RANK ONE BRIDGELAND STABLE MODULI SPACES ON A PRINCIPALLY POLARIZED ABELIAN SURFACE

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Abstract. We compute moduli spaces of Bridgeland stable objects on an irreducible principally polarized complex abelian surface $(\mathbb{T},\ell)$ corresponding to twisted ideal sheaves. We use Fourier-Mukai techniques to extend the ideas of Arcara and Bertram to express wall-crossings as Mukai flops and show that the moduli spaces are projective.

Introduction

Let $(\mathbb{T},\ell)$ be a principally polarized abelian surface over $\mathbb{C}$. We shall assume that $\operatorname{Pic}\mathbb{T}=\langle\ell\rangle$. We shall also fix a line bundle L with $c_1(L)=\ell$. Then the linear system $|\ell|$ consists of a unique smooth divisor D given as the zero set of the unique (up to scale) section of L. We can translate D to give a family of divisors which we shall denote by $D_x=\tau_xD\in|\tau^*_{-x}\ell|$. As observed in [Mac11], we can view these D_x as analogues of lines on the projective plane. They have the property that any two intersect in exactly 2 points (up to multiplicity) and any two points (or fat point) lies on exactly two of them. Given a 0-scheme $X\subset\mathbb{T}$ we will say that X is *collinear* if there is some x such that $X\subset D_x$.

Now consider objects of the (bounded) derived category $D(\mathbb{T})$ whose Chern characters are $(1,2\ell,4-n)$, for an integer $n\geqslant 0$. A torsion-free sheaf with such a Chern character takes the form $L^2\otimes\mathcal{P}_{\hat{x}}\otimes\mathcal{I}_X$, where $\mathcal{P}_{\hat{x}}$ is the flat line bundle corresponding to some point $\hat{x}$ of the dual torus $\hat{\mathbb{T}}$ and X is a 0-scheme of length n. We shall drop the tensor product signs in what follows. The Gieseker moduli space of such objects is a fine moduli space given by $\operatorname{Hilb}^n\mathbb{T}\times\hat{\mathbb{T}}$. We can view this asymptotically as a Bridgeland stable moduli space (see [Bri08]) in a certain abelian subcategory $\mathcal{A}_0$ (defined below). There is a 1-parameter family of stability conditions indexed by a positive real number t in $\mathcal{A}_0$. For some large t_0, if $t>t_0$ the moduli functor

$$\mathcal{M}_t^{(1,2\ell,4-n)}:\Sigma\mapsto\left\{a\in D^b(\mathbb{T}\times\Sigma):\begin{array}{c}\operatorname{ch}(a_\sigma)=(1,2\ell,4-n),\\ a_\sigma\text{ is }t\text{-stable, for all }\sigma\in\Sigma,\end{array}\right\}/\sim,\tag{0.1}$$

where $i_\sigma:\mathbb{T}\to\mathbb{T}\times\Sigma$ is the inclusion corresponding to $\sigma\in\Sigma$, $a_\sigma=Li^*a_\sigma$ and $\sim$ is the usual equivalence relation $a\sim a\otimes\pi_\Sigma^*M$, for any line bundle M on Σ, is represented by $\operatorname{Hilb}^n\mathbb{T}\times\hat{\mathbb{T}}$. We shall omit the superscript on $\mathcal{M}$ if the context is clear. As t decreases, we expect to cross walls as some of the objects become unstable. The object of this paper

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is to describe the resulting moduli spaces M_t for all $t > 0$ and all $n \geqslant 0$. In fact for $n < 3$, the torsion-free sheaves are t-stable for all $t > 0$. When $n > 3$ there is more than one moduli space and each moduli space is modified by a Mukai flop as we cross the wall. This is very similar to what is found in Arcara and Bertram [ABL07] where the case $\mathrm{ch}(a) = (0, H, H^2/2)$ is computed in the abelian category $\mathcal{A}_{H/2}$ (there H is some polarization). In that case, the situation is made complicated by the presence of "higher rank walls" and a complete picture is not given. In our case, there is only one higher rank wall (when $n = 5$) and we can give an explicit description of that case. Of course our results are much less general than those of [ABL07]. We pay this price in order to have a useful computational tool at our disposal which allows us to be more explicit in our constructions. However, Arcara and Bertram do prove that the resulting moduli spaces really do exist as smooth proper schemes representing 0.1. We discuss this in section 4.

That tool is the Fourier-Mukai transform. We choose to use the original such transform defined by Mukai in [Muk81] (see [Huy06] and [BBHR09] for an exposition of the theory), but shifted by [1] in $D(\hat{\mathbb{T}})$. We shall denote this by $\Phi : D(\mathbb{T}) \to D(\hat{\mathbb{T}})$. As is now well known, this is an equivalence of categories. It was used extensively in [Mac11] to understand how divisors in the linear system $|2\ell|$ intersect and we shall use several of those computations below. Pulling back the transform to include a parameter space Σ allows us to observe that Φ preserves moduli in the sense that if M together with a universal object $\mathbb{E}$ represents a moduli functor $\mathcal{M}$ on $\mathbb{T}$ then $\Phi(M)$ together with $\Phi_\Sigma(\mathbb{E})$ represents the pullback functor $\Phi_\Sigma^*(\mathcal{M}_t)$. But we can improve this using an observation of Huybrechts ([Huy08]). He showed that for any given Fourier-Mukai transform there is a choice of $\mathbb{R}$-polarizations β on $\mathbb{T}$ and β' and $\hat{\mathbb{T}}$ such that $\Phi : \mathcal{A}_\beta \xrightarrow{\sim} \mathcal{A}_{\beta'}$ and moreover $\Phi_\Sigma^*(\mathcal{M}_t) = \hat{\mathcal{M}}_{t'}$ for some t' depending on t, β and β' and where $\hat{\mathcal{M}}$ is the same functor as $\mathcal{M}$ but for $\hat{\mathbb{T}}$ and with $\Phi(\mathrm{ch}(a_\sigma))$ instead of $\mathrm{ch}(a_\sigma)$. In our case, $\beta = 0 = \beta'$ and $t' = 1/t$. We also have the formula that $\Phi(r, c\ell, \chi) = (-\chi, c\ell, -r)$. We can see from this why $n = 5$ is special for us as that is precisely the case when $\mathrm{ch}(a)$ is preserved by Φ. Immediately we can conclude that $\mathcal{M}_t$ is represented by $\Phi(M_{1/t})$ for all $t < 1/t_0$ (we shall see that $t_0 = \sqrt{3}$). From [Mac11] we know that some $L^2\mathcal{P}_{\hat{x}}\mathcal{I}_W$, where $|W| = 5$ are not WIT and so there are elements of M_t for small t which are not sheaves. However, it turns out that this is the only time that non-sheaves can arise.

We shall see that for any n there are $d = \lfloor \frac{n-1}{2} \rfloor$ walls except when $n = 5$ when there is an additional (so called, higher rank) wall. So there are $\lfloor \frac{n-1}{2} \rfloor + 1$ moduli spaces $M_0, \ldots$, M_d where M_0 corresponds to $t \gg 0$. Now M_0 is well known to be given by Gieseker stable sheaves (in this case, actually μ-stable) and so the usual GIT construction shows that it is projective. On the other hand, we shall see that $\Phi(M_d)$ are represented by sheaves as well (so long as $n > 3$) and hence, M_d is also projective. To show that the other spaces M_i are projective we observe that we can vary β and in a suitable range each moduli space corresponds to a moduli space of Bridgeland stable objects for t arbitrarily small. Then we can apply a suitable Fourier-Mukai transform to show that M_i is isomorphic to a Bridgeland stable moduli space of sheaves but now with t large which are again known to be projective. The difficult step here is to show that the transforms of the points of M_i are pure sheaves.

Finally, we look at the $n = 3$, $n = 4$ and $n = 5$ cases in more detail. In many ways, the $n = 3$ case is the most interesting. There is a single wall in that case and we show explicitly that the two moduli spaces are isomorphic. Crossing the wall corresponds to a birational transformation which replaces a $\mathbb{P}^1$-fibred codimension 1 subspace with its dual fibration. We will see explicitly that the resulting birational map between the two moduli spaces does not extend to an isomorphism (even though the spaces are actually isomorphic). It also turns out that for nearby $\mathcal{A}_s$ with $s > 0$ there is another wall and this time it is a codimension 0 wall.

