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Original citation:
Hernández-Hernández, Ma Elena and Kolokoltsov, V. N. (Vasiliĭ Nikitich). (2016) On the solution of two-sided fractional ordinary differential equations of Caputo type. Fractional Calculus and Applied Analysis, 19 (6). 1393–1413.

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http://dx.doi.org/10.1515/fca-2016-0072

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RESEARCH PAPER

ON THE SOLUTION OF TWO-SIDED FRACTIONAL ORDINARY DIFFERENTIAL EQUATIONS OF CAPUTO TYPE *

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Abstract

* *A paper presented at Workshop "FaF",*
Lorentz Center - Leiden, The Netherlands, May 17-20, 2016

This paper provides well-posedness results and stochastic representations for the solutions to equations involving both the right- and the left-sided generalized operators of Caputo type. As a special case, these results show the interplay between two-sided fractional differential equations and two-sided exit problems for certain Lévy processes.

MSC 2010: Primary 34A08; Secondary 35S15, 26A33, 60H30

Key Words and Phrases: two-sided fractional equations, generalized Caputo type derivatives, boundary point, stopping time, Feller process, Lévy process

1. Introduction

The successful use of classical fractional derivatives to describe, for example, relaxation phenomena, processes of oscillation, viscoelastic systems and diffusions in disordered media (anomalous diffusions) among others, have promoted an increasing research on the field of fractional differential equations. For an account of historical notes, applications and different methods to solve fractional equations we refer, e.g., to [7]-[10], [12], [14], [21]-[22], [25], [29]-[32], [34], [39], and references cited therein.

Apart from the different notions of fractional derivatives found in the literature (e.g., the Caputo, the Riemann-Liouville, the Grunwald-Letnikov, the Riesz, the Weyl, the Marchaud, and the Miller and Ross fractional derivatives), numerous generalizations (mostly from an analytical point of view) have been proposed by many authors, we refer, e.g., to [2], [18]-[19], [23]-[24], [33] for details. As for the *generalized fractional operators of Caputo type* considered in this work, they were introduced in [27] by one of the authors as generalizations (from a probabilistic point of view) of the classical Caputo derivatives of order $\beta \in (0,1)$ when applied to regular enough functions. These Caputo type operators can be thought of as the generators of Feller processes *interrupted* on the first attempt to cross certain boundary point (see precise definition later).

As a continuation of our previous works, which show a new link between stochastic analysis and fractional equations (see [16]-[17], [27]), this paper appeals to a probabilistic approach to study equations involving both left-sided and right-sided generalized operators of Caputo type. We address the boundary value problem for the *two-sided generalized linear equation* with Caputo type derivatives $-D_{a+*}^{(\nu_+)}$ and $-D_{b-*}^{(\nu_-)}$:

$$-D_{a+*}^{(\nu_+)}u(x) - D_{b-*}^{(\nu_-)}u(x) - Au(x) = \lambda u(x) - g(x), \quad x \in (a,b),$$
$$u(a) = u_a, \quad u(b) = u_b, \qquad (1.1)$$

where $\lambda \geq 0$, $u_a, u_b \in \mathbb{R}$ and g is a prescribed function on $[a,b]$. Notation $-A \equiv -A^{(\gamma,\alpha)}$ refers to the second order differential operator

$$-A^{(\gamma,\alpha)} := \gamma(\cdot)\frac{d}{dx} + \alpha(\cdot)\frac{d^2}{dx^2}. \qquad (1.2)$$

Equation (1.1) includes, as special cases, the fractional equations

$$D_{a+*}^{\beta_1}u(x) + D_{b-*}^{\beta_2}u(x) = g(x), \quad x \in (a,b), \quad \beta_1, \beta_2 \in (0,1), \qquad (1.3)$$
$$u(a) = u_a, \quad u(b) = u_b,$$

where $D_{a+*}^{\beta_1}$ and $D_{b-*}^{\beta_2}$ are the left- and the right-sided Caputo derivatives of order β_1 and β_2, respectively. There are relatively scarce results dealing with two-sided fractional ordinary equations. For example, to the best of our knowledge, the Riemann-Liouville version of (1.3) was analyzed (in the space of distributions) in [35]-[36], whereas the explicit solution to the two-sided fractional equation in (1.3) was just recently provided in [27].

Another special case of equation (1.1) is the *two-sided* equation:

$$c_1 D_{a+*}^{\beta_1}u(x) + c_2 D_{b-*}^{\beta_2}u(x) + \gamma(x)u'(x) + \lambda u(x) = g(x), \quad x \in (a,b), \quad (1.4)$$
$$u(a) = u_a, \quad u(b) = u_b.$$

If $c_1 > 0$, $c_2 = 0$, $\beta_1 = \frac{1}{2}$ and $\lambda = 1$, then the (one-sided) equation is known as the *Basset equation*, well-studied in the literature (see, e.g. [29]

and references therein). The one-sided case with $\beta_1 \in (0,1)$ (known as the *composite fractional relaxation equation*) was treated via the Laplace transform method in [15, Section 4], whereas the left-sided case with Caputo type and RL type operators was studied by the authors in [17].

Some other examples showing the relevance of left- and right-sided derivatives in mathematical modeling appear in the study of FPDE's on bounded domains, as well as in fractional calculus of variations, see, e.g.,[1], [3], [21], [31], [37].

In this paper we study the well-posedness of (1.1) by considering two types of solutions: *solutions in the domain of the generator* and *generalized solutions*. The first type is understood as a solution u that belongs to the domain of the two-sided operator seen as the generator of a Feller process. Since the existence of such a solution is quite restrictive once one imposes boundary conditions, the notion of generalized solution is introduced via the limit of approximating solutions taken from the domain of the generator.

Further, appealing to the relationship between two-sided equations and exit problems for Feller processes (already mentioned in [27]), we provide some explicit solutions to two-sided equations in the context of classical fractional derivatives. Even though exit problems for Lévy processes have been widely studied (see, e.g., [5]-[6], [28], [38]), to our knowledge fractional equations of the type in (1.3) and their connection with exit problems seem to be novel in the literature. We believe that the probabilistic solutions presented in this work can be used, for example, to obtain numerical solutions to classical fractional equations for which explicit solutions are unknown.

The paper is organized as follows. The next Section 2 sets standard notation and definitions. Section 3 gives a quick review about generalized Caputo type operators. Section 4 provides preliminary results concerning *two-sided generalized operators* and their connections with the generators of Feller processes. Then, Section 5 addresses the well-posedness for the RL type version of (1.1). The study of the Caputo type equation (1.1) is given in Section 6. Some examples are presented in Section 7. Finally, Section 8 contains the proofs of some key results established in Section 4.

2. Preliminaries

2.1. **Notation.** Let $\mathbb{N}$ and $\mathbb{R}$ be the set of positive integers and the real line, respectively. For any open set $A \subset \mathbb{R}$, notation $B(A)$, $C(A)$ and $C_\infty(A)$ denote the set of bounded Borel measurable functions, bounded continuous functions and continuous functions vanishing at infinity defined on A, respectively, equipped with the sup-norm $||h|| = \sup_{x \in A} |h(x)|$. The space of continuous functions on A with continuous derivatives up to and including order k is denoted by $C^k(A)$. This space is equipped with the

norm $||h||_{C^k} := ||h|| + \sum_{k=1} ||h^{(k)}||$. For functions defined on the closure $\bar{A}$ of A, notation $C^k(\bar{A})$ means the space of k times continuously differentiable functions up to the boundary. Further, spaces $C_0[a,b]$ and $C_0^k[a,b]$ stand for the space of continuos functions on $[a,b]$ vanishing at the boundary and the space of functions $C_0[a,b] \cap C^k[a,b]$, respectively.

