BOUNDARY QUANTUM KNIZHNIK-ZAMOLODCHIKOV EQUATIONS AND FUSION

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Abstract. In this paper we extend our previous results concerning Jackson integral solutions of the boundary quantum Knizhnik-Zamolodchikov equations with diagonal K-operators to higher-spin representations of quantum affine $\mathfrak{sl}_2$. First we give a systematic exposition of known results on R-operators acting in the tensor product of evaluation representations in Verma modules over quantum $\mathfrak{sl}_2$. We develop the corresponding fusion of K-operators, which we use to construct diagonal K-operators in these representations. We construct Jackson integral solutions of the associated boundary quantum Knizhnik-Zamolodchikov equations and explain how in the finite-dimensional case they can be obtained from our previous results by the fusion procedure.

Contents

1. Introduction

The boundary q-Knizhnik-Zamolodchikov (qKZ) equations have their origins in the representation theory through works of Cherednik [6, 7] and in quantum field theory and in statistical mechanics with special "integrable" boundary conditions, see, e.g., [1, 22, 18, 19, 37]. For detailed references see [32]. Their formulation involves solutions to the Yang-Baxter equation, the so-called R-operators or R-matrices, and solutions to the reflection equation, known as (boundary) K-operators or K-matrices.

1.1. **The boundary qKZ equations.** Let M^ℓ be the Verma module over quantum $\mathfrak{sl}_2$ with highest weight $\ell \in \mathbb{C}$. Then we will denote by $\mathcal{R}^{k\ell}(x)$ the operator acting in $M^k \otimes M^\ell$ which is the evaluation of the truncated universal R-matrix for quantum affine $\mathfrak{sl}_2$ acting in the tensor product of corresponding evaluation representations. It satisfies the Yang-Baxter equation:

$$\mathcal{R}^{k\ell}_{12}(x-y)\mathcal{R}^{km}_{13}(x-z)\mathcal{R}^{\ell m}_{23}(y-z) = \mathcal{R}^{\ell m}_{23}(y-z)\mathcal{R}^{km}_{13}(x-z)\mathcal{R}^{k\ell}_{12}(x-y) \tag{1.1}$$

This is an equation in $M^k \otimes M^\ell \otimes M^m$ and we are using the standard notations $R^{k\ell}_{12}(x) = R^{k\ell}(x) \otimes \mathrm{Id}_{M^m}$, etc. For details and references see Section 2.

Given the above R-operator $\mathcal{R}^{k\ell}(x)$, operators $\mathcal{K}^{+,\ell}(x)$ and $\mathcal{K}^{-,\ell}(x)$ acting in M^ℓ are called left and right K-operators if they satisfy the left and right reflection equations, respectively. These equations are also known as "boundary Yang Baxter equations" and were introduced in [34]. In the current setting they are given by

$$\begin{aligned}\mathcal{R}^{k\ell}(x-y)\mathcal{K}^{+,k}_1(x)\mathcal{R}^{\ell k}_{21}(x+y)\mathcal{K}^{+,\ell}_2(y) &= \\ = \mathcal{K}^{+,\ell}_2(y)\mathcal{R}^{k\ell}(x+y)\mathcal{K}^{+,k}_1(x)\mathcal{R}^{\ell k}_{21}(x-y),& \\ \mathcal{R}^{\ell k}_{21}(x-y)\mathcal{K}^{-,k}_1(x)\mathcal{R}^{k\ell}(x+y)\mathcal{K}^{-,\ell}_2(y) &= \\ = \mathcal{K}^{-,\ell}_2(y)\mathcal{R}^{\ell k}_{21}(x+y)\mathcal{K}^{-,k}_1(x)\mathcal{R}^{k\ell}(x-y).&\end{aligned} \tag{1.2}$$

These are equations in $M^k \otimes M^\ell$; we are using the notations $\mathcal{K}^{\pm,k}_1(x) = \mathcal{K}^{\pm,k}(x) \otimes \mathrm{Id}$, $\mathcal{K}^{\pm,\ell}_2(y) = \mathrm{Id} \otimes \mathcal{K}^{\pm,\ell}(y)$ and $\mathcal{R}^{\ell k}_{21}(x) := \mathcal{P}^{\ell k}\mathcal{R}^{\ell k}(x)\mathcal{P}^{k\ell}$, where $\mathcal{P}^{k\ell} : M^k \otimes M^\ell \to M^\ell \otimes M^k$ is the permutation operator $\mathcal{P}^{k\ell}(m^k \otimes m^\ell) = m^\ell \otimes m^k$ ($m^k \in M^k$, $m^\ell \in M^\ell$).

For $\underline{\ell} = (\ell_1, \ldots, \ell_N) \in \mathbb{C}^N$, consider the tensor product

$$M^{\underline{\ell}} = M^{\ell_1} \otimes \cdots \otimes M^{\ell_N}.$$

The boundary qKZ equations [6, 7] in $M^{\underline{\ell}}$ are given by the following compatible system of difference equations

$$f(\mathbf{t} + \tau \boldsymbol{e}_r) = \Xi^{\underline{\ell}}_r(\mathbf{t}; \xi_+, \xi_-; \tau) f(\mathbf{t}), \qquad r = 1, \ldots, N \tag{1.3}$$

for $M^{\underline{\ell}}$-valued meromorphic functions $f(\mathbf{t})$ in $\mathbf{t} \in \mathbb{C}^N$, where $\tau \in \mathbb{C}^\times$ and $\{\boldsymbol{e}_r\}_r$ is the standard orthonormal basis of $\mathbb{R}^N$. Here

$$\begin{aligned}\Xi^{\underline{\ell}}_r(\mathbf{t}; \xi_+, \xi_-; \tau) := {}& \mathcal{R}^{\ell_r\ell_{r+1}}_{r,r+1}(t_r - t_{r+1} + \tau) \cdots \mathcal{R}^{\ell_r\ell_N}_{r,N}(t_r - t_N + \tau) \\ & \times \mathcal{K}^{+,\ell_r}_r(t_r + \frac{\tau}{2})\mathcal{R}^{\ell_N\ell_r}_{N,r}(t_N + t_r) \cdots \mathcal{R}^{\ell_{r+1}\ell_r}_{r+1,r}(t_{r+1} + t_r) \\ & \times \mathcal{R}^{\ell_{r-1}\ell_r}_{r-1,r}(t_{r-1} + t_r) \cdots \mathcal{R}^{\ell_1\ell_r}_{1,r}(t_1 + t_r)\mathcal{K}^{-,\ell_r}_r(t_r) \\ & \times \mathcal{R}^{\ell_r\ell_1}_{r,1}(t_r - t_1) \cdots \mathcal{R}^{\ell_r\ell_{r-1}}_{r,r-1}(t_r - t_{r-1})\end{aligned} \tag{1.4}$$

is the (boundary) transport operator on $M^{\underline{\ell}}$, depending meromorphically on $\mathbf{t} \in \mathbb{C}^N$. The compatibility of the system (1.3) is guaranteed by the conditions

$$\Xi^{\underline{\ell}}_r(\mathbf{t} + \mathbf{e}_s\tau; \xi_+, \xi_-; \tau)\Xi^{\underline{\ell}}_s(\mathbf{t}; \xi_+, \xi_-; \tau) = \Xi^{\underline{\ell}}_s(\mathbf{t} + \mathbf{e}_r\tau; \xi_+, \xi_-; \tau)\Xi^{\underline{\ell}}_r(\mathbf{t}; \xi_+, \xi_-; \tau),$$

for $r, s = 1, \ldots, N$, which themselves are consequences of the quantum Yang-Baxter and reflection equations (1.1-1.2). In this paper we construct explicit Jackson integral solutions of (1.3) when the left and right K-operators $\mathcal{K}^{\pm,\ell}(x)$ are of the form $\mathcal{K}^{\xi_\pm,\ell}(x)$ with $\mathcal{K}^{\xi,\ell}(x)$ ($\xi \in \mathbb{C}$) an explicit one-parameter family of K-operators diagonal with respect to the weight basis of M^ℓ.

1.2. **Finite-dimensional representations and fusion.** When $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ the representation M^ℓ is no longer irreducible; it has an infinite-dimensional subrepresentation and an irreducible finite-dimensional quotient representation V^ℓ. When some of the ℓ_s's in the boundary qKZ equations are in $\frac{1}{2}\mathbb{Z}_{\geq 0}$ the equations descend to the tensor product of corresponding quotient modules.

For $k, \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, the tensor product of the associated evaluation modules $V^k(x) \otimes V^\ell(y)$ becomes reducible for special values of $x, y \in \mathbb{C}$ [5]. Owing to this degeneracy,

R-operators acting in (tensor products of) higher-dimensional evaluation modules can be obtained from corresponding objects acting in (tensor products of) lower-dimensional evaluation modules through a process called fusion [24, 15]. We extend this representation-theoretic approach to fusion of K-operators in Section 4. Such R- and K-operators can then be generalized to R- and K-operators associated with modules $M^\ell(x)$ for arbitrary $\ell \in \mathbb{C}$ by means of an analytical continuation. This will allow us to establish the above reflection equation (1.2) for a larger class of K-operators than hitherto has been done. In particular, we obtain the diagonal K-operators $\mathcal{K}^{\xi,\ell}(x)$ from this fusion approach applied to Cherednik's [7] diagonal K-matrix associated to $V^{\frac{1}{2}}$. The $\mathcal{K}^{\xi,\ell}(x)$ are closely related to the family of K-operators constructed in [11] using the q-Onsager algebra.

For other approaches to fusion of K-operators, see e.g. [13, 25, 21, 26, 28, 38].

1.3. **Main result.** In [32] we constructed q-integral solutions to (1.3) when all $\ell_s = \frac{1}{2}$. In this case the corresponding irreducible quotient spaces are two-dimensional and (1.3) reduces to an equation in $(\mathbb{C}^2)^{\otimes N}$. The main result of this paper is the construction of q-integral solutions to (1.3) for arbitrary $\ell_s \in \mathbb{C}$. For $\ell_s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ it gives Jackson integral solutions in the tensor product of corresponding irreducible representations V^{ℓ_s}. Our main result (Theorem 6.2) can be summarized as follows.

Theorem 1.1. *Let $\xi_+, \xi_- \in \mathbb{C}$ and let $g_{\xi_+,\xi_-}(x)$, $h(x)$ and $F^\ell(x)$ be meromorphic functions in $x \in \mathbb{C}$ satisfying the functional equations*

$$g_{\xi_+,\xi_-}(x+\tau) = \frac{\sinh(\xi_- - x - \frac{\eta}{2})\sinh(\xi_+ - x - \frac{\tau}{2} - \frac{\eta}{2})}{\sinh(\xi_- + x + \tau - \frac{\eta}{2})\sinh(\xi_+ + x + \frac{\tau}{2} - \frac{\eta}{2})} g_{\xi_+,\xi_-}(x),$$
$$h(x+\tau) = \frac{\sinh(x+\tau)\sinh(x+\eta)}{\sinh(x)\sinh(x+\tau-\eta)} h(x),$$
$$F^\ell(x+\tau) = \frac{\sinh(x+\tau-\ell\eta)}{\sinh(x+\tau+\ell\eta)} F^\ell(x).$$

Given fixed generic $\mathbf{x}_0 \in \mathbb{C}^S$, and fixed parameters ξ_+, ξ_-, η, τ in a suitable parameter domain (see Section 6), the $M^{\underline{\ell}}$-valued sum

$$f_S^{\underline{\ell}}(\mathbf{t}) := \sum_{\mathbf{x} \in \mathbf{x}_0 + \tau\mathbb{Z}^S} \Big(\prod_{i=1}^S g_{\xi_+,\xi_-}(x_i)\Big)\Big(\prod_{1\leq i<j\leq S} h(x_i+x_j)h(x_i-x_j)\Big)$$
$$\times \Big(\prod_{r=1}^N \prod_{i=1}^S F^{\ell_r}(t_r+x_i)F^{\ell_r}(t_r-x_i)\Big)\Big(\prod_{i=1}^S \overline{\mathcal{B}}^{\xi_-}(x_i;\mathbf{t})\Big)\Omega$$

is a solution of the boundary qKZ equations (1.3)*, meromorphic in $\mathbf{t} \in \mathbb{C}^N$. Here, $\overline{\mathcal{B}}^{\xi}(x;\mathbf{t})$ are matrix elements of the boundary quantum monodromy matrix and $\Omega = m_1^{\ell_1} \otimes \cdots \otimes m_1^{\ell_N}$ is the tensor product of highest-weight vectors $m_1^{\ell_s} \in M^{\ell_s}$ (see Section 5 for details).*

Explicit formulae for functions g_{ξ_+,ξ_-}, h and F^ℓ are given in Section 6. We will discuss integral (not Jackson integral) solutions in a forthcoming paper. It yields a complete system of solutions to the boundary qKZ equations.

Theorem 1.1 gives for $\ell_s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ Jackson integral solutions of the boundary qKZ equations taking values in

$$V^{\underline{\ell}} = V^{\ell_1} \otimes \cdots \otimes V^{\ell_S}.$$

These can alternatively be obtained from a fusion procedure applied to the Jackson integral solutions when all $\ell_s = \frac{1}{2}$ derived earlier in [32] (see Subsection 8.3). It seems though that the result for continuous spin $\ell_s \in \mathbb{C}$ (Theorem 1.1) cannot be obtained from half-integer spins by analytic continuation.

1.4. **Outline of the paper.** In Section 2 and Section 3 we overview solutions to the quantum Yang-Baxter equation corresponding to quantum $\mathfrak{sl}_2$ and their fusion, following [23, 24, 15]. Reflection equations and the fusion of K-operators are discussed in Section 4. The boundary monodromy matrices defined in terms of these R- and K-operators are introduced in Section 5, as are the off-shell Bethe vectors $\big(\prod_{i=1}^S \overline{\mathcal{B}}^{\xi_-}(x_i;\mathbf{t})\big)\Omega$. In Section 6 we state and discuss the main theorem on the Jackson integral solutions of the boundary qKZ equations with continuous spins; its proof is given in Section 7. In Section 8, we show that the boundary qKZ equations (1.3) acting on $V^{\underline{\ell}}$ ($\ell_s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$) and the associated Jackson integral solutions of the boundary qKZ equations can be obtained from the special case when all $\ell_s = \frac{1}{2}$ by fusion.

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2. Quantum affine $\mathfrak{sl}_2$ and R-operators

In this section we discuss basic facts on quantum affine $\mathfrak{sl}_2$ and its associated evaluation R-and L-operators, following [17, 15]. We use slightly different conventions compared to [17, 15] in order to obtain a direct match with the R-and L-operators of the 6-vertex model (see Subsection 2.5).

2.1. **Quantum affine algebra $\mathfrak{sl}_2$ and the universal R-matrix.** We fix $\eta \in \mathbb{C}$ such that $p := \mathrm{e}^\eta$ is not a root of unity. We write $p^x := \mathrm{e}^{\eta x}$ for $x \in \mathbb{C}$.

Set $\mathfrak{h} = \mathbb{C}h_0 \oplus \mathbb{C}h_1$. Quantum affine $\mathfrak{sl}_2$ is the Hopf algebra $\widehat{\mathcal{U}}_\eta := \mathcal{U}_\eta(\widehat{\mathfrak{sl}}_2)$ over $\mathbb{C}$ with generators e_i, f_i $(i = 0, 1)$, p^h $(h \in \mathfrak{h})$ and with defining relations

$$p^0 = 1, \qquad p^{h+h'} = p^h p^{h'},$$
$$p^h e_i p^{-h} = p^{\alpha_i(h)} e_i, \quad p^h f_i p^{-h} = p^{-\alpha_i(h)} f_i, \quad [e_i, f_j] = \delta_{i,j}\frac{p^{h_i} - p^{-h_i}}{p - p^{-1}},$$
$$e_i^3 e_j - (p^2 + 1 + p^{-2}) e_i^2 e_j e_i + (p^2 + 1 + p^{-2}) e_i e_j e_i^2 - e_j e_i^3 = 0, \quad i \neq j,$$
$$f_i^3 f_j - (p^2 + 1 + p^{-2}) f_i^2 f_j f_i + (p^2 + 1 + p^{-2}) f_i f_j f_i^2 - f_j f_i^3 = 0, \quad i \neq j$$

for $i, j = 0, 1$ and $h, h' \in \mathfrak{h}$. Here α_i are linear functionals on $\mathfrak{h}$ satisfying $\alpha_j(h_i) = a_{ij}$ with Cartan matrix

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The comultiplication Δ and the counit ϵ are determined by their action on generators:

$$\begin{aligned}\Delta(p^h) &= p^h \otimes p^h,\\ \Delta(e_i) &= e_i \otimes 1 + p^{-h_i} \otimes e_i,\\ \Delta(f_i) &= f_i \otimes p^{h_i} + 1 \otimes f_i\end{aligned}$$

and

$$\epsilon(p^h) = 1, \quad \epsilon(e_i) = 0, \quad \epsilon(f_i) = 0.$$

The antipode is determined by $S(p^h) = p^{-h}$, $S(e_i) = -p^{h_i}e_i$ and $S(f_i) = -f_i p^{-h_i}$.

The extension $\widetilde{\mathcal{U}}_\eta$ of this algebra by generators $p^{\lambda d}$ ($\lambda \in \mathbb{C}$) such that $[p^{\lambda d}, p^h] = 0$ and $p^{\lambda d} e_i = p^{\lambda \delta_{i,0}} e_i p^{\lambda d}$, $p^{\lambda d} f_i = p^{-\lambda\delta_{i,0}} f_i p^{\lambda d}$ is a quantized Kac-Moody algebra. The corresponding Lie algebra has a non-degenerate scalar product and there is a universal R-matrix $R \in \widetilde{\mathcal{U}}_\eta \hat{\otimes} \widetilde{\mathcal{U}}_\eta$ [14]. It has the form

$$R = \exp(\eta(c \otimes d + d \otimes c))\mathcal{R}$$

where $c = h_0 + h_1$ and $\mathcal{R} \in \widehat{\mathcal{U}}_\eta \hat{\otimes} \widehat{\mathcal{U}}_\eta$. In the category of modules where c acts by zero (zero-level representations), the element $\mathcal{R}$ satisfies all properties of the universal R-matrix:

$$\begin{aligned}\mathcal{R}\Delta(a) &= \Delta^{\mathrm{op}}(a)\mathcal{R},\\ (\Delta \otimes \mathrm{Id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, &\qquad (\mathrm{Id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}.\end{aligned}$$

Here, Δ^{op} the opposite comultiplication. See also [15, Lecture 9] for further details (note though that we have a different convention for the comultiplication).

2.2. **Evaluation representations.** We write $\mathcal{U}_\eta \subset \widehat{\mathcal{U}}_\eta$ for the Hopf subalgebra generated by e_1, f_1 and $p^{\lambda h_1}$ ($\lambda \in \mathbb{C}$). It is the quantized universal enveloping algebra of $\mathfrak{sl}_2$.

Let $\ell \in \mathbb{C}$ and $M^\ell := \bigoplus_{n=1}^{\infty} \mathbb{C} m_n^\ell$ be a left $\mathcal{U}_\eta$-module with the action given by

$$\begin{aligned}\pi^\ell(p^{\lambda h_1}) m_n^\ell &= p^{2\lambda(\ell+1-n)} m_n^\ell,\\ \pi^\ell(e_1) m_n^\ell &= \frac{\sinh((n-1)\eta)\sinh((2\ell+2-n)\eta)}{\sinh(\eta)^2} m_{n-1}^\ell,\\ \pi^\ell(f_1) m_n^\ell &= m_{n+1}^\ell,\end{aligned}$$

where $m_0^\ell := 0$. The $\mathcal{U}_\eta$-module (π^ℓ, M^ℓ) is the Verma module with highest weight ℓ and highest weight vector m_1^ℓ.

If $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ the subspace $N^k := \bigoplus_{n=2k+2}^{\infty} \mathbb{C} m_n^k \subset M^k$ is a $\mathcal{U}_\eta$-submodule. We write $V^k := M^k/N^k$ for the resulting quotient $\mathcal{U}_\eta$-module. The cosets $v_n^k := m_n^k + N^k$ ($1 \leq n \leq 2k+1$) form a weight basis in V^k. The associated representation map will be denoted by $\overline{\pi}^k$ and for this representation of $\mathcal{U}_\eta$ we will write $(\overline{\pi}^k, V^k)$.

For each $x \in \mathbb{C}$ there exists a unique unit-preserving algebra homomorphism $\phi_x : \widehat{\mathcal{U}}_\eta \to \mathcal{U}_\eta$ satisfying

$$\begin{aligned}\phi_x(p^{\lambda h_0}) &= p^{-\lambda h_1}, &\qquad \phi_x(p^{\lambda h_1}) &= p^{\lambda h_1},\\ \phi_x(e_0) &= \mathrm{e}^{-x} f_1, &\qquad \phi_x(e_1) &= \mathrm{e}^{-x} e_1,\\ \phi_x(f_0) &= \mathrm{e}^{x} e_1, &\qquad \phi_x(f_1) &= \mathrm{e}^{x} f_1.\end{aligned}$$

Given a representation π of $\mathcal{U}_\eta$ on V we write $\pi_x := \pi \circ \phi_x$, which turns V in a representation of $\widehat{\mathcal{U}}_\eta$ called the evaluation representation. Sometimes we will denote it by $V(x)$.

In what follows we will work with evaluation representation $(\overline{\pi}^k_x, V^k)$ and (π^ℓ_x, M^ℓ), where $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{C}$.

