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Critical points for the Hausdorff dimension of pairs of pants

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Abstract We study the dependence of the Hausdorff dimension of the limit set of a hyperbolic Fuchsian group on the geometry of the associated Riemann surface. In particular, we study the type and location of extrema subject to restriction on the total length of the boundary geodesics. In addition, we compare different algorithms used for numerical computations.

Keywords Closed geodesics · Hausdorff dimension · Fuchsian group · Limit set · Numerical estimates · Riemann surface · Selberg zeta function

Mathematics Subject Classification (2000) 37M25 · 37F35 · 37L30 · 20H10 · 11M36 · 37C30

1 Introduction

The dependence of the Hausdorff dimension of dynamically defined sets on the underlying dynamics has been studied by many different authors in many different settings. In the case of limit sets Λ of convex cocompact Fuchsian groups this question is intimately connected with the spectrum of the Laplacian, and aspects of this problem have been studied rigorously by a number of authors, including Phillips–Sarnak [10], Pignataro–Sullivan [11] and McMullen [9], and experimentally by Gittins–Peyerimhoff–Stoiciu–Wirosoetisno [6]. In this note we will concentrate on the simplest case of a convex cocompact Fuchsian group, namely the one corresponding to a pair of pants, i.e., a Fuchsian group Γ generated by reflection in three disjoint geodesics in the hyperbolic plane. Each pair of pants are described

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up to isometry by the lengths $b_1, b_2, b_3 > 0$ of three closed boundary geodesics fixed by pairs of reflections.

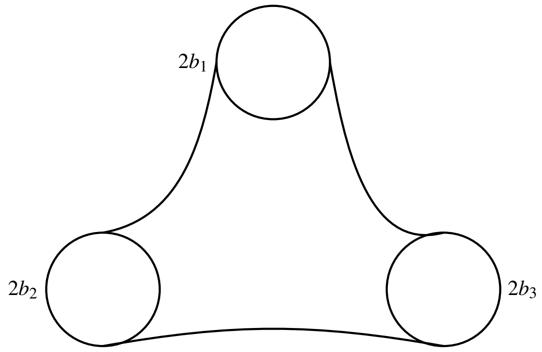

Fig. 1 A pair of pants with geodesic boundary curves of lengths $2b_1$, $2b_2$, $2b_3$

We can consider the function $\dim : \mathbb{R}^3_+ \to \mathbb{R}$ which associates to each pair of pants parameterised by $\underline{b} = (b_1, b_2, b_3) \in \mathbb{R}^3_+$ the Hausdorff dimension $\dim(\underline{b}) = \dim_H(\Lambda_{\underline{b}})$ of the associated limit set $\Lambda_{\underline{b}}$. Given any $b > 0$ we will also be considering the behaviour of the restriction $\dim :\ \Delta_b \to \mathbb{R}_+$ to the simplex

$$\Delta_b = \{\underline{b} = (b_1, b_2, b_3) \in \mathbb{R}^3_+ : b_1 + b_2 + b_3 = b\}$$

We will also be interested in the extension of dim to the closure $\overline{\Delta}_b$ and its restriction to the boundary $\partial\Delta_b$. Our starting point is the following simple but useful result.

Theorem 1 *Let* $b > 1$.

1. *The map* $\dim : \Delta_b \to \mathbb{R}_+$ *is real analytic,*
2. $\dim : \overline{\Delta}_b \to \mathbb{R}_+$ *is continuous, and*
3. $\dim : \partial\Delta_b \to \mathbb{R}_+$ *is real analytic.*

Theorem 1 is a folklore fact, but we include a simple proof of the first part in §3 using a slightly different viewpoint; and we give proofs in the same spirit of Part 2 and Part 3 (as Theorem 5) in §6.

There has been much interest historically in the behaviour of $\dim(\underline{b})$ in a neighbourhood of the boundary of Δ. The case of a symmetric pair of pants (i.e., $b_1 = b_2 = b_3 = b/3$) was studied by McMullen and the limiting case of the Hecke group (i.e., $b_1 = 0$ and $b_2 = b_3 = b/2$) was studied by Phillips–Sarnak [10] and Pignataro–Sullivan [11].

The study of $\dim : \Delta_b \to \mathbb{R}$ restricted to simplices Δ_b seems to have begun with Gittins et al who used a numerical method to describe empirically the function $\dim(\cdot)$ providing b is sufficiently large. Their experiments were carried out using an algorithm described by McMullen. In fact, more accurate values can be obtained near the centre of the simplex using a comparable amount of computation but a different algorithm based on the famous Selberg zeta functions, as originally described in [8]. In particular, the dimension $\dim(\underline{b})$ occurs as a zero of the Selberg zeta function

$$Z_{\underline{b}}(s) = \prod_{\gamma} \prod_{m=0}^{\infty} \left(1 - e^{-(s+m)\lambda(\gamma)}\right)$$

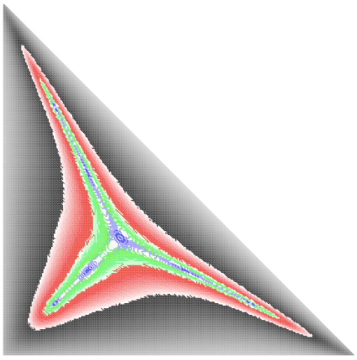

Fig. 2 A contour plot for $b = 9$ with the seven critical points in the theorem 2.

where γ denotes a closed geodesic of length $\lambda(\gamma)$. In §3 we will show that we can write

$$Z_{\underline{b}}(s) = 1 + \sum_{n=1}^{\infty} a_{2n}(s, \underline{b}), \tag{1}$$

where $a_{2n}(s, \underline{b})$ is given by a simple explicit expression defined in terms of the lengths of closed geodesics (of word length at most $2n$). This provides an efficient method for computing the dimension (which also provides explicit bounds, see §9). One of the original motivations for this note was to compare the relative efficiency of these two approaches in the context of these canonical examples.

Example 1 When $b = \frac{9}{2}$ and $b_1 = b_2 = b_3 = \frac{3}{2}$ then we can estimate $\dim(\underline{b}) = 0 \cdot 667232\ldots$ which is empirically accurate to six decimal places and uses a truncation of the series (1) to $n \leq 8$.

Based on their empirical results, Gittins et al proposed that there were four particular points, including the centre of the simplex, which were local minima. We first show the following.

Theorem 2 *Let $b > 0$. The following points in the simplex are critical points for the function* $\dim : \Delta_b \to \mathbb{R}$*:*

1. *The centre* $(\frac{b}{3}, \frac{b}{3}, \frac{b}{3})$*;*
2. *The points* $(\frac{2b}{3}, \frac{b}{6}, \frac{b}{6})$, $(\frac{b}{6}, \frac{2b}{3}, \frac{b}{6})$ *and* $(\frac{b}{6}, \frac{b}{6}, \frac{2b}{3})$*; and*
3. *The points* $(\frac{b}{2}, \frac{b}{4}, \frac{b}{4})$, $(\frac{b}{4}, \frac{b}{2}, \frac{b}{4})$ *and* $(\frac{b}{4}, \frac{b}{4}, \frac{b}{2})$.

