Hide-and-Seek Games on a Network, Using Combinatorial Search Paths

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Abstract

This paper introduces a new search paradigm to hide-and-seek games on networks. The Hider locates at any point on any arc. The Searcher adopts a 'combinatorial' path when searching the network: a sequence of arcs, each adjacent to the last, and traced out at unit speed. In previous literature the Searcher was allowed 'simple motion', any unit speed path, including ones which turn around inside an arc. The new approach more closely models real problems such as search for IEDs using vehicles which can only turn around at particular locations on a road. The search game is zero sum, with the time taken to find the Hider as the payoff.

Using a lemma giving an upper bound for the expected search time on a semi Eulerian network, we solve the search game on a network Q_3 consisting of two nodes connected by three arcs of arbitrary lengths. When two Q_3 networks with unit length arcs are linked by two small central arcs incident at the start node, one of these arcs must be traversed at least three times in an optimal search. This property holds for both combinatorial paths and simple motion paths, and the latter makes it a counterexample to a conjecture of S. Gal which said that two traversals were always sufficient.

1 Introduction

Network search games are zero sum games where the payoff to the maximizing Hider is the time taken for the Searcher to *find* him, to be located at the same point of the network. When the topic of 'search games with immobile hiders' was first introduced in a speculative final chapter in the classic text *Differential Games* by Rufus Isaacs (1965), the search paths available to the Searcher were what Isaacs called a 'simple motion', with "no other restriction save that its speed w is constant". In particular, he said that "paths with sharp corners are not outlawed". Of course Isaacs was speaking in a multidimensional context, but this means that for network search it has been allowed, since the early work of Gal (1979), for the Searcher to change directions inside an arc.

This paper initiates the study of network search games in which the allowed paths to search the network are the more traditional paths known in graph theory, computer science and operations research: sequences of arcs, each one sharing a common node with the previous one. Our new assumption, that the Searcher can only turn around at designated locations, more closely models real problems where the network is a road system, Hider is an IED (improvised explosive device), and the searcher is a large vehicle that can only turn around at intersections or other wider places. Since the networks studied in search games have given lengths for each arc, it makes sense to additionally say that these paths (which we call *combinatorial paths* as opposed to *simple motion paths*) will be traced out at unit speed. The assumption made here of combinatorial search paths simplifies the game by making the strategy sets finite, and enables us to study a wider class of networks. For weakly Eulerian networks (whose 2-connected components are Eulerian), the important result of Gal (2000) says that an optimal search strategy is to choose any Chinese Postman (one of minimal length $\bar{\mu}$) Tour (CPT) and traverse it equiprobably in either direction. Since a CPT path is a combinatorial path, it follows that for weakly Eulerian networks simple motion paths and combinatorial paths result in equivalent search games. That is why we deal here exclusively with networks which are not weakly Eulerian. For network search games with combinatorial paths, the class of networks that appears to be important are the semi Eulerian networks, those with exactly two nodes of odd degree. We study these networks here, as well as some made by combining such networks.