2.3. **Evaluation R-and L-operators.** We follow here [15, Lecture 9]. Fix $x, y \in \mathbb{C}$ with $\Re(x-y) \ll 0$. For $k, \ell \in \mathbb{C}$ the evaluation of the truncated universal R-matrix

$$\big(\pi^k_x \otimes \pi^\ell_y\big)(\mathcal{R})$$

is a linear operator on $M^k \otimes M^\ell$ which only depends on the difference $x-y$ of x and y. It acts on the tensor product of highest weight vectors as

$$\big(\pi^k_x \otimes \pi^\ell_y)(\mathcal{R}) m^k_1 \otimes m^\ell_1 = \alpha^{k\ell}(x-y) m^k_1 \otimes m^\ell_1$$

where $\alpha^{k\ell}(x-y)$ is invertible for generic p and $x-y$. Define

$$\mathcal{R}^{k\ell}(x-y) := \alpha^{k\ell}(x-y)^{-1}\big(\pi^k_x \otimes \pi^\ell_y\big)(\mathcal{R}).$$

The operator $\mathcal{R}^{k\ell}(x-y)$ intertwines the action of $\widehat{\mathcal{U}}_\eta$ with its opposite:

$$\mathcal{R}^{k\ell}(x-y)(\pi^k_x \otimes \pi^\ell_y)(\Delta(X)) = (\pi^k_x \otimes \pi^\ell_y)(\Delta^{\mathrm{op}}(X))\mathcal{R}^{k\ell}(x-y), \qquad X \in \widehat{\mathcal{U}}_\eta \tag{2.1}$$

and satisfies $\mathcal{R}^{k\ell}(x-y) m^k_1 \otimes m^\ell_1 = m^k_1 \otimes m^\ell_1$. These properties determine $\mathcal{R}^{k\ell}(x-y)$ uniquely for generic values of $x-y$.

The dependence of the operator $\mathcal{R}^{k\ell}(x-y)$ on x, y, k, ℓ is as a rational function in e^{x-y}, p^k and p^ℓ. Analytic continuation thus gives a well-defined linear operator $\mathcal{R}^{k\ell}(x-y)$ on $M^k \otimes M^\ell$ for generic values of $x-y$, which can be characterized by the same intertwining property (2.1) with respect to the action of $\widehat{\mathcal{U}}_\eta$.

Let $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and write $\mathrm{pr}^k : M_k \twoheadrightarrow V_k$ for the canonical map. For each $x \in \mathbb{C}$, it defines an intertwiner $\mathrm{pr}^k_x : M_k(x) \twoheadrightarrow V_k(x)$ of $\widehat{\mathcal{U}}_\eta$-modules. Note that for $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, there exists a unique linear map

$$L^{k\ell}(x-y) : V^k \otimes M^\ell \to V^k \otimes M^\ell$$

depending rationally on e^{x-y} and satisfying

$$(\mathrm{pr}^k \otimes \mathrm{Id}_{M^\ell})\mathcal{R}^{k\ell}(x-y) = L^{k\ell}(x-y)(\mathrm{pr}^k \otimes \mathrm{Id}_{M^\ell}).$$

Similarly, for $k, \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, there exists a unique linear map

$$R^{k\ell}(x-y) : V^k \otimes V^\ell \to V^k \otimes V^\ell$$

satisfying

$$(\mathrm{pr}^k \otimes \mathrm{pr}^\ell)\mathcal{R}^{k\ell}(x-y) = R^{k\ell}(x-y)(\mathrm{pr}^k \otimes \mathrm{pr}^\ell). \tag{2.2}$$

2.4. **Basic properties of evaluation R-and L-operators.** We follow [17] and for details [15, Lecture 9].

The basic properties of the universal R-matrix give the quantum Yang-Baxter equation

$$\mathcal{R}^{k\ell}_{12}(x-y)\mathcal{R}^{km}_{13}(x-z)\mathcal{R}^{\ell m}_{23}(y-z) = \mathcal{R}^{\ell m}_{23}(y-z)\mathcal{R}^{km}_{13}(x-z)\mathcal{R}^{k\ell}_{12}(x-y) \tag{2.3}$$

as linear operators on $M^k \otimes M^\ell \otimes M^m$. In addition, the operator $\mathcal{R}^{k\ell}(x-y)$ satisfies unitarity:

$$\mathcal{R}^{k\ell}(x-y)^{-1} = \mathcal{R}^{\ell k}_{21}(y-x),$$

where

$$\mathcal{R}^{\ell k}_{21}(x):=\mathcal{P}^{\ell k}\mathcal{R}^{\ell k}(x)\mathcal{P}^{k\ell}: M^k\otimes M^\ell\to M^k\otimes M^\ell$$

and $\mathcal{P}^{k\ell}: M^k\otimes M^\ell\to M^\ell\otimes M^k$ is the permutation operator.

Both properties descend naturally to the L-operators and finite R-operators. In particular, the familiar RLL-relations

$$R^{k\ell}_{12}(x-y)L^{km}_{13}(x-z)L^{\ell m}_{23}(y-z)=L^{\ell m}_{23}(y-z)L^{km}_{13}(x-z)R^{k\ell}_{12}(x-y) \tag{2.4}$$

for $k,\ell\in\frac{1}{2}\mathbb{Z}_{\geq 0}$ as well as the quantum Yang-Baxter equation for the R-operators $R^{k\ell}(x)$ ($k,\ell\in\frac{1}{2}\mathbb{Z}_{\geq 0}$) follow immediately from the quantum Yang-Baxter equation for $\mathcal{R}^{k\ell}$.

The next property of $\mathcal{R}^{k\ell}(x)$ is P-symmetry:

Lemma 2.1. *As linear maps on $M^k\otimes M^\ell$ we have for generic $x\in\mathbb{C}$,*

$$\mathcal{R}^{\ell k}_{21}(x)=\mathcal{R}^{kl}(x). \tag{2.5}$$

Proof. Write $T^{k\ell}(x)$ for the left-hand side of (2.5). Then clearly

$$T^{k\ell}(x)m_1^k\otimes m_1^\ell=m_1^k\otimes m_1^\ell=\mathcal{R}^{k\ell}(x)m_1^k\otimes m_1^\ell.$$

Hence it suffices to show that for generic x and y,

$$T^{k\ell}(x-y)(\pi_x^k\otimes\pi_y^\ell)(\Delta(X))=(\pi_x^k\otimes\pi_y^\ell)(\Delta^{op}(X))T^{k\ell}(x-y),\qquad\forall\, X\in\widehat{\mathcal{U}}_{e^\eta}.$$

This is clear for $X=p^h$ ($h\in\mathfrak{h}$). For $X=e_0,f_1$ it is a direct consequence of the identity

$$(\pi_y^k\otimes\pi_x^\ell)(\Delta^{op}(e_0))=(\pi_{-y}^k\otimes\pi_{-x}^\ell)(\Delta(f_1))$$

and (2.1). For the algebraic generators $X=e_1,f_0$ it follows similarly from (2.1) using the fact that

$$(\pi_y^k\otimes\pi_x^\ell)(\Delta^{op}(e_1))=(\pi_{-y}^k\otimes\pi_{-x}^\ell)(\Delta(f_0)).$$ □

Finally we discuss crossing symmetry. We start with crossing symmetry for L-operators:

Lemma 2.2. *Let $k\in\frac{1}{2}\mathbb{Z}_{\geq 0}$ and $\ell\in\mathbb{C}$. Let $w^k: V^k\xrightarrow{\sim}V^k$ be the linear isomorphism defined by*

$$w^k(v_n^k):=c_nv^k_{2k+2-n}$$

with $c_n\in\mathbb{C}^\times$ determined by the recursion $c_{n+1}:=-c_np^{2k+1-2n}$ and $c_1:=1$. Then

$$L^{k\ell}(-x)^{T_1}=\alpha^{k\ell}(x)\alpha^{k\ell}(x-\eta)(w^k\otimes\mathrm{Id}_{M^\ell})L^{k\ell}(x-\eta)(w^k\otimes\mathrm{Id}_{M^\ell})^{-1}$$

with T_1 the transpose in the first tensor component with respect to the weight basis.

Proof. For an evaluation module (π,V) over $\widehat{\mathcal{U}}_\eta$ we write (π^*,V^*) for the graded dual V^* of V with respect to the weight grading, with $\widehat{\mathcal{U}}_\eta$-action $(\pi^*(X)\phi)(v):=\phi(\pi(S(X))v)$. If $A:V\to V$ is a linear map, then we write $A^t:V^*\to V^*$ for the corresponding dual linear operator.

It follows from the identity $(S\otimes\mathrm{Id})(R)=R^{-1}$ that

$$\big((\overline{\pi}_x^k)^*\otimes\pi_y^\ell\big)(\mathcal{R})=\big((\overline{\pi}_x^k\otimes\pi_y^\ell)(\mathcal{R}^{-1})\big)^{t_1}. \tag{2.6}$$

Here t_1 means taking the dual with respect to the first component in the tensor product. Write $\{(v_n^k)^*\}$ for the basis of $(V^k)^*$ dual to the weight basis $\{v_n^k\}_n$ of V^k. We identify $V^k\simeq(V^k)^*$ by $v_n^k\mapsto(v_n^k)^*$ (the dual A^t of a linear operator $A: V^k\to V^k$ then corresponds to the transpose A^T of A with respect to the weight basis $\{v_n^k\}$ of V^k). Accordingly we interpret the map w^k as a linear map $w^k : V^k \to (V^k)^*$, in which case it defines an isomorphism $V^k(x-\eta) \xrightarrow{\sim} V^k(x)^*$ of $\widehat{\mathcal{U}}_\eta$-modules. Consequently

$$\begin{aligned} L^{k\ell}(-x+y)^{T_1} &= \alpha^{k\ell}(x-y)\big(\overline{\pi}_x^k \otimes \pi_y^\ell)(\mathcal{R}^{-1})\big)^{T_1} \\ &= \alpha^{k\ell}(x-y)\big((\overline{\pi}_x^k)^* \otimes \pi_y^\ell\big)(\mathcal{R}) \\ &= \alpha^{k\ell}(x-y)(w^k \otimes \mathrm{Id}_{M^\ell})(\overline{\pi}_{x-\eta}^k \otimes \pi_y^\ell)(\mathcal{R})(w^k \otimes \mathrm{Id}_{M^\ell})^{-1}, \end{aligned}$$

where we have used (2.6) for the second equality. This proves the desired result. □

Remark 2.3. For $k \in \mathbb{C}$ the canonical linear isomorphism $M^k \xrightarrow{\sim} (M^k)^{**}$ defines an isomorphism $M^k(x-2\eta) \xrightarrow{\sim} M^k(x)^{**}$ of $\widehat{\mathcal{U}}_\eta$-modules (cf. Lemma 2.2). It then follows from a double application of (2.6) (for arbitrary evaluation modules) that

$$\mathcal{R}^{k\ell}(x-2\eta) = \frac{\alpha^{k\ell}(x)}{\alpha^{k\ell}(x-2\eta)}\big(((\mathcal{R}^{k\ell}(x)^{-1})^{T_1})^{-1}\big)^{T_1}.$$

Note the difference with [15, Prop. 9.5.2], which involves an additional conjugation by a diagonal operator in the first tensor component.

2.5. **Explicit formulae for L-operators.** It is possible to compute $L^{\frac12\ell}(x)$ explicitly using the expression of the universal R-matrix (a comprehensive survey of this can be found in [4]). This leads to the formulae

$$\begin{aligned} L^{\frac12\ell}(x)(v_1^{\frac12} \otimes m_n^\ell) &= \frac{\sinh(x+(\frac32+\ell-n)\eta)}{\sinh(x+(\frac12+\ell)\eta)} v_1^{\frac12} \otimes m_n^\ell \\ &\quad + \mathrm{e}^{(\ell+\frac32-n)\eta} \frac{\sinh((n-1)\eta)\sinh((2\ell+2-n)\eta)}{\sinh(\eta)\sinh(x+(\frac12+\ell)\eta)} v_2^{\frac12} \otimes m_{n-1}^\ell \end{aligned}$$

and

$$\begin{aligned} L^{\frac12\ell}(x)(v_2^{\frac12} \otimes m_n^\ell) &= \mathrm{e}^{(-\ell-\frac12+n)\eta} \frac{\sinh(\eta)}{\sinh(x+(\ell+\frac12)\eta)} v_1^{\frac12} \otimes m_{n+1}^\ell \\ &\quad + \frac{\sinh(x+(-\frac12-\ell+n)\eta)}{\sinh(x+(\frac12+\ell)\eta)} v_2^{\frac12} \otimes m_n^\ell. \end{aligned}$$

Note that exponential factors can be removed by a similarity transformation. After this, the result coincides with the L-operator found in [23]. It follows from these formulae that the finite R-operator $R^{\frac12\frac12}(x)$ is the 6-vertex R-operator:

$$R^{\frac12\frac12}(x) = \frac{1}{\sinh(x+\eta)} \begin{pmatrix} \sinh(x+\eta) & 0 & 0 & 0 \\ 0 & \sinh(x) & \sinh(\eta) & 0 \\ 0 & \sinh(\eta) & \sinh(x) & 0 \\ 0 & 0 & 0 & \sinh(x+\eta) \end{pmatrix} \tag{2.7}$$

with respect to the ordered basis $(v_1^{\frac12} \otimes v_1^{\frac12}, v_1^{\frac12} \otimes v_2^{\frac12}, v_2^{\frac12} \otimes v_1^{\frac12}, v_2^{\frac12} \otimes v_2^{\frac12})$ of $V^{\frac12} \otimes V^{\frac12}$.

The crossing symmetry of the L-operators (Lemma 2.2) becomes

$$L^{\frac12\ell}(-x)^{T_1} = \vartheta^\ell(x)\sigma_1^y L^{\frac12\ell}(x-\eta)\sigma_1^y \tag{2.8}$$

as linear operators on $V^{\frac12} \otimes M^\ell$, where T_1 is the matrix transpose with respect to the weight basis in $V^{\frac12}$ and

$$\sigma^y := \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \vartheta^\ell(x) = \frac{\sinh(x-(\frac12-\ell)\eta)}{\sinh(x-(\frac12+\ell)\eta)}. \tag{2.9}$$

Formula (2.8) can be directly verified using the above explicit formulae for $L^{\frac{1}{2}\ell}(x)$.

3. Fusion of R-operators

We use the notations from Section 2. Fix a generic $\eta \in \mathbb{C}$ throughout this section and write $p = \mathrm{e}^{\eta}$.

3.1. **Tensor products of evaluation representations.** Let $k, \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. By [5, Thm. 4.8] the tensor product $\widehat{\mathcal{U}}_\eta$-module $V^k(x) \otimes V^\ell(y)$ is irreducible for generic $x, y \in \mathbb{C}$. For the fusion of R- and K-operators we need to focus on the special cases that the $\widehat{\mathcal{U}}_\eta$-module $V^k(x) \otimes V^\ell(y)$ is reducible.

For $k, \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ we write $P^{k\ell} : V^k \otimes V^\ell \to V^\ell \otimes V^k$ for the permutation operator. The following result should be compared with [5, Prop. 4.9]. The proof is by a straightforward computation.

Proposition 3.1. *Let $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.*
(i) *The linear map $\iota^k : V^{k+\frac{1}{2}} \hookrightarrow V^{\frac{1}{2}} \otimes V^k$, defined by*

$$\iota^k\big(v_n^{k+\frac{1}{2}}\big) = \mathrm{e}^{\frac{\eta}{2}(n-1)} v_1^{\frac{1}{2}} \otimes v_n^k + \mathrm{e}^{-\frac{\eta}{2}(n-2-2k)} \frac{\sinh((n-1)\eta)}{\sinh(\eta)} v_2^{\frac{1}{2}} \otimes v_{n-1}^k,$$

defines a $\widehat{\mathcal{U}}_\eta$-intertwiner $\iota_x^k : V^{k+\frac{1}{2}}(x) \hookrightarrow V^{\frac{1}{2}}(x - k\eta) \otimes V^k(x + \frac{\eta}{2})$.
(ii) *The linear map $j^k := P^{\frac{1}{2}k}\iota^k : V^{k+\frac{1}{2}} \hookrightarrow V^k \otimes V^{\frac{1}{2}}$ defines a $\widehat{\mathcal{U}}_\eta$-intertwiner*

$$j_x^k : V^{k+\frac{1}{2}}(x) \hookrightarrow V^k(x - \frac{\eta}{2}) \otimes V^{\frac{1}{2}}(x + k\eta).$$

Note that the intertwiners ι_x^k and j_x^k do not depend on x as linear maps. We add the subscript x to clarify the $\widehat{\mathcal{U}}_\eta$-action we are considering.

3.2. **Fusion operators.** It follows from Lemma 2.1 that the R-operators $R^{k\ell}(x)$ $(k, \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0})$ are P-symmetric. In the remainder of this section we focus on the fusion of the R-operators $R^{k\ell}(x)$ $(k, \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0})$.

For the fusion of the R-operators the interpretation of R-operators as intertwiners between tensor products of evaluation modules plays a crucial role. We need explicit expressions for its action in case that the tensor product of the evaluation modules is reducible.

Lemma 3.2. *For $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ the linear operators $R^{\frac{1}{2}k}(x)$ and $R^{k\frac{1}{2}}(x)$ are regular at $x = (k + \frac{1}{2})\eta$. The resulting linear maps $S^k := P^{\frac{1}{2}k}R^{\frac{1}{2}k}\big((k + \frac{1}{2})\eta\big)$ and $T^k := P^{k\frac{1}{2}}R^{k\frac{1}{2}}\big((k + \frac{1}{2})\eta\big)$, which we will view as $\widehat{\mathcal{U}}_\eta$-intertwiners*

$$S_x^k : V^{\frac{1}{2}}(\mathrm{e}^{x+k\eta}) \otimes V^k(\mathrm{e}^{x-\frac{\eta}{2}}) \to V^k(\mathrm{e}^{x-\frac{\eta}{2}}) \otimes V^{\frac{1}{2}}(\mathrm{e}^{x+k\eta}),$$
$$T_x^k : V^k(\mathrm{e}^{x+\frac{\eta}{2}}) \otimes V^{\frac{1}{2}}(\mathrm{e}^{x-k\eta}) \to V^{\frac{1}{2}}(\mathrm{e}^{x-k\eta}) \otimes V^k(\mathrm{e}^{x+\frac{\eta}{2}})$$

are explicitly given by

$$S^k(v_1^{\frac{1}{2}} \otimes v_n^k) = \frac{\sinh((2k+2-n)\eta)}{\sinh((2k+1)\eta)} \mathrm{e}^{-\frac{\eta}{2}(n-1)} j^k(v_n^{k+\frac{1}{2}}),$$
$$S^k(v_2^{\frac{1}{2}} \otimes v_n^k) = \frac{\sinh(\eta)}{\sinh((2k+1)\eta)} \mathrm{e}^{\frac{\eta}{2}(n-2k-1)} j^k(v_{n+1}^{k+\frac{1}{2}}).$$

$$T^k(v_n^k \otimes v_1^{\frac{1}{2}}) = \frac{\sinh((2k+2-n)\eta)}{\sinh((2k+1)\eta)} \mathrm{e}^{-\frac{\eta}{2}(n-1)} \iota^k(v_n^{k+\frac{1}{2}}),$$
$$T^k(v_n^k \otimes v_2^{\frac{1}{2}}) = \frac{\sinh(\eta)}{\sinh((2k+1)\eta)} \mathrm{e}^{\frac{\eta}{2}(n-2k-1)} \iota^k(v_{n+1}^{k+\frac{1}{2}}).$$

Proof. By P-symmetry we have $R^{k\frac{1}{2}}(x) = P^{\frac{1}{2}k} R^{\frac{1}{2}k}(x) P^{k\frac{1}{2}}$, and Proposition 3.1 gives $\iota^k = P^{k\frac{1}{2}} j^k$. So it suffices to prove the statement for S^k. Using the fact that $(\mathrm{Id}_{V^{\frac{1}{2}}} \otimes \mathrm{pr}^k) L^{\frac{1}{2}k}(x) = R^{\frac{1}{2}k}(x)(\mathrm{Id}_{V^{\frac{1}{2}}} \otimes \mathrm{pr}^k)$, Remark 2.5 gives explicit formulae for S^k. Comparing those formulae with the explicit formulae for j_x^k (see Proposition 3.1) now leads to the desired result. □

3.3. **The fusion formula for the R- and L-operators.** The fusion formulae for the R-operators $R^{k\ell}(x)$ ($k, \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$) and L-operators $L^{k\ell}(x)$ ($k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, $\ell \in \mathbb{C}$) follow directly from the representation-theoretic considerations of the previous subsection. Recall the linear map $\iota^k : V^{k+\frac{1}{2}} \hookrightarrow V^{\frac{1}{2}} \otimes V^k$ from Proposition 3.1.

Proposition 3.3. *For $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{C}$ we have the fusion formula*

$$(\iota^k \otimes \mathrm{Id}_{M^\ell}) L^{k+\frac{1}{2},\ell}(x-y) = L_{13}^{\frac{1}{2}\ell}(x-k\eta-y) L_{23}^{k\ell}(x+\frac{\eta}{2}-y)(\iota^k \otimes \mathrm{Id}_{M^\ell})$$

as linear maps $V^{k+\frac{1}{2}} \otimes M^\ell \to V^{\frac{1}{2}} \otimes V^k \otimes M^\ell$.

