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Exact holomorphic differentials on a quotient of the Ree curve.

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Abstract

We produce several families of exact holomorphic differentials on a quotient X of the the Ree curve in characteristic 3, defined by $X: \; y^q - y = x^{q_0}(x^q - x) \; /\mathbb{F}_q$, (where $q_0 = 3^s$, $s \geqslant 1$ and $q = 3q_0^2$). We conjecture that they span the whole space of exact holomorphic differentials, and prove this in the cases $s = 1$ and $s = 2$, by calculating the kernel of the Cartier operator.

§ 1 Introduction

For a nonsingular projective curve C over a field K, $H^0(C, \Omega^1)$ is the K-vector space of holomorphic (i.e. everywhere regular) differentials on C (defined over K). Such a differential is said to be *exact* if it is of the form df for some function $f \in K(C)$. If f is constant then $df = 0$. If f is non-constant then f necessarily has at least one pole, and if K has characteristic 0 then df will have a pole in the same place. So in characteristic 0, there are no non-zero exact holomorphic differentials. But in characteristic $p > 0$, where p^{th} powers differentiate to 0, a pole whose order is a multiple of p might disappear upon differentiation, and there can be non-zero exact holomorphic differentials. Inside $H^0(C, \Omega^1)$, the subspace of exact holomorphic differentials is the kernel of the Cartier operator $\mathfrak{C}$. It seems like a natural problem, given a curve C/K with $\text{char}(K) = p > 0$, to calculate this subspace. There are at least two further ways to motivate this problem.

First, $\dim_K H^0(C, \Omega^1)^{\mathfrak{C}=0} = \dim_K \text{Hom}(\alpha_p, J_C[p])$, the so-called a-number of the Jacobian J_C of C [LO, Equation 5.2.8]. Here $J_C[p]$ is the sub-group-scheme of p-torsion,

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and $\alpha_p \simeq \mathrm{Spec}(K[x]/x^p)$ is the group-scheme of p^{th}-roots of 0. Note that each element $b \in K$ gives an endomorphism of α_p, via $x \mapsto bx$, so $\mathrm{Hom}(\alpha_p, J_C[p])$ is naturally a K-vector space. This a-number is an important invariant of $J_C[p]$. For instance, $a + f \leq g$, where $f = \dim_{\mathbb{F}_p}(J_C[p](\overline{K}))$ is the "p-rank" of J_C. For more, see [FGMPW], [LO].

Second, suppose that K is finite, that E/K is an elliptic curve, and consider E as a "constant" elliptic curve over the global field $K(C)$. Then morphisms from C to E, defined over K, are identified with $K(C)$-rational points of E. It may happen that α_p is isomorphic to a subgroup-scheme of $E[p]$, necessarily $\ker F : E \to E^{(p)}$, the kernel of the p^{th}-power Frobenius morphism. In fact, this is always true when E is supersingular (in which case the kernel of the Verschiebung $V : E^{(p)} \to E$ is also isomorphic to α_p). Then the Selmer group for the isogeny F is identified with the flat cohomology $H^1(C, \alpha_p)$, which in turn is identified with $H^0(C, \Omega^1)^{\mathfrak{C}=0}$, [U, Proposition 3.3(b)]. For a supersingular E/K, an invariant differential ω (on any isogenous curve) is exact, and if $\theta : C \to E^{(p)}$ is a $K(C)$-rational point on $E^{(p)}$ then the pullback $\theta^*(\omega)$ is an exact holomorphic differential on C. In this way $E^{(p)}(K(C))/F(E(K(C)))$ is embedded as a subgroup of $H^0(C, \Omega^1)^{\mathfrak{C}=0}$, and the cokernel is the F-torsion in the Shafarevich-Tate group of E/K.

Friedlander et. al. [FGMPW] calculate the space of exact holomorphic differentials on the Suzuki curve $C : y^q - y = x^{q_0}(x^q - x)$, with $q = 2q_0^2$, where $q_0 = 2^s$, $s \geq 1$, of genus $g = q_0(q-1)$, in order to determine the a-number of its Jacobian. The fact that the characteristic is 2 ensures that the exact differentials are simply those of the form $f^2\, dx$, and one just has to find a basis for those f with divisor bounded such that $f^2\, dx$ is holomorphic.

Gross [G] calculates the space of exact holomorphic differentials on the Hermitian curve $C : y^{q+1} = x^q + x \,/\mathbb{F}_{q^2}$, (where $q = p^f$ is a prime power), of genus $g = q(q-1)/2$, in order to bound from below the order of the Shafarevich-Tate group, and thereby to improve a bound for the sphere-packing density of the Mordell-Weil lattice $E(K(C))/\mathrm{const} \simeq \mathrm{Hom}_K(J_C, E)$. (This free, finitely-generated abelian group has on it an even integral quadratic form, given by twice the degree of a morphism. This construction of lattices is due to Elkies.) The finite group $\mathbb{F}_{q^2}^\times$ acts on $C/\mathbb{F}_{q^2}$ by the automorphisms $(x, y) \mapsto (\alpha^{q+1}x, \alpha y)$, and this abelian group action decomposes $H^0(C, \Omega)$ into one-dimensional pieces (spanned by $x^m y^n\, dy$ with $m, n \geq 0$ and $m + n \leq q-2$) on which the group acts by distinct characters. The Cartier operator necessarily permutes these one dimensional spaces, so to find its kernel one only needs to know which of these basis elements it kills, so again the calculation is relatively simple.

In both cases the Jacobian J_C is *isogenous* to E^g for a certain supersingular elliptic curve E. If J_C is *isomorphic* to E^g (i.e. "superspecial") then things are simple, as $J_C[p] \simeq E[p]^g$, so $a = g$, $f = 0$, and every holomorphic differential is exact. For the Hermitian curve, $J_C \simeq E^g$ if and only if $q = p$, while for the Suzuki curve it never happens. (We are grateful to R. Pries for correcting an error in an earlier version.) The fact that in general J_C is isogenous, but not isomorphic, to E^g is at the root of the subtlety of the situation.

The Suzuki curve enjoys automorphisms by the finite simple group $\mathrm{Sz}(q) = {}^2B_2(q)$, while the Hermitian curve is acted upon by a finite projective unitary group $\mathrm{PU}_3(\mathrm{q})$. The third family of Deligne-Lusztig curves (see [H]) comprises those acted upon by the finite simple Ree groups ${}^2G_2(q)$, where $q = 3q_0^2$, with $q_0 = 3^s$, $s \geq 1$. The function field of the Ree curve $C'/\mathbb{F}_q$ is given by $\mathbb{F}_q(C') = \mathbb{F}_q(x, y_1, y_2)$, with

$$y_1^q - y_1 = x^{q_0}(x^q - x) \tag{1}$$

$$y_2^q - y_2 = x^{q_0}(y_1^q - y_1). \tag{2}$$

The genus of C' is $g = \frac{3}{2}q_0(q-1)(q+q_0+1)$, and it has $1 + q^3$ $\mathbb{F}_q$-rational points, including one point at infinity. Let X (of genus $\frac{3}{2}q_0(q-1)$) be the non-singular model of the function field of the affine curve defined by (1). It is a quotient of $C'/\mathbb{F}_q$, via the map $\pi : C' \to X$ such that $(x, y_1, y_2) \mapsto (x, y_1)$. From now on, for the sake of simplicity we will replace y_1 by y, so (the function field of) X is defined by the equation

$$y^q - y = x^{q_0}(x^q - x). \tag{3}$$

We seek the exact holomorphic differentials for the quotient X rather than for the Ree curve itself, since it seems to be a more tractable problem. (We are grateful to the referee for pointing out that for the Ree curve, even an explicit basis for $H^0(C', \Omega^1)$ is not known.) Furthermore, X is a direct analogue of the Suzuki curve, being defined by an equation that looks the same. For both X and the Suzuki curve there is an action of $\mathbb{F}_q^\times$, by $(x, y) \mapsto (\alpha x, \alpha^{q_0+1}y)$, but unlike the Hermitian case, this does not decompose $H^0(X, \Omega^1)$ into one-dimensional eigenspaces. Despite this, the Suzuki curve may be dealt with fairly easily because the characteristic is only 2. The source of the extra difficulty in characteristic 3 is identified in Remark 1, in Section 3. For more on quotients of X, see [CO1, CO2].

Conjecture 1.1. *Let $q_0 = 3^s, q = 3q_0^2$, with $s \geq 1$, and let $X/\mathbb{F}_q$ be a complete nonsingular model of the curve $y^q - y = x^{q_0}(x^q - x)$. The dimension of the space $H^0(X, \Omega^1)^{\mathfrak{C}=0}$, of exact holomorphic differentials on X, is*

$$d := \frac{2q_0}{27}\left(14q_0^2 + 9\right) + \frac{1}{12}\left(11q_0^2 + 9\right).$$

Theorem 1.2. *d is a lower bound for the dimension of $H^0(X, \Omega^1)^{\mathfrak{C}=0}$ (i.e. for the a-number of the Jacobian of X), for all $s \geq 1$.*

Theorem 1.3. *In the cases $s = 1$ and $s = 2$, d is equal to the dimension of $H^0(X, \Omega^1)^{\mathfrak{C}=0}$.*

We should make some remark on one of the motivating problems. The zeta function of $X/\mathbb{F}_q$ is

$$\frac{(1 + 3q_0t + qt^2)^{q_0(q-1)}(1 + qt^2)^{q_0(q-1)/2}}{(1-t)(1-qt)}.$$

Let $E_1, E_2/\mathbb{F}_q$ be elliptic curves with zeta functions

$$\frac{(1 + 3q_0t + qt^2)}{(1-t)(1-qt)} \quad \text{and} \quad \frac{(1 + qt^2)}{(1-t)(1-qt)},$$

respectively. Letting $L_i = \mathrm{Hom}_K(J_X, E_i)$, for $i = 1, 2$, we find that $\mathrm{rank}(L_1) = 2q_0(q-1)$ and $\mathrm{rank}(L_2) = q_0(q-1)$, and that J_X is isogenous to $E_1^{q_0(q-1)} \times E_2^{q_0(q-1)/2}$. This isogeny is not an isomorphism, and for $s = 1$ or 2 a simple reason is that our calculations show that $a \neq g$. In what follows, we omit details, but see the proofs of Propositions 11.11 and 14.10 of [G] for the method by which ranks are calculated, minimal norms bounded from below and determinants bounded from above. Note that for the refined upper bound for the determinant of the lattice, a *lower* bound for $\dim H^0(X, \Omega^1)^{\mathfrak{C}=0}$ is required, which is precisely what we have. Let the centre density of a lattice L of rank n be $\delta := \frac{(\min/4)^{n/2}}{(\det L)^{1/2}}$, where min is the minimal norm. This is the sphere-packing density divided by the volume of a unit n-dimensional sphere. For L_1, for $s = 1$ we have $n = 156$ and find $\log_2(\delta) \geq -80.9$, while for $s = 2$ we have $n = 4356$ and find $\log_2(\delta) \geq 710$. For comparison, looking at records for known dense lattices in nearby dimensions in Table 1.3 of [CS], for $n = 150$, $\log_2(\delta) = 113.06$ and for $n = 4098$, $\log_2(\delta) = 11279$. For L_2 our bounds are even worse compared to record known lattices. Still, it seems to be an interesting problem to determine invariants of these lattices. In the case of the Hermitian curve, the precise determinants are calculated in [D1], while the structure of the Shafarevich-Tate group is obtained in [D2].