A more general study of the relation between wall crossing and Fourier-Mukai transforms is given in [MYY11].

Notation

P, Q, Y, Z, W	0-schemes of lengths 1, 2, 3, 4 and 5, respectively
$\mathcal{I}_X$	ideal sheaf of general 0-scheme X
L	fixed choice of polarizing line bundle with $c_1(L) = \ell$
$D(\mathbb{T})$	bounded derived category of coherent sheaves on $\mathbb{T}$
$(r, c\ell, \chi)$	typical Chern character of an object of $D(\mathbb{T})$
$\mathbb{T} \cong \hat{\mathbb{T}}$	canonical identification via our choice of L.
a, b, d, e, ...	arbitrary objects of $D(\mathbb{T})$
A^i, B^i, D^i, E^i, ...	cohomology of a, b, d, e,...

This last piece of notation is to avoid clutter with $H^i(-)$.

1. Stability Conditions on Abelian Surfaces

Following Bridgeland [Bri08], we consider a special collection of stability conditions on our abelian surface $(\mathbb{T}, \ell)$. These arise as tilts of $\mathrm{Coh}_{\mathbb{T}}$ and are parametrized by a complex Kähler class $\beta + i\omega$. We will take $\omega = t\ell$ and $\beta = s\ell$. Then we define a torsion theory by:

$$\begin{aligned} F_s &= \{E \in \mathrm{Coh}_{\mathbb{T}} : E \text{ is TF}, \ \mu_+(E) \leqslant 2s\} \\ T_s &= \{E \in \mathrm{Coh}_{\mathbb{T}} : E \text{ is torsion or } \mu_-(E/\operatorname{tors}(E)) > 2s\} \end{aligned}$$

We let the associated tilted abelian subcategories be denoted by $\mathcal{A}_s$. Explicitly,

$$\mathcal{A}_s = \{a \in D(\mathbb{T}) : A^i = 0, \ i \neq -1, 0, \ A^{-1} \in F_s, \ A^0 \in T_s\}$$

(recalling our notational convention that $A^i = H^i(a)$). This carries a 1-parameter family of stability conditions whose charge is

$$\begin{aligned} Z_{s,t}(a) &= \langle e^{(s+it)\ell}, \mathrm{ch}(a) \rangle \\ &= -\chi + 2sc - r(s^2 - t^2) + 2it(c - rs), \end{aligned}$$

where $\mathrm{ch}(a) = (r, c\ell, \chi)$. Recall for an abelian surface that the top part of the Chern character of a is equal to the Euler character $\chi(a)$. For a quick proof that this defines a stability condition see [ABL07, Cor 2.1]. Then $Z_{s,t}$ provides us with a Bridgeland stability condition on $\mathcal{A}_s$. We can then declare an object $a \in \mathcal{A}_s$ to be (Bridgeland) *t-stable* provided for each proper subobject $b \to a$ in $\mathcal{A}_s$, we have $\mu_t(b) < \mu_t(a)$, where the *t-slope* $\mu_t(a)$ is given by

$$-\frac{\Re Z_{s,t}(\mathrm{ch}(a))}{\Im Z_{s,t}(\mathrm{ch}(a))} = \frac{\chi - 2sc + r(s^2 - t^2)}{2t(c - rs)}.$$

We view this as taking values in $\mathbb{R} \cup \{\infty\}$, taking an infinite value precisely when the denominator vanishes. As an example of how this works we prove the following easy generalization of [ABL07, Lemma 3.2]

Lemma 1.1. *If E is a μ-stable torsion-free sheaf which is not locally-free and $\mu(E) \leqslant 2s$ then $E[1] \in \mathcal{A}_s$ is not t-stable for any $t > 0$.*

Proof. Observe that we have $\mu_+(E) = \mu(E) < 2s$ and so $E \in F_s$. Hence, $E[1] \in \mathcal{A}_s$. But if X is the 0-scheme of the singularity set of E then we have a sheaf short exact sequence

$$0 \to E \to E^{**} \to \mathcal{O}_X \to 0.$$

Note that E^{**} is still μ-stable and of the same slope as E and so $E^{**}[1] \in \mathcal{A}_s$. Then

$$0 \to \mathcal{O}_X \to E[1] \to E^{**}[1] \to 0$$

is short exact in $\mathcal{A}_s$. But $\mu_t(\mathcal{O}_X) = \infty$ and so cannot be less than $\mu_t(E)$ for any t. $\square$

We also prove

Lemma 1.2. *The objects a of $\mathcal{A}_s$ with infinite t-slope are given by the short exact sequence (in $\mathcal{A}_s$)*

$$0 \to E[1] \to a \to \mathcal{O}_X \to 0$$

where X is a 0-scheme (possibly empty) and E is a μ-semistable torsion-free sheaf of slope $2s$ or is the zero sheaf.

Proof. Suppose first that $r(A^0) > 0$. Then let $\mu(A^{-1}) = c/r$ and $\mu(A^0) = c'/r'$. For $\mu_t(a) = \infty$ we require $\mu(a) = 2s$. But $\mu(a) = \frac{c'-c}{r'-r} > \frac{c'}{s}$ because $c'/r' > c/r$ (this is a characterising property of slope functions: if $x \to y \to z$ is a short exact sequence then $\mu(y) > \mu(z)$ implies $\mu(x) > \mu(y)$). But $c'/r' > 2s$ as $A^0 \in T_s$. This contradiction implies that $r' = 0$. But then $2s = (c' - c)/(-r) = \mu(A^{-1}) - c'/r \leqslant 2s$ with equality only if $\mu(A^{-1}) = 2s$ and $c' = 0$ as required. $\square$

Finally in this section we make the following useful observations (left as exercises for the reader).

Proposition 1.3.

(1) *(Schur's lemma) If $a \in \mathcal{A}_s$ is t-stable for some $t > 0$ then* $\mathrm{Hom}(a, a)$ *consists of automorphisms.*
(2) *If $E \in \mathcal{A}_s \cap \mathrm{Coh}_{\mathbb{T}}$ and there is some t_0 such that for all $t > t_0$, E is t-stable then E must be torsion-free. (In fact, any torsion subsheaf must eventually t-destabilise it).*

(3) *More generally, if* $E \in \mathcal{A}_s \cap \operatorname{Coh}_{\mathbb{T}}$ *then there is some* t_0 *such that for all* $t > t_0$, E *is* t*-stable if and only if* E *is (twisted) Gieseker stable.*
(4) *If* $E[1] \in \mathcal{A}_0 \cap \operatorname{Coh}_{\mathbb{T}}[1]$ *has* $c_1(E) = 0$ *then* E *is* μ*-semistable.*
(5) *If* $E \in \mathcal{A}_0 \cap \operatorname{Coh}_{\mathbb{T}}$ *satisfies* $c_1(E) = \ell$ *then* $E/\operatorname{tors}(E)$ *is* μ*-semistable (or zero).*

For the rest of this paper we will be interested purely in the case $0 \leqslant s < 1$. These have slopes
$$\frac{4-n-4s+s^2-t^2}{2t(2-s)}.$$

Our starting point is the following well known theorem (see for example [HL10]) translated into our context:

Theorem 1.4. *There is some real number* $t_0 > 0$ *such that for all* $t > 0$, $\mathcal{M}_t^{(1,2\ell,4-n)}$ *is represented by the projective space* $\operatorname{Hilb}^n \mathbb{T} \times \hat{\mathbb{T}}$. *A universal sheaf* $\mathbb{E}_t$ *is given by* $\pi_1^* L^2 \otimes \pi_{13}^* \mathcal{P} \otimes \pi_{12}^* \mathbb{I}_{\mathcal{Z}}$, *where* $\mathcal{P}$ *is the Poincaré bundle over* $\mathbb{T} \times \hat{\mathbb{T}}$, π_i *and* π_{ij} *is the projections from* $\mathbb{T} \times \operatorname{Hilb}^n \mathbb{T} \times \hat{\mathbb{T}}$ *to the* i^{th} *and* ij^{th} *factors respectively, and* $\mathcal{I}_{\mathcal{Z}}$ *is the ideal sheaf of the tautological universal subscheme* $\mathcal{Z} \subset \mathbb{T} \times \operatorname{Hilb}^n \mathbb{T}$.