Proof. Using the fact that

$$\big(\overline{\pi}_x^{\frac{1}{2}} \otimes \overline{\pi}_y^k \otimes \pi_z^\ell\big)(\mathcal{R}_{13}\mathcal{R}_{23}) = \big(\overline{\pi}_x^{\frac{1}{2}} \otimes \overline{\pi}_y^k \otimes \pi_z^\ell\big)((\Delta \otimes \mathrm{Id})(\mathcal{R}))$$

and the intertwining property of ι_x^k (see Proposition 3.1), gives

$$L_{13}^{\frac{1}{2}\ell}(x-k\eta-y) L_{23}^{k\ell}(x+\frac{\eta}{2}-y)(\iota_x^k \otimes \mathrm{Id}_{M^\ell}) = (\iota_x^k \otimes \mathrm{Id}_{M^\ell}) L^{k+\frac{1}{2},\ell}(x-y)$$

as linear maps $V^{k+\frac{1}{2}}(x) \otimes M^\ell(y) \to V^{\frac{1}{2}}(x-k\eta) \otimes V^k(x+\frac{\eta}{2}) \otimes M^\ell(y)$. The result follows now immediately. □

Remark 3.4. Proposition 3.3 leads for $k, \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ to the fusion formula

$$(\iota^k \otimes \mathrm{Id}_{V^\ell}) R^{k+\frac{1}{2},\ell}(x-y) = R_{13}^{\frac{1}{2}\ell}(x-k\eta-y) R_{23}^{k\ell}(x+\frac{\eta}{2}-y)(\iota^k \otimes \mathrm{Id}_{V^\ell})$$

for the R-operators.

Remark 3.5. Another approach to fusion formulae for L-operators (originating from [24]) is by specialization of the RLL relations (2.4) at values of $x-y$ for which $R_{12}^{k\ell}(x-y)$ is not invertible. For instance, in the present setting (2.4) gives

$$\begin{aligned}(T^k \otimes \mathrm{Id}_{M^\ell}) L_{13}^{k\ell}(x+\frac{\eta}{2}-y) L_{23}^{\frac{1}{2}\ell}(x-k\eta-y) = \\ = L_{13}^{\frac{1}{2}\ell}(x-k\eta-y) L_{23}^{k\ell}(x+\frac{\eta}{2}-y)(T^k \otimes \mathrm{Id}_{M^\ell}),\end{aligned}$$

which shows directly that the operator $L_{13}^{\frac{1}{2}\ell}(x-k\eta-y)L_{23}^{k\ell}(x+\frac{\eta}{2}-y)$ restricts to a linear endomorphism on the image of $T^k \otimes \mathrm{Id}_{M^\ell}$. The resulting linear operator is equivalent to the fused L-operator $L^{k+\frac{1}{2},\ell}(x-y)$ in view of Lemma 3.2.

4. The reflection equation, fusion of K-operators and diagonal K-operators

4.1. **Reflection equations.** A collection of linear maps $\mathcal{K}^\ell(x) : M^\ell \to M^\ell$ is called a family of higher-spin K-operators if they satisfy the reflection equations in $M^k \otimes M^\ell$:

$$\mathcal{R}^{k\ell}(x-y)\mathcal{K}_1^k(x)\mathcal{R}^{k\ell}(x+y)\mathcal{K}_2^\ell(y) = \mathcal{K}_2^\ell(y)\mathcal{R}^{k\ell}(x+y)\mathcal{K}_1^k(x)\mathcal{R}^{k\ell}(x-y). \tag{4.1}$$

Remark 4.1. The natural representation-theoretic forms of the reflection equations (4.1) involve $\mathcal{R}_{21}^{\ell k}(x) = \mathcal{P}^{\ell k}\mathcal{R}^{\ell k}(x)\mathcal{P}^{k\ell}$, cf. (1.2). However, the P-symmetry (2.5) of the R-operators has the simplifying effect that all R-operators can be put into the form $\mathcal{R}^{k\ell}$ and consequently the distinction between left and right versions of reflection equations disappears (cf. [34]).

Suppose that for $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ there exists a (necessarily unique) linear map $K^k(x) : V^k \to V^k$ such that

$$\mathrm{pr}^k \circ \mathcal{K}^k(x) = K^k(x) \circ \mathrm{pr}^k.$$

Then the equations (4.1) naturally give rise to (semi-)finite-dimensional versions which will also be referred to as reflection equations. More precisely, when $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ equation (4.1) projects to the following equation in $V^k \otimes M^\ell$:

$$L^{k\ell}(x-y)K_1^k(x)L^{k\ell}(x+y)\mathcal{K}_2^\ell(y) = \mathcal{K}_2^\ell(y)L^{k\ell}(x+y)K_1^k(x)L^{k\ell}(x-y); \tag{4.2}$$

Furthermore, when $k, l \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ equation (4.1) then projects to the following equation in $V^k \otimes V^\ell$:

$$R^{k\ell}(x-y)K_1^k(x)R^{k\ell}(x+y)K_2^\ell(y) = K_2^\ell(y)R^{k\ell}(x+y)K_1^k(x)R^{k\ell}(x-y). \tag{4.3}$$

Just as solutions to the quantum Yang-Baxter equation are related to the representation theory of quantized universal enveloping algebras, solutions to the reflection equation (K-operators) are related to co-ideal subalgebras of quantized universal enveloping algebras. We will discuss it briefly in Subsection 4.4.

4.2. **K-matrices for spin-$\frac{1}{2}$.** With respect to the 6-vertex R-operator $R^{\frac{1}{2}\frac{1}{2}}(x)$ (see (2.7)), the general diagonal solution of (4.3) (for $k = \ell = \frac{1}{2}$) is given by Cherednik's [7] one-parameter family

$$K^{\xi,\frac{1}{2}}(x) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\sinh(\xi - x)}{\sinh(\xi + x)} \end{pmatrix}$$

written with respect to the basis $(v_1^{\frac{1}{2}}, v_2^{\frac{1}{2}})$ of $V^{\frac{1}{2}}$. To simplify notations we will use $R(x)$ for $R^{\frac{1}{2}\frac{1}{2}}(x)$ and $K^\xi(x)$ for $K^{\xi,\frac{1}{2}}(x)$. In other words, this matrix acts on the weight basis as

$$K^\xi(x)v_1^{\frac{1}{2}} = v_1^{\frac{1}{2}}, \qquad K^\xi(x)v_2^{\frac{1}{2}} = \frac{\sinh(\xi - x)}{\sinh(\xi + x)}v_2^{\frac{1}{2}}.$$

Remark 4.2. The proof that $K^\xi(x)$ satisfies (4.3) for $k = \ell = 1/2$ reduces to the identity

$$\sum_{\epsilon_1,\epsilon_2 \in \{\pm 1\}} \epsilon_1\epsilon_2 \frac{\sinh(\xi + \epsilon_1 x)\sinh(\xi + \epsilon_2 y)}{\sinh(\epsilon_1 x + \epsilon_2 y)} = 0$$

cf. [32].

The reflection operator $K^\xi(x)$ satisfies the boundary crossing symmetry:

$$\mathrm{Tr}_2\Big(R_{12}(2x-2\eta)P_{12}K_2^\xi(x)\Big) = \frac{\sinh(\xi+x-\eta)\sinh(2x)}{\sinh(\xi+x)\sinh(2x-\eta)}K_1^\xi(x-\eta), \tag{4.4}$$

where Tr_2 is the partial trace over the second tensor component of $V^{\frac12}\otimes V^{\frac12}$ and $P = P^{\frac12\frac12}$. The identity (4.4) is equivalent to the trigonometric identity

$$\sinh(\xi+x)\sinh(x-z)+\sinh(\xi-x)\sinh(x+z) = \sinh(\xi-z)\sinh(2x). \tag{4.5}$$

In Lemma 7.8 we prove a multivariate extension of (4.5), which plays an important role in the proof of the main result (Theorem 6.2).

A three-parameter family of solutions $K^{\frac12}(x)$ of (4.3) (with $k=\ell=\frac12$) is known, see [12, 29].

4.3. **Fusion formula for K-operators when $k,\ell\in\frac12\mathbb{Z}_{\geq 0}$.** Notwithstanding Remark 4.1, in order to put formulas in the natural representation-theoretic form, we will sometimes use the notation $R_{21}^{\ell k}(x)$. The intertwining property of the R-operator $R^{k\ell}(x)$ gives

$$R_{21}^{k\ell}(x-y)(\pi_{-x}^\ell\otimes\pi_{-y}^k)(\Delta^{op}(X)) = (\pi_{-x}^\ell\otimes\pi_{-y}^k)(\Delta(X))R_{21}^{k\ell}(x-y), \quad \forall\, X\in\widehat{\mathcal{U}}_\eta.$$

Proposition 4.3. *Suppose that the $K^{\frac12}(x)$ are complex-linear operators on $V^{\frac12}$ depending meromorphically on $x\in\mathbb{C}$ and satisfying the reflection equation*

$$R_{21}^{\frac12\frac12}(x-y)K_1^{\frac12}(x)R^{\frac12\frac12}(x+y)K_2^{\frac12}(y) = K_2^{\frac12}(y)R_{21}^{\frac12\frac12}(x+y)K_1^{\frac12}(x)R^{\frac12\frac12}(x-y) \tag{4.6}$$

as linear operators on $V^{\frac12}\otimes V^{\frac12}$. Then there exist unique complex-linear operators $K^k(x)$ on V^k for $k\in\frac12\mathbb{Z}_{\geq 2}$ satisfying

$$j^kK^{k+\frac12}(x) = P^{\frac12 k}K_1^{\frac12}(x-k\eta)R^{\frac12 k}(2x-(k-\frac12)\eta)K_2^k(x+\frac{\eta}{2})\iota^k \tag{4.7}$$

for all $k\in\frac12\mathbb{Z}_{\geq 2}$. Furthermore,

$$R_{21}^{\ell k}(x-y)K_1^k(x)R^{k\ell}(x+y)K_2^\ell(y) = K_2^\ell(y)R_{21}^{\ell k}(x+y)K_1^k(x)R^{k\ell}(x-y) \tag{4.8}$$

as linear operators on $V^k\otimes V^\ell$ for all $k,\ell\in\frac12\mathbb{Z}_{>0}$.

Remark 4.4. We will always set $K^0(x) := \mathrm{Id}_{V^0}$. Then formulae (4.7) and (4.8) are trivially satisfied for $k=0$ and/or $\ell=0$.

Remark 4.5. Fusion of K-operators has been studied before in various different contexts, see, e.g., [13, 25, 21, 28, 26, 27, 38].

Proof of Proposition 4.3. Let $m\in\frac12\mathbb{Z}_{\geq 0}$ and suppose that the K-operators $K^k(x)$ have been constructed for $k\leq m$ satisfying (4.7) for $k<m$ and satisfying (4.8) for $k,l\leq m$.

Consider (4.8) for $\ell=\frac12$ and $k=m$, and replace x by $x+\frac{\eta}{2}$ and y by $x-m\eta$. Then we obtain

$$\begin{aligned}S^mK_2^m(x+\frac{\eta}{2})\check{R}^{m\frac12}(2x-(m-\frac12)\eta)K_2^{\frac12}(x-m\eta) =\\ =P^{\frac12 m}K_1^{\frac12}(x-m\eta)R^{\frac12 m}(2x-(m-\frac12)\eta)K_2^m(x+\frac{\eta}{2})T^m\end{aligned}$$

with $\check{R}^{k\ell}(x) := P^{k\ell}R^{k\ell}(x)$ (see Lemma 3.2 for the definition of S^m and T^m). Since the images of the linear maps T^m and ι^m coincide by Lemma 3.2, it follows that the image of the linear map

$$P^{\frac{1}{2}m}K_1^{\frac{1}{2}}(x-m\eta)R^{\frac{1}{2}m}(2x-(m-\frac{1}{2})\eta)K_2^m(x+\frac{\eta}{2})\iota^m$$

is contained in the image of S^m. By Lemma 3.2 again, the image of S^m coincides with the image of j^m, hence there exists a unique linear operator $K^{m+\frac{1}{2}}(x)$ on $V^{m+\frac{1}{2}}$ such that

$$j^m K^{m+\frac{1}{2}}(x) = P^{\frac{1}{2}m}K_1^{\frac{1}{2}}(x-m\eta)R^{\frac{1}{2}m}(2x-(m-\frac{1}{2})\eta)K_2^m(x+\frac{\eta}{2})\iota^m.$$

It remains to show that (4.8) is valid for $k,\ell \in \frac{1}{2}\mathbb{Z}_{\leq 0}$ and $k,\ell \leq m+\frac{1}{2}$. It suffices to consider the case that $k = m+\frac{1}{2}$ and/or $\ell = m+\frac{1}{2}$. We divide it into the following three cases:

(1) $(k,\ell) = (m+\frac{1}{2},\ell)$ with $\ell \leq m$.
(2) $(k,\ell) = (k,m+\frac{1}{2})$ with $k \leq m$.
(3) $(k,\ell) = (m+\frac{1}{2},m+\frac{1}{2})$.

If the reflection equation (4.8) is proved for case (1), then (2) follows from (1) using the unitarity of the R-operator, and (3) follows from (1) and (2) by taking $\ell = m+\frac{1}{2}$ in the following proof of (1).

Proof of (1): Suppose $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $\ell \leq m$. Using the fusion formulae of the R- and K-operators we obtain

$$\begin{aligned}
&R_{21}^{\ell,m+\frac{1}{2}}(x-y)K_1^{m+\frac{1}{2}}(x)R^{m+\frac{1}{2},\ell}(x+y)K_2^{\ell}(y) = \\
&\quad = (\iota^m\otimes \mathrm{Id}_{V^\ell})^{-1}R_{31}^{\ell\frac{1}{2}}(x-m\eta-y)R_{32}^{\ell m}(x+\frac{\eta}{2}-y)(\iota^m\otimes \mathrm{Id}_{V^\ell}) \\
&\quad \times (j^m\otimes \mathrm{Id}_{V^\ell})^{-1}P_{12}^{\frac{1}{2}m}K_1^{\frac{1}{2}}(x-m\eta)R_{12}^{\frac{1}{2}m}(2x-(m-\frac{1}{2})\eta)K_2^m(x+\frac{\eta}{2}) \\
&\quad \times R_{13}^{\frac{1}{2}\ell}(x-m\eta+y)R_{23}^{m\ell}(x+\frac{\eta}{2}+y)K_3(y)(\iota^m\otimes \mathrm{Id}_{V^\ell}),
\end{aligned}$$

where the sublabels $1,2,3$ in the right-hand side stand for the first, second and third tensor component in $V^{\frac{1}{2}}\otimes V^m\otimes V^\ell$ and the sublabels $1,2$ in the left-hand side stand for the first and second tensor component in $V^{m+\frac{1}{2}}\otimes V^\ell$. Using $P^{m\frac{1}{2}}j^m = \iota^m$ the expression simplifies to

$$\begin{aligned}
&(\iota^m\otimes \mathrm{Id}_{V^\ell})^{-1}R_{31}^{\ell\frac{1}{2}}(x-m\eta-y)K_1^{\frac{1}{2}}(x-m\eta) \\
&\qquad \times R_{32}^{\ell m}(x+\frac{\eta}{2}-y)R_{12}^{\frac{1}{2}m}(2x-(m-\frac{1}{2})\eta)R_{13}^{\frac{1}{2}\ell}(x-m\eta+y) \\
&\qquad \times K_2^m(x+\frac{\eta}{2})R_{23}^{m\ell}(x+\frac{\eta}{2}+y)K_3^{\ell}(y)(\iota^m\otimes \mathrm{Id}_{V^\ell}).
\end{aligned}$$

Using the quantum Yang-Baxter equation in the second line the expression can be rewritten as

$$\begin{aligned}
&(\iota^m\otimes \mathrm{Id}_{V^\ell})^{-1}R_{31}^{\ell\frac{1}{2}}(x-m\eta-y)K_1^{\frac{1}{2}}(x-m\eta) \\
&\qquad \times R_{13}^{\frac{1}{2}\ell}(x-m\eta+y)R_{12}^{\frac{1}{2}m}(2x-(m-\frac{1}{2})\eta) \\
&\qquad \times R_{32}^{\ell m}(x+\frac{\eta}{2}-y)K_2(x+\frac{\eta}{2})R_{23}^{m\ell}(x+\frac{\eta}{2}+y)K_3^{\ell}(y)(\iota^m\otimes \mathrm{Id}_{V^\ell}).
\end{aligned}$$

Applying the reflection equation to the last line leads to the expression

$$\begin{aligned}&(\iota^m\otimes \mathrm{Id}_{V^\ell})^{-1}R_{31}^{\ell\frac{1}{2}}(x-m\eta-y)K_1^{\frac{1}{2}}(x-m\eta)R_{13}^{\frac{1}{2}\ell}(x-m\eta+y)K_3^\ell(y)\\ &\times R_{12}^{\frac{1}{2}m}(2x-(m-\frac{1}{2})\eta)R_{32}^{\ell m}(x+\frac{\eta}{2}+y)\\ &\times K_2^m(x+\frac{\eta}{2})R_{23}^{m\ell}(x+\frac{\eta}{2}-y)(\iota^m\otimes \mathrm{Id}_{V^\ell}).\end{aligned}$$

Now applying the reflection equation to the first line gives

$$\begin{aligned}&(\iota^m\otimes \mathrm{Id}_{V^\ell})^{-1}K_3^\ell(y)R_{31}^{\ell\frac{1}{2}}(x-m\eta+y)K_1^{\frac{1}{2}}(x-m\eta)\\ &\qquad\times R_{13}^{\frac{1}{2}\ell}(x-m\eta-y)R_{12}^{\frac{1}{2}m}(2x-(m-\frac{1}{2})\eta)R_{32}^{\ell m}(x+\frac{\eta}{2}+y)\\ &\qquad\times K_2^m(x+\frac{\eta}{2})R_{23}^{m\ell}(x+\frac{\eta}{2}-y)(\iota^m\otimes \mathrm{Id}_{V^\ell}).\end{aligned}$$

Applying the quantum Yang-Baxter equation to the second line leads to

$$\begin{aligned}&(\iota^m\otimes \mathrm{Id}_{V^\ell})^{-1}K_3^\ell(y)R_{31}^{\ell\frac{1}{2}}(x-m\eta+y)R_{32}^{\ell m}(x+\frac{\eta}{2}+y)\\ &\qquad\times K_1^{\frac{1}{2}}(x-m\eta)R_{12}^{\frac{1}{2}m}(2x-(m-\frac{1}{2})\eta)K_2^m(x+\frac{\eta}{2})\\ &\qquad\times R_{13}^{\frac{1}{2}\ell}(x-m\eta-y)R_{23}^{m\ell}(x+\frac{\eta}{2}-y)(\iota^m\otimes \mathrm{Id}_{V^\ell}).\end{aligned}$$

The fusion formulae for the R- and K-operators and the fact that $P^{m\frac{1}{2}}j^m=\iota^m$ show that the last expression equals

$$K_2^\ell(y)R_{21}^{\ell,m+\frac{1}{2}}(x+y)K_1^{m+\frac{1}{2}}(x)R_{12}^{m+\frac{1}{2},\ell}(x-y),$$

where the sublabels 1 and 2 stand for the first and second tensor component in $V^{m+\frac{1}{2}}\otimes V^\ell$. This completes the proof of the reflection equation for case (1). □

4.4. **Reflection equation and coideal subalgebras.** Here we briefly discuss the representation-theoretical meaning of reflection equations, cf., e.g., [9, 10, 8]. Let $\mathcal{A}\subseteq\widehat{\mathcal{U}}_\eta$ be a left coideal subalgebra, i.e. it is a unital subalgebra of $\widehat{\mathcal{U}}_\eta$ satisfying $\Delta(\mathcal{A})\subseteq\widehat{\mathcal{U}}_\eta\otimes\mathcal{A}$. If M is a $\widehat{\mathcal{U}}_\eta$-module, we write $M|_{\mathcal{A}}$ for the $\mathcal{A}$-module obtained by restricting the action of $\widehat{\mathcal{U}}_\eta$ on M to $\mathcal{A}$.

Suppose that for $k,\ell\in\frac{1}{2}\mathbb{Z}_{>0}$ we have $\mathcal{A}$-intertwiners

$$K^k(x):V^k(x)|_{\mathcal{A}}\to V^k(-x)|_{\mathcal{A}},\qquad K^\ell(x):V^\ell(x)|_{\mathcal{A}}\to V^\ell(-x)|_{\mathcal{A}}. \tag{4.9}$$

Then the left and right sides of the reflection equation (4.8) are $\mathcal{A}$-intertwiners $\big(V^k(x)\otimes V^\ell(y)\big)|_{\mathcal{A}}\to\big(V^k(-x)\otimes V^\ell(-y)\big)|_{\mathcal{A}}$. Consequently, if $\big(V^k(x)\otimes V^\ell(y)\big)|_{\mathcal{A}}$ is an irreducible $\mathcal{A}$-module for generic x and y, then Schur's lemma implies the reflection equation (4.8) up to a constant. Such examples of K-operators have been constructed with $\mathcal{A}$ the q-Onsager algebra, cf., e.g., [8, 9, 10, 11].

