Asymptotic Analyses of the Start-Up Stage of Couette Flow Subjected to Different Boundary Conditions

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Abstract: In this article, the process for reaching "developed" stage was investigated under both imposed shear stress and specified velocity boundary conditions. Four specific situations are investigated. These are (1) constant shear stress, (2) linearly increasing shear stress from zero shear, (3) constant velocity and (4) linearly increasing velocity from stationary. Analytical solutions of velocity distributions under these four situations were obtained. A dimensionless viscosity, defined as the ratio of the measured viscosity calculated based on the measuring principle of Couette-type viscometer to the true viscosity of fluid was proposed to describe the initial transient period. We define the "developed" stage when the dimensionless viscosity is 1% away from its final value or when it reaches 1.01. By analyzing Stokes' first problem, compact models of the dimensionless viscosity were expressed and exact quantitative relations among the initial values of dimensionless viscosity under these four specific situations were found. Time periods for Couette flow to reach the "developed" stage was calculated. The development time is the shortest under the constant velocity boundary and is the longest under the linearly increasing shear stress boundary.

Key words: Couette flow; boundary conditions; development time; dimensionless viscosity

Nomenclature

d	the distance between two parallel plates, m
u	fluid velocity, $\mathrm{m/s}$
x	vertical spatial coordinate, m
t	time, s
u_0	initial velocity imposed on the moving plate, $\mathrm{m/s}$
α	acceleration of the bottom wall, $\mathrm{m/s^2}$
u_d	velocity after the flow field has developed, $\mathrm{m/s}$
u_w	imposed wall velocity, $\mathrm{m/s}$
t_d	time period to reach developed stage, i.e., development time, s
w	the velocity during the developing period, $\mathrm{m/s}$
t^*	dimensionless time, dimensionless time
$\mathcal{L}$	symbol of Laplace transform
$\mathcal{L}^{-1}$	symbol of inverse Laplace transform
s	a complex quantity
U	Laplace transform in s domain of u

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Greek Symbols

μ_M	measured viscosity based on measuring principle, $\mathrm{Pa \cdot s}$
τ_M	measured shear stress, Pa
τ_w	imposed wall shear stress, Pa
$\dot{\gamma}_M$	measured shear rate, $\mathrm{s^{-1}}$
$\tilde{\mu}$	dimensionless viscosity, dimensionless
μ	true viscosity of fluid, $\mathrm{Pa \cdot s}$
ν	kinematic viscosity of fluid, $\mathrm{m^2/s}$
τ_0	the initial shear stress imposed at inner surface of moving plate, Pa
ε	the growth rate of imposed shear stress, Pa/s
β_n	eigen value, dimensionless
ξ_n	eigen value, dimensionless
$\tilde{\mu}_\tau$	$\tilde{\mu}$ under imposed shear stress condition
$\tilde{\mu}_u$	$\tilde{\mu}$ under imposed velocity boundary condition
$\tilde{\mu}_{\tau=\tau_0}$	$\tilde{\mu}$ under the constant shear stress condition
$\tilde{\mu}_{\tau=\varepsilon t}$	$\tilde{\mu}$ under the condition of linearly increasing shear stress with time from 0
$\tilde{\mu}_{u=u_0}$	$\tilde{\mu}$ under the condition of constant velocity
$\tilde{\mu}_{u=\alpha t}$	$\tilde{\mu}$ under the condition of linearly increasing velocity with time from 0

1. INTRODUCTION

Viscosity measurements are usually conducted using flow between two cylinders. The outer cylinder is kept stationary while the inner cylinder rotates. The inner cylinder's rotation is induced through (1) imposed wall shear stress τ_w or (2) imposed wall velocity u_w. Due to the large radius of curvature of the experimental setup, the flows between the two cylinders are commonly modeled as Couette flow between two parallel plates separated by a distance d as shown in Fig. 1 (a).

Initial start-up where one plate starts suddenly at a constant velocity and the other one is kept at rest has been investigated. Analytical solutions of the time-dependent velocity distributions have been reported by many researchers[1–3]. Based on the fundamental issue, now scientists have carried out further research on unsteady Couette flow subjected to many special conditions, such as the transient Couette flow with applied pressure gradients[4], unsteady Couette flow of non-Newtonian fluid[5–10] and transient Couette flow through a porous medium or in a magnetic field[11–13]. These researchers considered the moving plate as an imposed velocity boundary. Ting[14] studied the unsteady Couette flows of a second grade fluid where the moving plate was subjected to constant tangential surface force. Bernardin and Nouar [15] investigated the transient flow in Taylor-Couette system where the moving wall was subjected to constant torque boundary.