Using Proposition 1.3(3) again and the observation in the introduction about the Fourier-Mukai transform preserving moduli, we also have non-empty fine projective moduli spaces $M_t^{(n-4,2\ell,-1)}$ for $n \geqslant 4$ and so we also see that $\mathcal{M}_t^{(1,2\ell,4-n)}$ is represented by this space for all t less than some t_1.

The situation for $n < 3$ is cleared up by the following proposition.

Proposition 1.5. *(see* [AB09]*) The following holds in* $\mathcal{A}_s$ *for all* $t > 0$ *and all* $0 \leqslant s < 1$.
(1) *For all integers* $m > 0$ L^m *is* t*-stable.*
(2) *For all integers* $m \leqslant 0$, $L^m[1]$ *is* t*-stable.*
(3) *If* $E \in \mathcal{A}_s \cap \operatorname{Coh}_{\mathbb{T}}$ *has* $c_1(E) = \ell$ *and* $r(E) = 1$ *and is torsion-free then* E *is* t*-stable.*
(4) *For all 0-schemes* $X \subset \mathbb{T}$ *with* $|X| < 3$, $L^2 I_X$ *is* t*-stable for* $s = 0$.
(5) *If* $E \in \mathcal{A}_0$ *is a pure torsion sheaf with* $c_1(E) = \ell$ *then* E *is* t*-stable.*

Proof. Case (1) is treated in [AB09, Proposition 3.6(b)] but we can give a more direct proof by observing that a destabilizing object must be a sheaf $K \to L^m$. Now assume that K is μ-semistable and $K \to L^m$ is non-zero. If its Chern character is $(r, c\ell, \chi)$ then we can re-arrange $\mu_t(K) - \mu_t(L^m) \geqslant 0$ to
$$t^2 \leqslant \frac{(m-s)(\chi - c(s+m) + mrs)}{rm - c}$$
But $m - s > 0$ and $rm - c > 0$ (to ensure $\operatorname{Hom}(K, L^m) \neq 0$). Now $r(\chi - c(s+m) + mrs) = (r\chi - c^2) + (c - rs)(c - rm)$. The second term is negative as $K \in T_s$. The first term is non-positive by Bogomolov. So every factor K' of the μ-Harder-Narasimhan filtration of K has $\mu_t(K') < \mu_t(L^m)$ and so K cannot destabilize L^m. (2) is similar and we leave as an exercise for the reader.

For (3) observe that if $k \to E$ is supposed to destabilize E then the image of $K^0 \to E$ must have slope equal to E and so Q^0 is supported on points, where $q = E/k$ in $\mathcal{A}_s$. But

$K^{-1} = 0$ and we are left with a long exact sequence (in $\mathrm{Coh}_{\mathbb{T}}$)

$$0 \to Q^{-1} \to K \to E \to Q^0 \to 0.$$

Now assume that $s = 0$. Then $\deg(Q^{-1}) = \deg(K) - 2 < 0$ since if it equalled to 0, q would have infinite slope if $s = 0$ and could not destabilize E. But this implies $\deg(K) = 0$ and this can only happen if K is supported in dimension 0, which is impossible as E is torsion-free. This also applies if E is pure rank 0 as well and so we have (5) as well.

Returning to (3) with $0 < s < 1$, we have just shown that there are no walls intersecting the line $s = 0$. Each wall is a semicircle with centre on the s-axis. Let $\mathrm{ch}(K) = (r, c\ell, \chi)$. Then the destabilizing condition is

$$\chi(1-s) + s^2(c-r) - \chi(E)(c-rs) > 0. \tag{1.1}$$

But for $Q^{-1} \in F_s$ we must have $c - 1 \leqslant s(r-1) < r - 1$. The centre of the semicircular wall has

$$s = -\frac{1}{2}\frac{\chi - r\chi(E)}{r-c}$$

Since there are no walls at $s = 0$ we have $\chi < r\chi(E)$. Then the destabilizing condition 1.1 becomes

$$0 < -\chi(E)(c - rs - r(1-s)) + s^2(c-r) = (-\chi(E) + s^2)(c-r).$$

This is a contradiction unless $\chi(E) = 1$. But this is dealt with in (1).

For (4) we proceed as follows (this will be typical of such proofs). We suppose $L^2\mathfrak{I}_X$ is not t-stable. Then there must exist destabilising subobjects $k \to L^2\mathfrak{I}_X$. Let the quotient (in $\mathcal{A}_0$) be q as above. Again $K^{-1} = 0$. Now $K = K^0$ must be torsion-free (because $Q^{-1} \in F_0$) and so has positive degree. Let the Chern character of K be $(r, c\ell, \chi)$. Then the fact that it destabilizes gives us the inequality

$$2\chi + (n-4)c \geqslant (2r - c)t^2.$$

But $\deg(Q^{-1}) \leqslant s < 1$ and so $\deg(K/Q^{-1}) \geqslant 2c$. But $r(K/Q^{-1}) = 1$ and so $\deg(K/Q^{-1}) = 2$ or 4. In the latter case, if $c = 2$ then $c_1(q) = 0$ but then q cannot destabilize after all. If $c = 1$ then K must be μ-semistable by Prop 1.3(5) and so $\chi \leqslant 1$ by the Bogomolov inequality. But $2\chi \geqslant 4 - n > 1$ for $n < 3$. So $\chi = 1$ and $n = 2$. But this only destabilizes if $t = 0$ which is impossible. This contradiction shows that no such K can exist. ☐

Note that we only used $n < 3$ at the very end so we see more generally that the only possible destabilising subobject must be a μ-semistable sheaf of degree 2. Moreover $\chi \geqslant 4 - n$.

2. Identifying the Candidate Stable Objects

Now we look for which objects may be representatives of points of our moduli spaces. In other words, we find objects a with Chern character $(1, 2\ell, 4-n)$ which are t-stable for some $t > 0$. In this section we start by assuming assuming $s = 0$.

Proposition 2.1. *Suppose $e \in \mathcal{A}_0$ with $\mathrm{ch}(e) = (1, 2\ell, 4-n)$ is t-stable for some $t > 0$. Then, either*

(1) *e is a torsion-free sheaf E, i.e. $E = L^2\mathcal{I}_X\mathcal{P}_{\hat{x}}$ for some $X \in \operatorname{Hilb}^n \mathbb{T}$ and $\hat{x} \in \hat{\mathbb{T}}$, or*
(2) *e is a sheaf E with torsion, in which case, $\operatorname{tors}(E)$ is a line bundle supported on some D_x of degree $4 - n + m$ and $E/\operatorname{tors}(E) \cong L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}}$ for some $\hat{x} \in \hat{\mathbb{T}}$ and $X' \in \operatorname{Hilb}^m \mathbb{T}$, where $0 \leqslant m < (n-2)/2$, or*
(3) *e is a two-step complex with $E^{-1} \cong L^{-1}\mathcal{P}_{\hat{x}}$ for some $\hat{x} \in \hat{\mathbb{T}}$ and E^0 a μ-stable locally-free sheaf with $\operatorname{ch}(E^0) = (2, \ell, 0)$ only when $n = 5$.*

Proof. We have already seen that if e is a torsion-free sheaf then it is t-stable for large enough t. So we assume that e is not a torsion-free sheaf.

Now suppose e is a sheaf E with torsion subsheaf $\operatorname{tors}(E)$. Since sheaves supported on 0-schemes have infinite slope any such subsheaf S of $\operatorname{tors}(E)$ would destabilize E for all t as $E/S \in T_0$. Observe also that E is not a torsion sheaf and so $\deg(\operatorname{tors}(E)) = 2$. Hence, $\operatorname{tors}(E)$ is supported on a translate of D and locally-free on its support. Suppose it has degree d (so $\chi(\operatorname{tors}(E)) = d - 1$). Let $F = E/\operatorname{tors}(E)$. Then $\operatorname{ch}(F) = (1, \ell, 5 - n - d)$ and F is torsion-free. So $F \cong L\mathcal{P}_{\hat{x}}\mathcal{I}_{X'}$, where $|X|' = n + d - 4 = m$. Then $d = 4 - n + m$. But $\mu_t(\operatorname{tors}(E)) = (d-1)/2t$ and this will always destabilize E if $m \geqslant (n-2)/2$. So we require $m < (n-2)/2$. Note that such E cannot be t-stable for $t \geqslant \sqrt{n - 2 + 2m}$ as they are destabilized by their own torsion.

