On the effects of changing mortality patterns on investment, labour and consumption under uncertainty

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Abstract

In this paper we extend the consumption-investment life cycle model for an uncertain-lived agent, proposed by Richard (1974), to allow for flexible labor supply. We further study the consumption, labor supply and portfolio decisions of an agent facing age-dependent mortality risk, as presented by UK actuarial life tables spanning the time period from 1951-2060 (including mortality forecasts). We find that historical changes in mortality produces significant changes in portfolio investment (more risk taking), labour (decrease of hours) and consumption level (shift to higher level) contributing up to 5% to GDP growth during the period from 1980 until 2010.

Keywords: Life-Cycles, Portfolio Investment, Flexible Labor, Age-Dependent Mortality Rates and Uncertain Lifetime

JEL Subject Classification: G11; J11; J22; C61;

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1 Introduction

Lifetime consumption and investment models for infinitely lived agents have been considered by various authors, including Merton (1969 and 1971), Bodie, Merton and Samuelson (1992) as well as Bodie et al. (2004). The setup in all of these contributions is very similar, they all study the problem of maximizing expected discounted utility under consideration of a utility function which includes consumption and in some cases leisure, over the life time of a representative agent. Bodie, Merton and Samuelson (1992) considered an exogenously given retirement age and left it as an open question, to determine the optimal retirement age within an optimal stopping context. This problem has now been considered by Dybvig and Liu (2010). Zhang (2010) considered retirement age as exogenously given, but allowed for fully flexible labour supply, in essence including retirement as an option for the agent. Davis, Kubler and Willen (2006) argued that equity holdings over the life cycle in the classical Merton (1969 and 1971) models were unrealistically high and emphasize the aspect of borrowing costs/constraints. They did not account for flexible labour and utility from leisure however.

In reality of course agents are not infinitely lived. Richard (1974) extended Merton's (1971) model to allow for a finitely lived agent with a random time of death. He introduced a bequest motive and life insurance into the model and considered the problem of optimal investment into the life insurance product. Using the Hamilton-Jacobi-Bellman framework he derived analytic expressions for the optimal portfolio rule, consumption rate and life insurance under constant relative risk aversion (CRRA). In that case, mortality enters into the optimal portfolio rule, which is the same fraction as in Merton (1971) but multiplied by the ratio of total

wealth (including mortality dependent human wealth, i.e. future incomes until death) to financial wealth. However, Richard (1974) did not consequently study how changes in mortality affected his optimal portfolio rule. Optimal consumption in Richard (1974) is a time dependent fraction of total wealth, but from the expression derived, it is not clear how consumption shifts and how consumption growth is affected as a consequence of the changes in mortality. Richard (1974) did not allow for flexible labour decisions either.

Milevsky and Young (2007) modified the framework presented by Richard (1974) to take account of some institutional issues related to the purchase and payout of annuities. In fact their focus was on the optimal annuitization when the agent is already retired and does not receive any labour income. Milevsky and Young (2007) did account for mortality, but by using a Hamilton-Jacobi-Bellman approach, real mortality data do not enter their model directly, but through a suitably parametrized Gompertz-Makeham hazard rate function. In addition, the resulting Hamilton-Jacobi-Bellman equation has been linearized, leading to approximate solutions only. A semi-analytic solution was presented for the case of constant force of mortality only. Huang et. al. (2012) study a Yaari (1965) et. al. framework with stochastic force of mortality but focus on consumption only, leaving out stochastic investment and labor. In conclusion, neither Richard (1974) nor Milevsky and Young (2007) or Huang et. al. (2012) did study the effects that historical changes in mortality rates cause on the agent's optimal strategy.

Pang and Warshawsky (2010) also studied a portfolio problem, involving riskless and risky assets as well as annuities. Agents in their model have exposure to mortality risk as well as uninsured health care costs. Their model was in discrete time and no attempt was made to solve the model analytically. Instead the model

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was solved numerically and results are based on simulation. Their main observation is that health spending risk drives the agent's portfolio to shift from risky assets to safer assets. As in Richard (1974) and Milevsky and Young (2007), their study does not involve an investigation on how changes in actually observed and predicted mortality rates affects the agent's optimal strategy.

More recently, Gahramanov and Tang (2013) have presented a paper in which they considered the retirement problem in a continuous time model with time varying mortality. However their work differs from ours, in that they focused on the retirement problem with mortality being given by an explicit analytic function as in Feigenbaum (2008). Furthermore, they did not allow for investment into risky assets.

One of the main contributions of this article is the inclusion of time varying, general mortality risk into a continuous time stochastic life time consumption model, where a representative agent chooses consumption, labor supply and portfolio investment into a risk-less and a risky asset and in consequence a rigorous study on how historically observed changes in the mortality patterns affect the agent's decisions of portfolio selection, consumption and labor supply. We adopt a CRRA type of utility function measuring utility from consumption against dis-utility from supplying labor. We assume no bequest motive, and in consequence the agent's optimal life insurance strategy is to contract their respective wealth to be transferred to a life insurance company at the time of their death in exchange for a fairly priced annuity as proposed by Yaari (1964 and 1965) and Blanchard (1985).[1] To solve our model, rather than using the Hamilton-Jacobi-Bellman framework, which

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[1]Yaari (1964 and 1965) and Blanchard (1985) did not consider risky investment and Blanchard (1985) only considered a constant mortality rate.

seems less flexible in the context of general time varying mortality curves, we use a combination of Martingale techniques that have evolved from the Mathematical Finance literature, see for example the exposition in Korn (2001) and Zhang (2008) or the original work by Pliska (1986), Karatzas (1987) and Cox and Huang (1988).[2] The use of these methods enables us to derive analytic expressions for the optimal consumption, labour supply and portfolio investment process in the presence of mortality risk. We are further able to derive a compact form for the Euler equation of consumption growth. As a first result we find that the effect of mortality risk on consumption and labour supply is through the Lagrange multiplier of the associated static constrained optimization problem only, and as such it shifts consumption and labor supply, but has no effect on the Euler equation. This effect was not observed in Richard (1974). Mortality risk also affects optimal portfolio investment, but in a more subtle way than in Richard (1974) due to the presence of flexible labour.

Generally, the presence of mortality in a lifetime consumption context leads to a number of interesting effects and trade-offs, which so far existing models have not been able to capture and quantify. Longer life expectancy will emphasize the aspect of pre-cautionary savings for old age. In addition fear of death might encourage people to consume their goods sooner than later (while still alive). Both of these mechanisms cause an effect where an increase in mortality increases current consumption. However, when life expectancy increases, longer (working) lives will increase human wealth and thus increase current consumption and investment.

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[2]By considering the mortality rates obtained from the Office for National Statistics as deterministic piecewise linear functions, it is possible to solve the model via the Hamilton-Jacobi-Bellman equation. This requires to solve the corresponding PDEs on 110 intervals, each according to one year between 1951 and 2060 and gluing the solutions together at the respective boundaries.

This mechanism works in the opposite direction, i.e. an increase in mortality contributes to a decrease in current consumption. Finally, risk taking behaviour in investment will also be altered, as long-term investment horizons will increase in length and thus making risky assets more attractive.

We also derive a closed-form expression for the elasticity of consumption with respect to the mortality rate. Using realistic parameters we find that this elasticity is negative, within the range of 0 (i.e zero mortality rate) to -0.53 (equivalent to a mortality rate of 0.002 which corresponds roughly to the mortality rate of a 39 year old UK male). In the empirical part of the paper we have used actual and forecasted mortality curves as obtained from statistical life tables supplied by the UK's Office for National Statistics covering the years from 1951 until 2060. Substituting these curves into our model we observe that keeping all other parameters constant, changes in the mortality curves from 1980 to 2010 lead to a shift in consumption upwards of roughly 5%, contributing to a total of approximately 100% in real GDP growth in the UK from 1980 to 2010.[3] We also observe that optimal labour supply in effect of the same changes of the mortality curves is reduced by 4%, from about 40.2 hours to 38.7 hours per week from 1980 to 2010. Finally, portfolio investment into the risky asset is increased by a factor of roughly 6%, financing the reduction in labour and increase in consumption. Therefore, we conclude that historical changes in mortality risk do indeed have significant impact on consumption spending, labour supply and portfolio investment.

The remainder of the paper is organized as follows. In section 2, we set up our model and derive some basic equations, while in section 3, we consequently

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[3]Historical data for real GDP have been obtained via $https : //docs.google.com/spreadsheet/ ccc?key = 0AonYZs4MzlZbcGhOdG0zTG1EWkVPX1k1VWR6LTd1U3c\#gid = 1$. The shift in consumption over the whole data period from 1951 until 2060 is about 12 %.

proceed by using Martingale methods in order to transform the dynamic problem into a constrained static problem, which allows us to solve the dynamic problem explicitly. Section 4 contains both theoretical and empirically founded examples, while the main conclusions are summarized in section 5.