The proof of Theorem 2 appears in §4. We will also prove the following.

Theorem 3 *The centre of the simplex* $(\frac{b}{3}, \frac{b}{3}, \frac{b}{3})$ *is a local minimum for* $\dim : \Delta_b \to \mathbb{R}$ *for b sufficiently large.*

The proof of Theorem 3 is presented in §5. The method of proof uses explicit bounds on the Selberg zeta function $Z_{\underline{b}}(s)$ which appear in §9.

Even with the use of our more efficient algorithm to plot $\dim : \Delta_b \to \mathbb{R}$ for smaller values of b, the plots still seemed to support the conjecture that the four critical points from (1) and (2) of Theorem 3 are local minima. However, in contrast to these results and Theorem 3, we expect that $(\frac{b}{3}, \frac{b}{3}, \frac{b}{3})$ is a local maximum for b sufficiently small (see comments in §10 for some heuristic justification).

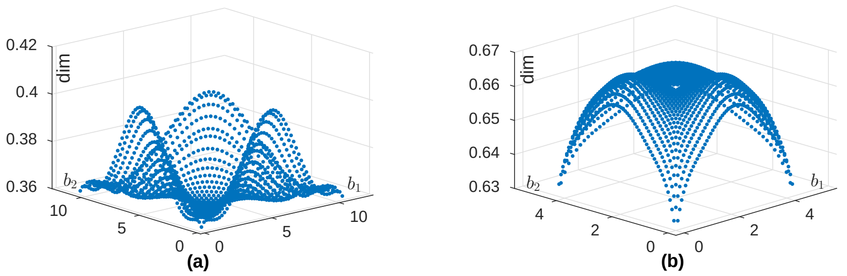

Fig. 3 (a) The plot of dimension when b is large ($b = 11$); (b) The plot of dimension when b is smaller ($b = 4.5$).

2 Hyperbolic Geometry

A Fuchsian group Γ is a discrete subgroup of the isometries $\mathrm{Isom}(\mathbb{H}^2)$ of the hyperbolic plane $\mathbb{H}^2 = \{z = x+iy : y > 0\}$ with respect to the Poincaré metric

$$ds^2 = \frac{dx^2+dy^2}{y^2}.$$

The orientation preserving isometries in $\mathrm{Isom}(\mathbb{H}^2)$ are linear fractional transformations $z \mapsto \frac{az+b}{cz+d}$ with $ad-bc = 1$. It is often convenient to identify these with matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$. (The orientation reversing isometries in $\mathrm{Isom}(\mathbb{H}^2)$ are linear fractional transformations $z \mapsto \frac{az+b}{cz+d}$ with $ad-bc = -1$ and correspond to matrices with determinant -1.)

Definition 1 The limit set $\Lambda = \Lambda_\Gamma$ is the compact set of accumulation points in the Euclidean norm for $\Gamma i = \{gi : g \in \Gamma\}$.

The limit set lies in the boundary $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$.

It is sometimes convenient to use the equivalent Poincaré disk model for hyperbolic plane, where $\mathbb{D}^2 = \{z = x+iy : |z| < 1\}$, and the Poincaré metric in this case is of the form

$$ds^2 = \frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}.$$

In this model limit set lies in the boundary $\partial\mathbb{D} \cup \{\infty\}$, which is the unit circle.

In the present context the limit set Λ is a Cantor set. It is an interesting question to estimate the size of the set Λ via its Hausdorff dimension and its dependence on the surface Γ. The original approach to these problems was through the work of Patterson and Sullivan on measures on Λ. This has been considered by a number of authors in particular special cases:

Example 2 ($b_1 = b_2 = b_3$) The case of a symmetric pair of pants (i.e., $b_1 = b_2 = b_3 = b/3$) was studied in [9]. In the case that b tends to zero or b tends to infinity we can deduce from a result of McMullen [9] an asymptotic estimate for the dimension at the central point:

1. there exists $c_1 > 0$ such that $\dim(\frac{b}{3}, \frac{b}{3}, \frac{b}{3}) \sim \frac{c_1}{b}$ as $b \to +\infty$; and
2. there exists $c_2 > 0$ such that $\dim(\frac{b}{3}, \frac{b}{3}, \frac{b}{3}) \sim 1 - c_2 b$ as $b \to 0$.

Numerically, we can estimate the first constant as $c_1 = 0.6924\ldots$.

Example 3 *($b_1 = 0$ and $b_2 = b_3$)* The case of the Hecke group $\Gamma_\varepsilon = \langle -\frac{1}{z}, z+2+\varepsilon \rangle$, where $\varepsilon > 0$ (i.e., $b_1 = 0$ and $b_2 = b_3 = \frac{1}{2}b$) was studied by Phillips–Sarnak [10] and Pignataro–Sullivan [11] who established an asymptotic formulae for the dimension of the limit set of the form $1 - \dim(0, \frac{b}{2}, \frac{b}{2}) \sim \frac{b}{2}$ as $b \to 0$.

3 Selberg zeta function

We can consider the pair of pants $V = \mathbb{H}^2/\Gamma$ where Γ is the convex cocompact group generated by reflections in three disjoint circles. For each conjugacy class in Γ we can associate a unique closed geodesic γ, and we let $\lambda(\gamma) = \lambda_{\underline{b}}(\gamma)$ denote its length. By analogy with the familiar presentation of the Selberg zeta function for compact manifolds without boundary we can define the following.

Definition 2 We can formally define the Selberg zeta function by

$$Z_{\underline{b}}(s) = \prod_m \prod_\gamma (1 - e^{-(s+m)\lambda_{\underline{b}}(\gamma)})$$

where the first product is taken over closed geodesics γ of length $\lambda_{\underline{b}}(\gamma)$.

There is a well known bijection between closed geodesics and *cyclically reduced words* $\underline{i} = (i_1, \cdots, i_{2n}) \in \{1,2,3\}^{2n}$, for $n \geq 1$, with

1. $i_k \neq i_{k+1}$ for $1 \leq k \leq 2n-1$, and
2. $i_1 \neq i_{2n}$.

Namely, to any closed geodesic on a pair of pants one can associate a periodic cutting sequence, which defines a conjugacy class in $\pi_1(V)$. We shall denote the composition of $2n$-reflections with respect to the geodesics $\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_{2n}}$ by $R_{\underline{i}} = R_{i_1} \cdots R_{i_{2n}}$.

Theorem 4 **(after Ruelle)** *The infinite product $Z_{\underline{b}}(s)$ converges to a non-zero analytic function for $\Re(s) > \dim(\underline{b})$ and extends as an analytic function to $\mathbb{C}$ with a simple zero at $s = \dim(\underline{b})$. Furthermore, we can expand*

$$Z(s) = 1 + \sum_{n=1}^{\infty} a_{2n}(s), \tag{2}$$

where

1. *$a_{2n}(s)$ depends only on the lengths $\lambda(\gamma)$ of closed geodesics γ corresponding to cyclically reduced words of length $2n$; and*
2. *There exists $C > 0$ and $0 < \theta < 1$ such that $|a_{2n}(s)| \leq C^n \theta^{n^2}$.*

The original proof was in the context of Anosov flows, which would correspond to the geodesic flows on closed surfaces. The geodesic flow on (the recurrent part of) a pair of pants is a more general hyperbolic flow, nevertheless, the method of proof easily adapts. Since the proof of this theorem is a little technical we will postpone it until §9, including explicit estimates on $C > 0$ and $0 < \theta < 1$ and showing their dependence on b. However, for the present it suffices to show how the result above provides a proof of the first part of Theorem 1. We begin by considering the values of the dimension where $0 < b_1, b_2, b_3 < 1$, i.e., $\underline{b} \in \mathrm{int}(\Delta)$.

