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Sharing the cost of risky projects

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Abstract

Users share the cost of unreliable non rival projects (items). For instance, industry partners pay today for R&D that may or may not deliver a cure to some viruses, agents pay for the edges of a network that will cover their connectivity needs, but the edges may fail, etc.. Each user has a binary inelastic need that is served if and only if certain subsets of items are actually functioning. We ask how should the cost be divided when individual needs are heterogenous. We impose three powerful separability properties:

Independence of Timing ensures that the cost shares computed ex ante are the expectation, over the random realization of the projects, of shares computed ex post. *Cost Additivity* together with *Separability Across Projects* ensure that the cost shares of an item depend only upon the service provided by that item for a given realization of all other items.

Combining these with fair bounds on the liability of agents with more or less flexible needs, and of agents for whom an item is either indispensable or useless, we characterize two rules: the *Ex Post Service* rule is the expectation of the equal division of costs between the agents who end up served; the *Needs Priority* rule splits the cost first between those agents for whom an item is critical ex post, or if there are no such agents between those who end up being served.

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1 Introduction

A group of users share a set A of *non rival* items $a, b, \cdots$ and are jointly liable to cover their costs; users only differ in the way these items fulfill their objective

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needs (known to the benevolent dictator): all items are not equally useful to all agents. Just from the description of these needs, we wish to define item per item liability shares for all the liable users.

For instance each item is an edge in a communication network, and each user wants to connect a certain pair of nodes of the network: many different paths achieve this, so an agent's needs are described by a *set of subsets of items.* Or each agent has a machine infected by a certain subset of viruses, and each item is a software that neutralizes a certain subset of viruses: several combinations of softwares will be enough to clean a given agent's machine. Or our agents want access to some e-database where the provider knows costs of the different parts (items in the database) and is trying to split costs more fairly based on actual usage rather than conventional flat fees.

In general we assume that the agents "take over" an exogenous set of items (e.g., links in a network) with given costs (e.g., for maintenance and exploitation), and this set is not optimally designed to fit their needs (we are not at the construction stage). Each item has a given cost (e. g.,each link has its own maintenance cost), and this is relevant to the fairness of the overall division of costs, for the usual reasons: which links are more helpful to fulfill the needs of which agents should be a consideration, and justifies deviations from simple benchmarks like equal split (flat fees).

We allow for an arbitrarily complex pattern of personal needs,[1] yet the axiomatic analysis remains tractable because we eschew all the standard microeconomic subtleties coming from heterogenous preferences and the interplay between these and the optimal choice of items to produce. Specifically

- needs are binary: each agent gets "service" or not;
- demand is fully inelastic: each agent has a large willingness to pay for service, and exactly how large is irrelevant;
- items to be paid for are exogenous: their set is not the result of any decision process, optimizing or otherwise, to which our agents participated; all costs must be paid even if some items could be dispensed with;
- costs of items are additive; although it is not possible to pay only for a subset of items, the liability shares are item specific so the cost distribution across items matters.

We are effectively taxing our agents according to their benefits, when individual "benefits" are captured by the probability of receiving service. As documented in the next Section, this "bare bones" approach to the division of costs is increasingly important in microeconomic distributive justice, because of the practical impossibility to measure benefits in terms of welfare and the fact that exogenous

[1] Recall that no closed form expression of the number of inclusion-monotonic subsets of 2^A is known.

costs, not influenced by individual preferences, are often the norm (see e. g., [14] and references there).[2]

We introduce a new dimension to the original model in [9] by assuming that items may fail, that they have *limited reliability,* and we show that this has a significant impact on the definition of fairness.

If the edges of the network represent trade routes, safe commodity transportation was never perfect for many centuries (piracy on the high sea is still a factor today). Associations of merchants (e.g., the German Hansa, see [11]) have throughout history shared the cost of keeping routes safe in a probabilistic sense, taking into account the individual trade patterns of their members. Firms pooling resources in R&D ventures like the development of drugs or softwares face a similar random return to their investment: the cost of the research must be paid upfront, but ex post the needs of only a subset of the partners will be met.

To stress that the relevant items succeed or fail randomly, we speak of individual "projects". Two important observations are: 1) the risk dimension introduces a tradeoff between the *ex ante* and *ex post* viewpoint on fairness; and 2) the impact of more or less flexible needs becomes ambiguous.

Ex-post versus Ex-ante: Consider a simple example with two projects a, b each of cost 1, such that each project succeeds with independent and equal probability p. Ann is served if (and only if) at least one project works, which we write as $\mathcal{D}^{Ann} = \{a \vee b\}$ while Bob is served only if both projects work, $\mathcal{D}^{Bob} = \{a \wedge b\}$. One approach is to divide cost in proportion to *ex ante* probabilities of service, $p(2-p)$ for Ann and p^2 for Bob, which suggests the shares $(1 - \frac{p}{2}, \frac{p}{2})$ for each item. We can focus instead on the *ex post* realization of the projects, and share costs equally if both are served or if none is served, whereas only Ann should pay if she is served but Bob is not. This results in the expected cost shares

$$(p^2 + (1-p)^2)(\frac{1}{2}, \frac{1}{2}) + 2p(1-p)(1,0) = (\frac{1}{2} + p(1-p), \frac{1}{2} - p(1-p)) \qquad (1)$$

so Ann pays more than Bob in both cases, but less so (resp. more so) in the ex post interpretation if $p \leq \frac{1}{2}$ (resp. $p \geq \frac{1}{2}$).

In the ex post approach we simply take the expectation of the cost shares in the *deterministic* problems generated once successes and failures are realized. This is the meaning of our axiom *Independence of Timing* (IT) below.

