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The Beautiful Art of Mathematics*

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Abstract

Mathematicians frequently use aesthetic vocabulary and sometimes even describe themselves as engaged in producing art. Yet aestheticians, in so far as they have discussed this at all, have often downplayed the ascriptions of aesthetic properties as metaphorical. In this paper I argue firstly that the aesthetic talk should be taken literally, and secondly that it is at least reasonable to classify some mathematics as art.

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1 Introduction

One might think it strange if there were a field of human activity in which the practitioners regularly describe themselves as motivated by aesthetic considerations, appraise each others' work using apparently aesthetic vocabulary, and even explicitly describe what they are doing as 'art'; yet aestheticians show no interest in the field.

But such is, or was until recently, the peculiar position of mathematics. What literature that does exist on this topic — and it is rather little — has consisted mostly of scattered remarks made by mathematicians reflecting on their subject, with not much written in a systematic way by philosophers. The situation is starting to change, but the subject of the aesthetics of mathematics is still in its infancy.

My aim here is primarily to argue that there are indeed genuinely interesting aesthetic issues here, and that mathematics is a perfectly good topic for aestheticians to discuss.

For the most part I will give an overview of the issues; many of them deserve a far more lengthy treatment than I have space for here. But I shall be arguing in a little more detail for two specific theses. Firstly, that the aesthetic vocabulary used in discussing mathematics should be taken literally. If that isn't so, the aesthetics of mathematics is a pseudo-subject, and attempts to nurture it into maturity are misguided. Secondly, and more tentatively, I shall outline how it could be argued that mathematics is sometimes an art.

2 The use of aesthetic vocabulary

Here are some quotations from mathematicians:

> I think it is correct to say that [the mathematician's] criteria of selection, and also those of success, are mainly aesthetical. [von Neumann, 1947]

> A mathematician, like a painter or poet, is a maker of patterns . . . The mathematician's patterns, like the painter's or poets, must be *beautiful* . . . Beauty is the first test: there is no permanent place in the world for ugly mathematics. [Hardy, 1941, pp. 84–5]

> I am interested in mathematics only as a creative art [Hardy, 1941, p. 115].

> The mathematician's best work is art, a high perfect art, as daring as the most secret dreams of imagination, clear and limpid. Mathematical genius and artistic genius touch one another. (Mittag-Leffler, quoted in Rose and De Pillis [1988])

> I like to look at mathematics almost more as an art than as a science; for the activity of the mathematician, constantly creating as he is, guided though not controlled by the external world of senses, bears a resemblance, not fanciful I believe, but real, to the activities of the artist, of a painter, let us say. [Bôcher, 1904, p. 133]

> Why are numbers beautiful? It's like asking why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you.

I know numbers are beautiful. If they aren't beautiful, nothing is.

(Erdös, quoted in Devlin [2000, p. 140])

Such instances are not at all exceptional. Indeed, as Rota [1997, p. 171] observes, 'whereas painters and musicians are likely to be embarrassed by references to the beauty of their work, mathematicians instead like to engage in discussions of the beauty of mathematics'.

A little statistical evidence can be found in the empirical study by Zeki et al. [2014]. Of the 14 mathematicians in the study, all but one reported emotional responses to equations. (They were not asked explicitly whether they thought equations could be beautiful, but the data is suggestive.)

3 Examples

It might be useful to have some examples of (putative) mathematical beauty in front of us. Here are three of my own favourites.

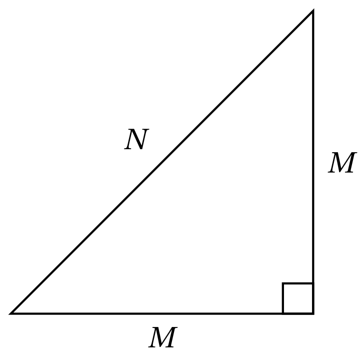
Euler's famous formula

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$$

has appeared high up in some polls, coming 5th out of 24 in the survey Wells [1990] of readers of the *Mathematical Intelligencer* and 9th out of 60 in the study Zeki et al. [2014] of postgraduate and postdoctoral mathematicians.[1]

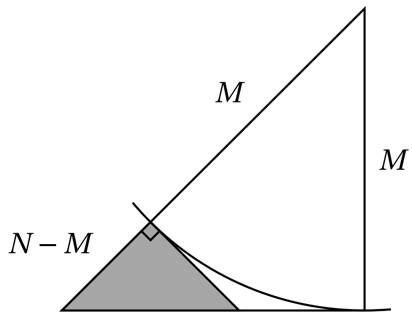
[1]Instructions for each survey were respectively 'give each theorem a score for beauty between 0 and 10, inclusive' and 'rate the beauty of each equation on a scale from -5 (ugly), through zero (neutral) up to +5 (beautiful)'. The formula $e^{i\pi}=-1$ (or a re-arrangement of it) came top in both surveys. Wells reports, though (p. 38), that some awarded it a low score; he speculates that, remarkable though this equation is, those who have a reasonable familiarity with complex analysis may find it too obvious to score it highly.