We now give a simple proof of Part 1 of Theorem 1 using the Selberg zeta function and Theorem 4.

Proof (of Part 1 of Theorem 1) By Theorem 4 we have that

1. The function $\mathbb{R} \times \Delta \ni (t, \underline{b}) \mapsto Z(t, \underline{b})$ is real analytic;
2. For each $\underline{b}$ we have that $\frac{\partial Z(t,\underline{b})}{\partial t}|_{t=0} \neq 0$; and
3. The Hausdorff dimension $\delta = \dim(\Lambda_{\underline{b}})$ of $\Lambda_{\underline{b}}$ is the unique positive solution to the equation $Z_{\underline{b}}(\delta) = 0$.

It is an immediate consequence of the analyticity of the zeta function and the implicit function theorem that we have the result.

4 Critical Points

In this section we will prove Theorem 2. Our approach is completely elementary, making use of the analyticity of $\dim(\underline{b})$ and the symmetry in the coordinate space.

We begin by giving some simple lemmas that will be used in the proof.

Lemma 1 *For* $\underline{b} = (b_1, b_2, b_3) \in \Delta$*:*

1. $\dim(b_1, b_2, b_3) = \dim(b_2, b_3, b_1) = \dim(b_3, b_1, b_2)$
2. $\dim(b_1, b_2, b_3) = \dim(b_1, b_3, b_2)$

Proof We can see from the geometry that the dimension is invariant under permutations of the coordinates and the result follows by symmetry.

The following general results are elementary exercises in calculus.

Lemma 2 *Let* $f\colon \mathbb{R}^n \to \mathbb{R}$ *be a real analytic function.*

1. Assume there exists a neighbourhood U *of a point* x_0 *and non-zero vectors* $v_1, v_2 \in \mathbb{R}^n$ *with* $x_0 + v_1, x_0 + v_2 \in U$ *and such that* $f(x_0 + \varepsilon v_1) = f(x_0 + \varepsilon v_2)$ *for* $\varepsilon > 0$ *small enough. Then the directional derivative* $D_{(v_1 - v_2)} f = \lim_{\delta \to 0} \frac{f(x + \delta(v_1 - v_2))}{\delta}$ *vanishes at* x_0.
2. Assume that there exist n *linearly independent vectors* $w_1, \ldots, w_n$ *such that directional derivatives in these directions are degenerate at the point* x_0*. Then* x_0 *is a critical point.*

In order to understand the other critical points, we recall the following simple lemma, relating values along lines from the centre of the simplex to the corners. It appears as Proposition 4.2, in [6].

Lemma 3 (Gittins et al) *Let* $2(b_1 + b_2) = b$*. Then*

$$\dim(2b_1, b_2, b_2) = \dim(2b_2, b_1, b_1).$$

For the reader's convenience we provide a short proof in §8.

We now turn to the proof of Theorem 2 and begin with the proof of part (1). Observe that the dimension of the limit set is invariant with respect to any permutation on coordinates. In particular, this implies that for $\varepsilon > 0$ sufficiently small,

$$\dim_H\left(\frac{b+\varepsilon}{3}, \frac{b+\varepsilon}{3}, \frac{b-2\varepsilon}{3}\right) = \dim_H\left(\frac{b-2\varepsilon}{3}, \frac{b+\varepsilon}{3}, \frac{b+\varepsilon}{3}\right) = \dim_H\left(\frac{b+\varepsilon}{3}, \frac{b-2\varepsilon}{3}, \frac{b+\varepsilon}{3}\right).$$

Then any pair of the three vectors $v_1 = (1, 1, -2)$, $v_2 = (-2, 1, 1)$, $v_3 = (1, -2, 1)$ satisfy the conditions of Lemma 2 where $n = 3$ and $f = \dim$. Since $w_1 := v_1 - v_2 = (3, 0, -3)$

and $w_2 := v_1 - v_3 = (0,3,-3)$ are independent the lemma follows from the second part of Lemma 2. This completes the proof of part (1) of the theorem.

We now turn to the proof of part (2). We can consider $\underline{b} = (\frac{b}{2}, \frac{b}{4}, \frac{b}{4})$, the other cases being similar. We first prove that at these points the derivative is zero along the line to the centre. By Lemma 3

$$\dim\Big(\frac{b+\varepsilon}{2}, \frac{b-2\varepsilon}{4}, \frac{b-2\varepsilon}{4}\Big) = \dim\Big(\frac{b-2\varepsilon}{2}, \frac{b+2\varepsilon}{4}, \frac{b+2\varepsilon}{4}\Big).$$

Thus by Lemma 1

$$D_{\bar{v}_1} \dim\Big(\frac{b}{2}, \frac{b}{4}, \frac{b}{4}\Big) = 0, \quad \text{where } v_1 :\ = \Big(1, -\frac{1}{2}, -\frac{1}{2}\Big).$$

We next show that at these points the derivative is zero in the orthogonal direction to the median. By a symmetry argument, based on dim being invariant under reflecting in the median of the simplex Δ,

$$\dim\Big(\frac{b}{2}, \frac{b}{4} + \varepsilon, \frac{b}{4} - \varepsilon\Big) = \dim\Big(\frac{b}{2}, \frac{b}{4} - \varepsilon, \frac{b}{4} + \varepsilon\Big).$$

Thus by Part 1 of Lemma 2

$$D_{\bar{v}_2} \dim\Big(\frac{b}{2}, \frac{b}{4}, \frac{b}{4}\Big) = 0, \quad \text{where } v_2 :\ = (0,2,-2).$$

Two vectors v_1 and v_2 satisfy the conditions of part 2 of Lemma 2. This completes the proof of part (2).

We now proceed to the proof of part (3). It suffices to consider $\underline{b} = (\frac{2b}{3}, \frac{b}{6}, \frac{b}{6})$, the other cases being similar. We first prove that at these points the derivative is zero along the line to the centre. By Lemma 3

$$\dim\Big(\frac{b+t}{2}, \frac{b-2t}{4}, \frac{b-2t}{4}\Big) = \dim\Big(\frac{b-2t}{2}, \frac{b+2t}{4}, \frac{b+2t}{4}\Big).$$

Thus when we differentiate at $t = \frac{b}{3}$ we have that

$$D_{\bar{v}_1} \dim\left(\frac{2b}{3}, \frac{b}{6}, \frac{b}{6}\right) = D_{\bar{v}_1} \dim\left(\frac{b}{3}, \frac{b}{3}, \frac{b}{3}\right) = 0, \quad \text{where } v_1 :\ = \Big(1, -\frac{1}{2}, -\frac{1}{2}\Big).$$

We next show that at these points the derivative is zero in the orthogonal direction to the line to the centre. By a symmetry argument, based on the invariance of dim under reflecting in the median we have that

$$\dim\Big(\frac{2d}{3}, \frac{d}{6} + \varepsilon, \frac{d}{6} - \varepsilon\Big) = \dim\Big(\frac{2d}{3}, \frac{d}{6} - \varepsilon, \frac{d}{6} + \varepsilon\Big).$$

Thus by Part 1 of Lemma 2

$$D_{\bar{v}_2}\Big(\frac{2d}{3}, \frac{d}{6}, \frac{d}{6}\Big) = 0, \quad \text{where } v_2 :\ = (0,2,-2).$$

The two vectors v_1 and v_2 satisfy the conditions of part 2 of Lemma 2, and the result follows. This completes the proof of part (3).