2   The Model

Let us consider a representative uncertain lived agent financing consumption through labour income and investment into one risky and one riskless asset, who wishes to maximize her/his expected life-time utility. The agent also has the option at any time to contract his wealth at time of death to be returned to a life insurance company in exchange for payment of an annuity, which will be determined below. As in Blanchard (1985) and Yaari (1965) Case C there is no bequest motive. Mathematically, the agent is trying to solve the following maximization problem:

$$\max_{\pi, C, L, A} \mathbb{E} \left(\int_0^\tau e^{- \int_0^t \rho_s ds} u(C_t, L_t) dt \right). \tag{1}$$

Here τ denotes the time of death, C_t denotes instantaneous consumption, L_t denotes instantaneous labour supply and π_t is the fraction of financial wealth invested into the risky asset. The control A_t takes values in $[0, 1]$ and represents the fraction of wealth at time of death that is contracted to the life insurance company.[4]   As in Yaari (1965) Case C, we assume that the agent is required to have positive wealth at time of death. This is guaranteed if the agent contracts all of her/his wealth to the life insurance company. This is also optimal as otherwise the agent would forsake payments from the annuity, whereas the agent would

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[4]The subscript t denotes 'at time t' throughout, unless otherwise stated.

not benefit from leftover wealth after death (no bequest motive). In fact, any admissible strategy (π, C, L, A) will be dominated by the corresponding strategy $(\pi, C, L, 1)$, where all wealth at time of death is contracted to the life insurance company. In consequence, we can solve problem (1) assuming $A \equiv 1$ and remove A as a control. This is done in the following. In our setup time 0 stands for an arbitrarily chosen reference time and can be viewed as the starting age, when the agent enters the labor market. In our main empirical example we will assume that this is at age 25, when the agent starts making rational decisions about consumption, labour and investment. However, we will also consider different ages. We assume that $C_t \geq 0$, $L_t \geq 0$ and π_t are chosen by the agent depending on information contained in the sigma algebra $\mathcal{F}_t$ which will be introduced below. The investment assets available to the agent will also be introduced below. The time preference rate ρ_s of the agent is assumed to be a deterministic and positive function, while the time of death will be considered as random, with

$$\mathbb{P}\left(\tau \in [t, t+dt) | \tau \geq t\right) = \nu_t dt, \tag{2}$$

where ν_t is the time dependent instantaneous mortality rate. Intuitively, the mortality rate ν_t describes the likelihood of the agent aged t dying in the interval $[t, t+dt)$ given she/he is still alive at time t. This rate can be easily obtained from actuarial life tables and in general differs regionally, historically and by gender. We assume that ν_t is a deterministic function of time.[5] Under this assumption,

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[5]The variable ν_t is also referred to as force of mortality. Models with stochastic force of mortality have been considered in the actuarial literature. In this literature it is typically assumed that $\mathbb{P}(\tau > t | \tau > s, \mathcal{F}_s) = \mathbb{E}\left[e^{-\int_s^t \nu_s ds} \Big| \mathcal{F}_s\right]$ where ν_s is a stochastic process. Often ν_s is assumed to follow a diffusion or jump process. Lee and Carter (1992), Cairns et. al. (2006), Wills and Sherris (2010) as well as Huang et. al. (2012) are important contributions to this literature. Most of our results could in principle be generalized to cases, where this conditional expectation

the agent's likelihood of surviving until age t is given by

$$\mathbb{P}(\tau > t) = e^{-\int_0^t \nu_s ds}. \tag{3}$$

We assume that the random time τ is independent of any of the economic state variables[6], and hence we obtain[7]

$$\mathbb{E}\left(\int_0^\tau e^{-\int_0^t \rho_s ds} u(C_t, L_t)dt\right) = \mathbb{E}\left(\int_0^\infty e^{-\int_0^t (\rho_s+\nu_s)ds} u(C_t, L_t)dt\right).$$

We can then write (1) as

$$\max_{\pi,C,L} \mathbb{E}\left(\int_0^\infty e^{-\int_0^t \hat{\rho}_s ds} u(C_t, L_t)dt\right), \tag{4}$$

with

$$\hat{\rho}_t = \rho_t + \nu_t \tag{5}$$

being the mortality adjusted discount rate.

Let us now specify the investment assets in our model. We assume that the economy features one risk-less asset modeled as

$$dB_t = B_t r_t dt, \tag{6}$$

becomes an ordinary expectation, e.g. when $e^{-\int_s^t \nu_s ds}$ is independent of $\mathbb{F}_s$ for all $t > s$.

[6]This is a common simplifying assumption, however there is research linking mortality to economic variables such as stock market growth and volatility, see for example Yap. et. al (2016) and Dacorogna and Meitner (2015).

[7]Note that we have $\mathbb{E}(\mathbf{1}_{\{t<\tau\}}) = \mathbb{P}(\tau > t)$ and therefore $\mathbb{E}\left(\int_0^\tau e^{-\int_0^t \rho_s ds} u(C_t, L_t)dt\right) = \mathbb{E}\left(\int_0^\infty e^{-\int_0^t \rho_s ds} u(C_t, L_t) \cdot \mathbf{1}_{\{t<\tau\}}dt\right)$

and one risky asset

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t). \tag{7}$$

Here W_t denotes a standard Brownian motion and we denote with $\mathcal{F}_t$ the filtration it generates. The parameters r_t, μ_t and σ_t are allowed to vary deterministically in time. As in Blanchard (1985) we assume the existence of fairly priced life insurance, which replaces a bequest motive: "In the absence of a bequest motive, and if negative bequests are prohibited, agents will contract to have their wealth (positive or negative) returned to the life insurance company contingent on their death." In other words, the agent uses life insurance to hedge against the possibility of dying (unexpectedly) with positive or negative wealth, which would otherwise incur costs through forsaken consumption or penalties. The modeling framework in this article assumes that the representative agent[8] represents "a large number of identical agents", and as such life insurance contracts can be offered risk-less (on average) by life insurance companies. We assume that the market for life insurance contracts is competitive[9], and hence free entry and exit will result in a expected zero profit condition, which in turn implies that the fair pricing of the insurance contract obliges/entitles the holder to a payment of

$$X_t \nu_t dt \tag{8}$$

per infinitesimal time interval dt, where X_t denotes the wealth of the agent at time

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[8]The existence of a representative agent (as in a general equilibrium framework) is discussed in Karatzas (1997) Chapter 3. Under fairly general conditions for a pure exchange economy with finite horizon existence of a representative agent is demonstrated.

[9]Kwok et. al. (2016) study a duopolistic framework of two insurers. They study the optimal portfolio allocation of the insurer, which includes a claims process. They find that the market structure has an important impact on the demand of longevity bonds.

t. Note that (8) represents a payment to be made by the agent to the insurance company, in case the agent has any debt, i.e. $X_t < 0$[10] and otherwise presents an income, i.e. payment from the insurance company to the agent, in exchange for the agent giving up his wealth to the insurance company at the time of her/his death.

As we have seen at the beginning of this section it is optimal for the agent to contract all wealth at time of death into life insurance. As such one part of the optimal allocation problem has already been solved and the problem has been reduced. However from an asset-management perspective, it can be useful to consider life-insurance from a different angle, that of an actuarial note. In fact Yaari (1965) also introduces the notion of an actuarial note, which pays a continuous interest rate until the time of death of a nominated person. Unlike a perpetual bond, which pays interest forever, the actuarial note stops paying interest and expires worthless at the time of death. In a perfectly competitive market where mortality risk can be fully diversified on the insurer side (in the absence of systematic shocks this is approximately the case due to the law of large numbers), the actuarial note would be required to pay an interest of $r_t + \nu_t$.[11] Our economy then consists of three investment assets, a perpetual bond (classical money market account), a risky asset and an actuarial note. Note that there is no contradiction in having two bond-like contracts paying different rates. While the actuarial note pays an advanced interest which is higher than that of the perpetual bond, it only pays this rate for a finite period. The perpetual bond pays a lower

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[10]Payment structures such as this can be found in many financial contracts. Payment protection insurance is one specific example.