For a second example, here is a proof that $\sqrt{2}$ is irrational, remarkably not discovered until quite recently [Apostol, 2000]. The usual proof[2] is algebraic; this is a geometric variation. Suppose $\sqrt{2}$ is rational, say it is M/N where the fraction is in its lowest terms. Then there is an isosceles right-angled triangle with integer sides which is the smallest one possible:



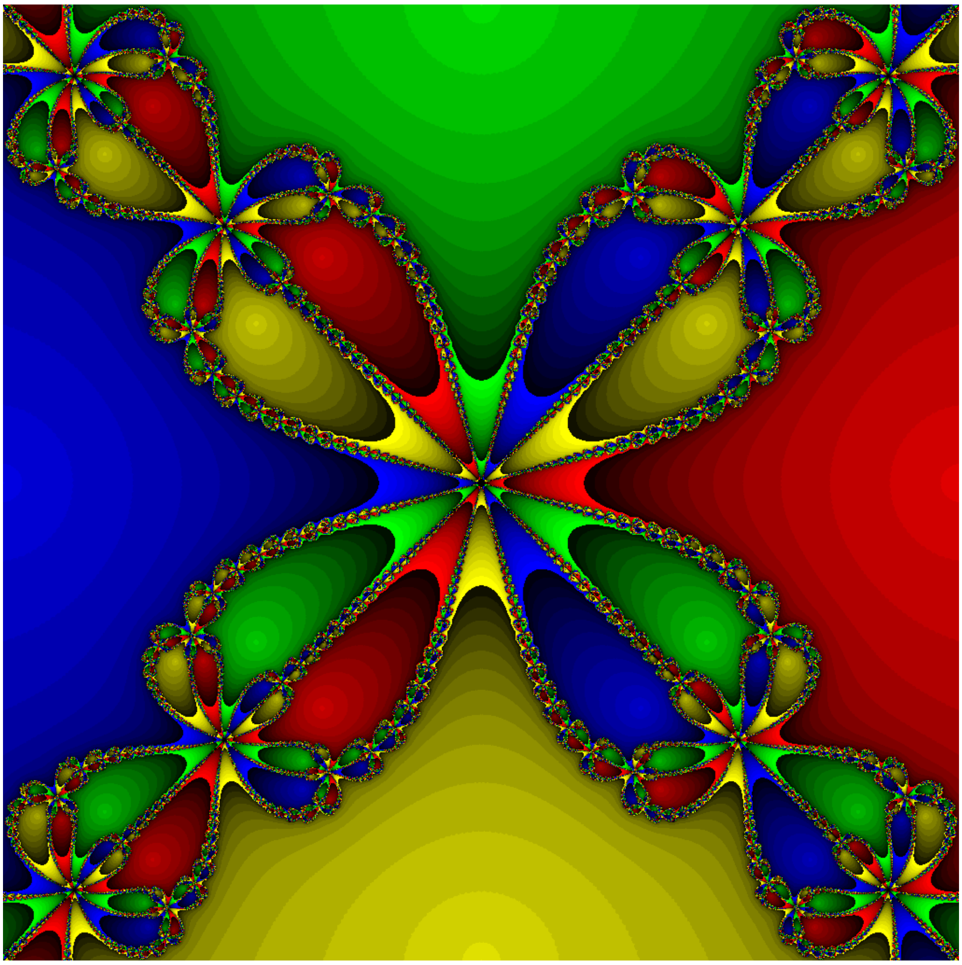
We can show, for a contradiction, that there is another, smaller similar triangle, also with integer sides. Construct the circle with centre and radius as shown; we claim the shaded triangle is isosceles and right-angled with integer sides. It is clearly right-angled; it is isosceles since it shares an angle of 45° with the larger triangle; and its hypotenuse is of integer length since it equals M minus the length of one of the shorter sides, $N - M$ (tangents from a point to a circle are equal).

[2]Which, incidentally, is actually cited as a paradigm of mathematical beauty by Hardy [1941, p. 94].



Here is a third example. The equation $z^4 = 1$ has 4 roots (± 1 and $\pm i$). The Newton-Raphson method is an elementary iterative technique for finding the roots of an equation; given an approximation to a root, it (almost always) returns a better approximation, converging on the root if applied repeatedly.