Muzychka and Yovanovich [16] illustrated the relationship between the unsteady Couette flow and Stokes' first problem by conducting asymptotic analysis on transient Couette flow of Newtonian fluid. They proposed a compact model to describe the time-dependent shear stress on the internal surface of the moving plate. But only the imposed constant velocity boundary condition was considered. In practice, the choice of boundary condition varies in the application of Couette flow. Transient Couette flow under step velocity and step shear stress boundary conditions were discussed in literature [17]. However, the initial conditions of step flow were dynamic and the time-dependent boundary condition was not considered. Connection and comparison of the transient stage of Couette flow starting from rest under different boundary conditions should be worthy of attention.

In this article, asymptotic analysis is conducted and time periods for transient Couette flow starting from rest to reach developed stage are compared quantitatively for both imposed shear stress and imposed velocity boundary conditions. Four specific situations are considered. These are (1) constant shear stress, (2) linearly increasing shear stress with time from zero shear, (3) constant velocity and (4) linearly increasing velocity with time from stationary.

The remainder of this article is divided into seven sections. The problem analyzed in this article is outlined and the definition of the dimensionless viscosity is presented in the next section. Mathematical descriptions of the problem which include the governing equation, the initial condition and the boundary conditions are listed in section 3. The velocity distributions under different boundary conditions are obtained in section 4. Solutions of the dimensionless viscosities are presented in section 5. Dimensionless viscosity for $t \to 0$ are obtained by solving the Stokes' first problem and compact models of the dimensionless viscosity are presented in section 6. This is followed by discussion of the results. Some concluding remarks are given to conclude the article.

2. PROBLEM DESCRIPTION

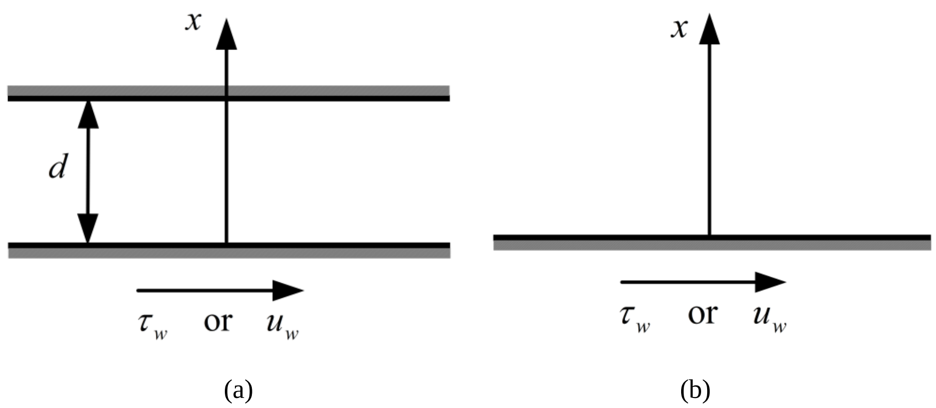

Fig. 1 (a)Schematic of the Couette flow problem (b) Schematic of the Stokes' first problem

Fig. 1(a) shows the schematic of Couette flow. The flows between two parallel plates separated by a distance d start from rest by imposing different conditions on the bottom wall while the top wall remains stationary. The velocity field of the Newtonian fluid between the parallel plates is also a function of vertical spatial coordinate x and time t.

Based on the viscosity measuring principle of Couette type viscometer, measured viscosity is described as

$$\mu_M = \frac{\tau_M}{\dot{\gamma}_M} \tag{1}$$

where the measured shear stress τ_M is the actual shear stress at the inner surface of the moving plate which has been defined in literature [17] and the measured shear rate $\dot{\gamma}_M$ is calculated by the actual velocity of moving plate and the distance between two parallel plates as

$$\dot{\gamma}_M \equiv \frac{u(x=0,t)}{d} \tag{2}$$

A dimensionless viscosity $\tilde{\mu}$ which the same meaning as the relative coefficient of viscosity in literature [17] is defined as

$$\tilde{\mu}(t) \equiv \frac{\mu_M}{\mu} = -\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} \cdot \frac{d}{u(x=0,t)} \tag{3}$$

When a constant wall velocity is imposed, the shear stress at the moving wall develops from an infinite value (due to the velocity discontinuity as a result of the sudden jump in the wall velocity) to its steady-state value. When a constant shear force is imposed, the wall velocity develops from its initial stationary value to the developed value. Similar with literature[17], in this article, we use the dimensionless viscosity $\tilde{\mu}$ to quantify the transient period of a Couette flow under different boundary conditions. During the initial transient period, the measured viscosity is different from the actual fluid viscosity. Once the fluid velocity has developed, the measured viscosity equals with the true viscosity of fluid resulting in a dimensionless viscosity of $\tilde{\mu}(t) = 1$.