The fusion formula (4.7) is compatible with this representation-theoretic perspective in the following sense. Assume that $K^{\frac{1}{2}}(x):V^{\frac{1}{2}}(x)|_{\mathcal{A}}\to V^{\frac{1}{2}}(-x)|_{\mathcal{A}}$ and $K^k(x):V^k(x)|_{\mathcal{A}}\to V^k(-x)|_{\mathcal{A}}$ are $\mathcal{A}$-intertwiners. Then the right-hand side of (4.7), which can be written as

$$K_2^{\frac{1}{2}}(x-k\eta)\check{R}^{\frac{1}{2}k}(2x-(k-\frac{1}{2})\eta))K_2^k(x+\frac{\eta}{2})\iota_x^k$$

with $\check{R}^{k\ell}(x) := P^{k\ell}R^{k\ell}(x)$, is an $\mathcal{A}$-intertwiner

$$V^{k+\frac{1}{2}}(x)|_{\mathcal{A}} \to \big(V^k(-x-\frac{\eta}{2}) \otimes V^{\frac{1}{2}}(-x+k\eta)\big)|_{\mathcal{A}}.$$

It follows that the corresponding fused K-operator $K^{k+\frac{1}{2}}(x) : V^{k+\frac{1}{2}} \to V^{k+\frac{1}{2}}$, characterized by

$$j^k_{-x}K^{k+\frac{1}{2}}(x) = K^{\frac{1}{2}}_2(x-k\eta)\check{R}^{\frac{1}{2}k}(2x-(k-\frac{1}{2})\eta))K^k_2(x+\frac{\eta}{2})\iota^k_x,$$

becomes an intertwiner

$$K^{k+\frac{1}{2}}(x) : V^{k+\frac{1}{2}}(x)|_{\mathcal{A}} \to V^{k+\frac{1}{2}}(-x)|_{\mathcal{A}}$$

of $\mathcal{A}$-modules.

4.5. **Diagonal K-operators.**

Proposition 4.6. *The K-operator $K^{\xi,\ell}(x) : V^\ell \to V^\ell$ ($\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$) obtained by recursively fusing $K^\xi(x) = K^{\xi,\frac{1}{2}}(x)$ using* (4.7) *acts on the weight basis as*

$$K^{\xi,\ell}(x)v^\ell_n = C^\ell_n(x;\xi)v^\ell_n, \qquad 1 \leq n \leq 2\ell+1, \tag{4.10}$$

where

$$C^\ell_n(x;\xi) := \prod_{j=1}^{n-1}\frac{\sinh(\xi-x+(\ell+\frac{1}{2}-j)\eta)}{\sinh(\xi+x+(\ell+\frac{1}{2}-j)\eta)} \tag{4.11}$$

for $n \in \mathbb{Z}_{>1}$ and $C^\ell_1(x;\xi) = 1$.

Remark 4.7. The K-operators $K^{\xi,\ell}(x)$ coincide with an appropriate limit of the explicit $\mathcal{A}$-intertwiner $V^\ell(x)|_{\mathcal{A}} \to V^\ell(-x)|_{\mathcal{A}}$ for the q-Onsager coideal subalgebra $\mathcal{A} \subset \widehat{\mathcal{U}}_\eta$ derived in [11]. This is to be expected from the representation-theoretic context of the fusion procedure of K-operators, cf. Section 4.4.

Proof of Proposition 4.6. By induction with respect to ℓ. By the fusion formula (4.7) for K-operators it suffices to show that

$$C^{\ell+\frac{1}{2}}_n(x;\xi)j^\ell(v^{\ell+\frac{1}{2}}_n) = P^{\frac{1}{2}\ell}K^{\xi,\frac{1}{2}}_1(x-\ell\eta)R^{\frac{1}{2}\ell}(2x-(\ell-\frac{1}{2})\eta)K^{\xi,\ell}_2(x+\frac{\eta}{2})\iota^\ell(v^{\ell+\frac{1}{2}}_n) \tag{4.12}$$

with $K^{\xi,\ell}(x)$ satisfying (4.10). Both sides can be computed using the the explicit actions of the maps on the standard bases. It follows that the desired identity (4.12) is equivalent to

$$\begin{aligned}C^{\ell+\frac{1}{2}}_n(x;\xi) &= \frac{\sinh(2x+(2-n)\eta)}{\sinh(2x+\eta)}C^\ell_n(x+\frac{\eta}{2};\xi)+\\ &\quad+\frac{\sinh((n-1)\eta)}{\sinh(2x+\eta)}C^\ell_{n-1}(x+\frac{\eta}{2};\xi),\\ C^{\ell+\frac{1}{2}}_n(x;\xi) &= \frac{\sinh(\xi-x+\ell\eta)}{\sinh(\xi+x-\ell\eta)}\bigg(\frac{\sinh((2\ell+2-n)\eta)}{\sinh(2x+\eta)}C^\ell_n(x+\frac{\eta}{2};\xi)+\\ &\qquad+\frac{\sinh(2x+(n-1-2\ell)\eta)}{\sinh(2x+\eta)}C^\ell_{n-1}(x+\frac{\eta}{2};\xi)\bigg)\end{aligned}$$

for $1 \leq n \leq 2\ell+1$. These follow easily from the trigonometric identity (4.5). □

Definition 4.8. *For $\ell \in \mathbb{C}$ define the linear operator $\mathcal{K}^{\xi,\ell}(x)$ on M^ℓ by*

$$\mathcal{K}^{\xi,\ell}(x) m_n^\ell = C_n^\ell(x;\xi) m_n^\ell, \qquad n \geq 1.$$

Here functions $C_n^\ell(x;\xi)$ are defined in (4.11).

Note that if $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $\mathrm{pr}^k : M^k \to V^k$ is the projection from the Verma module to the corresponding finite-dimensional irreducible quotient V^k, then

$$\mathrm{pr}^k \circ \mathcal{K}^{\xi,k}(x) = K^{\xi,k}(x) \circ \mathrm{pr}^k, \tag{4.13}$$

where $K^{\xi,k}(x) : V^k \to V^k$ is the K-operator obtained by fusion in the previous subsection.

Proposition 4.9. *Let $\xi \in \mathbb{C}$ then the operators $\mathcal{K}^{\xi,k}(x)$ satisfy the reflection equation:*

$$\mathcal{R}^{k\ell}(x-y)\mathcal{K}_1^{\xi,k}(x)\mathcal{R}^{k\ell}(x+y)\mathcal{K}_2^{\xi,\ell}(y) = \mathcal{K}_2^{\xi,\ell}(y)\mathcal{R}^{k\ell}(x+y)\mathcal{K}_1^{\xi,k}(x)\mathcal{R}^{k\ell}(x-y) \tag{4.14}$$

for all $k, \ell \in \mathbb{C}$.

Remark 4.10. From the observations in Subsection 4.1 it follows from Proposition 4.9 that for $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{C}$, the K-operators $K^{\xi,k}(x)$ and $\mathcal{K}^{\xi,\ell}(x)$ satisfy (4.2).

Proof of Proposition 4.9. For $k, \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ denote by $d_{n,r;s}^{k,\ell}(\mathrm{e}^x)$ the matrix elements of $R^{k\ell}(x)$ in the weight basis:

$$R^{k\ell}(x) v_n^k \otimes v_r^\ell = \sum_s d_{n,r;s}^{k,\ell}(\mathrm{e}^x) v_{n-s}^k \otimes v_{r+s}^\ell \tag{4.15}$$

for $1 \leq n \leq 2k+1$, $1 \leq r \leq 2\ell+1$ and $s \in \mathbb{Z}$ such that $1 \leq n-s \leq 2k+1$ and $1 \leq r+s \leq 2\ell+1$. Similarly, we write for $k, \ell \in \mathbb{C}$

$$\mathcal{R}^{k\ell}(x) m_n^k \otimes m_r^\ell = \sum_s c_{n,r;s}(\mathrm{e}^x; p^{2k}, p^{2\ell}) m_{n-s}^k \otimes m_{r+s}^\ell, \qquad n, r \in \mathbb{Z}_{>0} \tag{4.16}$$

with the sum running over the integers s such that $n-s, r+s \geq 1$. The coefficients $c_{n,r;s}(\mathrm{e}^x; p^{2k}, p^{2\ell})$ are rational functions in e^x, p^{2k} and $p^{2\ell}$.

Let $n, r \in \mathbb{Z}_{>0}$ satisfying $n-s, r+s \in \mathbb{Z}_{>0}$. Then we have

$$c_{n,r;s}(\mathrm{e}^x; \mathrm{e}^{2\eta k}, \mathrm{e}^{2\eta\ell}) = d_{n,r;s}^{k,\ell}(\mathrm{e}^x) \tag{4.17}$$

for sufficiently large $k, \ell \in \frac{1}{2}\mathbb{Z}_{>0}$ by (2.2).

Note furthermore that the dependence of $C_n^k(x;\xi)$ on k is by a rational dependence on p^{2k}. To emphasize it, we write $C_n(x;\xi;p^{2k}) := C_n^k(x;\xi)$ for the remainder of the proof.

The equation (4.14) we want to prove is equivalent to the following identities: for all $n, r \in \mathbb{Z}_{>0}$ and $t \in \mathbb{Z}$ satisfying $1-r \leq t \leq n-1$,

$$\begin{aligned}\sum_{s=1-r}^{n-1} & c_{n-s,r+s;t-s}(\mathrm{e}^{x-y}; p^{2k}, p^{2\ell}) C_{n-s}(x;\xi;p^{2k}) \\ & \times c_{n,r;s}(\mathrm{e}^{x+y}; p^{2k}, p^{2\ell}) C_n(y;\xi;p^{2\ell}) = \\ = \sum_{s=1-r}^{n-1} & C_{r+t}(y;\xi;p^{2\ell}) c_{n-s,r+s;t-s}(\mathrm{e}^{x+y}; p^{2k}, p^{2\ell}) \\ & \times C_{n-s}(x;\xi;p^{2k}) c_{n,r;s}(\mathrm{e}^{x-y}; p^{2k}, p^{2\ell}).\end{aligned}$$

Since these identities depend rationally on p^{2k} and $p^{2\ell}$, it suffices to prove them for $k,\ell\in\frac{1}{2}\mathbb{Z}_{\geq 0}$ sufficiently large. But then they follow from (4.17) and the "finite" reflection equations

$$R^{k\ell}(x-y)K_1^{\xi,k}(x)R^{k\ell}(x_y)K_2^{\xi,\ell}(y)=K_2^{\xi,\ell}(y)R^{k\ell}(x+y)K_1^{\xi,k}(x)R^{k\ell}(x-y)$$

for $k,\ell\in\frac{1}{2}\mathbb{Z}_{\geq 0}$. □

5. Boundary monodromy operators and Bethe vectors

5.1. **Monodromy matrices.** In order to formulate our (Jackson integral) solutions to the boundary qKZ equations in $M^{\underline{\ell}}=M^{\ell_1}\otimes\cdots\otimes M^{\ell_N}$ we need to introduce (off-shell) Bethe vectors for the reflecting chain, which in turn are defined using boundary monodromy operators. Boundary monodromy operators are linear operators acting on the extended tensor product $V^{\frac{1}{2}}\otimes M^{\underline{\ell}}$; the component $V^{\frac{1}{2}}$ is called auxiliary space and the component $M^{\underline{\ell}}$ state space. From now on we restrict our attention to the case that the K-matrices are diagonal (cf. Subsection 4.5).

The definition of the boundary monodromy operators involves the L-operators

$$L^{\ell}(x):=L^{\frac{1}{2}\ell}(x):V^{\frac{1}{2}}\otimes M^{\ell}\to V^{\frac{1}{2}}\otimes M^{\ell}$$

for $\ell\in\mathbb{C}$. They provide the link between the integrable structure on the auxiliary space and the integrable structure on the state space and satisfy the RLL commutation relations (2.4) (with $k=\ell=\frac{1}{2}$ and $R^{\frac{1}{2}\frac{1}{2}}(x)$ the 6-vertex R-operator) as well as the "mixed" reflection equations (4.2) (with $k=\frac{1}{2}$, $K^{\frac{1}{2}}(x)=K^{\xi}(x)$ and $\mathcal{K}^{\ell}(x)=\mathcal{K}^{\xi,\ell}(x)$). In addition,

$$L^k(x)L^{\ell}(x+y)\mathcal{R}^{k\ell}(y)=\mathcal{R}^{k\ell}(x)L^{\ell}(x+y)L^k(x) \tag{5.1}$$

as linear operators on $V^{\frac{1}{2}}\otimes M^k\otimes M^{\ell}$. The L-operators $L^{\ell}(x)$, together with the integrable data $K^{\xi}(x)$ and $R(x)$ on the auxiliary space, define an integrable quantum spin chain with diagonal reflecting ends (see [34]). It is the inhomogeneous Heisenberg XXZ spin chain with continuous spins.

Let S_N be the symmetric group in N letters. For $\sigma\in S_N$ define the linear operator $T_\sigma(x;\mathbf{t})=T_\sigma^{\underline{\ell}}(x;\mathbf{t})$ on $V^{\frac{1}{2}}\otimes M^{\underline{\ell}}$ by

$$\begin{aligned}T_\sigma(x;\mathbf{t}):=&L^{\ell_{\sigma(1)}}(x-t_{\sigma(1)})\cdots L^{\ell_{\sigma(N)}}(x-t_{\sigma(N)})\\=&\begin{pmatrix}A_\sigma(x;\mathbf{t}) & B_\sigma(x;\mathbf{t})\\ C_\sigma(x;\mathbf{t}) & D_\sigma(x;\mathbf{t})\end{pmatrix},\end{aligned} \tag{5.2}$$

where in the last equality we have written $T_\sigma(x;\mathbf{t})$ as a $\mathrm{End}(M^{\underline{\ell}})$-valued matrix with respect to the ordered basis $(v_1^{\frac{1}{2}},v_2^{\frac{1}{2}})$ of $V^{\frac{1}{2}}$. The special case $T(x;\mathbf{t}):=T_e(x;\mathbf{t})$ with $e\in S_N$ the neutral element is the (A-type) monodomy operator. We write the corresponding matrix coefficients as $A(x;\mathbf{t})=A_e(x;\mathbf{t}),\ldots,D(x;\mathbf{t})=D_e(x;\mathbf{t})$.

The operators $T_\sigma(x;\mathbf{t})$ satisfy the commutation relations

$$R_{00'}(x-y)T_{\sigma,0}(x;\mathbf{t})T_{\sigma,0'}(y;\mathbf{t})=T_{\sigma,0'}(y;\mathbf{t})T_{\sigma,0}(x;\mathbf{t})R_{00'}(x-y) \tag{5.3}$$

as linear operators on $V^{\frac{1}{2}}\otimes V^{\frac{1}{2}}\otimes M^{\underline{\ell}}$, where $T_{\sigma,0}(x;\mathbf{t})$ is the operator $T_\sigma(x;\mathbf{t})$ acting on the first and third tensor leg and $T_{\sigma,0'}(y;\mathbf{t})$ the operator $T_\sigma(y;\mathbf{t})$ on the second and third tensor leg, while $R_{00'}(x-y)$ is the action of $R(x-y)$ on the tensor product $V^{\frac{1}{2}}\otimes V^{\frac{1}{2}}$ of the auxiliary spaces only.

Similarly, for $\sigma \in S_N$ we define $\mathcal{U}^\xi_\sigma(x;\mathbf{t})=\mathcal{U}^{\xi;\underline{\ell}}_\sigma(x;\mathbf{t})$ by

$$\begin{aligned}\mathcal{U}^\xi_\sigma(x;\mathbf{t}) :=& T_\sigma(x;\mathbf{t})^{-1}K^\xi(x)^{-1}T_\sigma(-x;\mathbf{t})\\ =&\begin{pmatrix}\mathcal{A}^\xi_\sigma(x;\mathbf{t}) & \mathcal{B}^\xi_\sigma(x;\mathbf{t})\\ \mathcal{C}^\xi_\sigma(x;\mathbf{t}) & \mathcal{D}^\xi_\sigma(x;\mathbf{t})\end{pmatrix}\end{aligned} \tag{5.4}$$

as a linear operator on $V^{\frac{1}{2}} \otimes M^{\underline{\ell}}$ (here $K^\xi(x)^{-1}$ only acts on the auxiliary space component of the tensor product). Then $\mathcal{U}^\xi(x;\mathbf{t}) := \mathcal{U}^\xi_e(x;\mathbf{t})$ is the boundary monodromy operator [34] associated to the K-operator K^ξ. The operators $\mathcal{U}^\xi_\sigma(x;\mathbf{t})$ satisfy the commutation relations

$$\begin{aligned}&R_{00'}(y-x)\mathcal{U}^\xi_{\sigma,0}(x;\mathbf{t})R_{00'}(-x-y)\mathcal{U}^\xi_{\sigma,0'}(y;\mathbf{t}) =\\ &\qquad =\mathcal{U}^\xi_{\sigma,0'}(y;\mathbf{t})R_{00'}(-x-y)\mathcal{U}^\xi_{\sigma,0}(x;\mathbf{t})R_{00'}(y-x)\end{aligned} \tag{5.5}$$

as linear operators on $V^{\frac{1}{2}} \otimes V^{\frac{1}{2}} \otimes M^{\underline{\ell}}$ with the same notational conventions as for (5.3). One of the consequences of these commutation relations is the commutativity of the operators $\mathcal{B}^\xi_\sigma$:

$$[\mathcal{B}^\xi_\sigma(x;\mathbf{t}),\mathcal{B}^\xi_\sigma(y;\mathbf{t})]=0.$$

Remark 5.1. Boundary transfer operators were constructed in [34] in the context of quantum integrable models with boundaries. In the present context the boundary transfer operator is the linear operator on $M^{\underline{\ell}}$ defined as

$$\begin{aligned}\mathcal{T}^{\xi_+,\xi_-}(x;\mathbf{t}) :=& \mathrm{Tr}_{V^{\frac{1}{2}}}\left(K^{\xi_+}(x-\eta)\mathcal{U}^{\xi_-}(x;\mathbf{t})\right)\\ =&\mathcal{A}^{\xi_-}(x;\mathbf{t})+\frac{\sinh(\xi_+-x+\eta)}{\sinh(\xi_++x-\eta)}\mathcal{D}^{\xi_-}(x;\mathbf{t}),\end{aligned}$$

where $\xi_+,\xi_- \in \mathbb{C}$. It is a commuting family of operators:

$$[\mathcal{T}^{\xi_+,\xi_-}(x;\mathbf{t}),\mathcal{T}^{\xi_+,\xi_-}(y;\mathbf{t})]=0.$$

In a similar way one can define boundary transfer operators acting on the same state space $M^{\underline{\ell}}$ but involving higher-spin representations V^k ($k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$) in the auxiliary space, similar to the situation for periodic boundary conditions (see for example, the lectures [31]). We will describe their properties in a separate publication.

5.2. **The pseudo-vacuum and the Bethe vectors.** We write

$$L^\ell(x)=\begin{pmatrix}A^\ell(x) & B^\ell(x)\\ C^\ell(x) & D^\ell(x)\end{pmatrix}$$

with respect to the ordered basis $(v_1^{\frac{1}{2}}, v_2^{\frac{1}{2}})$ of the auxiliary space. The matrix coefficients are linear operators on M^ℓ. Explicitly they are given by

$$\begin{aligned}A^\ell(x)m_n^\ell &= \frac{\sinh(x+(\frac{3}{2}+\ell-n)\eta)}{\sinh(x+(\frac{1}{2}+\ell)\eta)}m_n^\ell,\\ B^\ell(x)m_n^\ell &= \frac{\sinh(\eta)}{\sinh(x+(\frac{1}{2}+\ell)\eta)}\mathrm{e}^{(-\ell-\frac{1}{2}+n)\eta}m_{n+1}^\ell,\\ C^\ell(x)m_n^\ell &= \frac{\sinh((n-1)\eta)\sinh((2\ell+2-n)\eta)}{\sinh(\eta)\sinh(x+(\frac{1}{2}+\ell)\eta)}\mathrm{e}^{(\ell+\frac{3}{2}-n)\eta}m_{n-1}^\ell,\\ D^\ell(x)m_n^\ell &= \frac{\sinh(x+(-\frac{1}{2}-\ell+n)\eta)}{\sinh(x+(\frac{1}{2}+\ell)\eta)}m_n^\ell,\end{aligned} \tag{5.6}$$

where m_0^ℓ should be read as zero. Note that

$$A^\ell(x)m_1^\ell = m_1^\ell, \qquad D^\ell(x)m_1^\ell = \vartheta^\ell(-x)m_1^\ell, \qquad C^\ell(x)m_1^\ell = 0,$$
$$\mathcal{R}^{k\ell}(x)(m_1^k \otimes m_1^\ell) = m_1^k \otimes m_1^\ell, \qquad \mathcal{K}^{\xi,\ell}(x)m_1^\ell = m_1^\ell.$$

Here $\vartheta^\ell(x)$ is defined in (2.9). Set

$$\Omega := m_1^{\ell_1} \otimes m_1^{\ell_2} \otimes \cdots \otimes m_1^{\ell_N} \in M^{\underline{\ell}}. \tag{5.7}$$

Note that

$$A_\sigma(x;\mathbf{t})\Omega = \Omega, \qquad D_\sigma(x;\mathbf{t})\Omega = \Big(\prod_{r=1}^N \vartheta^{\ell_r}(t_r - x)\Big)\Omega \tag{5.8}$$

for all $\sigma \in S_N$. The vector Ω will play the role of the pseudo-vacuum vector, from which off-shell Bethe vectors are generated by repeatedly applying operators $\mathcal{B}^\xi(x_i;\mathbf{t})$, cf. [34].