Suppose we colour each point of the complex plane according to which root it converges to — what do we get? A reasonable guess would be that each point converges to the *nearest* root, and indeed that's what happens if we start near to a root. But the full behaviour is quite extraordinary:



Of course, the mathematical beauty here is distinct from (though perhaps related to) the beauty of the picture.[3]

Some whole areas of mathematics are sometimes cited as particularly beautiful: for example number theory and complex analysis (an area that stands out in my own memory of studying mathematics as an undergraduate). In contrast, the theory of differential equations, which has the appearance of a ragbag of disparate techniques, has been cited as particularly ugly: 'this is botany, not

[3]Multiple coloured pictures of this sort may be found by typing 'Newton fractal' into a search engine.

mathematics', Sawyer [1961, p. 145].[4]

Finding necessary and sufficient conditions for beauty is not something many aestheticians think is possible.[5] However, in the mathematical case, a number of features have come up quite frequently in discussion (for example Wells [1990], Hardy [1941], Rota [1997]).

Many of these features appear in Hutcheson [1726]. Hutcheson considers that the key to beauty is *uniformity amidst variety* (I.II.III). He devotes an entire section (I.III) to 'the Beauty of Theorems', claiming there is no kind of beauty 'in which we shall see such an amazing variety with uniformity' (I.III.I). How widely this idea is applicable to beauty in general may be debatable, but in my view Hutcheson is definitely on to something in the case of mathematics. An interplay between simplicity and complexity is typical of paradigmatic examples of mathematical beauty. In the Newton-Raphson example, a very simple equation generates a very complex pattern. In the theory of numbers, the simplest building blocks exhibit endlessly intricate behaviour. Another example of the same phenomenon is Conway's Game of Life, where there is a striking contrast between the simplicity of the rules of generation and the complexity of the pattens they create.

Another aspect picked out by Hutcheson is *surprise*, though he is careful to note it is not a sufficient condition for beauty. This seems to be playing a part in all three examples I have given; in no case would one, on seeing them for the first time, anticipate what is coming.

[4]Sawyer rightly observes that the comparison is unfair to botany, which of course aims at more than a collection of specimens.

[5]For example, McMahon [2007, p. 40] writes: '...there are no necessary or sufficient conditions for beauty when these conditions are construed as properties an object must have in order to be beautiful.' See also Sircello [1975, p. 44].

Hardy [1941, §§14–18] mentions *seriousness*,[6] which he analyses as combining *generality* and *depth*; the best theorems are not isolated facts, but concern, or are generalizable, to a variety of cases, and have far-reaching consequences. (Hutcheson also draws attention (I.III.V) to cases where 'one theorem contains a great multitude of corollaries deducible from it', and gives an example from Euclid; this seems also related to the 'uniformity amidst variety' idea.) The best arguments are *economical*; for example, a proof which argued by considering many similar cases could not be beautiful.[7]

A final aspect concerns a certain kind of *understanding*. For example, Rota [1997, p. 180] talks about *enlightenment*, contrasting it with cases where one merely follows the steps of a proof without grasping its 'sense'.[8] The geometric proof of the irrationality of $\sqrt{2}$ above is an example of this; it makes it clear, almost obvious, *why* $\sqrt{2}$ is irrational, by making *visible* the method of infinite descent. (These criteria can sometimes pull in different directions; Barker [2009, p. 66] gives the example of the (second) recursion theorem, which has a short, elegant proof which, however, makes it hard to see to why the theorem is true.[9])

4 The objects of mathematical beauty

If there is beauty in mathematics, what exactly is beautiful?

[6]Hardy writes of beauty and seriousness as the two criteria by which mathematics is to be judged, but he is quite explicit (§11) that they are not independent: '...the beauty of a mathematical theorem *depends* a great deal on its seriousness'. He apparently regards seriousness as either a component, or at least a necessary condition, of beauty in mathematics.

[7]An extreme case is the first published proof of the four-colour theorem [Appel et al., 1977a,b] which required the checking of 1936 different cases by computer.

[8]Rota's view (p. 181) is that talk of mathematical beauty is really indirect talk about enlightenment, a concept he (somewhat implausibly) claims mathematicians dislike and avoid discussing directly because it admits of degrees.

[9]There are slick proofs of the incompleteness theorems which have the same property.

Taking the examples above, in the first case we have an *equation* or *theorem*. The beauty does not seem to depend on the exact syntactic formulation (for example, it would not matter greatly if the left hand side were replaced by a verbal description of the sum). This suggests that it is what the equation expresses, rather than the syntactic equation itself, that is really what is beautiful here.

In the second case, although the theorem itself may also be beautiful, it is the *proof* which is the main focus. Again, what seems important is not the exact words and pictures used, but the ideas they express.

