Geometric phase around exceptional points

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A wave function picks up, in addition to the dynamic phase, the geometric (Berry) phase when traversing adiabatically a closed cycle in parameter space. We develop a general multidimensional theory of the geometric phase for (double) cycles around exceptional degeneracies in non-Hermitian Hamiltonians. We show that the geometric phase is exactly π for symmetric complex Hamiltonians of arbitrary dimension and for nonsymmetric non-Hermitian Hamiltonians of dimension 2. For nonsymmetric non-Hermitian Hamiltonians of higher dimension, the geometric phase tends to π for small cycles and changes as the cycle size and shape are varied. We find explicitly the leading asymptotic term of this dependence, and describe it in terms of interaction of different energy levels.

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Non-Hermitian dissipative terms enter a quantum system Hamiltonian when studying nonisolated systems, e.g., effective Hamiltonians describing decay of unstable states. It turned out that the non-Hermitian physics differs dramatically from the Hermitian physics in the presence of degeneracies (energy level crossings), even if the non-Hermitian system is close to the Hermitian one [1,2]. The most important degeneracy intrinsic to non-Hermitian Hamiltonians is the exceptional point (EP), at which two eigenvalues and corresponding eigenvectors coalesce, as opposed to the diabolic point (DP) degeneracy of Hermitian operators, at which the eigenvalues coalesce while the eigenvectors remain different. EP degeneracies have been observed in laser induced ionization of atoms [3], microwave cavities [4,5], and in "crystals of light" [6]. Similar phenomena (where the Hamiltonian is substituted by a different system operator) are encountered in optics of absorptive media [7], acoustics [8], electronic circuits [9], and mechanical systems [10,11].

A wave function of a quantum system, whose parameters undergo adiabatic cyclic evolution, acquires a complex factor dependent only on the loop in parameter space and, thus, called geometric or Berry phase [12]. Geometric phases in non-Hermitian systems were studied in [2,7,13–19]. In such systems, it is important whether or not EP is inside the closed path. For Hamiltonians given by specific 2×2 matrices, the geometric phase for a (double) cycle around EP was found to be exactly π. Later this result was verified experimentally in [5]. So far, EPs have been observed in decaying systems described by symmetric effective Hamiltonians. This is the case when the corresponding isolated system is time reversal (described by a real symmetric Hamiltonian). Time-irreversal interactions, e.g., with external magnetic field, break the symmetry of the effective Hamiltonian.

We should note that the existing theoretical studies for the geometric phase around EPs rely on the possibility of reducing the system to the two-dimensional form. However, one should be aware that the geometric phase is generally not preserved under such a reduction, as this reduction is given by a parameter dependent change of basis. For example, we mention the change of geometric phase under the parameter-dependent magnetic gauge transformation [14].

In this paper, we develop a general multidimensional theory for geometric phases around EPs. We show that, for symmetric complex Hamiltonians of arbitrary dimension and for general non-Hermitian Hamiltonians of dimension 2, the geometric phase is exactly π. However, for nonsymmetric non-Hermitian Hamiltonians of a higher dimension, the geometric phase generally diverges from π as the cycle size increases. We explicitly find the leading term of this divergence. It describes the background influence of energy levels not involved in the EP degeneracy. We note that the divergence from π is related to irreversible Hermitian terms, rather than to non-Hermitian dissipative terms.

Let $H(X)$ be a non-Hermitian complex Hamiltonian smoothly dependent on a vector of m real parameters $X=(X_1,\ldots,X_m)$. For simplicity, we consider Hamiltonians represented by non-Hermitian complex matrices of arbitrary dimension, but the results are valid in infinite dimensional case as well. Let $E_n(X)$ be the eigenvalues of $H(X)$ (labeled n), and $|\psi_n(X)\rangle$ be the corresponding eigenvectors. In multiparameter space, a set of EPs defines a smooth surface of codimension 2 [20]. For clarity, we assume that the number of parameters is three (then EPs form a curve), keeping in mind that the results below are valid for any number of parameters. Consider the EP curve, corresponding to the coincidence of the levels $E_n=E_{n+1}$ and the eigenvectors $|\psi_n(X)\rangle=|\psi_{n+1}(X)\rangle$. Let $C=\{X(t):0\leqslant t\leqslant T\}$ be a cycle making one turn around this EP curve in parameter space, see Fig. 1. We assume that there are no degeneracies (multiple eigenvalues) at points of the cycle C, as well as there are no other EP curves inside C. We note that EP is the only generic codimension 2 degeneracy for complex non-Hermitian Hamiltonians smoothly dependent on parameters [20]. Thus, strictly speaking, EP is the only degeneracy that can be encircled by C in generic systems.