5 Proof of Theorem 2

Recall that we want to show that the point $\left(\frac{b}{3}, \frac{b}{3}, \frac{b}{3}\right)$ is a local minimum providing b is sufficiently large. To achieve this we will show that for large b a natural function for which $\dim(\underline{b})$ is a solution can be approximated by a particularly simple function for which the corresponding zero is easily seen to be a local minimum. The function we choose is the Selberg zeta function and the content of the proof is to show that this approximation is uniform in a suitable sense.

Lemma 4 *We can write*

$$Z_{\underline{b}}(s) = 1 + a_2(s,\underline{b}) + \psi(s,\underline{b})$$

where:

1. $a_2(s,\underline{b}) = -2(e^{-sb_1} + e^{-sb_2} + e^{-sb_3})$ *when* $\underline{b} = (b_1, b_2, b_3)$ *; and*
2. $\psi\left(\frac{s}{b}, \underline{b}\right)$ *tends to zero uniformly as* $b \to \infty$*,*
   (a) *for* s *in a fixed complex neighbourhood* $[0,1] \subset U \subset \mathbb{C}$ *and*
   (b) $\underline{x} = (x_1, x_2, x_3) := \underline{b}/b$ *in a compact region* $K \subset \Delta$ *of the standard simplex*

$$\Delta = \{\underline{\xi} = (\xi_1, \xi_2, \xi_3) : \xi_1 + \xi_2 + \xi_3 = 1\}.$$

Proof Recall that by (2) we can write

$$Z_{\underline{b}}(s) = 1 + a_2(s,\underline{b}) + \sum_{n=2}^{\infty} a_{2n}(s,\underline{b})$$

and so we denote

$$\psi(s,\underline{b}) := \sum_{n=2}^{\infty} a_{2n}(s).$$

By construction,

$$a_2(s,\underline{b}) = -2(e^{-sb_1} + e^{-sb_2} + e^{-sb_3}).$$

By the bounds on the zeta function in Theorem 4 (see also §9) we know there exists $C > 0$ and $0 < \theta < 1$ such that $|a_{2n}(s)| \leq C^n \theta^{n^2}$, where C and θ can be chosen independent of $s \in U$ and $\underline{x} \in \Delta$. However, we use the more detailed description of these bounds given in §9 to provide the more explicit estimate on the dependence on $C = C(\underline{b})$ and $\theta = \theta(\underline{b})$ that are required. More precisely, we see that for large b we can bound $\theta = O(1/b)$ and $C = O(1/b)$. In particular, we have that

$$|\psi(s,\underline{b})| = O\left(\sum_{n=2}^{\infty} C^n \theta^{n^2}\right) = O(1/b^4)$$

for all $s \in U$ and $x = \underline{b}/b \in \Delta$. This suffices to prove the lemma.

Writing $0 < x_i = b_i/b < 1$ with $x_1 + x_2 + x_3 = 1$, as above, we can then write

$$a_2(s/b, b) = -2\sum_{i=1}^{3} e^{-sx_i}.$$

It is easy to see by convexity that if d_0 is the solution to

$$e^{-d_0/3} + e^{-d_0/3} + e^{-d_0/3} = \frac{1}{2}$$

then for (nearby) $(x_1,x_2,x_3) \in \Delta - \left\{\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right\}$ we have that $e^{-d_0x_1}+e^{-d_0x_2}+e^{-d_0x_3} > \frac{1}{2}$. In particular, the solution $d_1 = d_1(x_1,x_2,x_3) > 0$ to $e^{-d_1x_1}+e^{-d_1x_2}+e^{-d_1x_3} = \frac{1}{2}$ therefore satisfies $d_1 > d_0$ and we deduce that $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ is a local minimum (with non-zero Hessian) for $1+a_2(s,\underline{b})$. By choosing b sufficiently large and applying Lemma 4 we complete the proof of Theorem 2.

6 Boundary behaviour

We begin with a few observations on the behaviour of the dimension $\dim(\underline{b})$ as $\underline{b}$ approaches the boundary of Δ_b.

Proposition 1 (after Beardon [1]) *We have that if $\underline{b} \in \partial\Delta_b$ then $\frac{1}{2} \le \dim(\underline{b}) < 1$.*

Proof The observation that if $\underline{b} \in \partial\Delta_b$ then $\dim(\underline{b}) \ge \frac{1}{2}$ is due to Beardon, although it also possible to give an alternative proof by inducing on the boundary. The observation that $\underline{b} \in \partial\Delta_b$ then $\dim(\underline{b}) < 1$ is easily seen by introducing an extra circle into any gap on the boundary of the Poincaré disk and observing that the dimension of the limit set corresponding to reflections in this larger collection of circles would necessarily be strictly larger.

We let $L_i = \{\underline{b} = (b_1,b_2,b_3) \in \Delta_b : b_i = 0 \text{ and } 0 < b_j < b \text{ for } j \neq i\}$, for $i = 1,2,3$, denote the three one-dimensional boundary segments.

The next theorem is a more precise statement of Theorem 1, Part 3.

Theorem 5 *For each $i = 1,2,3$ we have that $L_i \ni \underline{b} \mapsto \dim(\underline{b})$ is analytic.*

We now outline a proof of Theorem 5 using the viewpoint we have developed. Without loss of generality, we can use the upper half plane model and consider the limit set Λ corresponding to the maps (on the extended real line)

$$\begin{aligned} S &: z \to -\frac{1}{z} \\ T_a &: z \to a - z \\ T_{-c} &: z \to -c - z \end{aligned}$$

(where $a,c > 2$). These are three transformations given by: reflection in the unit circle, and the reflections in the lines $\Re(z) = a/2$ and $\Re(z) = -c/2$, respectively. Up to a Möbius transformation, this is the same limit set as for the pair of pants corresponding to $b_1 = 0$, say. Moreover, we can write $a = a(b_2)$ and $c = c(b_3)$ where these clearly have an analytic dependence on $b_2, b_3 > 0$.