In Section 2 we introduce useful functions u and v on X, and find a basis for the space of holomorphic differentials on X, comprising certain elements of the form $x^a y^b u^c v^d\, dx$, with $0 \leq b \leq 2$ and various restrictions on the other exponents. In Section 3 we introduce the Cartier operator, and calculate its action on the 81 differentials of the form $\omega = x^\alpha y^\beta u^\gamma v^\delta\, dx$ with $0 \leq \alpha, \beta, \gamma, \delta \leq 2$. Since $\mathfrak{C}(f^3\omega) = f\mathfrak{C}(\omega)$, this determines which of our basis elements are exact. In Section 4 we prove Theorem 1.2 by producing as many exact holomorphic differentials as we can. In Section 5 we consider a natural action of the group $\mathbb{F}_q^\times$ on $H^0(X, \Omega^1)$, and show that the eigenspaces are of dimensions $\frac{3q_0 \pm 1}{2}$. The Cartier operator permutes these eigenspaces, so we may consider its kernel on each separately. In Section 6, we prove Theorem 1.3 by calculating these kernels in the cases $s = 1$ and $s = 2$. Throughout the paper, we work over $K = \mathbb{F}_q$, but the calculations look exactly the same over any extension.

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§ 2 The holomorphic differentials on X

Recall that $X/\mathbb{F}_q$ is a nonsingular projective model of the affine curve $y^q - y = x^{q_0}(x^q - x)$, where $q_0 = 3^s$ and $q = 3^{2s+1}$, $s \geq 1$.

Proposition 2.1. *$X/\mathbb{F}_q$ is irreducible with a single point at infinity (i.e. in the complement of the affine curve), denoted by P_∞. The rational functions on $X/\mathbb{F}_q$, defined by*

$$1,\ x,\ y,\ u = x^{3q_0+1} - y^{3q_0},\ \textit{and}\ v = x^2 y^{3q_0} - u^{3q_0}$$

are regular on $X \setminus \{P_\infty\}$. At P_∞, the pole orders of these functions are

$$-\mathrm{ord}_\infty(1) = 0, \quad -\mathrm{ord}_\infty(x) = q, \quad -\mathrm{ord}_\infty(y) = q + q_0,$$

$$-\mathrm{ord}_\infty(u) = q + 3q_0, \quad -\mathrm{ord}_\infty(v) = 2q + 3q_0 + 1.$$

The element $\frac{xu}{v}$ is a uniformizer at P_∞.

We omit the proof, since it is elementary, and essentially identical to that of Lemma 1.8 of [HS].

The pullback of u to the Ree curve C' is among the functions defined by Pederson [P], who calls it ω_1. (All rational functions on C' that do not involve y_2 may be considered functions on X.) We have had to introduce v here for our purposes, but it is analogous to the function on the Suzuki curve denoted f_{q+2q_0+1} by Hansen and Stichtenoth [HS].

From the above definitions of u and v, one can easily verify the following relations in $\mathbb{F}_q(X)$:

$$y^3 = x^2u + v\,; \tag{4}$$

$$u^{q_0} = x^{q_0}x - y\,; \tag{5}$$

$$v^{q_0} = x^{2q_0}y - u\,. \tag{6}$$

Proposition 2.2. *The curve X has genus $g = \frac{3}{2}q_0(q-1)$. The differential dx has divisor* $\mathrm{div}(dx) = (2g-2)P_\infty$.

Proof. If $(\alpha, \beta) \in X(\overline{\mathbb{F}}_q) \setminus \{P_\infty\}$, then β is one of the q roots of the equation

$$y^q - y = \alpha^{q_0}(\alpha^q - \alpha).$$

If $h(y) := y^q - y - \alpha^{q_0}(\alpha^q - \alpha)$, then h and $\frac{dh}{dy}$ have no common roots, so all the roots of $y^q - y = \alpha^{q_0}(\alpha^q - \alpha)$ are distinct. Thus we have q distinct points for which $(x - \alpha)$ is zero. But $-\mathrm{ord}_\infty(x-\alpha) = -\mathrm{ord}_\infty(x) = q$, so all of these zeros are simple zeros, and hence

$$\mathrm{ord}_{(\alpha,\beta)}dx = \mathrm{ord}_{(\alpha,\beta)}d(x-\alpha) = 0,$$

showing that dx has no zeros on $X(\overline{\mathbb{F}}_q) - \{P_\infty\}$. Since $\deg(\mathrm{div}(dx)) = 2g-2$,

$$\mathrm{div}(dx) = (2g-2)P_\infty. \tag{7}$$

Now we find the genus. From $v = x^2y^{3q_0} - u^{3q_0}$ we get

$$\begin{aligned} &dv = 2xy^{3q_0}dx \\ &\Rightarrow \mathrm{ord}_\infty(dx) = \mathrm{ord}_\infty(dv) - \mathrm{ord}_\infty(x) - \mathrm{ord}_\infty(y^{3q_0}). \end{aligned} \tag{8}$$

Since $-\mathrm{ord}_\infty(v)$ is coprime to 3, $-\mathrm{ord}_\infty(dv) = -\mathrm{ord}_\infty(v) + 1 = 2q + 3q_0 + 2$. Putting this in (8) gives

$$\mathrm{ord}_\infty(dx) = 3q_0(q-1) - 2. \tag{9}$$

But (7) shows that $\mathrm{ord}_\infty(dx) = 2g-2$. Comparing with (9) gives

$$g = \frac{3}{2}q_0(q-1)$$

$\square$

Define a set I of indices $(a,b,c,d) \in \mathbb{Z}^4$ by the following conditions:

1. $a, b, c, d \geq 0$.
2. $a+b+c+2d \leq 3q_0 - 1$.
3. If $a+b+c+2d = 3q_0 - 2$ then $0 \leq c \leq 2q_0 - 2$. Writing $c = 2q_0 - 2 - i$, where $0 \leq i \leq 2q_0 - 2$, either (i) $b + 3d < 2 + 3i$ and $d \leq \frac{q_0+i}{2}$ or (ii) $b + 3d = 2 + 3i$ and $0 \leq d \leq q_0 - 2$.
4. If $a+b+c+2d = 3q_0 - 1$ then $0 \leq c \leq q_0 - 2$. Writing $c = q_0 - 2 - j$, $b + 3d \leq 2 + 3j$.

Lemma 2.3. *The differential $x^a y^b u^c v^d\, dx$ is holomorphic if and only if $(a,b,c,d) \in I$.*

This can be checked using Propositions 2.1 and 2.2

Proposition 2.4. *Define $J = \{(a,b,c,d) \in I \mid 0 \leq b \leq 2$ and $0 \leq c, d \leq q_0 - 1\}$. Then $\{x^a y^b u^c v^d\, dx \mid (a,b,c,d) \in J\}$ is a basis for $H^0(X, \Omega^1)$.*

Proof. The holomorphic differentials $x^a y^b u^c v^d dx$, for $(a,b,c,d) \in J$, have distinct orders at P_∞. (See the proof of Proposition 3.7 of [FGMPW], which is the characteristic 2 analogue.) Hence they are linearly independent. If one counts the elements of J, there are exactly g of them, hence the corresponding differentials must form a basis for $H^0(X, \Omega^1)$. $\square$

§ 3 Exact differentials and the Cartier operator

Let K be a field of characteristic $p > 0$, and C/K any nonsingular projective curve. We will use the (non-linear) Cartier operator $\mathfrak{C}$, which maps the space Ω_C of meromorphic differentials on C to itself.

Proposition 3.1. *Some of the properties of the Cartier operator are as follows [S, Section 10].*
(1) If ν is not a p^th power in the function field $K(C)$, then every $f \in K(C)$ can be expressed as

$$f = f_0^p + f_1^p \nu + \ldots + f_{p-1}^p \nu^{p-1} \tag{10}$$

for suitable $f_i \in K(C)$. We define

$$\mathfrak{C}(f\, d\nu) = f_{p-1} d\nu. \tag{11}$$

(2) $\mathfrak{C}$ is well-defined, independent of the choice of ν.
(3) $\mathfrak{C}$ is additive: $\mathfrak{C}(\omega_1+\omega_2)=\mathfrak{C}(\omega_1)+\mathfrak{C}(\omega_2)$ for all $\omega_1,\omega_2\in\Omega_C$.
(4) For any differential ω on C and $g\in K(C)$, $\mathfrak{C}(g^p\omega)=g\mathfrak{C}(\omega)$.
(5) $\mathfrak{C}(\omega)=0$ if and only if ω is exact.

Proposition 3.2. *The exact meromorphic differentials on $X/\mathbb{F}_q$ are precisely those of the form*

$$\omega=(f_0^3+f_1^3x)dx, \quad \text{for } f_0,\ f_1\in\mathbb{F}_q(X). \tag{12}$$

Proof. It is clear that x is not a cube in $\mathbb{F}_q(X)$, as if it were then so would be y, by (3), so every function in $\mathbb{F}_q(X)$ would be a cube, which is not true of course, as $\mathbb{F}_q(X)$ is not perfect. (Alternatively, as suggested by the referee, we can see from Proposition 2.1 that the pole order of v is not divisible by 3.) Now from Proposition 3.1(1) we can write every function $f\in\mathbb{F}_q(X)$ as

$$f=f_0^3+f_1^3x+f_2^3x^2, \tag{13}$$

and $\mathfrak{C}(f\,dx)=f_2\,dx$, so by Proposition 3.1(5), $\omega=f\,dx$ is exact if and only if it is of the form

$$\omega=(f_0^3+f_1^3x)dx.$$

(Observe how one can "integrate" such a differential without trying to divide by zero.) ☐

***Remark 1**:* There is a possibility that cancelation of poles takes place between the two terms $f_0^3\,dx$ and $f_1^3x\,dx$. The difficulty this causes, in determining which exact differentials are holomorphic, does not arise in the characteristic 2 case, where there is only one term ($f_0^2\,d\nu$) in the expression for an exact differential. To find a basis for the exact holomorphic differentials on the Suzuki curve, one simply takes $\{f_0^2\,d\,x\}$, where f_0 runs through a basis for $\mathcal{L}((g-1)P_\infty)=\{f\in K(C)|\mathrm{div}(f)\geq-(g-1)P_\infty\}$.

Recall (from Proposition 2.4), a basis for the space of holomorphic differentials on X, involving certain $x^ay^bu^cv^ddx$. Each is a cube times one of the 81 things given by $x^\alpha y^\beta u^\gamma v^\delta dx$, with $0\leq\alpha,\ \beta,\ \gamma,\ \delta\leq 2$. So if we know how the Cartier operator applies to these 81 differentials then we know how it applies to any element in the basis.

Proposition 3.3. *If $\omega=x^\alpha y^\beta u^\gamma v^\delta dx$, with $0\leq\alpha,\ \beta,\ \gamma,\ \delta\leq 2$, then $\mathfrak{C}(\omega)$ is given by Table 1 below.*

	ω/dx	$\mathfrak{C}(\omega)/dx$
1	1	0
2	x	0
3	y	0
4	u	0
5	v	$x^{\frac{2q_0}{3}}u^{\frac{q_0}{3}}+v^{\frac{q_0}{3}}$
6	x^2	1