For convenience, to construct our solutions to the boundary qKZ equations we will use a different normalization for $\mathcal{B}_\sigma^\xi(x;\mathbf{t})$:

$$\overline{\mathcal{B}}_\sigma^\xi(x;\mathbf{t}) := \Big(\prod_{r=1}^N \frac{\sinh(x - t_r - \ell_r\eta)}{\sinh(x - t_r + \ell_r\eta)}\Big)\frac{\sinh(\xi - x - \frac{\eta}{2})\sinh(2x)}{\sinh(2x+\eta)}\mathcal{B}_\sigma^\xi(x + \frac{\eta}{2};\mathbf{t}).$$

The change from $\mathcal{B}$ to $\overline{\mathcal{B}}$ does not affect the commutativity:

$$[\overline{\mathcal{B}}_\sigma^\xi(x;\mathbf{t}), \overline{\mathcal{B}}_\sigma^\xi(y;\mathbf{t})] = 0.$$

Hence, the following operator is well-defined for all $\mathbf{x} = (x_1, \ldots, x_S)$ with $S \in \mathbb{Z}_{\geq 0}$:

$$\overline{\mathcal{B}}_\sigma^{\xi,(S)}(\mathbf{x};\mathbf{t}) := \prod_{j=1}^S \overline{\mathcal{B}}_\sigma^\xi(x_j;\mathbf{t}).$$

We will write $\overline{\mathcal{B}}^\xi(x;\mathbf{t}) := \overline{\mathcal{B}}_e^\xi(x;\mathbf{t})$ and $\overline{\mathcal{B}}^{\xi,(S)}(\mathbf{x};\mathbf{t}) := \overline{\mathcal{B}}_e^{\xi,(S)}(\mathbf{x};\mathbf{t})$ when $\sigma = e$ is the identity element of S_N. The associated off-shell Bethe vectors are the vectors $\overline{\mathcal{B}}^{\xi,(S)}(\mathbf{x};\mathbf{t})\Omega \in M^{\underline{\ell}}$.

6. Jackson integral solutions of the boundary qKZ equations

We recall the notion of mero-uniformly convergent sums for scalar-valued functions (cf. [33]), which can be extended to $M^{\underline{\ell}}$-valued functions in an obvious manner.

Definition 6.1. *Let $\mathcal{C} \subset \mathbb{C}^M$ be a discrete subset and $w(\mathbf{x};\mathbf{t})$ ($\mathbf{x} \in \mathcal{C}$) a weight function with values depending meromorphically on $\mathbf{t} \in \mathbb{C}^N$. Suppose that for all $\mathbf{t}_0 \in \mathbb{C}^N$, there exists an open neighbourhood $U_{\mathbf{t}_0} \subset \mathbb{C}^N$ of $\mathbf{t}_0$ and a nonzero holomorphic function $v_{\mathbf{t}_0}$ on $U_{\mathbf{t}_0}$ such that*

1. *$v_{\mathbf{t}_0}(\mathbf{t})w(\mathbf{x};\mathbf{t})$ is holomorphic in $\mathbf{t} \in U_{\mathbf{t}_0}$ for all $\mathbf{x} \in \mathcal{C}$,*
2. *the sum $\sum_{\mathbf{x}\in\mathcal{C}} v_{\mathbf{t}_0}(\mathbf{t})w(\mathbf{x};\mathbf{t})$ is absolutely and uniformly convergent for $\mathbf{t} \in U_{\mathbf{t}_0}$.*

Then there exists a unique meromorphic function $f(\mathbf{t})$ in $\mathbf{t} \in \mathbb{C}^N$ satisfying

$$v_{\mathbf{t}_0}(\mathbf{t})f(\mathbf{t}) = \sum_{\mathbf{x}\in\mathcal{C}} v_{\mathbf{t}_0}(\mathbf{t})w(\mathbf{x};\mathbf{t})$$

for $\mathbf{t} \in U_{\mathbf{t}_0}$ *and* $\mathbf{t}_0 \in \mathbb{C}^N$. *We will write*

$$f(\mathbf{t}) = \sum_{\mathbf{x} \in \mathcal{C}} w(\mathbf{x};\mathbf{t})$$

and we will say that the sum converges mero-uniformly.

We are now in a position to present our main theorem. For a meromorphic function h of one variable, write $h(x \pm y) = h(x+y)h(x-y)$.

Theorem 6.2. *Let* $\xi_+, \xi_- \in \mathbb{C}$ *and let* $g_{\xi_+,\xi_-}(x)$, $h(x)$ *and* $F^\ell(x)$ *be meromorphic functions in* $x \in \mathbb{C}$ *satisfying the functional equations*

$$\begin{aligned} g_{\xi_+,\xi_-}(x+\tau) &= \frac{\sinh(\xi_- - x - \frac{\eta}{2})\sinh(\xi_+ - x - \frac{\tau}{2} - \frac{\eta}{2})}{\sinh(\xi_- + x + \tau - \frac{\eta}{2})\sinh(\xi_+ + x + \frac{\tau}{2} - \frac{\eta}{2})} g_{\xi_+,\xi_-}(x), \\ h(x+\tau) &= \frac{\sinh(x+\tau)\sinh(x+\eta)}{\sinh(x)\sinh(x+\tau-\eta)} h(x), \\ F^\ell(x+\tau) &= \frac{\sinh(x+\tau-\ell\eta)}{\sinh(x+\tau+\ell\eta)} F^\ell(x). \end{aligned}$$

Fix generic $\mathbf{x}_0 \in \mathbb{C}^S$ *and suppose that the* $M^{\underline{\ell}}$*-valued sum*

$$\begin{aligned} f_S^{\underline{\ell}}(\mathbf{t}) := \sum_{\mathbf{x} \in \mathbf{x}_0 + \tau\mathbb{Z}^S} \Big(\prod_{i=1}^S g_{\xi_+,\xi_-}(x_i)\Big)\Big(\prod_{1 \le i < j \le S} h(x_i \pm x_j)\Big) \\ \times \Big(\prod_{r=1}^N \prod_{i=1}^S F^{\ell_r}(t_r \pm x_i)\Big)\overline{\mathcal{B}}^{\xi_-,(S)}(\mathbf{x};\mathbf{t})\Omega \end{aligned}$$

converges mero-uniformly in $\mathbf{t} \in \mathbb{C}^N$. *Then* $f_S^{\underline{\ell}}$ *is a solution of the boundary qKZ equations* (1.3).

Theorem 6.2 generalizes the main result of [32] from 2-dimensional representations of quantum $\mathfrak{sl}_2$ to arbitrary Verma modules. The proof of Theorem 6.2 follows roughly the line of reasoning of the spin-$\frac{1}{2}$ case [32], although considerably more technical difficulties need to be overcome. The proof is given in Section 7.

We now make the solutions concrete. We set $q := \mathrm{e}^\tau$ and we assume that $\Re(\tau) < 0$, so that $|q| < 1$. Solutions g_{ξ_+,ξ_-}, h and F^ℓ of the resulting functional relations can now be expressed in terms of q-Gamma functions or, equivalently, in terms of q-shifted factorials

$$\big(x;q\big)_\infty := \prod_{i=0}^\infty (1 - q^i x).$$

We write $\big(x_1,\ldots,x_s;q\big)_\infty := \prod_{i=1}^s \big(x_i;q\big)_\infty$ for products of q-shifted factorials. As solutions of the functional equations we take

$$\begin{aligned} g_{\xi_+,\xi_-}(x) &= \mathrm{e}^{(\frac{2(\xi_-+\xi_+-\eta)}{\tau}+1)x} \frac{\big(q^2\mathrm{e}^{2(x+\xi_-)-\eta}, q\mathrm{e}^{2(x+\xi_+)-\eta}; q^2\big)_\infty}{\big(\mathrm{e}^{2(x-\xi_-)+\eta}, q\mathrm{e}^{2(x-\xi_+)+\eta}; q^2\big)_\infty}, \\ h(x) &= \mathrm{e}^{-\frac{2\eta x}{\tau}}(1-\mathrm{e}^{2x})\frac{\big(q^2\mathrm{e}^{2(x-\eta)};q^2\big)_\infty}{\big(\mathrm{e}^{2(x+\eta)};q^2\big)_\infty}, \\ F^\ell(x) &= \mathrm{e}^{\frac{2\ell\eta x}{\tau}} \frac{\big(q^2\mathrm{e}^{2(x+\ell\eta)};q^2\big)_\infty}{\big(q^2\mathrm{e}^{2(x-\ell\eta)};q^2\big)_\infty}. \end{aligned} \tag{6.1}$$

With these choices for the solutions of the functional equations and the assumption that $\Re(\tau) < 0$, it is readily established (cf. [32, Subsections 3.4 and 3.5]) that the solution $f_S^{\underline{\ell}}(\mathbf{t})$ defined in Theorem 6.2 converges mero-uniformly in $\mathbf{t} \in \mathbb{C}^N$ for generic $\mathbf{x}_0 \in \mathbb{C}^S$ when $\Re(\eta) \geq 0$ and

$$\Re\big(2\xi_+ + 2\xi_- + 2\big(2\sum_{r=1}^{N}\ell_r - 1\big)\eta + \tau\big) < 0. \tag{6.2}$$

7. Proof of the main result

7.1. **Preliminary steps.** Let S_N be the symmetric group in N letters and $\sigma \in S_N$. We view

$$L^{\ell_{\sigma(1)}}(x - t_{\sigma(1)})L^{\ell_{\sigma(2)}}(x - t_{\sigma(2)}) \cdots L^{\ell_{\sigma(N-1)}}(x - t_{\sigma(N-1)}) \tag{7.1}$$

as a linear operator on $V^{\frac{1}{2}} \otimes M^{\underline{\ell}}$ acting trivially on the tensor component of $M^{\underline{\ell}}$ labelled by $\sigma(N)$. Write

$$\begin{pmatrix} \widehat{A}_\sigma(x;\mathbf{t}) & \widehat{B}_\sigma(x;\mathbf{t}) \\ \widehat{C}_\sigma(x;\mathbf{t}) & \widehat{D}_\sigma(x;\mathbf{t}) \end{pmatrix}$$

for the operator (7.1), written as a matrix with respect to the ordered basis $(v_1^{\frac{1}{2}}, v_2^{\frac{1}{2}})$ of $V^{\frac{1}{2}}$. The operators $\widehat{A}_\sigma(x;\mathbf{t}), \ldots, \widehat{D}_\sigma(x;\mathbf{t})$ act on $M^{\underline{\ell}}$. They act trivially on the $\sigma(N)$-th tensor component of $M^{\underline{\ell}}$ and do not depend on $t_{\sigma(N)}$.

For $\sigma \in S_N$, $J \subseteq \{1, \ldots, S\}$ and $\boldsymbol{\epsilon} \in \{\pm\}^S$ we write

$$\begin{aligned} \mathcal{Y}_\sigma^{\xi,\boldsymbol{\epsilon},J}(\mathbf{x};\mathbf{t}) := \Big(\prod_{i=1}^{S} \epsilon_i \sinh(\xi - \epsilon_i x_i - \frac{\eta}{2}) \prod_{r=1}^{N} \frac{\sinh(\epsilon_i x_i - t_r - \ell_r\eta)}{\sinh(\epsilon_i x_i - t_r + \ell_r\eta)}\Big) \\ \times \Big(\prod_{1 \leq i < j \leq S} \frac{\sinh(\epsilon_i x_i + \epsilon_j x_j + \eta)}{\sinh(\epsilon_i x_i + \epsilon_j x_j)}\Big) Y_\sigma^J((-\epsilon_1 x_1 - \frac{\eta}{2}, \ldots, -\epsilon_S x_S - \frac{\eta}{2});\mathbf{t}) \end{aligned}$$

with

$$Y_\sigma^J(\mathbf{x};\mathbf{t}) := \Big(\prod_{i\in J} \frac{\sinh(x_i - t_{\sigma(N)} + (\frac{1}{2} - \ell_{\sigma(N)})\eta)}{\sinh(x_i - t_{\sigma(N)} + (\frac{1}{2} + \ell_{\sigma(N)})\eta)}\Big) \prod_{(i,j)\in J\times J^c} \frac{\sinh(x_i - x_j + \eta)}{\sinh(x_i - x_j)}$$

and $J^c := \{1, \ldots, S\} \setminus J$ (empty products are equal to one). Similarly to the spin-$\frac{1}{2}$ case (see [32, Cor. 4.3]) we have the explicit expression

$$\begin{aligned} \overline{\mathcal{B}}_\sigma^{\xi,(S)}(\mathbf{x};\mathbf{t})\Omega = \sum_{\boldsymbol{\epsilon}\in\{\pm 1\}^S} \sum_{J\subseteq\{1,\ldots,S\}} \mathcal{Y}_\sigma^{\xi,\boldsymbol{\epsilon},J}(\mathbf{x};\mathbf{t}) \\ \times \Big(\prod_{j\in J^c} B^{\ell_{\sigma(N)}}(-\epsilon_j x_j - \frac{\eta}{2} - t_{\sigma(N)})\Big)\Big(\prod_{i\in J} \widehat{B}_\sigma(-\epsilon_i x_i - \frac{\eta}{2};\mathbf{t})\Big)\Omega \end{aligned}$$

of the Bethe vector (see [32, Cor. 4.3]). For $r \in \{1, \ldots, N-1\}$ write $s_r \in S_N$ for the simple neighbouring transposition $r \leftrightarrow r+1$. In [32, Lemma 5.4] the condition that the function $f_S^{\underline{\ell}}(\boldsymbol{t})$ with $\ell = (\frac{1}{2}, \cdots, \frac{1}{2})$ satisfies the boundary qKZ equations is re-written as a system of equations involving the weight functions $\mathcal{Y}_\sigma^{\xi,\boldsymbol{\epsilon},J}$ where $\sigma = s_r \cdots s_{N-1}$ for some $r \in \{1, \ldots, N\}$. This directly generalizes to the following result in the current higher-spin context.

Lemma 7.1. *Provided mero-uniform convergence,*

$$f_S^{\ell}(\mathbf{t}) := \sum_{\mathbf{x}\in\mathbf{x}_0+\tau\mathbb{Z}^S} w^{(S)}(\mathbf{x};\mathbf{t};\xi_+,\xi_-)\overline{\mathcal{B}}^{\xi_-,(S)}(\mathbf{x};\mathbf{t})\Omega$$

satisfies the boundary qKZ equations (1.3) *iff*

$$\begin{aligned}\sum_{\mathbf{x},\boldsymbol{\epsilon},J} & w^{(S)}(\mathbf{x};\mathbf{t};\xi_+,\xi_-)\Big(\prod_{i=1}^{S}\frac{\sinh(\pm x_i+t_r+\ell_r\eta)}{\sinh(\pm x_i+t_r-\ell_r\eta)}\Big)\mathcal{Y}^{\xi_-,\boldsymbol{\epsilon},J}_{s_r\cdots s_{N-1}}(\mathbf{x};e_r\mathbf{t})\\ & \times\mathcal{K}^{\xi_+,\ell_r}(t_r+\frac{\tau}{2})\\ & \times\Big(\prod_{j\in J^c}B^{\ell_r}(-\epsilon_j x_j-\frac{\eta}{2}+t_r)\Big)\Big(\prod_{i\in J}\widehat{B}_{s_r\cdots s_{N-1}}(-\epsilon_i x_i-\frac{\eta}{2};\mathbf{t})\Big)\Omega\end{aligned}\tag{7.2}$$

equals

$$\begin{aligned}\sum_{\mathbf{x},\boldsymbol{\epsilon},J} & w^{(S)}(\mathbf{x};\mathbf{t}+\tau\boldsymbol{e}_r;\xi_+,\xi_-)\mathcal{Y}^{\xi_-,\boldsymbol{\epsilon},J}_{s_r\cdots s_{N-1}}(\mathbf{x};\mathbf{t}+\tau\boldsymbol{e}_r)\\ & \times\Big(\prod_{j\in J^c}B^{\ell_r}(-\epsilon_j x_j-\frac{\eta}{2}-t_r-\tau)\Big)\Big(\prod_{i\in J}\widehat{B}_{s_r\cdots s_{N-1}}(-\epsilon_i x_i-\frac{\eta}{2};\mathbf{t})\Big)\Omega\end{aligned}\tag{7.3}$$

for $r=1,\ldots,N$, where the summations are over $\mathbf{x}\in\mathbf{x}_0+\tau\mathbb{Z}^S$, $\boldsymbol{\epsilon}\in\{\pm\}^S$ and over subsets $J\subseteq\{1,\ldots,S\}$ (recall that $J^c=\{1,\ldots,S\}\backslash J$).

We fix $S\geq 1$ and suppress it from the notations. For $d\in\{0,\ldots,S\}$ set $\mathcal{L}_r^{(d)}(\mathbf{t})$ and $\mathcal{R}_r^{(d)}(\mathbf{t})$ for (7.2) and (7.3) respectively, with the sums running over $\mathbf{x}\in\mathbf{x}_0+\tau\mathbb{Z}^S$, $\boldsymbol{\epsilon}\in\{\pm\}^S$ and over subsets $J\subseteq\{1,\ldots,S\}$ of cardinality $S-d$. The strategy of the proof of Theorem 6.2 is to determine sufficients conditions on the weight function $w^{(S)}(\mathbf{x};\mathbf{t};\xi_+,\xi_-)$ so that

$$\mathcal{L}_r^{(d)}(\mathbf{t})=\mathcal{R}_r^{(d)}(\mathbf{t})\tag{7.4}$$

for all $d\in\{0,\ldots,S\}$ and $r\in\{1,\ldots,N\}$. We will call d the *depth*.

Remark 7.2. In the study [32] of Jackson integral solutions for the spin-$\frac{1}{2}$ representations the terms $\mathcal{L}_r^{(d)}(\mathbf{t})$ and $\mathcal{R}_r^{(d)}(\mathbf{t})$ are automatically zero if $d\geq 2$, cf. [32, Rem. 5.5]. When M^{ℓ_s} are highest weight modules with $\ell_s\in\mathbb{C}$ we have to deal with the terms $\mathcal{L}_r^{(d)}(\mathbf{t})$ and $\mathcal{R}_r^{(d)}(\mathbf{t})$ for any depth $d\in\{0,\ldots,S\}$.

7.2. **Depth zero.** Completely analogous to the spin-$\frac{1}{2}$ case (see [32, §5.1]) we have the following result.

Lemma 7.3. *Suppose that*

$$w^{(S)}(\mathbf{x};\mathbf{t};\xi_+,\xi_-)=\Big(\prod_{r=1}^{N}\prod_{i=1}^{S}F^{\ell_r}(t_r\pm x_i)\Big)G^{(S)}_{\xi_+,\xi_-}(\mathbf{x})$$

with $G^{(S)}_{\xi_+,\xi_-}(\mathbf{x})$ independent of $\mathbf{t}$. If

$$F^{\ell_r}(x+\tau)=\frac{\sinh(x+\tau-\ell_r\eta)}{\sinh(x+\tau+\ell_r\eta)}F^{\ell_r}(x)$$

for $r=1,\ldots,N$ then, provided mero-uniform convergence,

$$\mathcal{L}_r^{(0)}(\mathbf{t};\xi_+,\xi_-)=\mathcal{R}_r^{(0)}(\mathbf{t};\xi_+,\xi_-)\tag{7.5}$$

for $r=1,\ldots,N$.

In the remainder of the section we assume that the weight function $w^{(S)}(\mathbf{x};\mathbf{t};\xi_+,\xi_-)$ is of the form as specified in Lemma 7.3.

7.3. **The remaining depths.** We have the setup that

$$f_S^{\boldsymbol{\ell}}(\mathbf{t})=\sum_{\mathbf{x}\in\mathbf{x}_0+\tau\mathbb{Z}^S} w^{(S)}(\mathbf{x};\mathbf{t};\xi_+,\xi_-)\overline{\mathcal{B}}^{\xi_-,(S)}(\mathbf{x};\mathbf{t})\Omega$$

with the sum converging mero-uniformly in $\mathbf{t}\in\mathbb{C}^N$ and with weight function of the form

$$w^{(S)}(\mathbf{x};\mathbf{t};\xi_+,\xi_-)=\Big(\prod_{r=1}^N\prod_{i=1}^S F^{\ell_r}(t_r\pm x_i)\Big)G^{(S)}_{\xi_+,\xi_-}(\mathbf{x}) \tag{7.6}$$

with $G^{(S)}_{\xi_+,\xi_-}(\mathbf{x})$ independent of $\mathbf{t}$ and with the F^ℓ satisfying

$$F^\ell(x+\tau)=\frac{\sinh(x+\tau-\ell\eta)}{\sinh(x+\tau+\ell\eta)}F^\ell(x). \tag{7.7}$$

We are now going to show that conditions on the weight factor $G^{(S)}_{\xi_+,\xi_-}(\mathbf{x})$ as stated in Theorem 6.2 imply that (7.4) is valid for $d\in\{1,\ldots,S\}$ and $r\in\{1,\ldots,N\}$. Combined with Lemma 7.3 and Lemma 7.1, this will complete the proof of Theorem 6.2.

Since the $\xi_\pm$ are fixed throughout this subsection, we will suppress $\xi_\pm$ from the notations; in particular, we write $w^{(S)}(\mathbf{x};\mathbf{t})$ for $w^{(S)}(\mathbf{x};\mathbf{t};\xi_+,\xi_-)$. We also suppress $S\in\mathbb{Z}_{\geq 1}$ from the notations.

If $J\subseteq\{1,\ldots,S\}$, $\epsilon\in\{\pm\}^S$ and $\mathbf{x}\in\mathbf{x}_0+\tau\mathbb{Z}^S$ then we write $\mathbf{x}_J:=(x_j)_{j\in J}$ and $\boldsymbol{\epsilon}_J:=(\epsilon_j)_{j\in J}$. Conversely, for given $\boldsymbol{\epsilon}_J$ and $\boldsymbol{\epsilon}_{J^c}$ the associated S-tuple of signs will be denoted by $\boldsymbol{\epsilon}$ (and similarly for $\mathbf{x}$).