Interpreting flexible needs: In the example above Ann's needs are more flexible than Bob's, because for each realization of the projects such that Bob is served, she is too. On the one hand she is more likely to be served than Bob and this suggests she should pay more than him, as in the two divisions above. On the other hand Ann is less dependent upon any particular project than Bob hence her liability toward each project should be lower . The former argument is especially convincing if the common probability p is low, because then Bob

[2]By contrast, Wicksell's "benefit principle" is generally abandoned for the taxation of "macro" public services, because of its pervasive regressivity.

stands much less of a chance of service than Ann. The latter argument is more convincing if reliability is high, for instance $p = 1$, because then they are both served and paying for two projects instead of one is only useful to Bob: none of the two divisions above reflects this, but the *Needs Priority rule* does (see Section 3).

The inputs to our *cost allocation rules* are the cost of each project in the set A, the individual needs of each agent i described by a set $\mathcal{D}^i$ of subsets of A, and the random distribution of successes and failures (on which we place no restriction). The output are individual cost liabilities for each project. We impose three powerful separability properties on such rules, none of them conveying a particular concept of fairness. Together they characterize a simple family of rules, on which the impact of additional normative properties is transparent.

After Independence of Timing (above) we assume *Cost Additivity* (CA), one of the most common requirements in the cost sharing literature of the last three decades (see e.g. [13]). CA means that we must assign *liability shares* toward the cost of an project independently of its actual cost. Say that the two projects a, b are paid for by Ann and Chris with $\mathcal{D}^{Ann} = \{a \vee b\}$ and $\mathcal{D}^{Chris} = \{a\}$ (Chris needs a to succeed and has no use for b). Clearly Ann should be more liable for b than Chris, while Chris has a bigger stake in a than Ann: we want Ann to pay more than Chris for b and vice versa for a. CA allows us to implement these two normative choices irrespective of the actual costs of the two projects.

The third property, *Separability Across Projects* (SAP), ensures that in a deterministic problem (where each project either works or fails for sure) the cost shares of a given project a depend only upon the partition of the agents into three groups: those who are served if and only if a is available, those who are served even when a fails, and those who are not served even if a succeeds.

We propose and axiomatize two cost allocation rules, that differ mainly in their treatment of needs flexibility. The *Ex-post Service* rule, illustrated in example (1), divides total cost equally between all agents served ex post (if no one is served it divides costs equally among everybody). Although it is cost additive, this rule ignores differences in the costs of different projects. The *Needs Priority* rule, defined in Section 3, is more subtle: it computes the liability project per project and uses the information contained in the 3-partition mentioned in the previous paragraph.

Both rules meet IT, CA and SAP. To characterize each rule we use four properties which, unlike the previous three, have a clear fairness content. *No Charge for No Service* says that an agent who is never served pays nothing, at least if there is always at least one other agent who is served. Both rules meet this mild requirement. *Liable for Flexibility* insists that if Ann is easier to serve than Bob, she will not pay less than him: it drives a wedge between our two benchmark rules, as Needs Priority fails it while Ex-post Service meets it.

Say that agent i needs an project *single-mindedly* if every other project is useless to her: *Liable for Single-minded Needs* requires agent i to pay the largest individual share of the project. Finally *Useless Free* says that an agent does not pay for an project if it is useless to her (never needed to serve her needs), while another agent j needs it single-mindedly.

Among the rules meeting Cost Additivity, Separability Across projects, Independence of Timing and No Charge for no Service, the Ex-post Service rule is captured by the combination of Liable for Flexibility and Liable for Single-minded Needs, while the Needs Priority rule is captured by Useless Free.

Contents: After reviewing the relevant literature in Section 2, we define our model in Section 3. Section 4 introduces the three independence properties and characterizes the resulting family of cost sharing rules: Proposition 1. Within this family Section 5 characterizes the Ex-post service and the Needs Priority rules with the help of the additional fairness tests just mentioned, and checks that the axioms are independent.

2 Relation to the literature

In its formal simplicity our model is a relative of the large literature on the *rationing* (aka *bankruptcy*) problem of fairly dividing (exogenous) costs or benefits with respect to (objective) claims or liabilities: [18], [1], [13], [19]. See also the literature on the *airport* problem ([20]) and the more general minimal cost spanning tree problem (mcst) (e.g., [3], [10], [6]). There as here participants in the division process are entirely described by objective parameters (legally valid claims in the bankruptcy problem; verifiable stand-alone costs in the airport problem; verifiable connection costs in the mcst problem) with no relation to standard microeconomic characteristics like preferences, risk aversion, etc.. In such streamlined environment, it is possible to understand the complex interplay of a handful of normative principles.

Our model extends earlier work by Moulin and Laigret ([14]) and Hougaard and Moulin ([9]), both in the deterministic case where all projects always work.

In [14] each project is indispensable to at least one agent, so no project is redundant. One division method stands out, dubbed the *Equal Need* solution: it divides the cost of any project equally between all agents needing this project for service. Our Needs Priority rule applies the "*equal need equal share*" principle of fairness in our richer model and coincides with Equal Need on deterministic problems with no redundant project.

In [9] we allow for redundant projects and axiomatize the family of *Counting* rules, of which a typical rule goes as follows. For each project agents pay in proportion to the fraction of their minimal service sets including the project over the total number of their minimal service sets.[3] In the deterministic case this not only provides cost shares for redundant projects, but also more compelling shares for non-redundant problems than the Equal-Need solution. However, the *Expected Counting Liability* rule defined in Section 3 (by taking expectations over deterministic problems) typically violates SAP.

Independence of Timing was introduced by Myerson ([16]) in the context of risky axiomatic bargaining. Several recent papers discuss similarly a risky

[3] We call $D \in \mathcal{D}^i$ *minimal* if $D \diagdown \{a\} \notin \mathcal{D}^i$ for all $a \in D$.

version of the rationing problem: Ertemel and Kumar ([5]) characterize versions of our Ex-post and Ex-Ante Service rules in a state contingent claims framework; while Habis and Herings ([7]) and Xue ([21]) consider stochastic versions of the uniform gains rule.