7	y^2	$x^{\frac{2q_0}{3}}$
8	u^2	x^{2q_0}
9	v^2	$x^{\frac{5q_0}{3}+1}\,u^{\frac{q_0}{3}} + x^{q_0+1}\,v^{\frac{q_0}{3}} - x^{\frac{2q_0}{3}}\,y\,u^{\frac{q_0}{3}} - y\,v^{\frac{q_0}{3}}$
10	xy	$x^{\frac{q_0}{3}}$
11	xu	x^{q_0}
12	xv	0
13	yu	$x^{\frac{4q_0}{3}}$
14	yv	$-x^{\frac{2q_0}{3}}\,u^{\frac{2q_0}{3}} - u^{\frac{q_0}{3}}\,v^{\frac{q_0}{3}}$
15	uv	$-x^{\frac{4q_0}{3}}\,u^{\frac{2q_0}{3}} + x^{\frac{2q_0}{3}}\,u^{\frac{q_0}{3}}\,v^{\frac{q_0}{3}} - v^{\frac{2q_0}{3}}$
16	x^2y	$-u^{\frac{q_0}{3}}$
17	x^2u	$-x^{\frac{2q_0}{3}}\,u^{\frac{q_0}{3}} - v^{\frac{q_0}{3}}$
18	x^2v	$-x^{q_0}x + y$
19	xy^2	$x^{\frac{q_0}{3}}\,u^{\frac{q_0}{3}}$
20	y^2u	$-x^{\frac{2q_0}{3}}\,v^{\frac{q_0}{3}}$
21	y^2v	$u^{\frac{2q_0}{3}}\,v^{\frac{q_0}{3}}$
22	xu^2	$x^{\frac{5q_0}{3}}\,u^{\frac{q_0}{3}} + x^{q_0}\,v^{\frac{q_0}{3}}$
23	yu^2	$x^{\frac{4q_0}{3}}\,v^{\frac{q_0}{3}}$
24	u^2v	$x^{2q_0}y - u$
25	xv^2	$x^{\frac{4q_0}{3}+1}\,u^{\frac{2q_0}{3}} - x^{\frac{2q_0}{3}+1}\,u^{\frac{q_0}{3}}\,v^{\frac{q_0}{3}} + x\,v^{\frac{2q_0}{3}}$
26	yv^2x	$x^{q_0+1}\,u^{\frac{q_0}{3}}\,v^{\frac{q_0}{3}} + x^{\frac{2q_0}{3}}\,y\,u^{\frac{2q_0}{3}} + x^{\frac{q_0}{3}+1}\,v^{\frac{2q_0}{3}} + y\,u^{\frac{q_0}{3}}\,v^{\frac{q_0}{3}}$
27	uv^2x	$x^{\frac{4q_0}{3}}\,y\,u^{\frac{2q_0}{3}} - x^{\frac{2q_0}{3}}\,y\,u^{\frac{q_0}{3}}\,v^{\frac{q_0}{3}} + y\,v^{\frac{2q_0}{3}}$
28	xyu	$x^{q_0}\,u^{\frac{q_0}{3}} - x^{\frac{q_0}{3}}\,v^{\frac{q_0}{3}}$
29	xyv	$-x^{\frac{4q_0}{3}+1} + x^{\frac{q_0}{3}}\,y$
30	xuv	$-x^{2q_0+1} + x^{q_0}y$
31	yuv	$-x^{\frac{2q_0}{3}}\,u^{\frac{2q_0}{3}}\,v^{\frac{q_0}{3}} + u^{\frac{q_0}{3}}\,v^{\frac{2q_0}{3}}$
32	x^2y^2	$u^{\frac{2q_0}{3}}$
33	x^2u^2	$x^{\frac{4q_0}{3}}\,u^{\frac{2q_0}{3}} - x^{\frac{2q_0}{3}}\,u^{\frac{q_0}{3}}\,v^{\frac{q_0}{3}} + v^{\frac{2q_0}{3}}$
34	x^2v^2	$x^{2q_0+2} + x^{q_0+1}\,y + y^2$
35	y^2u^2	$x^{\frac{2q_0}{3}}\,v^{\frac{2q_0}{3}}$
36	y^2v^2	$x^{\frac{5q_0}{3}+1}\,y - x^{\frac{2q_0}{3}}\,y^2 + x^{\frac{q_0}{3}+1}\,u^{\frac{q_0}{3}}\,v^{\frac{2q_0}{3}} - y\,u^{\frac{2q_0}{3}}\,v^{\frac{q_0}{3}}$
37	u^2v^2	$x^{2q_0}\,y^2 + x^{q_0+1}\,u + yu$
38	x^2yu	$x^{\frac{2q_0}{3}}\,u^{\frac{2q_0}{3}} + u^{\frac{q_0}{3}}\,v^{\frac{q_0}{3}}$
39	x^2yv	$-x^{q_0+1}\,u^{\frac{q_0}{3}} + x^{\frac{q_0}{3}+1}\,v^{\frac{q_0}{3}} - y\,u^{\frac{q_0}{3}}$
40	x^2uv	$-x^{\frac{5q_0}{3}+1}\,u^{\frac{q_0}{3}} - x^{q_0+1}\,v^{\frac{q_0}{3}} - x^{\frac{2q_0}{3}}\,y\,u^{\frac{q_0}{3}} - y\,v^{\frac{q_0}{3}}$
41	xy^2u	$-x^{\frac{q_0}{3}}\,u^{\frac{q_0}{3}}\,v^{\frac{q_0}{3}}$
42	xy^2v	$x^{\frac{2q_0}{3}+1}\,v^{\frac{q_0}{3}} + x^{\frac{q_0}{3}}\,y\,u^{\frac{q_0}{3}}$
43	y^2uv	$x^{\frac{4q_0}{3}}\,y\,u^{\frac{q_0}{3}} + x^{\frac{2q_0}{3}}\,y\,v^{\frac{q_0}{3}} - u^{\frac{2q_0}{3}}\,v^{\frac{2q_0}{3}}$
44	xyu^2	$x^{q_0}\,u^{\frac{q_0}{3}}\,v^{\frac{q_0}{3}} + x^{\frac{q_0}{3}}\,v^{\frac{2q_0}{3}}$
45	xu^2vx	$x^{\frac{5q_0}{3}}\,y\,u^{\frac{q_0}{3}} + x^{q_0}\,y\,v^{\frac{q_0}{3}}$

46	yu^2v	$-x^{\frac{4q_0}{3}}\,y\,v^{\frac{q_0}{3}}+u^{\frac{q_0}{3}+1}$
47	xyv^2	$-x^{\frac{4q_0}{3}+1}y+x^{\frac{2q_0}{3}+1}u^{\frac{2q_0}{3}}v^{\frac{q_0}{3}}+x^{\frac{q_0}{3}}y^2-x\,u^{\frac{q_0}{3}}v^{\frac{2q_0}{3}}$
48	xuv^2	$x^{2q_0+1}y+x^{q_0}x^2+xu$
49	yuv^2	$-x^{\frac{4q_0}{3}}y^2+x^{\frac{2q_0}{3}}y\,u^{\frac{2q_0}{3}}v^{\frac{q_0}{3}}+x^{\frac{q_0}{3}+1}u-y\,u^{\frac{q_0}{3}}v^{\frac{2q_0}{3}}$
50	$xyuv$	$-x^{\frac{4q_0}{3}+1}v^{\frac{q_0}{3}}+x^{q_0}y\,u^{\frac{q_0}{3}}-x^{\frac{q_0}{3}}y\,v^{\frac{q_0}{3}}$
51	x^2y^2u	$x^{\frac{2q_0}{3}}y-u^{\frac{2q_0}{3}}v^{\frac{q_0}{3}}$
52	x^2y^2v	$x^{\frac{q_0}{3}+1}u^{\frac{q_0}{3}}v^{\frac{q_0}{3}}+y\,u^{\frac{2q_0}{3}}$
53	x^2yu^2	$x^{\frac{4q_0}{3}}y+x^{\frac{2q_0}{3}}u^{\frac{2q_0}{3}}v^{\frac{q_0}{3}}-u^{\frac{q_0}{3}}v^{\frac{2q_0}{3}}$
54	x^2u^2v	$x^{\frac{4q_0}{3}}y\,u^{\frac{2q_0}{3}}-x^{\frac{2q_0}{3}}y\,u^{\frac{q_0}{3}}v^{\frac{q_0}{3}}+y\,v^{\frac{2q_0}{3}}$
55	x^2yv^2	$x^{\frac{4q_0}{3}+2}v^{\frac{q_0}{3}}+x^{q_0+1}y\,u^{\frac{q_0}{3}}-x^{\frac{q_0}{3}+1}y\,v^{\frac{q_0}{3}}-y^2u^{\frac{q_0}{3}}$
56	x^2uv^2	$x^{\frac{5q_0}{3}+1}y\,u^{\frac{q_0}{3}}+x^{q_0+1}y\,v^{\frac{q_0}{3}}-x^{\frac{2q_0}{3}}y^2u^{\frac{q_0}{3}}-y^2v^{\frac{q_0}{3}}$
57	xy^2u^2	$x^{\frac{5q_0}{3}}y+x^{\frac{q_0}{3}}u^{\frac{q_0}{3}}v^{\frac{2q_0}{3}}$
58	y^2u^2v	$x^{\frac{2q_0}{3}}y\,v^{\frac{2q_0}{3}}-u^{\frac{2q_0}{3}+1}$
59	xy^2v^2	$-x^{\frac{4q_0}{3}+1}y\,u^{\frac{q_0}{3}}+x^{\frac{q_0}{3}}y^2u^{\frac{q_0}{3}}+x\,u^{\frac{2q_0}{3}}v^{\frac{2q_0}{3}}$
60	y^2uv^2	$-x^{\frac{4q_0}{3}+1}y^2u^{\frac{q_0}{3}}+x^{\frac{q_0}{3}+1}u^{\frac{q_0}{3}+1}+y\,u^{\frac{2q_0}{3}}v^{\frac{2q_0}{3}}$
61	xu^2v^2	$x^{\frac{5q_0}{3}}y^2u^{\frac{q_0}{3}}+x^{q_0}y^2v^{\frac{q_0}{3}}-x^{\frac{2q_0}{3}+1}u^{\frac{q_0}{3}+1}-x\,u\,v^{\frac{q_0}{3}}$
62	yu^2v^2	$x^{\frac{4q_0}{3}}y^2v^{\frac{q_0}{3}}+x^{q_0+1}u^{\frac{q_0}{3}+1}-x^{\frac{q_0}{3}+1}u\,v^{\frac{q_0}{3}}-y\,u^{\frac{q_0}{3}+1}$
63	x^2yuv	$-x^{q_0+1}u^{\frac{q_0}{3}}v^{\frac{q_0}{3}}+x^{\frac{2q_0}{3}}y\,u^{\frac{2q_0}{3}}-x^{\frac{q_0}{3}+1}v^{\frac{2q_0}{3}}+y\,u^{\frac{q_0}{3}}v^{\frac{q_0}{3}}$
64	xy^2uv	$-x^{\frac{2q_0}{3}+1}v^{\frac{2q_0}{3}}-x^{\frac{q_0}{3}}y\,u^{\frac{q_0}{3}}v^{\frac{q_0}{3}}$
65	xyu^2v	$x^{q_0}y\,u^{\frac{q_0}{3}}v^{\frac{q_0}{3}}+x^{\frac{q_0}{3}}y\,v^{\frac{2q_0}{3}}$
66	$xyuv^2$	$x^{\frac{4q_0}{3}+1}y\,v^{\frac{q_0}{3}}+x^{q_0}y^2u^{\frac{q_0}{3}}-x^{\frac{q_0}{3}}y^2v^{\frac{q_0}{3}}-x\,u^{\frac{q_0}{3}+1}$
67	$x^2y^2u^2$	$-x^{\frac{4q_0}{3}}y\,u^{\frac{q_0}{3}}+x^{\frac{2q_0}{3}}y\,v^{\frac{q_0}{3}}+u^{\frac{2q_0}{3}}v^{\frac{2q_0}{3}}$
68	$x^2y^2v^2$	$x^{\frac{2q_0}{3}+2}v^{\frac{2q_0}{3}}-x^{\frac{q_0}{3}+1}y\,u^{\frac{q_0}{3}}v^{\frac{q_0}{3}}+y^2u^{\frac{2q_0}{3}}$
69	$x^2u^2v^2$	$x^{\frac{4q_0}{3}}y^2u^{\frac{2q_0}{3}}-x^{\frac{2q_0}{3}}y^2u^{\frac{q_0}{3}}v^{\frac{q_0}{3}}+y^2v^{\frac{2q_0}{3}}$
70	$y^2u^2v^2$	$x^{\frac{2q_0}{3}}y^2v^{\frac{2q_0}{3}}-x^{\frac{q_0}{3}+1}u^{\frac{q_0}{3}+1}v^{\frac{q_0}{3}}+y\,u^{\frac{2q_0}{3}+1}$
71	x^2y^2uv	$-x^{\frac{5q_0}{3}+1}y+x^{\frac{2q_0}{3}}y^2-x^{\frac{q_0}{3}+1}u^{\frac{q_0}{3}}v^{\frac{2q_0}{3}}-y\,u^{\frac{2q_0}{3}}v^{\frac{q_0}{3}}$
72	x^2yu^2v	$x^{\frac{4q_0}{3}}y^2+x^{\frac{2q_0}{3}}y\,u^{\frac{2q_0}{3}}v^{\frac{q_0}{3}}-x^{\frac{q_0}{3}+1}u-y\,u^{\frac{q_0}{3}}v^{\frac{2q_0}{3}}$
73	x^2yuv^2	$x^{q_0+1}y\,u^{\frac{q_0}{3}}v^{\frac{q_0}{3}}-x^{\frac{2q_0}{3}}y^2u^{\frac{2q_0}{3}}+x^{\frac{q_0}{3}+1}y\,v^{\frac{2q_0}{3}}-y^2u^{\frac{q_0}{3}}v^{\frac{q_0}{3}}$
74	xy^2u^2v	$x^{\frac{5q_0}{3}}y^2-x^{\frac{2q_0}{3}+1}u+x^{\frac{q_0}{3}}y\,u^{\frac{q_0}{3}}v^{\frac{2q_0}{3}}$
75	xy^2uv^2	$x^{\frac{2q_0}{3}+1}y\,v^{\frac{2q_0}{3}}-x^{\frac{q_0}{3}}y^2u^{\frac{q_0}{3}}v^{\frac{q_0}{3}}+x\,u^{\frac{2q_0}{3}+1}$
76	xyu^2v^2	$x^{q_0}y^2u^{\frac{q_0}{3}}v^{\frac{q_0}{3}}+x^{\frac{2q_0}{3}+1}u^{\frac{2q_0}{3}+1}+x^{\frac{q_0}{3}}y^2v^{\frac{2q_0}{3}}+x\,u^{\frac{q_0}{3}+1}v^{\frac{q_0}{3}}$
77	$x^2y^2u^2v$	$-x^{\frac{4q_0}{3}}y^2u^{\frac{q_0}{3}}+x^{\frac{2q_0}{3}}y^2v^{\frac{q_0}{3}}-x^{\frac{q_0}{3}+1}u^{\frac{q_0}{3}+1}+y\,u^{\frac{2q_0}{3}}v^{\frac{2q_0}{3}}$
78	$x^2y^2uv^2$	$x^{\frac{5q_0}{3}+1}y^2+x^{\frac{2q_0}{3}}v+x^{\frac{q_0}{3}+1}y\,u^{\frac{q_0}{3}}v^{\frac{2q_0}{3}}-y^2u^{\frac{2q_0}{3}}v^{\frac{q_0}{3}}$
79	$x^{2y}u^2v^2$	$-x^{\frac{4q_0}{3}+2}u+x^{\frac{4q_0}{3}}v+x^{\frac{2q_0}{3}}y^2u^{\frac{2q_0}{3}}v^{\frac{q_0}{3}}+x^{\frac{q_0}{3}+1}y\,u-y^2u^{\frac{q_0}{3}}v^{\frac{2q_0}{3}}$
80	$xy^2u^2v^2$	$x^{\frac{5q_0}{3}+2}u+x^{\frac{5q_0}{3}}v-x^{\frac{2q_0}{3}+1}y\,u+x^{\frac{q_0}{3}}y^2u^{\frac{q_0}{3}}v^{\frac{2q_0}{3}}4-x\,u^{\frac{2q_0}{3}+1}v^{\frac{q_0}{3}}$
81	$x^2y^2u^2v^2$	$-x^{\frac{4q_0}{3}+2}u^{\frac{q_0}{3}+1}-x^{\frac{4q_0}{3}}u^{\frac{q_0}{3}}v+x^{\frac{2q_0}{3}}v^{\frac{q_0}{3}+1}+x^{\frac{q_0}{3}+1}y\,u^{\frac{q_0}{3}+1}+$ $y^2u^{\frac{2q_0}{3}}v^{\frac{2q_0}{3}}$

