PA}) :\equiv \neg \mathrm{Pr}^\diamond_{\mathbf{PA}}(\overline{\ulcorner 0 = 1 \urcorner})$$

can be characterized as

$$\mathbf{I\Sigma}_1 \vdash \mathrm{RFN}^\diamond_{\Sigma_1}(\mathbf{PA}) \leftrightarrow \forall_x(F_{\varepsilon_0}(x)\!\downarrow \rightarrow \mathrm{RFN}_{\Sigma_1}(\mathbf{I\Sigma}_{x+1})) \quad (5)$$

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and

$$\mathbf{I\Sigma}_1 \vdash \mathrm{Con}^{\diamond}(\mathbf{PA}) \leftrightarrow \forall_x (F_{\varepsilon_0}(x)\downarrow \to \mathrm{Con}(\mathbf{I\Sigma}_{x+1})).$$

As the last equivalence reveals the notion of slow **PA**-proof comes from the article [11] by S.-D. Friedman, Rathjen and Weiermann: These authors introduce the slow consistency statement

$$\mathrm{Con}^*(\mathbf{PA}) \equiv \forall_x (F_{\varepsilon_0}(x)\downarrow \to \mathrm{Con}(\mathbf{I\Sigma}_x))$$

and show that we have

$$\mathbf{PA} + \mathrm{Con}^*(\mathbf{PA}) \nvdash \mathrm{Con}(\mathbf{PA}). \tag{6}$$

It has been pointed out by Michael Rathjen [16] that slow provability satisfies the Gödel-Löb conditions, provably so in $\mathbf{I\Sigma}_1$. In many respects it thus behaves as the usual provability predicate for Peano Arithmetic. The index shift between our $\mathrm{Con}^{\diamond}(\mathbf{PA})$ and the formula $\mathrm{Con}^*(\mathbf{PA})$ of [11] has been introduced to improve the bounds on proof sizes that we are about to establish.
The central ingredient to our bounds on proof sizes is a computational analysis of slow reflection. Since this analysis is independent and somewhat technical we defer it to Section 3 below. In the present section we will only use the following result of this analysis:

Theorem 3.10. *For any provably total function g of $\mathbf{PA} + \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$ there is a number N such that we have*

$$g(F_{\varepsilon_0}(n \dot{-} 1)) \leq F_{\varepsilon_0}(n) \qquad \text{for all } n \geq N.$$

In particular any provably total function of the theory $\mathbf{PA} + \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$ is eventually dominated by F_{ε_0}.

The reader who prefers to see all proofs in order may go through Section 3 now and return to this point afterwards. In the rest of this section we show how results about proof sizes in fragments of Peano Arithmetic can be deduced. It is worth observing that a weaker version of Theorem 3.10 suffices for these applications: Namely, it would be enough to bound the provably total functions of $\mathbf{I\Sigma}_1 + \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$ rather than those of $\mathbf{PA} + \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$. However, as a result in its own right Theorem 3.10 is certainly more satisfying with the stronger base theory. Let us now investigate the size of proofs of the formulas $F_{\varepsilon_0}(\overline{n})\downarrow$. Afterwards we will come to the slightly more subtle case of the Paris-Harrington Principle:

Lemma 2.2. *There is a number N such that we have*

$$p > \langle F_{\varepsilon_0}(n \dot{-} 1), F_{\varepsilon_0}(n \dot{-} 1)\rangle \quad \text{for any slow } \mathbf{PA}\text{-proof } p \text{ of } F_{\varepsilon_0}(\overline{n})\downarrow \text{ with } n \geq N.$$

To avoid misunderstanding we recall that $\langle \cdot, \cdot \rangle$ denotes the Cantor pairing.

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Proof. We apply Proposition 1.3 to the proof predicate $\mathrm{Proof}^{\diamond}_{\mathbf{PA}}(p,\varphi)$, the theory $\mathbf{T} = \mathbf{I\Sigma}_1 + \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$, the formula $\psi(x,y) \equiv F_{\varepsilon_0}(x) = y$ (so that F_ψ is the function F_{ε_0}), and the function $n \mapsto \langle F_{\varepsilon_0}(n \dot{-} 1), F_{\varepsilon_0}(n \dot{-} 1)\rangle$ at the place of f. Let us verify the assumptions of Proposition 1.3: By (5) we have

$$\mathbf{I\Sigma}_1 + \mathrm{RFN}_{\Sigma_1}(\mathbf{PA}) \vdash \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA}),$$

where $\mathrm{RFN}_{\Sigma_1}(\mathbf{PA})$ denotes the usual uniform Σ_1-reflection principle for Peano Arithmetic. This shows that the theory $\mathbf{I\Sigma}_1 + \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$ is sound. Next, using [3, Proposition 2.5] we have

$$n \leq F_{\varepsilon_0}(n \dot{-} 1) \leq \langle F_{\varepsilon_0}(n \dot{-} 1), F_{\varepsilon_0}(n \dot{-} 1)\rangle.$$

Finally, consider an arbitrary function g that is provably total in the theory $\mathbf{I\Sigma}_1 + \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$. We have to show that there is a number N such that we have

$$g(\langle F_{\varepsilon_0}(n \dot{-} 1), F_{\varepsilon_0}(n \dot{-} 1)\rangle) \leq F_{\varepsilon_0}(n) \qquad \text{for all } n \geq N.$$

This follows from Theorem 3.10, applied not to g itself but rather to the function $m \mapsto g(\langle m, m\rangle)$, which is still provably total in the theory $\mathbf{I\Sigma}_1 + \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$. Now Proposition 1.3 gives us precisely the claim. □

It is easy to deduce bounds for proofs in the fragments of Peano Arithmetic:

Theorem 2.3. *There is a number N such that for all $n \geq N$ no proof of the statement $F_{\varepsilon_0}(\overline{n})\downarrow$ in the theory $\mathbf{I\Sigma}_n$ can have code less than or equal to $F_{\varepsilon_0}(n \dot{-} 1)$.*

Proof. We can assume that the bound N in Lemma 2.2 is bigger than zero. Let us show that the present result holds with the same bound: Aiming at a contradiction, suppose that $q \leq F_{\varepsilon_0}(n \dot{-} 1)$ is an $\mathbf{I\Sigma}_n$-proof of the formula $F_{\varepsilon_0}(\overline{n})\downarrow$, for some $n \geq N$. By definition $\langle q, F_{\varepsilon_0}(n \dot{-} 1)\rangle$ is a slow $\mathbf{PA}$-proof of $F_{\varepsilon_0}(\overline{n})\downarrow$. Thus the inequality

$$\langle q, F_{\varepsilon_0}(n \dot{-} 1)\rangle \leq \langle F_{\varepsilon_0}(n \dot{-} 1), F_{\varepsilon_0}(n \dot{-} 1)\rangle$$

contradicts Lemma 2.2. □

To deduce corresponding results for instances of the Paris-Harrington Principle, recall the function $(n,k) \mapsto \sigma(n,k)$ defined just before Corollary 1.5 above. We need to link this function to the function F_{ε_0}:

Lemma 2.4 ([3])**.** *We have*

$$F_{\varepsilon_0}(n) \leq \sigma(n+2, 10^{35n^2}) \leq \sigma(n+3, 8) \qquad \textit{for all } n \geq 15.$$

Proof. This is the result of [3, Lemma 3.6, Theorem 3.10], except that [3] works with a slightly different version of fundamental sequences, setting

$$\{\beta + \omega^\gamma \cdot (k+1)\}(n) = \beta + \omega^\gamma \cdot k + \omega^\delta \cdot n \quad \text{in case } \gamma = \delta + 1.$$