Letters $\mathbf{P}$ and $\mathbf{E}$ are reserved for the probability and the mathematical expectation, respectively. For a stochastic process $X_x = (X_x(t))_{t \geq 0}$ with state space A, the subscript x in $X_x(t)$ means that the process starts at $x \in A$, so that notation $\mathbf{E}\left[f\left(X_x(t)\right)\right]$ is understood as $\mathbf{E}\left[f\left(X(t)\right) | X(0) = x\right]$. All the processes considered in this paper are assumed to be defined on a fixed complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2.2. **Feller processes: basic definitions.** Let $\{T_t\}_{t \geq 0}$ be a strongly continuous semigroup of linear bounded operators on a Banach space $(B, ||\cdot||_B)$, i.e. $\lim_{t \to 0} ||T_t f - f||_B = 0$ for all $f \in B$. Its (infinitesimal) generator L with domain $\mathfrak{D}_L$, shortly $(L, \mathfrak{D}_L)$, is defined as the (possibly unbounded) operator $L : \mathfrak{D}_L \subset B \to B$ given by the strong limit

$$Lf := \lim_{t \downarrow 0} \frac{T_t f - f}{t}, \quad f \in \mathfrak{D}_L, \tag{2.1}$$

where the domain of the generator $\mathfrak{D}_L$ consists of those $f \in B$ for which the limit in (2.1) exists in the norm sense. We also recall that, if L is a closed operator, then a linear subspace $\mathcal{C}_L \subset \mathfrak{D}_L$ is called a *core* for the generator L if the operator L is the closure of the restriction $L\big|_{\mathcal{C}_L}$ [13, Chapter 1, Section 3]. If additionally $T_t \mathcal{C}_L \subset \mathcal{C}_L$ for all $t \geq 0$, then $\mathcal{C}_L$ is said to be an *invariant core*. The *resolvent operator* R_λ of the semigroup $\{T_t\}_{t \geq 0}$ is defined (for any $\lambda > 0$) as the Bochner integral (see, e.g., [11, Chapter 1], [13, Chapter 1])

$$R_\lambda g := \int_0^\infty e^{-\lambda t} T_t g \, dt, \quad g \in B. \tag{2.2}$$

By taking $\lambda = 0$ in (2.2), one obtains the *potential operator* denoted by $R_0 g$ (whenever it exists).

We say that a (time-homogeneous) Markov process $X = (X(t))_{t \geq 0}$ taking values on $A \subset \mathbb{R}^d$ is a *Feller process* (see, e.g., [25, Section 3.6]) if its semigroup $\{T_t\}_{t \geq 0}$, defined by

$$T_t f(x) := \mathbf{E}\left[f\left(X(t)\right) | X(0) = x\right], \quad t \geq 0, \ x \in A, \ f \in B(A),$$

gives rise to a *Feller semigroup* when reduced to $C_\infty(A)$, i.e. it is a strongly continuous semigroup on $C_\infty(A)$ and it is formed by positive linear contractions $(0 \leq T_t f \leq 1$ whenever $0 \leq f \leq 1)$.

3. Generalized fractional operators of Caputo type and RL type

The generalized Caputo type operators introduced in [27] are defined in terms of a function $\nu : \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}^+$ satisfying the condition:

(H0) The function $\nu(x,y)$ is continuous as a function of two variables and continuously differentiable in the first variable. Furthermore,

$$\sup_x \int \min\{1,|y|\}\nu(x,y)dy < \infty, \quad \sup_x \int \min\{1,|y|\}\Big|\frac{\partial}{\partial x}\nu(x,y)\Big|dy < \infty,$$

and

$$\lim_{\delta\to 0}\sup_x \int_{|y|\leq\delta} |y|\nu(x,y)dy = 0.$$

DEFINITION 3.1. Let $a, b \in \mathbb{R}$ with $a < b$. For any function ν satisfying the condition (H0), the operators $-D_{a+*}^{(\nu)}$ and $-D_{b-*}^{(\nu)}$ defined by

$$\left(-D_{a+*}^{(\nu)}h\right)(x) = \int_0^{x-a}(h(x-y)-h(x))\nu(x,y)dy$$
$$+ (h(a)-h(x))\int_{x-a}^{\infty}\nu(x,y)dy, \quad (3.1)$$

for functions $h : [a,\infty) \to \mathbb{R}$, and by

$$\left(-D_{b-*}^{(\nu)}h\right)(x) = \int_0^{b-x}(h(x+y)-h(x))\nu(x,y)dy$$
$$+ (h(b)-h(x))\int_{b-x}^{\infty}\nu(x,y)dy, \quad (3.2)$$

for functions $h : (-\infty,b] \to \mathbb{R}$, are called *the generalized left-sided Caputo type operator* and *the generalized right-sided Caputo type operator*, respectively. The values a and b will be referred to as the *terminals* of the corresponding operators.

REMARK 3.1. The sign $-$ appearing in the previous notation is introduced to comply with the standard notation of fractional derivatives.

Due to assumption (H0), the operators (3.1)-(3.2) are well defined at least on the space of continuously differentiable functions (with bounded derivative).

REMARK 3.2. The *left-sided* (resp. *right-sided*) *generalized Riemann-Liouville type operator* $-D_{a+}^{(\nu)}$ (resp. $-D_{b-}^{(\nu)}$) is defined by setting $h(a) = 0$ (resp. $h(b) = 0$) in (3.1) (resp. (3.2)). Hence,

$$-D_{a+*}^{(\nu)}h(x) = -D_{a+}^{(\nu)}[h-h(a)](x) \quad \text{and} \quad -D_{b-*}^{(\nu)}h(x) = -D_{b-}^{(\nu)}[h-h(b)](x).$$

3.0.1. **Particular cases.** For smooth enough functions h, the standard analytical definitions of the left-sided Caputo derivative D_{a+*}^{β} and the right-sided Caputo derivatives D_{b-*}^{β} of order $\beta \in (0,1)$ (see, e.g., [10, Definition 2.2, Definition 3.1]) can be rewritten as (see, e.g., [27, Appendix])

$$\left(D_{a+*}^{\beta}h\right)(x) = \frac{\beta}{\Gamma(1-\beta)}\int_0^{x-a}\frac{h(x-y)-h(x)}{y^{1+\beta}}dy - \frac{h(x)-h(a)}{\Gamma(1-\beta)(x-a)^{\beta}},$$
(3.3)

and

$$\left(D_{b-*}^{\beta}h\right)(x) = \frac{\beta}{\Gamma(1-\beta)}\int_0^{b-x}\frac{h(x+y)-h(x)}{y^{1+\beta}}dy - \frac{h(x)-h(b)}{\Gamma(1-\beta)(b-x)^{\beta}}.$$
(3.4)

Hence, for h regular enough, $D_{a+*}^{\beta}h$ (resp. $-D_{b-*}^{\beta}h$) is a particular case of $-D_{a+*}^{(\nu)}h$ (resp. $-D_{b-*}^{\beta}h$) obtained by taking the function

$$\nu(x,y) \equiv \nu(y) = -\frac{\beta}{\Gamma(1-\beta)y^{1+\beta}}, \quad \beta \in (0,1).$$
(3.5)

REMARK 3.3. Other examples of generalized operators $-D_{a+*}^{(\nu)}$ include the *fractional derivatives of variable order*, as well as the *generalized distributed order fractional derivatives* (see [16], [27] for precise definitions).

4. Two-sided operators of RL type and Caputo type

Given two functions ν_+ and ν_- satisfying condition (H0), define the function $\nu : \mathbb{R} \times \mathbb{R} \setminus \{0\} \to \mathbb{R}^+$ associated with ν_+ and ν_- by setting

$$\nu(x,y) := \nu_+(x,y), \quad y > 0, \qquad \nu(x,y) := \nu_-(x,-y), \quad y < 0. \quad (4.1)$$

Define the *two-sided operator of RL type* $-L_{[a,b]}$ and the *two-sided operator of Caputo type* $-L_{[a,b]*}$ by

$$\left(-L_{[a,b]}f\right)(x) := \left(-D_{a+}^{(\nu_+)}f\right)(x) + \left(-D_{b-}^{(\nu_-)}f\right)(x) + \left(-A^{(\gamma,\alpha)}f\right)(x),$$
(4.2)

and

$$\left(-L_{[a,b]*}f\right)(x) := \left(-D_{a+*}^{(\nu_+)}f\right)(x) + \left(-D_{b-*}^{(\nu_-)}f\right)(x) + \left(-A^{(\gamma,\alpha)}f\right)(x).$$
(4.3)

Notation $-A^{(\gamma,\alpha)}$ stands for the differential operator given in (1.2). We will see that the operator $-L_{[a,b]*}$ can be thought of as the generator of a Feller process on $[a,b]$, whereas $-L_{[a,b]}$ is related to the generator of a killed process. For that purpose, let us introduce an additional definition for the regularity of the boundary (see, e.g., [26, Chapter 6]).