Table 1: The Cartier Operator

Proof. In order to apply the Cartier operator, we first express each $f = x^\alpha y^\beta u^\gamma v^\delta$, with $0 \le \alpha,\ \beta,\ \gamma, \delta \le 2$, in the form $f = f_0^3 + f_1^3 x + f_2^3 x^2$. For this recall (5) and the definitions of u and v:

$$y = u^{q_0} - x^{q_0}\, x\,; \tag{14}$$

$$u = x^{3q_0}\, x - y^{3q_0}\,; \tag{15}$$

$$v = y^{3q_0}\, x^2 - u^{3q_0}\,. \tag{16}$$

Using these relations, we can express all of the above 81 monomials in the form

$$f = f_0^3 + f_1^3 x + f_2^3 x^2.$$

For 1 and x, it is obvious that $f_2^3 = 0$, therefore $\mathfrak{C}(dx) = \mathfrak{C}(x\,dx) = 0$. Similarly from (14),

$$y\,dx = [(u^{\frac{q_0}{3}})^3 - (x^{\frac{q_0}{3}})^3\, x]\, dx. \tag{17}$$

In the above equation $f_0 = u^{\frac{q_0}{3}}$ and $f_1 = x^{\frac{q_0}{3}}$, while $f_2 = 0$, so consequently $\mathfrak{C}(ydx) = 0$. Similarly, $C(u\,dx) = 0$. Also, (16) shows that

$$v\,dx = [(-u^{q_0})^3 + (y^{\frac{q_0}{3}})^3\, x^2]\, dx. \tag{18}$$

Thus $f_2 = y^{\frac{q_0}{3}}$, so $\mathfrak{C}(vdx) = y^{\frac{q_0}{3}}\, dx$, which shows that $v\,dx$ is not exact. However if we consider $xv\,dx$, then from

$$xv\,dx = [(xy^{\frac{q_0}{3}})^3 - (u^{q_0})^3\, x]\, dx, \tag{19}$$

it is clear that $f_2 = 0$, therefore $\mathfrak{C}(xv\,dx) = 0$, so $xv\,dx$ is an exact differential.
We give just one more complicated example to illustrate how the results in the table were obtained. We show how to calculate entry 35.

From (14) and (15), we have

$$\begin{aligned} y^2 u^2 dx = {} & [x^{7q_0+3}\, u^{q_0} + x^{5q_0+3}\, y^{3q_0} + y^{6q_0}\, u^{2q_0}]\, dx \\ & + [x^{8q_0+3} + x^{3q_0}\, y^{3q_0}\, u^{2q_0} + x^{q_0}\, y^{6q_0}\, u^{q_0}] x\, dx \\ & + [x^{6q_0}\, u^{2q_0} + x^{4q_0}\, y^{3q_0}\, u^{q_0} + x^{2q_0}\, y^{6q_0}]\, x^2 dx. \end{aligned}$$

From the above equation, f_2^3 for y^2u^2 is

$$f_2^3 = [x^{2q_0}\, u^{\frac{2q_0}{3}} + x^{\frac{4q_0}{3}}\, y^{q_0}\, u^{\frac{q_0}{3}} + x^{\frac{2q_0}{3}}\, y^{2q_0}]^3.$$

Hence

$$\mathfrak{C}(y^2\, u^2 dx) = [x^{2q_0}\, u^{\frac{2q_0}{3}} + x^{\frac{4q_0}{3}}\, y^{q_0}\, u^{\frac{q_0}{3}} + x^{\frac{2q_0}{3}}\, y^{2q_0}]\, dx. \tag{20}$$

The above expression involves $y^{q_0}dx$ and $y^{2q_0}dx$, which are not in the basis for $H^0(X,\Omega^1)$ in Proposition 2.4, so we need to replace them using (4):

$$\begin{aligned} & y^3 = x^2u+v \\ \Rightarrow \qquad & (y^3)^{\frac{q_0}{3}} = (x^2u+v)^{\frac{q_0}{3}} \\ \Rightarrow \qquad & y^{q_0} = (x^2u)^{\frac{q_0}{3}} + (v)^{\frac{q_0}{3}}. \end{aligned} \tag{21}$$

Putting these values in (20), we have

$$\begin{aligned} \mathfrak{C}(y^2u^2dx) =& [x^{2q_0}\, u^{\frac{2q_0}{3}} + x^{\frac{4q_0}{3}}\, u^{\frac{q_0}{3}}(x^{\frac{2q_0}{3}}\, u^{\frac{q_0}{3}} + v^{\frac{q_0}{3}}) + x^{\frac{2q_0}{3}}(x^{\frac{4q_0}{3}}\, u^{\frac{2q_0}{3}} - x^{\frac{2q_0}{3}}\, u^{\frac{q_0}{3}}\, v^{\frac{q_0}{3}} + v^{\frac{2q_0}{3}})]\, dx \\ =& [x^{2q_0}\, u^{\frac{2q_0}{3}} + x^{2q_0}\, u^{\frac{2q_0}{3}} + x^{\frac{4q_0}{3}}\, u^{\frac{q_0}{3}}\, v^{\frac{q_0}{3}} + x^{2q_0}\, u^{\frac{2q_0}{3}} - x^{\frac{4q_0}{3}}\, u^{\frac{q_0}{3}}\, v^{\frac{q_0}{3}} + x^{\frac{2q_0}{3}}\, v^{\frac{2q_0}{3}}]\, dx \\ =& x^{\frac{2q_0}{3}}\, v^{\frac{2q_0}{3}}\, dx. \end{aligned}$$

□

The table shows in particular that among the 81 differentials considered, the only exact ones are dx, $x\,dx$, $y\,dx$, $u\,dx$ and $xv\,dx$. The exactness of these differentials may be verified directly: $dx = d(x)$, $x\,dx = d(-x^2)$, $y\,dx = d(-u^{q_0}x - x^{q_0}x^2)$, $u\,dx = d(x^{2q_0}xy - v^{q_0}x)$ and $xv\,dx = d(x^3y^{q_0}x + u^{q_0}x^2)$. The expressions for $y\,dx$, $u\,dx$ and $xv\,dx$, may be verified easily using (4), (5) and (6).

§ 4 Proof of Theorem 1.2

We define certain classes **Ai**, **Bi**, **Ci**, **Di** of exact holomorphic differentials. That denoted "$g\,dx$" consists of all differentials $f^3g\,dx$ with $f = x^\alpha u^\gamma v^\delta$; $0 \le \gamma$, $\delta \le \frac{q_0}{3} - 1$, with whatever further condition on α, γ and δ is necessary to make the differentials $\omega = f^3gdx$ holomorphic. We will make these conditions explicit later, while counting the number of differentials in each of these classes.

A1: dx,
A2: $x\,dx$,
A3: $y\,dx$,
A4: $u\,dx$,
A5: $xv\,dx$.

Although y^3dx is an exact holomorphic differential, it is not in the basis in Proposition 2.4. However, from (4) we may express it in terms of the basis differentials as $y^3dx = (x^2u+v)dx$. This illustrates the fact that an exact differential may be a linear combination of non-exact basis elements. Similarly we need to express each of the following exact holomorphic differentials in terms of the basis elements:

$$\begin{array}{llll} & u^{q_0}dx, & u^{2q_0}dx, & v^{q_0}dx \\ y^3dx, & y^3u^{q_0}dx, & y^3u^{2q_0}dx, & y^3v^{q_0}dx \\ y^6dx, & y^6u^{q_0}dx, & y^6u^{2q_0}dx, & y^6v^{q_0}dx. \end{array}$$

In fact, we need to look at all of the above multiplied by each of 1, x, y, u and xv (55 possibilities). Although y^9dx is an exact holomorphic differential, we do not consider y^9dx, since

$$y^9dx = x^6u^3dx + v^3dx,$$

which is a linear combination of differentials in class **A1**. In a similar way we can ignore the differentials involving the higher cubic powers of y. Thus we obtain the following classes.