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With this definition, descending to the n-th member of the fundamental sequence can introduce a coefficient (bounded by) n. In our case the new coefficients are bounded by $n+1$. The overall bound $\sigma(n+2, 10^{23n^2})$ of [3, Lemma 3.6] then increases to our $\sigma(n+2, 10^{35n^2})$.
Let us describe the concrete changes that are necessary (the reader will have to consult [3] for context): First, the bound of [3, Proposition 2.9] increases from $|T_{k,c,n}| \leq (n+1)_k^c$ to $|T_{k,c,n}| \leq (n+2)_k^c$. At the same time the rather generous bound $|T_{k,c,n}| \leq 2_{k-1}^{(n^{6c})}$ of [3, Proposition 2.10] remains valid without change. Thus [3, Lemma 3.1] remains valid, and so does [3, Lemma 3.2.1]. A small change is required in [3, Lemma 3.2.2]: We need to weaken the condition $g(x_0, \dots, x_{n-1}) \leq x_0$ to $g(x_0, \dots, x_{n-1}) \leq x_0 + 1$. It is easy to see that g is then controlled by an $(n+1, 10^5)$-algebra (instead of an $(n+1, 10^4)$-algebra). Consequently, [3, Lemma 3.2.3] now constructs an $(n+1, 10^{5c})$-algebra. One can check that [3, Lemma 3.4] remains valid in spite of the prior changes: The bound of [3, Lemma 3.2.3] is still strong enough for the base case of the proof; in the step, the bound is generous enough to accomodate the fact that G_3 is now an $(n+2, 10^5)$-algebra. It follows that [3, Theorem 3.5] remains unchanged: For $n, k \geq 1$ the function $F_{\omega_n^k}$ is captured by an $(n+2, 10^{n\cdot(12n+2k+8)})$-algebra. Parallel to [3, Lemma 3.6] we can now deduce the desired bound: We have $\{\omega_{n+1}\}(n) = \omega_n^{n+1}$ and thus $F_{\varepsilon_0}(n) = F_{\omega_{n+1}}(n) = F_{\omega_n^{n+1}}(n)$ (as opposed to $F_{\varepsilon_0}(n) = F_{\omega_n^n}(n)$ in the original [3, Lemma 3.6]). Let G_0 be an $(n+2, 10^{14n^2+20n})$-algebra that captures $F_{\omega_n^{n+1}}$. Let G_1 be an $(n+2, 7)$-algebra such that $\min(S) \geq 2n+3$ holds whenever S is suitable for G_1. In view of

$$7 \cdot 10^{14n^2+20n} \leq 10^{14n^2+20n+1} \leq 10^{35n^2} \quad \text{(for } n \geq 1)$$

we can choose an $(n+2, 10^{35n^2})$-algebra G which simulates G_0 and G_1. If S is suitable for G then we have

$$max(S) \geq s_2 > s_1 \geq F_{\omega_n^{n+1}}(s_0) \geq F_{\omega_n^{n+1}}(n) = F_{\varepsilon_0}(n).$$

This means that the restriction

$$G \restriction_{[F_{\varepsilon_0}(n)]^{n+2}} \colon [F_{\varepsilon_0}(n)]^{n+2} \to 10^{35n^2}$$

admits no suitable set. Thus we have $F_{\varepsilon_0}(n) < \sigma(n+2, 10^{35n^2})$.
It remains to check $\sigma(n+2, 10^{35n^2}) \leq \sigma(n+3, 8)$. This is parallel to the proof of [3, Theorem 3.10]: Observe that we have

$$F_3^{n+1}(n+2) \geq F_3(n) \geq 2^{2^n} \geq 2^{4\cdot 35n^2} \geq 10^{35n^2} \qquad \text{for } n \geq 15.$$

Thus by [3, Lemma 3.9] each $(n+2, 10^{35n^2})$-algebra can be simulated by an $(n+3, 8)$-algebra, and this implies the claim. Note that the condition $n \geq 15$ could easily be replaced by a smaller bound. □

This implies the following result, which we will need in our applications:

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Corollary 2.5. *For any provably total function g of $\mathbf{I\Sigma}_1 + \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$ there is a number N such that we have*

$$g(F_{\varepsilon_0}(n \dot{-} 1)) \leq \sigma(n+2, 10^{35n^2}) \leq \sigma(n+3, 8) \qquad \textit{for all } n \geq N.$$

Proof. This follows from Theorem 3.10 and Lemma 2.4. □

Similar to Lemma 2.2, slow proofs of certain instances of the Paris-Harrington Principle must be long:

Lemma 2.6. *The following holds:*

(a) There is a number K' such that we have $p > \langle F_{\varepsilon_0}(n \dot{-} 1), F_{\varepsilon_0}(n \dot{-} 1)\rangle$ for any slow $\mathbf{PA}$-proof p of $\exists_N \mathrm{PH}(\overline{10^{35n^2}}, \overline{n+3}, \overline{n+2}, N)$ with $n \geq K'$.

(b) There is a number K' such that we have $p > \langle F_{\varepsilon_0}(n \dot{-} 1), F_{\varepsilon_0}(n \dot{-} 1)\rangle$ for any slow $\mathbf{PA}$-proof p of $\exists_N \mathrm{PH}(8, \overline{n+4}, \overline{n+3}, N)$ with $n \geq K'$.

Proof. We only show (a). The proof of (b) is similar and somewhat easier. Compared to the proof of Lemma 2.2, the main subtlety is that the formulas

$$\varphi_n :\equiv \exists_N \mathrm{PH}(\overline{10^{35n^2}}, \overline{n+3}, \overline{n+2}, N)$$

are not of the form $\varphi(\overline{n})$, i.e. parametrized by the n-th numeral. To make Proposition 1.3 applicable we need to preprocess proofs of these formulas, as sketched in Remark 1.4: Let $e(x) = z$ be a Σ_1-formula such that we have

$$\mathbb{N} \vDash e(\overline{n}) = \overline{k} \qquad \Leftrightarrow \qquad k = 10^{35n^2}$$

and $\mathbf{I\Sigma}_1 \vdash \forall_x \exists_z\, e(x) = z$. In view of the latter, the witnesses to all unbounded quantifiers of the Σ_1-formula $\exists_z\, e(\overline{n}) = z$ are bounded by a primitive recursive function in n. By the proof of Σ_1-completeness there is a primitive recursive function $p_e : \mathbb{N}^2 \to \mathbb{N}$ such that $p_e(n,k)$ is an $\mathbf{I\Sigma}_k$-proof of $e(\overline{n}) = \overline{10^{35n^2}}$. Next, let $\psi(x,y)$ be a Σ_1-formula with

$$\mathbf{I\Sigma}_1 \vdash \psi(x,y) \leftrightarrow \exists_z (e(x) = z \wedge \mathrm{PH}(z, x+3, x+2, y)). \tag{7}$$

Following Remark 1.4, we need a primitive recursive function $h : \mathbb{N} \to \mathbb{N}$ which transforms a slow $\mathbf{PA}$-proof of φ_n into a slow $\mathbf{PA}$-proof of $\exists_y \psi(\overline{n}, y)$. Let us first construct a primitive recursive function $h' : \mathbb{N}^2 \to \mathbb{N}$ such that $h'(k,q)$ is an $\mathbf{I\Sigma}_{k+1}$-proof of $\exists_y \psi(\overline{n}, y)$ if q is an $\mathbf{I\Sigma}_{k+1}$-proof of φ_n: Given a proof q as described, we can read off its end formula φ_n and then the number n. Recall that $p_e(n, k+1)$ is an $\mathbf{I\Sigma}_{k+1}$-proof of $e(\overline{n}) = \overline{10^{35n^2}}$. Combining this with q and introducing an existential quantifier yields an $\mathbf{I\Sigma}_{k+1}$-proof of

$$\exists_z (e(\overline{n}) = z \wedge \exists_N \mathrm{PH}(z, \overline{n+3}, \overline{n+2}, N)).$$