Let $|\Psi_n(0)\rangle=|\psi_n(X(0))\rangle$ and $|\Psi_{n+1}(0)\rangle=|\psi_{n+1}(X(0))\rangle$ be the interacting quantum states at $t=0$. After traversing the

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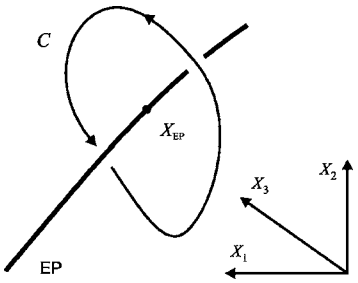

FIG. 1. Cycle around EP in parameter space.

cycle C once, the states interchange (up to the phase multiplier) [17]. When making two turns, both states return to their initial values picking up, in addition to the usual dynamical phase $\delta_n=-\frac{1}{\hbar}\int_0^{2T}E_n(t)dt$, a geometric phase γ_n [12]: $|\Psi_{n,n+1}(2T)\rangle=e^{i(\delta_n+\gamma_n)}|\Psi_{n,n+1}(0)\rangle$. Note that, due to the interchanging of the states, we have $\delta_n=\delta_{n+1}$ and $\gamma_n=\gamma_{n+1}$. For non-Hermitian systems, the geometric phase is given by the integral [13,14]

$$\gamma_n=\gamma_{n+1}=i\oint_{2C}\frac{\langle\tilde{\psi}_n(X)|d\psi_n(X)\rangle}{\langle\tilde{\psi}_n(X)|\psi_n(X)\rangle}, \tag{1}$$

where $\langle\tilde{\psi}_n(X)|$ is the left eigenvector corresponding to $E_n(X)$. The integral in Eq. (1) is evaluated over the cycle C traversed twice in the increasing time direction (we denote this by $2C$). The right and left eigenvectors are orthogonal at EP [21], which means that the denominator of the integral expression in Eq. (1) is zero at EP.

First, consider complex symmetric Hamiltonians: $H(X)=H^T(X)$. In this case, the left and right eigenvectors are complex conjugate: $\langle\tilde{\psi}_{n,n+1}(X)|=\overline{\langle\psi_{n,n+1}(X)|}$. By using this property, we write Eq. (1) in the form

$$\gamma_n=i\oint_{2C}\frac{\overline{\langle\psi_n(X)|}d\psi_n(X)\rangle}{\overline{\langle\psi_n(X)|}\psi_n(X)\rangle}=\frac{i}{2}\oint_{2C}d\ln\overline{\langle\psi_n(X)|}\psi_n(X)\rangle. \tag{2}$$

The phase γ_n is equal to the change of the complex quantity $\ln\overline{\langle\psi_n(X)|}\psi_n(X)$ over the double cycle $2C$. This change depends only on the number of turns made by the complex number $\overline{\langle\psi_n(X)|}\psi_n(X)\rangle$ around zero in complex plane, where zero is a branch point of the complex logarithm function. In one turn around zero, the logarithm changes by $\pm2\pi i$ [22].

Since the geometric phase γ_n does not depend on the form and size of the cycle, we evaluate γ_n by considering small cycles C around a point X_{EP} of the EP curve. At X_{EP}, two eigenvalues of the Hamiltonian $H_{EP}=H(X_{EP})$ coalesce: $E_{EP}=E_n(X_{EP})=E_{n+1}(X_{EP})$. E_{EP} has a single eigenvector $|\chi_0^{EP}\rangle=|\psi_n(X_{EP})\rangle=|\psi_{n+1}(X_{EP})\rangle$ and an associated vector $|\chi_1^{EP}\rangle$ determined by [10]

$$H_{EP}|\chi_0^{EP}\rangle=E_{EP}|\chi_0^{EP}\rangle,\quad H_{EP}|\chi_1^{EP}\rangle=E_{EP}|\chi_1^{EP}\rangle+|\chi_0^{EP}\rangle. \tag{3}$$