B1: $y^3dx = (x^2u + v)dx$,
B2: $y^3ydx = (x^2yu + yv)dx$,
B3: $y^3udx = (x^2u^2 + uv)dx$,
B4: $y^3xvdx = (x^3uv + xv^2)dx$,
B5: $y^6dx = (x^3xu^2 - x^2uv + v^2)dx$,
B6: $y^6ydx = (x^3xyu^2 - x^2yuv + yv^2)dx$,
B7: $u^{q_0}xdx = (x^{q_0}x^2 - xy)dx$,
B8: $u^{q_0}ydx = (x^{q_0}xy - y^2)dx$,
B9: $u^{q_0}udx = (x^{q_0}xu - yu)dx$,
B10: $u^{q_0}xvdx = (x^{q_0}x^2v - xyv)dx$,
B11: $u^{q_0}y^3ydx = (x^{q_0+3}yu + x^{q_0}xyv - x^2y^2u - y^2v)dx$,
B12: $u^{q_0}y^3udx = (x^{q_0+3}u^2 + x^{q_0}xuv - x^2yu^2 - yuv)dx$,
B13: $u^{q_0}y^3xvdx = (x^{q_0+3}xuv + x^{q_0}x^2v^2 - x^3yuv - xyv^2)dx$,
B14: $u^{q_0}y^6ydx = (x^{q_0+3}x^2yu^2 - x^{q_0+3}yuv + x^{q_0}xyv^2 - x^3xy^2u^2 + x^2y^2uv - y^2v^2)dx$.
Similarly with the help of the remaining possibilities (among those 55, stated earlier), we define some other classes of exact holomorphic differentials as follows.

$$\begin{aligned} y^6udx &= (x^3u^3x - x^2u^2v + uv^2)dx \\ &= x^3u^3xdx - (x^2u^2v - uv^2)dx. \end{aligned}$$

Since x^3u^3xdx is already on **A2**, a new list of exact differentials is (denoted by)
$y^6udx \Rightarrow$**C1**: $(x^2u^2v - uv^2)dx$.
Note that discarding x^3u^3xdx reduces the pole order, allowing more possibilities for f^3.

Similarly we just state the other such classes. All of the the following classes are obtained by discarding one (previously defined) differential, except for **C14**, which is obtained by discarding two differentials, belonging to **B8** and **C4**.

$$\begin{aligned} u^{2q_0}xdx \Rightarrow\ &\mathbf{C2}:\ (x^{q_0}x^2y + xy^2)dx, \\ u^{2q_0}udx \Rightarrow\ &\mathbf{C3}:\ (x^{2q_0}v - x^{q_0}xyu - y^2u)dx, \\ v^{q_0}xdx \Rightarrow\ &\mathbf{C4}:\ (x^{q_0}y^2 - xu)dx, \\ v^{q_0}udx \Rightarrow\ &\mathbf{C5}:\ (x^{2q_0-3}xyv - x^{q_0-3}x^2y^2u - x^{q_0-3}y^2v + u^2)dx, \end{aligned}$$

$$
\begin{aligned}
v^{q_0}xvdx &\Rightarrow \ \mathbf{C6}: \ (x^{q_0}x^2y^2u + x^{q_0}y^2v - x^3u^2 - xuv)dx,\\
y^3u^{2q_0}xdx &\Rightarrow \ \mathbf{C7}: \ (x^{q_0+3}xyu + x^{q_0}x^2yv + x^3y^2u + xy^2v)dx,\\
y^3u^{2q_0}udx &\Rightarrow \ \mathbf{C8}: \ (x^{2q_0}x^2uv + x^{2q_0}v^2 - x^{q_0+3}yu^2 - x^{q_0}xyuv - x^2y^2u^2 - y^2uv)dx,\\
y^3v^{q_0}udx &\Rightarrow \ \mathbf{C9}: \ (x^{2q_0}yuv + x^{2q_0-3}xyv^2v - x^{q_0}xy^2u^2 + x^{q_0-3}x^2y^2uv - x^{q_0-3}y^2v^2 + u^3x^2 + u^2v)dx,\\
y^3v^{q_0}xvdx &\Rightarrow \ \mathbf{C10}: \ (x^{q_0+3}xy^2u^2 - x^{q_0}x^2y^2uv + x^{q_0}y^2v^2 - x^3u^3x^2 + x^3u^2v - xuv^2)dx,\\
y^6u^{q_0}udx &\Rightarrow \ \mathbf{C11}: \ (x^{q_0+3}u^2v - x^{q_0}xuv^2 - x^2yu^2v + yuv^2)dx,\\
y^6u^{2q_0}xdx &\Rightarrow \ \mathbf{C12}: \ (x^{q_0+6}yu^2 - x^{q_0+3}xyuv + x^{q_0}x^2yv^2 + x^3x^2y^2u^2 - x^3y^2uv + xy^2v^2)dx,\\
y^6u^{2q_0}udx &\Rightarrow \ \mathbf{C13}: \ (x^{2q_0+3}xu^2v - x^{2q_0}x^2uv^2 + x^{q_0+3}yu^2v - x^{q_0}xyuv^2 + x^2y^2u^2v - y^2uv^2)dx,\\
y^6v^{q_0}udx &\Rightarrow \ \mathbf{C14}: \ (x^{2q_0}x^2yu^2v - x^{2q_0}yuv^2 - u^3x^2v + u^2v^2)dx.
\end{aligned}
$$

Among the 55 possibilities stated earlier, we may discard the remaining ones, as they are linear combinations of previously determined classes, e.g. $y^3xdx = (x^3u + xv)dx$, which is a linear combination of two exact holomorphic differentials already present in **A4** and **A5**.

Recall from Proposition 3.2 that the *exact* differentials on X are of the form

$$\omega = (f_0^3 + f_1^3 x)dx, \quad \text{for } f_0,\, f_1 \in \mathbb{F}_q(X).$$

There is a possibility that both f_0^3dx and f_1^3xdx are not holomorphic (e.g. $f_i = x^{\alpha_i}y^{\beta_i}u^{\gamma_i}v^{\delta_i}$ with $\alpha_i + \beta_i + \gamma_i + 2\delta_i > q_0 - 1$) but that cancelation of poles allows the sum to be holomorphic. This motivates us to consider the non-holomorphic exact differentials

$$
\begin{array}{lllll}
u^{q_0}v^{q_0}dx, & v^{2q_0}dx, & u^{2q_0}v^{q_0}dx, & u^{q_0}v^{2q_0}dx, & u^{2q_0}v^{2q_0}dx,\\
y^3u^{q_0}v^{q_0}dx, & y^3v^{2q_0}dx, & y^3u^{2q_0}v^{q_0}dx, & y^3u^{q_0}v^{2q_0}dx, & y^3u^{2q_0}v^{2q_0}dx,\\
y^6u^{q_0}v^{q_0}dx, & y^6v^{2q_0}dx, & y^6u^{2q_0}v^{q_0}dx, & y^6u^{q_0}v^{2q_0}dx, & y^6u^{2q_0}v^{2q_0}dx,
\end{array}
$$

and similarly each of the above multiplied by x, (hence 30 possibilities altogether). Among these possibilities, here we state only those which lead us to some new classes **Di**'s of exact holomorphic differentials. The rest can be discarded, as they lead only to linear combinations of **Ai**'s, **Bi**'s, **Ci**'s and **Di**'s.

We begin with $u^{q_0}\, v^{q_0}xdx$.

$$
\begin{aligned}
u^{q_0}\, v^{q_0}xdx &= (x^{3q_0}x^2y - x^{2q_0}xy^2 - x^{q_0}x^2u + xyu)dx\\
&= x^{2q_0}(x^{q_0}x^2y + xy^2)dx + (x^{2q_0}xy^2 - x^{q_0}x^2u + xyu)dx,
\end{aligned}
$$

where $(x^{q_0}x^2y + xy^2)dx \in \mathbf{C2}$, so we have a new class **D1** given as follows,

D1: $(x^{2q_0}xy^2 - x^{q_0}x^2u + xyu)dx$.

(One may check that this exact differential is holomorphic, using Lemma 2.3.) In a similar way we construct the following classes,

$$
\begin{aligned}
u^{q_0}\, v^{2q_0} dx &= (x^{5q_0} xy^2 - x^{4q_0} y^3 + x^{3q_0} xyu - x^{2q_0} y^2 u + x^{q_0} xu^2 - yu^2) dx \\
&= (x^{5q_0} xy^2 - x^{4q_0} x^2 u - x^{4q_0} v + x^{3q_0} xyu - x^{2q_0} y^2 u + x^{q_0} xu^2 - yu^2) dx \\
&= x^{3q_0} \mathbf{D1} - x^{2q_0} \mathbf{C3} - x^{2q_0-3} \mathbf{C7} - \mathbf{D2} + x^{q_0} \mathbf{D3},
\end{aligned}
$$

where
D2: $(x^{2q_0} y^2 u + yu^2) dx$,
D3: $(x^{2q_0-3} x^2 yv + x^{q_0-3} xy^2 v + xu^2) dx$.
One may check using Table 1 that each of **D2** and **D3** is exact. In fact, **D3**, shows up in the calculation of the kernel of the Cartier operator in the case $s = 1$ (see Section 6 below). This is what suggested the decomposition $-\mathbf{D2} + x^{q_0} \mathbf{D3}$, and that we should seek similar decompositions as follows.

$$
\begin{aligned}
y^3\, u^{q_0}\, v^{2q_0} dx = (&x^{5q_0+3} y^2 u + x^{5q_0} xy^2 v - x^{4q_0+3} xu^2 + x^{4q_0} x^2 uv - x^{4q_0} v^2 \\
&+ x^{3q_0+3} yu^2 + x^{3q_0} xyuv - x^{2q_0} x^2 y^2 u^2 - x^{2q_0} y^2 uv + x^{q_0+3} u^3 \\
&+ x^{q_0} xu^2 v - u^3 x^2 y - yu^2 v) dx \\
= \; &x^{3q_0+3} \mathbf{D2} - x^{2q_0} \mathbf{C8} - x^{2q_0-3} \mathbf{C12} + x^{q_0+3} u^3 \mathbf{A1} + x^{3q_0} \mathbf{D4} - \mathbf{D5} + x^{q_0} \mathbf{D6},
\end{aligned}
$$

where
D4: $(x^{2q_0} xy^2 v - x^{q_0+3} xu^2 - x^{q_0} x^2 uv + xyuv) dx$,
D5: $(x^{2q_0} x^2 y^2 u^2 + x^{2q_0} y^2 uv + u^3 x^2 y + yu^2 v) dx$,
D6: $(x^{2q_0} xyuv + x^{2q_0-3} x^2 yv^2 + x^{q_0} x^2 y^2 uv + x^{q_0-3} xy^2 v^2 + xu^2 v) dx$.
Similarly by expanding $y^6 u^{q_0}\, v^{2q_0} dx$, we have

$$
\begin{aligned}
&(x^{5q_0+3} x^2 y^2 u^2 - x^{5q_0+3} y^2 uv - x^{5q_0} xy^2 v^2 - x^{4q_0+6} u^3 - x^{4q_0} v^3 + x^{3q_0+3} u^3 x^2 y \\
&- x^{3q_0+3} yu^2 v + x^{3q_0} xyuv^2 - x^{2q_0+3} u^3 xy^2 + x^{2q_0} x^2 y^2 u^2 v - x^{2q_0} y^2 uv^2 + x^{q_0+3} u^3 x^2 u \\
&- x^{q_0+3} u^3 v + x^{q_0} xu^2 v^2 - x^3 u^3 xyu + u^3 x^2 yv - yu^2 v^2) dx \\
= \; &x^{3q_0+3} \mathbf{D5} - x^{2q_0} \mathbf{C13} - x^{4q_0+6} u^3 \mathbf{A1} - x^{4q_0} v^3 \mathbf{A1} - x^{q_0+3} u^3 \mathbf{B1} - x^3 u^3 \mathbf{D1} + x^{3q_0} \mathbf{D7} + \mathbf{D8} + x^{q_0} \mathbf{D9},
\end{aligned}
$$

where
D7: $(x^{2q_0+3} y^2 uv + x^{2q_0} xy^2 v^2 + x^{q_0+3} xu^2 v - x^{q_0} x^2 uv^2 + x^3 yu^2 v + xyuv^2) dx$,
D8: $(x^{2q_0} x^2 y^2 u^2 v - x^{2q_0} y^2 uv^2 - x^{q_0+3} u^3 x^2 u + u^3 x^2 yv - yu^2 v^2) dx$,
D9: $(x^{2q_0+3} yu^2 v - x^{2q_0} xyuv^2 + x^{q_0} x^2 y^2 u^2 v - x^{q_0} y^2 uv^2 + x^3 u^3 v + xu^2 v^2) dx$.