It is not unreasonable to assume that $\overline{n+3}$ (resp. $\overline{n+2}$) is the same term as $\overline{n} + 3$ (resp. $\overline{n} + 2$). Even if not, there are primitive recursive functions which

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map a pair (k, n) to $\mathbf{I\Sigma}_{k+1}$-proofs of $\overline{n+3} = \overline{n}+3$ and $\overline{n+2} = \overline{n}+2$. We then apply the equality axioms and prefix the existentially quantified N, giving an $\mathbf{I\Sigma}_{k+1}$-proof of

$$\exists_y \exists_z (e(\overline{n}) = z \wedge \mathrm{PH}(z, \overline{n}+3, \overline{n}+2, y)).$$

Invoking the equivalence (7) we get the desired proof $h'(k,q)$ of $\exists_y \psi(\overline{n}, y)$. Now to construct h, assume that $p = \langle q, M\rangle$ is a slow $\mathbf{PA}$-proof of φ_n. By definition there is an $m \leq M$ such that q is an $\mathbf{I\Sigma}_{m+1}$ proof of φ_n and such that we have $F_{\varepsilon_0}(m) = M$. Recall that the relation $F_{\varepsilon_0}(x) = y$ is primitive recursively decidable, and that F_{ε_0} is strictly monotone. Thus we can primitive recursively determine the unique m with the stated property. Now it suffices to set

$$h(p) := \langle h'(m,q), M\rangle.$$

We need to increase h to make it monotone and ensure $h(p) \geq p$. Clearly, the increased function still satisfies the following: If p is a slow $\mathbf{PA}$-proof of φ_n then there is a slow $\mathbf{PA}$-proof of $\exists_y \psi(\overline{n}, y)$ below $h(p)$.
Now we apply Proposition 1.3 to the proof predicate $\mathrm{Proof}^{\diamond}_{\mathbf{PA}}(p, \varphi)$, the theory $\mathbf{T} = \mathbf{I\Sigma}_1 + \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$, the Σ_1-formula $\psi(x,y)$ defined above, and the function $n \mapsto h(\langle F_{\varepsilon_0}(n \dot{-} 1), F_{\varepsilon_0}(n \dot{-} 1)\rangle)$ at the place of f. In view of (7) we have

$$\mathbb{N} \vDash \psi(\overline{n}, \overline{m}) \quad \Leftrightarrow \quad \mathbb{N} \vDash \mathrm{PH}(\overline{10^{35n^2}}, \overline{n+3}, \overline{n+2}, \overline{m}),$$

so that F_ψ is the function $n \mapsto \sigma(n+2, 10^{35n^2})$. Concerning the assumptions of Proposition 1.3, in view of $h(p) \geq p$ (see also the proof of Lemma 2.2) we have

$$h(\langle F_{\varepsilon_0}(n \dot{-} 1), F_{\varepsilon_0}(n \dot{-} 1)\rangle) \geq n \quad \text{for all } n.$$

Coming to the other assumption, let g be any provably total function of $\mathbf{I\Sigma}_1 + \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$. We must show that $n \mapsto g(h(\langle F_{\varepsilon_0}(n \dot{-} 1), F_{\varepsilon_0}(n \dot{-} 1)\rangle))$ is eventually dominated by the function $n \mapsto \sigma(n+2, 10^{35n^2})$. To see this one applies Corollary 2.5 to the function $m \mapsto g(h(\langle m, m\rangle))$, which is still provably total in the theory $\mathbf{I\Sigma}_1 + \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$. Having verified the assumptions Proposition 1.3 gives us a bound K' such that we have

$$p' > h(\langle F_{\varepsilon_0}(n \dot{-} 1), F_{\varepsilon_0}(n \dot{-} 1)\rangle)$$

whenever p' is a slow $\mathbf{PA}$-proof of $\exists_y \psi(\overline{n}, y)$ with $n \geq K'$. To deduce the claim of (a), let p be a slow $\mathbf{PA}$-proof of $\exists_N \mathrm{PH}(\overline{10^{35n^2}}, \overline{n+3}, \overline{n+2}, N)$, still with $n \geq K'$. As we have seen above, this implies that there is a slow $\mathbf{PA}$-proof of $\exists_y \psi(\overline{n}, y)$ below $h(p)$. By the bound that we have just established we must have

$$h(p) > h(\langle F_{\varepsilon_0}(n \dot{-} 1), F_{\varepsilon_0}(n \dot{-} 1)\rangle).$$

Since h is monotone this does indeed imply $p > \langle F_{\varepsilon_0}(n \dot{-} 1), F_{\varepsilon_0}(n \dot{-} 1)\rangle$. $\square$

We can derive the central result of the paper, claim (1) from the introduction:

Theorem 2.7. *The following holds:*

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(a) *There is a number K such that for all $n \geq K$ no proof of the formula $\exists_N \mathrm{PH}(\overline{10^{35(n \dot{-} 2)^2}}, \overline{n+1}, \overline{n}, N)$ in the theory $\mathbf{I\Sigma}_{n \dot{-} 2}$ can have Gödel number less than or equal to $F_{\varepsilon_0}(n \dot{-} 3)$.*

(b) *There is a number K such that for all $n \geq K$ no proof of the formula $\exists_N \mathrm{PH}(\overline{8}, \overline{n+1}, \overline{n}, N)$ in the theory $\mathbf{I\Sigma}_{n \dot{-} 3}$ can have Gödel number less than or equal to $F_{\varepsilon_0}(n \dot{-} 4)$.*

Proof. We only write out the proof for (a), the proof of (b) being completely parallel: Let K' be the bound from Lemma 2.6, and set $K := \max\{K'+2, 3\}$. Consider an arbitrary $n \geq K$ and a proof q of $\exists_N \mathrm{PH}(\overline{10^{35(n-2)^2}}, \overline{n+1}, \overline{n}, N)$ in the theory $\mathbf{I\Sigma}_{n-2}$. It follows that the pair $\langle q, F_{\varepsilon_0}(n-3)\rangle$ is a slow **PA**-proof of $\exists_N \mathrm{PH}(\overline{10^{35(n-2)^2}}, \overline{n+1}, \overline{n}, N)$. Lemma 2.6 yields

$$\langle q, F_{\varepsilon_0}(n-3)\rangle > \langle F_{\varepsilon_0}(n-3), F_{\varepsilon_0}(n-3)\rangle.$$

Since the Cantor pairing is monotone we get $q > F_{\varepsilon_0}(n-3)$, as desired. □

By claim (2) from the introduction both $\exists_N \mathrm{PH}(\overline{10^{35(n \dot{-} 2)^2}}, \overline{n+1}, \overline{n}, N)$ and $\exists_N \mathrm{PH}(\overline{8}, \overline{n+1}, \overline{n}, N)$ have short proofs in $\mathbf{I\Sigma}_{n \dot{-} 1}$. The fragment $\mathbf{I\Sigma}_{n \dot{-} 2}$ in part (a) of the theorem is thus optimal. Concerning (b), it is currently open whether $\exists_N \mathrm{PH}(\overline{8}, \overline{n+1}, \overline{n}, N)$ has a short proof in $\mathbf{I\Sigma}_{n \dot{-} 2}$. In any case the parameters of the Paris-Harrington Principle leave room for variation: For example, the bounds established by Loebl and Nešetřil [17] (with shorter proofs than in [3]) lead to similar results.