The third case is more difficult to pinpoint — perhaps it is a *fact* which is beautiful? What seems to be beautiful is that such a richly complex pattern can be generated by such a simple equation.

In fact most of the cases cited in the literature are either theorems or proofs. The surveys carried out by Wells [1990] and Zeki et al. [2014] concentrate on theorems; Zangwill's objections discussed below focus on proofs, as do the collectors of beautiful proofs 'from the book' in Aigner et al. [2010]. (An entire area, such as Galois theory or complex analysis, is a collection or sequence of theorems and their proofs.)

Interestingly, there are not many claims in the literature that *mathematical objects* are themselves beautiful. Although the Erdös quotation above suggest that numbers are literally beautiful, mathematicians do not usually refer to particular integers, or π, as beautiful. Perhaps Erdös should be interpreted as meaning that the *totality* of numbers, or the number *structure*, is beautiful, but even that would be contrary to the way most mathematicians talk. Some geometrical figures are cited as beautiful, but this is perhaps visual rather than mathematical beauty.

An unusual suggestion in Rota [1997, p. 171] is that a *definition* can be beautiful. He gives the example of the notion of *category*, which facilitates the study of mathematical structure at an extreme level of abstraction. The definition of *compactness* in topology might provide another example. When one first encounters this, one is puzzled as to why such an apparently complex property deserves a label; but doing so makes possible beautifully simple proofs of various theorems.

Rota (pp. 173–4) is careful to distinguish the beauty of a theory from the beauty of any particular exposition of it, and cites Galois theory as an example of a beautiful theory, none of the expositions of which have succeeded in matching its beauty. He is surely right to make the distinction; moreover, of the two, it is again the beauty of the theory itself, of the *content*, that seems by far the more significant. (It is notable that the word 'elegant', rather than 'beautiful', is often used when discussing particular presentations; for example Rota (p. 74) uses it when describing expositions of the Lebesgue integral.)

The paradigmatic cases, then, all seem to be, roughly speaking at least, *propositional* (a proof being naturally thought of as a sequence of propositions, though this may be arguable in the case of a visual proof).

5 Is this genuinely aesthetic?

If the subject of the aesthetics of mathematics is to get off the ground, it had better be that case that the judgments referred to above are genuinely aesthetic. Yet some have denied this. Thus Rom Harré has written

> ... quasi-aesthetic appraisals are not a queer sort of aesthetic appraisal but simply not aesthetic appraisals at all ... the satisfaction that we call peculiarly aesthetic is absent from the mathematical situation. [Harre, 1958, p. 136]

Other writers to express hostility to, or scepticism about, the use of aesthetic vocabulary in this context being literal are Zangwill [2001] and Todd [2008].

The burden of proof, it seems to me, is really on the deniers. The testimony of a large number of mathematicians, who are using this vocabulary without irony, is itself a *prima facie* case in favour of their experiences being genuinely aesthetic. Perhaps it is the case that only a small proportion of the population talk in this way,[10] but this hardly seems relevant. The Zeki et al. [2014] study revealed that the same areas of the brain fire when mathematicians contemplate equations they find beautiful as when they appreciate beautiful pieces of music or art, though this is suggestive rather than conclusive.

There is a sense in which nothing is more convincing than one's own introspection. If I reflect on my own experience in contemplating the examples above, it seems to belong to the same distinctive class as that involved in appreciating art and music. But this is rather unsatisfactory as a means of collectively reaching a conclusion on the matter. What actual arguments can be adduced pro and con?

Harré notes (p. 135) the relative paucity of the vocabulary used about mathematics, compared with other areas — we use 'beautiful' and 'elegant', but not

[10] In contrast to the 14 mathematicians in the Zeki et al. [2014] survey mentioned above, 9 of the 12 *non*-mathematicians questioned denied having an emotional response to beautiful theorems; on the other hand, Hardy [1941, p. 87] cites the popularity of chess, bridge and puzzles of various sorts as evidence that the ability to appreciate mathematics is in fact quite widespread.

'charming', 'delightful', 'lovely' or 'handsome'. (Incidentally I think Harré is wrong about 'lovely'.) Todd [2008, p. 71] also appeals to this feature. I do not think a huge amount can be drawn from this alone, however. The range of aesthetic experiences obtainable from mathematics is no doubt less wide than those to be obtained from painting, music or literature, but this hardly shows that the experiences themselves, or the resulting appraisals, are not aesthetic.