DEFINITION 4.1. For a domain $D \subset \mathbb{R}$ with boundary ∂D, a point $x_0 \in \partial D$ is said to be *regular in expectation* for a Markov process X (or for its generator) if $\mathbf{E}\left[\tau_D(x)\right] \to 0$, as $x \to x_0$, $x \in D$, where $\tau_D(x) := \inf\{t \geq 0 : X_x(t) \notin D\}$, with the usual convention that $\inf\{\varnothing\} = \infty$.

THEOREM 4.1. *Let ν be a function satisfying assumption (H0). Suppose that $\gamma \in C_0^3[a, b]$, $\alpha \in C^3[a, b]$ with derivative $\alpha' \in C_0[a, b]$ and α being a positive function. Then,*

(i) the operator $(-L_{[a,b]*}, \hat{\mathfrak{D}}_*)$ generates a Feller process $\hat{X}$ on $[a, b]$ with a domain $\hat{\mathfrak{D}}_*$ such that

$$\left\{f \in C^2[a, b] : f' \in C_0[a, b]\right\} \subset \hat{\mathfrak{D}}_*. \tag{4.4}$$

(ii) The points $\{a, b\}$ are regular in expectation for $(-L_{[a,b]*}, \hat{\mathfrak{D}}_*)$. Further, the first exit time $\hat{\tau}_{(a,b)}(x)$ from the interval (a, b) of $\hat{X}_x$, $x \in (a, b)$, has a finite expectation.

P r o o f. See proof in Section 8. □

Stopped and killed processes. To introduce the notion of solutions to the equation (1.1) we are interested in, we need the stopped version of $\hat{X}$.

THEOREM 4.2. *Suppose that the assumptions of Theorem 4.1 hold. Let $\hat{X}_x$ be the process started at $x \in (a, b)$ generated by $(-L_{[a,b]*}, \hat{\mathfrak{D}}_*)$.*

(i) *The process $X_x^{[a,b]*}$ defined by $X_x^{[a,b]*}(s) := \hat{X}_x(s \wedge \hat{\tau}_{(a,b)}(x))$, $s \geq 0$, is a Feller process on $[a, b]$. If the operator $(-L_{stop}, \mathfrak{D}_{[a,b]*}^{stop})$ denotes the generator of $X^{[a,b]*}$, then for any $f \in \hat{\mathfrak{D}}_*$ satisfying $\left(-L_{[a,b]*}f\right)(x) = 0$ for $x \in \{a, b\}$, it holds that $f \in \mathfrak{D}_{[a,b]*}^{stop}$ and $-L_{stop}f = -L_{[a,b]*}f$.*

(ii) *The process $X_x^{[a,b]}$ defined by $X_x^{[a,b]}(s) := X_x^{[a,b]*}(s)$ for $s < \hat{\tau}_{(a,b)}(x)$ is a Feller (sub-Markov) process on (a, b). If $(-L_{kill}, \mathfrak{D}_{[a,b]}^{kill})$ denotes the generator of $X^{[a,b]}$, then for any $f \in \mathfrak{D}_{[a,b]*}^{stop}$ satisfying $f(x) = 0$ for $x \in \{a, b\}$, it holds that $f \in \mathfrak{D}_{[a,b]}^{kill}$ and $-L_{kill}f = -L_{[a,b]}f$.*

P r o o f. See proof in Section 8. □

REMARK 4.1. The operator $-L_{[a,b]*}$ can be obtained from the generator $(L, \mathfrak{D}_L)$ of a Feller process, say X_x, given by

$$(Lf)(x) = \int_{-\infty}^{\infty} (f(x + y) - f(x))\nu(x, y)dy + \gamma(x)f'(x) + \alpha(x)f''(x),$$

$$\tag{4.5}$$

by modifying it in such a way that it forces the jumps aimed to be out of the interval (a, b) to land at the nearest (boundary) point (see also [27]). If, instead, the process is killed upon leaving (a, b), then the corresponding process has a generator related to the operator $-L_{[a,b]}$. Thus, when starting at the same state $x \in (a, b)$, it holds that the paths of the processes X_x, $\hat{X}_x$, $X_x^{[a,b]*}$ and $X_x^{[a,b]}$ coincide before their first exit time from the interval (a, b). Hence, the first exit time in all cases will always be denoted by $\tau_{(a,b)}(x)$. We refer to the processes X_x, $\hat{X}_x$, $X_x^{[a,b]*}$ and $X_x^{[a,b]}$ as the *underlying* process, the *interrupted* process, the *stopped* process and the *killed* process, respectively.

5. Two-sided equations involving RL type operators

Let us now study the equation (1.1) for which we will also use the notation $\left(-L_{[a,b]*}, \lambda, g, u_a, u_b\right)$. We shall start with the boundary value problem with zero boundary conditions: $u_a = 0 = u_b$. Thus, due to the relationship between Caputo and RL type operators (see Remark 3.2), the two-sided Caputo type operator $-L_{[a,b]*}$ can be replaced with the RL type operator $-L_{[a,b]}$, so that the equation $\left(-L_{[a,b]}, \lambda, g, 0, 0\right)$ will be called the two-sided RL type equation.

DEFINITION 5.1. (**Solutions to RL type equations**) Let $g \in B[a, b]$ and $\lambda \geq 0$. A function $u \in C_0[a, b]$ is said to solve the linear equation of RL type $(-L_{[a,b]}, \lambda, g, 0, 0)$ as (i) a *solution in the domain of the generator* if u is a solution belonging to $\mathfrak{D}_{[a,b]}^{kill}$; (ii) a *generalized solution* if for all sequence of functions $g_n \in C_0[a, b]$ such that $\sup_n ||g_n|| < \infty$ uniformly on n and $\lim_{n\to\infty} g_n \to g$ a.e., it holds that $u(x) = \lim_{n\to\infty} w_n(x)$ for all $x \in [a, b]$, where w_n is the unique solution (in the domain of the generator) to the RL type problem $(-L_{[a,b]}, \lambda, g_n, 0, 0)$.

DEFINITION 5.2. For $g \in B[a, b]$ and $\lambda \geq 0$, we say that the equation $(-L_{[a,b]}, \lambda, g, 0, 0)$ is *well-posed in the generalized sense* if it has a unique generalized solution according to Definition 5.1.

THEOREM 5.1. (**Well-posedness**) *Let ν be a function defined in terms of two functions ν_+ and ν_- via the equalities in (4.1). Let $\lambda \geq 0$ and assume that the assumptions of Theorem 4.1 hold. Let $\hat{R}_\lambda$ denote the resolvent operator (or the potential operator if $\lambda = 0$) of the process $\hat{X}_x$.*

(i) *If $g \in C_0[a, b]$ and $\left(\hat{R}_\lambda g\right)(x) = 0$ for $x \in \{a, b\}$, then there exists a unique solution in the domain of the generator, $u \in C_0[a, b]$, to*

the two-sided RL type equation $(-L_{[a,b]}, \lambda, g, 0, 0)$ given by $u(x) = R_\lambda^{[a,b]} g(x)$, where $R_\lambda^{[a,b]}$ denotes the resolvent operator (or potential operator if $\lambda = 0$) of the process $X_x^{[a,b]}$.