3. The Provably Total Functions of Slow Reflection

The goal of this section is to provide a proof of Theorem 3.10, which we already used (but did not prove) in the previous section. We will need the following characterization of uniform Σ_1-reflection over the fragments of Peano Arithmetic:

Proposition 3.1. *We have*

$$\mathbf{I\Sigma}_1 \vdash \forall_x (F_{\omega_x}\!\downarrow \leftrightarrow \mathrm{RFN}_{\Sigma_1}(\mathbf{I\Sigma}_x)).$$

Proof. It is known that the equivalence $F_{\omega_n}\!\downarrow \leftrightarrow \mathrm{RFN}_{\Sigma_1}(\mathbf{I\Sigma}_n)$ for fixed n is provable in $\mathbf{I\Sigma}_1$ (and in weaker theories): A model-theoretic proof can be found in [18] or [14, Proposition 6.8]. For a proof-theoretic approach (via iterated reflection principles) we refer to [19, Theorem 1, Proposition 7.3, Remark 7.4]. The author has found no fully explicit argument that the formalization is uniform in n. We provide a detailed proof of this fact in [20]: This is a proof-theoretic argument, formalizing the infinitary proof system from [21] by the method of [22]. □

Using this result and (5) we can view slow reflection as a statement about the fast-growing hierarchy of functions:

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Corollary 3.2. *We have*

$$\mathbf{I\Sigma}_1 \vdash \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA}) \leftrightarrow \forall_x (F_{\varepsilon_0}(x)\!\downarrow \to F_{\omega_{x+1}}\!\downarrow).$$

Note that the "index shift", stemming from the definition of slow proof, is indeed optimal: In view of $F_{\varepsilon_0}(x) \simeq F_{\omega_{x+1}}(x)$ we can deduce

$$\mathbf{I\Sigma}_1 \vdash \forall_x (F_{\varepsilon_0}(x)\!\downarrow \to F_{\omega_{x+2}}\!\downarrow) \to \forall_y F_{\varepsilon_0}(y)\!\downarrow$$

by induction on y. Thus a stronger slow reflection statement would collapse into the usual notion of Σ_1-reflection over Peano Arithmetic. This explains why our bounds on proof size are relatively sharp.
Our next goal is to transform the Π_2-statement $\forall_x (F_{\varepsilon_0}(x)\!\downarrow \to F_{\omega_{x+1}}\!\downarrow)$ into a formula which defines a unary function.

Definition 3.3. The inverse $F^{-1}_{\varepsilon_0}$ of the function F_{ε_0} (see [11, Definition 3.2]) is given by

$$F^{-1}_{\varepsilon_0}(x) := \max(\{z \leq x \,|\, \exists_{w \leq x} F_{\varepsilon_0}(z) = w\} \cup \{0\}).$$

Note that the Δ_0-definition of F_{ε_0} yields a Δ_0-definition of $F^{-1}_{\varepsilon_0}$. To define a slow variant $F^{\diamond}_{\varepsilon_0}$ of the function F_{ε_0} we set

$$F^{\diamond}_{\varepsilon_0}(x) := F_{\omega_{F^{-1}_{\varepsilon_0}(x)+1}}(x),$$

which has the Σ_1-definition

$$F^{\diamond}_{\varepsilon_0}(x) = y \quad \Leftrightarrow \quad \exists_z (z = F^{-1}_{\varepsilon_0}(x) \wedge \exists_\alpha (\alpha = \omega_{z+1} \wedge F_\alpha(x) = y)).$$

Clearly, z is bounded by x. In view of (4) the code of α is bounded by a polynomial in x. Thus the given definition of $F^{\diamond}_{\varepsilon_0}$ is Δ_0 in $\mathbf{I\Sigma}_1$.

We remark that the idea behind $F^{\diamond}_{\varepsilon_0}$ is similar to Simmons' slow variant of the Ackermann function in [23, Paragraph 2]. Let us now connect $F^{\diamond}_{\varepsilon_0}$ with the slow reflection principle:

Proposition 3.4. *We have*

$$\mathbf{I\Sigma}_1 \vdash \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA}) \leftrightarrow F^{\diamond}_{\varepsilon_0}\!\downarrow.$$

Proof. By Corollary 3.2 the claim of the proposition is equivalent to

$$\mathbf{I\Sigma}_1 \vdash \forall_x (F_{\varepsilon_0}(x)\!\downarrow \to F_{\omega_{x+1}}\!\downarrow) \leftrightarrow F^{\diamond}_{\varepsilon_0}\!\downarrow.$$

To show the direction "$\rightarrow$" we work in $\mathbf{I\Sigma}_1$ and assume that the formula $\forall_x (F_{\varepsilon_0}(x)\!\downarrow \to F_{\omega_{x+1}}\!\downarrow)$ holds. We have to prove $F^{\diamond}_{\varepsilon_0}(x)\!\downarrow$ for an arbitrary x. The finitely many $x < F_{\varepsilon_0}(0)$ are treated by Σ_1-completeness. For $x \geq F_{\varepsilon_0}(0)$ the set $\{z \leq x \,|\, \exists_{w \leq x} F_{\varepsilon_0}(z) = w\}$ is non-empty, so $F^{-1}_{\varepsilon_0}(x) =: z$ is an element of this set. In particular it follows that $F_{\varepsilon_0}(z)$ is defined. Then the assumption $\forall_x (F_{\varepsilon_0}(x)\!\downarrow \to F_{\omega_{x+1}}\!\downarrow)$ tells us that $F_{\omega_{z+1}}$ is total. Thus $F_{\omega_{z+1}}(x)$ is defined, as

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required for $F^{\diamond}_{\varepsilon_0}(x){\downarrow}$.
For the direction "$\leftarrow$", assume that the function $F^{\diamond}_{\varepsilon_0}$ is total, let x be arbitrary, and assume that $F_{\varepsilon_0}(x)$ is defined. We have to prove that $F_{\omega_{x+1}}$ is total. By [11, Lemma 2.3] it suffices to show that $F_{\omega_{x+1}}(y)$ is defined for arbitrarily large y. Since $F_{\varepsilon_0}(x)$ was assumed to be defined, we may consider an arbitrary y above this value. Then we have $x \leq F^{-1}_{\varepsilon_0}(y) =: z$. Invoking the totality of $F^{\diamond}_{\varepsilon_0}$ we learn that $F^{\diamond}_{\varepsilon_0}(y) = F_{\omega_{z+1}}(y)$ is defined. It follows by [11, Lemma 2.4, Proposition 2.12, Lemma 2.3] that $F_{\omega_{x+1}}(y)$ is defined (and has value at most $F^{\diamond}_{\varepsilon_0}(y)$). $\square$

By the parenthesis at the end of the proof, the function $F^{\diamond}_{\varepsilon_0}$ dominates $F_{\omega_{x+1}}$ for values above $F_{\varepsilon_0}(x)$. In other words, $F^{\diamond}_{\varepsilon_0}$ eventually dominates any provably total function of Peano Arithmetic. In particular we have

$$\mathbf{PA} \nvdash \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA}).$$

