A note on the shrinking sector problem for surfaces of variable negative curvature

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Abstract

Given the universal cover $\widetilde{V}$ for a compact surface V of variable negative curvature and a point $\widetilde{x}_0 \in \widetilde{V}$ we consider the set of directions $\widetilde{v} \in S_{\widetilde{x}_0}\widetilde{V}$ for which a narrow sector in the direction $\widetilde{v}$, and chosen to have unit area, contains exactly k points from the orbit of the covering group. We can consider the size of the set of such $\widetilde{v}$ in terms of the induced measure on $S_{\widetilde{x}_0}\widetilde{V}$ by any Gibbs measure for the geodesic flow. We show that for each k the size of such sets converges as the sector grows narrower and describe these limiting values. The proof involves recasting a similar result by Marklof and Vinogradov, for the particular case of surfaces of constant curvature and the volume measure, by using the strong mixing property for the geodesic flow, relative to the Gibbs measure.

In memorium, Dmitri Victorovich Anosov

1 Introduction

We begin by recalling the famous classical circle problem of Gauss for Euclidean space. In 1834, Gauss estimated the number of lattice points

$$\#\{(x,y) \in \mathbb{Z}^2 : x^2 + y^2 \leq r^2\} = \pi r^2 + O(r)$$

as r tends to infinity. In this case the principal term is trivial and the difficulty is finding a good error term. In the corresponding hyperbolic circle problem one can consider the Poincaré disk $\mathbb{D}^2 = \{z = x + iy \in \mathbb{C} : |z| < 1\}$ with the Poincaré metric d.

Let $\Gamma < \mathrm{Isom}(\mathbb{D}^2, d)$ be a uniform Fuchsian group Γ then we can consider the orbit $\Gamma 0 = \{\gamma 0 : \gamma \in \Gamma\}$ and count those points in $\Gamma 0$ with $t - s \leq d(0, \gamma 0) \leq t$ (see Figure 1(a)).

More generally, let $S_0\mathbb{D}^2$ denote the unit tangent vectors based at the origin and for $v \in S_0\mathbb{D}^2$ and $0 < \theta \leq \pi$ we denote by $S(v, \theta) \subset \mathbb{D}^2$ the fixed sector bounded by (radial) geodesics from 0 at an angle θ from v (see Figure 1(b)). In the hyperbolic circle problem it is much harder to establish the principle term than in the Euclidean case. We recall two asymptotic formulae for these hyperbolic counting problems.

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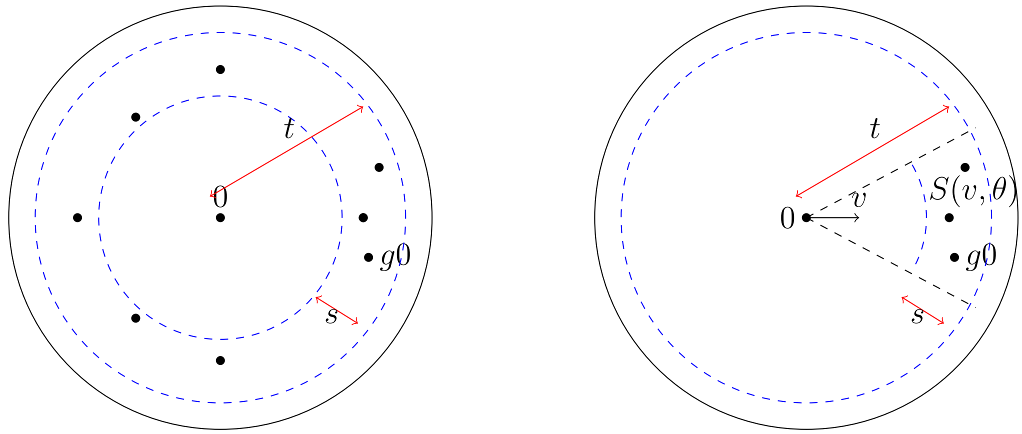

Figure 1: (a) Points in the orbit $\Gamma 0$ in the annulus $t - s \le d(0, z) \le t$; (b) Points in the orbit $\Gamma 0$ in the intersection of the annulus $t - s \le d(0, z) \le t$ and the sector $S(v, \theta)$

Theorem 1.1 (Hyperbolic Circle and Sector Problems, Huber **?** and Nicholls **?**). *There exists $C > 0$ such that for any $s > 0$ and $0 < \theta \le \pi$:*

1. $\#\{\gamma \in \Gamma : t - s \le d(0, \gamma 0) \le t\} \sim Ce^t \left(1 - e^{-s}\right)$ *as* $t \to +\infty$; *and*
2. $\#\{\gamma \in \Gamma : t - s \le d(0, \gamma 0) \le t$ *and* $\gamma 0 \in S(v, \theta)\} \sim \frac{\theta}{\pi} Ce^t \left(1 - e^{-s}\right)$ *as* $t \to +\infty$.