The Engineering literature discusses at length the optimal and fair allocation of network resources (see e.g., [12], [17]). It focuses on traffic optimization when network capacity is exceeded and communication channels may fail, thus the thrust is completely different. A recent paper by Bergantinos and Martinez ([4]) has some resemblance with our problem though. They consider cost sharing in trees where each agent (node) is a source with individual production and demand and argue that if production is unreliable it creates asymmetric liabilities for edges which matter for the fair division of costs.

Finally, we do not follow the familiar approach (e.g., [13], or [8]) attaching a cooperative game to the problem at hand, then applying the Shapley value or some other solution concept to compute cost shares. As explained in Section 3, in our problem the game in question would come from the family of *expected* Stand Alone costs. The first objection to this approach (as already argued in [14] and [9]) is that it violates both axioms CA and SAP (Stand Alone costs are not linear in costs). Next it runs counter to the spirit of our model where costs are not optimized: we must divide the total cost of all projects, even if only a subset of these would be enough to cover all needs.

3 The model

The finite set A contains the risky projects and each project $a \in A$ has a non-negative cost c_a. If successful, the projects are public goods that can be consumed without rivalry by the agents in N. Nature chooses the set $X, \varnothing \subseteq X \subseteq A$, of successful projects. Irrespective of the realization of X, all agents must share the total cost $c_A = \sum_{a \in A} c_a$.

The needs of agent $i \in N$ are described by a non empty set $\mathcal{D}^i \subseteq 2^A$ of *service sets*: agent i is *served* if and only if the subset of projects actually working is in $\mathcal{D}^i$. Each set $\mathcal{D}^i$ is inclusion monotonic: if $D \in \mathcal{D}^i$ any superset of D is also in $\mathcal{D}^i$. Two extreme service sets that will play a role in the axioms are $\mathcal{D}^i = \varnothing$ for an agent who never gets service, and $\mathcal{D}^i = 2^A$ for an agent who gets service irrespective of the realized outcome. The service sets of each agent $\mathcal{D}^i$ allow for very complex expressions of personal need where, for instance, projects a, b jointly substitute for project c, etc..

Definition 1: A cost allocation problem under risk is a list (Q, p, c) where $Q = (N, A, \{\mathcal{D}^i\}_{i \in N})$ is the profile of needs, $p \in \Delta(2^A)$ is the probability distribution of the set X of successful projects, and $c \in \mathbb{R}_+^A$ is the vector of project costs.

In the first example of the Introduction, A is the set of edges of an undirected communication network, each with limited reliability, and agents have different routing/connectivity needs. These could be: at least one path between two specific nodes; or a path avoiding certain edges and/or nodes (for privacy concerns); or two edge-disjoint paths (to allow two independent communications); etc..

Or V is a set of *(biological or digital)* viruses and project a is a software that (if successful) will neutralize the subset $V(a) \subseteq V$ of viruses. Agent i is infected by the set of viruses $T_i \subseteq V$, so she is cured if and only if the set X of successful softwares is such that $T_i \subseteq \cup_{a \in X} V(a)$.[4]

Definition 2: A cost allocation rule assigns to any problem (Q, p, c) a vector of cost shares $\widetilde{y}(Q, p, c) \in \mathbb{R}_+^N$ satisfying 1. Budget-balance: $\sum_{i \in N} \widetilde{y}_i = c_A$; 2. Symmetry in N, i.e., $\mathcal{D}^i = \mathcal{D}^j$ implies $\widetilde{y}_i = \widetilde{y}_j$; and 3. Symmetry in A, i.e., cost shares do not change when exchanging the cost of two projects entering symmetrically in Q.[5]

Our first two examples are two simple rules ignoring the differences in cost across projects. The first one charges the users in proportion to their *ex-ante* probability of being served. Hence agent i's share in the **Ex-ante Service rule** $\widetilde{y}^{xa}$ is:

$$\widetilde{y}_i^{xa}(Q, p, c) = \begin{cases} \frac{p(X \in \mathcal{D}^i)}{\sum_{j \in N} p(X \in \mathcal{D}^j)} c_A & \text{if } p(\cup_j \mathcal{D}^j) > 0 \\ \frac{1}{|N|} c_A & \text{if } p(\cup_j \mathcal{D}^j) = 0 \end{cases} \tag{2}$$

In a network context, the Ex-ante Service rule shares total cost in proportion to the *network reliability* index discussed in the Operations Research literature (e.g., [2]).

The second rule charges equally the agents served *ex-post*, i.e., those in $S(Q; X) = \{i \in N | X \in \mathcal{D}^i\}$ and computes the expectation of these shares. We use the notation $e[S]$ for the vector in the simplex $\Delta(N)$ over N such that $e[S]_i = \frac{1}{|S|}$ if $i \in S$ and $e[S]_i = 0$ otherwise. Importantly we set $e[\varnothing] = e[N] = (\frac{1}{|N|})_{i \in N}$. Cost shares of the **Ex-post Service rule** $\widetilde{y}^{xp}$ are:

$$\widetilde{y}^{xp}(Q, p, c) = (\sum_{X \in 2^A} p(X) \times e[S(Q, X)]) c_A \tag{3}$$

Note that an agent i with zero probability of service, $p(X \in \mathcal{D}^i) = 0$, is not charged under the Ex-ante rule (provided $p(\cup_j \mathcal{D}^j) > 0$), whereas under the Ex-post rule she will be charged some positive share if $p(\cup_j \mathcal{D}^j) < 1$ because $S(Q; X)$ is empty with positive probability.

The next two rules are more subtle and relate to cost shares for individual projects.

For the **Needs Priority rule**, if project a fails its cost is shared equally between all agents served ex-post. If a succeeds, Needs Priority charges only those agents who are served *if and only if* a is provided, and if there are no

[4]It is easy to see that this model encompasses our model: any cost allocation problem in Definition 1 can be written as a virus problem for an appropriate choice of V and T_i.