Since slow reflection implies slow consistency this was already known by [11, Proposition 3.3]. It is important that the argument we just gave does not formalize in Peano Arithmetic: To show that $F^{\diamond}_{\varepsilon_0}$ dominates $F_{\omega_{x+1}}$ we had to know that $F_{\varepsilon_0}(x)$ is defined. If this was different then $F^{\diamond}_{\varepsilon_0}{\downarrow}$ would imply $F_{\varepsilon_0}{\downarrow}$, contradicting the result that we are about to prove.
Recall that our goal is to bound the provably total functions of the theory $\mathbf{PA}+\mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$, or equivalently those of $\mathbf{PA}+F^{\diamond}_{\varepsilon_0}{\downarrow}$. It is a classical result that any provably total function of Peano Arithmetic is dominated by some function F_α with $\alpha < \varepsilon_0$ from the fast-growing hierarchy. To analyse $\mathbf{PA}+\mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$ we build an analogous hierarchy on top of $F^{\diamond}_{\varepsilon_0}$:

Definition 3.5. By induction on $\alpha < \varepsilon_0$ we define functions $F^{\diamond}_{\varepsilon_0+\alpha}$: Set

$$\begin{aligned} F^{\diamond}_{\varepsilon_0+0}(n) &:= F^{\diamond}_{\varepsilon_0}(n), \\ F^{\diamond}_{\varepsilon_0+\alpha+1}(n) &:= (F^{\diamond}_{\varepsilon_0+\alpha})^{n+1}(n), \\ F^{\diamond}_{\varepsilon_0+\alpha}(n) &:= F^{\diamond}_{\varepsilon_0+\{\alpha\}(n)}(n) \quad \text{for } \alpha \text{ limit}, \end{aligned}$$

where the superscript $n+1$ denotes the number of iterations and $\{\alpha\}(n)$ refers to the fundamental sequence of α, as defined at the beginning of Section 2.

To make use of this hierarchy we will need some monotonicity properties. These will involve the "step down"-relation from [3, Section 2] (with slightly different fundamental sequences) or [11, Section 2]: We write $\beta \to_n \gamma$ to express that there is a sequence $\langle \delta_0, \ldots, \delta_k \rangle$ of ordinals with $\delta_0 = \beta$, $\delta_k = \gamma$ and $\{\delta_i\}(n) = \delta_{i+1}$ for all $i < k$. The following properties are familiar from the usual fast-growing hierarchy:

Lemma 3.6. *For all numbers m, n and ordinals $\alpha, \beta < \varepsilon_0$ the following holds:*

(i) We have $n \leq n^2 < F^{\diamond}_{\varepsilon_0+\alpha}(n)$.

(ii) If $m \leq n$ then $F^{\diamond}_{\varepsilon_0+\alpha}(m) \leq F^{\diamond}_{\varepsilon_0+\alpha}(n)$.

(iii) If $\alpha \to_n \beta$ then $F^{\diamond}_{\varepsilon_0+\beta}(n) \leq F^{\diamond}_{\varepsilon_0+\alpha}(n)$.

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Proof. We repeat the well-known proof for the usual fast-growing hierarchy (see [3, Proposition 2.5]), with minor modifications in the base case: Claim (i) is shown by induction on α. For $\alpha = 0$ we have

$$n^2 < F_{\omega_{F_{\varepsilon_0}^{-1}(n)+1}}(n) = F^{\diamond}_{\varepsilon_0}(n)$$

by [14, Proposition 5.4]. Successor and limit case are easy. Claims (ii) and (iii) are shown by a simultaneous induction on α. Concerning $\alpha = 0$ it is easy to see that $m \leq n$ implies $F_{\varepsilon_0}^{-1}(m) \leq F_{\varepsilon_0}^{-1}(n)$. Then

$$F^{\diamond}_{\varepsilon_0}(m) = F_{\omega_{F_{\varepsilon_0}^{-1}(m)+1}}(m) \leq F_{\omega_{F_{\varepsilon_0}^{-1}(n)+1}}(m) \leq F_{\omega_{F_{\varepsilon_0}^{-1}(n)+1}}(n) = F^{\diamond}_{\varepsilon_0}(n)$$

follows by [11, Lemma 2.3, Proposition 2.12]. Claim (iii) is trivial for $\alpha = 0$. In case $\alpha = \gamma + 1$ claim (ii) holds by

$$\begin{aligned} F^{\diamond}_{\varepsilon_0+\alpha}(m) = (F^{\diamond}_{\varepsilon_0+\gamma})^{m+1}(m) \leq (F^{\diamond}_{\varepsilon_0+\gamma})^{m+1}(n) \leq \\ \leq (F^{\diamond}_{\varepsilon_0+\gamma})^{n+1}(n) = F^{\diamond}_{\varepsilon_0+\alpha}(n), \end{aligned}$$

due to the induction hypothesis and claim (i). Concerning (iii) note that $\{\alpha\}(n) = \gamma$ forces $\beta = \gamma$ or $\gamma \to_n \beta$. Thus

$$F^{\diamond}_{\varepsilon_0+\beta}(n) \leq F^{\diamond}_{\varepsilon_0+\gamma}(n) \leq (F^{\diamond}_{\varepsilon_0+\gamma})^{n+1}(n) = F^{\diamond}_{\varepsilon_0+\alpha}(n)$$

follows by the induction hypothesis and claim (i). Let us come to (ii) for a limit ordinal α: By [11, Proposition 2.12] we have $\{\alpha\}(n) \to_m \{\alpha\}(m)$. Then

$$F^{\diamond}_{\varepsilon_0+\alpha}(m) = F^{\diamond}_{\varepsilon_0+\{\alpha\}(m)}(m) \leq F^{\diamond}_{\varepsilon_0+\{\alpha\}(n)}(m) \leq F^{\diamond}_{\varepsilon_0+\{\alpha\}(n)}(n) = F^{\diamond}_{\varepsilon_0+\alpha}(n)$$

uses the induction hypothesis of both (iii) and (ii). As for (iii), note that $\alpha \to_n \beta$ implies $\beta = \{\alpha\}(n)$ or $\{\alpha\}(n) \to_n \beta$. Thus

$$F^{\diamond}_{\varepsilon_0+\beta}(n) \leq F^{\diamond}_{\varepsilon_0+\{\alpha\}(n)}(n) = F^{\diamond}_{\varepsilon_0+\alpha}(n)$$

follows from the induction hypothesis. □

To approach Theorem 3.10 we bound the functions $F^{\diamond}_{\varepsilon_0+\alpha}$ in terms of the usual fast-growing hierarchy:

Lemma 3.7. *Consider numbers l, m, n with $m > 0$ and an ordinal $\alpha \leq \omega_m$ which satisfy $(F_{\omega_m+\alpha})^l(n) \leq F_{\varepsilon_0}(m)$. Then we have*

$$(F^{\diamond}_{\varepsilon_0+\alpha})^l(n) \leq (F_{\omega_m+\alpha})^l(n).$$

Proof. We argue by transfinite induction on α with a side induction on l. The base $l = 0$ of the side induction amounts to the trivial inequality $n \leq n$. So let us come to the side induction step $l \rightsquigarrow l + 1$: There we have the assumption $(F_{\omega_m+\alpha})^{l+1}(n) \leq F_{\varepsilon_0}(m)$. Abbreviating $N := (F^{\diamond}_{\varepsilon_0+\alpha})^l(n)$ our task is to show $F^{\diamond}_{\varepsilon_0+\alpha}(N) \leq (F_{\omega_m+\alpha})^{l+1}(n)$. We will use some well-known monotonicity properties of the fast-growing hierarchy, which can be found in [11, Section 2] (or