On the pro side, it is a perfectly sensible activity for a mathematician to search for better and better proofs of a result already known to be true. This seems to show that mathematicians aim at more than the pursuit of truth. What else is it? One answer is that they seek proofs that are *explanatory*; that give understanding as to why a theorem holds, with promise perhaps of further developments and applications. But another answer, consistent with the first, that has been given is that the motivation is aesthetic:

> Much research for new proofs of theorems already correctly established is undertaken simply because the existing proofs have no aesthetic appeal.[11] [Kline, 1964, p. 470]

Sensory dependence

A direct challenge to the idea of aesthetics in mathematics comes from the idea that aesthetic qualities are tied up with *perception*. For example Levinson writes

> My proposal is that aesthetic properties ... are *higher order ways of appearing*, dependent in systematic fashion on lower-order ways of

[11] Hutcheson [1726] seems to be making a similar point in the second paragraph of I.III.V.

appearing . . . [Levinson, 2005, p. 218]

Similarly Zangwill [2001] has argued that 'sensory properties are necessary for aesthetic properties', which entails that no abstract objects have any aesthetic properties, and hence (assuming for the moment a platonistic conception) that no mathematical proofs, theorems or objects can be beautiful.

Actual arguments for the sensory dependence thesis seem hard to come by. It is a natural view perhaps, given the historical concentration of aestheticians on the visual arts and, to a lesser extent, music. Indeed the etymology of 'aesthetic' suggests dependence on perceptual properties. For Zangwill the thesis fits into a wider project of aesthetic formalism.

Here, maybe, is an unfortunate by-product of the neglect of mathematics as a topic for aestheticians! Had mathematics had the discussion it deserved, perhaps no-one would have been tempted by this thesis. But the position seems implausible independently of any mathematical considerations. For example, on literature: Zangwill believes that the content of a literary work — that is, 'what the work means, the story it tells, the characters it portrays, the emotions it evokes, the ideas it involves, and so on' (p. 135) — have no aesthetic value. And so, for example, if we read *War and Peace* in English, virtually all[12] of its aesthetic properties are literally 'lost in translation'. This seems to me to be close to a *reductio ad absurdum* of the position.[13]

Zangwill is one of the few aestheticians to have discussed the case of mathematics explicitly. In his view, ascriptions of beauty etc. to mathematics are *metaphorical*. In support of this, he notes that a proof has a *purpose*, and 'our

[12]A few may remain, for example names may be sonically well-chosen for their characters.

[13]Zangwill himself does express some doubts (p. 137) as to the correctness of the sensory dependence theory in the case of literature.

admiration of a good proof... turns solely on its effectiveness in attaining this end, or else its having properties which make attaining the end likely. Could a proof be elegant if it was invalid, or did not possess properties which tend to make proofs valid?... Surely not' (p. 141).

Perhaps, though, Zangwill imagines an opponent replying, this very purpose can give rise to *dependent* beauty, that is, a proof might be 'beautiful in the *way* it fulfilled its function', like the way 'a building is beautiful *as* a thing with a certain function or a painting is beautiful *as* a representation of something' (p. 141).

However, he rejects this on the grounds that in genuine aesthetic cases, beauty and function can 'come apart'. Thus, 'a building might aesthetically express the function of being a library but not actually function well as a library. It might even have no potential to function well as a library (p. 141).' In contrast 'there cannot be proofs... which which are disfunctional yet beautiful or elegant' (p. 142).

Even if we grant him the possibility that the library may have dependent beauty of the sort described without actually functioning well as a library, Zangwill seems to have overlooked that *some* dependent beauty may depend on actual success in fulfilling the function.

His argument goes through, therefore, only if (i) the only possible mathematical beauty is dependent beauty to be cashed out in terms of effectiveness of a proof in fulfilling its function, (ii) genuine cases of dependent beauty arise entirely from *expressing* a function, rather than actually *fulfilling* it, and (iii) in the mathematical case there is no possibility of 'expressing' the function coming apart from fulfilling it. But all of these seem highly contentious.

With regard to (i), while proofs have purposes, it's less clear that theorems or mathematical objects do (although perhaps a theorem has some kind of representational purpose, as I discuss below). In addition, it seems misconceived to set things up in this way: there is surely more to the (purported) beauty of a proof than its simple effectiveness, or else any two correct proofs of the same theorem would be on a par. (ii) seems false; a library could have dependent beauty in virtue of the way it actually functioned as library, and a painting in virtue of accurately depicting its subject. And (iii) is also dubious; a proof might perhaps be strictly invalid but still contain valuable ideas which made it beautiful.[14] Overall, therefore, Zangwill's remarks are unconvincing. (For a more detailed critique of Zangwill's view, see Barker [2009].)