At the very initial stage of a transient Couette flow, that is $t \to 0$, the velocity field has not penetrated far enough into the fluid to reach the stationary wall. Thus, the solution governed by Stokes' first problem (Fig. 1b) is valid soon after Couette flow commences[16].

After bottom plate moves suddenly in Fig. 1(a), the fluid velocity varies nonlinearly in space resulting in transient velocity gradient along coordinate x. This initial transient process of Couette flow varies with imposed boundary conditions. The objectives of this article are (1) to compare the initial transient stages of Couette flow quantitatively and (2) to estimate the length of these initial transient periods under different boundary conditions.

3. MATHEMATICAL DESCRIPTIONS

3.1 Governing Equation, Initial Condition and Boundary Condition at the Stationary Wall

In this section, we present the materials common to both types of bottom wall boundary conditions. These will be used in the solutions of the two types of problems.

From literature [17], governing equation can be written as

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \tag{4}$$

where t is the time, x is the vertical spatial coordinate, and ν is the kinematic viscosity of the fluid.

The fluid is initially stationary at $t = 0$. This can be written as

$$u(x, t=0) = 0 \text{ for } 0 \le x \le d \tag{5}$$

The top wall is always stationary which is the same as literature [17] and can also be written as

$$u(x = d, t) = 0 \text{ for } t > 0 \tag{6}$$

3.2 Bottom Wall Boundary Conditions

Imposed shear stress on the bottom wall: When a shear stress is imposed on the bottom wall, the bottom wall shear stress can be written as

$$\tau_w = \tau(x = 0, t) = -\mu \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = \tau_0 + \varepsilon t \tag{7}$$

Imposed velocity on the bottom wall: For the imposed wall velocity condition, the velocity of moving plate follows a function described by Eq.(8).

$$u_w = u(x = 0, t) = u_0 + \alpha t \tag{8}$$

4 VELOCITY DISTRIBUTIONS OF UNSTEADY COUETTE FLOWS

4.1 Imposed Wall Shear Stress τ_w

It was observed experimentally that after an initial developmental time period, saying $t = t_d$, the velocity can be described by

$$u(x, t > t_d) \equiv u_d(x,t) = \frac{\tau_0 + \varepsilon t}{\mu}(d - x) \tag{9}$$

As shown in literature [17], the complete velocity field $u(x,t)$ could be regarded as a superposition of the developing solution $w(x,t)$ and the developed solution $u_d(x,t)$. And $u(x,t)$ can be written as

$$u(x,t) = w(x,t) + \frac{\tau_0 + \varepsilon t}{\mu}(d - x) \tag{10}$$

Based on Eqs. (4) to (7) and Eq.(10), we can obtain that

$$\begin{cases} \dfrac{\partial w}{\partial t} = \nu \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\varepsilon}{\mu}(x - d), & 0 < x < d, \quad t > 0 \\ \left.\dfrac{\partial w(x,t)}{\partial x}\right|_{x=0} = w(x = d, t) = 0, & t > 0 \\ w(x, t = 0) = \dfrac{\tau_0}{\mu}(x - d), & 0 \le x \le d \end{cases} \tag{11}$$

The fluid velocity $u(x,t)$ is solved by seeking the solution w from Eq.(11). Finally, $u(x,t)$ can be expressed as

$$u(x,t) = -\frac{2}{\mu d}\sum_{n=0}^{\infty}\cos\beta_n x\left\{\frac{\tau_0}{\beta_n^2}\exp(-\nu\beta_n^2 t) + \frac{\varepsilon}{\nu\beta_n^4}[1-\exp(-\nu\beta_n^2 t)]\right\} + \frac{\tau_0+\varepsilon t}{\mu}(d-x) \tag{12}$$

where β_n represents the eigen values described as

$$\beta_n = \frac{(2n+1)\pi}{2d}, \quad n = 0,1,2\cdots \tag{13}$$

4.2 Imposed Wall Velocity u_w

After an initial developmental time period saying $t = t_d$, the developed velocity field can be described by