The limit set Λ generated by these three reflections will also have the same dimension $\dim(\underline{b})$ as the limit set Λ_0 generated by the countable family of transformations given by inducing (with repeat to the reflection S in the unit circle). More precisely, we can denote

$$\begin{aligned} U_1^{(n)}(z) &:= S \circ (T_a \circ T_{-c})^n(z) = \frac{-1}{z+n(a+c)} \\ U_2^{(n)}(z) &:= S \circ (T_a \circ T_{-c})^n \circ T_a(z) = \frac{1}{z-a-n(a+c)} \\ U_3^{(n)}(z) &:= S \circ T_c \circ (T_a \circ T_{-c})^n(z) = \frac{1}{z+b+n(a+c)} \end{aligned}$$

for $n \geq 1$, and define

$$\Lambda_0 = \left\{ \lim_{l \to +\infty} U_{i_1}^{(n_1)} \circ \cdots \circ U_{i_l}^{(n_l)}(0) : \text{where } n_1, \cdots, n_l \geq 1 \text{ and } i_1, \cdots, i_l \in \{1,2,3\} \right\}$$

It is easy to see that these maps are strictly contracting (i.e., $\max_i \sup_{n,z\in\Lambda} |(U_i^{(n)'}(z)| < 1$). Moreover, if we choose $0 < \varepsilon < \frac{\min\{a,c\}}{2} - 1$ we observe that if $B(0,1) \subset \mathbb{C}$ denotes the unit ball then $\overline{U_i^{(n)}(B(0,1))} \subset B(0,1)$ for $n \geq 1$ and $i \in \{1,2,3\}$.

To show the analyticity of the dimension it is convenient to characterize it in terms of the following operator.

Lemma 5 *If $\mathscr{B}$ denotes the Banach space of bounded analytic functions on $B(0,1)$ with the supremum norm then the operator $\mathscr{L}_t : \mathscr{B} \to \mathscr{B}$ defined by*

$$\mathscr{L}_t w(z) = \sum_{n=1}^{\infty} \sum_{i=1}^{3} \left(U_i^{(n)'}(z) \right)^t w\left(U_i^{(n)}(z) \right)$$

is a nuclear operator.

We refer to §9 for the definition of nuclear operator. The operator is well defined for $t > \frac{1}{2}$. Moreover, the operator has an isolated maximal positive eigenvalue $e^{P(t)}$ (cf. [13]), where P is the pressure function, $P(t) = \log \lambda_t$, where λ_t is the maximal eigenvalue for $\mathscr{L}_t$. Using analytic perturbation theory, we deduce that the map $(t, b_2, b_3) \mapsto (t, a, c) \mapsto P(t)$ is analytic. Since the dimension d is characterized by $P(d) = 0$, then since one can readily check $\frac{\partial}{\partial t} P(t) \neq 0$ it follows from the implicit function theorem that the dimension $\dim(\underline{b})$ depends analytically on (b_2, b_3). This completes the proof of Theorem 5.

Now we explain a proof of Theorem 1, part (2). We can also use the construction above to see that $\dim : \Delta_{\underline{b}} \to \mathbb{R}$ extends continuously to the boundary. Let us consider $\underline{b} = (b_1, b_2, b_3) \in \Delta_b$ and assume for definiteness that $b_1 \to 0$ and b_2, b_3 remain bounded away from zero. This corresponds to T_a and T_b being replaced by reflections in (large) circles. Furthermore, although the maps $T_a \circ T_{-c}$ are no longer translations, the corresponding induced maps $U_i^{(n)}$ still satisfy $\overline{U_i^{(n)} B(0,1)} \subset B(0,1)$ and have an analytic dependence on $\underline{b}$. The proof of the continuity part of Theorem 1 follows from this.

7 The efficiency of the algorithm

In this section we will compare the two algorithms used to compute the dimension in a number of examples. The first method is that of McMullen, as used in the article [6].

McMullen's approach The zeta function $Z(s)$ can be approximated by determinants $\det(I - B_n(s))$, where $B_n(s)$ is a finite matrix indexed by allowed strings of generators $R_{i_0} R_{i_1} \ldots R_{i_{n-1}}$, say. The entries

1. vanish (equal to zero), if the row is indexed by $R_{i_0} \ldots R_{i_{n-1}}$ and the column is indexed by $R_{j_0} \ldots R_{j_{n-1}}$ and $R_{i_1} \ldots R_{i_{n-1}} \neq R_{j_0} \ldots R_{j_{n-2}}$
2. are equal to $(R_{i_0} \ldots R_{i_{n-1}})'(x_{i_0 \ldots i_{n-1}})^{-s}$, if the row is indexed by $R_{i_0} \ldots R_{i_{n-1}}$ and the column is indexed by $R_{j_0} \ldots R_{j_{n-1}}$ and $R_{i_1} \ldots R_{i_{n-1}} = R_{j_0} \ldots R_{j_{n-2}}$, where $x_{i_0 \ldots i_{n-1}}$ is the repelling fixed point on $\partial \mathbb{D}^2$ for $R_{i_0} \ldots R_{i_{n-1}}$.

This is an implementation of the approach of McMullen in [9]. The approach in [9] was based on characterising the dimension in terms of the largest eigenvalue of the transfer operator with the objective of numerically computing the dimension. The method we presented leads to better approximations in the case of "moderate hyperbolicity" corresponding large b; however for smaller values of b this advantage is often lost.

The zeta function approach The second method is to use the Selberg zeta function approach, as described in the present article. More precisely, we compute approximations to $Z(s)$ by truncations of the series in (2) to give expressions in terms of finitely many closed geodesics.

In the tables below, "time" refers to computational time (in milliseconds) obtained when using the Matlab environment on a laptop with Intel Core 2 Duo processor. In Table 1, we show the estimates for $b_1 = b_2 = b_3 = \frac{3}{2}$ (and $b = \frac{9}{2}$) and for $b_1 = b_2 = b_3 = 4$ (and $b = 12$) using the McMullen approach.

b	$\underline{b}$	$N = 14$		$N = 16$		$N = 18$	
		dim	time	dim	time	dim	time
4.5	(1.5,1.5,1.5)	0.667462	5.905	0.667307	26.569	0.667254	118.082
12	(4,4,4)	0.33455	5.85	0.334543	26.07	0.334542	116.209

Table 1 Estimates for $b_1 = b_2 = b_3 = \frac{3}{2}$ (and $b = \frac{9}{2}$) and for $b_1 = b_2 = b_3 = 4$ (and $b = 12$), using the first algorithm.

In Table 2 we show estimates for the same values, but this time using this new method. The empirical improvement in the estimates is easy to observe with better convergence in a shorter time.

b	$\underline{b}$	$N = 14$		$N = 16$		$N = 18$	
		dim	time	dim	time	dim	time
4.5	(1.5,1.5,1.5)	0.668836	2.487	0.667232	9.042	0.667232	41.513
12	(4,4,4)	0.334541	2.312	0.334541	8.921	0.334541	35.312

Table 2 Estimates for $b_1 = b_2 = b_3 = \frac{3}{2}$ (and $b = \frac{9}{2}$) and for $b_1 = b_2 = b_3 = 4$ (and $b = 12$) using the Selberg zeta function method.

In these examples we are computing the dimension at the centre of the simplex. If we consider the estimates on the dimension at points which are closer to the boundary then the situation is slightly different.