(ii) For any $g \in B[a, b]$, the equation $(-L_{[a,b]}, \lambda, g, 0, 0)$ has a unique generalized solution $u \in C_0[a, b]$ given by

$$u(x) = \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} e^{-\lambda t} g\left(X_x(t)\right) dt\right], \qquad (5.1)$$

where $\tau_{(a,b)}(x)$ denotes the first exit time from the interval (a, b) of the underlying process X_x generated by the operator (4.5).

(iii) The solution in (5.1) depends continuously on the function g in topology.

P r o o f. (i) Theorem 4.1 implies that $(-L_{[a,b]*}, \hat{\mathfrak{D}}_*)$ generates a Feller process $\hat{X}$ and a strongly continuous semigroup on $C[a, b]$. Then, the resolvent equation $-L_{[a,b]*}u = \lambda u - g$ has a unique solution $u \in \hat{\mathfrak{D}}_*$ given by the resolvent operator $\hat{R}_\lambda g$ for $\lambda > 0$ and for any $g \in C[a, b]$ [11, Theorem 1.1]. In particular, the latter statement holds for $g \in C_0[a, b]$ such that $\left(\hat{R}_\lambda g\right)(x) = 0$ for $x \in \{a, b\}$. Further, Theorem 4.2 implies that $\hat{R}_\lambda g = R_\lambda^{[a,b]} g$, so that $-L_{[a,b]*}u = -L_{[a,b]}u$. Hence, u is a solution to $(-L_{[a,b]}, g, \lambda, 0, 0)$ belonging to $\mathfrak{D}_{[a,b]}^{kill}$, as required.

Since $\tau_{(a,b)}(x) := \inf\{t \geq 0 : X_x^{[a,b]}(t) \notin (a, b)\}$ is the lifetime of the process $X_x^{[a,b]}$, the definition of $R_\lambda^{[a,b]}$ and Fubini's theorem imply

$$R_\lambda^{[a,b]} g(x) = \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} e^{-\lambda t} g\left(X_x^{[a,b]}(t)\right) dt\right], \qquad (5.2)$$

yielding (5.1) as the paths of $X_x^{[a,b]}$ and X_x coincide before the time $\tau_{(a,b)}(x)$. If $\lambda = 0$, then setting $\lambda = 0$ in (5.2) implies (as $\tau_{(a,b)}(x)$ has a finite expectation) that

$$||R_0^{[a,b]} g|| \leq \sup_{x \in [a,b]} \mathbf{E}\left[\tau_{(a,b)}(x)\right] < +\infty.$$

Therefore, the potential operator $R_0^{[a,b]} g$ provides the unique solution for $\lambda = 0$ belonging to the domain $\mathfrak{D}_{[a,b]}^{kill}$, [11, Theorem 1.1'].

(ii) Take $g \in B[a, b]$ and any sequence $\{g_n\}$ satisfying Definition 5.1. Fubini's theorem and the dominated convergence theorem applied to (5.2) imply the convergence of $\lim_{n\to\infty} R_\lambda^{[a,b]} g_n(x) =: u(x)$, which in turn implies that u is the unique generalized solution to $(-L_{[a,b]}, \lambda, g, 0, 0)$.

(iii) Follows from the fact that, for any $\lambda \geq 0$, the equality (5.1) implies

$$||u - u_n|| \leq ||g - g_n|| \sup_{x \in [a,b]} \mathbf{E}\left[\tau_{(a,b)}(x)\right], \tag{5.3}$$

for the solutions u and u_n to equations $(-L_{[a,b]}, \lambda, g, 0, 0)$ and $(-L_{[a,b]}, \lambda, g_n, 0, 0)$, respectively. $\qquad\square$

6. **Two-sided equations involving Caputo type operators**

We now turn our attention to the well-posedness for the Caputo type equation with general boundary conditions. We will use that both operators $-L_{[a,b]*}$ and $-L_{[a,b]}$ coincide on functions h vanishing on $\{a, b\}$.

Suppose that u solves (1.1). Take any function $\phi \in \mathfrak{D}_{[a,b]*}^{stop}$ satisfying $\phi(a) = u_a$ and $\phi(b) = u_b$. By Theorem 4.2 we can take, for example, $\phi \in C^2[a,b]$ such that $\phi' \in C_0[a,b]$ with $\left(-L_{[a,b]*}\phi\right)(x) = 0$ for $x \in \{a, b\}$ and $\phi(a) = u_a$ and $\phi(b) = u_b$. Define $w(x) := u(x) - \phi(x)$, $x \in [a,b]$, then
$$-L_{[a,b]}w(x) = -L_{[a,b]*}w(x) = -L_{[a,b]*}u(x) + L_{[a,b]*}\phi(x),$$
as w vanishes at the boundary. Hence,
$$\begin{aligned}
-L_{[a,b]}w(x) &= \lambda u(x) - g(x) + L_{[a,b]*}\phi(x), \\
&= \lambda w(x) + \lambda\phi(x) - g(x) + L_{[a,b]*}\phi(x), \tag{6.1}
\end{aligned}$$
yielding the RL type equation $(-L_{[a,b]}, \lambda, g - L_{[a,b]*}\phi - \lambda\phi, 0, 0)$ for the function w. Therefore, if w is the (possibly generalized) solution to (6.1), then $u = w + \phi$ can be considered as a generalized solution to the Caputo type equation (1.1). This motivates the definition below.

DEFINITION 6.1. (**Solutions to Caputo type equations**) Let $g \in B[a,b]$ and $\lambda \geq 0$. A function $u \in C[a,b]$ is said to solve the linear equation (1.1) as (i) a *solution in the domain of the generator* if u is a solution belonging to $\mathfrak{D}_{[a,b]*}^{stop}$; (ii) a *generalized solution* if u can be written as $u = \phi + w$, where w is the (possibly generalized) solution to the RL type problem
$$(-L_{[a,b]}, \lambda, g - L_{[a,b]*}\phi - \lambda\phi, 0, 0)$$
with $\phi \in C^2[a,b]$ satisfying that $\phi' \in C_0[a,b]$, $\left(-L_{[a,b]*}\phi\right)(x) = 0$ in $\{a, b\}$, $\phi(a) = u_a$ and $\phi(b) = u_b$.

DEFINITION 6.2. For $g \in B[a,b]$ and $\lambda \geq 0$. We say that the two-sided linear equation (1.1) is *well-posed in the generalized sense* if it has a unique generalized solution according to Definition 6.1.

THEOREM 6.1. *If a generalized solution $u = w + \phi$ exists for the Caputo type linear equation (1.1) with w and ϕ as in Definition 6.1, then the solution u is unique and independent of ϕ.*

P r o o f. Suppose that there are two different solutions u_j for $j \in \{1, 2\}$ to equation (1.1). Then, $u_j = w_j + \phi_j$, where w_j is the unique solution (possibly generalized) to the RL type equation $(-L_{[a,b]},\ \lambda,\ g - L_{[a,b]*}\phi_j - \lambda\phi_j,\ 0,\ 0)$ for some ϕ_j satisfying the conditions stated in Definition 6.1. Define $u(x) := u_1(x) - u_2(x)$ for $x \in [a, b]$, then

$$-L_{[a,b]}u(x) = -L_{[a,b]*}u(x) = -L_{[a,b]*}u_1(x) + L_{[a,b]*}u_2(x) = \lambda u(x).$$

Hence, u solves the RL type equation $(-L_{[a,b]}, \lambda, g = 0, 0, 0)$ whose unique solution (by Theorem 5.1) is $u \equiv 0$, which implies the uniqueness and so the independence of ϕ. □

THEOREM 6.2. (**Well-posedness**) *Let* $\lambda \geq 0$. *Suppose that the assumptions of Theorem 5.1 hold.*

(i) *For any* $g \in B[a, b]$, *the two-sided equation (1.1) is well-posed in the generalized sense. The solution admits the stochastic representation*

$$u(x) = u_a \mathbf{E}\left[e^{-\lambda\tau_{(a,b)}(x)}\mathbf{1}_{\{X_x(\tau_{(a,b)}(x))\leq a\}}\right]$$

$$+ u_b \mathbf{E}\left[e^{-\lambda\tau_{(a,b)}(x)}\mathbf{1}_{\{X_x(\tau_{(a,b)}(x))\geq b\}}\right] + \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} e^{-\lambda t}g\left(X_x(t)\right)dt\right],$$

(6.2)

where $\tau_{(a,b)}(x)$ *and* X_x *are as in Theorem 5.1.*

(ii) *If* $g \in C[a, b]$ *satisfying that* $g(a) = \lambda u_a$, $g(b) = \lambda u_b$ *and* $\lambda\hat{R}_\lambda g(x) = g(x)$ *for* $x \in \{a, b\}$, *then the solution (6.2) belongs to* $\mathfrak{D}_{[a,b]*}^{stop}$.