A variation on this problem is where the angle $\theta = \theta(t)$ of the sector is allowed to tend to zero as $t \to +\infty$. More precisely, we denote by $R(v, t) := S(v, \theta(t)) \cap \{z \in \mathbb{D}^2 : t - s \le d(0, z) < t\}$ the region which is the intersection of the annulus and the shrinking sector with angles $\pm\theta(t)$ to v (see Figure 2 (a)). We make the particular choice $\theta(t) = (8\pi(\sinh^2(t/2) - \sinh^2(s/2)))^{-1}$ so that we have the normalisation $\mathrm{Area}_{\mathbb{D}^2}(R(v, t)) = 1$. Let $\lambda := \mu_{S_0\mathbb{D}^2}$ be the normalised Haar measure on the fibre $S_0\mathbb{D}^2$, i.e., the natural volume on $S_0\mathbb{D}^2$.

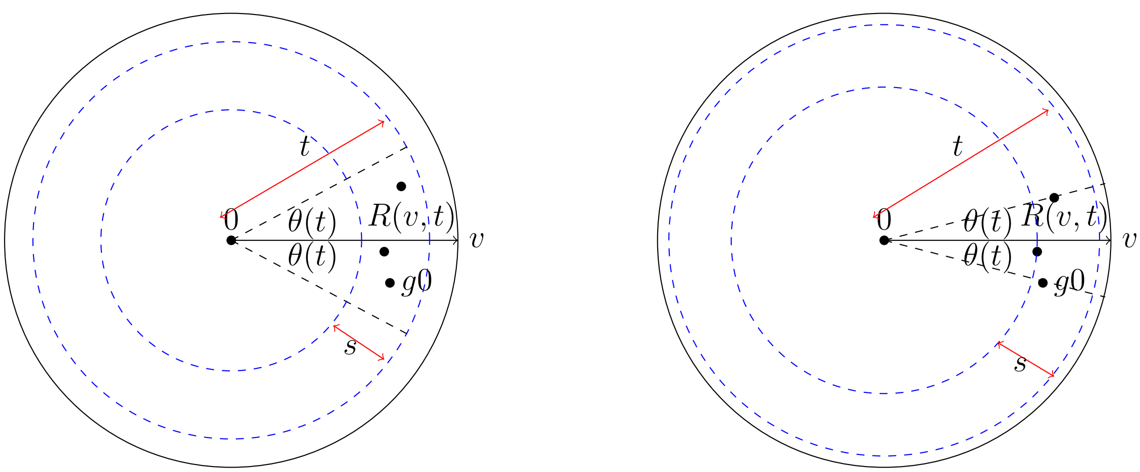

Figure 2: (a) The region $R(v, t)$ contained in a sector with angle $2\theta(t)$; (b) Since the region $R(v, t)$ has constant area it becomes thinner as t increases.

Theorem 1.2 (Shrinking Sector Theorem, Marklof-Vinogradov **?**)**.** *For each $k \in \mathbb{N}_0$ the following limit exists:*

$$\mathbb{P}_k := \lim_{t \to +\infty} \lambda \left(\left\{ v \in S_0 \mathbb{D}^2 \, : \, \#\{R(v,t) \cap \Gamma 0\} = k \right\} \right)$$

In the above, we have considered only those points that lie in annuli of width $s > 0$ (which proves convenient in the proof). In the hyperbolic setting, unlike the Euclidean setting, most of the area of the disk lies in annuli near the boundary. By letting $s \to +\infty$ and applying the dominated convergence theorem we see from the proof in §4 that the corresponding result follows where the annulus is replaced by a ball of radius t.

The proof by Marklof and Vinogradov of Theorem **??** essentially uses the equidistribution property of horocycles. In order to generalise the result to variable curvature and Gibbs measures it is more convenient to use the strong mixing property of the Gibbs measure for the geodesic flow. Of course, the equdistribution result is itself a consequence of the mixing property in the constant curvature case, so our approach should be viewed as being closer to being a reformulation of part of their proof, rather than being fundamentally new.

Acknowledgement. I am grateful to Jens Marklof for enlightening discussions on his work with I. Vinogradov, and for posing the question of generalising the same to variable curvature and Gibbs measures, during his lectures at MSRI in 2015.

2 A generalisation to variable curvature

We want to formulate a suitable generalisation of Theorem **??** to the case of *variable* negative curvature. In this more general setting, it is natural to begin with a compact surface M with variable negative curvature and let $\widetilde{M}$ be the universal cover for M. In particular, $\widetilde{M}$ is equipped with the Riemannian metric $d_{\widetilde{M}}(\cdot,\cdot)$ lifted from M. (In the special case that M has constant curvature $\kappa = -1$ then this setting reduces to the previous setting, where $\widetilde{M} = \mathbb{D}^2$ with the Poincaré metric.)