In Table 3, we show the estimates for $b_1 = b_2 = 0.5$, $b_3 = 3.5$ (and $b = \frac{9}{2}$) and for $b_1 = b_2 = 0.5$, $b_3 = 11$ (and $b = 12$) using the McMullen method.

b	$\underline{b}$	$N = 14$		$N = 16$		$N = 18$	
		dim	time	dim	time	dim	time
4.5	(0.5,0.5,3.5)	0.690859	5.859	0.682198	26.267	0.676990	117.638
12	(0.5,0.5,11)	0.439097	5.827	0.425927	26.284	0.401183	116.020

Table 3 Estimates for $b_1 = b_2 = 0.5$, $b_3 = 3.5$ (and $b = \frac{9}{2}$) and for $b_1 = b_2 = 0.5$, $b_3 = 11$ (and $b = 12$) using the McMullen method.

Finally, in Table 4 we show estimates for these same values, but this time using this new method. Empirically we obtain poor estimates on the dimension that are strictly greater than 1, which is impossible.

		dim	time	dim	time	dim	time
b	$\underline{b}$	$N=14$		$N=16$		$N=18$	
4.5	(0.5,0.5,3.5)	1.89414	48.721	1.762531	219.539	1.502638	733.793
12	(0.5,0.5,11)	1.892794	48.137	1.76106	212.104	1.499075	728.036

Table 4 Estimates for $b_1 = b_2 = 0.5$, $b_3 = 3.5$ (and $b = \frac{9}{2}$) and for $b_1 = b_2 = 0.5$, $b_3 = 11$ (and $b = 12$) using the Selberg zeta function method.

In particular, we see that Selberg zeta function algorithm appears more efficient in the case of $\underline{b}$ nearer the centre of the simplex. In this case the empirical approximations work particularly well and convergence appears faster than with McMullen algorithm. On the other hand, when (b_1, b_2, b_3) is close to the boundary of the simplex, the Selberg Zeta function algorithm is not applicable, as the group is not hyperbolic enough to achieve convergence. Indeed, if we consider the terms $a_n(s)$ for different values of the exponent s we notice that the coefficients decrease very slow, even with 18 matrices (see Table 5).

s	a_2	a_4	a_6	a_8	a_{10}	a_{12}	a_{14}	a_{16}	a_{18}
-0.05	-12.567	62.156	-174.142	320.462	-419.280	409.496	-308.455	183.393	-87.5483
0.05	-9.769	41.354	-103.342	173.389	-209.591	190.774	-134.735	75.436	-34.022
0.15	-8.498	32.653	-75.931	120.470	-139.242	122.168	-83.676	45.650	-2.0137
0.25	-7.765	27.704	-60.512	90.943	-100.205	84.217	-55.460	29.174	-12.436
0.35	-7.232	24.169	-49.68	70.524	-73.616	58.755	-36.814	18.454	-7.505
0.45	-6.783	21.307	-41.235	55.195	-54.398	41.037	-24.327	11.546	-4.449
0.55	-6.379	18.856	-34.363	43.339	-40.268	28.652	-16.027	7.181	-2.613
0.65	-6.004	16.710	-28.678	34.070	-29.824	19.998	-10.544	4.454	-1.528
0.75	-5.653	14.816	-23.947	26.794	-22.093	13.955	-6.931	2.759	-0.892
0.85	-5.324	13.139	-20.0	21.08	-16.367	9.737	-4.555	1.708	-0.520
0.95	-5.014	11.653	-16.704	16.578	-12.125	6.793	-2.993	1.057	-0.303
1.05	-4.722	10.335	-13.953	13.041	-8.982	4.740	-1.967	0.654	-0.177

Table 5 Coefficients of $\zeta(s,1)$ for $b_1 = b_2 = 0.7$ and $b_3 = 10.6$ demonstrate poor convergence of the Selberg zeta function method.

8 Proof of Lemma 3

In this section we recall the proof of Lemma 3, corresponding to Proposition 4.2 of [6].

Proof Denote by γ_1, γ_2 and γ_3 the original three disjoint geodesics. Without loss of generality, we may assume that one these, γ_1, say, is a diameter in the Poincaré disk model and $\rho(\gamma_1, \gamma_2) = \rho(\gamma_1, \gamma_3) = b_2$, $\rho(\gamma_2, \gamma_3) = 2b_1$. Consider the two geodesics that are images of γ_2 and γ_3 with respect to the reflection in R_1 in γ_1, and denote these $\gamma_4 := R_1\gamma_2$ and $\gamma_5 := R_1\gamma_3$. The free group generated by $R_{\gamma_2}, R_{\gamma_3}, R_{\gamma_4}, R_{\gamma_5}$ has index 2 in the original group and is therefore a normal subgroup:

$$\langle R_{\gamma_2}, R_{\gamma_3}, R_{\gamma_4}, R_{\gamma_5} \rangle \triangleleft \langle R_{\gamma_1}, R_{\gamma_2}, R_{\gamma_3} \rangle$$

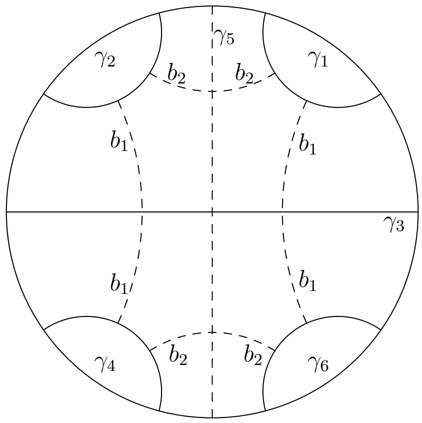

Fig. 4 Proof of Lemma 3. Original geodesics $\gamma_1, \gamma_2, \gamma_3$ and supplementary geodesics $\gamma_4, \gamma_5, \gamma_6$.

Indeed, by direct calculation, $R_{\gamma_1} R_{\gamma_2} R_{\gamma_1} = R_{\gamma_4}$ and $R_{\gamma_1} R_{\gamma_3} R_{\gamma_1} = R_{\gamma_5}$. Hence the dimensions of the associated limit sets are equal. Now observe that

$$\rho(\gamma_1, \gamma_2) = \rho(\gamma_1, \gamma_4) = \rho(\gamma_1, \gamma_3) = \rho(\gamma_1, \gamma_5) = b_2$$

and, moreover, by construction,

$$\rho(\gamma_2, \gamma_4) = \rho(\gamma_3, \gamma_5) = 2b_2$$

Finally, consider a sixth geodesic γ_6, that is taken to be the diameter in the Poincaré disk model, orthogonal to γ_1. Then $\rho(\gamma_6, \gamma_5) = \rho(\gamma_6, \gamma_3) = b_1$. Thus the group $\langle R_6, R_5, R_3 \rangle$ is generated by reflections with respect to the geodesics $\gamma_6, \gamma_5, \gamma_3$ with pairwise distances b_1, b_1, and $2b_2$. By the same argument as above, $\langle R_{\gamma_2}, R_{\gamma_3}, R_{\gamma_4}, R_{\gamma_5} \rangle$ is its normal subgroup of index 2 and therefore the dimensions of the associated limit sets are equal.