[11]Suitably structured actuarial notes are the basis for mortality linked securities and longevity bonds, a multi billion pounds market. We refer to Wong et. al (2015) and Dong and Wong (2015).

interest, but for an infinite period. For an individual investor the non-existence of a bequest motive then implies that she/he always prefers the actuarial note over the perpetual bond, and the optimal allocation in the perpetual bond is zero In fact, denoting with π_t the fraction of wealth invested into the risky asset and with w_t the wage rate, the dynamics of the wealth process is described by

$$dX_t = X_t \left\{ (r_t + \nu_t)dt + \pi_t \left[(\mu_t - r_t)dt + \sigma_t dW_t \right] \right\} - C_t dt + w_t L_t dt, \qquad (9)$$

with $X_0 = x \geq 0$. Equation (9) shows that in difference to the classical wealth equation, the drift rate r_t is replaced by the yield of the actuarial note $r_t + \nu_t$.[12]

As the analysis above has shown, the problem (1) of the finitely lived agent subject to constraint (9), is equivalent to the problem (4) of an infinitely lived agent subject to constraint (9), where the discount rate as well as the drift of the wealth process have been adjusted to accommodate the mortality risk. In order for the stochastic optimal control problem to be well defined and to simplify our arguments, we assume that the deterministic functions r_t and ν_t as well as there (possibly piecewise) derivatives are bounded, the wage process w_t is bounded on every interval $[0, T]]$ and the controls π_t, C_t and L_t are progressively measurable

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[12]The two approaches, using either life insurance or actuarial notes are equivalent. In fact if no life insurance were available but an actuarial note paying a yield of $r_t + \nu_t$ the investor could use a portfolio of perpetual bonds and actuarial notes to create an interest income of $X_t \nu_t$ as in equation (8) and hence construct the payoff structure that Blanchard (1985) uses. In this interpretation the extra term $X_t \nu_t$ is to be understood as an additional income from the insurance contract rather than the interest of a financial asset. This income would persist even if the agent would invest all of her/his wealth into the risky asset.

L^2-processes on every interval $[0, T]$.[13] We further define

$$\hat{r}_t = r_t + \nu_t \tag{10}$$

$$\hat{\mu}_t = \mu_t + \nu_t \tag{11}$$

and note that the market price of financial risk

$$\theta_t = \frac{\mu_t - r_t}{\sigma_t} = \frac{\hat{\mu}_t - \hat{r}_t}{\sigma_t} \tag{12}$$

is unaffected by mortality risk ν_t.

3   Martingale Approach

In order to apply Martingale methods to solve the problem discussed in the previous section, we define a stochastic discount factor $\hat{H}_t$ adjusting for the mortality risk via

$$d\hat{H}_t = -\hat{H}_t \left(\hat{r}_t dt + \theta_t dW_t \right)$$

$$\hat{H}_0 = 1. \tag{13}$$

For the moment, this discount factor is purely for convenience. Note that the stochastic discount factor features the mortality adjusted rate $\hat{r}_t$ and the classical

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[13]These assumptions are stronger then is actually needed. Following Korn (2001) Chapter 2.2 it would be sufficient to assume that for all $T > 0$ the integrals $\int_0^T \pi_t^2 dt$, $\int_0^T C_t dt$ and $\int_0^T L_t dt$ are finite $\mathbb{P}$-almost sure, as this guarantees that the stochastic differential equations (9) has a unique solution, which can be obtained via the method of the variation of constants. However, the optimal solution derived in the following will meet our stronger assumptions and the stronger assumptions simplify our argument.

market price of risk θ_t in it. We can write $\hat{H}_t$ as

$$\hat{H}_t \;\; = e^{-\int_0^t \nu_s ds} H_t, \tag{14}$$

where H_t is the classical stochastic discount factor and is defined by [14]

$$H_t \;\; = \;\; e^{-\int_0^t \left(r_s + \frac{1}{2}\theta_s^2\right)ds - \int_0^t \theta_s dW_s}. \tag{15}$$

Hence the stochastic discount factor $\hat{H}_t$ splits up into two components with $e^{-\int_0^t \nu_s ds}$ adjusting for mortality risk and H_t adjusting for financial risk.

Applying the Itô product rule, it is easy to verify that

$$d(\hat{H}_t X_t) = \hat{H}_t X_t \left(\pi_t \sigma_t - \theta_t\right) dW_t - \hat{H}_t C_t dt + \hat{H}_t w_t L_t dt. \tag{16}$$

Integrating (16) from t to ∞ and imposing the following transversality condition[15]

$$\lim_{u \to \infty} \mathbb{E}(\hat{H}_u X_u) = 0 \tag{17}$$

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[14]See for example Korn (2001). Note that the optimization problem (9) is independent of any discount factor and carried out under the measure $\mathbb{P}$. However, within the economy defined in section 2, H_t represents the unique discount factor related to the unique martingale measure $\mathbb{Q}$ for this economy. The discount factor $\hat{H}_t$ represents the discount factor corresponding to an economy, where the perpetual bond is replaced by the actuarial note. These discount factors are the only ones in the respective economies that do not permit arbitrage and are hence are tied to an equilibrium condition.

[15]The corresponding deterministic version of this transversality condition appears in Blanchard (1985) on page 227, and prevents the case where an agent takes up more and more debt, while being covered by life insurance. The analogue equation in Yaari (1965) is equation (29) on page 146.

we obtain

$$-\hat{H}_t X_t = \int_t^\infty \hat{H}_s X_s (\pi_s \sigma_s - \theta_s) dW_s - \int_t^\infty \hat{H}_s C_s ds + \int_t^\infty \hat{H}_s w_s L_s ds. \qquad (18)$$

Denoting the conditional expectation with respect to $\mathcal{F}_t$ as $\mathbb{E}_t$ we obtain

$$X_t = \mathbb{E}_t \left[\int_t^\infty \frac{\hat{H}_s}{\hat{H}_t} C_s ds \right] - \mathbb{E}_t \left[\int_t^\infty \frac{\hat{H}_s}{\hat{H}_t} w_s L_s ds \right]. \qquad (19)$$

At time $t = 0$ we obtain the static budget constraint

$$\mathbb{E} \left(\int_0^\infty \hat{H}_s C_s ds \right) = x + \mathbb{E} \left(\int_0^\infty \hat{H}_s w_s L_s ds \right). \qquad (20)$$

The intuition behind equation (20) is that expected stochastically discounted consumption needs to be equal to initial wealth plus expected stochastically discounted wage income, where the discount factor takes both market risk and mortality risk into account.

We can now conclude that problem (4) subject to the dynamic constraint (9) and transversality condition (17) is equivalent to problem (4) with the static budget constraint (20). In order to solve the latter problem we introduce the Lagrange function

$$\begin{aligned} \mathcal{L}(\lambda, C_t, L_t) &= \mathbb{E} \left(\int_0^\infty e^{-\int_0^t \hat{\rho}_s ds} u(C_t, L_t) dt \right) \\ &\quad + \lambda \left\{ x + \mathbb{E} \left(\int_0^\infty \hat{H}_s w_s L_s ds \right) - \mathbb{E} \left(\int_0^\infty \hat{H}_s C_s ds \right) \right\}. \end{aligned} \qquad (21)$$

In order to proceed to a closed form solution, we need to specify the utility function

$u(C_t, L_t)$ at this point. We define

$$u(C_t, L_t) := \frac{C_t^{1-\gamma}}{1-\gamma} - b_t \frac{L_t^{1+\eta}}{1+\eta}, \tag{22}$$

The intuition behind (22) is to weigh up benefits from consumptions against costs from labour in a constant relative risk aversion (CRRA) manner. The deterministic function b_t measures the relative cost of labour, which may vary between age groups. We assume $\gamma > 0$, $\gamma \neq 1$, $\eta > 0$ and that the range of b_t is a compact subset of $(0, \infty)$, consistent with decreasing marginal benefits from consumption and increasing marginal costs of labour.[16]

Differentiating the Lagrange function (21), we obtain the following first order conditions

$$C_t^{-\gamma} = \frac{\partial u}{\partial C_t} = \lambda e^{\int_0^t \hat{\rho}_s ds} \hat{H}_t \tag{23}$$

$$-b_t L_t^{\eta} = \frac{\partial u}{\partial L_t} = -\lambda e^{\int_0^t \hat{\rho}_s ds} \hat{H}_t w_t. \tag{24}$$

Using (5) and (14), we obtain from (23) and (24) that

$$C_t^{-\gamma} = \lambda e^{\int_0^t \rho_s ds} H_t \tag{25}$$

$$-b_t L_t^{\eta} = -\lambda e^{\int_0^t \rho_s ds} H_t w_t. \tag{26}$$

The mortality component ν_t hence cancels out in the time dependent component of consumption and labour supply above. It can therefore be concluded that the

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[16]The case $\gamma = 1$ corresponds to the case where utility from consumption is logarithmic and can in fact be obtained from the following results by considering the limit for $\gamma \to 1$. The same holds for the case $\eta = 0$, which corresponds to linear costs of labour, which has interesting implications on the elasticity of consumption, as discussed in section 4.

mortality risk will have no effect on the expected growth rate of consumption $\frac{d}{dt}\mathbb{E}_t\left(\frac{dC_t}{C_t}\right)$. This is a major difference to Richard (1974). The reason is likely that Richard (1974) used a non-zero bequest motive and life insurance that pays off at the time of death, rather than annuities as used in our framework. Nevertheless, as we will see below, consumption is not unaffected by mortality changes. In fact, mortality changes will affect the value of the Lagrange multiplier λ and hence shift consumption to different levels. Furthermore, dynamic changes in the mortality curves may indeed contribute to consumption growth as explored in the next section. The optimal consumption and labour supply can be easily derived from (25) and (26) as

$$C_t^* = \lambda^{-\frac{1}{\gamma}}e^{-\frac{1}{\gamma}\int_0^t \rho_s ds}H_t^{-\frac{1}{\gamma}} \tag{27}$$

$$L_t^* = \lambda^{\frac{1}{\eta}}e^{\frac{1}{\eta}\int_0^t \rho_s ds}(H_t w_t)^{\frac{1}{\eta}}b_t^{-\frac{1}{\eta}}. \tag{28}$$

Compulsory retirement at a given age T can also be easily implemented in our model, by choosing $w_s = 0$ for all $s > T$. We further expect b_t, the dis-utility from labour, to be sharply increasing at old ages, decreasing labour supply in later life stages. This has been adopted for our empirical analysis.