[5]Two projects a, b enter symmetrically in Q if each service constraint $\mathcal{D}^i$ is invariant by the operation in 2^A switching a and b (if X contains one and not the other, exchange them in X; otherwise do nothing).

such agents, it charges equally all agents served ex post. Using the notation $e[S^1; S^2] = e[S^1]$ if $S^1 \neq \varnothing$, and $e[\varnothing; S^2] = e[S^2]$, we define $\widetilde{y}^{np}$ as follows:

$$\widetilde{y}^{np}(Q,p,c) = \sum_{a\in A}\Big(\sum_{\varnothing\subseteq X\subseteq\mathcal{A}} p(X)\times e[S(Q;X)\diagdown S(Q;X\diagdown a); S(Q;X)]\Big)c_a \qquad (4)$$

For comparison with the Ex-Post rule we go back first to the two-project two-agent example discussed in the introduction where $\mathcal{D}^{Ann} = \{a \vee b\}$ and $\mathcal{D}^{Bob} = \{a \wedge b\}$. Note that the success of each project is critical to Bob but not to Ann. Start with the deterministic case where all projects work, $p(A) = 1$, where the difference between the two rules is especially stark: Needs Priority gives Ann a free ride on both projects and Bob pays everything; while Ex-post Service ignores all differences in needs and split costs equally. Both outcomes are extremist: the former is too hard to Bob, because Ann still needs *some* project to work, and the latter too easy on him since Ann's flexibility is not rewarded. Now assume projects a and b have IID probability of success 2/3. Then Ex-post Service picks cost shares $(13/18, 5/18)$: Ann pays everything when exactly one project works (probability 4/9) and costs are split equally otherwise. Interestingly the Needs Priority rule splits costs equally. Consider project a: if both projects work it is critical to Bob only; if only one project works it is critical to Ann only; and if no project works they split equally. As the probability of two projects working, and that of just one working are equal to 4/9, we end up with equal split. But if the IID probability of success is above (resp. below) 2/3, then Bob (resp. Ann) pays more.

The second example in the Introduction has $\mathcal{D}^{Ann} = \{a \vee b\}$ and $\mathcal{D}^{Chris} = \{a\}$. In the deterministic case the two rules are again very coarse: Needs Priority charges all of a to Chris and shares b equally; while Ex-post Service splits all costs equally. But with IID probability of success 2/3, they are much more nuanced. Ex-post Service splits costs as $(11/18, 7/18)$: Ann pays more because she is easier to serve. Needs Priority shares a as $(7/18, 11/18)$ and b as $(11/18, 7/18)$: it makes sense to view Chris as more liable for project a than Ann, and while the opposite is true for project b.

The second rule is a risky version of the Counting rule discussed in [9], we call it the **Expected Counting Liability rule**. A set $D \in \mathcal{D}^i$ is a *minimal service set* of agent i if $D\diagdown\{a\} \notin \mathcal{D}^i$ for all $a \in D$. The Counting Index $\varphi_i(a; A, Q)$ of agent i for project a is the ratio of the number of i's minimal service sets containing a to the total number of i's minimal service sets; the (deterministic) Counting rule in [9] divides the cost of each a in proportion to these ratios. In our probabilistic model we project service needs for any realization X as $\mathcal{D}^i(X) = \{D \subseteq X | D \subseteq \mathcal{D}^i\}$, and take expectations over X. Thus agent i's share in $\widetilde{y}^{ecl}$ is:

$$\widetilde{y}^{ecl}_i(Q,p,c) = \sum_{a\in A}\Big(\sum_{\emptyset\subseteq X\subseteq A} p(X)\times \frac{\frac{|\overline{\mathcal{D}}^i(a,X)|}{|\overline{\mathcal{D}}^i(X)|}}{\sum_{j\in N}\frac{|\overline{\mathcal{D}}^j(a,X)|}{|\overline{\mathcal{D}}^j(X)|}}\Big)c_a, \text{ for all } i. \qquad (5)$$

where $\overline{\mathcal{D}}^i(a, X)$ is the set of i's *minimal* service sets containing project a given realizations X, and with the convention $\frac{0}{0} = \frac{1}{n}$.

Recall the example $\mathcal{D}^{Ann} = \{a \vee b\}$ and $\mathcal{D}^{Bob} = \{a \wedge b\}$ with both projects having IID probabilities of success 2/3. The Expected Counting Liability rule gives cost shares $(0.54, 0.46)$ for each project. Consider for instance project a: if both projects fail or only b works, Ann and Bob share equally; if only a works Ann pays the full cost; and if both projects work shares are $(1/3, 2/3)$ (Ann has counting index 1/2 and Bob 1).

In the next example where $\mathcal{D}^{Ann} = \{a \vee b\}$ and $\mathcal{D}^{Chris} = \{a\}$, the cost shares differ for the two projects. Still assuming IID probabilities of success 2/3, Ann and Chris share equally project a except when both projects work. Then Ann's counting index is 1/2 and Chris' is 1, so they share as $(1/3, 2/3)$, hence the final shares $(0.43, 0.57)$. For project b we get the shares $(0.83, 0.17)$.

Our last cost allocation rule applies the Shapley value to the standard cooperative game where the stand-alone cost of a coalition $S \subset N$ in problem (Q, p, c) is defined as follows. Set $\mathcal{D}^S = \cap_{i \in S} \mathcal{D}^i$ to be the set of subsets of projects serving every agent $i \in S$, and let $\mathcal{D}^S(X) = \{D \subseteq X | D \subseteq \mathcal{D}^S\}$ be the projection of $\mathcal{D}^S$ onto the realization $X \subseteq A$. Then the Stand Alone cost of S given realization X is

$$c(S, X) = \begin{cases} \min\{c_D | D \in \mathcal{D}^S(X)\} & \text{if } \mathcal{D}^S(X) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and the expected Stand Alone cost of $S \subsetneq N$ is

$$\bar{c}(S) = \sum_{\emptyset \subseteq X \subseteq A} p(X) \times c(S, X) \tag{6}$$

while $\bar{c}(N) = c_A$ is the total cost that has to be covered.