9 Additional bounds for Theorem 4

The convergence of the series (2.2) in Theorem 4 follows from general estimates of Ruelle and Fried, based on earlier ideas of Grothendieck. They can be formulated in terms of a family of bounded linear operators $\mathscr{L}_s : \mathscr{A} \to \mathscr{A}$ on a Banach space $\mathscr{A}$. We begin with a general definition

Definition 3 We say that an operator $T : \mathscr{B} \to \mathscr{B}$ on a Banach space $\mathscr{B}$ is *nuclear* if there exist

1. a sequence of vectors $w_n \in \mathscr{B}$, $n \geq 1$;
2. a sequence of linear functionals $\nu_n \in \mathscr{B}^*$, $n \geq 1$; and
3. $C > 0$ and $0 < \lambda < 1$ such that $\|w_n\|_{\mathscr{B}} \|\nu_n\|_{\mathscr{B}^*} \leq C\lambda^n$,

such that we can write $T(w) = \sum_{n=1}^{\infty} w_n \nu_n(w)$, for $w \in \mathscr{B}$.

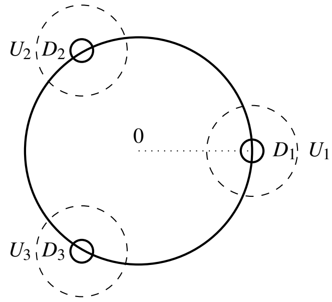

Fig. 5 Three circles of reflection on the unit circle and neighbourhoods

In the present context we want to apply this to the Banach space $\mathscr{A}$ of bounded analytic functions on the union of disjoint discs $U_j = \{z \in \mathbb{C} : |z - z_j| < r_j\} \subset \mathbb{C}$, $(j = 1,2,3)$ and $r_j > 0$ sufficiently small.

We want to consider the transfer operators $\mathscr{L}_s : \mathscr{A} \to \mathscr{A}$ defined by reflections R_j in the boundary of much smaller disks $D_j \subset U_j$ given by

$$\mathscr{L}_s w(z) = \sum_{j \neq l} |R_j'(z)|^s w(R_j z) \text{ for } z \in U_l.$$

Lemma 6 (after Ruelle [13], Grothendieck [7]) *The operators $\mathscr{L}_s$ are nuclear. Furthermore, we can denote $a_n(s) = \sum_{i_1 < \cdots < i_n} \det\left([\nu_{i_u}(w_{i_v})]_{u,v=1}^n\right)$ and write*

$$Z(s,\underline{b}) = 1 + \sum_{n=1}^{\infty} a_n(s).$$

Moreover, the following simple (and easily proved) identity is very useful in explicitly bounding $|a_n(s)|$.

Lemma 7 (Euler) *Given $0 < \lambda < 1$ we have*

$$\prod_{n=1}^{\infty} (1 + \lambda^n z) = 1 + \sum_{n=1}^{\infty} c_n(\lambda) z^n \text{ where } c_n(\lambda) = \frac{\lambda^{n(n+1)/2}}{(1-\lambda)(1-\lambda^2)\cdots(1-\lambda^n)}.$$

In particular, since the nuclearity of the operator means that $|\nu_i(w_i)| \leq C\lambda^i$, comparing the two lemmas above gives that

$$a_n(s) = \sum_{i_1 < \cdots < i_n} \det\left([\nu_{i_u}(w_{i_v})]_{u,v=1}^n\right) \leq \frac{C(s)^n \lambda^{n(n+1)/2} n^{n/2}}{(1-\lambda)(1-\lambda^2)\cdots(1-\lambda^n)} \tag{3}$$

where $n^{n/2}$ bounds the absolute value of the determinant of any $n \times n$ matrix whose entries bounded by 1.

9.1 Asymptotic bounds

In order to understand the asymptotic dependence of the bounds $C = C(\underline{b})$ and $\lambda = \lambda(\underline{b})$ on $\underline{b}$ it is convenient to map the unit disk to the upper half plane $\mathbb{H}^2 = \{z = x + iy : y > 0\}$ by $S(z) = \frac{1}{i}\frac{z-1}{z+1}$.

Furthermore, without loss of generality we can make the following simplifying assumptions:

1. The image disks $V_i = S(U_i)$ $(i = 1,2,3)$ can be chosen to remain independent of b provided only the values $(x_1,x_2,x_3) := (b_1/b, b_2/b, b_3/b)$ remain in a bounded region in the unit simplex Δ away from boundary $\partial\Delta$.
2. We can assume, after applying a Möbius map, if necessary, that V_1 is centred at 0 , V_2 is centred at 1 and V_3 is centred at -1.

The images $E_i := S(D_i)$ $(i = 1,2,3)$ of the original circle of reflection are now circle in which we now reflect in the transformed picture. We will concentrate on the case of E_1 and E_2, the others being similar. By assumption, we have that V_1 intersects the real axis at $z_1 = -r_1$ and $z_2 = r_1$ and E_2 intersects the real axis at $w_1 = 1 - r_2$, $w_2 = 1 + r_2$ (cf. Figure 6).

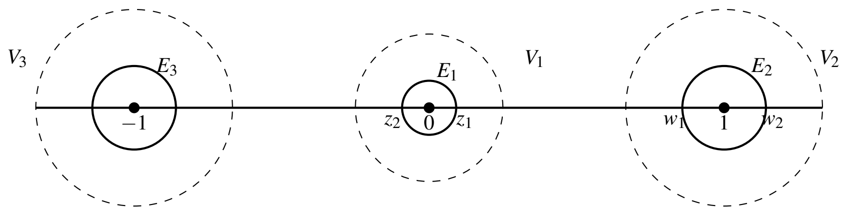

Fig. 6 Three circles and their neighbourhoods moved to the real line

We now come to a useful geometric lemma. Given $z_1 < w_1 < w_2 < z_2$ we define the cross ratio by

$$[z_1, w_1, w_2, z_2] = \frac{(z_1 - w_2)}{(z_1 - w_1)} \frac{(w_1 - z_2)}{(w_2 - z_2)}.$$

We recall the following simple result (cf. [2] §7.23).

Lemma 8 *Let L_1, L_2 be two disjoint geodesics with boundary end points z_1, z_2 and w_1, w_2. The distance b between L_1 and L_2 satisfies $[z_1, w_1, w_2, z_2] \tanh^{-2}(b/2) = 1$.*

We can apply this in the present setting to deduce the following corollary.

Corollary 1 *As $b \to +\infty$ we have that $|z_1 - z_2|^2 + |w_1 - w_2|^2 = e^{-b}\left(\frac{1}{2} + o(1)\right)$. In particular, we have that $r_1, r_2, r_3 = O(e^{-b})$*

Finally, we observe that

1. For each $i = 1,2,3$, the images $\cup_{j \neq i} R_i(V_j) \subset S(D_i) \subset V_i$ $(i = 1,2,3)$ are contained in a small disk of radius $O(e^{-b})$ and thus by definition we can choose $\lambda = O(e^{-b})$.
2. We can bound $C = \sup_{z \in D_i} |R_j'(z)| = O(\text{diam}(E_i)) = O(e^{-b})$.