We denote by $\Gamma < \mathrm{Isom}(\widetilde{M})$ the covering group for M. Then Γ is isomorphic to $\pi_1(M)$ and $\widetilde{M}/\Gamma = M$. We let $S_{\widetilde{x}_0}\widetilde{M}$ denote the unit tangent vectors based at a point $\widetilde{x}_0 \in \widetilde{M}$ and consider a subarc $B \subset S_{\widetilde{x}_0}\widetilde{M}$ and the corresponding sector $S \subset \widetilde{M}$, i.e., the union of geodesics starting from $\widetilde{x}_0$ in the directions in B.

Given $\widetilde{x}_0 \in \widetilde{M}$, we want to consider the orbit $\Gamma\widetilde{x}_0 = \{\gamma\widetilde{x}_0 \, : \, \gamma \in \Gamma\}$.

Theorem 2.1 (Margulis **?**, Sharp **?**)**.** *There exists $h > 0$ and $C = C(S)$ such that for any $s > 0$:*

1. *$\#\{\gamma \in \Gamma \, : \, t - s \le d(\widetilde{x}_0, \gamma\widetilde{x}_0) \le t\} \sim C\left(1 - e^{-hs}\right)e^{ht}$ as $t \to +\infty$; and*
2. *There exists a probability measure $m_{\widetilde{x}_0}$ on $S_{\widetilde{x}_0}\widetilde{V}$ such that*

$$\#\{\gamma \in \Gamma \, : \, t - s \le d(\widetilde{x}_0, \gamma\widetilde{x}_0) \le t \text{ and } \gamma 0 \in S\} \sim C m_{\widetilde{x}_0}(B)\left(1 - e^{-hs}\right)e^{ht}$$

as $t \to +\infty$.

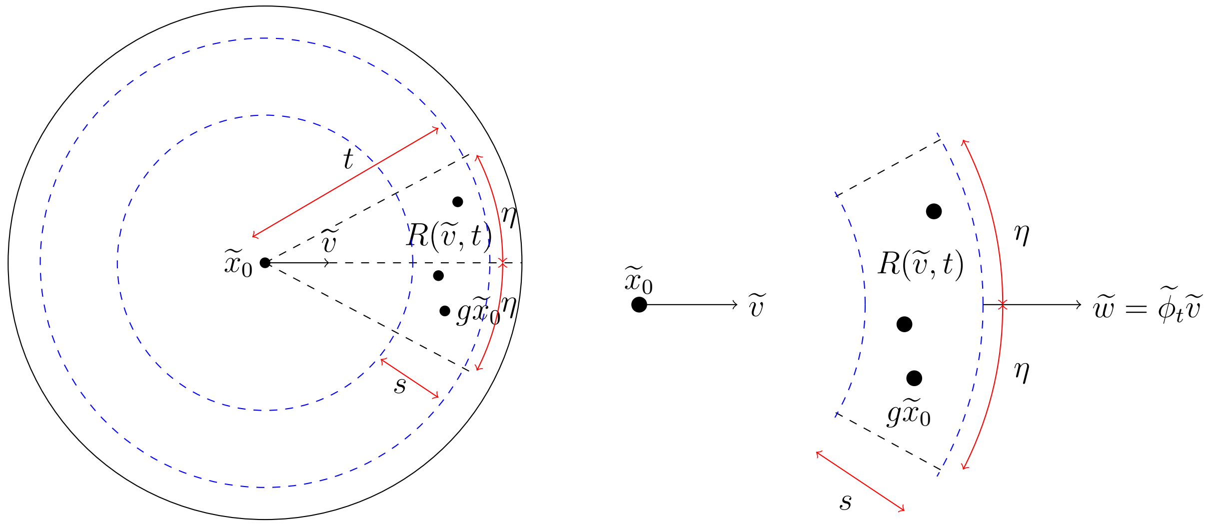

Figure 3: (a) The sector-like region $R(\widetilde{v}, t)$ in variable curvature is defined in terms of $\widetilde{v}$ and the arc length 2η; (b) The associated vector $\widetilde{w} = \widetilde{\phi}_t\widetilde{v}$.

The value h is precisely the topological entropy of the associated geodesic flow. We next want to consider a suitable analogue of the family of shrinking sectors.

In the more general setting of variable curvature one needs to define a suitable analogue of a sector associated to a vector $\widetilde{v} \in S_{\widetilde{x}_0}\widetilde{M}$. It transpires that it is more natural to define these sets in terms of the distance on the perimeter of the circle, rather than using angles in $S_{\widetilde{x}_0}\widetilde{M}$. To simplify the proofs it is also more convenient to initially study the position of the sector that lies in an annulus.

Definition 2.2. *Given $t > s > 0$ and $\widetilde{v} \in S_{\widetilde{x}_0}\widetilde{M}$ we can consider a region in $\widetilde{M}$ bounded by:*

1. *an arc in the circle of radius t centred at $\widetilde{x}_0$, of length 2η and centred at the projection to $\widetilde{M}$ of $\widetilde{w} = \widetilde{\phi}_t\widetilde{v}$;*
2. *an arc in the circle of radius $t - s$ centred at $\widetilde{x}_0$; and*
3. *two segments of geodesics emanating from $\widetilde{x}_0$,*

as in Figure 3. Finally, we choose η so that the area of the projection is equal to 1. We denote this region by $R(\widetilde{v}, t)$.