(iii) *The solution to (1.1) depends continuously on the function* g *and on the boundary conditions* $\{u_a, u_b\}$.

P r o o f. *(i)* Theorem 4.1 implies that the operator $(-L_{[a,b]*}, \hat{\mathfrak{D}}_*)$ generates a Feller process $\hat{X}$ on $[a, b]$ and also ensures that $\tau_{(a,b)}(x)$ has a finite expectation. Let us take any function $\phi \in C^2[a, b]$ satisfying the conditions from Definition 6.1. Then (by Theorem 5.1) the generalized solution w to the RL type equation $(-L_{[a,b]}, g - \lambda\phi - L_{[a,b]*}\phi, \lambda, 0, 0)$ is given by $w = I - II$, where

$$I := \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} e^{-\lambda t}g\left(X_x^{[a,b]}(t)\right)dt\right],$$

$$II := \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} e^{-\lambda t}(\lambda + L_{[a,b]*})\phi\left(X_x^{[a,b]}(t)\right)dt\right].$$

Thus, $u = w + \phi$ is (by definition) the generalized solution to (1.1). Using the martingale

$$Y(r) := e^{-\lambda r} \phi\left(X_x^{[a,b]*}(r)\right) + \int_0^r e^{-\lambda s}(\lambda + L_{[a,b]*})\phi\left(X_x^{[a,b]*}(s)\right) ds$$

and the stopping time $\tau_{(a,b)}(x)$, Doob's stopping theorem yields

$$II = \phi(x) - \mathbf{E}\left[e^{-\lambda\tau_{(a,b)}(x)}\phi\left(X_x^{[a,b]*}\left(\tau_{(a,b)}(x)\right)\right)\right]$$

which in turn implies

$$u(x) = \mathbf{E}\left[e^{-\lambda\tau_{(a,b)}(x)}u\left(X_x^{[a,b]*}\left(\tau_{(a,b)}(x)\right)\right)\right]$$
$$+ \quad \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} e^{-\lambda t}g\left(X_x^{[a,b]*}(t)\right) dt\right], \qquad (6.3)$$

as $\phi\left(X_x^{[a,b]*}\left(\tau_{(a,b)}(x)\right)\right) = u\left(X_x^{[a,b]*}\left(\tau_{(a,b)}(x)\right)\right)$ by assumption. Finally, since at the random time $\tau_{(a,b)}(x)$ the process $X_x^{[a,b]*}$ takes either the value a or the value b, the first term in the r.h.s of (6.3) can be written as

$$\mathbf{E}\left[e^{-\lambda\tau_{(a,b)}(x)}u\left(X_x^{[a,b]*}\left(\tau_{(a,b)}(x)\right)\right)\right]$$
$$= u_a\mathbf{E}\left[e^{-\lambda\tau_{(a,b)}(x)}\mathbf{1}_{\{X_x(\tau_{(a,b)}(x))\leq a\}}\right] + u_b\mathbf{E}\left[e^{-\lambda\tau_{(a,b)}(x)}\mathbf{1}_{\{X_x(\tau_{(a,b)}(x))\geq b\}}\right],$$

where X_x is the underlying process (see (4.5)), which yields the result (6.2). (ii) Take $g \in C[a,b]$ such that $\lambda\hat{R}_\lambda g(x) = g(x)$ for $x \in \{a,b\}$. Item (i) above ensures that the solution is given by $u = w + \phi$, where w is a RL type solution and ϕ is a function satisfying the conditions given in Definition 6.1. By Theorem 5.1, w belongs to $\mathfrak{D}_{[a,b]}^{kill}$ whenever

$$g(a) = \lambda u_a + (-L_{[a,b]*}\phi)(a) \quad \text{and} \quad g(b) = \lambda u_b + (-L_{[a,b]*}\phi)(b).$$

But, by Theorem 4.2, $(-L_{[a,b]*}\phi)(a) = (-L_{[a,b]*}\phi)(b) = 0$ because $\phi \in \mathfrak{D}_{[a,b]*}^{stop}$. Further, assumption $\lambda\hat{R}_\lambda g(x) = g(x)$ in $\{a,b\}$ implies $-L_{[a,b]*}u(x) = 0$ for $x \in \{a,b\}$, which in turn implies $-L_{[a,b]*}u = -L_{stop}u$. Hence, Theorem 4.2 guarantees that $u \in \mathfrak{D}_{[a,b]*}^{stop}$ whenever $g(a) = \lambda u_a$ and $g(b) = \lambda u_b$, as required.

(iii) Follows from the representation (6.2) and from (5.3). $\qquad\square$

Case $-A$ vanishing or $-A = \gamma(\cdot)\frac{d}{dx}$. For these cases, an additional assumption is needed to guarantee the regularity in expectation of the boundary points $\{a,b\}$.

(H1) There exist a constant $C > 0$ and $q \in (0, 1)$ such that

$$\int_{-\infty}^{0} \min(|y|, \epsilon)\nu(a, y)dy > C\epsilon^q \quad \text{and} \quad \int_{0}^{\infty} \min(y, \epsilon)\nu(b, y)dy > C\epsilon^q.$$

THEOREM 6.3. *Let $\lambda \geq 0$. Assume that the function ν associated with ν_+ and ν_- (defined via the equalities in (4.1)) satisfies assumptions (H0) and (H1). Then, Theorem 5.1 and Theorem 6.2 also hold with $\alpha \equiv 0$ and with either $\gamma \equiv 0$ or $\gamma \in C_0^1[a, b]$.*

P r o o f. Since the reasoning is same as before, we omit the details. $\square$

To finish this section, let us consider the following result related to the exit time of Feller processes from bounded intervals and generalized fractional equations of Caputo type. Let X_x be the process generated by (4.5). Define $\Pi_a(x)$ and $\Pi_b(x)$ as the event that the process X_x leaves the interval (a, b) through the lower boundary a, and through the upper boundary b, respectively, i.e.

$$\Pi_a(x) := \left\{ X_x\left(\tau_{(a,b)}(x)\right) \leq a \right\} \quad \text{and} \quad \Pi_b(x) := \left\{ X_x\left(\tau_{(a,b)}(x)\right) \geq b \right\}.$$

Let $H^D(x, \cdot)$ be the *potential measure* for the process X_x (see, e.g. [6]) defined by

$$H^D(x, dy) := \mathbf{E}\left[\int_0^\infty \mathbf{1}_{\{X_x(t)\in dy\}} \mathbf{1}_{\{\forall s \leq t, X_x(s)\in D\}} dt\right].$$

COROLLARY 6.1. *Under the assumptions of Theorem 6.2, the generalized solution to the two-sided equation (1.1) with $\lambda = 0$ rewrites as*

$$u(x) = u_a \Pi_a(x) + u_b \Pi_b(x) + \int_a^b g(y) H^{(a,b)}(x, dy). \tag{6.4}$$

In particular, $u(x) = \mathbf{E}\left[\tau_{(a,b)}(x)\right]$ is the generalized solution to the two-sided equation with $g = -1$ and $u_a = u_b = 0$. Further, $u(x) = \Pi_a(x)$ is the generalized solution to the equation with $g = 0$, $u_a = 1$ and $u_b = 0$, whereas $u(x) = \Pi_b(x)$ solves the equation with $g = 0$, $u_a = 0$ and $u_b = 1$.