We can try to construct more exact holomorphic differentials using the classes **Ci**'s, e.g. $u^{q_0} \mathbf{C6}$, $y^3 u^{q_0} \mathbf{C6}$ and $y^6 u^{q_0} \mathbf{C6}$ etc., but they only give us the same classes as discussed above, so we can ignore them. It can be observed that in each class, containing differentials of the form $f^3 g dx$, the pole order of g allows multiplication by $f^3 = y^3$ and y^6, but considering such possibilities never gives us any new classes, e.g.

$$
y^6 \mathbf{D2} = x^3 u^3 \mathbf{D1} - \mathbf{D8}, \tag{22}
$$

Given these thwarted attempts to construct more, we begin to suspect that the whole space of exact holomorphic differentials on the curve X is spanned by those in the classes **Ai**, **Bi**, **Ci** and **Di**.

The elements we have produced are all of the form $f^3 g\,dx$, where gdx can be any of the 42 forms given by the **Ai**'s, **Bi**'s, **Ci**'s and **Di**'s. Next we introduce certain restrictions on the monomial $f = x^\alpha u^\gamma v^\delta$, to make $f^3 gdx$ a linear combination of elements of the basis in Proposition 2.4, then we count the number of such elements in each of the 42 classes. We state these restrictions, for each class, in the following table. (It is taken as read that $\alpha \geq 0$ and $0 \leq \gamma, \delta \leq \frac{q_0}{3} - 1$.)

Classes	Restrictions on α, γ and δ.
A1, **A2**, **A3**, **A4**	$\alpha + \gamma + 2\delta \leq q_0 - 1$
A5, **B1**, **B2**, **B3**	$\alpha + \gamma + 2\delta \leq q_0 - 2$
B4, **B5**, **B6**, **C1**	$\alpha + \gamma + 2\delta \leq q_0 - 3$
B7, **B8**, **B9**, **C4**	Either $\alpha + \gamma + 2\delta \leq \frac{2q_0}{3} - 2$ or $\alpha + \gamma + 2\delta = \frac{2q_0}{3} - 1$ with $\gamma + \delta \leq \frac{q_0}{3} - 1$
B10, **C2**	$\alpha + \gamma + 2\delta \leq \frac{2q_0}{3} - 2$
B11, **C6**	Either $\alpha + \gamma + 2\delta \leq \frac{2q_0}{3} - 3$ or $\alpha + \gamma + 2\delta = \frac{2q_0}{3} - 2$ with $\gamma + \delta \leq \frac{q_0}{3} - 1$
B12	Either $\alpha + \gamma + 2\delta \leq \frac{2q_0}{3} - 3$ or $\alpha + \gamma + 2\delta = \frac{2q_0}{3} - 2$ with $\gamma + \delta \leq \frac{q_0}{3} - 2$
B13, **C7**, **C11**	$\alpha + \gamma + 2\delta \leq \frac{2q_0}{3} - 3$
B14, **C10**	Either $\alpha + \gamma + 2\delta \leq \frac{2q_0}{3} - 4$ or $\alpha + \gamma + 2\delta = \frac{2q_0}{3} - 3$ with $\gamma + \delta \leq \frac{q_0}{3} - 2$
C3, **C5**, **D3**	$\alpha + \gamma + 2\delta \leq \frac{q_0}{3} - 1$
C8, **C9**, **D1**, **D2**, **D4**, **D6**	$\alpha + \gamma + 2\delta \leq \frac{q_0}{3} - 2$
C12	$\alpha + \gamma + 2\delta \leq \frac{2q_0}{3} - 4$
C13, **C14**, **D5**, **D7**, **D8**, **D9**	$\alpha + \gamma + 2\delta \leq \frac{q_0}{3} - 3$

Table 2: Restrictions on f.

The following table summarises the results of counting the number of elements in each class.

Classes	Number of differentials
A1, **A2**, **A3**, **A4**	$\frac{q_0^2}{2 \cdot 3^2}(q_0+3)$
A5, **B1**, **B2**, **B3**	$\frac{q_0^2}{2 \cdot 3^2}(q_0+1)$
B4, **B5**, **B6**, **C1**	$\frac{q_0^2}{2 \cdot 3^2}(q_0-1)$
B7, **B8**, **B9**, **C4**	$\frac{q_0+3}{2^3 \cdot 3^4}(14q_0^2+21q_0+27)$
B10, **C2**	$\frac{q_0+3}{2^3 \cdot 3^4}(14q_0^2-15q_0+27)$
B11, **C6**	$\frac{q_0-3}{2^3 \cdot 3^4}(14q_0^2+15q_0+27)+\frac{q_0}{2 \cdot 3^2}(q_0+3)^2$
B12	$\frac{q_0-3}{2^3 \cdot 3^4}(14q_0^2+51q_0+27)$
B13, **C7**, **C11**	$\frac{q_0-3}{2^3 \cdot 3^4}(14q_0^2+15q_0+27)$
B14, **C10**	$\frac{q_0-3}{2^3 \cdot 3^4}(14q_0^2-3q_0-27)$
C3, **C5**, **D3**	$\frac{q_0+3}{2^3 \cdot 3^4}(2q_0^2+21q_0+27)$
C8, **C9**, **D1**, **D2**, **D4**, **D6**	$\frac{q_0^2-9}{2^3 \cdot 3^4}(2q_0+9)$
C12	$\frac{q_0-3}{2^3 \cdot 3^4}(14q_0^2-39q_0-27)$
C13, **C14**, **D5**, **D7**, **D8**, **D9**	$\frac{q_0^2-9}{2^3 \cdot 3^4}(2q_0-9)$

Table 3: Number of differentials in each class.

Lemma 4.1. *Let* $\omega_1 = f_1^3 g_1 dx$ *and* $\omega_2 = f_2^3 g_2 dx$ *be two holomorphic differentials, (where* $f_i = x^{\alpha_i} u^{\gamma_i} v^{\delta_i}$*;* $\alpha_i \geq 0$, $0 \leq \gamma_i, \delta_i \leq \frac{q_0}{3} - 1$*) such that* $\operatorname{ord}_\infty(\omega_1) = \operatorname{ord}_\infty(\omega_2)$*. Then*

$$-[\operatorname{ord}_\infty(g_1) - \operatorname{ord}_\infty(g_2)] = 3kq + 9lq_0 + 3m, \tag{23}$$

where $k, l, m \in \mathbb{Z}$*, with* $|m| < \frac{q_0}{3}$ *and* $|l| < \frac{2q_0}{3} - 1$*.*

Proof. Let ω_1 and ω_2 be as above.

$$\begin{aligned} &\operatorname{ord}_\infty(\omega_1) = \operatorname{ord}_\infty(\omega_2) \\ &\Leftrightarrow -\operatorname{ord}_\infty(g_1) - (-\operatorname{ord}_\infty(g_2)) = -\operatorname{ord}_\infty(f_2^3) - (-\operatorname{ord}_\infty(f_1^3)). \end{aligned} \tag{24}$$

Since $f_i = x^{\alpha_i} u^{\gamma_i} v^{\delta_i}$,

$$\begin{aligned} -\mathrm{ord}_\infty(f_i^3) &= 3[(\alpha_i + \gamma_i + 2\delta_i)q + 3(\gamma_i + \delta_i)q_0 + \delta_i] \\ \Rightarrow -\mathrm{ord}_\infty(f_2^3) - (-\mathrm{ord}_\infty(f_1^3)) &= 3[(\alpha_2 + \gamma_2 + 2\delta_2)q + 3(\gamma_2 + \delta_2)q_0 + \delta_2] \\ &\quad - 3[(\alpha_1 + \gamma_1 + 2\delta_1)q + 3(\gamma_1 + \delta_1)q_0 + \delta_1] \\ &= 3[kq + 3lq_0 + m], \text{ such that } k,\, l \text{ and } m \in \mathbb{Z}, \end{aligned}$$

with l and m satisfying the conditions stated above. □

Proof. of Theorem 1.2.

Each of the classes **Ai**, **Bi**, **Ci** and **Di** consists of exact holomorphic differentials $\omega = f^3 g dx$, for certain monomials f in x, u and v. With the help of Lemma 4.1, it can be verified that no two such differentials have the same orders at P_∞, e.g. from **B1** and **B2** we have $g_1 = x^2 u + v$, with $-\mathrm{ord}_\infty(g_1) = 3q + 3q_0$, and $g_2 = x^2 yu + yv$, with $-\mathrm{ord}_\infty(g_2) = 4q + 4q_0$. Then $-\mathrm{ord}_\infty(g_1) - (-\mathrm{ord}_\infty(g_2)) = -q - q_0$, which is not of the form $3kq + 9lq_0 + 3m$ with $|m| < \frac{q_0}{3}$, and $|l| < \frac{2q_0}{3} - 1$. We can show the same for each other pair of classes. In particular, all these differentials are different, i.e. there is no overlap between the classes. To count them we add up all the numbers of elements in the 42 classes, as listed individually in the table, giving us a lower bound for the dimension of the space of exact holomorphic differentials. It is precisely

$$\frac{2q_0}{27}\left(14q_0^2 + 9\right) + \frac{1}{12}\left(11q_0^2 + 9\right),$$

so we have proved Theorem 1.2. □

Conjecture 1.1 is that this lower bound is the exact dimension.

Remark 2: The above sum must obviously be an integer, but we can also see this directly from the formula. Of course $\frac{2q_0}{3^3}(14q_0^2 + 9)$ is an integer since the numerator is divisible by 3^3 (as $3|q_0$). But also $\frac{1}{3\cdot 4}(11q_0^2 + 9) \in \mathbb{Z}$, since $11q_0^2 + 9 \equiv 11 \times 1 + 9 \equiv 0 \pmod 4$, and $11q_0^2 + 9$ is obviously divisible by 3.

Remark 3: For any large s, comparing the (conjectured) dimension d of the space of exact holomorphic differentials to the genus $g = \frac{3}{2}q_0(q-1)$ shows that it is approximately a quarter of g. To be precise, $\lim\limits_{s\to\infty} \frac{d}{g} = \frac{56}{243}$.

§ 5 A group action on $H^0(X, \Omega^1)$

Let $\zeta \in \mathbb{F}_q$ be a primitive $(q-1)$st root of unity. Define an $\mathbb{F}_q$-linear automorphism θ of the function field $\mathbb{F}_q(X)$ by $\theta(x) = \zeta\, x$, $\theta(y) = \zeta^{q_0+1}\, y$. (This preserves the equation

$y^q - y = x^{q_0}(x^q - x)$.) This θ then also acts naturally as an automorphism of the curve $X/\mathbb{F}_q$, and as an $\mathbb{F}_q$-linear transformation of $H^0(X, \Omega)$.

$$\theta : \begin{cases} x \mapsto \zeta\, x \\ y \mapsto \zeta^{q_0+1}\, y \\ u \mapsto \zeta^{3q_0+1}\, u \\ v \mapsto \zeta^{3q_0+3}\, v \\ dx \mapsto \zeta\, dx. \end{cases} \tag{25}$$

For each $i (\mathrm{mod}(q-1))$, let $\mathcal{A}_i$ be the subspace of $H^0(X, \Omega^1)$ on which θ acts as multiplication by ζ^i. Note that this is independent of the choice of ζ, in fact it is an isotypic component for the action on $H^0(X, \Omega^1)$ of the cyclic group $G \simeq \mathbb{F}_q^\times$ generated by θ. Since the order of G is coprime to the characteristic, we have

$$H^0(X, \Omega^1) = \bigoplus_{i (\mathrm{mod}(q-1))} \mathcal{A}_i \tag{26}$$

The following lemma (suggested by the referee) is immediate.