In the variable curvature case $\eta(\widetilde{v}, t)$ typically depends on $\widetilde{v}$ as well as t. This is in contrast to the constant curvature case where $\eta(\widetilde{v}, t)$ is independent of $\widetilde{v}$. Let $\lambda = \mu_{S_{\tilde{x}_0}\widetilde{M}}$ be a normalised volume on the fibre $S_{\widetilde{x}_0}\widetilde{M}$ induced from the Liouville measure μ on SM.

Theorem 2.3 (Shrinking Sector Theorem for variable curvature)**.** *For each $k \in \mathbb{N}_0$ we have that the following limit exists:*

$$\mathbb{P}_k := \lim_{t\to+\infty} \lambda\left(\left\{\widetilde{v} \in S_{\tilde{x}_0}\widetilde{M} : \#\{R(\widetilde{v}, t) \cap \Gamma\tilde{x}_0\} = k\right\}\right).$$

In the special case of a surface of constant negative curvature this reduces to Theorem **??**.

The proof of Theorem **??** is very robust. It uses only a simple geometric argument involving the hyperbolicity of the geodesic flow and the mixing property of the Liouville measure. In particular, it readily generalises to other Gibbs measures and slightly more general settings (see §4 and §5, respectively).

Remark 2.4. For each $s > 0$ exists $k_0 = k_0(s)$ such that $\mathbb{P}_k = 0$ for $k \geq k_0$. However, as $s \to +\infty$ then $k_0(s) \to +\infty$.

Remark 2.5. Theorem **??** remains true if instead of $\#\{R(\widetilde{v}, t) \cap \Gamma\tilde{x}_0\} = k$ one counts $\#\{R(\widetilde{v}, t) \cap \Gamma\widetilde{y}_0\}$ for any other point $\widetilde{y}_0 \in M$.

3 Geodesic and Anosov flows

In this section we want to introduce the more dynamical setting which will be used in proving Theorem **??**. We begin by recalling the definition of an Anosov flow.

Definition 3.1. *A C^∞ flow $\phi_t : N \to N$ on a compact manifold N is an Anosov flow if*

1. *there is a $D\phi_t$-invariant continuous splitting $TN = E^0 \oplus E^s \oplus E^u$ such that E^0 is one dimensional and tangent to the flow direction, and there exists $C, \lambda > 0$ such that $\|D\phi_t|E^s\| \leq Ce^{-\lambda t}$ and $\|D\phi_{-t}|E^u\| \leq Ce^{-\lambda t}$, for $t \geq 0$; and*
2. *$\phi_t : N \to N$ is transitive.*

Anosov flows provide the correct dynamical context, as is shown by the following classical result.

Proposition 3.2 (Anosov **?**)**.** *The geodesic flow $\phi_t : SM \to SM$ on the three dimensional unit tangent bundle SM of a compact surface M is an Anosov flow.*

We next collect together here some basic results on Anosov geodesic flows which will be of use later.

3.1 Foliations

There are several natural foliations of SM each of which are preserved by the geodesic flow.

Definition 3.3. *For each $w \in SM$ we can associate a strong stable manifold defined by*

$$W^{ss}(w) = \{w' : d(\phi_t w, \phi_t w') \to 0 : t \to +\infty\}.$$

We can similarly define for each $w \in SM$ a strong unstable manifold defined by

$$W^{su}(w) = \{w' : d(\phi_{-t} w, \phi_{-t} w') \to 0 : t \to +\infty\}.$$

The following result on the regularity of these foliations in the case of geodesic flows on negatively curved surfaces is also well known.

Lemma 3.4 (Hirsch-Pugh, **?**). *Let $\phi_t : SM \to SM$ be a geodesic flow on a compact negatively curved surface.*

1. *Each strong stable manifold $W^{ss}(w)$ is a C^∞ one dimensional immersed submanifold and the family $\mathcal{F}^{ss} = \{W^{ss}(w)\}_{w \in SM}$ of such leaves forms a C^1 foliation of the manifold M.*

2. *Each weak unstable manifold $W^{wu}(w)$ is a C^∞ two dimensional immersed submanifold and the family $\mathcal{F}^{wu} = \{W^{wu}(w)\}_{w \in SM}$ of such leaves forms a C^1 foliation of the manifold M.*

For geodesic flows, the following trivial geometric fact will be useful later.

Lemma 3.5. *For each $v \in S_{x_0}M$ the one dimensional tangent fibre $T_v(S_{x_0}M)$ doesn't coincide with any of the one dimensional fibres E_v^0, E_v^s or E_v^u.*

To formulate the next statement, it is convenient to introduce some standard notation:

Definition 3.6. *Let M be a compact surface with negative curvature with universal cover $\widetilde{M}$ and unit tangent bundle SM.*

1. *Let $\rho : SM \to M$ denote the canonical fibre projection from SM to M.*

2. *Let $\pi : \widetilde{M} \to M$ denote the canonical covering projection from $\widetilde{M}$ to M.*

The hyperbolicity of the geodesic flow means that the effect of ϕ_t on the individual fibres $S_{x_0}M$ is to stretch them and flatten them towards the weak unstable manifolds. The following is then easily seen (see Figure 4).