Now suppose that e is not a sheaf. Let $\operatorname{ch}(E^{-1}) = (r, c\ell, \chi)$ with $r \geqslant 1$. Then $\operatorname{ch}(E^0) = (r+1, (2+c)\ell, 4 - n + \chi)$. then $c < 0$ (because if $c = 0$, $E^{-1}[1]$ would destabilize e for all t). But $2 + c > 0$ and so $c = -1$ is the only possible value and E^{-1} must be μ-semistable. Indeed, if D was a potential μ-destabilising object then $\deg(D) = 0$ and the composite $\mathcal{A}_0$-injection $D[1] \to E^{-1}[1] \to E$ would destabilise E for all $t > 0$; contradiction. Thus, by Bogomolov, we have $\chi \leqslant 1$ and E is t-stable for some $t > 0$ if and only if for some $t > 0$, $\mu_t(E) < \mu_t(E^0)$ which is equivalent to $0 < (2r+1)t^2 < 4 - n + 2$. This implies $n < 6$.

Now let $F = E^0/\operatorname{tors}(E^0)$. Then $c_1(F) = c_1(E)$ as $c_1(E)$ is minimal in T_0 and $\chi(F) = 4 - n + \chi - p \leqslant 0$ by Bogomolov and Prop 1.3(5), for some $p \geqslant 0$. But composing $\mathcal{A}_0$-surjections $e \to E^0 \to F$, we see that there must exist t such that $\mu_t(F) - \mu_t(e) > 0$. This can only happen if $4 - n + 2\chi - 2p > 0$. But $\chi - p \leqslant n - 4$ and so $n - 4 > 0$. Hence, $n = 5$ is the only possibility.

When $n = 5$ we have $\chi(E^{-1}) = \chi = 1$ which can only happen if $r(E^{-1}) = 1$. Then $E^{-1} \cong L^{-1}\mathcal{P}_{\hat{x}}$ for some $\hat{x} \in \hat{\mathbb{T}}$. We also have $s = 0$ and $\operatorname{ch}(E) = (2, \ell, 0)$. Such a μ-semistable sheaf must be μ-stable and locally-free. □

So we see that if $n \neq 5$, only sheaves can be t-stable for some t; all other objects are t-unstable for all t.

The proposition does not prove that cases (2) and (3) do actually arise. To show that (3) does arise we use the Fourier-Mukai transform. Observe that $E^{-1}[1] \to e$ will destabilize if $t \geqslant 1/\sqrt{3}$. We now compute the Fourier-Mukai transform of these objects.

Proposition 2.2. *Suppose $e \in \mathcal{A}_s$ has $\operatorname{ch}(E^{-1}) = (1, -\ell, 1)$ and $\operatorname{ch}(E^0) = (2, \ell, 0)$ with E^0 torsion-free. Then $\Phi(e)$ is a torsion-free sheaf.*

Proof. We use the spectral sequence $\Phi^{p+q}(e) \Leftarrow \Phi^p(E^q)$. We have $\Phi(E^{-1}) \cong \tau_{\hat{x}}^*\hat{L}[-1]$ (see [Muk81] or [Mac11]) and $\Phi(E^0)$ is a torsion sheaf of rank 1 supported on some D_x of degree -1. Then the spectral sequence has only two non-zero terms $E_2^{1,-1} \cong \tau_{\hat{x}}^* \hat{L}$ and $E_2^{0,0} \cong \Phi(E^0)$. So we have a short exact sequence (in $\mathcal{A}_0$):

$$0 \to \tau_{\hat{x}}^* \hat{L} \to \Phi(e) \to \Phi(E^0) \to 0$$

and so $\Phi(e)$ is in $\mathcal{A}_{\mathbb{T}} \cap \mathrm{Coh}_{\mathbb{T}}$. To see that it is torsion-free observe that any torsion must be supported on D_x with degree less than -1. Then $\Phi(e)/\operatorname{tors}(\Phi(e))$ would have Euler characteristic bigger than 1 which is impossible for a torsion-free sheaf or rank 1 and degree 2. □

So $\Phi(e)$ takes the form $\hat{L}^2 \mathcal{P}_x \mathcal{I}_{\hat{X}}$ for some $\hat{X} \in \mathrm{Hilb}^5 \hat{\mathbb{T}}$ and $x \in \mathbb{T} = \hat{\hat{\mathbb{T}}}$. But for $1/t$ sufficiently large this is $1/t$-stable and so e is t-stable for t sufficiently small. Hence, case (3) does arise (but only if $n = 5$).

For case (2), consider a torsion sheaf G supported on D_x of rank 1 and degree $4 - n + m$ and some $X' \in \mathrm{Hilb}^m \mathbb{T}$, for some $m < (n-2)/2$. Observe that

$$\chi(L\mathcal{I}_{X'}, G) = 1 - n + m < -n/2 < 0$$

and so $\mathrm{Ext}^1(L\mathcal{I}_{X'}, G) \neq 0$ and hence there are non-trivial extensions

$$0 \to G \to E \to L\mathcal{I}_{X'} \to 0.$$

G will destabilize E if $t \geqslant \sqrt{n-2+2m}$. If $t < \sqrt{n-2+2m}$ then we need to check that E can be chosen to be t-stable. As before there must be a sheaf $K \in T_0$ and an injection $K \to E$ in $\mathcal{A}_0$ which destabilizes. Let the quotient be q. Now both G and $L\mathcal{I}_{X'}$ are t-stable (by Proposition 1.5(3)). We can assume that K is itself t-stable by picking the first Jordan-Hölder co-factor of the first Harder-Narasimhan factor. Then $\mathrm{Hom}(K, G) = 0$ and so $\mathrm{Hom}(K, L\mathcal{I}_{X'}) \neq 0$. Note that $\deg(Q^{-1}) = 0$ as $r(K/Q^{-1}) = 1$ and $\mathrm{Hom}(K/Q^{-1}, L\mathcal{I}_{X'}) \neq 0$. But then $\mu_t(K/Q^{-1}) \geqslant \mu_t(K)$ and $K/Q^{-1} \to E$ injects in $\mathcal{A}_0$ which is impossible given the choice of K. So $q = Q^0 = Q$ is a sheaf and $r(K) = 1$, $\deg(K) = 2$. So $K \cong L\mathcal{I}_{X''}$ for some $X'' \supset X'$. But this can only destabilize for $t^2 < n - 2 - 2|X''|$. Hence, for $\sqrt{n-4-2m} < t < \sqrt{n-2-2m}$, E must be t-stable. So again, case (2) does arise for all $n > 2$.

Finally, let us consider the torsion free sheaves of the form $E = L^2 \mathcal{P}_{\hat{x}} \mathcal{I}_X$. The argument at the end of the proof of Proposition 1.5 shows that any destabilizing object of E must be a torsion-free sheaf of degree 2. In other words, there is some 0-scheme X' of length m and a map $L\mathcal{P}_{\hat{y}} \mathcal{I}_{X'} \to L^2 \mathcal{P}_{\hat{x}} \mathcal{I}_X$. As a sheaf map this injects with quotient G, a torsion sheaf of rank 1 supported on some D_x of degree $4 - n - |X'|$. Now this destabilizes only when $t < \sqrt{n-2-2m}$. The existence of such a destabilizing subsheaf can be described geometrically. The following refines Proposition 1.5(4).

Proposition 2.3. *Let X be a 0-scheme of length n. Suppose $X'' \subset X$ is a collinear subscheme of maximal length. Then $E = L^2 \mathcal{P}_{\hat{x}} \mathcal{I}_X$ is t-stable for all $t > \sqrt{\max(0, 2|X''| - n - 2)}$.*

Proof. The existence of X'' is equivalent to the existence of a non-zero map from $L\mathcal{P}_{\hat{y}} \mathcal{I}_{X'} \to E$ where $X' = X \setminus X''$ and $\hat{y}$ is some element of $\hat{\mathbb{T}}$. The maximality assumption implies that $|X'|$ is least among such maps and so E is t-stable for all $t^2 > n - 2 - 2|X'| = 2|X''| - n - 2$. □

Note that the codimension of such loci in $\mathrm{Hilb}^n\,\mathbb{T}$ is $|X''|-2$. Collecting these results together, we can state the following.