We will now derive an analytic expression for the Lagrange multiplier λ and by doing this identify the mortality dependence in (27) and (28). In order to proceed, we need to make assumptions about the dynamics of the wage rate w_t. We assume for simplicity that the wage rate grows at a time-dependent growth rate $a_t \geq 0$,

i.e.[17]

$$dw_t = w_t a_t dt. \tag{29}$$

It can then be shown that

$$\lambda^{-\frac{1}{\gamma}} \int_0^\infty e^{-\frac{1}{\gamma} \int_0^t \left(\rho_s + (\gamma-1)\left(r_s + \frac{\theta_s^2}{2\gamma}\right)\right) ds} \cdot e^{-\int_0^t \nu_s ds} dt \tag{30}$$

$$= x + \lambda^{\frac{1}{\eta}} w_0^{\frac{\eta+1}{\eta}} \int_0^\infty e^{\frac{1}{\eta} \int_0^t \left(\rho_s - (\eta+1)\left(r_s - a_s - \frac{\theta_s^2}{2\eta}\right)\right) ds} \cdot e^{-\int_0^t \nu_s ds} \cdot b_t^{-\frac{1}{\eta}} dt.$$

The details of this computation are presented in Appendix A. Note that the computability of the integrals above depends on the deterministic functions ρ_s, r_s, θ_s, a_s, b_s and ν_s. If for example these are all constant, then it is straightforward to compute all the integrals in (30) explicitly. However, it will still not be possible to solve (30) analytically for λ, as the equation $\lambda^a = x + \lambda^b$ can not be solved explicitly for λ in the generic case. On the other hand, in the most general case, it is a simple exercise to compute the integrals and λ from (30) numerically.

The non-existence of a bequest motive on the other hand can also be interpreted as that the representative agent is born at time $t = 0$ without any means, i.e. $x = 0$. In this case we obtain an explicit solution for the Lagrange multiplier

$$\lambda = \left(\left(w_0^{\frac{\eta+1}{\eta}} \right) \frac{\int_0^\infty e^{\frac{1}{\eta} \int_0^t \left(\rho_s - (\eta+1)\left(r_s - a_s - \frac{\theta_s^2}{2\eta}\right)\right) ds} \cdot e^{-\int_0^t \nu_s ds} \cdot b_t^{-\frac{1}{\eta}} dt}{\int_0^\infty e^{-\frac{1}{\gamma} \int_0^t \left(\rho_s + (\gamma-1)\left(r_s + \frac{\theta_s^2}{2\gamma}\right)\right) ds} \cdot e^{-\int_0^t \nu_s ds} dt} \right)^{-\frac{\gamma\eta}{\gamma+\eta}}. \tag{31}$$

We can see clearly in (31) the mortality dependence of the Lagrange multiplier.

---

[17]The derivation in Appendix A shows that it would not cause any trouble, if w_t would be assumed to be stochastic, following a geometric Brownian motion, e.g. $dW_t = w_t(a_t dt + \varphi_t dW_t)$ with a deterministic function φ_t which would enter equation (30). However we found it difficult to calibrate for φ_t as it adds an extra layer of indeterminacy to the calibration and hence choose wage growth to be deterministic.

The classical model without mortality risk is obtained by setting $\nu_s \equiv 0$. Noticing once more that $\mathbb{P}(\tau > t) = e^{-\int_0^t \nu_s ds}$ we can see that compared to the classical case without mortality the integrands in the nominator and denominator in (31) are weighted by the probability of survival. We will later show that under realistic choices of parameters, the inclusion of mortality risk will have significant quantitative effects.

Let us summarize the results so far in the following theorem:

Theorem 3.1. *The optimal strategies of consumption C_t^* and labour supply L_t^* of the agent optimizing (1) under the dynamic constraint (9) and transversality condition (17) are given by*

$$
C_t^* = e^{-\frac{1}{\gamma}\int_0^t \rho_s ds} H_t^{-\frac{1}{\gamma}} \left(\left(w_0^{\frac{\eta+1}{\eta}}\right) \frac{\int_0^\infty e^{\frac{1}{\eta}\int_0^u \left(\rho_s - (\eta+1)\left(r_s - a_s - \frac{\theta_s^2}{2\eta}\right)\right)ds} \cdot e^{-\int_0^u \nu_s ds} \cdot b_u^{-\frac{1}{\eta}} du}{\int_0^\infty e^{-\frac{1}{\gamma}\int_0^u \left(\rho_s + (\gamma-1)\left(r_s + \frac{\theta_s^2}{2\gamma}\right)\right)ds} \cdot e^{-\int_0^u \nu_s ds} du} \right)^{\frac{\eta}{\gamma+\eta}}
$$

$$
L_t^* = e^{\frac{1}{\eta}\int_0^t \rho_s ds} (H_t w_t)^{\frac{1}{\eta}} b_t^{-\frac{1}{\eta}} \left(\left(w_0^{\frac{\eta+1}{\eta}}\right) \frac{\int_0^\infty e^{\frac{1}{\eta}\int_0^u \left(\rho_s - (\eta+1)\left(r_s - a_s - \frac{\theta_s^2}{2\eta}\right)\right)ds} \cdot e^{-\int_0^u \nu_s ds} \cdot b_u^{-\frac{1}{\eta}} du}{\int_0^\infty e^{-\frac{1}{\gamma}\int_0^u \left(\rho_s + (\gamma-1)\left(r_s + \frac{\theta_s^2}{2\gamma}\right)\right)ds} \cdot e^{-\int_0^u \nu_s ds} du} \right)^{-\frac{\gamma}{\gamma+\eta}}.
$$

As indicated above, perhaps the most important observation from Theorem 3.1 is that the mortality risk enters consumption and labour only through the time independent factor in the large brackets on the right-hand sides of the relevant expressions. As such changes in mortality at any age shift consumption and labour supply over the whole life cycle.

We will now turn our attention to the optimal investment strategy π_t^*. The computations in Appendix B show that the optimal wealth X_t^* can be written as

$$
X_t^* = f_t C_t^* - g_t w_t L_t^*, \tag{32}
$$

18

where f_t and g_t are deterministic functions. Let us also note at this point that the expression $e^{-\int_t^s \nu_u du}$ is equal to $\mathbb{P}(\tau > s | \tau > t)$, the probability of survival until s given that the agent is still alive at time $t < s$.

Using the representation (32) together with (13) we compute

$$d\left(\hat{H}_t X_t^*\right) = \hat{H}_t f_t dC_t^* - \hat{H}_t g_t w_t dL_t^* + X_t^* d\hat{H}_t + (\ldots)dt. \tag{33}$$

The terms indicated by $(\ldots)$ in front of dt will be irrelevant for the following, which is why we omit them. In fact we will only be interested in the diffusion term, i.e. the expression in front of dW_t, within the expression (33). To identify this term, we compute

$$dC_t^* = -\frac{1}{\gamma}C_t^* H_t^{-1} dH_t + (\ldots)dt \tag{34}$$

$$dL_t^* = \frac{1}{\eta}L_t^* H_t^{-1} dH_t + (\ldots)dt \tag{35}$$

Furthermore, using (14) and

$$dH_t = -H_t(r_t dt + \theta_t dW_t), \tag{36}$$

we eventually obtain

$$
\begin{aligned}
d\left(\hat{H}_t X_t^*\right) &= \frac{1}{\gamma}\hat{H}_t f_t C_t^* \theta_t dW_t + \frac{1}{\eta}\hat{H}_t g_t w_t L_t^* \theta_t dW_t - \hat{H}_t X_t^* \theta_t dW_t + (\ldots)dt \\
&= \left\{ g_t \hat{H}_t w_t L_t^* \left(\frac{\eta+1}{\eta}\right) - f_t \hat{H}_t C_t^* \left(\frac{\gamma-1}{\gamma}\right) \right\} \theta_t dW_t + (\ldots)dt, \tag{37}
\end{aligned}
$$

where for the second equality we used (32). Since the diffusion term in the repre-

sentation (37) must coincide with the diffusion term in the representation (16) for $X_t = X_t^*$, we obtain by noticing (32) once more and solving for π_t^*:

Theorem 3.2. *The optimal investment strategy of the agent optimizing (1) under the dynamic constraint (9) and transversality condition (17) is given by*

$$\pi_t^* = \frac{1}{\gamma}\frac{\mu_t - r_t}{\sigma_t^2} + g_t \cdot \left(\frac{1}{\gamma} + \frac{1}{\eta}\right)\frac{\mu_t - r_t}{\sigma_t^2} \cdot \frac{w_t L_t^*}{X_t^*}. \tag{38}$$

with $g_t =: \int_t^\infty e^{-\int_t^s \left(\frac{\eta+1}{\eta}\left(r_u - a_u - \frac{\theta_u^2}{2\eta}\right) - \frac{1}{\eta}\rho_u\right)du} \left(\frac{b_s}{b_t}\right)^{-\frac{1}{\eta}} \cdot e^{-\int_t^s \nu_u du} ds.$

Note that the function g_t in (38) depends on mortality risk, and as such the proportion invested into the risky asset does so as well. In fact expression (38) represents a modification of the classical Merton (1969) rule $\pi_t = \frac{1}{\gamma}\frac{\mu_t - r_t}{\sigma_t^2}$, where the adjustment for flexible labour supply, wages and mortality risk is given by the term $g_t \cdot \left(\frac{1}{\gamma} + \frac{1}{\eta}\right)\frac{\mu_t - r_t}{\sigma_t^2} \cdot \frac{w_t L_t^*}{X_t^*}$. It can be observed that when the mortality curves shift down, i.e. mortality rates decrease uniformly, the proportion of total wealth invested into the risky asset increases. The rule also shares strong similarities with the rule obtained in Richard (1974), with the addition of an extra term reflecting the flexible labour and a different expression for the multiplier g_t, which is due to a different model setup.

An empirical comparative analysis of this expression will follow in the next section.

We have already indicated above that the consumption growth is not directly affected by the inclusion of the mortality risk. Nevertheless we believe that it is interesting to derive the Euler equation for consumption growth at this point.

Computing the term in front of dt in (34) explicitly, we obtain

$$dC_t^* = \frac{1}{\gamma}\theta_t C_t^* dW_t + \frac{1}{\gamma}\left(r_t - \rho_t + \frac{\gamma+1}{2\gamma}\theta_t^2\right)C_t^* dt. \tag{39}$$

Dividing (39) by C_t^* and taking the expectation,[18] we obtain

$$\frac{d}{dt}\mathbb{E}\left(\frac{dC_t^*}{C_t^*}\right) = \frac{1}{\gamma}\left(r_t - \rho_t + \frac{\gamma+1}{2\gamma}\theta_t^2\right). \tag{40}$$

As expected, the consumption Euler equation does not depend on the mortality risk parameter ν_t. This can be attributed to the full insurance against loss of life. But the uncertainty attached to the financial market does affect the individual's consumption decision (see the third term in the bracket of (40)). [19]

4 Examples and Empirical Analysis

Before considering the case of historical time dependent mortality rates, we start with a toy example to highlight some of the fundamental forces and tradeoffs at play. In this toy example only, all parameters, including the mortality rate ν_t, are assumed to be constant. Furthermore, assuming that the representative agent is born without any initial wealth[20], i.e. $x = 0$, we can compute the Lagrange multiplier $\lambda(\nu)$ in (31) as a function of the mortality rate ν ($\nu_t \equiv \nu$, for all t)

---

[18]Note that due to Theorem 3.1. the stochastic integral in (39) is indeed a martingale.

[19]We refer to Zhang (2010) for details on how the individual adjusts consumption according to financial risk. Note that Zhang (2010) did not take account of the mortality risk.

[20]This is a consequence of no bequest motive as discussed earlier.

explicitly:

$$\lambda(\nu) = w_0^{\left(\frac{\eta+1}{\eta}\right)} b^{\left(\frac{\gamma}{\gamma+\eta}\right)} \left(\frac{\nu + \frac{\rho}{\gamma} + \frac{\gamma-1}{\gamma}\left(r + \frac{\theta^2}{2\gamma}\right)}{\nu - \frac{\rho}{\eta} + \frac{\eta+1}{\eta}\left(r - a - \frac{\theta^2}{2\eta}\right)} \right)^{-\frac{\gamma\eta}{\gamma+\eta}}. \tag{41}$$

In the following we will compute the elasticity of consumption with respect to the mortality risk ν. This elasticity represents the percentage change in consumption for each percentage change in the mortality rate. It is rather simple to verify by using (27) and (41) that

$$\frac{\frac{dC_t^*(\nu)}{C_t^*(\nu)}}{\frac{d\nu}{\nu}} = -\frac{1}{\gamma} \frac{\frac{d\lambda(\nu)}{\lambda(\nu)}}{\frac{d\nu}{\nu}}. \tag{42}$$

That is, the elasticity of consumption is a constant fraction of the elasticity of the Lagrange multiplier.[21] The constant factors w_0 and b do not affect the elasticities with respect to the mortality risk.

Using (42) it is then a tedious but straightforward exercise to verify that

$$\frac{\frac{dC_t^*(\nu)}{C_t^*(\nu)}}{\frac{d\nu}{\nu}} = \frac{\left(1 - \frac{\nu + \frac{\rho}{\gamma} + \frac{(\gamma-1)\left(r+\frac{\theta^2}{2\gamma}\right)}{\gamma}}{\left((\nu-\frac{\rho}{\eta}+\frac{(\eta+1)(r-a-\frac{\theta^2}{2\eta})}{\eta})\right)}\right)\eta\nu}{(\gamma+\eta)\left(\nu + \frac{\rho}{\gamma} + \frac{(\gamma-1)\left(r+\frac{\theta^2}{2\gamma}\right)}{\gamma}\right)}. \tag{43}$$

It can be concluded from (43) that for general parameters

$$\left. \frac{\frac{dC_t^*(\nu)}{C_t^*(\nu)}}{\frac{d\nu}{\nu}} \right|_{\nu=0} = 0. \tag{44}$$

---

[21] Note that dC_t^* above is the change in consumption in effect of a change in mortality, and that equation (42) is a priori unrelated to the consumption Euler equation (40), where the change dC_t^* is in effect of a change in time t.

This means that at the mortality rate $\nu = 0$ there is no first order effect on consumption by increasing the mortality rate. The two effects of increasing current consumption because of fear of death in the future and decreasing consumption because of a decrease in human wealth exactly offset each other. The same neutrality holds for linear costs of labour. It can also be easily verified that in the limit for $\eta \to 0$, expression (43) converges to 0 as well, i.e.

$$
\left. \frac{\frac{dC_t^*(\nu)}{C_t^*(\nu)}}{\frac{d\nu}{\nu}} \right|_{\eta \to 0} = 0, \tag{45}
$$

and independent of γ. The elasticity (43) can also be interpreted as a sensitivity which measures the error in consumption forecast with regards to errors in the determination of the mortality rate ν. The results (44) and (45) are hence reassuring in a way that they indicate some form of stability, i.e. no first order effects at $\nu = 0$ or $\eta = 0$. The numerical example below will show that for realistic parameters this sensitivity will remain well below 1 in absolute value, meaning that a 1% error in mortality (in relative terms) will cause less than a 1% error in consumption forecast.

Neutrality ceases to hold however, when the mortality rate is positive and $\eta > 0$, as the following numerical example shows.[22]

---

[22]For the case $\gamma = 1$, which corresponds to logarithmic utility from consumption, we obtain in the limit

$$
\left. \frac{\frac{dC_t^*(\nu)}{C_t^*(\nu)}}{\frac{d\nu}{\nu}} \right|_{\gamma \to 1} = \frac{\left(1 - \frac{\nu + \rho}{\nu - \frac{\rho}{\eta} + \frac{(\eta+1)\left(r - a - \frac{\theta^2}{2\eta}\right)}{\eta}} \right) \eta \nu}{(\eta + 1)(\nu + \rho)},
$$

and observe, that even in this case, for $\nu \neq 0$, the elasticity of consumption with respect to mortality is non-zero, except in the case where $\eta = 0$, i.e. linear costs of labour.