The left eigenvector $\langle\tilde{\chi}_0^{EP}|=\langle\tilde{\psi}_n(X_{EP})|=\langle\tilde{\psi}_{n+1}(X_{EP})|$ and associated vector $\langle\tilde{\chi}_1^{EP}|$ are determined by

$$\langle\tilde{\chi}_0^{EP}|H_{EP}=E_{EP}\langle\tilde{\chi}_0^{EP}|,\quad \langle\tilde{\chi}_1^{EP}|H_{EP}=E_{EP}\langle\tilde{\chi}_1^{EP}|+\langle\tilde{\chi}_0^{EP}|. \tag{4}$$

Recall that $\langle\tilde{\chi}_0^{EP}|\chi_0^{EP}\rangle=0$. Additionally, we impose the normalization conditions

$$\langle\tilde{\chi}_1^{EP}|\chi_0^{EP}\rangle=\langle\tilde{\chi}_0^{EP}|\chi_1^{EP}\rangle=1,\quad \langle\tilde{\chi}_1^{EP}|\chi_1^{EP}\rangle=0. \tag{5}$$

In the neighborhood of X_{EP}, we have [23]

$$|\psi_{n,n+1}(X)\rangle=|\chi_0^{EP}\rangle\pm\sqrt{\mu}|\chi_1^{EP}\rangle+o(\sqrt{\|X-X_{EP}\|}),$$

$$\langle\tilde{\psi}_{n,n+1}(X)|=\langle\tilde{\chi}_0^{EP}|\pm\sqrt{\mu}\langle\tilde{\chi}_1^{EP}|+o(\sqrt{\|X-X_{EP}\|}), \tag{6}$$

where μ is the linear scalar function of parameters

$$\mu(X)=\sum_{j=1}^{m}\langle\tilde{\chi}_0^{EP}|\partial H/\partial X_j|\chi_0^{EP}\rangle(X_j-X_j^{EP}), \tag{7}$$

with the derivatives taken at X_{EP}; the equation $\mu=0$ gives the tangent of the EP curve in parameter space [23]. By using Eqs. (5) and (6), and the property $\langle\tilde{\chi}_{0,1}^{EP}|=\overline{\langle\chi_{0,1}^{EP}|}$ for symmetric matrices, we obtain

$$\overline{\langle\psi_n(X)|}\psi_n(X)\rangle=2\sqrt{\mu}+o(\sqrt{\|X-X_{EP}\|}). \tag{8}$$

The complex number μ makes one turn around zero in complex plane for one cycle C in parameter space. Hence, $\overline{\langle\psi_n(X)|}\psi_n(X)\rangle$ makes a single closed loop around zero in complex plane for the double cycle $2C$. As a result, the complex logarithm function in (2) changes by $\pm2\pi i$, and we obtain $\gamma_n=\pm\pi$. The sign depends on the direction of the cycle in complex plane; it does not influence the final result, since the phase is determined up to the additional term $2\pi k$ for any integer k.

There is the geometric phase analogy between EPs of complex symmetric Hamiltonians and DPs of real symmetric Hamiltonians. For real symmetric Hamiltonians, just like for complex symmetric Hamiltonians, the geometric phase is "produced" only by the degeneracies: It is π if the degeneracy is encircled, and zero otherwise [14]. Such phases, which do not depend on the shape (geometry) of the cycle, are called topological [24]. The major difference between complex and real cases is that the cycle should be traversed twice for EP and once for DP. When a complex symmetric perturbation is given to a real symmetric Hamiltonian, DP splits into two EPs [25]. One can say that each EP takes one-half of the geometric phase of DP (counted per single cycle).

Now, consider nonsymmetric non-Hermitian Hamiltonians. We study the local structure of the EP degeneracy by means of the versal deformation theory of matrices [20,26]. The eigenvectors $|\psi_n(X)\rangle$ and $|\psi_{n+1}(X)\rangle$ are nonsmooth functions of parameters at X_{EP}. However, together they define a two-dimensional invariant linear subspace, which smoothly depends on parameters. This invariant linear subspace can be given by two vectors $|\chi_0(X)\rangle$ and $|\chi_1(X)\rangle$ smoothly dependent on parameters: $|\chi_{0,1}(X)\rangle$ are linear combinations of $|\psi_n(X)\rangle$ and $|\psi_{n+1}(X)\rangle$ and satisfy the equations (see [26])