We analyze the following zero sum game. A Searcher wishes to minimize the time taken to find a Hider on a known connected network Q. Starting at a designated node O, he chooses a combinatorial path, a sequence of adjacent arcs, each traversed at unit speed. Each arc A of Q has a given length μ_A and the network Q has total length denoted by μ. The immobile Hider can choose to hide at any point H of Q. The arcs are not directed but it is useful to give them an arbitrary orientation so that any point on arc A is determined by its distance d_A from the back end of A. A mixed strategy for the Hider is a probability measure h on Q. It is clear that two measures h and h' are equivalent for the Hider (have the same expected search times) if for every arc A, (i) $h\left(A\right) = h'\left(A\right)$, and (ii) the mean of each measure is at the same distance d_A from the back end of A. The last condition simply says that the center of gravity of the measures are the same. Thus a mixed strategy for the Hider is determined by the numbers $h(A)$ and d_A for all arcs A. This game $G(Q,O)$ is a zero sum game between the minimizing Searcher and the maximizing Hider and has a value $V = V(Q,O)$. The existence of a value follows from the fact that the game is essentially one with finitely many pure strategies for each player. For the Hider, the $2m$ pure strategies (where m is the number of arcs) correspond to hiding at the back or front end ('next' to the corresponding node but not on it) of each arc, and hence hiding at distance d_A on arc A is a convex combination of the two pure strategies on A. (We can either interpret $d_A = 0$ and μ_A as ideal points of Q, next to a node but not on it; or we can disallow such values of d_A and have only $\varepsilon-$optimal strategies for the Hider. (In practice this is not a problem.) For the Searcher, the m arcs can be labelled A_1 to A_m in the order of their first appearance in an undominated search path. (A pure strategy is said to be dominated by another if the other one does better against any pure strategy of the opponent.) There are at most $m!$ ways of ordering the first appearances. Between the first appearance of A_i and of A_{i+1} in the path there are no new points of Q covered, so the intervening arcs must be taken from the subnetwork consisting of arcs A_j, $j \leq i$, and form a shortest path in this subnetwork. A shortest path between two nodes of a network cannot include any arc more than once, so between A_i and A_{i+1} there can be at most $i-1$ connecting arcs. Thus there are at most m first appearance arcs and at most $0+1+2+\cdots+(m-1)$ connecting arcs, and hence the length of any undominated search path is bounded above by $m(m+1)/2$. This estimate is likely much too high, but in any case establishes finite length and hence a finite number of undominated search paths. Note that, unlike the case for Chinese Postman Tours, undominated search paths may include a given arc more than twice (a new result of this paper).

This note initiates the study of the search game with combinatorial search paths for networks which are not weakly Eulerian. It is to be hoped that this approach will enable progress on the basic game to more general networks (as opposed to newer variations), which has been sparse since 2000.

For a background in the area of Search Games, see the texts of Gal (1980), Garnaev (2000), Alpern & Gal (2003) and the edited volume Alpern et al (2013). Articles surveying more recent work are Gal (2011,2013), Alpern (2011b) and Lidbetter (2013). Many new versions of network search games have also been recently introduced: arbitrary searcher starting point (Dagan-Gal 2008), find-and-fetch search (Alpern, 2011a), search on windy networks (Alpern 2010, Alpern and Lidbetter (2014)), an expanding search region rather than a path (Alpern and Lidbetter 2013), search at nodes of a lattice (Zoroa et al 2013), search for a Hider at nodes with searching costs (Baston and Kikuta, 2015), and two-speed search for a small object (Alpern and Lidbetter 2015). Computational approaches to determining optimal strategies are given in Anderson and M. Aramendia (1990). The original article in the field was Gal (1979), extended by Reijnierse and Potter (1993).

2 Searching a Semi Eulerian Network

Note that for Eulerian networks, the value is already known: tracing an Eulerian tour equiprobably in either direction is optimal, with $V = \mu/2$. It is not possible to have only one node of odd degree, so the next type of network to consider is one with exactly two nodes of odd degree. Such a network is called $semi - Eulerian$ (or sometimes $traversable$, because it has an Eulerian path). An example of such a network is shown below in Figure 1, together with a distinguished path P as in the following Lemma.

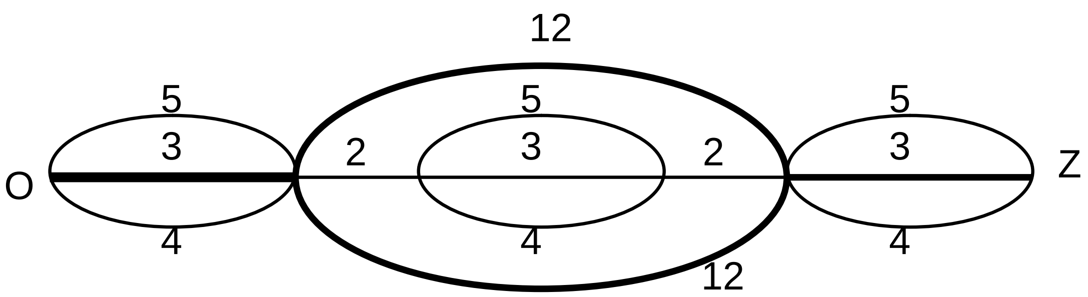

Figure 1. Network Q with path P thick (top or bottom), $a = 3 + 12 + 3 = 18$.