Lemma 3.7. *For each $\widetilde{v} \in S_{x_0}\widetilde{M}$, the subarc of the circle of radius t in part 1 of Defintion 2.2 is arbitrarily close to a pre image $\pi^{-1}W^{ss}(w)$ (where $v = \rho(\widetilde{v})$ and $w = \phi_t v$) for sufficiently large t.*

3.2 Gibbs measures on N

We now consider a standard general class of flow invariant measures for the Anosov flow $\phi_t : N \to N$.

Definition 3.8. *Given a Hölder continuous function $F : N \to \mathbb{R}$ we define the associated Gibbs measure $\mu = \mu_F$ to be the unique invariant probability measure such that*

$$h(\phi, \mu) + \int F d\mu = \sup \left\{ h(\phi, \nu) + \int F d\nu \, : \, \nu = \phi\text{-invariant probability measure} \right\}$$

where $h(\phi, \nu)$ denotes the entropy of the time-one Anosov flow.

The Gibbs measures μ are always non-atomic and fully supported. We recall two best known Gibbs measures measure on N.

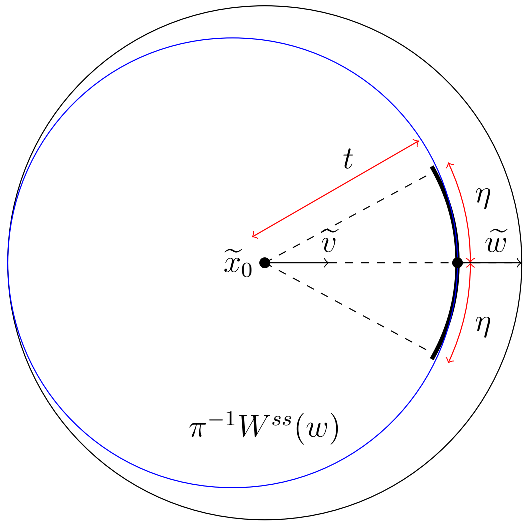

Figure 4: The circular arc of length 2η is approximated by a lift $\pi^{-1}W^{ss}(w)$

1. The Sinai-Ruelle-Bowen measure μ is the unique ϕ_t-invariant probability measure which is equivalent to the Riemannian volume on the unstable manifolds of ϕ_t.

2. The Bowen-Margulis measure m_{BM} is the unique ϕ_t-invariant probability measure with entropy $h(\phi_t, m_{BM}) = h(\phi_t)$.

In the particular case of geodesic flows $\phi_t : SM \to SM$ the Sinai-Ruelle-Bowen measure μ coincides with the Liouville measure, which is the unique ϕ_t-invariant probability measure equivalent to the Riemannian volume on SM.

One of the most important distinctions between constant and variable curvature is that in the constant curvature case the natural Liouville-Haar measure μ coincides with the Bowen-Margulis measure or measure of maximal entropy, which is not the case for surfaces of variable curvature.

3.3 The strong mixing property

In this section we want to recall a simple mixing result which is the main ingredient in the proof of Theorem **??**.

Definition 3.9. *We say that a ϕ_t-invariant probability measure ν on SM is strong mixing if for any Borel measurable sets $A, B \subset SM$ we have that $\nu(\phi_t A \cap B) \to \nu(A)\nu(B)$ as $t \to +\infty$.*

This property holds for Gibbs measures **?**.

Proposition 3.10 (Strong mixing property)**.** *Let μ be a Gibbs measure (for a Hölder continuous function) then μ is strong mixing.*

We will apply this result with A being a neighbourhood of $S_{x_0}M$.

3.4 Induced measures on $S_{x_0}M$

Let $h_t : SM \to SM$ be a horocycle flow whose orbits are leaves of the C^1 stable foliation and whose parameterization comes from the induced Riemannian metric. We can choose a neighbourhood $A = A(\epsilon, \delta)$ of the one dimensional curve $S_{x_0}M$ of the form

$$S_{x_0}M \subset A := \phi_{[-\epsilon,\epsilon]} h_{[-\delta,\delta]} S_{x_0}Ms = \{\phi_u h_v(x) : v \in S_{x_0}M, -\epsilon < u < \epsilon \text{ and } -\delta < v < \delta\}$$

for small $\epsilon, \delta > 0$ (cf. Lemma **??**). The basic idea is that we are "fattening up" $S_{x_0}M$ in both the strong stable direction and flow direction (see Figure 5).