Theorem 2.4. *In the* 1*-parameter family of stability conditions* $(\mathcal{A}_0,\mu_t)$ *the moduli functor* $\mathcal{M}_t^{(1,2\ell,4-n)}$ *has* $\lfloor (n-1)/2\rfloor$ *walls for all* $n>0$ *except for* $n=5$ *when there are* 3 *walls. The highest wall is at* $t=\sqrt{n-2}$ *and, except for* $|X|=5$*, the lowest is at* $\sqrt{1+(n+1 \mod 2)}$

So the generating series for the number of walls is

$$\frac{x^3\left(1+x^2-x^3-x^4+x^5\right)}{(1+x)\,(1-x)^2}$$

We can extend this to s in the interval $(0,1)$ by observing that any further destabilizing objects for $L^2\mathfrak{I}_X$ with Chern characters $(r,c\ell,\chi)$ would result in a destabilizing condition of the form

$$\begin{aligned} 0<\chi(2-s)+(s^2+n-4)(c-2r) &= \\ -(2r-c)\left(s+\frac{1}{2}\frac{\chi+(n-4)r}{2r-c}\right)^2 &+2\,\chi+(n-4)c+\frac{1}{4}\frac{(\chi+(n-4)r)^2}{2r-c} \end{aligned}$$

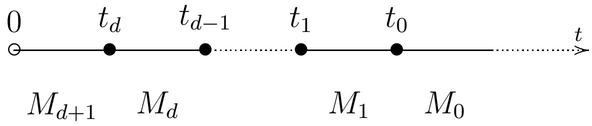
Note that $c/r<2$ as the destabilizing object must be a sheaf K in T_s for $0<s<1$ and the kernel of the map $K\to L^2\mathfrak{I}_X$ is in F_s. Since we require the centre to be in $(0,1)$ we have $\chi<-(n-4)r$. But this contradicts the destabilizing inequality. Combining this with Proposition 1.5(3), we have the following.

Proposition 2.5. *For all* $n\geqslant 4$*, the only walls associated to the Chern character* $(1,2\ell,4-n)$ *in the region* $0\leqslant s<2$ *are those which intersect* $s=0$*.*

The situation for $n=3$ is different (see section 5.1 below).

3. Projectivity of the Moduli Spaces

If we number the walls $i=0,\dots,d=\lfloor (n-3)/2\rfloor$ from the greatest t downwards then we have $\lfloor (n+1)/2\rfloor$ potential moduli spaces M_i, with $M_0=\mathrm{Hilb}^n\,\mathbb{T}\times\hat{\mathbb{T}}$ (and analogously for $n=5$).



Theorem 3.1. *For any* $t>0$*, the moduli space of* t*-stable objects with Chern character* $(1,2\ell,4-n)$ *in* $\mathcal{A}_0$ *is a smooth complex projective variety for each positive integer* n*.*

The fact that M_i are fine moduli spaces given by smooth varieties will follow from key results in [ABL07] (generalized a little to cover our case) and we will deal with this in the next section. We first show that the spaces M_i are projective. We shall assume in this section that $n\geqslant 4$. The case $n=3$ will be dealt with as a special case in section 5.1 below.

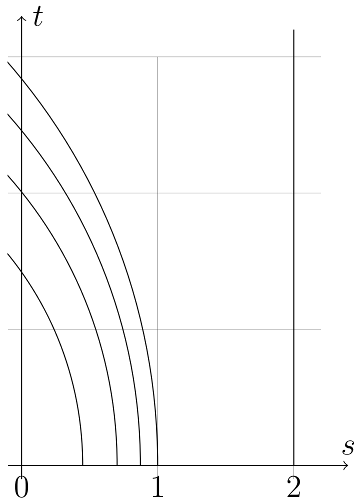


FIGURE 1. Chamber and walls for $n = 10$

The trick is to consider the region $t > 0$ and $0 \leqslant s < 1$ in the set of stability conditions. Proposition 2.5 tells us that, for a given $n \geqslant 4$ there are no further walls. The condition for a wall is given by

$$t^2 + \left(s + \frac{n-m-3}{2}\right)^2 - \left(\frac{n-m-3}{2}\right)^2 - (n-2-2\,m) = 0$$

corresponding to destabilising sheaves $L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}}$ with $|X'| = m$. The resulting semicircles are illustrated in Figure 1 for the case $n = 10$.

The semicircles intersect the $t = 0$ axis in distinct points (as can be easily checked) and so for each moduli space M_i we can always find a rational number $s = q_i$ which lies between the i^{th} and $i+1^{\text{st}}$ wall on $t = 0$. Now let Φ_{-q_i} be the Fourier-Mukai transform given by a universal sheaf $\mathbb{E}$ over $\mathbb{T}\times\hat{\mathbb{T}}$ whose restriction $E_{\hat{x}} = \mathbb{E}|_{\mathbb{T}\times\{\hat{x}\}}$ satisfies $\mathrm{ch}(E_{\hat{x}}) = (a^2, -ab\ell, b^2)$ where $b/a = q_i$ written in its lowest form. Then $\Phi_{-q_i}(\mathcal{A}_{q_i}) = \mathcal{A}_{r_i}$, where $r_i = c/a$ and $c_1(E_x) = c\ell$, where $E_x = \mathbb{E}|_{x\times\hat{\mathbb{T}}}$ is the restriction to the other factor. Moreover, $e \in \mathcal{A}_{q_i}$ is t-stable for $t \ll 1$ if and only if $\Phi_{-q_i}(e)$ is t-stable for $t \gg 0$ in $\mathcal{A}_{r_i}$. Since, our Chern characters $(1, 2\ell, 4-n)$ are primitive we know that the moduli space $M_t^{\Phi_{-q_i}(1,2\ell,4-n)}$ is a fine moduli space of torsion-free sheaves for $t \gg 0$ and is projective provided it is non-empty. Consequently it will follow that M_i is also projective. Since the codimension of the non-torsion-free sheaf locus in M_i is greater than $n/2 - 1$ the non-emptyness of $M_t^{\Phi_{-q_i}(1,2\ell,4-n)}$ will follow from the following.

Proposition 3.2. *Let $0 \leqslant q < 1$ be a rational number. If $n > 3$ there is some $X \in \mathrm{Hilb}^n\,\mathbb{T}$ such that $\Phi_{-q}(L^2\mathcal{I}_X)$ is a torsion-free sheaf in $\mathcal{A}_r = \Phi_{-q}(\mathcal{A}_q)$.*

Before the proof we recall a few facts and definitions about Fourier-Mukai transforms. We say that an object e is Φ_{-q}-WIT_i if the cohomologies $\Phi^j_{-q}(e)$ in $\mathrm{Coh}_{\hat{\mathbb{T}}}$ are zero for all

$j \neq i$. If $a \in \mathcal{A}_q$ then $\Phi^1_{-q}(A^0) = 0 = \Phi^{-1}_{-q}(A^{-1})$ (since $\Phi_{-q}(a) \in \mathcal{A}_r$). We denote the inverse transform by $\hat{\Phi}_{-q}$.

Proof. Let X be an arbitrary element of $\operatorname{Hilb}^n \mathbb{T}$. Observe first that $\operatorname{ch}(E_{\hat{x}} \otimes L^2\mathcal{I}_X) = (a^2, a(2a-b)\ell, (4-n)a^2 + b^2 - 4ab)$. So $\chi(E_{\hat{x}} \otimes L^2 I_X) < 0$ for $n > 3$ (in fact, this also works for $n = 3$ for a suitable choice of a and b but this case is not required). Consider the structure sequence,

$$0 \to L^2\mathcal{I}_X \to L^2 \xrightarrow{f} \mathcal{O}_X \to 0.$$

Now $\Phi_{-q}(L^2)[-1]$ is a sheaf of rank $(2-q)^2a^2$. Note also that $\Phi_{-q}(\mathcal{O}_X)[-1]$ is a sheaf. Let $k = \Phi_{-q}(L^2\mathcal{I}_X)$. Our aim is to show that we can find X so that k is a torsion-free sheaf. Suppose for a contradiction that $K^{-1} \neq 0$.