Lemma 1 *Let Q be a semi Eulerian network with O and Z its two nodes of odd degree. Let P be a path from O to Z of minimum length a such that $Q - P$ is connected. Then*

$$V(Q, O) \leq \bar{V} \equiv \left(a^2 + \mu^2\right)/2\mu. \tag{1}$$

A strategy guaranteeing this expected search time is as follows: with probability $p =$

$(a+\mu)\,/\,(2\mu)$, *first traverse* P *from* O *to* Z *and then follow an Eulerian tour of* $Q-P$ *from* Z, *equiprobably in either direction; with probability* $1-p$, *first follow an Eulerian tour of* $Q-P$ *from* O, *equiprobably in either direction, and then traverse* P *from* O *to* Z. *To obtain this expected time, the Hider must hide near* Z *when hiding on* P.

Proof. If $H \in P$, at distance ε from Z, the expected search time T satisfies

$$T \leq p\,(a-\varepsilon) + (1-p)\,((\mu - a) + (a-\varepsilon)) = \frac{1}{2\mu}\left(a^2 + \mu^2 - 2\varepsilon\mu\right) \leq \bar{V},$$

with equality only if H is next to Z, that is, as $\varepsilon \to 0$. If $H \in Q-P$, which is an Eulerian network of total length $\mu - a$, we have

$$T \leq p\left(a + \frac{\mu - a}{2}\right) + (1-p)\left(\frac{\mu - a}{2}\right) = \bar{V}.$$

∎

Of course there may be no path P for which $Q-P$ is connected, in which case the Lemma does not apply. A natural question that arises from this general bound is when is it tight. The next two sections give examples where it is tight and where it is not. Also note that when the bound is tight the Searcher can restrict to adopting Eulerian paths. In Section 4 we show that sometimes an optimal search strategy for a semi Eulerian network requires the use of paths which are not Eulerian.

Of course any upper bound on the value of a game with combinatorial search is also an upper bound for the traditional version of the game, with simple motion, so Lemma 1 applies to those games as well. For semi Eulerian networks like those covered by Lemma 1, the length $\bar{\mu}$ of a minimal tour is given by $\mu + a$, as arcs of total length a must be traversed twice. The upper bound on the value of $\bar{V}$ is better than that established by Gal (1979) for arbitrary networks of $\bar{\mu}/2$, because $\bar{V}$ can be written as $\mu/2 + (a/2)\ \ (a/\mu)$ whereas Gal's formula gives a higher upper bound of $(\mu + a)/2$, or $\mu/2 + (a/2)$. This shows that networks satisfying the assumptions of Lemma 1 cannot be weakly Eulerian, where the $\bar{\mu}/2$ bound is tight. See Gal (2000).

3 A Three Arc Network

We now consider what is possibly the simplest non weakly Eulerian network, the so called 'three arc network' $Q_3 = Q_3(a, b, c)$ consisting of two nodes, the start node O and another node Z, and three arcs A, B, C of lengths $a \leq b \leq c$. The 'arbitrary orientation' of arcs for notation is from O to Z. Clearly Q_3 is of the form of the Lemma, taking for P the single arc A. To show that $V = \bar{V}$ we must find a suitable hiding strategy. We derive the optimal hiding strategy under the assumption that $V = \bar{V}$ and that the Hider locates in each arc with a probability proportional to its length. From the analysis of Lemma 1, we know that when hiding on A, the Hider must be close to Z, that is, his distance d_A from O along A must be close to a. We denote the hiding points on B and C

as H_B and H_C respectively and label their distances from O along these arcs as $f = d_B$ and $d = d_C$. Clearly $\mu = a + b + c$. See Figure 2.

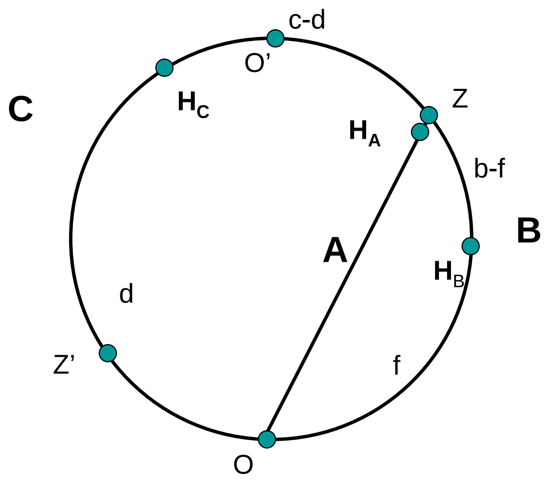

Figure 2. The 'Three-arc' Network

$Q_3(a, b, c)$.