It is convenient to define the following weights.

Definition 7.4. *For $r\in\{1,\ldots,N\}$, $\epsilon\in\{\pm\}^S$ and a subset $J\subseteq\{1,\ldots,S\}$ we write*

$$m_r^{\boldsymbol{\epsilon},J}(\mathbf{x};\mathbf{t}):=\Big(\prod_{i=1}^S\frac{\sinh(\pm x_i-t_r+\ell_r\eta)}{\sinh(\pm x_i-t_r-\ell_r\eta)}\Big)\frac{\mathcal{Y}^{\xi_-,\boldsymbol{\epsilon},J}_{s_r\cdots s_{N-1}}(\mathbf{x};\mathbf{t})}{\prod_{j\in J^c}\sinh(-\epsilon_jx_j-t_r+\ell_r\eta)}$$

for $\mathbf{x}\in\mathbf{x}_0+\tau\mathbb{Z}^S$.

It follows by a straightforward computation that

$$\begin{aligned}
m_r^{\boldsymbol{\epsilon},J}(\mathbf{x};\mathbf{t})&=\Big(\prod_{j\in J^c}\Big(\epsilon_j\frac{\sinh(\xi_--\epsilon_jx_j-\frac{\eta}{2})}{\sinh(-t_r-\epsilon_jx_j-\ell_r\eta)}\prod_{\substack{s=1\\ s\neq r}}^N\frac{\sinh(t_s-\epsilon_jx_j+\ell_s\eta)}{\sinh(t_s-\epsilon_jx_j-\ell_s\eta)}\Big)\Big)\\
&\times\Big(\prod_{(i,j)\in J\times J^c}\frac{\sinh(\epsilon_jx_j\pm x_i+\eta)}{\sinh(\epsilon_jx_j\pm x_i)}\Big)\Big(\prod_{\substack{i,i'\in J:\\ i<i'}}\frac{\sinh(\epsilon_ix_i+\epsilon_{i'}x_{i'}+\eta)}{\sinh(\epsilon_ix_i+\epsilon_{i'}x_{i'})}\Big)\\
&\times\Big(\prod_{\substack{j,j'\in J^c:\\ j<j'}}\frac{\sinh(\epsilon_jx_j+\epsilon_{j'}x_{j'}+\eta)}{\sinh(\epsilon_jx_j+\epsilon_{j'}x_{j'})}\Big)\\
&\times\prod_{i\in J}\Big(\epsilon_i\sinh(\xi_--\epsilon_ix_i-\frac{\eta}{2})\prod_{\substack{s=1\\ s\neq r}}^N\frac{\sinh(t_s-\epsilon_ix_i+\ell_s\eta)}{\sinh(t_s-\epsilon_ix_i-\ell_s\eta)}\Big).
\end{aligned} \tag{7.8}$$

Lemma 7.5. *Fix $d \in \{1, \ldots, S\}$ and $r \in \{1, \ldots, N\}$. Suppose that for all subsets $J \subseteq \{1, \ldots, S\}$ of cardinality $S - d$ and for all $\mathbf{x}_J$ and $\boldsymbol{\epsilon}_J$,*

$$C_{d+1}^{\ell_r}(t_r + \frac{\tau}{2}; \xi_+) \sum_{\mathbf{x}_{J^c}, \boldsymbol{\epsilon}_{J^c}} w^{(S)}(\mathbf{x}; \mathbf{t}; \xi_+, \xi_-) m_r^{\boldsymbol{\epsilon}, J}(\mathbf{x}; e_r \mathbf{t}) = \\ = \sum_{\mathbf{x}_{J^c}, \boldsymbol{\epsilon}_{J^c}} w^{(S)}(\mathbf{x}; \mathbf{t}; \xi_+, \xi_-) m_r^{\boldsymbol{\epsilon}, J}(\mathbf{x}; \mathbf{t} + \tau \boldsymbol{e}_r).$$

Then $\mathcal{L}_r^{(d)}(\mathbf{t}) = \mathcal{R}_r^{(d)}(\mathbf{t})$.

Proof. Recall that

$$\mathcal{K}^{\xi_+, \ell_r}(t_r + \frac{\tau}{2}) m_{d+1}^{\ell_r} = C_{d+1}^{\ell_r}(t_r + \frac{\tau}{2}; \xi_+) m_{d+1}^{\ell_r},$$

see Definition 4.8. Since

$$\Big(\prod_{j \in J^c} B^{\ell_r}(-\epsilon_j x_j - \frac{\eta}{2} + u)\Big) m_1^{\ell_r} = \frac{\sinh^d(\eta) \mathrm{e}^{\eta(\frac{d^2}{2} - \ell_r d)}}{\prod_{j \in J^c} \sinh(-\epsilon_j x_j + u + \ell_r \eta)} m_{d+1}^{\ell_r}$$

by (5.6) we thus have

$$\mathcal{K}^{\xi_+, \ell_r}(t_r + \frac{\tau}{2}) \Big(\prod_{j \in J^c} B^{\ell_r}(-\epsilon_j x_j - \frac{\eta}{2} + t_r)\Big) m_1^{\ell_r} = \\ = \frac{\sinh^d(\eta) C_{d+1}^{\ell_r}(t_r + \frac{\tau}{2}; \xi_+) \mathrm{e}^{\eta(\frac{d^2}{2} - \ell_r d)}}{\prod_{j \in J^c} \sinh(-\epsilon_j x_j + t_r + \ell_r \eta)} m_{d+1}^{\ell_r},$$

$$\Big(\prod_{j \in J^c} B^{\ell_r}(-\epsilon_j x_j - \frac{\eta}{2} - \tau - t_r)\Big) m_1^{\ell_r} = \\ = \frac{\sinh^d(\eta) \mathrm{e}^{\eta(\frac{d^2}{2} - \ell_r d)}}{\prod_{j \in J^c} \sinh(-\epsilon_j x_j - \tau - t_r + \ell_r \eta)} m_{d+1}^{\ell_r}.$$

Taking the expressions (7.2) and (7.3) for $\mathcal{L}_r^{(m)}(\mathbf{t})$ and $\mathcal{R}_r^{(m)}(\mathbf{t})$ into account we conclude that $\mathcal{L}_r^{(d)}(\mathbf{t}) = \mathcal{R}_r^{(d)}(\mathbf{t})$ if for all subsets $J \subseteq \{1, \ldots, S\}$ of cardinality $S - d$ and for all $\mathbf{x}_J$ and $\boldsymbol{\epsilon}_J$,

$$C_{d+1}^{\ell_r}(t_r + \frac{\tau}{2}; \xi_+) \sum_{\mathbf{x}_{J^c}, \boldsymbol{\epsilon}_{J^c}} w^{(S)}(\mathbf{x}; \mathbf{t}; \xi_+, \xi_-) m_r^{\boldsymbol{\epsilon}, J}(\mathbf{x}; e_r \mathbf{t}) = \\ = \sum_{\mathbf{x}_{J^c}, \boldsymbol{\epsilon}_{J^c}} w^{(S)}(\mathbf{x}; \mathbf{t} + \tau \boldsymbol{e}_r; \xi_+, \xi_-) \\ \times \Big(\prod_{i=1}^{S} \frac{\sinh(\pm x_i - t_r - \tau - \ell_r \eta)}{\sinh(\pm x_i - t_r - \tau + \ell_r \eta)}\Big) m_r^{\boldsymbol{\epsilon}, J}(\mathbf{x}; \mathbf{t} + \tau \boldsymbol{e}_r).$$

The lemma now follows from the fact that

$$w^{(S)}(\mathbf{x}; \mathbf{t} + \tau \boldsymbol{e}_r; \xi_+, \xi_-) = \Big(\prod_{i=1}^{S} \frac{\sinh(\pm x_i - t_r - \tau + \ell_r \eta)}{\sinh(\pm x_i - t_r - \tau - \ell_r \eta)}\Big) w^{(S)}(\mathbf{x}; \mathbf{t}; \xi_+, \xi_-),$$

which is a direct consequence of the specific form (7.6), (7.7) of the weight function $w^{(S)}(\mathbf{x}; \mathbf{t})$. □

In the remainder of this subsection we fix $d \in \{1,\ldots,S\}$, $r \in \{1,\ldots,N\}$, a subset $J \subseteq \{1,\ldots,S\}$ of cardinality $S-d$, as well as $\mathbf{x}_J$ and $\boldsymbol{\epsilon}_J$, which we all suppress from the notations. Set for $\boldsymbol{\epsilon}_{J^c} \in \{\pm\}^d$,

$$\Lambda_{r,\boldsymbol{\epsilon}_{J^c}}(\mathbf{t}) := \sum_{\mathbf{x}_{J^c}} w^{(S)}(\mathbf{x};\mathbf{t};\xi_+,\xi_-) m_r^{\boldsymbol{\epsilon},J}(\mathbf{x}; e_r\mathbf{t}),$$
$$\Upsilon_{r,\boldsymbol{\epsilon}_{J^c}}(\mathbf{t}) := \sum_{\mathbf{x}_{J^c}} w^{(S)}(\mathbf{x};\mathbf{t};\xi_+,\xi_-) m_r^{\boldsymbol{\epsilon},J}(\mathbf{x}; \mathbf{t}+\tau\boldsymbol{e}_r).$$

In view of the previous lemma, the desired identity (7.4) follows if

$$C_{d+1}^{\ell_r}(t_r+\frac{\tau}{2};\xi_+) \sum_{\boldsymbol{\epsilon}_{J^c}\in\{\pm\}^d} \Lambda_{r,\boldsymbol{\epsilon}_{J^c}}(\mathbf{t}) = \sum_{\boldsymbol{\epsilon}_{J^c}\in\{\pm\}^d} \Upsilon_{r,\boldsymbol{\epsilon}_{J^c}}(\mathbf{t}). \tag{7.9}$$

We write $m_r(\mathbf{x}_{J^c};\mathbf{t})$ for $m_r^{\boldsymbol{\epsilon},J}(\mathbf{x};\mathbf{t})$ with $\boldsymbol{\epsilon}_{J^c}$ the d-tuple $(-,-,\cdots,-)$ of minus signs,

$$\begin{aligned} m_r(\mathbf{x}_{J^c};\mathbf{t}) = (-1)^d \Big(\prod_{j\in J^c} \Big(\frac{\sinh(\xi_- + x_j - \frac{\eta}{2})}{\sinh(-t_r + x_j - \ell_r\eta)} \prod_{\substack{s=1\\ s\neq r}}^N \frac{\sinh(t_s + x_j + \ell_s\eta)}{\sinh(t_s + x_j - \ell_s\eta)} \Big) \Big) \\ \times \Big(\prod_{(i,j)\in J\times J^c} \frac{\sinh(x_j \pm x_i - \eta)}{\sinh(x_j \pm x_i)} \Big) \Big(\prod_{\substack{i,i'\in J:\\ i<i'}} \frac{\sinh(\epsilon_i x_i + \epsilon_{i'} x_{i'} + \eta)}{\sinh(\epsilon_i x_i + \epsilon_{i'} x_{i'})} \Big) \\ \times \Big(\prod_{\substack{j,j'\in J^c:\\ j<j'}} \frac{\sinh(x_j + x_{j'} - \eta)}{\sinh(x_j + x_{j'})} \Big) \\ \times \Big(\prod_{i\in J} \Big(\epsilon_i \sinh(\xi_- - \epsilon_i x_i - \frac{\eta}{2}) \prod_{\substack{s=1\\ s\neq r}}^N \frac{\sinh(t_s - \epsilon_i x_i + \ell_s\eta)}{\sinh(t_s - \epsilon_i x_i - \ell_s\eta)} \Big) \Big). \end{aligned} \tag{7.10}$$

Lemma 7.6. *Suppose that for all $i \in \{1,\ldots,S\}$,*

$$\begin{aligned} G_{\xi_+,\xi_-}(\mathbf{x}-\tau\boldsymbol{e}_i) = \frac{\sinh(\xi_- + x_i - \frac{\eta}{2})\sinh(\xi_+ + x_i - \frac{\tau}{2} - \frac{\eta}{2})}{\sinh(\xi_- - x_i + \tau - \frac{\eta}{2})\sinh(\xi_+ - x_i + \frac{\tau}{2} - \frac{\eta}{2})} \\ \times \Big(\prod_{\substack{i'=1\\ i'\neq i}}^S \frac{\sinh(x_i \pm x_{i'} - \tau)\sinh(x_i \pm x_{i'} - \eta)}{\sinh(x_i \pm x_{i'} - \tau + \eta)\sinh(x_i \pm x_{i'})} \Big) G_{\xi_+,\xi_-}(\mathbf{x}). \end{aligned} \tag{7.11}$$

Then

$$\Lambda_{r,\boldsymbol{\epsilon}_{J^c}}(\mathbf{t}) = (-1)^{\#J_+^c} \sum_{\mathbf{x}_{J^c}} w^{(S)}(\mathbf{x};\mathbf{t};\xi_+,\xi_-) q_{J_+^c}(\mathbf{x}_{J^c}; t_r) m_r(\mathbf{x}_{J^c}; e_r\mathbf{t}),$$
$$\Upsilon_{r,\boldsymbol{\epsilon}_{J^c}}(\mathbf{t}) = (-1)^{\#J_+^c} \sum_{\mathbf{x}_{J^c}} w^{(S)}(\mathbf{x};\mathbf{t};\xi_+,\xi_-) q_{J_+^c}(\mathbf{x}_{J^c}; -t_r-\tau) m_r(\mathbf{x}_{J^c}; \mathbf{t}+\tau\boldsymbol{e}_r)$$

with $J^c_+ := \{j\in J^c\,|\,\epsilon_j=+\}$, $J^c_- := J^c\setminus J^c_+$ *and*

$$\begin{aligned} q_{J^c_+}(\mathbf{x}_{J^c};t_r) :=&\Big(\prod_{\substack{j,j'\in J^c_+:\\ j<j'}}\frac{\sinh(x_j+x_{j'}-\tau-\eta)}{\sinh(x_j+x_{j'}-\tau+\eta)}\Big)\\ &\times\Big(\prod_{j\in J^c_-}\prod_{j'\in J^c_+}\frac{\sinh(x_{j'}-x_j-\eta)\sinh(x_{j'}+x_j-\tau)}{\sinh(x_{j'}-x_j)\sinh(x_{j'}+x_j-\tau+\eta)}\Big)\\ &\times\Big(\prod_{j\in J^c_+}\frac{\sinh(\xi_++x_j-\frac{\tau}{2}-\frac{\eta}{2})\sinh(t_r+x_j+\ell_r\eta)}{\sinh(\xi_+-x_j+\frac{\tau}{2}-\frac{\eta}{2})\sinh(t_r-x_j+\tau+\ell_r\eta)}\Big). \end{aligned}$$

Proof. The formula for $\Lambda_{r,\boldsymbol{\epsilon}_{J^c}}(\mathbf{t})$ is correct if $\boldsymbol{\epsilon}_{J^c}$ is the d-tuple $(-,-\cdots,-)$ of minus signs since $q_\emptyset(\mathbf{x}_{J^c};t_r)=1$ (empty products are equal to one by convention).

Fix $\boldsymbol{\epsilon}_{J^c}\in\{\pm\}^d$ and $I\subset J^c_+$. Write $\boldsymbol{\epsilon}^{I,-}_{J^c}$ for the d-tuple of signs obtained from $\boldsymbol{\epsilon}_{J^c}$ by replacing $\epsilon_i=+$ by $-$ for all $i\in I$. Similarly, we write $\boldsymbol{\epsilon}^{I,-}$ for the S-tupe of signs obtained from $\boldsymbol{\epsilon}$ by replacing $\epsilon_i=+$ by $-$ for all $i\in I$.

Fix $k\in J^c_+$ and rewrite $\Lambda_{r,\boldsymbol{\epsilon}_{J^c}}(\mathbf{t})$ as

$$\Lambda_{r,\boldsymbol{\epsilon}_{J^c}}(\mathbf{t})=\sum_{\mathbf{x}_{J^c}}w^{(S)}(\mathbf{x}-\tau\boldsymbol{e}_k;\mathbf{t};\xi_+,\xi_-)m_r^{\boldsymbol{\epsilon},J}(\mathbf{x}-\tau\boldsymbol{e}_k;e_r\mathbf{t}).$$

By the assumptions on $w^{(S)}(\mathbf{x};\mathbf{t})$ we have

$$w^{(S)}(\mathbf{x}-\tau\boldsymbol{e}_k;\mathbf{t};\xi_+,\xi_-)=\beta_k(\mathbf{x};\mathbf{t})w^{(S)}(\mathbf{x};\mathbf{t};\xi_+,\xi_-)$$

with

$$\begin{aligned} \beta_k(\mathbf{x};\mathbf{t}) :=&\Big(\prod_{s=1}^N\frac{\sinh(t_s+x_k+\ell_s\eta)\sinh(t_s-x_k+\tau-\ell_s\eta)}{\sinh(t_s+x_k-\ell_s\eta)\sinh(t_s-x_k+\tau+\ell_s\eta)}\Big)\\ &\times\frac{\sinh(\xi_-+x_k-\frac{\eta}{2})\sinh(\xi_++x_k-\frac{\tau}{2}-\frac{\eta}{2})}{\sinh(\xi_--x_k+\tau-\frac{\eta}{2})\sinh(\xi_+-x_k+\frac{\tau}{2}-\frac{\eta}{2})}\\ &\times\Big(\prod_{\substack{k'=1\\ k'\neq k}}^S\frac{\sinh(x_k\pm x_{k'}-\tau)\sinh(x_k\pm x_{k'}-\eta)}{\sinh(x_k\pm x_{k'}-\tau+\eta)\sinh(x_k\pm x_{k'})}\Big). \end{aligned}$$

In addition, by a direct computation using (7.8),

$$\beta_k(\mathbf{x};\mathbf{t})m_r^{\boldsymbol{\epsilon},J}(\mathbf{x}-\tau\boldsymbol{e}_k;e_r\mathbf{t})=-\gamma_k^{\boldsymbol{\epsilon}_{J^c}}(\mathbf{x}_{J^c};t_r)m_r^{\boldsymbol{\epsilon}^{\{k\},-},J}(\mathbf{x};e_r\mathbf{t})$$

with

$$\begin{aligned} \gamma_k^{\boldsymbol{\epsilon}_{J^c}}(\mathbf{x}_{J^c};t_r) :=&\Big(\prod_{j\in J^c\setminus\{k\}}\frac{\sinh(x_k+\epsilon_jx_j-\eta)\sinh(x_k-\epsilon_jx_j-\tau)}{\sinh(x_k+\epsilon_jx_j)\sinh(x_k-\epsilon_jx_j-\tau+\eta)}\Big)\\ &\times\frac{\sinh(\xi_++x_k-\frac{\tau}{2}-\frac{\eta}{2})\sinh(t_r+x_k+\ell_r\eta)}{\sinh(\xi_+-x_k+\frac{\tau}{2}-\frac{\eta}{2})\sinh(t_r-x_k+\tau+\ell_r\eta)}. \end{aligned}$$

Hence

$$\Lambda_{r,\boldsymbol{\epsilon}_{J^c}}(\mathbf{t})=-\sum_{\mathbf{x}_{J^c}}w^{(S)}(\mathbf{x};\mathbf{t};\xi_+,\xi_-)\gamma_k^{\boldsymbol{\epsilon}_{J^c}}(\mathbf{x}_{J^c};t_r)m_r^{\boldsymbol{\epsilon}^{\{k\},-},J}(\mathbf{x};e_r\mathbf{t}).$$

This in particular proves the desired expression of $\Lambda_{r,\boldsymbol{\epsilon}_{J^c}}(\mathbf{t})$ if $\epsilon_k=+$ and $\epsilon_j=-$ for $j\in J^c\setminus\{k\}$.

The formula for arbitrary $\boldsymbol{\epsilon}_{J^c} \in \{\pm\}^d$ follows by an induction argument with respect to $\#J_+^c$ using the following observation. For a subset $I \subseteq J_+^c$ set

$$\begin{aligned}\widetilde{q}_I(\mathbf{x}_{J^c};t_r) :=& \Big(\prod_{\substack{i,i'\in I:\\ i<i'}} \frac{\sinh(x_i+x_{i'}-\tau-\eta)}{\sinh(x_i+x_{i'}-\tau+\eta)}\Big)\\ &\times\Big(\prod_{i\in I}\frac{\sinh(\xi_+ + x_i - \frac{\tau}{2}-\frac{\eta}{2})\sinh(t_r+x_i+\ell_r\eta)}{\sinh(\xi_+ - x_i + \frac{\tau}{2}-\frac{\eta}{2})\sinh(t_r-x_i+\tau+\ell_r\eta)}\Big)\\ &\times \prod_{(i,j)\in I\times J^c\setminus I}\frac{\sinh(x_i+\epsilon_jx_j-\eta)\sinh(x_i-\epsilon_jx_j-\tau)}{\sinh(x_i+\epsilon_jx_j)\sinh(x_i-\epsilon_jx_j-\tau+\eta)}.\end{aligned}$$

Then $\widetilde{q}_\emptyset(\mathbf{x}_{J^c};t_r)=1$, $\widetilde{q}_{J_+^c}(\mathbf{x}_{J^c};t_r)=q_{J_+^c}(\mathbf{x}_{J^c};t_r)$ and for a subset $I\subset J_+^c$ and $k\in J_+^c\setminus I$,

$$\frac{\widetilde{q}_{I\cup\{k\}}(\mathbf{x}_{J^c};t_r)}{\widetilde{q}_I(\mathbf{x}_{J^c}-\tau\boldsymbol{e}_k;t_r)} = \gamma_k^{\boldsymbol{\epsilon}_{J^c}^{I,-}}(\mathbf{x}_{J^c};t_r).$$

The alternative expression for $\Upsilon_{r,\boldsymbol{\epsilon}_{J^c}}(\mathbf{t})$ follows from a similar computation, now using the observation that for $k\in J_+^c$,

$$\beta_k(\mathbf{x};\mathbf{t})m_r^{\boldsymbol{\epsilon},J}(\mathbf{x}-\tau\boldsymbol{e}_k;\mathbf{t}+\tau\boldsymbol{e}_r) = -\gamma_k^{\boldsymbol{\epsilon}_{J^c}}(\mathbf{x}_{J^c};-t_r-\tau)m_r^{\boldsymbol{\epsilon}^{\{k\},-},J}(\mathbf{x};\mathbf{t}+\tau\boldsymbol{e}_r). \quad \square$$

Note that (7.11) is satisfied if

$$G_{\xi_+,\xi_-}(\mathbf{x}) = \Big(\prod_{i=1}^S g_{\xi_+,\xi_-}(x_i)\Big)\prod_{1\le i<i'\le S}h(x_i\pm x_{i'})$$

with g_{ξ_+,ξ_-} and h as in Theorem 6.2.