$$u(x,t>t_d) \equiv u_d(x,t) = \frac{u_0+\alpha t}{d}(d-x) \tag{14}$$

Similar with Eq.(10), the complete velocity field can be written as

$$u(x,t) = w(x,t) + \frac{u_0+\alpha t}{d}(d-x) \tag{15}$$

Based on Eqs. (4) to (6), (8) and (15), we can find that

$$\begin{cases} \dfrac{\partial w}{\partial t} = \nu\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\alpha}{d}(x-d), & 0<x<d, \quad t>0 \\ w(x=0,t) = w(x=d,t) = 0, & t>0 \\ w(x,t=0) = \dfrac{u_0}{d}(x-d), & 0\le x\le d \end{cases} \tag{16}$$

The fluid velocity $u(x,t)$ under imposed velocity condition could be found by seeking the solution of w from Eq.(16). We can describe $u(x,t)$ under imposed velocity condition as

$$u(x,t) = -\frac{2}{d}\sum_{n=1}^{\infty}\sin\xi_n x\left\{\frac{u_0}{\xi_n}\exp(-\nu\xi_n^2 t) + \frac{\alpha}{\nu\xi_n^3}[1-\exp(-\nu\xi_n^2 t)]\right\} + \frac{u_0+\alpha t}{d}(d-x) \tag{17}$$

where ξ_n represents the eigen values described as

$$\xi_n = \frac{n\pi}{d}, \quad n = 1,2,3 \ \cdots \tag{18}$$

5. SOLUTIONS OF $\tilde{\mu}(t)$

In this paper, we use the dimensionless viscosity $\tilde{\mu}(t)$ defined in Eq.(3) to represent the transient state of the initial Couette flow process.

5.1 $\tilde{\mu}(t)$ under Imposed Wall Shear Stress Condition

When the shear stress is imposed, using Eqs. (7) and (12), the dimensionless viscosity given in Eq. (3) can be evaluated as

$$\tilde{\mu}_\tau(t) = \frac{1}{1 - \frac{2}{(\tau_0 + \varepsilon t)d^2}\sum_{n=0}^{\infty}\left\{\frac{\tau_0}{\beta_n^2}\exp(-\nu\beta_n^2 t) + \frac{\varepsilon}{\nu\beta_n^4}[1 - \exp(-\nu\beta_n^2 t)]\right\}} \tag{19}$$

where $\tilde{\mu}_\tau(t)$ is the dimensionless viscosity for the imposed shear stress boundary condition. Here we argued two situations under the imposed shear stress boundary condition. These are (1) constant shear stress, $\tau_w = \tau_0$ and (2) linear increasing shear stress from zero shear, $\tau_w = \varepsilon t$.

Constant wall shear stress: When $\varepsilon = 0$, $\tau_w = \tau_0$. Equation (19) reduces to

$$\tilde{\mu}_{\tau=\tau_0}(t) = \frac{1}{1 - \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\exp\left[-\frac{(2n+1)^2\pi^2}{4}t^*\right]} \tag{20}$$

where $\tilde{\mu}_{\tau=\tau_0}(t)$ is the dimensionless viscosity for the constant shear stress condition and $t^* = \nu t/d^2$ is a dimensionless time. The expression under constant shear stress boundary condition has been reported in literature [17] as a special form of step shear stress boundary conditions.

Linear increasing shear stress from 0: When $\tau_0 = 0$, we have $\tau_w = \varepsilon t$, which means Eq.(19) reduces to

$$\tilde{\mu}_{\tau=\varepsilon t}(t) = \frac{1}{1 - \frac{32}{\pi^4 t^*}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}\left\{1 - \exp\left[-\frac{(2n+1)^2\pi^2}{4}t^*\right]\right\}} \tag{21}$$

where $\tilde{\mu}_{\tau=\varepsilon t}(t)$ is the dimensionless viscosity for the condition of linearly increasing shear stress with time from zero shear.

Remarks: From Eqs.(20) and (21), it is clear that the dimensionless viscosity is larger than unity during the initial transient and approaches unity with time. We shall examine the time required for the dimensionless viscosity to reach unity or the time the measured viscosity is equal to the fluid viscosity later in this article.