The Hider wants to ensure that when touring the circle $CB = BC$ from either O or Z, the Searcher is indifferent between the two directions. For any circle, this is the case when the center of gravity (mean) of the distribution, considered as being on the line segment obtained by cutting the circle at the starting point, is located at the antipodal point to the start (at a distance from the left side of half the circumference). When starting the tour of BC at O, this means that

$$\frac{b}{b+c}(f)+\frac{c}{b+c}(b+c-d)=\frac{b+c}{2}, \text{ or}$$

$$d=\frac{2bf-b^2+c^2}{2c}. \tag{2}$$

With this relation between d and f we also have that the mean of the distribution is at Z' when cutting at Z.

Recall that in the search strategy of the Lemma 1, the shortest path P from O to Z (which for Q_3 is the single arc A) is always searched first or last, never in the middle. To ensure that CBA is better than CAB (after C) we need

$$\frac{b}{\mu}(b-f)+\frac{a}{\mu}(b+a)\leq\frac{a}{\mu}(0)+\frac{b}{\mu}(a+f), \text{ or}$$

$$f\geq\frac{a^2+b^2}{2b}. \tag{3}$$

The condition (3) also ensures that BCA is better than BAC (after B). Finally, we want to make sure that the shortest path from O to H_B is the direct one via B rather than via A and Z. (We will need this for example to exclude consideration of $ACAB$.) This requires that

$$f\leq a+(b-f), \text{ or}$$

$$f \leq \frac{a+b}{2}. \tag{4}$$

This condition (4) together with (2) implies

$$\begin{aligned} 2d &\leq c + \left(\frac{b}{c}\right) a \leq c + a, \text{ or} \\ a + (c - d) &\geq d. \end{aligned}$$

$$a + (c - d) \geq d. \tag{5}$$

The distance from O to H_C via A and Z is given by the left side of (5), so it is at least d. Hence the condition (4) also ensures that the shortest path from O to H_C is the direct one via C.

Proposition 2 *For the search game on the three arc network $Q_3(a, b, c)$ (where A, B, C are the arcs of respective lengths $a \leq b \leq c$), the value of the search game with combinatorial search paths is given by $V = (a^2 + \mu^2) / (2\mu)$. An optimal strategy for the Searcher is the following, where $p = (a + \mu) / (2\mu)$: with probability p first traverse arc A from O to Z and then tour the cycle BC from Z, equiprobably in either direction. With complementary probability $1 - p$, first tour the cycle BC from O, equiprobably in either direction and then traverse A from O to Z. For the Hider, it is ε-optimal to hide in each arc with a probability proportional to its length, and at a distance x from O on* *arc X, given by*

$$d_A = a - \varepsilon, \text{ that is, at the end of } A \text{ at } Z, \tag{6}$$

$$d_B = f, \text{ for } \frac{1}{2b}\left(a^2+b^2\right) \le f \le \frac{(a+b)}{2}, \text{ and} \tag{7}$$

$$d_C = d = \frac{1}{2c}\left(2bf - b^2 + c^2\right). \tag{8}$$

More generally, any hider distribution with mean on arc X at distance d_X as above is $\varepsilon-$optimal.

In particular if the lengths of the two smaller arcs are equal, then it is optimal to hide at the ends of these arcs at Z and at distance $\left(b^2+c^2\right)/(2c)$ from O on the long arc. If all three arcs have the same length, then hide equiprobably at the ends of each arc at Z.