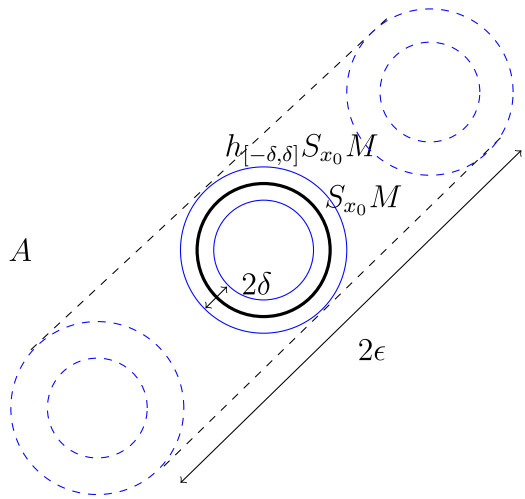

Figure 5: The set $S_{x_0}M$ is a one dimensional embedded circle in SM. Applying $h_{[-\delta,\delta]}$ gives a two dimensional embedded annulus. Then applying $\phi_{[-\epsilon,\epsilon]}$ gives the three dimensional neighbourhood A of $S_{x_0}M$.

Given any Gibbs measure μ we can induce probability measures $\mu_{x_0}^{\epsilon,\delta}$ on $S_{x_0}M$ as follows.

Definition 3.11. *For sufficiently small $\epsilon, \delta > 0$ we define a family of equivalent probability measures $\mu_{x_0}^{\epsilon,\delta}(B)$ by*

$$\mu_{x_0}^{\epsilon,\delta}(B) = \frac{\mu\left(\phi_{[-\epsilon,\epsilon]} h_{[-\delta,\delta]} B\right)}{\mu\left(A(\epsilon,\delta)\right)}.$$

where $B \subset S_{x_0}M$ is a Borel subset. Moreover, the following limit exists and defines a probability measure μ_{x_0} on $S_{x_0}M$ by

$$\mu_{x_0}(B) = \lim_{\delta\to 0}\lim_{\epsilon\to 0} \mu_{x_0}^{\epsilon,\delta}(B)$$

In the case that μ is the Liouville measure then $\mu_{x_0}^{\epsilon,\delta}$ and μ_{x_0} will be absolutely continuous measures on $S_{x_0}M$ with C^1 densities. In the particular case that M has constant negative curvature we see by symmetry considerations that μ_{x_0} is Haar measure on $S_{x_0}M$.

Remark 3.12. Induced measures are usually only defined up to mutual equivalence, due to the freedom in choosing sub-sigma algebras. In the above concrete formulation the ambiguity comes from the freedom in the choice of the parameterisation h_t and the corresponding density $\mathcal{D} : S_{x_0}M \to \mathbb{R}$.

4 The proof of Theorem ??

We will actually prove a more general result for Gibbs measures μ. Let μ_{x_0} and $\mu_{x_0}^{\epsilon,\delta}$ be measures on $S_{x_0}M$ introduced in the previous section and let $\widetilde{\mu}_{x_0}^{\epsilon,\delta}$ and $\widetilde{\mu}_{x_0}$, respectively, be the lifts to $S_{\tilde{x}_0}\widetilde{M}$.

Theorem 4.1 (Generalized version for Gibbs measures)**.** *For each $k \in \mathbb{N}_0$ we have that the following limit exists:*

$$\mathbb{P}_k := \lim_{t\to+\infty} \widetilde{\mu}_{x_0}\left(\left\{\widetilde{v} \in S_{\tilde{x}_0}\widetilde{M} \,:\, \#\{R(\widetilde{v},t) \cap \Gamma\tilde{x}_0\} = k\right\}\right).$$

In the particular case that μ is Liouville measure, Theorem **??** reduces to Theorem **??** stated in §2.

We can use the canonical projection $\pi : \widetilde{M} \to M$ to translate the formulation of the statement in Theorem **??** to a corresponding one down on the compact surface M. More precisely, we denote the image of $R(\widetilde{v},t) \subset \widetilde{M}$ under π by $Q(v,t) := \pi(R(\widetilde{v},t)) \subset M$. Then each $Q(v,t)$ can be visualised as a strip wrapped around the surface M like a bandage. We can then reformulate the counting function in terms of SM as follows:

Lemma 4.2. *$\mathcal{Q}(v,t) := \#\{R(\widetilde{v},t) \cap \Gamma\tilde{x}_0\}$ is the number of times that the point $x_0 \in M$ is covered by $Q(v,t)$ (see Figure 6 (b)).*

The proof of Theorem **??** will make use of two partitions of the unit tangent bundle SM. We first partition SM into pieces corresponding to level sets of the counting function as follows:

Lemma 4.3. *For $t > 0$ and sufficiently large $N \geq 1$ we can divide $SM = \cup_{k=0}^{N} F_{k,t}$ into sets*

$$F_{k,t} = \{w \in SM \,:\, \mathcal{Q}(\phi_{-t}w, t) = k\}$$

for $0 \leq k \leq N$.

We observe that one can choose N to be uniformly bounded in t since, for example, M has a non-zero injectivity radius.

The strategy of the proof is to approximate $Q(v,t)$ by a more dynamically defined region $P(w) \subset M$ (where $w = \phi_t v$) and then to use the strong mixing property.