Lemma 5.1.

$$x^a y^b u^c v^d\, dx \in \mathcal{A}_i \iff a + b(q_0+1) + c(3q_0+1) + d(3q_0+3) + 1 \equiv i \pmod{q-1}.$$

Lemma 5.2. $\mathfrak{C}(\mathcal{A}_{3i}) \subseteq \mathcal{A}_i$

Proof. Let $\omega_{3i} \in \mathcal{A}_{3i}$. It follows from the definition in Proposition 3.1 that $\mathcal{C}$ commutes with automorphisms of the curve. Hence

$$\begin{aligned} \theta\mathfrak{C}(\omega_{3i}) &= \mathfrak{C}(\theta\omega_{3i}) \\ &= \mathfrak{C}(\zeta^{3i}\omega_{3i}) \\ &= \zeta^i\mathfrak{C}(\omega_{3i}). \end{aligned}$$

From the above it is clear that $\mathfrak{C}(\omega_{3i}) \in \mathcal{A}_i$, hence $\mathfrak{C}(\mathcal{A}_{3i}) \subseteq \mathcal{A}_i$. □

Remark 4: Since $3^{2s+1} = q \equiv 1\, (\mathrm{mod}(q-1))$, $\mathfrak{C}$ therefore permutes the $\mathcal{A}_i$ in cycles of length dividing $2s+1$.

Equation (26) and Lemma 5.2 easily imply the following.

Proposition 5.3.

$$\ker(\mathfrak{C}) = \bigoplus_{i (\mathrm{mod}(q-1))} \ker(\mathfrak{C}|_{\mathcal{A}_i})$$

We are indebted to the referee for showing us what to use for the proof of the following.

Proposition 5.4.

$$\dim(\mathcal{A}_i) = \begin{cases} \frac{3q_0+1}{2} & \text{if } i \text{ is odd,} \\ \\ \frac{3q_0-1}{2} & \text{if } i \text{ is even.} \end{cases}$$

Proof. Consider the projection $\pi : X \to Y$, where Y is the quotient of X by the group G of automorphisms generated by θ. Recall that G is cyclic of order $q-1$, so π is a morphism of degree $q-1$. Recall also that $y^q - y = x^{q_0}(x^q - x)$, that $\theta(x)$ (i.e. the pullback $\theta^*(x)$) is ζx and $\theta(y) = \zeta^{q_0+1}y$. If $P = (\alpha, \beta) \in X(\overline{\mathbb{F}_q})$ with $\alpha \neq 0$, then $\theta^k(P) = (\zeta^k\alpha, \zeta^{k(q_0+1)}\beta)$, so the stabiliser of P under the action of G is trivial, and P is not a ramification point for π. At the other extreme, $P_0 := (0,0)$ and P_∞ are fixed points for the action of G, with ramification index $(q-1)$. There remain $(q-1)$ points $P_\beta := (0,\beta)$ for $\beta \in \mathbb{F}_q - \{0\}$. We have $\theta(P_\beta) = P_{\beta\zeta^{q_0+1}}$. Since $q-1 = (q_0+1)(3q_0-3)+2$, so that g.c.d.$(q_0+1, q-1) = 2$, it follows that these points form two orbits of size $(q-1)/2$ for the action of G, and all have ramification index 2. There are four branch points for π, with inverse images of sizes $1, 1, (q-1)/2, (q-1)/2$.

Now we are ready to find the genus $g(Y)$ of Y, using Hurwitz's formula $2g(X) - 2 = \deg(\pi)(2g(Y)-2) + \sum_{P \in X(\overline{\mathbb{F}_q})}(e_P - 1)$, for the tamely ramified cover $\pi : X \to Y$, where the e_P are the ramification indices. Since $g(X) = \frac{3}{2}q_0(q-1)$, we get

$$3q_0(q-1) - 2 = (q-1)(2g(Y)-2) + 2(q-2) + (q-1) = (q-1)(2g(Y)+1) - 2.$$

Hence $g(Y) = \frac{3q_0-1}{2}$. Since $\mathcal{A}_0 = \pi^* H^0(Y, \Omega^1)$, this proves the case $i = 0$ of the proposition.

To prove the other cases, we turn to Lemma 4.3 of Bouw [B], who credits it to Kani [K]. Let $Q_1 = \pi(P_\infty), Q_2 = \pi(P_0), Q_3$ and Q_4 be the branch points of π. The sizes of the inverse images are $n_1 = n_2 = q-1$ and $n_3 = n_4 = (q-1)/2$. In Bouw's notation, $\ell = q-1$ and we have numbers b_j and a_j for $1 \leq j \leq 4$. Since $\mathrm{ord}_{P_\infty}(x) = q$, x is a uniformiser at each of the q points P_β for $\beta \in \mathbb{F}_q$. At P_∞, xu/v is a uniformiser, by Proposition 2.1. Since $\theta^*(x) = \zeta x$ while $\theta^*(xu/v) = \zeta^{-1}xu/v$ (using (25)), we find that $b_1 = -1$ while $b_2 = b_3 = b_4 = 1$. Now a_i is defined to be the multiple of n_i such that $0 \leq a_i < \ell$ and $a_i b_i/n_i \equiv 1 \pmod{\ell/n_i}$. It follows that $a_1 = q-2, a_2 = 1$ and $a_3 = a_4 = (q-1)/2$.

If we define L_i, for $0 \leq i < q-1$, to be the subspace of $H^1(X, O_X)$ on which θ acts as multiplication by ζ^i, then according to Lemma 4.3 of [B] in our case,

$$\dim L_i = g(Y) - 1 + \sum_{j=1}^{4} \left\langle \frac{ia_i}{q-1} \right\rangle,$$

where $\langle a \rangle$ is the fractional part of a. This gives

$$\dim L_i = g(Y) - 1 + \left\langle \frac{i(q-2)}{q-1} \right\rangle + \left\langle \frac{i}{q-1} \right\rangle + 2\left\langle \frac{i((q-1)/2)}{q-1} \right\rangle$$

$$= \frac{3q_0 - 1}{2} - 1 + \frac{q-1-i}{q-1} + \frac{i}{q-1} + 2\langle i/2\rangle = \frac{3q_0-1}{2} + 2\langle i/2\rangle$$

$$= \begin{cases} \frac{3q_0-1}{2} & i \text{ even;} \\ \frac{3q_0+1}{2} & i \text{ odd.} \end{cases}$$

By Serre duality, $\dim\mathcal{A}_i = \dim L_{q-1-i}$, and since $q-1$ is even, the proposition follows.

□

Remark 5: One may check, using Proposition 2.4 and Lemma 5.1, that the following $\frac{3q_0+1}{2}$ differentials belong to our basis for $H^0(X, \Omega^1)$ and to $\mathcal{A}_1$, so must form a basis for $\mathcal{A}_1$:

$$dx,\ x^2v^{q_0-1}dx,\ x^4uv^{q_0-2}dx,\ \ldots,\ x^{q_0-3}yu^{q_0-2}vdx,\ x^{q_0-1}yu^{q_0-1}dx,$$

$$yu^{\frac{q_0-1}{2}}v^{\frac{q_0-1}{2}}dx,\ x^2yu^{\frac{q_0+1}{2}}v^{\frac{q_0-3}{2}}dx,\ \ldots,\ x^{q_0-1}u^{q_0-1}dx.$$

§ 6 Proof of Theorem 1.3

$\mathbf{s = 1}$

The $\mathcal{A}_i$ are of size 5 for i odd, 4 for i even. The following shows, for a few of the $\mathcal{A}_i$'s, a basis (arranged in a lexicographical order).

$$\mathcal{A}_1 = \langle dx,\ yuvdx,\ x^2v^2dx,\ x^2yu^2dx,\ x^4uvdx\rangle,$$
$$\mathcal{A}_2 = \langle xdx,\ xyuvdx,\ x^3v^2dx,\ x^3yu^2dx\rangle,$$

$$\mathcal{A}_3 = \langle yv^2dx,\ y^2u^2dx,\ x^2dx,\ x^2yuvdx,\ x^4yu^2dx\rangle,$$
$$\mathcal{A}_4 = \langle xyv^2dx,\ xy^2u^2dx,\ x^3dx,\ x^3yuvdx\rangle,$$
$$\mathcal{A}_5 = \langle ydx,\ y^2uvdx,\ x^2yv^2dx,\ x^2y^2u^2dx,\ x^4dx\rangle,$$
$$\mathcal{A}_6 = \langle xydx,\ xy^2uvdx,\ x^3y^2u^2dx,\ x^5dx\rangle.$$

These may be confirmed using Proposition 2.4 and Lemma 5.1, but the following indicates how these bases were generated in practice. We start with the basis for $\mathcal{A}_1$ given by Remark 5. In getting from $\mathcal{A}_1$ to $\mathcal{A}_2$ we have more-or-less multiplied by x, but we dropped x^5uvdx as it is not holomorphic. To get from $\mathcal{A}_2$ to $\mathcal{A}_3$, again it is mostly a case of multiplying by x. We have discarded the non-holomorphic x^4v^2dx, but have gained two by replacing x^4 (i.e. x^{q_0+1}) with y in both that and the holomorphic $x^4yu^2\,dx$.

As mentioned earlier, $\mathfrak{C}$ maps each $\mathcal{A}_i$ to $\mathcal{A}_{i/3}$, and permutes the $\mathcal{A}_i$ in cycles of length dividing $2s+1$. For $s=1$, one easily checks that it produces 8 length 3 cycles and 2 length 1 cycles (the latter for $i=0$ and $i=(q-1)/2$). For example

$$\mathcal{A}_1 \xrightarrow{\mathfrak{C}} \mathcal{A}_9 \xrightarrow{\mathfrak{C}} \mathcal{A}_3 \xrightarrow{\mathfrak{C}} \mathcal{A}_1.$$

Here we show calculations for a part of this cycle, $\mathcal{A}_9 \xrightarrow{\mathfrak{C}} \mathcal{A}_3$, where

$$\mathcal{A}_9 = \langle uv^2dx,\, y^2dx,\, x^2u^2vdx,\, x^4ydx,\, x^8dx\rangle$$

If $\omega \in \mathcal{A}_9$ then we can express ω as

$$\omega = \lambda_1^3(uv^2dx) + \lambda_2^3(y^2dx) + \lambda_3^3(x^2u^2vdx) + \lambda_4^3(x^4ydx) + \lambda_5^3(x^8dx), \text{ with } \lambda_i \in \mathbb{F}_q.$$

If $\omega \in \ker(\mathfrak{C})|_{\mathcal{A}_9}$ then $\mathfrak{C}(\omega) = 0$ shows

$$\begin{aligned} & \mathfrak{C}(\omega) = \mathfrak{C}(\lambda_1^3(uv^2dx) + \lambda_2^3(y^2dx) + \lambda_3^3(x^2u^2vdx) + \lambda_4^3(x^4ydx) + \lambda_5^3(x^8dx)) = 0 \\ \Rightarrow\quad & \lambda_1\mathfrak{C}(uv^2dx) + \lambda_2\mathfrak{C}(y^2dx) + \lambda_3\mathfrak{C}(x^2u^2vdx) + \lambda_4\mathfrak{C}(x^4ydx) + \lambda_5\mathfrak{C}(x^8dx) = 0 \\ \Rightarrow\quad & \lambda_1\mathfrak{C}(uv^2dx) + \lambda_2\mathfrak{C}(y^2dx) + \lambda_3\mathfrak{C}(x^2u^2vdx) + \lambda_4 x\mathfrak{C}(xydx) + \lambda_5 x^2\mathfrak{C}(x^2dx) = 0. \end{aligned} \tag{27}$$

We use Table 1 to substitute into (27). This gives

$$\begin{aligned} & \lambda_1(x^4yu^2 - x^2yuv + yv^2)dx + \lambda_2(x^2u^2vdx) + \lambda_3(x^4yu^2 - x^2yuv + yv^2)dx + \lambda_4(x^2dx) \\ & + \lambda_5(dx) = 0. \end{aligned}$$

Since $\mathfrak{C}(\omega) \in \mathcal{A}_3$, it can be expressed in terms of the generators of $\mathcal{A}_3$. The above equation becomes

$$\begin{aligned} & (\lambda_1 + \lambda_3)v^2ydx + (0)y^2u^2dx + (\lambda_2 + \lambda_4 + \lambda_5)x^2dx + (-\lambda_1 - \lambda_3)x^2yuvdx \\ & + (\lambda_1 + \lambda_3)x^4yu^2dx = 0 \end{aligned} \tag{28}$$

To find $\ker(\mathfrak{C}|_{\mathcal{A}_9})$, we have to find the null space of the associated matrix $\mathbf{M}_{9,3}$. We solve the following:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 2 & 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives

$$\begin{aligned} \lambda_1 &= -\lambda_3 \\ \lambda_5 &= -(\lambda_2 + \lambda_4). \end{aligned}$$

From the above we have the following three linearly independent exact holomorphic differentials:

$$(x^2u^2v - uv^2)dx\,; \tag{29}$$

$$(x^4y - y^2)dx\,; \tag{30}$$

$$(x^8 - x^4y)dx = x^3(x^3x^2 - xy)dx\,. \tag{31}$$

These basis elements actually belong to our earlier lists, specifically **C1**, **B8** and **B7** respectively.