Claim 1. $G = \hat{\Phi}_{-q}(K^{-1}[1])$ *is a torsion-free sheaf and there is a non-trivial map* $G \to L^2\mathcal{I}_X$ *which injects in* $\mathcal{A}_q$. *Moreover,* $\deg(G) > 2q \operatorname{rk}(G)$ *and* $\chi(G) > \operatorname{rk}(G)q^2$.

This follows be applying $\hat{\Phi}_{-q}$ to the triangle $k \to \Phi_{-q}(L^2) \to \Phi_{-q}(\mathcal{O}_X)$. We first take cohomology (in $\operatorname{Coh}_{\hat{\mathbb{T}}}$) and split the sequence via a sheaf Q:

$$\begin{tikzcd} 0 \arrow[r] & K^{-1} \arrow[r] & \Phi_{-q}(L^2)[-1] \arrow[rr] \arrow[dr] & & \Phi_{-q}(\mathcal{O}_X)[-1] \arrow[r] & K^0 \arrow[r] & 0. \\ & & & Q \arrow[ur] & & & \end{tikzcd}$$

This gives the following short exact sequence in $\mathcal{A}_q$:

$$0 \to \hat{\Phi}_{-q}(K^{-1}[1]) \to L^2 \to \hat{\Phi}_{-q}(Q[1]) \to 0.$$

Then we see that $\hat{\Phi}_{-q}(K^{-1}[1])$ is a torsion-free sheaf G and $G \in T_q$. Since G is Φ_{-q}-WIT$_{-1}$ we must have $\chi(E_{\hat{x}} \otimes G) > 0$. So $a^2\chi(G) > ab\deg(G) - b^2 \operatorname{rk}(G)$. But $G \in T_q$ so $\deg(G)/\operatorname{rk}(G) > 2q$. So we have $a^2\chi(G) > 2qab \operatorname{rk}(G) - b^2 \operatorname{rk}(G) = b^2 \operatorname{rk}(G)$.

Claim 2. *G also satisfies* $\deg(G) < 4 \operatorname{rk}(G)$.

We have a triangle (which is short exact in $\mathcal{A}_q$)

$$G \to L^2\mathcal{I}_X \to \hat{\Phi}_{-q}(K^0).$$

Then there is a surjection $L^2\mathcal{I}_X \to \hat{\Phi}^0_{-q}(K^0)$ in $\operatorname{Coh}_{\mathbb{T}}$. But $\hat{\Phi}^0_{-q}(K^0) \in T_q$ and is a torsion sheaf. Let the dimension of its support be w. Then $c_1(G) - (2-w)\ell = c_1(\hat{\Phi}_{-q}(K^0)) = c_1(\hat{\Phi}^{-1}_{-q}(K^0))$. But $\hat{\Phi}^{-1}_{-q}(K^0) \in F_q$. So $\deg(G) - 2(2-w) \leqslant 2q(\operatorname{rk}(G) - 1) < 2\operatorname{rk}(G) - 2$. Thus $\deg(G) < 2\operatorname{rk}(G) + 2(1-w) \leqslant 4\operatorname{rk}(G)$ as required.

Claim 3. *Fix* $0 < s < 1$ *(for the* $n = 5$ *case also assume s is larger than where the higher rank wall cross* $t = 0$*). There is some X such that* $\operatorname{Hom}(G, L^2\mathcal{I}_X) = 0$ *for all torsion-free sheaves* $G \in T_q$ *with* $\operatorname{ch}(G) = (r, c\ell, \chi)$ *such that* $\chi > rs^2$ *and* $2r > c > rs$.

Consider the numerator of $\mu_t(G) - \mu_t(L^2\mathcal{I}_X)$. This is given by

$$\chi(2-s) + s^2(c-2r) + (n-4)(c-rs) - (2r-c)t^2.$$

But

$$\chi(2-s) + s^2(c-2r) + (n-4)(c-rs) > rs^2(2-s) + s^2(rs-2r) = 0$$

So such a G must destabilize in $0 < s < 1$ and this is impossible unless $\mathrm{rk}(G) = 1$. But then we can pick X so that $\mathrm{Hom}(G, L^2\mathfrak{I}_X) = 0$ for all G as required.

Returning to the proof we see that K^{-1} must be zero as its transform cannot map non-trivially to $L^2\mathfrak{I}_X$. □

4. The Surgeries

It remains to show that M_i are smooth varieties which are fine moduli schemes representing the appropriate moduli functor 0.1. The proof proceeds in exactly the same way as [ABL07, Theorem 5.1] but the details are a little different. We first state a generalization of the Arcara Bertram construction. The details of the proofs are exactly the same as in [ABL07] and we omit them.

We state the following in generality for a general Bridgeland stability condition $(\mathcal{A}, Z)$ given by a fixed abelian subcategory $\mathcal{A} \subset D^b(S)$, where S is any K3 or abelian surface over $\mathbb{C}$.

Theorem 4.1 ([ABL07])**.** *Fix a Mukai vector v and suppose there is a path $p : \mathbb{R} \to U$ in the stability manifold for which $\mathcal{A}$ remains fixed. Suppose M is some fine moduli space of Bridgeland stable objects on S which is smooth and proper over $\mathbb{C}$ and represents the moduli of $p(t)$-stable objects for $t < 0$. Furthermore, suppose that M contains a sub-moduli space P whose objects a satisfy the following conditions*

1. *a becomes unstable for $t > 0$.*
2. *a are represented as short exact sequences*
$$0 \to e_1 \to a \to e_2 \to 0 \qquad \text{in } \mathcal{A}$$
for $e_1 \in B_1$ and $e_2 \in B_2$, where B_1 and B_2 are fine moduli spaces of such objects.
3. *For all $e_1 \in B_1$ and $e_2 \in B_2$ and non-trivial extensions*
$$0 \to e_2 \to b \to e_1 \to 0 \qquad \text{in } \mathcal{A}$$
b is Z-stable for all $Z \in U$.
4. $$N := \dim \mathrm{Ext}^1(e_1, e_2) - 1 > 1$$
for all $e_1 \in B_1$ and $e_2 \in B_2$.
5. *All $a' \in M \setminus P$ are $p(t)$-stable for all t.*

Then

1. *P is a projective bundle over $B_1 \times B_2$ with projective space fibres of dimension N.*
2. *There is a smooth proper variety $\mathrm{MF}(M, P)$ which is the Mukai flop of M along P and which is a fine moduli space of objects which are $p(t)$-stable for all $t > 0$.*

Note that the assumptions imply that $\mathrm{Hom}(e_1, e_2) = 0 = \mathrm{Hom}(e_2, e_1)$ because extensions on either side of the wall are stable for some t and so are simple. Hence, for the abelian surface case, $N = -\chi(e_1, e_2) - 1$.

The proof is exactly as given in [ABL07] except that Lemma 5.4 in that paper is not required. This is the only place the particular choices of β, B_1 and B_2 mattered. In fact, the lemma is unlikely to be true in our cases. The lemma is used to show that the constructed universal sheaf $\mathcal{U}$ on M_i satisfies

$$\mathcal{U}|_{\mathbb{T}\times P} \cong \mathcal{E}_i \otimes L$$

where $\mathcal{E}_i$ is the universal sheaf corresponding to the (fine) moduli space P and L is some line bundle pulled back from P. It would then allow us to assume L is trivial by choice of $\mathcal{U}$. But this is not needed for their argument.

This theorem applies to each of our walls because Proposition 1.5 (3) and (5) implies that the rank 1 walls satisfy the hypotheses of the theorem as e_1 takes the form $L\mathcal{I}_X\mathcal{P}_{\hat{x}}$ and e_2 is a pure torsion sheaf with $c_1(e_2) = \ell$. As we have already observed (and used) $N > 1$ for $n \geqslant 4$. The other hypotheses are met because no two walls intersect near $s = 0$. For the unique rank 2 wall (when $n = 5$) we have $e_2 = L^{-1}\mathcal{P}_{\hat{x}}[1]$ and so is t-stable by Proposition 1.5(1). Finally, e_1 is a μ-stable sheaf of Chern character $(2, \ell, 0)$. These are Fourier-Mukai transforms of pure torsion sheaves with $c_1 = \ell$. These are t-stable for all t by Proposition 1.5(5) again and so e_1 must also be t-stable (for all t and, in particular, for the values of t near the wall).