9.2 Explicit bounds

This provides explicit estimates on $a_n(s)$ via the constants C and $0 < \theta < 1$. Assume as above that, after applying a Möbius map if necessary, that the circles in which we reflect in $\mathbb{H}^2$ have centres $0, 1, -1$ and radii $r_1, r_2, r_3 > 0$. Given $b_1, b_2, b_3 > 0$ we can solve for the radii using the equations

$$\tanh^{-2}(b_1) = \frac{1-(r_1-r_2)^2}{1-(r_1+r_2)^2}, \tanh^{-2}(b_2) = \frac{1-(r_2-r_3)^2}{1-(r_2+r_3)^2} \text{ and } \tanh^{-2}(b_3) = \frac{1-(r_3-r_1)^2}{1-(r_3+r_1)^2}.$$

We can then choose any value $0 < \lambda < 1$ satisfying

$$\max\left\{\frac{r_1}{1+r_3}, \frac{r_1}{1+r_2}, \frac{r_3}{2-r_2}, \frac{r_3}{2-r_3}\right\} < \lambda < 1.$$

Of course, this bound may be improved by transforming the reflections to different circles under Möbius maps

10 Final remarks

In contrast to Theorem 2, there is a suggestion that for sufficiently small values of b we have that $(\frac{b}{3}, \frac{b}{3}, \frac{b}{3})$ is actually a local maximum, rather than a local minimum. These values of b appear to be beyond the reach of numerical experiments. It is necessary to use quadruple precision calculations in order to keep control of numerical error. Every quadruple precision number is stored in 16 bytes. There are exactly $2^{2n}+2$ closed geodesics of the word length $2n$, which means that one needs $16 \cdot (2^{2n}+2)$ bytes to store their lengths. For instance, it takes about 4GiB of RAM to store the lengths of geodesics of the word length 26. Using a contemporary computer with an Intel i7 processor, and a fast Fortran code, it is possible to compute an approximation to dimension for a single value of $\underline{b}$ using periodic points up to period 26 in about 4 hours. This allows us to consider values of b as small as $b = 3\log(\sqrt{2}) \approx 1.0397\ldots$[1], where a_{26} is of the order 10^{-7}. The centre still appears to be a local minimum, where $\dim(\log(\sqrt{2}), \log(\sqrt{2}), \log(\sqrt{2})) = 0.70721640\ldots$[2], while $\dim\big(\log(\sqrt{2})+0.02, \log(\sqrt{2})-0.01, \log(\sqrt{2})-0.01\big) = 0.70721999\ldots$.

The first piece of heuristic evidence supporting local maximum conjecture is based on the following standard observation.

Lemma 9 *Providing* $\dim(\overline{b}) > \frac{1}{2}$ *we have that* $\lambda = \lambda(\underline{b}) = \dim(\overline{b})(1-\dim(\overline{b}))$ *is the smallest eigenvalue for Laplacian* $-\Delta$.

For large b, following [11] and [4] we can assume that the eigenfunction $\psi_{\underline{b}}$ associated to the eigenvalue

$$\lambda(\underline{b}) = \inf \frac{\int |\nabla f|^2}{\int |f|^2}$$

will take values close to 1 on the convex core of the pair of pants and values close to 0 on the funnels. Moreover, $|\nabla \psi_b|$ is small except on hyperbolic collars for the short geodesics. By the collar lemma the thin part is a hyperbolic cylinder of width

$$\frac{1}{2}\log\left(\frac{\cosh(b_i/2)+1}{\cosh(b_i/2)-1}\right) \sim |\log b_i|$$

[1] The value is equal to three halves of the side length of the regular hyperbolic hexagon.

[2] Perhaps curiously, this value is close to $\sqrt{\frac{1}{2}} \approx 0.7071067\ldots$.

and area $b_i/(2\sinh(b_i/2)) \to 1$. As b tends to zero we can expect that $\lambda(\underline{b})$ can be compared with

$$\frac{1}{|\log b_1|} + \frac{1}{|\log b_2|} + \frac{1}{|\log b_2|}$$

subject to $b_1 + b_2 + b_3 = 1$, which has a local maximum at $b_1 = b_2 = b_3$.

The second indication comes from the following observation on lengths of closed geodesics. If we denote by a, b the generators for the fundamental corresponding to two of the boundary components then there is a one-one correspondence between closed geodesics and cyclically reduced words. In particular, interchanging a and b maps a closed geodesic $\gamma = \gamma_b$ to a reflected geodesic $R\gamma_b$. Let us change the length of the boundary curve b_1 corresponding to a to $b/3 + \varepsilon$ and the length of the boundary curve b_2 corresponding to b to $b/3 - \varepsilon$. Clearly some geodesic curves will get shorter (for example, those containing a larger proportion of generators b in their coding) while others will get longer (for example, those containing a larger proportion of generators a in their coding) but for $\dim(\underline{b})$ to decrease for small ε one would expect that "on average" closed geodesics get longer. To this end, for each closed geodesic γ we can associate its image $R\gamma$ and consider the behaviour of the dependence of the average of their lengths $a(\varepsilon) = (l(\gamma) + l(R\gamma))/2$. We claim that function $\varepsilon \mapsto a(\varepsilon)$ has a local minimum at $\varepsilon = 0$. By symmetry we see that $\varepsilon = 0$ is a critical point. Moreover, the dependence of $l(\gamma)$ (and $l(R\gamma)$) is strictly convex by [3] and [5].

Here we comment on a few problems which are related to the themes of this note.

1. We can also consider other zeros of $Z(s)$ which correspond to zeros of zeta function other than $\dim(\underline{b})$. These are frequently referred to as resonances. There are typically many different such zeros, as is shown in the empirical work of Borthwick, but it is potentially interesting to consider the behaviour of zeros closest to $\dim(\underline{b})$.
2. We can consider the case of the groups $\Gamma = \langle R_1, \cdots, R_n \rangle$ generated by n reflections. In this case the lengths of the boundary components alone may not be sufficient to describe a point in moduli space. However, the dimension of the limit set will still depend analytically on the metric.
3. We can consider the case of higher dimensions. It we consider four circles in $\mathbb{C}$ (the reflections there is generating a group) then we can assume without loss of generality that three of them will have centers on the unit circle, and have a similar parameterisation to the pair of pants. However, the fourth circle will introduce three more real dimensions (two given by the position of the centre and the third coming from the radii). Nevertheless, the dimension of the limit set will still depend analytically on the parameters.
4. We can consider the case that the pair of pants has variable negative curvature. In this case the moduli space would be infinite dimensional, but the dimension of the limit set will still depend analytically on the metric.
5. The determinant of the Laplacian $\det : \Delta_{\underline{b}} \to \mathbb{R}$ can be defined for infinite area surfaces via the work of Efrat, generalising the approach of Sarnak. In particular, we can express it in terms of the value $\frac{\partial}{\partial s} Z(s, \underline{b})|_{s=0}$. Furthermore, the symmetry argument we used for dim still applies in this context and we can deduce the following: *The point* $\underline{b} = \left(\frac{b}{3}, \frac{b}{3}, \frac{b}{3}\right)$ *is a critical point for* $\det : \Delta_{\underline{b}} \to \mathbb{R}$.

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