Note that the core of $(N, \bar{c})$ will often be empty due to redundancy.

4 Three Separability properties

4.1 Cost Additivity

Cost differences between projects should influence individual liabilities, as suggested already in the by now familiar two-project two-agent example with $\mathcal{D}^{Ann} = \{a \vee b\}$ and $\mathcal{D}^{Chris} = \{a\}$. Suppose that with probability 0.9 project a works but b fails, while with probability 0.1 project b works but a fails. The Ex-ante and Ex-post rules select approximately the same cost shares.[6] However if we learn that b is much more expensive than a, it makes sense to charge significantly more to Ann (as the Needs Priority rule will do) because she is more liable for b than Chris, and most of the costs come from b.

[6] Since Ann's ex ante probability of being served is 1, and Chris' is 0.9 we get $\widetilde{y}^{xa} = (0.53, 0.47)c_A$. Under the Ex-post rule if a works but b fails Ann and Bob share equally, while if b works but a fails only Ann is liable: so $\widetilde{y}^{xp} = (0.55, 0.45)c_A$.

Cost Additivity (CA): For all profile of needs Q, probability distribution p, and cost vectors c^1 and c^2

$$\widetilde{y}(Q,p,c^1+c^2)=\widetilde{y}(Q,p,c^1)+\widetilde{y}(Q,p,c^2).$$

Clearly a cost additive rule takes the form

$$\widetilde{y}(Q,p,c)=\sum_{a\in A} y(a;Q,p)c_a \tag{7}$$

where $\Delta(N)$ is the N-simplex and $y(a;Q,p)\in\Delta(N)$ specifies how the liability shares for project a.

The Ex-ante and the Ex-post rules are both cost additive, but they ignore the flexibility afforded by the decomposition (7). The Needs Priority and Expected Counting Liability rules are clearly of the form (7). However, the Shapley value of the Stand Alone game (6) is *not* a cost additive rule.

4.2 Independence of Timing

The Ex-ante and Ex-post rules react very differently to the arrival of new information about the outcome of some projects, leading to an update of the distribution p. Under the Ex-ante rule, this update typically upsets the distribution of shares. By contrast under the Ex-post rule computing the cost shares from the initial distribution p, or from the expectation of its updates, amounts to the same thing. Mathematically this is the property that cost shares are an affine function of probabilities.

Suppose the uncertainty about success of failure of the projects is described by a composed distribution $r=\lambda p+(1-\lambda)q$ (where $r,p,q\in\Delta(2^A)$ and $\lambda\in]0,1[$). We require that settling shares at time 0 given r, or waiting to compute shares till time 1 when either p or q is realized, yields the same shares (in expectation at time 0):

Independence of Timing (IT): For $p,q\in\Delta(2^A)$ and $\lambda\in[0,1]$

$$\widetilde{y}(Q,\lambda p+(1-\lambda)q,c)=\lambda\widetilde{y}(Q,p,c)+(1-\lambda)\widetilde{y}(Q,q,c)$$

This implies that payments are additively separable in the set of successful projects X (as in a classic von Neuman-Morgenstern setting).

Together CA and IT characterize the family of rules (7) with shares of the form

$$y(a;Q,p)=\sum_{\varnothing\subseteq X\subseteq A} p(X)\times y^*(a,Q,X) \tag{8}$$

where $y^*(a,Q,X)$ is the profile of liability shares for a when the *deterministic* outcome of the projects is that those in X succeed and only those.

The Ex-ante service rule (2) *fails* IT. The Ex-post service rule (3) is clearly of the form (8), so this rule meets IT. So do the Needs Priority, Expected Counting Liability, and Shapley value rules.

4.3 Separability Across Projects

Our third separability axiom bears on the impact of project b over the cost shares of other projects a, evaluated when project b is deterministic (always works or always fails) and costless. Costs shares should then depend on project b only through b's impact on the profile of needs.

Notation: given $b \in A$ let $\mathcal{D}^i(b=1) = \{D \subseteq A \diagdown b | D \cup \{b\} \in \mathcal{D}^i\}$ be the subsets of $A \setminus b$ for which adding project b would provide service for i, and let $\mathcal{D}^i(b=0) = \{D \subseteq A \diagdown b | D \in \mathcal{D}^i\}$ be those for which b is not needed, i.e., the projection of i's needs on $A \setminus b$. Let $Q(b=\delta) = (N, A \diagdown b, (\mathcal{D}^i(b=\delta))_{i \in N})$ for $\delta = 0, 1$, be the corresponding profiles of needs. Next let $p(b) = p(X \ni b)$ be the probability that b is successful and denote by $p^{b=1}$,$p^{b=0} \in \Delta(2^{A \diagdown b})$ the distributions $p^{b=1}(X) = p(X \cup b)$ and $p^{b=0}(X) = p(X)$ for all $X \in A \setminus b$. Finally $c(-b)$ is the projection of c on $\mathbb{R}_+^{A \diagdown b}$.

Separability Across Projects (SAP): for all b and $\delta = 0, 1$

$$\{p(b) = \delta \text{ and } c_b = 0\} \Longrightarrow \widetilde{y}(Q, p, c) = \widetilde{y}(Q(b=\delta), p^{b=\delta}, c(-b))$$

For a cost additive rule the axiom takes a simpler form: for all b and $\delta = 0, 1$

$$p(b) = \delta \Longrightarrow \; y(a; Q, p) = y(a; Q(b=\delta), p^{b=\delta}) \tag{9}$$

When b fails (resp. works) for sure, the shares of project a depend only upon the projected service profiles $Q(b=\delta)$ and probabilities $p^{b=\delta}$.

The Ex-ante, Ex-post Service, and Needs Priority rules satisfy SAP. The Expected Counting Liability and the Shapley value rules both fail SAP.