This completes the proof of our main Theorem 3.1.

5. Examples of the Moduli Spaces

Let us now consider the low values of n in more detail.

5.1. $\mathbf{n = 3.}$ In this case, the only possible value of m is zero and a non-trivial extension $0 \to G \to E \to L\mathcal{P}_{\hat{x}} \to 0$ has G with degree 1. The argument above proves that this is t-stable for $t < 1$ and t-unstable for $t > 1$. On the other hand, $L^2\mathcal{P}_{\hat{x}}\mathcal{I}_Y$ is t-stable for all $t > 1$ and all $(Y, \hat{x}) \in \operatorname{Hilb}^3 \mathbb{T} \times \hat{\mathbb{T}}$. If Y is not itself collinear then $L^2\mathcal{P}_{\hat{x}}\mathcal{I}_Y$ remains t-stable for all t. But, if Y is collinear then $L^2\mathcal{P}_{\hat{x}}\mathcal{I}_Y$ is destabilized by some $L\mathcal{P}_{\hat{y}}$. So we have one wall $t = 1$ and consequently two moduli spaces $M_{<1}$ and $M_{>1}$. The latter is just given by the twisted ideal sheaves. The former has a Zariski open subset corresponding to non-collinear length 3 0-schemes. The complement of this is a divisor in $M_{<1}$ and consists of sheaves with torsion subsheaves of the form G above. In particular, $M_{>1}$ and $M_{<1}$ are birationally equivalent. The existence of $M_{<1}$ as a fine moduli space will be established in the proof of the theorem below.

Note that $\operatorname{Ext}^1(L\mathcal{P}_{\hat{x}}, G)$ has dimension 2 for all $\hat{x}$ and such G. This is because $\chi(L, G) = -2$ but $\operatorname{Hom}(L\mathcal{P}_{\hat{x}}, G) = 0$ for all $\hat{x}$ and G. Indeed, any such map must factor through a subsheaf with $c_1 = \ell$ and $\chi \leqslant 0$. Then the kernel is torsion-free with degree 0 and $\chi \geqslant 1$, which is impossible. The moduli space of such G is isomorphic to $\mathbb{T} \times \hat{\mathbb{T}}$ given by $(x, \hat{x}) \mapsto \mathcal{O}_{D_x}(1)\mathcal{P}_{\hat{x}}$. Then the space of isomorphisms classes of these sheaves E is a $\mathbb{P}^1$ bundle over $\mathbb{T}\times \hat{\mathbb{T}} \times \hat{\mathbb{T}}$. On the other hand, we can also parametrize the points $L^2\mathcal{P}_{\hat{x}}\mathcal{I}_Y$ where Y is collinear by the dual bundle (corresponding to $\operatorname{Ext}^1(G, L) \cong \operatorname{Ext}^1(L, G)^*$ under Serre duality). In particular, the birational map given by identifying the points corresponding to non-collinear length 3-subschemes does not extend to an isomorphism of spaces. Nevertheless, the spaces are isomorphic.

Theorem 5.1. *The moduli spaces $M_{<1}$ and $M_{>1}$ exist and are isomorphic as smooth projective varieties.*

Proof. We know from Geometric Invariant Theory that $M_{>1}$ exists as a fine moduli space. To show that $M_{<1}$ exists we apply the Fourier-Mukai transform. This immediately tells us that $M_{<1}$ is isomorphic to $M_{>1}^{(-1,2\ell,-1)}$. We can give an explicit model for this space with a universal object as follows.

Claim 1. *The points of $M_{>1}^{(-1,2\ell,-1)}$ are given by objects $e \in \mathcal{A}_0$ such that $E^{-1} \cong L^{-2}\mathcal{P}_{\hat{x}}$ for some $\hat{x}$ and $E^0 \in \operatorname{Hilb}^3 \mathbb{T}$ such that $[e] \in \operatorname{Ext}^2(E^0, E^{-1})$ has maximal rank.*

By this last statement we mean that the composite of $[e]$ with any non-zero map $\mathcal{O}_x \to E^0$ from a skyscraper sheaf to $E^0 \cong \mathcal{O}_Y$ is also non-zero. Now the Mukai spectral sequence (as used at the end of the last section) gives us a long exact sequence of sheaves

$$0 \to \Phi^{-1}(e) \to H_Y \xrightarrow{f} \Phi^1(L^{-2}\mathcal{P}_{\hat{x}}) \to \Phi^0(e) \to 0,$$

where H_Y is the homogeneous bundle which is the Fourier-Mukai transform of $\mathcal{O}_Y$. Note that $\Phi^1(L^{-2}\mathcal{P}_{\hat{x}})$ is a rank 4 μ-stable vector bundle. Then f injects precisely when $[e]$ has maximal rank. Then maximal rank is precisely the condition for $\Phi(e)$ to be a sheaf.

On the other hand, all of the objects of $M_{<1}$ have transforms which are described by the claim. For the case when $L^2\mathcal{P}_{\hat{y}}\mathcal{I}_{Y'}$ has non-collinear Y' is given explicitly in [Mac11] Theorem 7.3. The other points of $M_{<1}$ are given as extensions

$$0 \to G \to E \to L\mathcal{P}_{\hat{y}} \to 0,$$

where G is (the direct image of) a line bundle of degree 1 on some D_x. But such G have $\Phi(G)$ of the same form and so applying Φ^* we have the exact sequence

$$0 \to \Phi^{-1}(E) \to \hat{L}^{-1}\mathcal{P}_{\hat{y}} \xrightarrow{g} \hat{G} \to \Phi^0(E) \to 0.$$

But g cannot surject as the kernel must be locally free. Hence, its image is a torsion sheaf supported on the support of $\hat{G}$. This implies that $\Phi^{-1}(E) \cong \hat{L}^{-2}\mathcal{P}_{\hat{x}}$, for some $\hat{x}$ and $\Phi^0(E) \in \operatorname{Hilb}^3 \hat{\mathbb{T}}$. This completes the proof of the claim.

Claim 2. *The isomorphism class of e is independent of $[e] \in \operatorname{Ext}^2(E^0, E^{-1})$.*

This statement is equivalent to saying that the isomorphism type of a quotient $\widehat{L^{-2}}/H_Y$ is independent of the (injective) map $H_Y \to \widehat{L^{-2}}$. But this follows because any two such maps g and g' are equivalent under the composition action of $\operatorname{Hom}(H_Y, H_Y)$ and so the two quotients $\operatorname{coker}(g)$ and $\operatorname{coker}(g')$ are isomorphic.

Now we see that $M_{>1}^{(-1,2\ell,-1)}$ is given by $\operatorname{Hilb}^3 \mathbb{T} \times \hat{\mathbb{T}}$ with universal sheaf $\pi_{12}^*\mathcal{O}_{\mathcal{Y}} \otimes \pi_{13}^*\mathcal{P}$ over $\mathbb{T} \times \operatorname{Hilb}^3 \mathbb{T} \times \hat{\mathbb{T}}$. In particular, $M_{<1}$ is isomorphic to $M_{>1} \cong \operatorname{Hilb}^3 \mathbb{T} \times \hat{\mathbb{T}}$. $\square$

In fact, the isomorphism can also be given as $\Phi \circ R\Delta$, where $R\Delta$ is the derived dual functor $\mathbf{R}\mathcal{H}om(-, \mathcal{O}_{\mathbb{T}})[1]$. This is because $R\Delta : M_{>1} \to M_{>1}^{(-1,2\ell,-1)}$. To see this observe that $R^{-1}\Delta(L^2\mathcal{I}_Y) \cong L^{-2}$ and $R^0\Delta(L^2\mathcal{I}_Y) \cong \mathcal{O}_Y$ and $\mathcal{O}_Y \to L^{-2}[2]$ must have maximal rank as taking the dual again gives a map $L^2 \to \mathcal{O}_Y$ which must surject to have come from $L^2\mathcal{I}_Y$.