We are now in a position to explain the construction of the approximation $P(w)$ to $Q(v,t)$ (where $w = \phi_t v$) which leads to the second partition of SM. Let $\lambda^{(2)}$ denote the normalised Riemannian volume $\lambda^{(2)}$ on M.

Definition 4.4. *We can associate to each point $w \in SM$:*

1. *A value $\eta' = \eta'(w) > 0$ (which depends continuously on w);*
2. *A piece $U(w) = h_{[-\eta',\eta']}(w) \subset W^{su}(w) \subset SM$ of the strong unstable manifold; and*
3. *A piece $W(w) = \phi_{[-s,0]}h_{[-\eta',\eta']}(w) \subset W^{wu}(w) \subset SM$ of the weak unstable manifold*

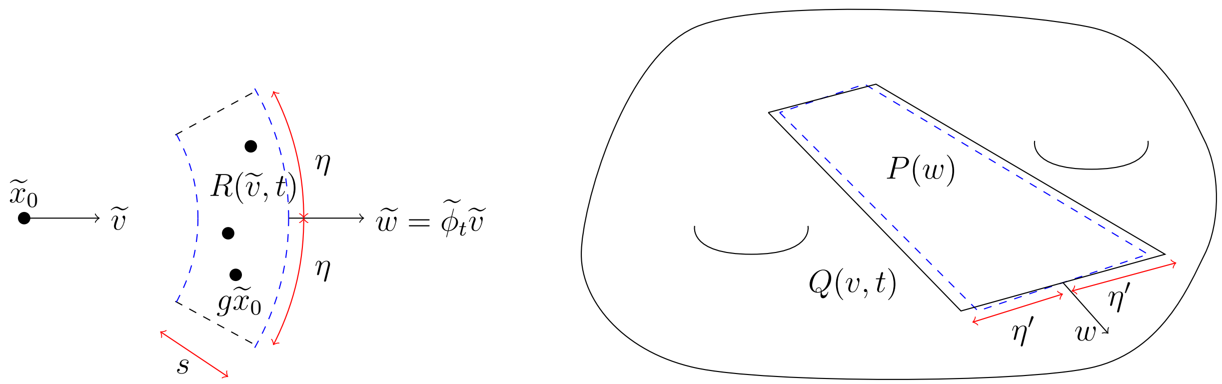

Figure 6: (a) The region $R(\widetilde{v}, t) \subset \widetilde{M}$; (b) The image $Q(v, t) \subset M$ (dotted lines) is approximated by $P(w) \subset M$ (solid line) with $w = \phi_t v$

where $\eta' = \eta'(w) > 0$ is chosen so that the canonical projection $P(w) = \rho(W(w))$ has unit area with respect to the volume element on M(i.e., $\lambda^{(2)}(P(w)) = 1$). (See Figure 6 (b))

We next want to introduce the second (more dynamical) partition of the unit tangent bundle SM. Given $w \in SM$ the set $P(w)$ can be visualised as a strip wrapping around the surface M (see Figure 6 (b)). Let $\mathcal{P}(w)$ denote the multiplicity of the number of times $P(w)$ covers $x_0 \in M$.

Definition 4.5. *For sufficiently large $N \geq 1$ we can partition $SM = \cup_{k=0}^N E_k$ into dynamically defined level sets*

$$E_k = \{w \in SM : \mathcal{P}(w) = k\}$$

for $0 \leq k \leq N$.

In order to compare these two partitions $SM = \cup_{k=0}^N E_k = \cup_{k=0}^N F_{k,t}$ we use the following straightforward property of Gibbs measures (which is clearly immediate in the particular case of the Liouville measure).

Lemma 4.6. *The sets E_k have the property that $\mu(\partial E_k) = 0$ (i.e., their boundaries have zero measure with respect to Gibbs measure). Moreover, $\lim_{\xi \to 0} \mu(B_\xi(\partial E_k)) = 0$ where $B_\xi(\partial E_k)$ is an ξ-neighbourhood of E_k.*

Proof. This follows easily from the useful fact that Gibbs measures are non-singular (with uniform bounds on the Radon-Nikodym derivative) under local homeomorphisms which preserve the strong stable and strong unstable manifolds. For hyperbolic transformations this was shown in **?** and the extension to hyperbolic flows is similar. If we assumed for a contradiction that $\mu(\partial E_k) > 0$ then we could arrange many disjoint images under such local homeomorphisms the sum of whose measures would contradict μ being a probability measure. □

For each k we want to relate the sets E_k and $F_{k,t}$ in the partitions above.

Lemma 4.7. *Let $k \geq 0$.*

1. *For all $\xi > 0$ there exists $t_0 > 0$ such that $F_{k,t} \subset B_\xi(E_k)$ and $B_\xi(E_k) \subset F_{k,t}$ for all $t > t_0$.*

2. $\lim_{t\to+\infty} \mu(F_{k,t}) = \mu(E_k)$.