5.2 $\tilde{\mu}(\mathrm{t})$ under Imposed Wall Velocity Conditions

When the wall velocity is imposed, using Eqs. (14) and (17), the dimensionless viscosity given in Eq.(3) can be written as

$$\tilde{\mu}_u(t) = 1 + \frac{2}{u_0 + \alpha t}\sum_{n=1}^{\infty}\left\{u_0 \exp(-\nu\xi_n^2 t) + \frac{\alpha}{\nu\xi_n^2}[1 - \exp(-\nu\xi_n^2 t)]\right\} \quad (22)$$

where $\tilde{\mu}_u(t)$ is the dimensionless viscosity subjected to an imposed velocity boundary condition. We again consider two situations for this boundary condition. These are (1) constant boundary velocity, $u_w = u_0$ and (2) linear increasing boundary velocity from stationary, $u_w = \alpha t$.

Constant velocity: When $\alpha = 0$, $u_w = u_0$. The velocity of moving plate is constant. From Eqs. (3), (18) and (22), the dimensionless viscosity can be expressed as

$$\tilde{\mu}_{u=u_0}(t) = 1 + 2\sum_{n=1}^{\infty}\exp(-n^2\pi^2 t^*) \quad (23)$$

where $\tilde{\mu}_{u=u_0}(t)$ is the dimensionless viscosity for the constant velocity condition. The result under constant velocity boundary condition is consistent with those reported in the literatures[16,18].

Linear increasing velocity from stationary: When $u_0 = 0$, $u_w = \alpha t$. The dimensionless viscosity can be obtained from Eqs. (3), (18) and (22), as

$$\tilde{\mu}_{u=\alpha t}(t) = 1 + \frac{2}{t^*\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}[1 - \exp(-n^2\pi^2 t^*)] \quad (24)$$

where $\tilde{\mu}_{u=\alpha t}(t)$ is the dimensionless viscosity for the condition of linearly increasing bottom wall velocity with time from stationary.

Remarks: Similar to the imposed wall shear stress situations, we can find that the dimensionless viscosity varies with time and reduces to unity over time.

6. SEEKING INITIAL $\tilde{\mu}(t)$ BY SOLVING STOKES' FIRST PROBLEM

When $t \to +\infty$, we can get $\tilde{\mu}(t) = 1$ from Eqs.(20), (21), (23), (24). But at the initial stage of unsteady Couette flow, that is $t \to 0$, it's difficult to figure out $\tilde{\mu}(t)$ because of the infinite series in these four equations. So, in this section, we obtain the very initial transient stage of $\tilde{\mu}(t)$ by solving the Stokes' first problem.

Compared with unsteady Couette flow, Stokes' first problem has the same moving boundary Eqs.(7) and (8) and governing equation Eq.(4). But the boundary condition Eq.(6) and initial condition Eq.(5) should be changed into

$$u(+\infty, t) = 0 \text{ for } t > 0 \quad (25)$$

and

$$u(x,0) = 0 \ \text{ for } \ 0 \leq x \leq +\infty \tag{26}$$

By applying Laplace transform to the velocity function we obtained

$$U(x,s) = \mathcal{L}[u(x,t)] \tag{27}$$

Then we use Laplace transform to Eq. (4). And based on the initial condition Eq. (5), we can get that

$$sU(x,s) = \nu \frac{\partial^2 U(x,s)}{\partial x^2} \tag{28}$$

The general solution of Eq.(28) is

$$U(x,s) = C_1 \exp\left(-\sqrt{\frac{s}{\nu}}x\right) + C_2 \exp\left(\sqrt{\frac{s}{\nu}}x\right) \tag{29}$$

Applying Laplace transform to boundary conditions, we can change the expressions into

$$U(+\infty, s) = 0 \tag{30}$$

and

$$\frac{\partial U(x=0,s)}{\partial x} = -\frac{\tau_0}{\mu s} - \frac{\varepsilon}{\mu s^2} \tag{31}$$

and

$$U(x=0,s) = \frac{u_0}{s} + \frac{\alpha}{s^2} \tag{32}$$

6.1 Imposed Wall Shear Stress τ_w

From Eqs.(29) to (31), we can express U as

$$U(x,s) = \left(\frac{\tau_0\sqrt{\nu}}{\mu s\sqrt{s}} + \frac{\varepsilon\sqrt{\nu}}{\mu s^2\sqrt{s}}\right)\exp\left(-\sqrt{\frac{s}{\nu}}x\right) \tag{33}$$

The velocity of moving boundary can be written as

$$u(x=0,t) = \mathcal{L}^{-1}[U(x=0,s)] = \left(2\frac{\tau_0\sqrt{\nu}}{\mu} + \frac{4\varepsilon\sqrt{\nu}t}{3\mu}\right)\sqrt{\frac{t}{\pi}} \tag{34}$$