$$H(X)|\chi_0(X)\rangle=s(X)|\chi_0(X)\rangle+p(X)|\chi_1(X)\rangle,$$

$$H(X)|\chi_1(X)\rangle = s(X)|\chi_1(X)\rangle + |\chi_0(X)\rangle. \quad (9)$$

Here, $s(X)=(E_n(X)+E_{n+1}(X))/2$ and $p(X)=(E_{n+1}(X)-E_n(X))^2/4$ are smooth scalar functions. At $X=X_{\mathrm{EP}}$, where $s(X_{\mathrm{EP}})=E_{\mathrm{EP}}$ and $p(X_{\mathrm{EP}})=0$, (9) yield the Jordan chain Eq. (3). Hence, $|\chi_0(X_{\mathrm{EP}})\rangle=|\chi_0^{\mathrm{EP}}\rangle$ is the eigenvector and $|\chi_1(X_{\mathrm{EP}})\rangle=|\chi_1^{\mathrm{EP}}\rangle$ is the associated vector of the double eigenvalue E_{EP}. By means of Eq. (9), the eigenvalues $E_{n,n+1}(X)$ and corresponding eigenvectors are found as

$$E_{n,n+1}(X) = s(X) \pm \sqrt{p(X)},$$

$$|\psi_{n,n+1}(X)\rangle = |\chi_0(X)\rangle \pm \sqrt{p(X)}|\chi_1(X)\rangle, \quad (10)$$

where two Riemann sheets of the complex square root correspond to $E_n(X)$ and $E_{n+1}(X)$. We remark that the function $\mu(X)$ in Eq. (7) is the linearization of $p(X)$ at X_{EP}. Similarly, the vectors $\langle\tilde{\chi}_{0,1}(X)|$ are introduced for the left eigenspace: They determine the left eigenvectors as

$$\langle\tilde{\psi}_{n,n+1}(X)| = \langle\tilde{\chi}_0(X)| \pm \sqrt{p(X)}\langle\tilde{\chi}_0(X)|, \quad (11)$$

and satisfy the orthonormality conditions

$$\langle\tilde{\chi}_0(X)|\chi_0(X)\rangle = \langle\tilde{\chi}_1(X)|\chi_1(X)\rangle = 0,$$

$$\langle\tilde{\chi}_1(X)|\chi_0(X)\rangle = \langle\tilde{\chi}_0(X)|\chi_1(X)\rangle = 1. \quad (12)$$

At EP, $\langle\tilde{\chi}_0^{\mathrm{EP}}|=\langle\tilde{\chi}_0(X_{\mathrm{EP}})|$ is the left eigenvector and $\langle\tilde{\chi}_1^{\mathrm{EP}}|=\langle\tilde{\chi}_1(X_{\mathrm{EP}})|$ is the left associated vector.

By using Eqs. (10)–(12) in Eq. (1), we obtain

$$\gamma_n = \frac{i}{2}\oint_{2C} d\ln\sqrt{p(X)} + i\oint_{2C}\frac{\langle\tilde{\chi}_0(X)|d\chi_0(X)\rangle + p(X)\langle\tilde{\chi}_1(X)|d\chi_1(X)\rangle}{2\sqrt{p(X)}} + \frac{i}{2}\oint_{2C}(\langle\tilde{\chi}_0(X)|d\chi_1(X)\rangle + \langle\tilde{\chi}_1(X)|d\chi_0(X)\rangle). \quad (13)$$

The double cycle $2C$ corresponds to a single cycle of the square root $\sqrt{p(X)}$ around zero in complex plane. Hence, the first integral in Eq. (13) equals $\pm 2\pi i$, where the sign depends on the direction of the cycle in complex plane. The second integral in Eq. (13) vanishes, since the square root in the denominator has opposite signs when traversing the first and second cycles. Finally, the third integral is the same for the first and second cycles. As a result, we have

$$\gamma_n = \pm\pi + i\oint_C(\langle\tilde{\chi}_0(X)|d\chi_1(X)\rangle + \langle\tilde{\chi}_1(X)|d\chi_0(X)\rangle). \quad (14)$$

Note that the integral in Eq. (14) is taken over one cycle C in the increasing time direction.

First, consider Hamiltonians given by 2×2 general complex matrices. According to Eq. (12), the 2×2 matrix $|1\rangle\langle\tilde{\chi}_1(X)|+|2\rangle\langle\tilde{\chi}_0(X)|$ is the inverse of $|\chi_0(X)\rangle\langle 1|+|\chi_1(X)\rangle\langle 2|$, where $|1\rangle=(1,0)$ and $|2\rangle=(0,1)$ are the unit vectors. Hence, components of the vectors $\langle\tilde{\chi}_{0,1}(X)|$ can be expressed explicitly in terms of the components of $|\chi_{0,1}(X)\rangle$. By using these expressions, we transform the integral in Eq. (14) to the form $\oint_C d\ln\det(|\chi_0(X)\rangle\langle 1|+|\chi_1(X)\rangle\langle 2|)$; it vanishes since the 2×2 matrix $|\chi_0(X)\rangle\langle 1|+|\chi_1(X)\rangle\langle 2|$ is everywhere nonsingular by definition. Hence, for 2×2 general non-Hermitian Hamiltonians, the geometric phase equals $\pm\pi$ and does not depend on the loop shape, similar to the case of symmetric complex Hamiltonians. This result justifies the existence of topological indices describing the polarization ellipses around C points in crystal optics [7].