By the explicit expression (7.10) of $m_r(\mathbf{x}_{J^c};\mathbf{t})$ we have

$$\begin{aligned}\widetilde{m}_r(\mathbf{x}_{J^c};\mathbf{t}) :=& m_r(\mathbf{x}_{J^c};e_r\mathbf{t})\prod_{j\in J^c}\sinh(t_r+x_j-\ell_r\eta)\\ =& m_r(\mathbf{x}_{J^c};\mathbf{t}+\tau\boldsymbol{e}_r)\prod_{j\in J^c}\sinh(-t_r-\tau+x_j-\ell_r\eta).\end{aligned} \tag{7.12}$$

Combined with Lemma 7.6, it follows that (7.9) is equivalent to

$$\begin{aligned}&\sum_{\mathbf{x}_{J^c}} w^{(S)}(\mathbf{x};\mathbf{t};\xi_+,\xi_-)\widetilde{m}_r(\mathbf{x}_{J^c};\mathbf{t})C_{d+1}^{\ell_r}(t_r+\frac{\tau}{2};\xi_+)\\ &\qquad\times\sum_{\boldsymbol{\epsilon}_{J^c}\in\{\pm\}^d}\frac{(-1)^{\#J_+^c}q_{J_+^c}(\mathbf{x}_{J^c};t_r)}{\prod_{j\in J^c}\sinh(t_r+x_j-\ell_r\eta)} =\\ &=\sum_{\mathbf{x}_{J^c}} w^{(S)}(\mathbf{x};\mathbf{t};\xi_+,\xi_-)\widetilde{m}_r(\mathbf{x}_{J^c};\mathbf{t})\sum_{\boldsymbol{\epsilon}_{J^c}\in\{\pm\}^d}\frac{(-1)^{\#J_+^c}q_{J_+^c}(\mathbf{x}_{J^c};-t_r-\tau)}{\prod_{j\in J^c}\sinh(-t_r-\tau+x_j-\ell_r\eta)}.\end{aligned}$$

Substituting the explicit expression (4.11) of $C_n^\ell(x;\xi)$, this is a direct consequence of the following lemma.

Lemma 7.7. *Let $J\subseteq\{1,\ldots,S\}$ be a subset of cardinality $S-d$ and $\boldsymbol{\epsilon}_{J^c}\in\{\pm\}^d$. Then the finite sum*

$$\Big(\prod_{n=1}^d\sinh(\xi_+-t_r-\frac{\tau}{2}+(\ell_r+\frac{1}{2}-n)\eta)\Big)\sum_{\boldsymbol{\epsilon}_{J^c}\in\{\pm\}^d}\frac{(-1)^{\#J_+^c}q_{J_+^c}(\mathbf{x}_{J^c};t_r)}{\prod_{j\in J^c}\sinh(t_r+x_j-\ell_r\eta)}$$

is invariant under the exchange of t_r *by* $-t_r-\tau$.

The proof of the lemma is given in the next subsection. It completes the proof of the main theorem (Theorem 6.2).

7.4. **Proof of Lemma 7.7.** Let $J\subseteq\{1,\ldots,S\}$ be a subset of cardinality $S-d$ and $\boldsymbol{\epsilon}_{J^c}\in\{\pm\}^d$. Choose an identification of the fixed subset J^c of cardinality d with $\{1,\ldots,d\}$. The choice of signs $\boldsymbol{\epsilon}_{J^c}\in\{\pm\}^d$ then is identified with choosing a subset $I\subseteq\{1,\ldots,d\}$ by the rule

$$I:=\{i\in\{1,\ldots,d\}\,|\,\epsilon_i=+\}.$$

Write $\xi=\xi_+-\frac{\eta}{2}$ and $\mathbf{x}=(x_1,\ldots,x_d)$. Then the statement in Lemma 7.7 is easily seen to be equivalent to the claim that

$$\begin{aligned}F(\mathbf{x};t):=\Big(\prod_{i=1}^d\frac{\sinh(\xi-t-\frac{\tau}{2}+(\ell+1-i)\eta)}{\sinh(t+x_i-\ell\eta)}\Big)\\ \times\sum_{I\subseteq\{1,\ldots,d\}}\Bigg\{(-1)^{\#I}\Big(\prod_{\substack{i,j\in I\\ i<j}}\frac{\sinh(x_i+x_j-\tau-\eta)}{\sinh(x_i+x_j-\tau+\eta)}\Big)\\ \times\Bigg(\prod_{i\in I}\frac{\sinh(\xi+x_i-\frac{\tau}{2})\sinh(t+x_i+\ell\eta)}{\sinh(\xi-x_i+\frac{\tau}{2})\sinh(t-x_i+\tau+\ell\eta)}\Bigg)\\ \times\prod_{(i,j)\in I\times I^c}\frac{\sinh(x_i-x_j-\eta)\sinh(x_i+x_j-\tau)}{\sinh(x_i-x_j)\sinh(x_i+x_j-\tau+\eta)}\Bigg)\Bigg\}\end{aligned}\tag{7.13}$$

satisfies

$$F(\mathbf{x};-t-\tau)=F(\mathbf{x};t).\tag{7.14}$$

By substituting $x_i\to x_i+\frac{\tau}{2}$ $(i=1,\ldots,d)$ and $t\to t-\frac{\tau}{2}$ and clearing denominators in (7.14), we obtain a trigonometric polynomial identity independent of τ. More precisely, for $i\in\{1,\ldots,d\}$ and $I\subseteq\{1,\ldots,d\}$ write $\epsilon_i^{(I)}=+$ if $i\in I$ and $\epsilon_i^{(I)}=-$ if $i\notin I$; also, write $x_i^{(I)}=x_i-\epsilon_i^{(I)}\frac{\eta}{2}$. For $I\subseteq\{1,\ldots,d\}$ we define

$$\begin{aligned}Q_I(\mathbf{x};t):=(-1)^{\#I}\Big(\prod_{i=1}^d\sinh(\xi+\epsilon_i^{(I)}x_i)\sinh(t+\epsilon_i^{(I)}x_i+\ell\eta)\Big)\\ \prod_{1\leq i<j\leq d}\sinh(x_i^{(I)}\pm x_j^{(I)})\end{aligned}$$

and write

$$V(\mathbf{x};t):=\Big(\prod_{i=1}^d\sinh(\xi-t+(\ell-i+1)\eta)\Big)\sum_{I\subseteq\{1,\ldots,d\}}Q_I(\mathbf{x};t).$$

Then (7.14) is equivalent to

$$V(\mathbf{x};t)=V(\mathbf{x};-t).\tag{7.15}$$

The identity (7.15) is a direct consequence of the following multivariate generalization of the trigonometric identity (4.5).

Lemma 7.8. *We have*

$$\sum_{I\subseteq\{1,\ldots,d\}} Q_I(\mathbf{x};t) = \Big(\prod_{1\le i<j\le d}\sinh(x_i\pm x_j)\Big)\prod_{i=1}^d \sinh(2x_i) \times (-1)^d\prod_{i=1}^d \sinh(\xi+t+(\ell-i+1)\eta). \tag{7.16}$$

Proof. Write $\mathbb{V}(\mathbf{x};t)$ for the left-hand side of (7.16). It is easy to see that

$$\mathbb{V}(\mathbf{x};t)\in\mathbb{C}[\mathrm{e}^{\pm 2x_1},\ldots,\mathrm{e}^{\pm 2x_d}],$$

since each term $Q_I(\mathbf{x};t)$ is a Laurent polynomial in $\mathrm{e}^{2x_1},\ldots,\mathrm{e}^{2x_d}$. We now first show that $\mathbb{V}(\mathbf{x};t)$ is anti-invariant with respect to the natural action of the Weyl group W of type C_d on $\mathbb{C}[\mathrm{e}^{\pm 2x_1},\ldots,\mathrm{e}^{\pm 2x_d}]$.

Let $W=\langle s_1,\ldots,s_d\rangle$ be the Weyl group of type C_d, with the simple reflections s_i $(i=1,\ldots,d)$ acting on $\mathbb{C}^d$ by permutations and sign flips: for $1\le i<d$ the simple reflection s_i acts on $(x_1,\ldots,x_d)\in\mathbb{C}^d$ by permuting x_i and x_{i+1}, and s_d acts by sending x_d to $-x_d$. The Weyl group W also acts on the power set of $\{1,\ldots,d\}$ by

$$s_iI=\begin{cases}(I\setminus\{i\})\cup\{i+1\}, & \text{if } i\in I,\, i+1\notin I,\\ (I\setminus\{i+1\})\cup\{i\}, & \text{if } i\notin I,\, i+1\in I,\\ I, & \text{otherwise}\end{cases}$$

for $1\le i<d$, and

$$s_dI=\begin{cases}I\setminus\{d\}, & \text{if } d\in I,\\ I\cup\{d\}, & \text{if } d\notin I.\end{cases}$$

Note that the action of W on the power set of $\{1,\ldots,d\}$ is transitive, and that the stabilizer subgroup of the empty set $\emptyset$ is equal to the symmetric group $S_d:=\langle s_1,\ldots,s_{d-1}\rangle$ in d letters.

By a direct computation we obtain the invariance property

$$Q_I(w\mathbf{x};t)=(-1)^{l(w)}Q_{w^{-1}I}(\mathbf{x};t),\qquad w\in W, \tag{7.17}$$

where $l(w)$ is the length of $w\in W$. It follows that

$$\mathbb{V}(\mathbf{x};t)=\frac{1}{d!}\sum_{w\in W}(-1)^{l(w)}Q_\emptyset(w^{-1}\mathbf{x};t),$$

in particular $\mathbb{V}(\mathbf{x};t)\in\mathbb{C}[\mathrm{e}^{\pm 2x_1},\ldots,\mathrm{e}^{\pm 2x_d}]$ is W-anti-invariant. Thus

$$\mathbb{V}(\mathbf{x};t)=Z(\mathbf{x};t)\delta(\mathbf{x}) \tag{7.18}$$

with the Weyl denominator

$$\delta(\mathbf{x}):=\Big(\prod_{1\le i<j\le d}\sinh(x_i\pm x_j)\Big)\prod_{i=1}^d\sinh(2x_i)$$

and with $Z(\mathbf{x};t)\in\mathbb{C}[\mathrm{e}^{\pm 2x_1},\ldots,\mathrm{e}^{2x_d}]$ W-invariant. A standard argument comparing degrees on both sides of (7.18) shows that $Z(\mathbf{x};t)$ is independent of $\mathbf{x}$. So

$$\mathbb{V}(\mathbf{x};t)=Z(t)\delta(\mathbf{x}) \tag{7.19}$$

for some constant $Z(t)$. We compute $Z(t)$ by evaluating both sides of (7.19) in

$$\mathbf{y} := (-\xi + (d-1)\eta, -\xi + (d-2)\eta, \ldots, -\xi).$$

By the explicit expression

$$Q_\emptyset(\mathbf{x};t) = \Big(\prod_{i=1}^{d} \sinh(\xi - x_i)\sinh(t - x_i + \ell\eta)\Big) \prod_{1\leq i<j\leq d} \sinh(x_i - x_j)\sinh(x_i + x_j + \eta)$$

it follows that $Q_\emptyset(w^{-1}\mathbf{y};t) = 0$ for $w \in W$ unless $w \in S_d$. Hence

$$\mathbb{V}(\mathbf{y};t) = \frac{1}{d!}\sum_{w\in S_d}(-1)^{l(w)}Q_\emptyset(w^{-1}\mathbf{y};t) = \frac{1}{d!}\sum_{w\in S_d} Q_{w\emptyset}(\mathbf{y};t) = Q_\emptyset(\mathbf{y};t),$$

and consequently

$$Z(t) = \frac{Q_\emptyset(\mathbf{y};t)}{\delta(\mathbf{y})} = (-1)^d \prod_{i=1}^{d} \sinh(\xi + t + (\ell - i + 1)\eta),$$

where the last equality follows from a straightforward computation. □

8. Fusion for the boundary qKZ equations and their solutions.

In this section we will show that, for $\underline{\ell} \in \frac{1}{2}\mathbb{Z}_{\geq 0}^N$, the solutions $f_S^{\underline{\ell}}(\mathbf{t})$ exhibited in Theorem 6.2 can be directly obtained using a fusion process from the spin-half solution $\big(\mathrm{pr}^{\frac{1}{2}}\big)^{\otimes N}\big(f_S^{(\frac{1}{2},\ldots,\frac{1}{2})}(\mathbf{t})\big)$ constructed before in [32]. Moreover, as we will see, arbitrary solutions of the boundary qKZ equations (1.3) in $M^{(\ell_1,\ldots,\ell_{s-1})}\otimes V^k \otimes V^{\frac{1}{2}} \otimes M^{(\ell_{s+1},\ldots,\ell_N)}$ can be naturally fused to obtain solutions in $M^{(\ell_1,\ldots,\ell_{s-1})} \otimes V^{k+\frac{1}{2}} \otimes M^{(\ell_{s+1},\ldots,\ell_N)}$.

8.1. **Notations.** In this section, we will slightly abuse notation when considering operators acting on a "mixed" N-fold tensor product made up of finite- and infinite-dimensional modules V^k ($k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$) and M^ℓ ($\ell \in \mathbb{C}$). For example, if $\ell_s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, there is a unique linear operator $\tilde{\Xi}_r^{\underline{\ell}}(\mathbf{t};\xi_+,\xi_-;\tau)$ on $M^{(\ell_1,\ldots,\ell_{s-1})}\otimes V^{\ell_s}\otimes M^{(\ell_{s+1},\ldots,\ell_N)}$ determined by

$$\begin{aligned}\tilde{\Xi}_r^{\underline{\ell}}(\mathbf{t};\xi_+,\xi_-;\tau)\Big(\mathrm{Id}_{M^{(\ell_1,\ldots,\ell_{s-1})}} \otimes \mathrm{pr}^{\ell_s} \otimes \mathrm{Id}_{M^{(\ell_{s+1},\ldots,\ell_N)}}\Big) &= \\ = \Big(\mathrm{Id}_{M^{(\ell_1,\ldots,\ell_{s-1})}} \otimes \mathrm{pr}^{\ell_s} \otimes \mathrm{Id}_{M^{(\ell_{s+1},\ldots,\ell_N)}}\Big)\Xi_r^{\underline{\ell}}(\mathbf{t};\xi_+,\xi_-;\tau);&\end{aligned}$$

we will denote the resulting operator $\tilde{\Xi}_r^{\underline{\ell}}(\mathbf{t};\xi_+,\xi_-;\tau)$ on $M^{(\ell_1,\ldots,\ell_{s-1})}\otimes V^{\ell_s}\otimes M^{(\ell_{s+1},\ldots,\ell_N)}$ simply by $\Xi_r^{\underline{\ell}}(\mathbf{t};\xi_+,\xi_-;\tau)$ as long as it is clear from context which tensor component we have projected onto its finite-dimensional quotient.

We will use this mild abuse of notation also when discussing the operators $T^{\underline{\ell}}(x;\mathbf{t})$, $\mathcal{U}^{\xi,\underline{\ell}}(x;\mathbf{t})$, $\mathcal{B}^{\xi,\underline{\ell}}(x;\mathbf{t})$, $\overline{\mathcal{B}}^{\xi,\underline{\ell}}(x;\mathbf{t})$ and $\overline{\mathcal{B}}^{\xi,(S),\underline{\ell}}(\mathbf{x};\mathbf{t})$. Similarly, we will use the notations $\Omega^{\underline{\ell}}$ and $f_S^{\underline{\ell}}(\mathbf{t})$ for those elements of $M^{(\ell_1,\ldots,\ell_{s-1})}\otimes V^{\ell_s}\otimes M^{(\ell_{s+1},\ldots,\ell_N)}$ that are actually given by $\mathrm{pr}^{\ell_s}\Omega^{\underline{\ell}}$ and $\mathrm{pr}^{\ell_s} f_S^{\underline{\ell}}(\mathbf{t})$, respectively.

To fuse the boundary qKZ transport operators $\Xi_r^{\underline{\ell}}(\mathbf{t}) := \Xi_r^{\underline{\ell}}(\mathbf{t};\xi_+,\xi_-;\tau)$, it is convenient to use the injection $j^k = P^{\frac{1}{2}k}\iota^k : V^{k+\frac{1}{2}} \hookrightarrow V^k \otimes V^{\frac{1}{2}}$ instead of ι^k. Let

$k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{C}$. The following "local" fusion relations in terms of j^k follow straightforwardly from Proposition 3.3 and (4.7) respectively,

$$(j^k \otimes \mathrm{Id}_{M^\ell})L^{k+\frac{1}{2},\ell}(x-y) = L_{23}^{\frac{1}{2}\ell}(x-k\eta-y)L_{13}^{k\ell}(x+\frac{\eta}{2}-y)(j^k \otimes \mathrm{Id}_{M^\ell}), \tag{8.1}$$

$$j^k K^{k+\frac{1}{2}}(x) = K_2^{\frac{1}{2}}(x-k\eta)R^{k\frac{1}{2}}(2x-(k-\frac{1}{2})\eta)K_1^k(x+\frac{\eta}{2})j^k. \tag{8.2}$$

Furthermore, in a similar way as we derived Proposition 3.3 and (8.1),

$$(j^k \otimes \mathrm{Id}_{M^\ell})L^{k+\frac{1}{2},\ell}(x-y) = L_{13}^{k\ell}(x-\frac{\eta}{2}-y)L_{23}^{\frac{1}{2}\ell}(x+k\eta-y)(j^k \otimes \mathrm{Id}_{M^\ell}). \tag{8.3}$$

Given $s = 1, \ldots, N$ and $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, denote

$$j_s^k := \mathrm{Id}_{M^{(\ell_1,\ldots,\ell_{s-1})}} \otimes j^k \otimes \mathrm{Id}_{M^{(\ell_{s+1},\ldots,\ell_N)}},$$

an injective map from $M^{(\ell_1,\ldots,\ell_{s-1})} \otimes V^{k+\frac{1}{2}} \otimes M^{(\ell_{s+1},\ldots,\ell_N)}$ to $M^{(\ell_1,\ldots,\ell_{s-1})} \otimes V^k \otimes V^{\frac{1}{2}} \otimes M^{(\ell_{s+1},\ldots,\ell_N)}$.

For the rest of this section, given $1 \leq s \leq N$ and $\underline{\ell} \in \mathbb{C}^N$ such that $\ell_s = k + \frac{1}{2}$ for $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, we write

$$\begin{aligned} \underline{\ell}' &= (\ell_1, \ldots, \ell_{s-1}, k, \frac{1}{2}, \ell_{s+1}, \ldots, \ell_N) \in \mathbb{C}^{N+1}, \\ \mathbf{t}' &= (t_1, \ldots, t_{s-1}, t_s + \frac{\eta}{2}, t_s - k\eta, t_{s+1}, \ldots, t_N), \end{aligned} \tag{8.4}$$

while $\mathbf{t} = (t_1, \ldots, t_{s-1}, t_s, t_{s+1}, \ldots, t_N)$ and $\underline{\ell} = (\ell_1, \ldots, \ell_N)$ with $\ell_s = k + \frac{1}{2}$.

8.2. **Fusion of transport operators.**

Proposition 8.1. *Let $1 \leq s \leq N$ and $\underline{\ell} \in \mathbb{C}^N$ such that $\ell_s = k + \frac{1}{2}$ for $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. For $1 \leq r \leq N$ we have*

$$j_s^k \Xi_r^{\underline{\ell}}(\mathbf{t}) = \begin{cases} \Xi_r^{\underline{\ell}'}(\mathbf{t}')j_s^k, & r < s, \\ \Xi_{s+1}^{\underline{\ell}'}(\mathbf{t}' + \mathbf{e}_s\tau)\Xi_s^{\underline{\ell}'}(\mathbf{t}')j_s^k, & r = s, \\ \Xi_{r+1}^{\underline{\ell}'}(\mathbf{t}')j_s^k, & r > s, \end{cases} \tag{8.5}$$

as linear operators $M^{(\ell_1,\ldots,\ell_{s-1})} \otimes V^{\ell_s} \otimes M^{(\ell_{s+1},\ldots,\ell_N)} \to M^{(\ell_1,\ldots,\ell_{s-1})} \otimes V^k \otimes V^{\frac{1}{2}} \otimes M^{(\ell_{s+1},\ldots,\ell_N)}$.