7. Examples

EXAMPLE 7.1. Consider the two-sided Caputo fractional equation

$$D_{-1+*}^{\beta} w(x) + D_{+1-*}^{\beta} w(x) = -\lambda w(x) + g(x), \quad x \in (-1, 1)$$
$$w(-1) = 0 = w(1). \tag{7.1}$$

By Theorem 6.3, the boundary value problem (7.1) is well-posed in the generalized sense for any $g \in B[-1, 1]$ with solution

$$w(x) = \mathbf{E}\left[\int_0^{\tau_{(-1,1)}(x)} e^{-\lambda t} g\left(X_x^\beta(t)\right) dt\right], \quad \lambda \geq 0,$$

where X_x^β is a symmetric stable process with exponent $\beta \in (0, 1)$ and

$$\tau_{(-1,1)}(x) := \inf\left\{t \geq 0 : X_x^\beta(t) \notin (-1, 1)\right\}.$$

Further, if $g = 1$ and $\lambda = 0$, then the *mean exit time* $\mathbf{E}\left[\tau_{(-1,1)}(x)\right]$ is the unique generalized solution to (7.1). Moreover, by Theorem 2.1 in [38], we obtain the explicit solution

$$w(x) = \frac{(1 - x^2)^{\beta/2}}{\Gamma(\beta + 1)}.$$

EXAMPLE 7.2. Consider now the two-sided Caputo fractional equation:

$$D_{-1+*}^\beta h(x) + D_{+1-*}^\beta h(x) = 0, \quad x \in (-1, 1) \quad \beta \in (0, 1),$$
$$h(-1) = 0, \quad h(1) = 1. \tag{7.2}$$

Corollary 6.1 gives the unique generalized solution

$$h(x) = \mathbf{P}\left[X_x^\beta(\tau_{(-1,1)}(x)) \in [1, \infty)\right],$$

which is given explicitly by [38, Formula 3.2]

$$h(x) = 2^{1-\beta} \frac{\Gamma(\beta)}{\Gamma(\beta/2)^2} \int_{-1}^x (1 - y^2)^{\frac{\beta}{2} - 1} dy. \tag{7.3}$$

Furthermore, again by Corollary 6.1, the equation

$$D_{-1+*}^\beta v(x) + D_{+1-*}^\beta v(x) = 0, \quad x \in (-1, 1), \quad \beta \in (0, 1),$$
$$v(-1) = 1, \quad v(1) = 0. \tag{7.4}$$

has solution $v(x) = 1 - h(x)$.

EXAMPLE 7.3. The two-sided Caputo fractional equation

$$D_{-1+*}^\beta u(x) + D_{1-*}^\beta u(x) = g(x) \quad x \in (-1, 1), \quad \beta \in (0, 1),$$
$$u(-1) = u_{-1}, \quad u(1) = u_1, \qquad u_{-1}, u_1 \in \mathbb{R}, \tag{7.5}$$

has a unique generalized solution (Corollary 6.1) which rewrites

$$u(x) = (u_1 - u_{-1})h(x) + u_{-1} + \int_{-1}^1 g(y) H_\beta^{(-1,1)}(x, y) dy,$$

where $h(x)$ is the function given in (7.3), and $H_\beta^{(-1,1)}(x,y)$ (the density of the potential measure of the process X_x^β) is given by [38]

$$H_\beta^{(-1,1)}(x,y) = 2^{-\beta}\pi_{-1/2}\frac{\Gamma(1/2)}{(\Gamma(\beta/2))^2}\int_0^z (r+1)^{-\frac{1}{2}}r^{\frac{\beta}{2}-1}|x-y|^{\beta-1}dr,$$

with $z = (1-x^2)(1-y^2)/(x-y)^2$.

REMARK 7.1.   Observe that all the explicit solutions w, v, h and u above are smooth solutions since they belong to $C[-1,1] \cap C^1(-1,1)$.

8. Proofs

Firstly, let us observe that for $f \in C^1[a,b]$, by setting $g(x) = f'(x)$ we can rewrite

$$-L_{[a,b]*}^{(\nu)}f(x) = M_*^{(\nu)}g(x) := \int_{a-x}^{b-x}\int_x^{x+y} g(z)dz\nu(x,y)dy+ \qquad (8.1)$$

$$+ \int_x^b g(z)dz\int_{b-x}^\infty \nu(x,y)dy + \int_x^a g(z)dz\int_{-\infty}^{a-x}\nu(x,y)dy.$$

8.1. Proof of Theorem 4.1.

P r o o f. (i) Let us approximate $-L_{[a,b]*}$ by a family of operators $(-L_{h*})_{h\in(0,1]}$ defined by

$$-L_{h*} := -L_{[a,b]*}^{(\nu_h)} - A^{(\gamma,\alpha)}, \qquad (8.2)$$

where $\nu_h(x,y) := \Phi_h(x,y)\nu(x,y)$ with $\Phi_h(x,y)$ being a smooth function on $[a,b]\times\mathbb{R}$, which equals 1 on the set $\{|y|>h, x\in[a+h,b-h]\}$ and vanishes near the boundary; and the operator $(-A^{(\gamma,\alpha)}, \mathfrak{D}_A)$ is the generator of a diffusion on $[a,b]$ with reflecting boundaries $\{a,b\}$ (see, e.g. [4, Chapter V, Section 6]) with a domain

$$\mathfrak{D}_A := \left\{f \in C[a,b] \, : \, -A^{(\gamma,\alpha)}f \in C[a,b], \, f'(a) = 0, \, f'(b) = 0\right\}.$$

Then, for each $h \in (0,1]$ the operator $-L_{h*}$ decomposes as a diffusion on $[a,b]$ perturbed by the bounded operator $-L_{[a,b]*}^{(\nu_h)}$ on $C[a,b]$, so that by perturbation theory (see, e.g., [26, Theorem 1.9.2]) the operator $(-L_{h*}, \mathfrak{D}_A)$ generates a Feller semigroup T_t^h on $C[a,b]$. This semigroup is the unique (bounded) solution to the evolution equation

$$\frac{d}{dt}f_t(x) = -L_{h*}f_t(x), \quad f_0 = f \in \mathfrak{D}_A. \qquad (8.3)$$

Moreover, due to the smoothness assumptions on γ, α and ν, the spaces $\{f \in C^j[a,b] \, : \, f' \in C_0[a,b]\}$ for $j \in \{2,3\}$ are invariant cores of $-L_{h*}$

[26, Theorem 1.9.2,(iii)]. Hence, if $f \in C^3[a,b]$ with $f' \in C_0[a,b]$, then $T_t^h f \in C^3[a,b]$ and $-L_{h*}T_t^h f \in C^1[a,b]$.