Proof. Note that $((a+b)/2) - ((a^2+b^2)/2b) = (a/2b)(b-a) \ge 0$, so that the feasible set for the parameter f given in (7) is not empty. We first evaluate the stated Hider strategies against the six Eulerian search paths using three (distinct) arcs:

$$ABC,\ ACB,\ BCA,\ CBA\ BAC,\ CAB. \tag{9}$$

First note that the first two have the same search times by (2), and similarly the third and fourth. Furthermore, the lower bound condition (3) on f guarantees that BAC is worse than BCA and CAB is worse than CBA. Hence it is enough to evaluate the two search paths ABC and BCA against the stated hiding strategy. For these, we have

$$
\begin{aligned}
ABC &: \quad \frac{a}{\mu}(a) + \frac{b}{\mu}(a + (b - f)) + \frac{c}{\mu}(a + b + d) = \left(a^2 + \mu^2\right) / (2\mu)\,, \\
BCA &: \quad \frac{b}{\mu}(f) + \frac{c}{\mu}(b + (c - d)) + \frac{a}{\mu}(a + b + c) = \left(a^2 + \mu^2\right) / (2\mu)\,.
\end{aligned}
$$

Next we must consider search paths with more than three arcs. These might be used, for example, when a is small and the Searcher would rather continue his search from O rather than Z or from Z rather than O. The shortest path between O and Z is arc A, so repeating other arcs than A is worse than repeating A. The longer paths we must consider are

$$ABAC, BAAC, ACAB,\ CAAB, AABC, AACB.$$

After A and B are searched (in either order), the shortest path from O to H_C is via arc C, because of condition (4). So we do not have to consider the first two paths. A similar argument shows we may ignore the next two paths. Finally, we consider the last two paths. The last two paths might make sense to use if the search time for the circle BC was smaller when starting from Z than when starting from O, but in fact in both cases it is $(b + c)/2$. $\blacksquare$

An anonymous referee has suggested that we mention the matrix form of the game on Q_3 that we alluded to above when observing that with combinatorial search paths the strategy sets are finite. We can consider that the maximizing Hider has six pure strategies that we can denote by A_O, A_Z, B_O, B_Z, C_O and C_Z. The subscript O corresponds to hiding arbitrarily close to node O on the indicated arc; Z to hiding near node Z. Hiding at an arbitrary point on say arc B can be written as a convex combination of hiding at B_O and B_Z. The undominated strategies for the Searcher are ABC, CBA, BAC, CAB, ACB, BCA (six having three arcs), $ABAC, BAAC, ACAB, CAAB, AABC, AACB$ (another six with four arcs). Note that the only arc which can be repeated in an undominated strategy is the shortest one, A. For example $CACB$ is dominated by $CAAB$ because arc B is search sooner in the later strategy. So the search game on the three arc network can be represented by a 6×12 matrix. Of course further analysis, as we did above, can reduce the number of columns (searcher strategies) that need to be considered.

Observe that for the network considered in this section, optimal search was concentrated on Eulerian paths which could be extended to Chinese Postman tours by returning to the start node after covering the whole network. Presumably this analysis can be extended to two nodes connected by an odd number of unequal length arcs.

4 The Double-Triple Network

We now give an example of a network DT, which we call the $double-triple$ network, which is semi Eulerian and satisfies $V < (a^2 + \mu^2)/(2\mu)$. The network DT consists of two ‘three arc’ networks Q_3 with unit length arcs, which are attached at an end of each. The start node is one of the nodes of degree 3. (In the following section we will consider a central start.) For this network, we have $\mu = 6$ and the value of a in Lemma 1 is $a = 2$. Label the three arcs at O as A and the three at the far end as B, using the symmetry of the network.

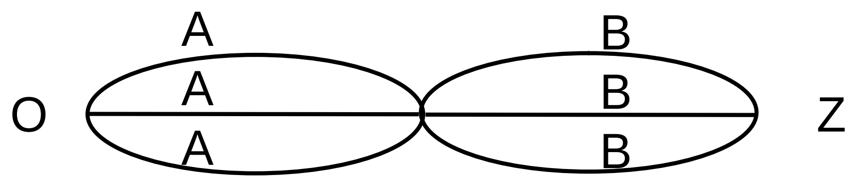

Figure 3. The Double-Triple Nework DT, with left start.

It turns out that, unlike the case for weakly Eulerian networks, the value of the double-triple network DT depends on the searcher starting point. In the next two subsections, we consider both an end start and a central start.

4.1 The Double-Triple Network with End Start

We first consider the case where the Searcher begins his search path from an end node, say the left end O. We find the following.