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[3, Section 2], with slightly different fundamental sequences). For example we have $(F_{\omega_m+\alpha})^l(n) < (F_{\omega_m+\alpha})^{l+1}(n)$, which allows us to apply the side induction hypothesis and obtain

$$N \leq (F_{\omega_m+\alpha})^l(n) < F_{\varepsilon_0}(m).$$

Now we distinguish the following cases:
Case $\alpha = 0$: Let us first show

$$F_{\varepsilon_0}^{-1}(N) + 1 \leq m. \tag{8}$$

Aiming at a contradiction, assume that we have $m \leq F_{\varepsilon_0}^{-1}(N)$. Observe that this implies $F_{\varepsilon_0}^{-1}(N) > 0$. Invoking the definition of $F_{\varepsilon_0}^{-1}$ we would then get

$$F_{\varepsilon_0}(m) \leq F_{\varepsilon_0}(F_{\varepsilon_0}^{-1}(N)) \leq N,$$

which contradicts $N < F_{\varepsilon_0}(m)$ from above. Now in view of (8) we obtain

$$F_{\varepsilon_0}^{\diamond}(N) = F_{\omega_{F_{\varepsilon_0}^{-1}(N)+1}}(N) \leq F_{\omega_m}(N) \leq (F_{\omega_m})^{l+1}(n),$$

which is the side induction step in the case $\alpha = 0$.
Case $\alpha = \beta + 1$: First observe

$$(F_{\omega_m+\beta})^{N+1}(N) = F_{\omega_m+\alpha}(N) \leq (F_{\omega_m+\alpha})^{l+1}(n) \leq F_{\varepsilon_0}(m).$$

This allows us to apply the main induction hypothesis with $N, N+1$ and β at the places of n, l and α, respectively. We get

$$\begin{aligned} F_{\varepsilon_0+\alpha}^{\diamond}(N) = (F_{\varepsilon_0+\beta}^{\diamond})^{N+1}(N) \leq (F_{\omega_m+\beta})^{N+1}(N) &= \\ &= F_{\omega_m+\alpha}(N) \leq (F_{\omega_m+\alpha})^{l+1}(n). \end{aligned}$$

Case α limit: The condition $\alpha \leq \omega_m$ implies $\omega_m + \{\alpha\}(N) = \{\omega_m + \alpha\}(N)$ (the ordinal ω_m meshes with α, see [11, Section 2]). Then we have

$$F_{\omega_m+\{\alpha\}(N)}(N) = F_{\omega_m+\alpha}(N) \leq (F_{\omega_m+\alpha})^{l+1}(n) \leq F_{\varepsilon_0}(m).$$

Now apply the main induction hypothesis with $N, 1$ and $\{\alpha\}(N)$ at the places of n, l and α, to get

$$\begin{aligned} F_{\varepsilon_0+\alpha}^{\diamond}(N) = F_{\varepsilon_0+\{\alpha\}(N)}^{\diamond}(N) \leq F_{\omega_m+\{\alpha\}(N)}(N) &= \\ &= F_{\omega_m+\alpha}(N) \leq (F_{\omega_m+\alpha})^{l+1}(n). \end{aligned}$$

We have thus completed the side induction step in all possible cases. □

From the lemma we can deduce the following result, which could be described as the "combinatorial half" of Theorem 3.10:

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Proposition 3.8. *For each $\alpha < \varepsilon_0$ there is a number N such that we have*

$$F^{\diamond}_{\varepsilon_0+\alpha}(F_{\varepsilon_0}(n \dot{-} 1)) \leq F_{\varepsilon_0}(n) \qquad \text{for all } n \geq N.$$

In particular $F^{\diamond}_{\varepsilon_0+\alpha}$ is eventually dominated by F_{ε_0}.

Proof. Consider some $\alpha < \varepsilon_0$. We shall see that the proposition holds for any $N > 0$ with $\omega_N \to_N \alpha + 1$. Let us first show that such a number N exists: As a first approximation take some $N_0 > 0$ with $\alpha < \omega_{N_0}$. From [3, Lemma 2.6] we get a number N with $\omega_{N_0} \to_N \alpha + 1$, and by [3, Corollary 2.4] we may assume $N \geq N_0$. By [11, Proposition 2.12] we have $\omega_N \to_N \omega_{N_0}$, and together this implies $\omega_N \to_N \alpha + 1$ as desired. To verify the proposition consider an arbitrary number $n \geq N$. We would like to apply the previous lemma with $n, 1$ and $F_{\varepsilon_0}(n-1)$ at the places of m, l and n, respectively. To do so we must verify the condition

$$F_{\omega_n+\alpha}(F_{\varepsilon_0}(n-1)) \leq F_{\varepsilon_0}(n) \tag{9}$$

of the lemma. Using [11, Lemma 2.7] we get $\omega_n + \alpha \to_n \omega_n$, and then

$$\begin{aligned} F_{\omega_n+\alpha}(F_{\varepsilon_0}(n-1)) = F_{\omega_n+\alpha}(F_{\omega_n}(n-1)) \leq F_{\omega_n+\alpha}(F_{\omega_n}(n)) \leq \\ \leq (F_{\omega_n+\alpha})^2(n) \leq (F_{\omega_n+\alpha})^{n+1}(n) = F_{\omega_n+\alpha+1}(n). \end{aligned}$$

The next step is to show $\omega_{n+1} \to_n \omega_n + \alpha + 1$: From [11, Lemma 2.10, 2.13] we get $\omega_{n+1} \to_n \omega^{\omega_{n-1}+1}$. In view of $\{\omega^{\omega_{n-1}+1}\}(1) = \omega_n + \omega_n$ we can use [11, Proposition 2.12] to obtain $\omega_{n+1} \to_n \omega_n + \omega_n$. Since ω_n meshes with ω_n it only remains to show $\omega_n \to_n \alpha + 1$. This follows from the above $\omega_N \to_N \alpha + 1$ using [3, Corollary 2.4] and [11, Proposition 2.12]. Now we get

$$F_{\omega_n+\alpha+1}(n) \leq F_{\omega_{n+1}}(n) = F_{\varepsilon_0}(n),$$

which completes the proof of (9). This allows us to apply the previous lemma, and we finally obtain

$$F^{\diamond}_{\varepsilon_0+\alpha}(F_{\varepsilon_0}(n-1)) \leq F_{\omega_n+\alpha}(F_{\varepsilon_0}(n-1)) \leq F_{\varepsilon_0}(n).$$

To deduce that $F^{\diamond}_{\varepsilon_0+\alpha}$ is eventually dominated by F_{ε_0} use $n \leq F_{\varepsilon_0}(n-1)$ and the fact that $F^{\diamond}_{\varepsilon_0+\alpha}$ is monotone. ☐

The previous proposition is complemented by the following result:

Proposition 3.9. *Any provably total function of* $\mathbf{PA} + \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$ *is eventually dominated by one of the functions $F^{\diamond}_{\varepsilon_0+\alpha}$ with $\alpha < \varepsilon_0$.*

Proof. By Proposition 3.4 the slow reflection principle $\mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$ is equivalent to the statement that the function $F^{\diamond}_{\varepsilon_0}$ is total. It is a classical result that any provably total function of Peano Arithmetic is eventually dominated by one of the functions F_α with $\alpha < \varepsilon_0$ from the fast-growing hierarchy. We need to see that this remains valid when one adds the base function $F^{\diamond}_{\varepsilon_0}$ (both as an axiom and as initial function of the fast-growing hierarchy). Indeed a general result to