For multidimensional non-Hermitian Hamiltonians, the integral in Eq. (14) is generally nonzero. Consider a cycle $C=\{X(t)=X_{\mathrm{EP}}+\epsilon\hat{X}(t):0\leqslant t\leqslant T\}$ making one turn around EP, where ϵ is a small positive parameter controlling size of the cycle. Formulae for derivatives of $|\chi_{0,1}(X)\rangle$ and $\langle\tilde{\chi}_{0,1}(X)|$ at X_{EP} are provided by the versal deformation method [26]. By using these formulae in Eq. (14), we obtain the asymptotic expression

$$\gamma_n = \pm\pi + ia\epsilon^2 + O(\epsilon^3), \quad (15)$$

where the complex constant a is given by the integral

$$a = \oint_C (2\langle\tilde{\chi}_0^{\mathrm{EP}}|H_1(G^{-3}-|\chi_1^{\mathrm{EP}}\rangle\langle\tilde{\chi}_1^{\mathrm{EP}}|)dH_1|\chi_0^{\mathrm{EP}}\rangle + \langle\tilde{\chi}_0^{\mathrm{EP}}|H_1G^{-2}dH_1|\chi_1^{\mathrm{EP}}\rangle + \langle\tilde{\chi}_1^{\mathrm{EP}}|H_1G^{-2}dH_1|\chi_0^{\mathrm{EP}}\rangle). \quad (16)$$

Here $H_1(\hat{X})=\Sigma_{j=1}^m(\partial H/\partial X_j)\hat{X}_j$ and $dH_1(\hat{X})=\Sigma_{j=1}^m(\partial H/\partial X_j)d\hat{X}_j$ with the partial derivatives taken at X_{EP}, and $G=H_{\mathrm{EP}}-E_{\mathrm{EP}}I+|\chi_1^{\mathrm{EP}}\rangle\langle\tilde{\chi}_1^{\mathrm{EP}}|$ is a nonsingular matrix (I is the identity operator). The correction term $ia\epsilon^2$ is determined by the information about the system at EP (this includes eigenvectors, associated vectors, and first derivatives of the Hamiltonian with respect to parameters) and by the cycle shape $\hat{X}(t)$. Details of the derivation of Eq. (16) will appear elsewhere [27].

The physical meaning of the constant (16) can be understood by using the eigenvector expansion of the unity and of the Hamiltonian at EP:

$$I = |\chi_0^{\mathrm{EP}}\rangle\langle\tilde{\chi}_1^{\mathrm{EP}}| + |\chi_1^{\mathrm{EP}}\rangle\langle\tilde{\chi}_0^{\mathrm{EP}}| + \sum_{k\neq n,n+1}|\psi_k^{\mathrm{EP}}\rangle\langle\tilde{\psi}_k^{\mathrm{EP}}|, \quad (17)$$

$$H_{\mathrm{EP}} = |\chi_0^{\mathrm{EP}}\rangle\langle\tilde{\chi}_0^{\mathrm{EP}}| + E_{\mathrm{EP}}(|\chi_0^{\mathrm{EP}}\rangle\langle\tilde{\chi}_1^{\mathrm{EP}}| + |\chi_1^{\mathrm{EP}}\rangle\langle\tilde{\chi}_0^{\mathrm{EP}}|) + \sum_{k\neq n,n+1}E_k^{\mathrm{EP}}|\psi_k^{\mathrm{EP}}\rangle\langle\tilde{\psi}_k^{\mathrm{EP}}|, \quad (18)$$