4.4 Putting the three properties together

We need additional notation to express the consequences of our three independence properties. Let $\mathcal{T}$ be the set of ordered 3-partitions $\tau = (T_-, T_0, T_+)$ of N where up to two subsets can be empty, and Θ be the set of triples of non negative integers $\theta = (\theta_-, \theta_0, \theta_+)$ of sum n. The cardinality of $\tau \in \mathcal{T}$ is $|\tau| = (|T_-|, |T_0|, |T_+|) \in \Theta$. Let Γ be the set of mappings γ from Θ into $\mathbb{R}_+^3$ such that

$$\theta_- \gamma_-(\theta) + \theta_0 \gamma_0(\theta) + \theta_+ \gamma_+(\theta) = 1 \text{ for all } \theta \in \Theta$$

Each γ in Γ defines a mapping g from $\mathcal{T}$ into $\Delta(N)$ as follows. For any $\tau \in \mathcal{T}$

$$g_i(\tau) = \gamma_\varepsilon(|\tau|) \text{ if } i \in T_\varepsilon \text{ , where } \varepsilon = -, 0, + \tag{10}$$

We write G for the set of such mappings g. Intuitively g assigns individual shares based only on the cardinality of the partition, and the membership in the partition. It is fully symmetric: if agents $i \in T_{\varepsilon_1}$ and $j \in T_{\varepsilon_2}$ swap membership (so the new partition has $T'_{\varepsilon_1} = T_{\varepsilon_1} + j - i$ etc..), their shares are swapped as well and other agents are unaffected.

Recall that $S(Q; X) = \{i \in N | X \in \mathcal{D}^i\}$ is the set of agents served if X is realized. When X contains a, we let $S_a(Q; X) = S(Q; X) \diagdown S(Q; X \diagdown \{a\})$ be the set of agents served at X for whom the success of project a is critical: they would not be served if project a had failed.

Given a profile of needs Q, an project a, and a subset of projects X, we define the partition $\tau(a; Q, X) \in \mathcal{T}$ as follows:

$$T_{-} = N \diagdown S(Q; X \cup \{a\}) \; ; \; T_{+} = S(Q; X \diagdown \{a\})$$

$$T_0 = S_a(Q; X) \text{ if } a \in X \; ; \; T_0 = S_a(Q; X \cup \{a\}) \text{ if } a \notin X$$

So T_{-} (resp. T_{+}) is the set of agents that are not served (resp. are served), irrespective of the success or failure of a; and T_0 is the set of agents who are served if and only if a succeeds.

Proposition 1: *A cost sharing rule is Cost Additive, Independent of Timing and Separable Across Projects, if and only if there exist two mappings $g^{-}, g^{+} \in G$* such that

$$y(a; Q, p) = \sum_{X: \varnothing \subseteq X \subseteq A \diagdown \{a\}} p(X) \times g^{-}(\tau(a; Q, X)) + \sum_{X: a \in X \subseteq A} p(X) \times g^{+}(\tau(a; Q, X)) \tag{11}$$

Proof We already observed that the combination of CA and IT characterizes the rules of the form (7), (8). It remains to describe the consequences of SAP and Symmetry in projects and agents.

Fix $Q, X \subseteq A$ and a, and recall that in equation (8) $y^*(a, Q, X) = y(a; Q, \delta_X)$ where $\delta_X \in \Delta(2^A)$ picks X with probability 1. When we apply SAP repeatedly to all projects b in $A \diagdown \{a\}$, with $b = 0$ if $b \notin X \cup \{a\}$ and $b = 1$ if $b \in X \diagdown \{a\}$, the profile of needs shrinks to

$$\overline{Q} = (N, \{a\}, \overline{\mathcal{D}}^i) \text{ where } \overline{\mathcal{D}}^i = \{D \subseteq \{a\} | D \cup (X \diagdown \{a\}) \in \mathcal{D}^i\}$$

the corresponding probability is

$$\overline{p}(a) = 1 \text{ if } a \in X \; ; \; \overline{p}(a) = 0 \text{ if } a \notin X$$

and we have

$$y^*(a; X; Q) = y(a; Q, \delta_X) = y(a; \overline{Q}, \overline{p})$$

As $\overline{p}$ is determined by a in or out of X, we can define g^{-} and g^{+} as follows:

$$y^*(a; X; Q) = g^{-}(a, \overline{Q}) \text{ if } a \notin X \; ; \; y^*(a; X; Q) = g^{+}(a, \overline{Q}) \text{ if } a \in X$$

Now each $\overline{\mathcal{D}}^i$ is a possibly empty subset of $2^{\{a\}}$, and three of these *four* subsets correspond to inclusion monotonic needs: the empty set, the singleton $\{a\}$, and $2^{\{a\}}$. Distinguish these three cases:

$\overline{\mathcal{D}}^i = \varnothing$ means: $\{a\} \cup (X \diagdown \{a\}) \notin \mathcal{D}^i \Longleftrightarrow X \cup \{a\} \notin \mathcal{D}^i \Longleftrightarrow i \in T_{-}$: agent i is not served at X, even if a outside X works;

$\overline{\mathcal{D}}^i = \{a\}$ means: $\{\{a\} \cup (X \diagdown \{a\}) \in \mathcal{D}^i \text{ and } (X \diagdown \{a\}) \notin \mathcal{D}^i\} \Longleftrightarrow i \in T_0$: agent i is served at X if a works, but not if a fails;

$\overline{\mathcal{D}}^i = 2^{\{a\}}$ means: $(X \diagdown \{a\}) \in \mathcal{D}^i \Longleftrightarrow i \in T_+$: agent i is served at X, even if we switch a from success to failure.

Thus $\overline{Q}$ is entirely described by the partition $\tau(a, Q, X) = (T_-, T_0, T_+)$ of N, and for $\varepsilon = -, +$ we can write g^ε as $g^\varepsilon(a, \tau(a; Q, X))$. Symmetry in A implies that the first a does not matter, and symmetry in N implies that g^ε takes the form (10) for some $\gamma \in \Gamma$, i.e., $g^\varepsilon \in G$. Thus we reach the desired form (11).