Differentiating (8.3) with respect to x, rearranging terms and using (8.1), yield the evolution equation for $g_t(x) = f_t'(x)$ given by

$$\frac{d}{dt}g_t(x) = -\boldsymbol{L}^{h,(1)}g_t(x), \tag{8.4}$$

where

$$-\boldsymbol{L}^{h,(1)}g(x) := -A^{(\gamma+\alpha',\alpha)}g(x) + \left[-L_{[a,b]}^{(\nu_h)} - M_*^{(\partial_x \nu_h)} + \gamma'(x)\right]g(x). \tag{8.5}$$

Since (by assumption) α' vanishes on $\{a,b\}$, the operator $-\boldsymbol{L}^{h,(1)}$ decomposes as a diffusion $-A^{(\gamma+\alpha',\alpha)}$ on $[a,b]$ (with reflecting boundaries) perturbed by the bounded operator K_h on $C[a,b]$ given by

$$K_h := -L_{[a,b]}^{(\nu_h)} - M_*^{(\partial_x \nu_h)} + \gamma'(\cdot).$$

Hence, $-\boldsymbol{L}^{h,(1)}$ generates a strongly continuous semigroup of contractions on $C[a,b]$, denoted by $T_t^{h,(1)}$. Due to the invariance of the space $\{f \in C^3[a,b] : f' \in C_0[a,b]\}$, it follows that $\frac{d}{dx}(T_t^h f)(x) = \left(T_t^{h,(1)}f'\right)(x)$ for f in the latter space. Now, the perturbation series representation for the semigroup $T_t^{h,(1)}$ [26, Equality 1.78, p. 52]) implies

$$||T_t^{h,(1)}f'|| \leq ||f'|| + \sum_{m=1}^{\infty}\frac{(t\,||K_h||)^m}{m!}||f'||. \tag{8.6}$$

Thus, as K_h is uniformly bounded in h due to the bounds from assumption (H0), the derivative $\frac{d}{dx}\left(T_t^h f\right)(x)$ is uniformly bounded in h and $t \leq t_0$ whenever $f \in C^3[a,b]$ with $f' \in C_0[a,b]$.

Let us now write (see [20, Lemma 19.26, p. 385])

$$(T_t^{h_1} - T_t^{h_2})f = \int_0^t T_{t-s}^{h_2}\left(-L_{h_1*} + L_{h_2*}\right)T_s^{h_1}f\,ds,$$

for $0 < h_2 \leq h_1 < 1$ and $f \in C^3[a,b]$ with $f' \in C_0[a,b]$. Since $T_t^{h_1}f$ is differentiable (with derivative uniformly bounded in h given by $T_t^{h_1,(1)}f'$), we can estimate (by mean value theorem)

$$\left|\left(-L_{h_1*} + L_{h_2*}\right)T_s^{h_1}f(x)\right| \leq \int_{h_2 \leq |y| \leq h_1}\left|T_s^{h_1}f(x+y) - T_s^{h_1}f(x)\right|\nu(x,y)dy$$

$$\leq \int_{h_2 \leq |y| \leq h_1}||T_s^{h_1,(1)}f'|||y|\nu(x,y)dy$$

$$= o(1)||T_s^{h_1,(1)}f'|| = o(1)||f||_{C^1}, \quad h_1 \to 0.$$

The last equality holds due to the assumption (H0) (i.e, the uniform bound of the first moment of ν and its tightness property). Therefore,

$$\left\| \left(T_t^{h_1} - T_t^{h_2} \right) f \right\| = o(1)t\|f\|_{C^1}. \tag{8.7}$$

Thus, for each $f \in C^3[a,b]$ with $f' \in C_0[a,b]$, the family $\{T_t^h f\}$ converges to a limiting family $\{T_t f\}$ as $h \to 0$. It follows then that the limiting family forms a strongly continuous semigroup of contractions on $C[a,b]$ (by standard approximation arguments). Now write

$$\frac{T_t f - f}{t} = \frac{T_t f - T_t^h f}{t} + \frac{T_t^h f - f}{t}.$$

Using the estimate (8.7), we conclude that $\{f \in C^3[a,b] : f' \in C_0[a,b]\}$ belongs to the domain of the generator, and that the generator is given by $-L_{[a,b]*}$ as

$$\lim_{t\downarrow 0} \frac{T_t f - f}{t} = \lim_{h\downarrow 0} \lim_{t\downarrow 0} \frac{T_t f - T_t^h f}{t} + \frac{T_t^h f - f}{t} = -L_{[a,b]*}f.$$

Now, take $f \in C^2[a,b]$ and $\{f_n\} \subset \{f \in C^3[a,b] : f' \in C_0[a,b]\}$ such that $f_n \to f$ uniformly as $n \to \infty$. Since the operator $-L_{[a,b]*}$ is closed [13, Corollary 1.6] and $-L_{[a,b]*}f_n \to g$ as $n \to \infty$ for some g, it follows that $g = -L_{[a,b]*}f$ and $f \in \hat{\mathfrak{D}}_*$. Therefore, the space $\{f \in C^2[a,b] : f' \in C_0[a,b]\}$ also belongs to the domain of the generator, as required.

(ii) Take the function $f_w(x) = (x - a)^w$ for some sufficiently small $w \in (0,1)$. We will prove that $\left(-L_{[a,b]*}f_w\right)(x) < 0$ for $x \in (a,c)$ and $c \in (a,b)$ (see method of Lyapunov functions, e.g., [26, Proposition 6.3.2]). Since

$$\left(-L_{[a,b]*}f_w\right)(x) = -L_{[a,b]*}^{(\nu)}f_w(x)+w\gamma(x)(x-a)^{w-1}+w(w-1)\alpha(x)(x-a)^{w-2},$$

when $\gamma(a) = 0$ and $\alpha(a) > 0$, then $\left(-L_{[a,b]*}f_w\right)(x) < 0$ as the first two terms in the r.h.s of the previous equality are dominated by the last term which tends to $-\infty$ as $x \to a$. The regularity for $x = b$ is proved analogously but with $f_w(x) = (b - x)^w$. Finally, Proposition 6.3.2 in [26] implies the finite expectation of $\hat{\tau}_{(a,b)}(x)$. □

8.2. **Proof of Theorem 4.2.**

P r o o f. (i) Theorem 4.1 implies that $(-L_{[a,b]*}, \hat{\mathfrak{D}}_*)$ generates a Feller process $\hat{X}_x$ on $[a,b]$ and ensures the regularity in expectation of the boundary points $\{a,b\}$. Hence, the stopped process $X_x^{[a,b]*} := \{\hat{X}_x(s \wedge \tau_{(a,b)}(x))\}_{s\geq 0}$ is also a Feller process on $[a,b]$ [26, Theorem 6.2.1, Chapter 6]. Let us denote by $(-L_{stop}, \mathfrak{D}_{[a,b]*}^{stop})$ the generator of the stopped process with a domain

denoted by $\mathfrak{D}_{[a,b]*}^{stop}$. By definition of $X_x^{[a,b]*}$ the states $\{a, b\}$ are absorbing, which implies that $(-L_{stop}f)(x) = 0$ for $x \in \{a, b\}$ and $f \in D_{[a,b]*}^{stop}$.

Take now $f \in \hat{\mathfrak{D}}_*$ such that $-L_{[a,b]*}f(x) = 0$ in $\{a, b\}$. Since the domain of the generator is given by the image of its resolvent operator (say $\hat{R}_\lambda$), given $f \in \hat{\mathfrak{D}}_*$ there exists $g \in C[a, b]$ such that $f = \hat{R}_\lambda g$.

Using that f solves the resolvent equation

$$\lambda \hat{R}_\lambda g + L_{[a,b]*}f = g,$$

and that (by assumption) $-L_{[a,b]*}f(x) = 0$ for $x \in \{a, b\}$, we get

$$f(a) = \hat{R}_\lambda g(a) = g(a)/\lambda \quad \text{and} \quad f(b) = \hat{R}_\lambda g(b) = g(b)/\lambda. \tag{8.8}$$

Moreover, Dynkin's formula implies

$$\hat{R}_\lambda g(x) = \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} e^{-\lambda s} g\left(\hat{X}_x(s)\right) ds\right] + \mathbf{E}\left[e^{-\lambda \tau_{(a,b)}(x)} f\left(\hat{X}_x(\tau_{(a,b)}(x))\right)\right]$$

for each $x \in (a, b)$. Using that the paths of the process $\hat{X}_x$ and $X_x^{[a,b]*}$ coincide before the first exit time $\tau_{(a,b)}(x)$, the previous expression becomes

$$\hat{R}_\lambda g(x) = \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} e^{-\lambda s} g\left(X_x^{[a,b]*}(s)\right) ds\right]$$
$$+ \mathbf{E}\left[e^{-\lambda \tau_{(a,b)}(x)} f(a)\mathbf{1}_{\{\tau_a < \tau_b\}} + f(b)\mathbf{1}_{\{\tau_b < \tau_a\}}\right],$$

where τ_a and τ_b denote the first exit time through the boundary point a and b, respectively. Finally, plugging (8.8) into the second term of the r.h.s of the last formula we get that $f = \hat{R}_\lambda g = R_\lambda^{[a,b]*}g$, where $R_\lambda^{[a,b]*}$ denotes the resolvent operator of $X^{[a,b]*}$. Therefore, $f \in \mathfrak{D}_{[a,b]*}^{stop}$ as there exits $g \in C[a, b]$ such that $f = R_\lambda^{[a,b]*}g$, which in turn implies that $-L_{stop}f = -L_{[a,b]*}f$.