Beauty and truth

The most serious threat to the literal interpretation of the aesthetic vocabulary arises from the observation that mathematicians are ultimately concerned with producing truths; hence, even if they describe themselves as pursuing beauty, it is dubious that they really mean it. Thus Todd [2008] is sceptical that aesthetic appraisals should be taken literally. What appear to be aesthetic judgments are, he suggests, really disguised epistemic ones.

He presents (p. 68) a dilemma for the literalist: either there *is* 'some important connection between epistemic and aesthetic factors in theory assessment' (a *conjunctive* view), in which case 'it is difficult to see what independent role aesthetic factors could play in theory assessment, or indeed what the dif-

[14]Euler's original proof of the $\frac{\pi^2}{6}$ formula, in which he lacked the relevant results on infinite products, might provide an example.

ference between aesthetic and empirical[15] criteria of assessment actually is'; or else 'they are essentially unconnected' (the *disjunctive* view), in which case 'problems also surround the mysterious role that aesthetic factors could play in theory assessment, particularly in respect of the problematic idea that theories could somehow be beautiful but not true'.

Surely *some* version of a conjunctive view is correct; beauty and truth are importantly connected (though the relationship is certainly more complex than identity, as Keats would have us believe). It does seem at least *roughly* right that truth (or validity in the case of a proof) is a *necessary* condition for beauty. The Euler proof mentioned earlier is invalid as it stands, but can be made rigorous by filling in some gaps. There could perhaps be a 'near miss' theorem that was untrue as stated but possessed some beauty — but it would be flawed, like a cracked vase, and the falsity certainly sharply reduces the aesthetic value.

Kivy [1991] suggests that the beauty of theories[16] should be thought of by analogy with representational painting:

> The scientist does not admire a theory's beauty and then admire it or not admire it for its truth. It is admired for how beautifully it is true; for how beautifully it represents nature. (p. 192)

If we take 'nature' in the case of mathematics to be 'mathematical reality', we have here, I think, a promising way to make sense of mathematical beauty. And one that is perfectly consistent with its relation to truth. For it seems the aesthetic value of a representational painting depends on the the success, the

[15]Todd's argument is framed in terms of empirical science, but is intended to apply to mathematics as well.

[16]Kivy's main focus is also on the empirical sciences, but again can be applied also to the mathematical case for most purposes.

truthfulness of the representation (of course, this should not be understood in a crude way as 'the more like a photograph, the better the painting'; but a successful painting says something illuminating and *true* about how its subject appears, or how one experiences seeing it).

Kivy goes too far in his 'conjunctive account'; he says that beauty and truth cannot be 'prised apart' (p. 193), and comes close to endorsing Keats at the end of his paper. But whether or not we can have beauty without truth, we can certainly, in mathematics, have truth without beauty.[17] Todd's charge that Kivy's conjunctive account does not keep the aesthetic sufficiently distinct from the epistemic is just.

It is not true though, as Todd claims (p. 66) that 'science just aims to get it right'. Truth isn't *all* there is. There is a position which avoids both the horns of Todd's dilemma: beauty and truth are neither independent, nor to be identified. Indeed, in the latter, more concessive, part of his paper, Todd countenances the possibility of 'explaining the aesthetic value of proofs and theories in terms of the *way in which* their epistemic content is conveyed' (p. 77), which suggests a position not far from Kivy's, though without the near-identification of the true and the beautiful.

Exactly what is the connection between beauty and truth is a large question! If, for example, seeking beauty is somehow to be a *guide* to finding the truth, it is an urgent matter to explain why.[18] I shall have a little more to say about it in

[17]Paradigmatic examples might be the kind of theorems we find in combinatorics, giving complex formulae for such things as the number of ways of tiling polygons with other polygons; in addition valid but unattractive proofs such as the one of the four-theorem cited earlier. This perhaps also serves as an example of quite a beautiful theorem (9th on Wells' survey) without a beautiful proof; Rota (p. 172) cites the prime number theorem, giving the asymptotic density of the primes, as another such example.

[18]James McAllister has developed, over a series of publications, an elaborate theory which

Section 7. But that mathematical beauty is enmeshed with truth does not seem a good reason to think that it is not really beauty at all.

6 Aesthetics and the metaphysics of mathematics

I have suggested above a way in which thinking about mathematics might have consequence for aesthetics, in telling against the sensory dependence thesis. An interesting questions is whether we might have interaction in the other direction: might aesthetic considerations have implications for more mainstream philosophy of mathematics?

If propositions are the locus of beauty, then this suggests that is no *easy* route from aesthetical considerations to conclusions about the ontology of mathematics. It seems Euler's $\frac{\pi^2}{6}$ theorem can have beauty whether one platonistically regards it as being about an externally existing realm of mind-independent mathematical objects or, alternatively, about a world of fictional objects created by human activity.