Proposition 3 *Consider the Double-Triple Network DT, with the Searcher starting point taken as the left end node O, as in Figure 3. DT has $\mu = 6$, $a = 2$, and hence*

$\bar{V} = 10/3 \simeq 3.33$. *The value of the search game is given by* $V = 29/9 \simeq 3.22$, *which is strictly less than* $\bar{V}$. *The optimal search strategy chooses a random path of the type* A, B, B, A, A, B *(a Eulerian path) with probability* $5/6$ *and a random path of the type* A, B, B, B, B, A, A *with probability* $1/6$. *By 'type' we mean that for an arc* $X \in \{A, B\}$ *a random untraversed one is chosen; if all have been traversed any can be chosen. An optimal hiding strategy hides near* Z *on a random adjacent arc* B *with probability* $2/3$ *and at the far end of a random arc* A *at* O *with probability* $1/3$.

Proof. First we show that $V \leq 29/9$. Suppose H is at distance x, $0 < x < 1$, from O. With probability $5/6$ the expected time for the Searcher to reach H is

$$\frac{1}{3}\left(x + (4 - x) + (4 + x)\right) = \frac{8 + x}{3} \leq 3 \text{ (with equality as } x \to 1).$$

With probability $1/6$, the expected search time is

$$\frac{1}{3}\left(x + (6 - x) + (6 + x)\right) = \frac{12 + x}{3} \leq \frac{13}{3}$$

Hence overall, the expected time to reach H is given by

$$\frac{5}{6}(3) + \frac{1}{6}\left(\frac{13}{3}\right) = \frac{29}{9}.$$

Next suppose H is at distance y, $0 < y < 1$ from Z. Then with probability $1/6$ the expected time to reach H is $1+V'$, where $V' = 5/3$ is the value of the game on the three arc game, from Proposition 2. With probability $5/6$, the time is given by

$$\frac{1}{3}\left((2-y)+(2+y)+(6-y)\right) = \frac{10-y}{3} \leq \frac{10}{3}, \text{(with equality as } y \to 0).$$

Hence overall, the expected time to reach H is at most

$$\frac{1}{6}\left(1+5/3\right) + \frac{5}{6}\left(\frac{10}{3}\right) = \frac{29}{9}.$$

If H is a node, the expected search time is strictly lower.

Next we show that $V \geq 29/9$. Clearly the first arc chosen is of type A. Then the Searcher can either (i) search the remaining arcs incident to O, so obtaining the full path $AAABBB$, or (ii) continue BB, to produce either of the full paths $ABBAAB$ or $ABBBBAA$. The expected search times for these three paths against the stated hiding strategy are as follows.

Path	Expected Search Time
$AAABBB$	$\frac{1}{9}(1+1+3)+\frac{2}{9}(4+4+6)=\frac{33}{9}\geq\frac{29}{9}$
$ABBAAB$	$\frac{1}{9}(1+3+5)+\frac{2}{9}(2+2+6)=\frac{29}{9}$
$ABBBBAA$	$\frac{1}{9}(1+5+7)+\frac{2}{9}(2+2+4)=\frac{29}{9}$

■

It is useful to note that in the semi Eulerian network DT, the Eulerian path P of Lemma 1 is no longer a single arc, but is of the form AB (two arcs). This fact is related to having a value less than $\bar{V}$.

4.2 The Double-Triple Network with Central Start

We now consider the DT network analyzed in the previous section, but with the start node between the two copies of the three-arc network. This considerably changes the optimal strategies but, surprisingly, not the value.

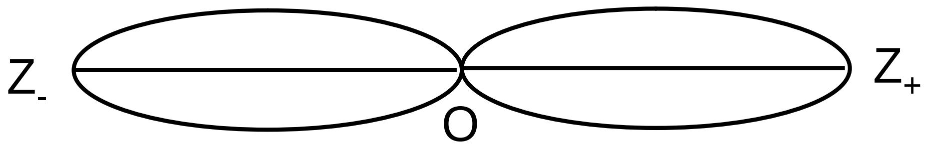

Figure 4. The DT network with central start.