Proof. The first part follows from the definitions of $P(w)$ and $Q(v,t)$ and Lemma **??** which imply, in particular, that these two sets are close in the Hausdorff topology. The second part follows from Lemma **??**. □

To complete the proof it only remains to approximate the partition $\{F_{k,t}\}$ by the partition $\{E_k\}$ and apply the strong mixing property, as follows. Let $\delta > 0$ and $\epsilon > 0$ be sufficiently small. By Lemma **??** we can bound

$$\begin{aligned}\limsup_{t\to\infty} \widetilde{\mu}_{\tilde{x}_0}^{\epsilon,\delta}\left(\left\{\widetilde{v} \in S_{\tilde{x}_0}\widetilde{M} : \#\{R(\widetilde{v},t) \cap \Gamma\tilde{x}_0\} = k\right\}\right) &= \limsup_{t\to\infty} \mu_{x_0}^{\epsilon,\delta}\left(\{v \in S_{x_0}M : \phi_t v \in F_{k,t}\}\right) \\ &\le \limsup_{t\to\infty} \mu_{x_0}^{\epsilon,\delta}\left(\{v \in S_{x_0}M : \phi_t v \in B_\xi(E_k)\}\right) \\ &= \limsup_{t\to\infty} \frac{\mu\left(A(\epsilon,\delta) \cap \phi_{-t}B_\xi(E_k)\right)}{\mu(A(\epsilon,\delta))} \\ &= \mu\left(B_\xi(E_k)\right),\end{aligned}$$

using Lemma **??** and the strong mixing property in Lemma **??**. Letting $\epsilon, \delta \to 0$ (and then letting $\xi \to 0$ and applying Lemma **??**) gives that

$$\limsup_{t\to\infty} \widetilde{\mu}_{x_0}\left(\left\{\widetilde{v} \in S_{\tilde{x}_0}\widetilde{M} : \#\{R(\widetilde{v},t) \cap \Gamma\tilde{x}_0\} = k\right\}\right) \le \mu\left(E_k\right).$$

A similar argument gives that

$$\liminf_{t\to+\infty} \widetilde{\mu}_{x_0}\left(\left\{\widetilde{v} \in S_{\tilde{x}_0}\widetilde{M} : \#\{R(\widetilde{v},t) \cap \Gamma\tilde{x}_0\} = k\right\}\right) \ge \mu\left(E_k\right).$$

This completes the proof of Theorem **??** with $\mathbb{P}_k = \mu(E_k)$, for $k \ge 0$.

5 Generalizations

The basic method of proof is straightforward and the same method leads to a number of simple generalisations, which we briefly describe.

1. If we assume that the surface M has finite area and negative curvature bounded away from zero then Theorem **??** still holds. In particular, the geodesic flow is still strong mixing for the Liouville measure by **?**.

2. The same general method can be used to prove results for d-dimensional manifolds ($d \ge 3$) with negative sectional curvatures in higher dimensions with suitable definitions of the shrinking sectors. For example, one can consider a region $R(\widetilde{v},t)$ which is the intersection of

(a) the set $\{\widetilde{x} \in \widetilde{M} : t - s < d(\widetilde{x}, \widetilde{x}_0) < t\}$ bounded between spheres of radius t and $t - s$ centred at $\widetilde{x}_0$; and

(b) the cone bounded by geodesics from $\widetilde{x}_0$ to the boundary of the ball $\{\widetilde{y} : d_{\widetilde{M}}(\widetilde{x}, \widetilde{y}) = t$ and $d_{\widetilde{M}}(\pi(\widetilde{\phi_t \widetilde{v}}), \widetilde{y}) < \eta\}$ contained in the larger sphere, where $\eta = \eta(\widetilde{v}, t)$ is chosen so that $\mathrm{vol}_{\widetilde{M}} R(\widetilde{v}, t) = 1$.

With these changes, the statements and proofs of Theorem **??** and Theorem **??** generalise to higher dimensions.

3. Let $\phi_t : M \to M$ be a three dimensional Anosov flow and let $S_1, S_2 \subset SM$ be compact one-dimensional submanifolds for which the tangents $T_{S_1}SM$ and $T_{S_2}SM$ are at no point tangent to either the unstable bundle or the stable bundle, respectively. Fix $s > 0$, then given $v \in S_1$, $t > 0$ and $\eta > 0$ we define a two dimensional subset $R(v,t) = \{\phi_{[t-s,t]}v' \in S_1M : d(\phi_t v, \phi_t v') < \eta\}$, say, where $\eta(v,t) > 0$ is chosen so that $\mathrm{vol}^{(2)}(R(v,t)) = 1$ (i.e., the two dimensional area is normalised). The analogues of the statements of Theorem **??** and Theorem **??** then hold for the counting function $\#\{R(v,t)) \cap S_2\}$. This can be generalized to Axiom A flows in a natural way and then covers the case of geodesic flows on convex co-compact surfaces, making a connection with the Patterson-Sullivan measure on the boundary at infinity.

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