23

For the analysis below we assume the following parameter values: $\rho = 0.06$; $\gamma = 2$; $r = 0.03$, $\mu = 0.09$, $\sigma = 0.35$; $a = 0.01$, $b = 0.5$ and $\eta = 3$. These values are of similar magnitude as those chosen in Milevsky and Young (2007) and Pang and Warshawsky (2010). Figure 1 shows the elasticity of consumption depending on the level of the mortality rate, for mortality rates ranging from 0 to 0.025. The mortality rate of 0.025 corresponds to a 72 year old male living in the UK in 2011, according to recent UK historical life tables published by the Office for National Statistics (2012).
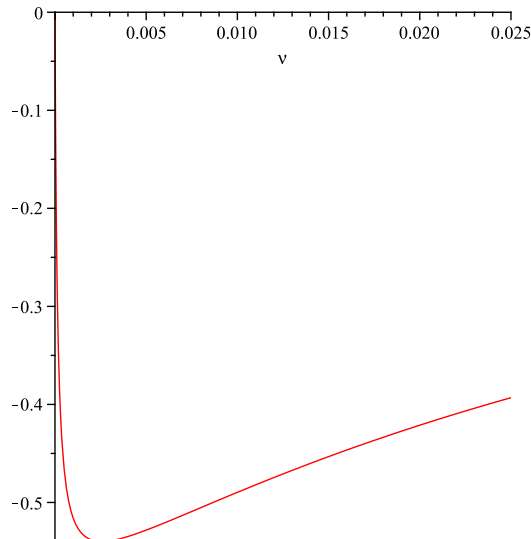
Figure 1: Elasticity of consumption with respect to mortality.

We observe that the elasticities are all negative, meaning that with increasing mortality consumption declines. In conclusion, here the human wealth mechanisms dominates the precautionary savings and fear of death mechanisms as indicated on page 4. Furthermore, the effect of a change in the mortality rate is strongest at about $\nu = 0.00157$, which corresponds to the mortality rate of a 39 year old male living in the UK in 2010. At that age, the elasticity of consumption is

approximately at -0.53, which can be loosely interpreted as saying that if the mortality rate of a 39 year old declines by 10%, then consumption will increase by about 5.3%. The mortality rate of a 39 year old male living in the UK in 1980 was 0.0017739, and hence declined over the period of 30 years between 1980 and 2010 by 12% inducing a growth in consumption of about 6.5%. If we look further down, at around pension age of 66 the mortality rate in 1980 was 0.03332 while in 2010 the mortality rate for the same age group was 0.01498, which means that the mortality rate has been reduced over the period by roughly 55%. The elasticity of consumption at that mortality rate is -0.44, so that the reduction in mortality of this age group affects consumption by approximately 24%. Real GDP over the period from 1980 to 2010 in the UK grew by about 100%. This implies that the simple analysis above, provides an indication that a reduction in mortality rates may have had a significant impact on real GDP, possibly explaining between $10\% - 25\%$ of its growth.[23]

We now consider the case of time dependent mortality rates. Figure 2 shows age dependent mortality rates for various years between 1960 and 2060 in the UK. The mortality rates from 2011 onward are projected.

The figure clearly shows that mortality rates are on very similar levels until about age 50, but then diverge. The following Figure 3 represents the mortality rate of different age groups over the period 1951-2060.

It can be seen that the mortality rates in the more senior age groups have decreased very significantly over the years, while in the more junior age groups up to age 30, the effect is far less significant.

---

[23]Worldbank data show that household final consumption expenditure in the UK has been fluctuating between 57% and 65% of GDP over the years 1980-2010, being rather stable at just below 65% during the last decade, see $http : //data.worldbank.org/indicator/NE.CON.PETC.ZS$
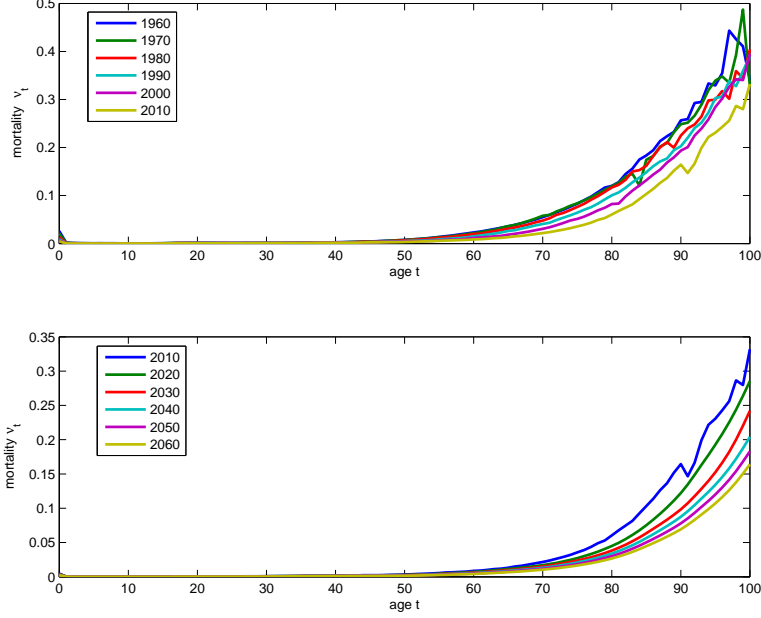
Figure 2: Age dependent mortality rates for selected years between 1960 and 2060.

For our next analysis we use the following set of parameters, which have been informed from relevant studies such as Milevsky and Young (2007) and Pang and Warshawsky (2010): $\rho = 0.05$, $\gamma = 2$, $r = 0.03$, $\mu = 0.09$, $\sigma = 0.5$, $a = 0.02$[24], $w = 80000$[25], $\eta = 3$. Figure 4 has been obtained by computing C_0^* in Theorem 3.1 with time dependent mortality rates obtained from UK life expectancy tables[26] for UK males from the years 1951 to 2060 with a starting age of 25, i.e. C_0^* represents the consumption at the reference time of a 25 year old.

---

[24]Note that a represents an age increment and not a salary inflation and is the same for all agents independent of their year of birth.

[25]This is the annual wage if an agent would work non-stop for 24 hours 7 days a week. Obviously, this is not optimal. For a hypothetical 40 hour working week, this value corresponds to approximately 19000 GBP annually, which is slightly above the 16400 GBP after tax average income in the UK according to recent OECD data.

[26]The computation of the integrals in Theorem 3.1 requires discretization. The step length has been chosen as one year in order to meet the frequency of the mortality data.
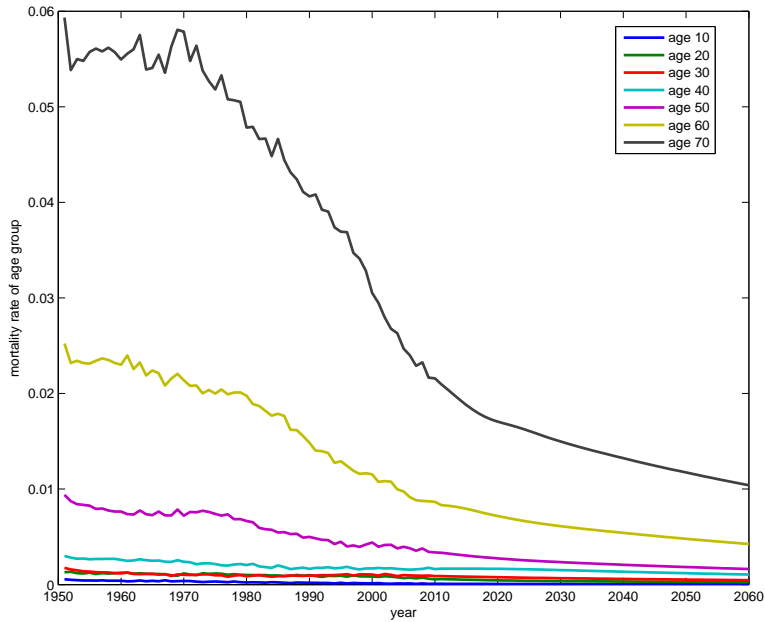
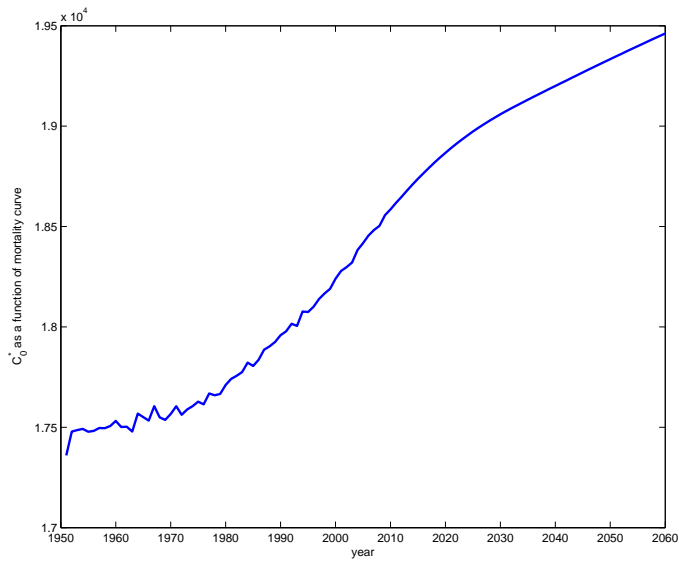Figure 3: Mortality rates for selected age groups between 1951 and 2060.