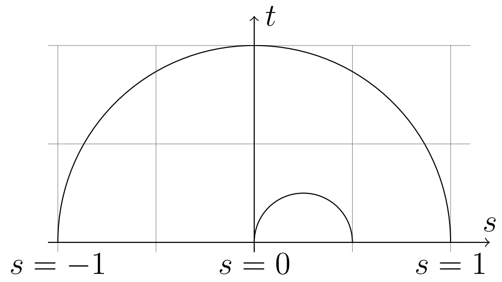


FIGURE 2. Chamber and walls for $n = 3$

Using the calculation in [Mac11] Theorem 7.3 we can write down the map $M_{>1} \to M_{<1}$ explicitly at a reduced 0-scheme $Y = \{p, q, y\}$ as

$$\begin{pmatrix} -1 & -1 & 0 & -1 \\ 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

thought of as acting on the "vector" $(p, q, y, \hat{x})$. In particular, it is not the extension of the birational map $M_{>1} - \to M_{<1}$.

For completeness observe that we have a fourth moduli space $M_{<1}^{(-1,2\ell,-1)}$. This is the Fourier-Mukai transform space of $M_{>1}$ and consists of generic points of the sort described in Claim 1 above but a codimension 1 subvariety consists of 2-step complexes with cohomology $L^{-1}\mathcal{P}_{\hat{y}}$ and G where G is a degree 1 line bundle supported on some D_x.

When $0 < s < 1$ there is a further wall due to destabilizing objects of the form $\Phi(L^{-2})[1]$. This corresponds to a "codimension 0" surgery. It is an exercise to check that there are no further destabilising objects for $-1 < s < 1$ and so the chamber and wall structure is as illustrated in Figure 2. Once we cross this additional wall the moduli space consists of objects e of the form

$$H_{\tilde{Y}}[1] \to e \to \Phi^1(L^{-2}\mathcal{P}_{\hat{x}})$$

Then the Fourier-Mukai transform under Φ of this space is exactly $\mathrm{Hilb}^3 \hat{\mathbb{T}} \times \mathbb{T}$ given by sheaves of the form $L^{-2}\mathcal{P}_x\mathcal{I}_{\tilde{Y}}$.

5.2. $\mathbf{n = 4.}$ Again there is only one wall, this time at $t = \sqrt{2}$. Just as for the length 3 case, it is the collinear length 4 0-schemes which correspond to non t-stable sheaves $L^2\mathcal{I}_Z$ as t crosses the wall. These live in a codimension 2 subvariety and so we can use the Arcara Bertram argument from [ABL07] to construct our moduli space $M_{<\sqrt{2}}$ as a Mukai flop of $M_{>\sqrt{2}}$. This is explained more fully in the next section.

In this case, the Fourier-Mukai transform gives us an isomorphism $M_{<\sqrt{2}} \cong M_{>1/\sqrt{2}}^{(0,2\ell,-1)}$ which consists of pure torsion sheaves of rank 1 and degree 3 supported on a translate of a divisor in the linear system $|2\ell|$. In particular, the moduli space $M_{<\sqrt{2}}$ is projective. The points of $M_{>1/\sqrt{2}}^{(0,2\ell,-1)}$ are harder to describe because this linear system has singular and reducible elements. For the Chern character $(0, 2\ell, -1)$ there is exactly one wall at $t = 1/\sqrt{2}$ and we need to glue in Fourier-Transforms of $L^2\mathcal{I}_Z$ (and their flat twists) corresponding to collinear Z. These are computed in [Mac11]. The objects are 2-step complexes with cohomology $L^{-1}\mathcal{P}_{-x}$ and $L\mathcal{P}_{-x+\Sigma Z}\mathcal{I}_{2x-\Sigma Z}$, where $Z \subset D_x$.

This should be compared with the situation in [ABL07]. The nearest such space (in their notation) is $H = 2\ell$ and we take $\mathcal{A}_2 \cong (- \otimes L)(\mathcal{A}_0)$. The corresponding Chern character is $(0, 2\ell, 4)$ rather than $(0, 2\ell, 3)$ as in our case. Of course, H is reducible and so their construction does not apply. But nevertheless, we obtain analogous data. There is a wall at $t = 1/2$ and glue in 2-step complexes whose -1 cohomology is (a twist of) L^{-1} and whose 0th cohomology is an extension of (a twist of) $L\mathcal{I}_y$ by $\mathcal{O}_z$.

5.3. $\mathbf{n = 5.}$ The length 5 case is special because of the higher rank wall which intersects $s = 0$. There are four moduli spaces corresponding to the 3 walls. The configuration is illustrated in Figure 3. The vertical lines indicate the walls. The horizontal lines indicate

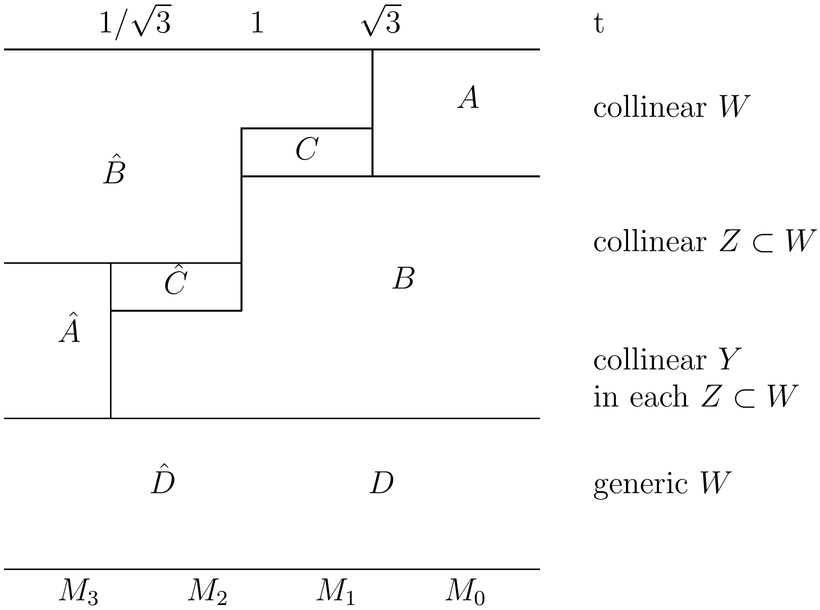


FIGURE 3. Diagram of surgeries for $n = 5$

strata in each moduli space. The letters A, B, C, D indicate sheaves (or 2-step complexes) of a particular type and their corresponding hatted letters are the Fourier-Mukai transformed spaces. To the right of a wall in regions A, B and D we have torsion-free sheaves characterised by the geometric property indicated. The codimensions of the spaces are A is codimension 3 in M_0, B is codimension 2 in M_0 and C is codimension 4 in M_1. In particular, C is codimension 1 in the replacement for A in M_1.

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References

[AB09] Daniele Arcara and Aaron Bertram. Reider's theorem and Thaddeus pairs revisited. arxiv:09043500, 2009.

[ABL07] Daniele Arcara, Aaron Bertram, and Max Lieblich. Bridgeland-stable moduli spaces for K-trivial surfaces. arxiv: 07082247, 2007.

[BBHR09] Claudio Bartocci, Ugo Bruzzo, and Daniel Hernández Ruipérez. *Fourier-Mukai and Nahm transforms in geometry and mathematical physics*, volume 276 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2009.

[Bri08] Tom Bridgeland. Stability conditions on $K3$ surfaces. *Duke Math. J.*, 141(2):241–291, 2008.

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.

[Huy06] Daniel. Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2006.

[Huy08] Daniel Huybrechts. Derived and abelian equivalence of $K3$ surfaces. *J. Algebraic Geom.*, 17(2):375–400, 2008.

[Mac11] A Maciocia. A Fourier-Mukai approach to the enumerative geometry of principally polarized abelian surfaces. arxiv: 1107.2549, 2011.

[Muk81] Shigeru Mukai. Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves. *Nagoya Math. J.*, 81:153–175, 1981.

[MYY11] Hiroki Minamide, Shintarou Yanagida, and Kota Yoshioka. Fourier-Mukai transforms and the wall-crossing behavior for Bridgeland's stability conditions. arXiv:1106.5217, June 2011.

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