where $E_k^{\mathrm{EP}}=E_k(X_{\mathrm{EP}})$, $|\psi_k^{\mathrm{EP}}\rangle=|\psi_k(X_{\mathrm{EP}})\rangle$, and $\langle\tilde{\psi}_k^{\mathrm{EP}}|=\langle\tilde{\psi}_k(X_{\mathrm{EP}})|$. Here we assume the normalization condition for the left and right eigenvectors $\langle\tilde{\psi}_k^{\mathrm{EP}}|\psi_k^{\mathrm{EP}}\rangle=1$. Recall that $\langle\tilde{\psi}_k^{\mathrm{EP}}|\chi_{0,1}^{\mathrm{EP}}\rangle=\langle\tilde{\chi}_{0,1}^{\mathrm{EP}}|\psi_k^{\mathrm{EP}}\rangle=0$ and $\langle\tilde{\psi}_{k'}^{\mathrm{EP}}|\psi_k^{\mathrm{EP}}\rangle=0$ if $k\neq k'$. Expression (18) represents the transformation of H_{EP} to the canonical Jordan form written in terms of eigenvectors and associated vectors [21]. By substituting Eqs. (17) and (18) into the expression for the matrix G, after a series of manipulations, we transform Eq. (16) to

$$a=\sum_{k\neq n,n+1}\oint_C\left(2\frac{\langle\tilde{\chi}_0^{\mathrm{EP}}|H_1|\psi_k^{\mathrm{EP}}\rangle\langle\tilde{\psi}_k^{\mathrm{EP}}|dH_1|\chi_0^{\mathrm{EP}}\rangle}{(E_k^{\mathrm{EP}}-E_{\mathrm{EP}})^3}+\frac{\langle\tilde{\chi}_1^{\mathrm{EP}}|H_1|\psi_k^{\mathrm{EP}}\rangle\langle\tilde{\psi}_k^{\mathrm{EP}}|dH_1|\chi_0^{\mathrm{EP}}\rangle}{(E_k^{\mathrm{EP}}-E_{\mathrm{EP}})^2}+\frac{\langle\tilde{\chi}_0^{\mathrm{EP}}|H_1|\psi_k^{\mathrm{EP}}\rangle\langle\tilde{\psi}_k^{\mathrm{EP}}|dH_1|\chi_1^{\mathrm{EP}}\rangle}{(E_k^{\mathrm{EP}}-E_{\mathrm{EP}})^2}\right). \tag{19}$$

The terms $\langle\tilde{\chi}_{0,1}^{\mathrm{EP}}|H_1|\psi_k^{\mathrm{EP}}\rangle$ and $\langle\tilde{\psi}_k^{\mathrm{EP}}|dH_1|\chi_{0,1}^{\mathrm{EP}}\rangle$ describe the interaction of the degenerate level E_{EP} with the levels E_k, $k\neq n,n+1$ at the EP. Thus, the change of the geometric phase with the cycle size and shape variation is due to the influence of the energy levels not involved in the EP degeneracy. One can see that, if the difference $E_k^{\mathrm{EP}}-E_{\mathrm{EP}}$ is big, the influence of the level E_k is proportional to $(E_k^{\mathrm{EP}}-E_{\mathrm{EP}})^{-2}$ and can be neglected. However, if $E_k^{\mathrm{EP}}-E_{\mathrm{EP}}$ is small, the change of the geometric phase due to the interaction with E_k grows proportionally to $(E_k^{\mathrm{EP}}-E_{\mathrm{EP}})^{-3}$ and may be big. In the extreme case $E_k^{\mathrm{EP}}-E_{\mathrm{EP}}\to 0$, i.e., near the triple degeneracy $E_n=E_{n+1}=E_k$, we have $a\to\infty$. Hence, triple degeneracies require special investigation.

Thus, for nonsymmetric non-Hermitian Hamiltonians, the deviation of the geometric phase from π is the multidimensional phenomenon, which cannot be captured in two-dimensional approximations. Asymptotic expression (15) with the coefficient (16) for the correction term was confirmed by numerical simulations for particular Hamiltonians of dimensions 3 and 4. This change of the geometric phase, which is intrinsic to nonsymmetric non-Hermitian Hamiltonians, can be verified in future experiments. One of the practical ways is extending recent microwave cavity experiments [5], where EP degeneracies were studied for complex symmetric Hamiltonians. The symmetry of the Hamiltonian can be broken by applying external magnetic field, as discussed in [2]. In general, this effect should exist when studying the decay of nearly degenerate resonant states for time-irreversal systems.

Based on the expansions of eigenvectors near EP, we have shown that in the general case the geometric phase integral can be evaluated by methods of complex analysis. This can be regarded as a response to Arnold [28] who suggested to develop a theory of "residues" to calculate the Berry phase.

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