So, for the imposed shear stress condition Eq.(7), initial velocity of moving plate in unsteady Couette flow issue is given by Eq.(34). The dimensionless viscosity at the very initial unsteady stage could be written as Eq.(35) and (36).

$$\tilde{\mu}_{\tau=\tau_0}(t\to 0) = \lim_{t\to 0}\frac{1}{1-\frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\exp\left[-\frac{(2n+1)^2\pi^2}{4}t^*\right]} = \frac{\sqrt{\pi}d}{2\sqrt{\nu t}} = \frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{t^*}} \tag{35}$$

and

$$\tilde{\mu}_{\tau=\varepsilon t}(t\to 0)=\lim_{t\to 0}\frac{1}{1-\frac{32}{\pi^4 t^*}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}\left\{1-\exp\left[-\frac{(2n+1)^2\pi^2}{4}t^*\right]\right\}}=\frac{3\sqrt{\pi}}{4}\frac{1}{\sqrt{t^*}} \tag{36}$$

6.2 Imposed Wall Velocity u_w

From Eqs. (29), (30) and (32), we obtain

$$U(x,s)=\left(\frac{u_0}{s}+\frac{\alpha}{s^2}\right)\exp\left(-\sqrt{\frac{s}{\nu}}x\right) \tag{37}$$

and

$$-\frac{\partial u(x,t)}{\partial x}\bigg|_{x=0}=-\mathcal{L}^{-1}\left[\frac{\partial U(x,s)}{\partial x}\bigg|_{x=0}\right]=\frac{1}{\sqrt{\nu}}\left(\frac{u_0}{\sqrt{\pi t}}+\frac{2\alpha\sqrt{t}}{\sqrt{\pi}}\right) \tag{38}$$

Based on Eqs.(3), (8) and (38), the dimensionless viscosity at the very initial unsteady stage could be written as

$$\tilde{\mu}_{u=u_0}(t\to 0)=\lim_{t\to 0}\left[1+2\sum_{n=1}^{\infty}\exp(-n^2\pi^2 t^*)\right]=\frac{d}{\sqrt{\nu\pi t}}=\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{t^*}} \tag{39}$$

$$\tilde{\mu}_{u=\alpha t}(t\to 0)=\lim_{t\to 0}\left\{1+\frac{2}{t^*\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-\exp(-n^2\pi^2 t^*)\right]\right\}=\frac{2d}{\sqrt{\nu\pi t}}=\frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{t^*}} \tag{40}$$

7. RESULTS and DISCUSSIONS

We presented expressions of $\tilde{\mu}(t)$ for four moving boundary conditions respectively. These are (1) constant wall shear stress, $\tau_w=\tau_0$, (2) linearly increasing wall shear stress with time from zero shear, $\tau_w=\varepsilon t$, (3) constant wall velocity, $u_w=u_0$ and (4) linearly increasing wall velocity with time from stationary, $u_w=\alpha t$. Further discussions will be carried out based on these four boundary conditions.

7.1 Comparison of $\tilde{\mu}(t)$ under different boundary conditions

Eq.(39) has been obtained by Muzychka and Yovanovich [16], so we regard $\tilde{\mu}_{u=u_0}$ as a reference scale. Obviously, when $t\to 0$, the exact quantitative relationship among $\tilde{\mu}(t)$ under different situation could be described as

$$\left.\begin{aligned}\tilde{\mu}_{u=\alpha t}(t)&=2\tilde{\mu}_{u=u_0}(t)\to+\infty\\ \tilde{\mu}_{\tau=\tau_0}(t)&=\frac{\pi}{2}\tilde{\mu}_{u=u_0}(t)\to+\infty\\ \tilde{\mu}_{\tau=\varepsilon t}(t)&=\frac{3\pi}{4}\tilde{\mu}_{u=u_0}(t)\to+\infty\end{aligned}\right\}\quad \text{when } t\to 0 \tag{41}$$

From Eq.(41), the relationship of $\tilde{\mu}(t)$ for different boundary conditions when $t\to 0$ can be seen. According to Eqs.(20), (21), (23) and (24), $\tilde{\mu}(t)$ is only a function of dimensionless time t^*. During the transient stage of Couette flows, the dimensionless viscosity $\tilde{\mu}(t)$ decreases from $+\infty$ to 1 with time.