(ii) Follows the same arguments as before, so that we omit the details. $\square$

Acknowledgements

References

[1] O.P. Agrawal, Formulation of Euler-Lagrange equations for fractional variational problems. *J. Math. Anal. Appl.* **272**, No 1 (2002), 368–379.

[2] O.P. Agrawal, Some generalized fractional calculus operators and their applications in integral equations. *Fract. Calc. Appl. Anal.* **15**, No 4 (2012), 700–711; DOI: 10.2478/s13540-012-0047-7; https://www.degruyter.com/view/j/fca.2012.15.issue-4/
issue-files/fca.2012.15.issue-4.xml.

[3] T.M. Atanackovic and B. Stankovic, On a differential equation with left and right fractional derivatives. *Fract. Calc. Appl. Anal.* **10**, No 2 (2007), 139–150; at http://www.math.bas.bg/∼fcaa.

[4] R.N. Bhattacharya and E.C. Waymire, *Stochastic Processes with Applications*. Wiley Ser. in Probability and Mathematical Statistics (1990).

[5] J. Bertoin, On the first exit time of a completely asymmetric stable process from a finite interval. *Bull. London Math. Soc.* **28** (1996), 514–520.

[6] R.M. Blumenthal, R.K. Getoor and D.B. Ray, On the distribution of first hits for the symmetric stable processes. *Trans. Amer. Math. Soc.* **99** (1961), 540–554.

[7] J.P. Bouchaud and A. Georges, Anomalous diffusion in disordered media: statistical mechanism, models and physical applications. *Phys. Rep.* **195** (1990), 127–293.

[8] A. Carpinteri and F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics.* Springer-Verlag, New York (1997), 291348.

[9] K. Diethelm, N.J. Ford, and A.D. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations. *Nonlinear Dynam.* **29**, No 1-4 (2002), 3–22.

[10] K. Diethelm, *The Analysis of Fractional Differential Equations, An Application-oriented Exposition Using Differential Operators of Caputo Type.* Lecture Notes in Math., Springer (2010).

[11] E.B. Dynkin, *Markov Processes*, Vol. I. Springer-Verlag (1965).

[12] J.T. Edwards, N.J. Ford, and A. C. Simpson, The numerical solutions of linear multi-term fractional differential equations: Systems of equations. *J. Comput. Appl. Math.* **148** (2001), 401–418.

[13] S.N. Ethier and T.G. Kurtz, *Markov Processes. Characterization and Convergence.* Wiley Ser. Probab. Math. Statist., Wiley, New York (2010).

[14] R. Gorenflo and F. Mainardi, Fractional calculus and stable probability distributions. *Arch. Mech.* **50**, No 3 (1998), 377–388.

[15] R. Gorenflo and F. Mainardi, *Fractional Calculus: Integral and Differential Equations of Fractional Order.* CISM Lecture Notes, International Centre for Mechanical Sciences, Italy (2008).

[16] M.E. Hernández-Hernández and V.N. Kolokoltsov, On the probabilistic approach to the solution of generalized fractional differential equations of Caputo and Riemann-Liouville type. *J. Fractional Calc. and Appl.* **7**, No 1 (2016), Article No 14, 147–175.

[17] M.E. Hernández-Hernández and V.N. Kolokoltsov, Probabilistic solutions to nonlinear fractional differential equations of generalized Caputo and Riemann-Liouville type. Submitted.

[18] R. Hilfer, Fractional time evolution. In: *Applications of Fractional Calculus in Physics*, R. Hilfer, Ed., World Scientific Publ. Co., Singapore, New Jersey, London and Hong Kong (2000), 87–130.

[19] S.L. Kalla, Operators of fractional integration. In: *Proc. Conf. Analytic Functions Kozubnik 1979*, Publ. as: Lecture Notes in Math. **798** (1980), 258–280.

[20] O. Kallenberg, *Foundations of Modern Probability*, 2nd Ed., Springer (Probability and Its Applications) (2001).

[21] P. Kamal, L. Fang, Y. Yubin and R. Graham, Finite difference method for two-sided space-fractional partial differential equations. In: I. Dimov, I. Farago, L. Vulkov (Eds.), *Finite Difference Methods, Theory and Applications, 6th International Conference* (2014), 307–314.

[22] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Math. Stud. **204** (2006).

[23] V.S. Kiryakova, *Generalized Fractional Calculus and Applications*. Pitman Res. Notes in Math. Ser. 301, Longman & J. Wiley, Harlow & N. York (1994).

[24] V. Kiryakova, A brief story about the operators of the generalized fractional calculus. *Fract. Calc. Appl. Anal.* **11**, No 2 (2008), 203–220; at http://www.math.bas.bg/~fcaa.

[25] V. N. Kolokoltsov, Generalized continuous-time random walks (CTRW), subordination by Hitting times and fractional dynamics. *Theory Probab. Appl.* **53**, No 4 (2009), 549–609.

[26] V.N. Kolokoltsov, *Markov Processes, Semigroups and Generators*. De-Gruyter Studies in Mathematics, Book 38 (2009).

[27] V.N. Kolokoltsov, On fully mixed and multidimensional extensions of the Caputo and Riemann-Liouville derivatives, related Markov processes and fractional differential equations. *Fract. Calc. Appl. Anal.* **18**, No 4 (2015), 1039–1073; DOI: 10.1515/fca-2015-0060; https://www.degruyter.com/view/j/fca.2015.18.issue-4/ issue-files/fca.2015.18.issue-4.xml.

[28] A.E. Kyprianou and A.R. Watson, Potentials of stable processes. In: Donati-Martin, C., Lejay, A. and Roualt, A., Eds. *Seminaire de Probabilites XLV1.* Springer, Switzerland (2014), 333–344.

[29] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics. *http://arxiv.org/abs/1201.0863v1*, 2012.

[30] M.M. Meerschaert and A. Sikorskii, *Stochastic Models for Fractional Calculus.* De Gruyter Studies in Mathematics **43** (2012).

[31] M.M. Meerschaert, C. Tadjeran, Finite difference approximations for two-sided space-fractional partial differential equations. *Appl. Numer. Math.* **56**, No 1 (2006), 80–90.

[32] I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications.* Mathematics in Science and Engineering **198**, Academic Press, Inc., San Diego (1999).

[33] F. Sabzikar, M.M. Meerschaert, J. Chen, Tempered fractional calculus. *J. Comput. Phys.* **293** (2015), 14–28.

[34] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications.* Gordon and Breach Science Publishers S. A. (1993).

[35] B. Stankovic, An equation in the left and right fractional derivatives of the same order. *Bull. Cl. Sci. Math. Nat. Sci. Math.* No 33 (2008), 83–90.

[36] B. Stankovic, Large linear equation with left and right fractional derivatives in a finite interval. *Bull. Cl. Sci. Math. Nat. Sci. Math.* No 36, (2011), 61-79.

[37] C. Torres, Existence of a solution for the fractional forced pendulum. *J. Appl. Math. Comput. Mech.* **13**, No 1 (2014), 125–142.

[38] S. Watanabe, On stable processes with boundary conditions. *J. Math. Soc. Japan* **14**, No 2 (1962), 170–198.

[39] G.M. Zaslavsky, Chaos, fractional kinetics, and anomalous transport. *Phys. Rep.* **371**, No 6 (2002), 461–580.

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