Proof. For the cases where $r \neq s$, simply by judiciously applying (8.1-8.3) to the right-hand side of (8.5) (see (1.4) for the definition of the transport operators). For $r = s$, the product of factors in $\Xi_{s+1}^{\underline{\ell}'}(\mathbf{t}' + \mathbf{e}_s\tau)\Xi_s^{\underline{\ell}'}(\mathbf{t}')$ can first be simplified using unitarity of the R-operator and the RLL-relations (2.4), yielding

$$
\begin{aligned}
\Xi^{\underline{\ell}'}_{s+1}(\mathbf{t}'+\mathbf{e}_s\tau)\Xi^{\underline{\ell}'}_{s}(\mathbf{t}') &= \left(\prod_{j=s+1}^{N} L^{\frac{1}{2}\ell_j}(t_s-t_j+\tau-k\eta)L^{k\ell_j}(t_s-t_j+\tau+\frac{\eta}{2})\right)\\
&\times K^{\xi_+,\frac{1}{2}}(t_s+\frac{\tau}{2}-k\eta)R^{k\frac{1}{2}}(2(t_s+\frac{\tau}{2})-(k-\frac{1}{2})\eta)K^{\xi_+,k}(t_s+\frac{\tau}{2}+\frac{\eta}{2})\\
&\times\left(\prod_{\substack{j=N\\ j\neq s}}^{1} L^{\frac{1}{2}\ell_j}(t_j+t_s-k\eta)L^{k\ell_j}(t_j+t_s+\frac{\eta}{2})\right)\\
&\times K^{\xi_-,\frac{1}{2}}(t_s-k\eta)R^{\frac{1}{2}k}(2t_s-(k-\frac{1}{2})\eta)K^{\xi_-,k}(t_s+\frac{\eta}{2})\\
&\times\left(\prod_{j=1}^{s-1} L^{\frac{1}{2}\ell_j}(t_s-t_j-k\eta)L^{k\ell_j}(t_s-t_j+\frac{\eta}{2})\right),
\end{aligned}
$$

where the ordering of the products over j is as prescribed. Now applying (8.1-8.2) yields (8.5) for the case $r=s$. $\square$

8.3. **Fusion of solutions.**

Proposition 8.2. *Let $1\le s\le N$ and $\underline{\ell}\in\mathbb{C}^N$ such that $\ell_s=k+\frac{1}{2}$ for $k\in\frac{1}{2}\mathbb{Z}_{\ge 0}$. Suppose that $f:\mathbb{C}^{N+1}\to M^{(\ell_1,\ldots,\ell_{s-1})}\otimes V^k\otimes V^{\frac{1}{2}}\otimes M^{(\ell_{s+1},\ldots,\ell_N)}$ is a meromorphic solution of the boundary qKZ equations,*

$$
\Xi^{\underline{\ell}'}_{r}(\mathbf{z})f(\mathbf{z})=f(\mathbf{z}+\tau\mathbf{e}_r),\qquad 1\le r\le N+1, \tag{8.6}
$$

where $\underline{\ell}'$ is given by (8.4). Suppose that f restricts to a meromorphic function on the hyperplane

$$
H:=\{\mathbf{z}\in\mathbb{C}^{N+1}\mid z_s-z_{s+1}=(k+\frac{1}{2})\eta\}.
$$

Then there exists a unique meromorphic function

$$
\mathrm{Fus}^{\underline{\ell}}_s(f):\mathbb{C}^N\to M^{(\ell_1,\ldots,\ell_{s-1})}\otimes V^{k+\frac{1}{2}}\otimes M^{(\ell_{s+1},\ldots,\ell_N)}
$$

satisfying

$$
j^k_s\mathrm{Fus}^{\underline{\ell}}_s(f)(\mathbf{t})=f(\mathbf{t}'), \tag{8.7}
$$

with $\mathbf{t}'$ given by (8.4). Furthermore, $\mathrm{Fus}^{\underline{\ell}}_s(f)$ is a meromorphic solution of the boundary qKZ equations (1.3) with values in $M^{(\ell_1,\ldots,\ell_{s-1})}\otimes V^{k+\frac{1}{2}}\otimes M^{(\ell_{s+1},\ldots,\ell_N)}$,

$$
\Xi^{\underline{\ell}}_{r}(\mathbf{t})\mathrm{Fus}^{k}_s(f)(\mathbf{t})=\mathrm{Fus}^{k}_s(f)(\mathbf{t}+\tau\mathbf{e}_r),\qquad 1\le r\le N. \tag{8.8}
$$

Proof. It follows from (8.6) with $r=s$ that $f(\mathbf{z})=\Xi^{\underline{\ell}'}_{s}(\mathbf{z}-\tau\mathbf{e}_s)f(\mathbf{z}-\tau\mathbf{e}_s)$. By assumption the left-hand side restricts to a meromorphic vector valued function on H. By the explicit expressions (1.4) for the transport operators, the operator $\Xi^{\underline{\ell}'}_{s}(\mathbf{z}-\tau\mathbf{e}_s)$ restricts to a meromorphic operator valued function on H, and

$$
\Xi^{\underline{\ell}'}_{s}(\cdot-\tau\mathbf{e}_s)|_H=R^{k\frac{1}{2}}((k+\frac{1}{2})\eta)Z(\cdot)
$$

for some meromorphic operator valued function Z on H. Hence $f|_H$ takes its values in the subspace $\mathrm{Im}(R^{k\frac{1}{2}}((k+\frac{1}{2})\eta))$ of $M^{(\ell_1,\ldots,\ell_{s-1})}\otimes V^k\otimes V^{\frac{1}{2}}\otimes M^{(\ell_{s+1},\ldots,\ell_N)}$. By

Lemma 3.2 we have $\mathrm{Im}(R^{k\frac{1}{2}}((k+\frac{1}{2})\eta)) \subseteq \mathrm{Im}(j_s^k)$. Since j_s^k is injective, we conclude that there exists a unique meromorphic function

$$\mathrm{Fus}_s^{\underline{\ell}}(f) : \mathbb{C}^N \to M^{(\ell_1,\ldots,\ell_{s-1})} \otimes V^{k+\frac{1}{2}} \otimes M^{(\ell_{s+1},\ldots,\ell_N)}$$

satisfying (8.7).

It remains to show that $\mathrm{Fus}_s^{\underline{\ell}}(f)$ satisfies the boundary qKZ equations (8.8). Since j^k is an injection, it suffices to prove that, for $r = 1, \ldots, N$,

$$j_s^k \Xi_r^{\underline{\ell}}(\mathbf{t})\mathrm{Fus}_s^{\underline{\ell}}(f)(\mathbf{t}) = j_s^k \mathrm{Fus}_s^{\underline{\ell}}(f)(\mathbf{t}+\tau\mathbf{e}_r). \tag{8.9}$$

For $r < s$ we have

$$j_s^k \Xi_r^{\underline{\ell}}(\mathbf{t})\mathrm{Fus}_s^{\underline{\ell}}(f)(\mathbf{t}) = \Xi_r^{\underline{\ell}'}(\mathbf{t}')f(\mathbf{t}') = f(\mathbf{t}'+\tau\mathbf{e}_r) = j_s^k \mathrm{Fus}_s^{\underline{\ell}}(f)(\mathbf{t}+\tau\mathbf{e}_r),$$

owing to (8.5), (8.7), the boundary qKZ equations (8.6) and (8.7) again. The case $r > s$ of (8.9) is proven similarly. Finally, for $r = s$ we have

$$\begin{aligned} j_s^k \Xi_s^{\underline{\ell}}(\mathbf{t})\mathrm{Fus}_s^{\underline{\ell}}(f)(\mathbf{t}) &= \Xi_{s+1}^{\underline{\ell}'}(\mathbf{t}'+\tau\mathbf{e}_s)\Xi_s^{\underline{\ell}'}(\mathbf{t}')f(\mathbf{t}') \\ &= \Xi_{s+1}^{\underline{\ell}'}(\mathbf{t}'+\tau\mathbf{e}_s)f(\mathbf{t}'+\tau\mathbf{e}_s) \\ &= f(\mathbf{t}'+\tau\mathbf{e}_s+\tau\mathbf{e}_{s+1}) = j_s^k \mathrm{Fus}_s^{\underline{\ell}}(f)(\mathbf{t}+\tau\mathbf{e}_s), \end{aligned}$$

where we have applied (8.5), (8.7), (8.6) twice, and finally (8.7) again. □

8.4. **Fusion of the Jackson integral solutions.** The special Jackson integral solutions of the boundary qKZ equations (see Theorem 6.2) are compatible with fusion in the following sense.

Proposition 8.3. *Let $1 \leq s \leq N$ and $\underline{\ell} \in \mathbb{C}^N$ such that $\ell_s = k + \frac{1}{2}$ with $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Let $\ell' \in \mathbb{C}^{N+1}$ be given by (8.4). Let*

$$f_S^{\underline{\ell}} : \mathbb{C}^N \to M^{(\ell_1,\ldots,\ell_{s-1})} \otimes V^{k+\frac{1}{2}} \otimes M^{(\ell_{s+1},\ldots,\ell_N)}$$

and

$$f_S^{\underline{\ell}'} : \mathbb{C}^{N+1} \to M^{(\ell_1,\ldots,\ell_{s-1})} \otimes V^k \otimes V^{\frac{1}{2}} \otimes M^{(\ell_s,\ldots,\ell_N)}$$

be the Jackson integral solutions of the boundary qKZ equations as given in Theorem 6.2, with $f_S^{\underline{\ell}}$ and $f_S^{\underline{\ell}'}$ having the same base point $\mathbf{x}_0 \in \mathbb{C}^S$, the same weight factors g_{ξ_+,ξ_-}, h and F^{ℓ_j} ($j \in \{1, \ldots, N\} \setminus \{s\}$) and with the remaining weight factors $F^{k+\frac{1}{2}}$, F^k and $F^{\frac{1}{2}}$ satisfying the compatibility condition

$$F^{k+\frac{1}{2}}(x) = F^k(x+\frac{\eta}{2})F^{\frac{1}{2}}(x-k\eta). \tag{8.10}$$

Then

$$f_S^{\underline{\ell}} = \mathrm{Fus}_s^{\underline{\ell}}(f_S^{\underline{\ell}'}).$$

Remark 8.4. Note that (8.10) is compatible with the difference equations that $F^\ell(x)$ satisfies (see Theorem 6.2). Note furthermore that the explicit choice (6.1) of $F^\ell(x)$ ($\ell \in \mathbb{C}$) satisfies (8.10).

Proof. By virtue of the fusion formulae (8.3) and (8.1), we have (cf. (5.2))

$$j_s^k T^{\underline{\ell}}(x;\mathbf{t}) = T^{\underline{\ell}'}(x;\mathbf{t}')j_s^k, \qquad j_s^k T^{\underline{\ell}}(x;\mathbf{t})^{-1} = T^{\underline{\ell}'}(x;\mathbf{t}')^{-1}j_s^k,$$

where we use the notations (8.4). Hence, owing to (5.4) we also have

$$j_s^k \mathcal{U}^{\xi,\underline{\ell}}(x;\mathbf{t}) = \mathcal{U}^{\xi,\underline{\ell}'}(x;\mathbf{t}')j_s^k. \tag{8.11}$$

The above three identities are as operators $V^{\frac{1}{2}}\otimes M^{(\ell_1,\ldots,\ell_{s-1})}\otimes V^{k+\frac{1}{2}}\otimes M^{(\ell_{s+1},\ldots,\ell_N)}$ $\to V^{\frac{1}{2}}\otimes M^{(\ell_1,\ldots,\ell_{s-1})}\otimes V^k\otimes V^{\frac{1}{2}}\otimes M^{(\ell_{s+1},\ldots,\ell_N)}$. Taking the appropriate matrix coefficients in (8.11) with respect to the auxiliary space, we obtain

$$j_s^k\mathcal{B}^{\xi,\underline{\ell}}(x;\mathbf{t})=\mathcal{B}^{\xi,\underline{\ell}'}(x;\mathbf{t}')j_s^k$$

as operators $M^{(\ell_1,\ldots,\ell_{s-1})}\otimes V^{k+\frac{1}{2}}\otimes M^{(\ell_{s+1},\ldots,\ell_N)}\to M^{(\ell_1,\ldots,\ell_{s-1})}\otimes V^k\otimes V^{\frac{1}{2}}\otimes M^{(\ell_{s+1},\ldots,\ell_N)}$.

Writing

$$\frac{\sinh(x-t_s-(k+\frac{1}{2})\eta)}{\sinh(x-t_s+(k+\frac{1}{2})\eta)}=\frac{\sinh(x-(t_s+\frac{\eta}{2})-k\eta)}{\sinh(x-(t_s+\frac{\eta}{2})+k\eta)}\frac{\sinh(x-(t_s-k\eta)-\frac{\eta}{2})}{\sinh(x-(t_s-k\eta)+\frac{\eta}{2})}$$

it follows that

$$j_s^k\overline{\mathcal{B}}^{\xi,\underline{\ell}}(x;\mathbf{t})=\overline{\mathcal{B}}^{\xi,\underline{\ell}'}(x;\mathbf{t}')j_s^k$$

and hence

$$j_s^k\overline{\mathcal{B}}^{\xi,(S),\underline{\ell}}(\mathbf{x};\mathbf{t})=\overline{\mathcal{B}}^{\xi,(S),\underline{\ell}'}(\mathbf{x};\mathbf{t}')j_s^k. \tag{8.12}$$

Since $j_s^k\Omega^{\underline{\ell}}=\Omega^{\underline{\ell}'}$ (see Proposition 3.1) it now follows from (8.10) that

$$j_s^kf_S^{\underline{\ell}}(\mathbf{t})=f_S^{\underline{\ell}'}(\mathbf{t}')=j_s^k\mathrm{Fus}_s^{\underline{\ell}}(f_S^{\underline{\ell}'})(\mathbf{t})$$

as meromorphic $M^{(\ell_1,\ldots,\ell_{s-1})}\otimes V^k\otimes V^{\frac{1}{2}}\otimes M^{(\ell_s,\ldots,\ell_N)}$ valued functions in $\mathbf{t}\in\mathbb{C}^N$, which proves the result. □

Remark 8.5. Note that $\sum_{r=1}^N\ell_r=\sum_{r=1}^{N+1}\ell_r'$ for $\underline{\ell}\in\mathbb{C}^N$ with $\ell_s=k$ and with $\underline{\ell}'$ given by (8.4). Hence the region of meromorphic convergence (6.2) for the solutions $f_S^{\underline{\ell}}$ and $f_S^{\underline{\ell}'}$ with weight factors (6.1) is compatible with fusion.

References

[1] Z. Bajnok, L. Palla, G. Takács, *On the boundary factor program*, Nucl. Phys. B **750** (2006), no. 3, 179–212.

[2] P. Baseilhac, S. Belliard, *The half-infinite XXZ chain in Onsager's approach*, Nucl. Phys. B **873** (2013), no. 3, 550–584.

[3] P. Baseilhac, T. Kojima, *Correlation functions of the half-infinite XXZ spin chain with a triangular boundary*, Nucl. Phys. B **880** (2014), 378–413.

[4] H. Boos, A. Göhmann, A. Klümper, K.S. Nirov, A.V. Razumov, *Exercises with the universal R-matrix*, J. Phys. A **43** (2010), no. 41, 415208, 35 pp.

[5] V. Chari, A. Pressley, *Quantum affine algebras*, Comm. Math. Phys. **142** (1991), 261–283.

[6] I. Cherednik, *Quantum Knizhnik-Zamolodchikov equations and affine root systems*, Comm. Math. Phys. **150** (1992), 109–136.

[7] I. Cherednik, *Factorizing particles on a half line, and root systems*, Theoret. and Math. Phys. **61** (1984), no. 1, 977-983.

[8] G.W. Delius, *Solitons, boundaries, and quantum affine algebras*, arXiv:math/0212317.

[9] G.W. Delius, N.J. MacKay, *Quantum group symmetry in sine-Gordon and affine Toda field theories on the half-line*, Comm. Math. Phys. **233** (2003), 173–190.

[10] G.W. Delius, N.J. MacKay, *Affine quantum groups*, arXiv:math/0607228.

[11] G.W. Delius, R.I. Nepomechie, *Solutions of the boundary Yang-Baxter equation for arbitrary spin*, J. Phys. A **35** (2002), no. 24, L341–L348.

[12] H.J. de Vega, A. Gonzalez-Ruiz, *Boundary K-matrices for the XYZ, XXZ and XXX spin chains*, J. Phys. A **27** (1994), no. 18, 6129–6137.

[13] J. Donin, P.P. Kulish, A.I. Mudrov, *On a universal solution to the reflection equation*, Lett. Math. Phys. **63** (2003), no. 3, 179–194.

[14] V. G. Drinfeld, *Quantum groups*, Proceedings ICM, Vol. 1,2 (Berkeley 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987.

[15] P.I. Etingof, I.B. Frenkel, A.A. Kirillov Jr., *Lectures on Representation Theory and Knizhnik-Zamolodchikov equations*, Math. Surveys and Monographs, Vol. 58, Amer. Math. Soc., 1998.

[16] G. Filali, N. Kitanine, *Spin chains with non-diagonal boundaries and trigonometric SOS model with reflecting end*, SIGMA **7** (2011), Paper 012, 22 pages.

[17] I. Frenkel, N. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Comm. Math. Phys. **146** (1992), 1–60.

[18] P. Di Francesco, *Boundary qKZ equation and generalized Razumov-Stroganov sum roles for open IRF models*, J. Stat. Mech. Theory Exp. 2005, no. 11, P11003, 18pp.

[19] P. Di Francesco, P. Zinn-Justin, *Quantum Kniahnik-Zamolodchikov equation: reflecting boundary conditions and combinatorics*, J. Stat. Mech. Theory Exp. 2007, no. 12, P12009, 30pp.

[20] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Encycl. Math. and Appl. **96**, Cambridge University Press 2004 (2nd edition).

[21] T. Inami, S. Odake, Y.-Z. Zhang, *Reflection K-Matrices of the* 19*-Vertex Model and XXZ Spin*-1 *Chain with General Boundary Terms*, Nucl. Phys. B **470** (1996), no. 3, 419–432.

[22] M. Jimbo, R. Kedem, H. Konno, T. Miwa, R. Weston, *Difference equations in spin chains with a boundary*, Nucl. Phys. B **448** (1995), no. 3, 429–456.

[23] P.P. Kulish; N. Yu. Reshetikhin, *Quantum linear problem for the sine-Gordon equation and higher representations* (Russian). Questions in quantum field theory and statistical physics, 2. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **101** (1981), 101–110, 207.

[24] P.P. Kulish, N.Yu. Reshetikhin, E.K. Sklyanin, *Yang-Baxter equations and representation theory. I*, Lett. Math. Phys. **5** (1981), no. 5, 393–403.

[25] P.P. Kulish, E.K. Sklyanin, *Algebraic structures related to reflection equations*, J. Phys. A **25** (1992), no. 22, 5963–5975.

[26] L. Mezincescu, R.I. Nepomechie, *Integrable open spin chains with non-symmetric R-matrices*, J. Phys. A: Math. Gen. **24** (1991), L17–L23.

[27] L. Mezincescu, R.I. Nepomechie, *Fusion procedure for open chains*, J. Phys. A: Math. Gen. **25** (1992), 2533–2543.

[28] L. Mezincescu, R.I. Nepomechie, V. Rittenberg, *Bethe ansatz solution to the Fateev-Zamolodchikov quantum spin chain with boundary terms*, Phys. Lett. A **147** (1990), no. 1, 70–78.

[29] R.I. Nepomechie, *Bethe ansatz solution of the open XXZ chain with nondiagonal boundaries*, J. Phys. A: Math. Gen. **37** (2004), 433–440.

[30] N. Reshetikhin, *Jackson-type integrals, Bethe vectors, and solutions to a difference analog of the Knizhnik-Zamolodchikov system*, Lett. Math. Phys. **26** (1992), 153–165.

[31] N. Reshetikhin, "Lectures on integrable models in statistical mechanics", In: "Exact methods in low-dimensional statistical physics and quantum computing", Proceedings of Les Houches School in Theoretical Physics, Oxford University Press, 2010.

[32] N. Resketikhin, J.V. Stokman, B.H.M. Vlaar, *Reflection quantum Knizhnik-Zamolodchikov equations and Bethe vectors*, arXiv:1305.1113.

[33] S.N.M. Ruijsenaars, *First order analytic difference equations and integrable quantum systems*, J. Math. Phys. **38**, no. 2 (1997), 1069–1146.

[34] E.K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. A: Math. Gen. **21** (1988), 2375–2389.

[35] F. Smirnov, *A general formula for soliton form factors in the quantum sine-Gordon model*, J. Phys. A **19**, no. 10 (1986), L575–L578.

[36] L.A. Takhtadzhan, L.D. Faddeev, *The quantum method of the inverse problem and the Heisenberg XYZ model*, Russian Math. Surveys **34**, no. 5 (1979), 11–68.

[37] R. Weston, *Correlation functions and the boundary qKZ equations in a fractured XXZ chain*, J. Stat. Mech. Theory Exp. 2011, no. 12, P12002, 28pp.

[38] Y. Zhou, *Fusion hierarchies with open boundaries and exactly solvable models*, in "Statistical models, Yang-Baxter equation and related topics, and Symmetry, statistical mechanical models and applications" (Tianjin, 1995), 351358, World Sci. Publ., River Edge, NJ, 1996.

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