This is confirmed by Fig. 2 which shows the variations of $\tilde{\mu}(t)$ with the dimensionless time t^*. Dots in different colors represent the exact solutions of $\tilde{\mu}(t)$ under different boundary conditions and the black solid line in figure means the corresponding fitting curves of exact solutions.

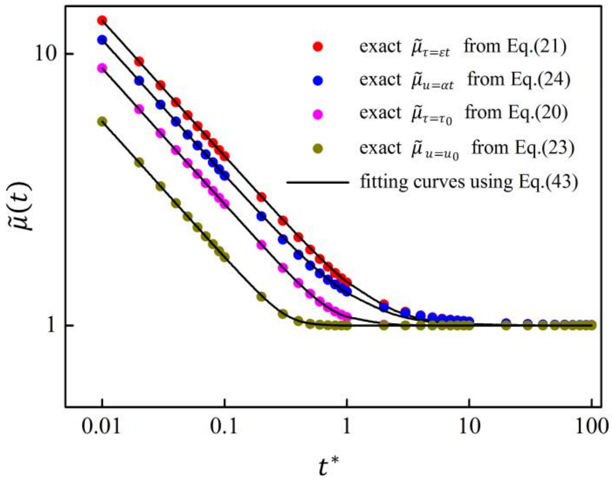

Fig. 2 The dimensionless viscosity $\tilde{\mu}(t)$ vs. dimensionless time t^*

According to Fig.2, we can obviously find that $\tilde{\mu}_{\tau=\varepsilon t}(t) > \tilde{\mu}_{u=\alpha t}(t) > \tilde{\mu}_{\tau=\tau_0}(t) > \tilde{\mu}_{u=u_0}(t)$ when $t \to 0$ and $\tilde{\mu}(t) = 1$ when $t \to +\infty$. As $\tilde{\mu}(t) = 1$ representing the velocity of the Couette flow reaches developed state, we can utilize this information to obtain the time period for the Couette flow to reach developed state for these four boundary conditions.

Fitting curves were obtained by applying the compact model mentioned in literature[16]. For the dimensionless viscosity, the compact model has the form shown in Eq.(42).

$$\tilde{\mu}(t) = \{[\tilde{\mu}(t \to 0)]^k + [\tilde{\mu}(t \to +\infty)]^k\}^{\frac{1}{k}} \tag{42}$$

where k is an dimensionless exponent. Using Eqs.(35), (36), (39), (40) and (42), we can express $\tilde{\mu}(t)$ in compact model shown as

$$\tilde{\mu}(t) = \begin{cases} \left[\left(\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{t^*}}\right)^{\frac{43}{5}} + 1\right]^{\frac{5}{43}}, & u_w = u_0 \\ \left[\left(\frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{t^*}}\right)^{\frac{33}{10}} + 1\right]^{\frac{10}{33}}, & u_w = \alpha t \\ \left[\left(\frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{t^*}}\right)^{\frac{17}{3}} + 1\right]^{\frac{3}{17}}, & \tau_w = \tau_0 \\ \left[\left(\frac{3\sqrt{\pi}}{4}\frac{1}{\sqrt{t^*}}\right)^{\frac{18}{5}} + 1\right]^{\frac{5}{18}}, & \tau_w = \varepsilon t \end{cases} \tag{43}$$

From Fig.2, we can find that the compact models in Eq.(43) fit the exact solutions perfectly.

7.2 Time period for Couette flow to reach developed state

According to the above research, we have already found that the time period for Couette flow to reach developed state is different for these four boundary conditions. As $\tilde{\mu}(t)$ decreases gradually from $+\infty$ to 1 during the initial stage of Couette flow, there should be a definite point to indicate the achievement of the developed state. Here, we defined that when $\tilde{\mu}(t) \leq 1.01$, Couette flow reaches its developed state. That is the velocity gradient of each layer is the same approximately. We can write this condition as

$$\tilde{\mu}(t^* = t_d^*) = 1.01 \tag{44}$$

where t_d^* is the dimensionless development time. Using Eq.(43), we can calculate t_d^* for the four boundary conditions respectively. The results are shown in Tab. 1.

Tab.1. dimensionless development time t_d^* under different boundary conditions

boundary conditions	$u_w = u_0$	$u_w = \alpha t$	$\tau_w = \tau_0$	$\tau_w = \varepsilon_0 t$
t_d^*	0.558	9.995	2.145	11.122

According to Tab.1, the time periods for Couette flow to reach developed state after boundary plate is moved suddenly are different under these four boundary conditions. The time period is shortest for the imposed velocity boundary condition $u_w = u_0$ and is the longest under imposed shear stress boundary condition $u_w = \varepsilon t$.