For the length-1 cycles, we find that $\ker(\mathfrak{C}|_{\mathcal{A}_{13}})$ is spanned by $(x^2u+v)dx \in \mathbf{B1}$, while $\ker(\mathfrak{C}|_{\mathcal{A}_0})$ is spanned by $x^3(x^2u^2+uv)dx \in \mathbf{B3}$ and $(x^3uv+xv^2)dx \in \mathbf{B4}$. From these and similar calculations, we find that the dimension of the space of exact holomorphic differentials $H^0(X,\Omega^1)^{\mathfrak{C}=0}$, for $s=1$, is 39 (compared with $g=117$), and the basis elements we find all lie on the lists **Ai**, **Bi**, **Ci**, **Di**.

$\mathbf{s}=2$

For $s=2$ we can proceed in a similar manner to that described for $s=1$, to find, inside each $\mathcal{A}_i$, a subset of our basis for $H^0(X,\Omega^1)$, of size $\frac{3q_0+1}{2}$ if i is odd, $\frac{3q_0-1}{2}$ if i is even. For example, the following shows our basis (arranged in a lexicographical order) for a few of the $\mathcal{A}_i$'s.

$$\begin{aligned}\mathcal{A}_1 =&\langle dx,\, yu^4v^4dx,\, x^2v^8dx,\, x^2yu^5v^3dx,\, x^4uv^7dx,\, x^4yu^6v^2dx,\, x^6u^2v^6dx,\, x^6yu^7vdx,\\ &x^8u^3v^5dx,\, x^8yu^8dx,\, x^{10}u^4v^4dx,\, x^{12}u^5v^3dx,\, x^{14}u^6v^2dx,\, x^{16}u^7vdx\rangle\\ \mathcal{A}_2 =&\langle xdx,\, xyu^4v^4dx,\, x^3v^8dx,\, x^3yu^5v^3dx,\, x^5uv^7dx,\, x^5yu^6v^2dx,\, x^7u^2v^6dx,\, x^7yu^7vdx,\\ &x^9u^3v^5dx,\, x^9yu^8dx,\, x^{11}u^4v^4dx,\, x^{13}u^5v^3dx,\, x^{15}u^6v^2dx\rangle\end{aligned}$$

$$\begin{aligned}\mathcal{A}_3 =&\langle yu^3v^5dx,\, y^2u^8dx,\, x^2dx,\, x^2yu^4v^4dx,\, x^4v^8dx,\, x^4yu^5v^3dx,\, x^6uv^7dx,\, x^6yu^6v^2dx,\\ &x^8u^2v^6dx,\, x^8yu^7vdx,\, x^{10}u^3v^5dx,\, x^{10}yu^8dx,\, x^{12}u^4v^4dx,\, x^{14}u^5v^3dx\rangle\\ \mathcal{A}_4 =&\langle xyu^3v^5dx,\, xy^2u^8dx,\, x^3dx,\, x^3yu^4v^4dx,\, x^5v^8dx,\, x^5yu^5v^3dx,\, x^7uv^7dx,\, x^7yu^6v^2dx,\\ &x^9u^2v^6dx,\, x^9yu^7vdx,\, x^{11}u^3v^5dx,\, x^{11}yu^8dx,\, x^{13}u^4v^4dx\rangle\\ \mathcal{A}_5 =&\langle yu^2v^6dx,\, y^2u^7vdx,\, x^2yu^3v^5dx,\, x^2y^2u^8dx,\, x^4dx,\, x^4yu^4v^4dx,\, x^6v^8dx,\, x^6yu^5v^3dx,\\ &x^8uv^7dx,\, x^8yu^6v^2dx,\, x^{10}y^2v^6dx,\, x^{10}yu^7vdx,\, x^{12}u^3v^5dx,\, x^{12}yu^8dx\rangle.\end{aligned}$$

We have altogether 242 $\mathcal{A}_i$'s, which give rise to 48 cycles of length 5 and 2 cycles of length 1. We deal with these cycles one by one, just like in the case $s=1$, finding each $\ker(\mathfrak{C}|_{\mathcal{A}_i})$ by solving a set of linear equations. However, the corresponding matrices will now be either 14×14 or 13×13. We therefore found their null spaces with the help of the computer package Maple. (For the details of these calculations see [F].)

In the case $s=1$, we observed that for each cycle of length 3, containing $\mathcal{A}_i$'s all of dimension either 4 or 5, the total contribution of the cycle to $\dim H^0(X,\Omega^1)^{\mathfrak{C}=\mathfrak{o}}$ is precisely $\dim(\mathcal{A}_i)$. However, in the case $s=2$ we found that, for the cycles of length 5, $\dim(\mathcal{A}_i)$ is only a lower bound for the contribution to $\dim H^0(X,\Omega^1)^{\mathfrak{C}=\mathfrak{o}}$ of the cycle containing $\mathcal{A}_i$. The length-5 cycles making the smallest contribution (i.e. 13) to the dimension are $\mathcal{A}_{40} \xrightarrow{\mathfrak{C}} \mathcal{A}_{94} \xrightarrow{\mathfrak{C}} \mathcal{A}_{112} \xrightarrow{\mathfrak{C}} \mathcal{A}_{118} \xrightarrow{\mathfrak{C}} \mathcal{A}_{120} \xrightarrow{\mathfrak{C}} \mathcal{A}_{40}$ and $\mathcal{A}_{122} \xrightarrow{\mathfrak{C}} \mathcal{A}_{202} \xrightarrow{\mathfrak{C}} \mathcal{A}_{148} \xrightarrow{\mathfrak{C}} \mathcal{A}_{130} \xrightarrow{\mathfrak{C}} \mathcal{A}_{124} \xrightarrow{\mathfrak{C}} \mathcal{A}_{122}$. Looking at the latter in more detail, the contributions are as follows:

$\ker(\mathfrak{C}|_{\mathcal{A}_{122}})$ is spanned by $x^3v^3udx \in \mathbf{A4}$, $x^9u^3udx \in \mathbf{A4}$, $v^3xvdx \in \mathbf{A5}$ and $x^6u^3xvdx \in \mathbf{A5}$.

$\ker(\mathfrak{C}|_{\mathcal{A}_{202}})$ is spanned by $x^3u^6(x^2u+v)dx \in \mathbf{B1}$, $v^6(x^{q_0}x^2y+xy^2)dx \in \mathbf{C2}$,

$\ker(\mathfrak{C}|_{\mathcal{A}_{148}})$ is spanned by $x^3u^3(x^3xu^2 - x^2uv + v^2)dx \in$ **B5** and $v^3(x^{2q_0-3}x^2yv + x^{q_0-3}xy^2v + xu^2)dx \in$ **D3**.

$\ker(\mathfrak{C}|_{\mathcal{A}_{130}})$ is spanned by $x^9v^3(x^2u + v)dx \in$ **B1** and $u^{15}u^3(x^2u + v)dx \in$ **B1**.

$\ker(\mathfrak{C}|_{\mathcal{A}_{124}})$ is spanned by $x^3v^3(x^2u+v)dx \in$ **B1**, $x^9u^3(x^2u+v)dx \in$ **B1**, and $u^3(x^{2q_0}xy^2 - x^{q_0}x^2u + xyu)dx \in$ **D1**.

The length-5 cycles making the largest contribution (i.e. 26) to the dimension are $\mathcal{A}_1 \xrightarrow{C} \mathcal{A}_{81} \xrightarrow{C} \mathcal{A}_{27} \xrightarrow{C} \mathcal{A}_9 \xrightarrow{C} \mathcal{A}_3 \xrightarrow{C} \mathcal{A}_1$ and $\mathcal{A}_{161} \xrightarrow{C} \mathcal{A}_{215} \xrightarrow{C} \mathcal{A}_{233} \xrightarrow{C} \mathcal{A}_{239} \xrightarrow{C} \mathcal{A}_{241} \xrightarrow{C} \mathcal{A}_{161}$.

The cycles of length 1 (containing $\mathcal{A}_{121}$ and $\mathcal{A}_{242}$) make smaller contributions:

$\ker(\mathfrak{C}|_{\mathcal{A}_{121}})$ is spanned by $v^3(x^2yu + yv)dx \in$ **B2** and $x^6u^3(x^2yu + yv)dx \in$ **B2**, while $\ker(\mathfrak{C}|_{\mathcal{A}_{242}})$ is spanned by $x^3v^6(x^2u^2 + uv)dx \in$ **B3**, $x^9u^3v^3(x^2u^2 + uv)dx \in$ **B3**, $x^{15}u^6(x^2u^2 + uv)dx \in$ **B3**, $v^6(x^3uv + xv^2)dx \in$ **B4**, $x^6u^3v^3(x^3uv + xv^2)dx \in$ **B4**, $x^{12}u^6(x^3uv + xv^2)dx \in$ **B4**, and $x^3u^6(x^3xyu^2 - x^2yuv + yv^2)dx \in$ **B6**.

All in all, we find that the dimension of the space of exact holomorphic differentials for X, when $s = 2$, is 837 (compared with $g = 3267$), which exactly matches with $\frac{2q_0}{3^3}(14q_0^2 + 9) + \frac{1}{12}(11q_0^2 + 9)$, (by putting $q_0 = 9$ and $q = 243$).

All the exact holomorphic differentials found by our calculations for $s = 1$ and $s = 2$ are accounted for by the classes **Ai**, **Bi**, **Ci** and **Di**. We checked directly that the numbers found in each class match those given by Table 3. Many of these turn out to be 0 in the case $s = 1$, and generally speaking, there are many more differentials in the earlier classes than in the later classes.

Now that we have proved Theorems 1.2 and 1.3, we address the question of why we might believe Conjecture 1.1. Originally, we only found the classes **Ai**, **Bi** and **Ci**, and thought that might be all, so why should we now believe that for every $s \geq 1$ the space of exact holomorphic differentials is spanned by those in these classes and the **Di**, aside from the fact that we can't find anything else? After finding the classes **Ai**, **Bi** and **Ci**, we then calculated the kernel of $\mathfrak{C}$ in the case $s = 1$ and found that, although almost everything we found was in one of these classes, the differential $(x^3x^2yv + xy^2v + xu^2)dx$, (obtained from $\mathcal{A}_{22} \xrightarrow{\mathfrak{C}} \mathcal{A}_{16}$) does not belong to any of them. This is what made us look for more, and discover the classes **Di**. The above differential belongs to **D3**. Our calculations for $s = 1$, the very first case we looked at, revealed what we had missed. Were we still missing anything after discovering the **Di**, it seems likely that our subsequent calculations for $s = 2$ would likewise have revealed it.

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