In the converse statement we only check that the allocation rule (11) meets SAP. Fix a problem (Q, p) and two outcomes a, b where $p(b) = 1$. Then for any X containing b, it is easy to check that $\tau(a; Q, X) = \tau(a; Q(b = 1), X \diagdown \{b\})$, and $p^{b=1}(X \diagdown \{b\}) = p(X)$: property (9) for $\varepsilon = 1$ follows. The proof for $\varepsilon = 0$ is similar. ■

Proposition 1 captures a large family of rules worthy of further axiomatic investigations: it includes rules that use the flexibility offered by Cost Additivity and assign individual cost shares to each project, such as the Needs Priority rule, and is closed by convex combinations.

Proposition 1 is a tight characterization result:

- If we drop IT the Ex-ante Service rule (2) meets all other axioms;
- If we drop SAP the Expected Counting Liability rule (5) meets all other axioms;
- If we drop CA consider the rule equal to Ex-post Service at all problems (Q, p, c) such that $c_A \leq 1$ and equal to Needs Priority if $c_A > 1$. It meets IT and SAP because both axioms compare cost shares at different problems with identical total cost shares.

5 Two special rules and their characterization

We now focus on the two rules defined in Section 3 satisfying CA, IT, and SAP. The Ex-post Service rule (3) takes the form (8) for

$$y^{xp}(a, Q, p) = \sum_{X \in 2^A} p(X) \times e[S(Q, X)]$$

which is the special case of (11) where

$$g^-(T_-, T_0, T_+) = e[T_+]; \;\; g^+(T_-, T_0, T_+) = e[T_0 \cup T_+]$$

The Needs Priority rule (4) takes the form (8) for

$$y^{np}(a, Q, p) = \sum_{\varnothing \subseteq X \subseteq \mathcal{A}} p(X) \times e[S(Q; X) \diagdown S(Q; X \diagdown a); S(Q; X)] \qquad (12)$$

which is the special case of (11) where

$$g^-(T_-, T_0, T_+) = e[T_+]; \; g^+(T_-, T_0, T_+) = e[T_0] \text{ if } T_0 \neq \varnothing \text{ , } g^+(T_-, \varnothing, T_+) = e[T_+]$$

Remark: The Ex-post Service rule and the Needs Priority rule are Population Monotonic: $\overline{y}_i(N, A, \mathcal{D}, p) \geq \overline{y}_i(N \cup \{j\}, A, \mathcal{D}, p)$ for all $j \notin N$. When utilization of the projects is excludable, this allows strategyproof elicitation of the agents' ex ante willingness to pay, as in [15].

5.1 Some simple fairness tests

We fix a problem (Q, p, c) throughout the definitions below. In the first two definitions $\widetilde{y} \in \mathbb{R}_+^N$ refers as in Definition 2 to total cost shares over all projects.

No Charge for No Service (NCNS): if $p(X \in \mathcal{D}^i) = 0$ while $p(X \in \cup_{j \in N} \mathcal{D}^j) = 1$, then $\widetilde{y}_i = 0$; if $p(X \in \mathcal{D}^j) = 0$ for all j then $\widetilde{y} = e[N] \times c_A$

If ex post at least one agent is served, then an agent never served pays nothing. It is compelling, and clearly satisfied for any rule of the form (11) (including both rules above) when g^+, g^- does not charge T_- provided $T_0 \cup T_+$ is non empty.

Liable for Flexibility (FLEX): if $\mathcal{D}^i \supseteq \mathcal{D}^j$, then $\widetilde{y}_i \geq \widetilde{y}_j$

This property drives a wedge between our two rules: the inclusion means that i' s needs are more flexible than j's, in particular i is more likely to be served, and the axiom assigns a (weakly bigger) liability to i than to j. While Ex-post Service clearly meets this property, Needs Priority fails it.

In the next two axioms we assume for simplicity that our allocation rule is Cost Additive, so we write it as in (7) in terms of the cost shares $y(a)$ for a particular project.[7] We call agent i *single-minded* on project a if $X \in \mathcal{D}^i \iff a \in X$: the success of a is all that matters to agent i.

Liable for Single-minded Needs (SMN): if i is single-minded on project a and $p(a \in X) = 1$, then $y_i(a) = \max_{j \in N} y_j(a)$

This property is uncontroversial and clearly true for both our rules.

We call project a *useless* to agent i if $D \in \mathcal{D}^i \Longrightarrow D \diagdown \{a\} \in \mathcal{D}^i$.

Useless is Free (UF): if a is useless to i while j is single minded on a, and $p(a \in X) = 1$, then $y_i(a) = 0$

Needs Priority meets this property: agent j is in T_0 for any X containing a, while agent i is in $T_- \cup T_+$. But Ex-post Service does not: even i for whom a is useless will contribute to the cost of a if she is served.

5.2 Characterization result

Theorem 2: *Among the rules satisfying Cost Additivity, Independence of Timing, and Separability Across Projects*
i) the ***Ex-post Service*** *rule is characterized by No Charge for No Service, Liable for Flexibility, and Liable for Single-minded Needs;*

[7] We could of course define the axioms for global cost shares, under the assumption that all items except a are costless.

*ii) the **Needs Priority** rule is characterized by No Charge for No Service, and Useless is Free.*

Proof We already checked that Ex-post Service meets NCNS, FLEX, and SMN; and that Needs Priority meets NCNS and UF. Conversely we fix a rule in the form (11) and we check the impact of the four fairness tests.

Step 1 NCNS implies that g^+ does not charge T_- (unless $T_- = N$) and $g^-(T_-, T_0, T_+) = e[T_+]$.