The Croce-Collingwood theory, according to which artworks are mental, has interesting parallels with the intuitionism of Brouwer, according to which mathematics consists of 'mental constructions'. And the theories run, I think, into parallel difficulties, that in each case what goes on in one's head seems to be answerable to something external, in a way that the theories struggle to account for; there are wrong ways to interpret a picture, and there are false mathematical propositions.

connect beauty and truth in science and mathematics, via what he calls the 'aesthetic induction' (see for example [1996] and [2005]). I do not have space to discuss McAllister's work here, but it is addressing exactly the questions I think need exploration.

Breitenbach [2015] expands some brief remarks of Kant into a worked out account of the beauty of mathematical proofs within a Kantian framework. According to Breitenbach's account, our experience of mathematical beauty is grounded in 'our felt awareness of the imaginative processes that lead to mathematical knowledge' (p. 2).

I cannot here discuss Breitenbach's intricate account in the detail it deserves. By focussing on mathematical demonstration as a human activity, Breitenbach is able to go some way towards accounting for the roles of *surprisingness* and *understanding* in mathematical beauty. An account focussing only on objects and their properties will surely struggle to do this.

On the other hand, the Kantian framework explicitly allows only *proofs*[19] to have beauty, and not theorems.[20] This is because, for Kant, a cognitive judgment, such as is involved in contemplating a theorem, differs essentially from an aesthetic one (in the first, but not the second, a 'synthesis of the sensory manifold' is 'subsumed under concepts' — see p. 6). This seems a serious weakness of the Kantian account, since the position that proofs but not theorems can be beautiful does not accord well with the experience and testimony of mathematicians. (It is not simply that we need to seek a different account for the beauty of theorems — such an account is *ruled out* in the Kantian system.) It would be an interesting project to see to what extent Breitenbach's insights could be preserved in a way that was less embedded in the Kantian approach.

[19]It is informal proofs that are intended here, and in particular in geometry. The question as to how far the account is extendible to other areas of mathematics is raised by Breitenbach herself (pp. 15–6).

[20]Breitenbach describes the contrast as existing between proofs, on the one hand, and mathematical objects and their properties, on the other, but that does not seem quite right. It is presumably not the (universal) properties such as primeness, straightness etc. which might be beautiful, but rather particular objects' having them, that is, something propositional.

What of someone who wanted to defend the beauty of mathematical theorems and proofs, but rejected propositions? One way out might perhaps be to argue that one can have an aesthetic experience *without an object*, analogously to adverbialist theories of perception. Perhaps it can simply be beautiful *that something is the case* without there being any object which is beautiful. I do not know whether a plausible theory could be developed along these lines, but it would be interesting if so and might have applications in other areas of aesthetics.[21]

7 Mathematics as art

Let us take stock. In arguing for the literalness of aesthetic appraisals in mathematics I made use of the analogy with representational painting, pointed out by Kivy, and the kinship with literature since the bearers of beauty are, in most cases, propositional. But one can now ask a further question: is mathematics, like painting and literature, an art? This question, of course, is separate from the question of whether mathematics has aesthetic properties. It could be that mathematical beauty exists but is natural beauty, like the beauty of a landscape or a flower.

Though several mathematicians, including some quoted above, have talked about mathematics as an art, as far as I know, no-one has explicitly defended this thesis philosophically. It certainly seems implausible that *all* mathematics should be art; in particular, a lot of applied mathematics will not be. But in this

[21] A referee suggests another route for the rejecter of propositions: to find beauty in the linguistic expressions. But as argued above (Section 4) mathematical beauty seems primarily located in the *content* of theorems and proofs, rather than the particular way that content is expressed.

mathematics is like several other activities (not all writing or drawing is art, for example).

How might one argue for the thesis that some mathematics, of the pure sort which its practitioners say is pursued for aesthetic reasons, is an art? In contrast to the case of beauty, a considerable amount of philosophical work has gone in to attempts to define art, without any great agreement (in this, of course, art is hardly exceptional). It would be an interesting task to assess whether mathematics counts as art according to some of the main theories that have been put forward, but that would hardly give us a conclusive answer, and is not something I shall attempt here.

In any case, perhaps there are no necessary and sufficient conditions for art. Wittgenstein's 'family resemblance' idea may be helpful here: I am cautiously inclined to think that the parallels, noted above, between mathematics and both representational painting and literature, combined with the genuinely aesthetic elements in mathematics for which I have already argued, suggest that mathematics is sometimes an art.