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this effect is shown as part of the proof of [24, Theorem 16]. However, in [24] the approach to the fast-growing hierarchy is somewhat different: The paper works with norms of ordinals rather than explicit fundamental sequences. This appears to be a technicality, but rather than working out a detailed comparison we take the more direct way and reprove [24, Theorem 16] in our setting:
As basis for our proof we take the analysis of the provably total functions of Peano Arithmetic in [21]. We assume that the reader has access to this paper. Note that the notation $\beta <_k \alpha$ in [21] refers to the same "step down"-relation that we write as $\alpha \to_k \beta$. First of all we need to extend the formalization of Peano Arithmetic in [21, Section 2] by the axiom $F^{\diamond}_{\varepsilon_0}\!\downarrow$. To do so, recall that the graph of $F^{\diamond}_{\varepsilon_0}$ is defined by a Δ_0-formula and is thus elementary. So the formal system of [21, Section 2] already contains a relation symbol $F^{\diamond}_{\varepsilon_0}(\cdot) = \cdot$ and defining axioms corresponding to its elementary definition. Using this relation symbol we extend the formal system by the new axiom $\forall_x \exists_y F^{\diamond}_{\varepsilon_0}(x) = y$. Next, we need to adapt the infinitary proof system of [21, Section 3]. This system contains a special relation symbol $\cdot \in \mathrm{N}$ which will be interpreted as a finite approximation to the set of natural numbers. The infinitary system contains an axiom which places zero in N and a rule which allows us to put in successors:

$$(\mathrm{N}) \qquad \text{if} \quad \vdash^{\alpha} \Gamma, n \in \mathrm{N} \quad \text{then} \quad \vdash^{\alpha+1} \Gamma, n+1 \in \mathrm{N}\,.$$

We need to add a new rule which gives access to values of the function $F^{\diamond}_{\varepsilon_0}$ (it is important that the increase in the ordinal bound is independent of n):

$$(\mathrm{N}^{\diamond}) \qquad \text{if} \quad \vdash^{\alpha} \Gamma, n \in \mathrm{N} \quad \text{then} \quad \vdash^{\alpha+1} \Gamma, F^{\diamond}_{\varepsilon_0}(n) \in \mathrm{N}\,.$$

Using this rule the embedding lemma is easily extended by a proof of the axiom $\forall_x \exists_y F^{\diamond}_{\varepsilon_0}(x) = y$ in the infinite system: Since the prime formula $F^{\diamond}_{\varepsilon_0}(n) = \overline{F^{\diamond}_{\varepsilon_0}(n)}$ is true we get $\vdash^1 n \notin \mathrm{N}, F^{\diamond}_{\varepsilon_0}(n) = \overline{F^{\diamond}_{\varepsilon_0}(n)}$ for each n. The axiom $\vdash^0 n \notin \mathrm{N}, n \in \mathrm{N}$ and the new rule $(\mathrm{N}^{\diamond})$ yield $\vdash^1 n \notin \mathrm{N}, F^{\diamond}_{\varepsilon_0}(n) \in \mathrm{N}$. Introducing a conjunction and an existential quantifier we obtain $\vdash^3 n \notin \mathrm{N}, \exists_{y \in \mathrm{N}} F^{\diamond}_{\varepsilon_0}(n) = y$. To keep the coefficients in the ordinal bound small we now apply accumulation: In view of $\omega \to_2 3$ we can conclude $\vdash^{\omega} n \notin \mathrm{N}, \exists_{y \in \mathrm{N}} F^{\diamond}_{\varepsilon_0}(n) = y$. By disjunction introduction and the ω-rule we arrive at $\vdash^{\omega+3} \forall_{x \in \mathrm{N}} \exists_{y \in \mathrm{N}} F^{\diamond}_{\varepsilon_0}(x) = y$. Using accumulation again we get

$$\vdash^{\omega \cdot 2} \forall_{x \in \mathrm{N}} \exists_{y \in \mathrm{N}} F^{\diamond}_{\varepsilon_0}(x) = y,$$

precisely as needed for the extended embedding lemma. It is straightforward to check that inversion, reduction and cut-elimination remain valid: In this respect the new rule $(\mathrm{N}^{\diamond})$ behaves just as the original rule (N). In the bounding lemma the bound $F_{\alpha}(k)$ is replaced by $F^{\diamond}_{\varepsilon_0+\alpha}(k)$:

> Assume that we have $\vdash^{\alpha} n_1 \notin \mathrm{N}, \dots, n_r \notin \mathrm{N}, \Gamma$ with cut rank 0, where Γ only contains closed positive $\Sigma_1(\mathrm{N})$-formulas. Then Γ is true in $F^{\diamond}_{\varepsilon_0+\alpha}(k)$ for $k = \max(\{2\} \cup \{3n_1, \dots, 3n_r\})$.

Recall that positive $\Sigma_1(\mathrm{N})$-formulas only contain the connectives $\vee, \wedge, \exists$ and do not contain subformulas of the form $n \notin \mathrm{N}$. A closed sequent is called

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true in m if the disjunction of its formulas is true under the interpretation $\mathrm{N} = \{n \,|\, 3n < m\}$ of the special relation symbol. To prove the bounding lemma one argues by induction on α and distinguishes cases according to the last rule of the deduction $\vdash^\alpha n_1 \notin \mathrm{N}, \dots, n_r \notin \mathrm{N}, \Gamma$. Using Lemma 3.6 this is straightforward and essentially as in [21]. Let us only consider the case of a deduction that ends in the new rule ($\mathrm{N}^\diamond$): Then Γ contains a formula of the form $F^\diamond_{\varepsilon_0}(n) \in \mathrm{N}$ and we have $\vdash^\beta n_1 \notin \mathrm{N}, \dots, n_r \notin \mathrm{N}, \Gamma, n \in \mathrm{N}$ with $\alpha = \beta + 1$. Since the premise of the rule contains no new formula of the form $m \notin \mathrm{N}$ the number k is unchanged and the induction hypothesis tells us that $\Gamma, n \in \mathrm{N}$ is true in $F^\diamond_{\varepsilon_0+\beta}(k)$. There are two possibilities: If Γ is true in $F^\diamond_{\varepsilon_0+\beta}(k)$ then it is also true in $F^\diamond_{\varepsilon_0+\alpha}(k) \geq F^\diamond_{\varepsilon_0+\beta}(k)$. Otherwise the formula $n \in \mathrm{N}$ must be true in $F^\diamond_{\varepsilon_0+\beta}(k)$, which means that we have $n \leq 3n < F^\diamond_{\varepsilon_0+\beta}(k)$. Using Lemma 3.6 we observe $3 \leq (F^\diamond_{\varepsilon_0+\beta})^2(k)$ and infer

$$\begin{aligned} 3 \cdot F^\diamond_{\varepsilon_0}(n) \leq 3 \cdot F^\diamond_{\varepsilon_0}(F^\diamond_{\varepsilon_0+\beta}(k)) \leq (F^\diamond_{\varepsilon_0+\beta})^2(k) \cdot (F^\diamond_{\varepsilon_0+\beta})^2(k) < \\ < (F^\diamond_{\varepsilon_0+\beta})^3(k) \leq (F^\diamond_{\varepsilon_0+\beta})^{k+1}(k) = F^\diamond_{\varepsilon_0+\alpha}(k). \end{aligned}$$