Fix $\tau \in \mathcal{T}$ with T_- and $T_0 \cup T_+$ both non empty. Fix A, a and $\widetilde{X}$ such that $a \in \widetilde{X}$ and construct a problem (Q, p) such that $p(\widetilde{X}) = 1$, and

- $\widetilde{X} \notin \mathcal{D}^i$ for $i \in T_-$
- $\widetilde{X} \in \mathcal{D}^i$ but $\widetilde{X} \diagdown \{a\} \notin \mathcal{D}^i$ for $i \in T_0$
- $\widetilde{X} \diagdown \{a\} \in \mathcal{D}^i$ for $i \in T_+$

so that $T_- = N \diagdown S(Q; \widetilde{X})$, $T_0 = S_a(Q; \widetilde{X})$, and $T_+ = S(Q; \widetilde{X} \diagdown \{a\})$ and (11) gives the shares $y(a, Q, p) = g^+(\tau)$. On the other hand NCNS implies that T_- pays nothing, so we conclude that g^+ does not charge T_-.

Next we fix $\tau \in \mathcal{T}$ and choose $A, a, \widetilde{X}$ (non empty) such that $a \notin \widetilde{X}$. We choose similarly a problem (Q, p) such that $p(\widetilde{X}) = 1$ and

- $T_- = N \diagdown S(Q; \widetilde{X} \cup \{a\})$
- $T_0 = S_a(Q; \widetilde{X} \cup \{a\})$
- $T_+ = S(Q; \widetilde{X})$

so that $y(a, Q, p) = g^-(\tau)$ while NCNS says that: if $T_+ \neq \varnothing$ then agents in $T_- \cup T_0$ pay nothing, while if $T_+ = \varnothing$ all agents share equally the cost of a. This is the desired conclusion.

Step 2 UF implies that g^+ does not charge T_+ if $T_0 \neq \varnothing$.

Fix $\tau \in \mathcal{T}$ with $T_0 \neq \varnothing$ and $A, a, \widetilde{X}$ s. t. $a \in \widetilde{X}$ and $\widetilde{X} \diagdown \{a\} \neq \varnothing$. Similarly to step 1 we construct (Q, p) s. t. $p(\widetilde{X}) = 1$, and

- $\widetilde{X} \notin \mathcal{D}^i$ for $i \in T_-$
- each $j \in T_0$ is single minded on a
- for each $i \in T_+$ we have $\widetilde{X} \diagdown \{a\} \in \mathcal{D}^i$ and in fact a is useless to i

so that $y(a, Q, p) = g^+(\tau)$ and by UF g^+ does not charge T_+.

Step 3 SMN implies that g^+ charges weakly more to agents in T_0 than to those in T_+.

Fix $\tau \in \mathcal{T}$ with T_0 and T_+ both non empty. The same construction as in step 2 (where for $i \in T_+$ we only need $\widetilde{X} \diagdown \{a\} \in \mathcal{D}^i$) gives $y(a, Q, p) = g^+(\tau)$ and by SMN $g_j^+(\tau) \geq g_i^+(\tau)$ for $j \in T_0$, $i \in T_+$.

Step 4 FLEX implies that g^+ charges weakly more to agents in T_+ than to those in T_0.

Fix $\tau \in \mathcal{T}$ with T_0 and T_+ both non empty. We use the same construction of $A, a, \widetilde{X}, (Q, p)$ as in step 2 for T_- and T_0, and for each $i \in T_+$ we choose

- $\mathcal{D}^i$ containing $\widetilde{X} \diagdown \{a\}$ as well as all supersets of $\{a\}$

Then $y(a, Q, p) = g^+(\tau)$ and by FLEX $g_i^+(\tau) \geq g_j^+(\tau)$ for $j \in T_0$, $i \in T_+$.

Step 5 Now statement $i)$ follows by combining Steps 1,3 and 4. while statement $ii)$ follows from Steps 1 and 2. ■

Both statements in Theorem 2 are tight characterization results: Because Proposition 1 is tight we only need to check that dropping any one of the additional fairness tests generates new rules in the form (11). For statement $i)$

Drop NCNS: the rule dividing total cost equally, irrespective of needs, meets FLEX and SMN.

Drop FLEX: the Needs Priority rule meets NCNS and SMN.

Drop SMN: consider the rule with $g^-(\tau) = e[T_+]$ and

$$g^+(\tau) = e[T_+] \text{ if } T_+ \neq \varnothing; \; = e[T_0] \text{ if } T_+ = \varnothing \neq T_0; \; = e[N] \text{ if } T_- = N$$

It meets NCNS by Step 1 in the above proof, and FLEX because if $\mathcal{D}^i \supseteq \mathcal{D}^j$ and i, j are not in the same element of τ, then j is below i (in the sense where T_- is below T_0 and T_0 is below T_+). It fails SMN when i is single-minded on project a and at the deterministic configuration (a, Q, X) with $a \in X$, we have $i \in T_0$ and $T_+ \neq \varnothing$.

For statement $ii)$

Drop NCNS: consider the rule

$$g^-(\tau) = g^+(\tau) = e[T_0] \text{ if } T_0 \neq \varnothing; \; = e[N] \text{ if } T_0 = \varnothing$$

It fails NCNS at the deterministic configuration (a, Q, X) when $a \notin X, i \in T_0$ and $T_+ \neq \varnothing$. UF holds true because if a is useless to i while j is single minded on a, and X contains a for sure, then T_0 contains j but not i so i does not pay for a.

Drop UF: the Ex-post Service rule meets all other axioms.

6 Compatibility of axioms and rules

	IT	CA	SAP	NCNS	FLEX	SMN	UF
Ex-ante Service	*NO*	+	+	+	+	+	*NO*
Ex-post Service	+	+	+	+	+	+	*NO*
Needs Priority	+	+	+	+	*NO*	+	+
Expected Counting Liability	+	+	*NO*	+	*NO*	+	+
Shapley value	+	*NO*	*NO*	*NO*	*NO*	*NO*	*NO*

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