The main consideration on the other side seems once again to be that truth plays too great a role in mathematics. But as in the discussion of beauty and truth above, the case of representational painting suggests this is hardly decisive. Not only is the representational accuracy of a painting no obstacle to its being art, it is (understood, as previously mentioned, in a non-simplistic way) essential to the aesthetic value of a painting.

The case of literature is more complicated. With prose at least, paradigmatic examples of literature-as-art tend to be fictional. (On some non-platonist views, mathematics itself is a kind of fiction and the objection loses its bite; for an explicit defence of a such a view, see Bueno [2009].) We are more likely to regard a novel as art than a work of biography, history or travel-writing.

There are several points to make here. Poetry always seems to be art, whether or not its subject matter is fictional. A landscape painting does not count as art only if the landscape depicted is fictional. Sometimes we do regard works of history or biography as art; and here, as in the case of representational painting, not only is the constraint to be truthful no obstacle to their being art, but its violation would be a serious flaw. Finally, even fiction is related to truth in an indirect way; the faithfulness of a novel to human nature in particular, and also to such things as the era in which it is set, is of direct relevance its aesthetic value.

Mathematics, then, is one of a family of activities which tell us how things are, in a way that is aesthetically valuable. It seems no travesty to call such a practice 'art'.

Hardy, who sees no contradiction between his platonism [1941, pp. 123–4] and explicitly stating that mathematics is an art (p. 115), raises an interesting issue which points to a difference between mathematics and the other arts I have been discussing. He quotes (p. 84) with approval Houseman's comment that 'poetry is not the thing said but a way of saying it', and of the lines from Richard II 'Not all the water in the rough rude sea/Can wash the balm from an anointed King' comments 'Could lines be better, and could ideas be at once more trite and more false?'. Mathematics works *only* with ideas, thinks Hardy, and is hence 'more permanent'.

Hardy does bring to light an important contrast here. In mathematics, the main aesthetic value lies with the thing represented, not the representation.

(Sometimes a particular presentation or formulation may be notable for its elegance, but it is mainly the theorems or proofs themselves that are seen to have the aesthetic value.) This seems to mark a distinction between mathematics and literature, and also representational art, where we talk of the beauty of the painting, not its subject.

This disanalogy might be seen to threaten the status of mathematics as art. But even if correct, it does not seem a conclusive reason why mathematics cannot be art. There can be art in selecting which pieces of (mathematical) reality to display, as du Sautoy discusses in a recent popular piece in which he is comparing mathematics and music:

> Most people's impression is that a mathematician's job is to establish proofs of all true statements about numbers and geometry . . . What is not appreciated is that mathematicians are actually engaged in making choices about what is being elevated to the mathematics that deserves performance in the seminar room or conference hall. The proof of Fermat's Last Theorem is considered one of the great mathematical opuses of the last century, while an equally complicated calculation is regarded as mundane and uninteresting. It is the narrative journey that the first proof takes you on that makes this proof worth telling. [du Sautoy, 2015, p. 50]

In any case, the contrast does not seem sharp. Arguably the most valued paintings have beautiful subjects, as well as being themselves beautiful representations; part of the what the artist is commended for is having successfully conveyed a beautiful part of reality. And it seems to me Hardy is wrong about the lines of Shakespeare; the lines *would* be finer, were the thought expressed by them true and insightful.[22] I argued above that part of the aesthetic value of literature is genuinely propositional, survives translation, and *contra* Houseman, is the 'thing said', rather than 'the way of saying it'. So the disanalogy with mathematics is less than Hardy suggests.[23]

8 Conclusion

In the course of this survey, I have argued firstly that aesthetic appraisals of mathematics should be taken literally. That the practioners of mathematics use aesthetic vocabulary apparently intending it to be understood non-metaphorically suggests the burden of proof is on those who deny the genuineness of the aesthetic appraisals. But neither an argument from sensory dependence, nor one maintaining that mathematics is too concerned with the pursuit of truth to be an aesthetic activity, seem convincing.

Moreover, mathematics seems to have enough in common with paradigmatic arts such as painting and literature that there is a case for counting at least some mathematics as itself an art.

I have only sketched how one might argue in more detail for these claims, but if I am correct, then mathematics is an area of human activity which deserves a lot more attention from aestheticians than it has so far had. And even if the claims are false, articulating exactly why promises to be illuminating in

[22]Indeed, Hardy himself comes close to recanting a few pages later (at the end of §11).

[23]The form of literature closest in analogy to a mathematical theorem is perhaps the *Wildean epigram*: for example, 'A man cannot be too careful in the choice of his enemies', from *The Picture of Dorian Gray*. We might note the element of surprise; the truth of the thought expressed; and that it survives translation.

clarifying our concepts of art and the aesthetic. Either way, there are plenty of interesting avenues for further exploration in the area.

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