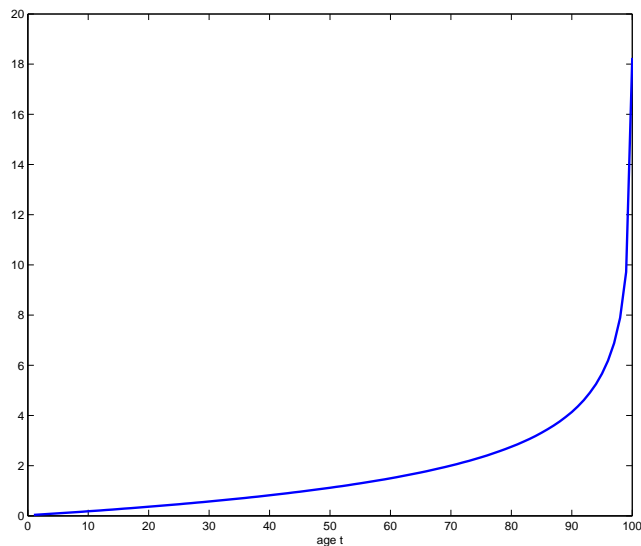Figure 4: Consumption under historical mortality at age 25.

Figure 5: Dis-utility from labour.

Figure 4 shows an upward trend, as expected. The overall change in consumption caused by the changing mortality curves over the 110 year period in this case is about 12%. The most prominent growth occurs in the 30 year period from 1980 until 2010. The growth in consumption caused by changing mortality patterns over this period is 5%, compared with the aforementioned 100% in real GDP growth over the same period. Changes in the mortality curves in this setting still seem to have a significant impact on GDP.

Figure 5 below shows the function representing the dis-utility from labour that was used in this analysis. As can be seen, disutility increases rapidly at old age. This is due to factors such as age related ailments or difficulties to obtain employment, if able and willing to work. The following figure 6 shows consumption for different years (from right to left) as a function of age. The curve at the top-boundary of the figure corresponds to the curve in figure 4, i.e. age 25.
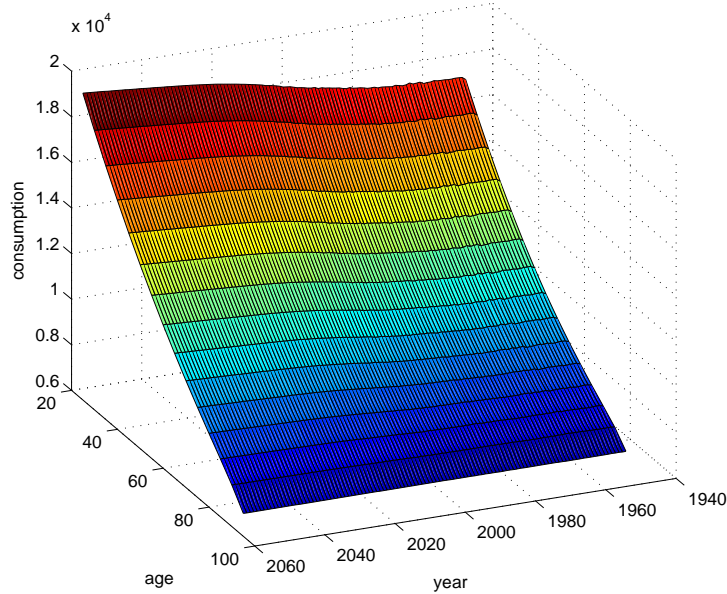
28

Figure 6: Consumption under historical mortality at different ages.

It can be observed that while consumption is decreasing with age, it is increasing across all ages under historical and predicted changes in mortality. We do not observe a consumption hump at a given age as discussed in Feigenbaum (2008), Feigenbaum and Li (2012), Kraft et. al (2016) and Gourinchas and Parker (2002). One reason for this is that our main borrowing constraint, which is expressed in the transversality condition (17) is less restrictive than those used in the literature above. A second reason is that we include life insurance in our model. As Feigenbaum (2008) states, a mortality induced consumption hump can no longer be observed if a pay-as-you-go social security system is incorporated into the model. Feigenbaum and Li (2012) argue among other things that age dependent wage uncertainty can contribute to the occurrence of a consumption hump, but also say that in comparison to previous studies, their results show a less pronounced hump.

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In principle our model would be able to cope with age dependent wage uncertainty, even though in order to derive explicit solutions in Theorem 3.1 we assumed that wage is deterministic, i.e. equation (29). As the focus of our paper is not on the consumption hump, we leave this interesting topic for future research.

Let us now have a look at the labour supply. With the same parameters as before, we compute labour supply from Theorem 3.1 for the above historical mortality curves and obtain the following Figure 7. Labour supply is expressed in terms of hours per week.
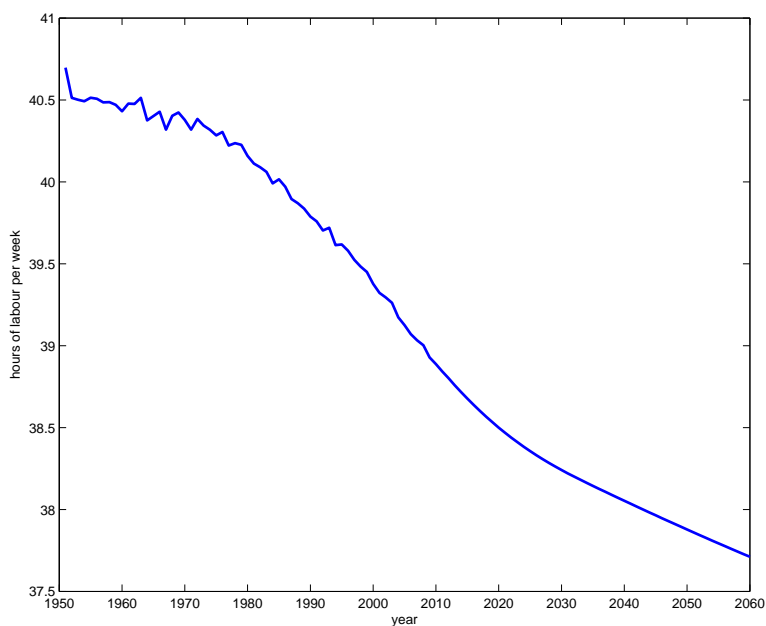

Figure 7: Labour supply under historical mortality in hours per week.

We observe a noticeable downward trend which can also be observed in reality. Specifically labour supply in our model decreases due to changing mortality curves between 1980 and 2010 by about 4% from 40.2 hours per week in 1980 to 38.6 hours in 2010.

Finally, let us look at portfolio investment. We have already indicated that the optimal portfolio investment strategy consists of a Merton type rule, which is adjusted by an additional term. We fix the ratio of wage income to initial wealth as 0.03 and choose $\rho = 0.06$, $\gamma = 0.95$, $r = 0.03$, $\mu = 0.045$, $\sigma = 0.5$, $a_t \equiv a = 0$, $\eta = 0.72$ and $b_t \equiv 3$ in order to make results comparable. The following figure 8 shows the percentage of (financial) wealth invested into the risky asset.
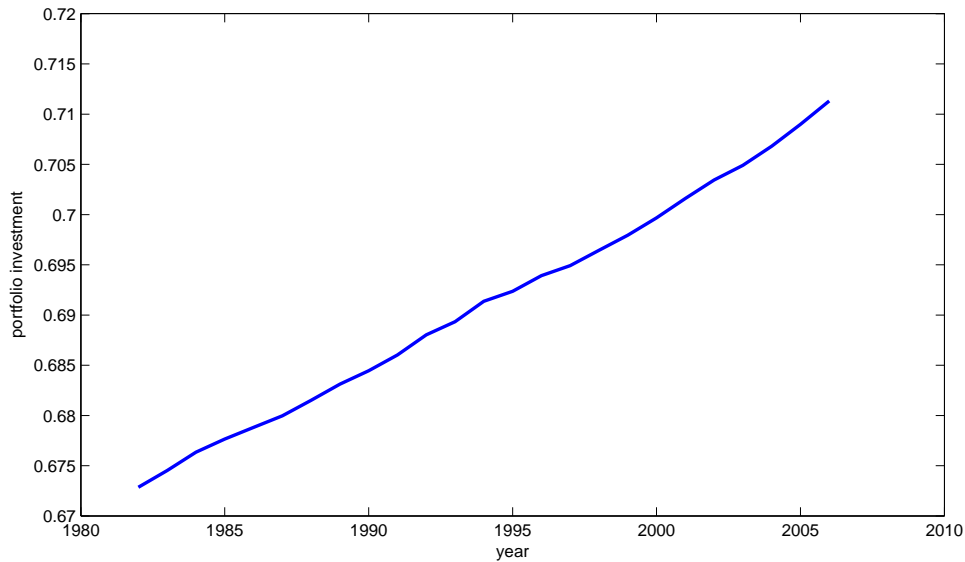

Figure 8: Portfolio investment under historical mortality.