Due to the symmetry of DT with central start, all the six unit length arcs are equivalent under automorphism and so the Hider has only a one parameter family of mixed strategies, namely to hide equiprobably on arcs and at distance x, $0 < x < 1$, from the end nodes $Z_{\pm}$ (or equivalently, at distance $1 - x$ from O. The Searcher must begin by going to an end, so without loss of generality assume he starts with A. Then he must continue with A. From this point there are two possibilities: (i) he can go back to Z_- and then search the right side, this is path $AAAABBB$, or (ii) he can continue with BB, and then by symmetry we can assume the remainder of the path is BBA,. giving full path $AABBBBA$. Note that while this network is semi Eulerian, Lemma 1 does not apply because the start node has even degree. The expected search times for these two potential optimal paths are as follows:

$$\begin{aligned}
AAAABBB,\ \frac{1}{6}\left((1-x)+(1+x)+(3-x)+(5-x)+(5+x)+(7-x)\right) &= \frac{22-2x}{6}. \\
AABBBBA,\ \frac{1}{6}\left((1-x)+(1+x)+(3-x)+(3+x+(5-x)+(7-x)\right) &= \frac{20-2x}{6}
\end{aligned}$$

Clearly it is optimal for the Hider to locate at the far end of the arc from O ($x \simeq 0$) and for the Searcher to search for example the top of the network first (including the middle) and then the bottom, any path equivalent to $AABBBBA$, always choosing equiprobably among untraversed arcs (of type A or B). We see that the value of the game for middle start on DT is 10/3, the same as we found in the previous section of end start. Summarizing this analysis we have the following.

Proposition 4 *For middle start on the network DT, we have $V = 10/3$, the optimal search strategy is to choose randomly among paths equivalent to $AABBAAB$, and for the Hider to locate equiprobably at the far ends (away from O) of the six unit arcs.*

5 The Double-Triple Network with Connecting Bridge

We now make an arbitrarily small modification to the DT network with central start as shown in Figure 4. We separate the two copies of Q_3 by two small arcs of arbitrarily small length ε which are incident to the central Searcher starting node O. We call the modified network DT^*, as shown in Figure 5. In the following two subsections, we analyze the search game on DT^* for combinatorial paths and simple motion paths.

5.1 Search with Combinatorial Paths

We begin our analysis under the assumption (as earlier in the paper) that the Searcher uses combinatorial paths. First note that the value of the search game on DT^*, which we denote by V^* remains (in the limit as $\varepsilon \to 0$) at $10/3$. The analysis is identical except now when going from O to Z the path is RB and when going directly from Z_- to Z_+ it is $ALRB$. Note that hiding on L or R is dominated, respectively, by hiding at Z_- or Z_+.

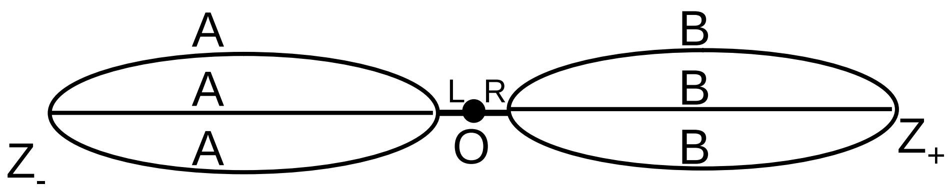

Figure 5. The DT^* network with central start, small central arcs L and R.

We now explain our reason for adding the additional infinitesimal arcs between the two copies of the three arc network. Note that by Proposition 4, for optimal searching in DT^*, one of the two central arcs *must be traversed three times*: once at the start, when leaving node O, and whenever there is a switch between an A and a B. In particular, the optimal search path $AABBBBA$ traverses arc L three times, at times $t = 0, 2$ and 6. Note that if instead of $AABBBBA$ the optimal path $AABBAAB$ is adopted, then this arc is traversed four times. This argument uses the fact that Proposition 4 gives all the optimal search paths.

Corollary 5 *Optimal search on the DT network, with the central arcs as in Figure 5, requires traversing one of the two central arcs at least three times.*

This may be seen as a counterexample to the conjecture of S. Gal (2005, Conjecture 37, p.207):

Optimal (minimax) strategies for searching a target on any graph never use trajectories which visit some arcs (or parts of arcs) more than twice.

However it is clear that this conjecture was made in the context of search paths which are allowed to change direction inside an arc (what we call simple motion paths), so the above analysis is only a counterexample when the Searcher must use combinatorial paths. But even with the simple motion paths assumed by Gal, the DT^* network with central start gives a counterexample to his conjecture, as we will see in the next section..