This means that Γ contains the formula $F^\diamond_{\varepsilon_0}(n) \in \mathrm{N}$ which is true in $F^\diamond_{\varepsilon_0+\alpha}(k)$. Now we can deduce the desired result as usual: Let g be a provably total function of the theory $\mathbf{PA} + F^\diamond_{\varepsilon_0}\!\downarrow$. From the given definition of g we can read off an elementary relation χ_g such that we have

$$\begin{aligned} g(m) = n \qquad &\Leftrightarrow \qquad \mathbb{N} \vDash \exists_z \chi_g(m, n, z), \\ &\mathbf{PA} + F^\diamond_{\varepsilon_0}\!\downarrow \vdash \forall_x \exists_{y,z} \chi_g(x, y, z). \end{aligned}$$

By embedding and cut elimination we get an ordinal $\alpha < \varepsilon_0$ and an infinitary deduction $\vdash^\alpha \forall_{x \in N} \exists_{y \in N} \exists_{z \in N} \chi_g(x, y, z)$ of cut rank 0. Inversion yields a deduction

$$\vdash^\alpha m \notin \mathrm{N}, \exists_{y \in N} \exists_{z \in N} \chi_g(m, y, z)$$

for each number m. Assume $m \geq 3$. By the bounding lemma there are numbers $n, k < F^\diamond_{\varepsilon_0+\alpha}(3m)$ such that $\chi_g(m, n, k)$ is true. Using Lemma 3.6 we get

$$g(m) < F^\diamond_{\varepsilon_0+\alpha}(3m) \leq F^\diamond_{\varepsilon_0+\alpha}(m^2) \leq (F^\diamond_{\varepsilon_0+\alpha})^2(m) \leq F^\diamond_{\varepsilon_0+\alpha+1}(m),$$

which shows that g is eventually dominated by $F^\diamond_{\varepsilon_0+\alpha+1}$. □

Putting pieces together we can deduce the main result of this section:

Theorem 3.10. *For any provably total function g of $\mathbf{PA} + \mathrm{RFN}^\diamond_{\Sigma_1}(\mathbf{PA})$ there is a number N such that we have*

$$g(F_{\varepsilon_0}(n \dot{-} 1)) \leq F_{\varepsilon_0}(n) \qquad \textit{for all } n \geq N.$$

In particular any provably total function of the theory $\mathbf{PA} + \mathrm{RFN}^\diamond_{\Sigma_1}(\mathbf{PA})$ is eventually dominated by F_{ε_0}.

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Proof. Consider a function g which is provably total in $\mathbf{PA} + \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$. The previous proposition provides an ordinal $\alpha < \varepsilon_0$ and a bound N such that we have

$$g(m) \leq F^{\diamond}_{\varepsilon_0+\alpha}(m) \quad \text{for all } m \geq N.$$

Increasing N if necessary Proposition 3.8 yields

$$F^{\diamond}_{\varepsilon_0+\alpha}(F_{\varepsilon_0}(n \dot{-} 1)) \leq F_{\varepsilon_0}(n) \quad \text{for all } n \geq N.$$

For $n \geq N$ we have $F_{\varepsilon_0}(n \dot{-} 1) \geq n \geq N$ and thus

$$g(F_{\varepsilon_0}(n \dot{-} 1)) \leq F^{\diamond}_{\varepsilon_0+\alpha}(F_{\varepsilon_0}(n \dot{-} 1)) \leq F_{\varepsilon_0}(n)$$

and

$$g(n) \leq F^{\diamond}_{\varepsilon_0+\alpha}(n) \leq F^{\diamond}_{\varepsilon_0+\alpha}(F_{\varepsilon_0}(n \dot{-} 1)) \leq F_{\varepsilon_0}(n),$$

as required for Theorem 3.10. □

We remark that the theorem implies

$$\mathbf{PA} + \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA}) \nvdash \mathrm{RFN}_{\Sigma_1}(\mathbf{PA}),$$

because the equivalence $\mathrm{RFN}_{\Sigma_1}(\mathbf{PA}) \leftrightarrow F_{\varepsilon_0}\downarrow$ is provable in Peano Arithmetic (even in $\mathbf{I\Sigma}_1$, as implied by Proposition 3.1). The analogous result for slow consistency has been proved in [11] (see also statement (6) in Section 2). In [12] we investigate the consistency strength of slow reflection (also for reflection formulas of complexity above Σ_1): In particular it is shown that $\mathbf{PA} + \mathrm{RFN}^{\diamond}_{\Sigma_1}(\mathbf{PA})$ does not even prove the consistency of Peano Arithmetic. Further results on slow provability can be found in work of Henk and Pakhomov [13]. To conclude this paper, let us rephrase our computational analysis in terms of subrecursive degree theory (see [24]):

Corollary 3.11. *The honest ε_0-elementary degree of $F^{\diamond}_{\varepsilon_0}$ is a non-zero degree strictly below the degree of F_{ε_0}.*

Proof. First we must verify that $F^{\diamond}_{\varepsilon_0}$ and F_{ε_0} are honest functions (in the sense of [24]). We already know that the two functions are monotone and have elementary graphs (since they can be defined by Δ_0-formulas). It remains to show that they dominate the function $n \mapsto 2^n$: By straightforward computations we see that $F_2(n) \geq 2^n$ holds for all n. Since $\omega_{m+1} \to_n 2$ holds for $n \geq 1$ and any m we obtain

$$F^{\diamond}_{\varepsilon_0}(n) = F_{\omega_{F^{-1}_{\varepsilon_0}(n)+1}}(n) \geq F_2(n) \geq 2^n \qquad \text{for } n \geq 1.$$

In the separate case $n = 0$ the inequality $F^{\diamond}_{\varepsilon_0}(0) \geq 2^0$ is immediate. Similarly one shows that $F_{\varepsilon_0}(n) \geq 2^n$ holds for all n. Now let us argue that the ε_0-elementary degree of $F^{\diamond}_{\varepsilon_0}$ is non-zero: In the discussion just after Proposition 3.4 above we have seen that $F^{\diamond}_{\varepsilon_0}$ eventually dominates any provably total function of Peano Arithmetic. Thus Peano Arithmetic cannot prove the totality of any

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honest representation of $F^{\diamond}_{\varepsilon_0}$. In the notation of [24, Section 10] this means that we do not have $F^{\diamond}_{\varepsilon_0} \leq_{\mathbf{PA}} \mathbf{0}$. Then [24, Theorem 16] tells us that we cannot have $F^{\diamond}_{\varepsilon_0} \leq_{\epsilon_0 E} \mathbf{0}$. In other words, the ε_0-elementary degree of $F^{\diamond}_{\varepsilon_0}$ is non-zero. Similarly Theorem 3.10 tells us that $F_{\varepsilon_0} \leq_{\mathbf{PA}} F^{\diamond}_{\varepsilon_0}$ must fail, so that $F_{\varepsilon_0} \leq_{\varepsilon_0 E} F^{\diamond}_{\varepsilon_0}$ must fail as well. Thus $F^{\diamond}_{\varepsilon_0}$ and F_{ε_0} do not have the same ε_0-elementary degree. On the other hand it is easy to see $F^{\diamond}_{\varepsilon_0} \leq_{\varepsilon_0 E} F_{\varepsilon_0}$: Since $F^{\diamond}_{\varepsilon_0}$ has an elementary graph and is dominated by F_{ε_0} it is even elementary in F_{ε_0}, by bounded minimization.

We remark that the use of [24, Theorem 16] is, in some sense, a detour: Rather than considering provability in Peano Arithmetic one could use the "Generalized Growth Theorem" [24, Theorem 13] in combination with Proposition 3.8 above. Then, however, one has the technical task to reconcile the different definitions of the fast-growing hierarchy in our paper and in [24]. □

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