As expected, we observe that over historical time, the agent invests more into the risky asset and less into the risk-less asset, primarily as a consequence of the reduced mortality and the increased life expectancy. This observation is similar to Pang and Warshawsky (2010) where the same effect occurs for reduced uncertainty in health care costs.

Figure 9 below displays investment into the risky asset under strategy (38) in excess of the classical Merton rule $\pi_t = \frac{1}{\gamma} \frac{\mu_t - r_t}{\sigma_t^2}$, depending on the age of the investor
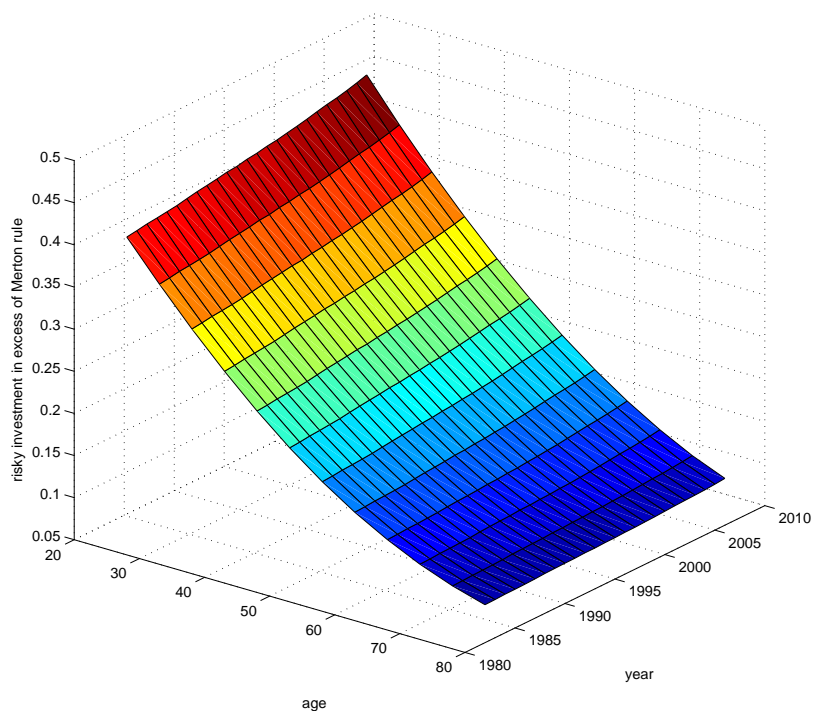
and year in history.[27]

Figure 9: Excess investment into risky asset as function of age and year in history.

We observe that among all ages of investors between 1980 to 2010 the proportions of wealth invested into the risky asset increase. Additional, fixing any year in history between 1980 and 2010, the proportion of wealth invested into the risky asset declines with the age of the investor. This is intuitive of course, as with the age of the investor, her/his mortality risk increases, and safer short term investments are sought. This effect can also be observed in reality, but none of the previous models, including Milevsky and Young (2007) and Pang and Warshawsky (2010), had been able to explain this feature through changes in the mortality rate.

---

[27]Note that the classical Merton (1969) set up does not include income and human wealth and that therefore this fraction refers to financial wealth only.

5 Conclusions

We have extended the consumption-investment life cycle model for an uncertain lived agent by Richard (1974) to include flexible labour. We have derived closed-form solutions for optimal consumption, labor supply and investment strategy and showed that the inclusion of mortality risk, and in fact the shape of the mortality risk curve, significantly affects the level of consumption as well as the decomposition of the investment portfolio. Our model is able to cope with historical mortality data which can be fed directly into our closed form solutions. An empirical analysis based on UK actuarial data from 1951 to 2060 supports our results. As such we observe that historical changes in mortality might indeed be responsible for about 5% of GDP growth in the period from 1980 until 2010, more risk taking and a reduction in labour.

Appendix

A. Computations to derive the Lagrange Multiplier

Substitution of (27) and (28) into (20) and using (14) we obtain

$$
\lambda^{-\frac{1}{\gamma}}\mathbb{E}\left(\int_0^\infty \left(e^{-\int_0^t \left(\nu_s + \frac{1}{\gamma}\rho_s\right)ds}\right) H_t^{\frac{\gamma-1}{\gamma}} dt\right) \tag{46}
$$
$$
= x + \lambda^{\frac{1}{\eta}} \quad \mathbb{E}\left(\int_0^\infty \left(e^{-\int_0^t \left(\nu_s - \frac{1}{\eta}\rho_s\right)ds}\right) b_t^{-\frac{1}{\eta}}(H_t w_t)^{\frac{\eta+1}{\eta}} dt\right).
$$

Then using that everything, except H_t and w_t, is deterministic, we obtain

$$\lambda^{-\frac{1}{\gamma}} \left(\int_0^\infty e^{-\int_0^t \left(\nu_s + \frac{1}{\gamma}\rho_s\right)ds} \mathbb{E}\left(H_t^{\frac{\gamma-1}{\gamma}} \right) dt \right) \tag{47}$$

$$= x + \lambda^{\frac{1}{\eta}} \left(\int_0^\infty e^{-\int_0^t \left(\nu_s - \frac{1}{\eta}\rho_s\right)ds} b_t^{-\frac{1}{\eta}} \mathbb{E}\left((H_t w_t)^{\frac{\eta+1}{\eta}} \right) dt \right).$$

Using (15) and (29) we can compute

$$\mathbb{E}\left(H_t^{\frac{\gamma-1}{\gamma}} \right) = e^{-\int_0^t \frac{\gamma-1}{\gamma}\left(r_s + \frac{\theta_s^2}{2\gamma}\right)ds}, \tag{48}$$

$$\mathbb{E}\left((H_t w_t)^{\frac{\eta+1}{\eta}} \right) = w_0^{\frac{\eta+1}{\eta}} e^{-\int_0^t \frac{\eta+1}{\eta}\left(r_s - a_s - \frac{\theta_s^2}{2\eta}\right)ds}. \tag{49}$$

Substituting this into (47) leads to the expression (30).

B. Computations to derive the Optimal Investment Rule

From (19) we obtain that the wealth process X_t^* under the optimal strategy (π_t^*, C_t^*, L_t^*) is given by

$$X_t^* = A_t - B_t, \tag{50}$$

with

$$A_t = C_t^* \mathbb{E}_t \left(\int_t^\infty \frac{\hat{H}_s}{\hat{H}_t} \frac{C_s^*}{C_t^*} ds \right) \tag{51}$$

$$B_t = (w_t L_t^*) \mathbb{E}_t \left(\int_t^\infty \frac{\hat{H}_s}{\hat{H}_t} \frac{(w_s L_s^*)}{(w_t L_t^*)} \right) ds. \tag{52}$$

In the following we will compute A_t and B_t. Substituting the expressions for C_t^* and L_t^* from Theorem 3.1. gives

$$
\begin{aligned}
A_t &= C_t^* \mathbb{E}_t \left(\int_t^\infty \frac{H_s}{H_t} \frac{C_s^*}{C_t^*} \cdot e^{-\int_t^s \nu_u du} ds \right) \\
&= C_t^* \mathbb{E}_t \left(\int_t^\infty \left(\frac{H_s}{H_t} \right)^{\frac{\gamma-1}{\gamma}} \cdot e^{-\int_t^s \left(\nu_u + \frac{1}{\gamma} \rho_u \right) du} ds \right) \\
&= C_t^* \int_t^\infty \mathbb{E}_t \left(\left(\frac{H_s}{H_t} \right)^{\frac{\gamma-1}{\gamma}} \right) \cdot e^{-\int_t^s \left(\nu_u + \frac{1}{\gamma} \rho_u \right) du} ds.
\end{aligned}
$$

Now, using that $\frac{H_s}{H_t}$ is independent of $\mathcal{F}_t$ and distributed like a geometric Brownian motion with time varying drift term, we obtain

$$A_t = C_t^* f_t, \qquad \text{with} \qquad (53)$$

$$f_t =: \int_t^\infty e^{-\int_t^s \left(\frac{\gamma-1}{\gamma} \left(r_u + \frac{\theta_u^2}{2\gamma} \right) + \frac{1}{\gamma} \rho_u \right) du} \cdot e^{-\int_t^s \nu_u du} ds.$$

Similarly we can compute

$$B_t = w_t L_t^* g_t, \qquad \text{with} \qquad (54)$$

$$g_t =: \int_t^\infty e^{-\int_t^s \left(\frac{\eta+1}{\eta} \left(r_u - a_u - \frac{\theta_u^2}{2\eta} \right) - \frac{1}{\eta} \rho_u \right) du} \left(\frac{b_s}{b_t} \right)^{-\frac{1}{\eta}} \cdot e^{-\int_t^s \nu_u du} ds.$$

Using (53) and (54) we then obtain the expression in (32).

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