5.2 Search with Simple Motion Paths

We now revert to the original definition of a search path as in the earlier literature, that is, simple motion paths. Our aim is to show that even in the original context, the network DT^* requires triple traversal of arcs L or R in an optimal search. Let V_+ denote the value of the search game on DT^* (with simple motion paths) when no arc (or part of any arc) can be traversed more than twice. With this restriction on the number of arc traversals, once the Searcher enters say the right-hand copy of Q_3, he must search all of it (tour it) before returning to search all of the left copy of Q_3. Let V_3 denote the value of the search game on the three arc network $Q_3 = Q_3\,(1,1,1)$, with simple motion paths. (V_3 is known, but for the moment we pretend not to know it).

We now give a lower bound on V_+ based on the Hider locating equiprobably in either copy of Q_3 and playing the optimal (or $\varepsilon-$optimal) strategy on each copy. If the Searcher begins his search by going to the copy of Q_3 containing the Hider, the expected search time is at least V_3. If he begins by first searching the other copy, he must first spend at least time 4 touring it, and then he must spend expected time at least V_3 to find the Hider in the copy of Q_3 he is hiding in – thus total expected time at least $4 + V_3$. Since the Hider chooses to locate equiprobably in either copy of Q_3, the overall expected search time is at least

$$V_+ \geq \frac{1}{2}V_3 + \frac{1}{2}(4 + V_3) = V_3 + 2.$$

Since Q_3 has length 3, it is easy to see that by hiding uniformly we have $V_3 \geq 3/2$, and hence

$$V_+ \geq \frac{3}{2} + 2 = \frac{7}{2} = 3.5 \ . \tag{10}$$

Denote by V value of the search game on $DT^{*\cdot}$ using simple motion paths and without any restriction on the number of traversals of an arc. Clearly V is bounded above by the value $V^* = 10/3$ of Proposition 4 where the searcher is restricted to combinatorial paths. It now follows from (10) that

$$V \leq 10/3 < V_+. \tag{11}$$

This says that the Searcher can do strictly better on DT^* if he is not restricted to traversing arcs at most twice. In particular, we have the following.

Proposition 6 *For the traditional search game (without the restriction to combinatorial* *paths) on the network DT^*, an optimal search strategy requires traversing at least one of the central arcs L or R at least three times.*

We have aimed to keep the proof of this result simple, so we have not referred here to the deep result of Pavlovic (1993), which showed that in fact $V_3 = (4 + \ln 2)/3 \simeq 1.5644$, following restricted proofs of this value by Gal (1980) and Bostock (1984). Hence the calculation (10) can be improved to show that $V_+ \geq 3.564$. Hence the restriction to doubly traversing arcs increases the expected search time for the DT^* network by at least $(3.564 - 3.334)/3.334 \simeq 6.8\%$.

It is of interest to note that Proposition 6 is essentially an existence proof, as we do not actually say what the optimal search strategy is for the network DT^*. We only show that it must involve pure search paths that thrice traverse a portion of arcs L or R. The existence of an optimal strategy for search games with simple motion search paths is given in Appendix 1 of Gal (1980).

6 Conclusions

While there have been many interesting recent variations on the classical network search game of Gal (1979), the generality of networks where the classical game is understood has not been greatly expanded since the extension to weakly Eulerian networks by Gal in 2000. In the classical setting, involving the 'simple motion' suggested by Isaacs (1965), even the symmetric three arc network is difficult to analyze. By replacing Isaac's simple motion by the more familiar notion of a 'combinatorial path', the notion used in computer science, graph theory and operations research, it has been possible here to expand the class of networks that can be analyzed. Hopefully this will be just the first of many papers to use the combinatorial path paradigm. Furthermore, if general results can be found in this context they could in theory be applied to the simple motion context by putting additional nodes of degree two into the network.

A rather serendipitous finding of this paper was a counterexample to the conjecture of S. Gal that optimal search of a network using simple motion paths never requires searching any part of an arc more than twice. Is three the new upper bound, or are there networks requiring arbitrarily many traversals of some arc? Another area for exploration is the characterization of networks for which the bound of Lemma 1 is tight. Another possible application of the restriction to combinatorial paths is the min-min search problem, called the *rendezvous search problem*, where the two players have the common aim of finding each other as soon as possible. See, for example, Alpern (1995, 2002), Baston (1999), Gal (1999), Howard (1999) and Chester and Tutuncu (2004).

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