8. CONCLUDING REMARKS

In this paper, based on the ideal model of Couette flow between two infinite parallel plates, the process for reaching developed state was investigated under both imposed velocity and shear stress boundary conditions. The dimensionless viscosity $\tilde{\mu}(t)$ is used as the parameter to represent the initial transient process. The specific results of research are as follows.

1. The exact solutions for the velocities when $\tau_w = \tau_0 + \varepsilon t$ and $u_w = u_0 + \alpha t$ were obtained as shown in Eqs.(12) and (17). The dimensionless viscosities $\tilde{\mu}(t)$ as functions of dimensionless time are also obtained.

2. When $t \to 0$, functional forms of $\tilde{\mu}(t)$ were obtained, as shown in Eqs.(35), (36), (39) and (40), and the exact quantitative relationship of $\tilde{\mu}(t)$ for these four boundary conditions was discovered. During the developing stage of Couette flows, we found $\tilde{\mu}_{\tau=\varepsilon t}(t) > \tilde{\mu}_{u=\alpha t}(t) > \tilde{\mu}_{\tau=\tau_0}(t) > \tilde{\mu}_{u=u_0}(t)$. Compact models of $\tilde{\mu}(t)$ for these four boundary conditions were proposed shown in Eq.(43).

3. During the transient stage, $\tilde{\mu}(t)$ decreases from $+\infty$ to 1 with time. The time periods for Couette flows needed to reach developed state are different under these four boundary conditions. The development time is the shortest under imposed velocity boundary condition $u_w = u_0$ and is the longest under imposed

shear stress boundary condition $\tau_w = \varepsilon t$.

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Reference

[1] T. Papanastasiou, G. Georgiou, A.N. Alexandrou, Viscous fluid flow, CRC Press, 2010.

[2] W. Graebel, Advanced fluid mechanics, Academic Press, 2007.

[3] M.E. Erdoğan, On the flows produced by sudden application of a constant pressure gradient or by impulsive motion of a boundary, Int. J. Non. Linear. Mech. 38 (2003) 781–797. doi:10.1016/S0020-7462(01)00133-0.

[4] A.A. Mendiburu, L.R. Carroccib, J.A. Carvalho, Analytical solution for transient one-dimensional Couette flow considering constant and time-dependent pressure gradients, Eng. Térmica (Thermal Eng. 8 (2009) 92–98.

[5] W. Shaowei, X. Mingyu, Exact solution on unsteady Couette flow of generalized Maxwell fluid with fractional derivative, Acta Mech. 187 (2006) 103–112. doi:10.1007/s00707-006-0332-9.

[6] T. Wenchang, P. Wenxiao, X. Mingyu, A note on unsteady flows of a viscoelastic fluid with the fractional Maxwell model between two parallel plates, Int. J. Non. Linear. Mech. 38 (2003) 645–650. doi:10.1016/S0020-7462(01)00121-4.

[7] M. Xu, W. Tan, Theoretical analysis of the velocity field, stress field and vortex sheet of generalized second order fluid with fractional anomalous diffusion, Sci. China Ser. A Math. 44 (2001) 1387–1399. doi:10.1007/BF02877067.

[8] T. Hayat, S. Asghar, A.M. Siddiqui, Some unsteady unidirectional flows of a non-Newtonian fluid, Int. J. Eng. Sci. 38 (2000) 337–346. doi:10.1016/S0020-7225(99)00034-8.

[9] D. Bose, U. Basu, Unsteady Incompressible Flow of a Generalised Oldroyed-B Fluid between Two Infinite Parallel Plates, World J. Mech. 03 (2013) 146–151. doi:10.4236/wjm.2013.32012.

[10] Q. Haitao, X. Mingyu, Some unsteady unidirectional flows of a generalized Oldroyd-B fluid with fractional derivative, Appl. Math. Model. 33 (2009) 4184–4191. doi:10.1016/j.apm.2009.03.002.

[11] M. Jana, S. Das, R.N. Jana, Unsteady Couette flow through a porous medium in a rotating system, Open J. Fluid Dyn. 2 (2012) 149–158.

[12] B.K. Jha, C.A. Apere, Time-dependent MHD Couette flow in a porous annulus, Commun. Nonlinear Sci. Numer. Simul. 18 (2013) 1959–1969. doi:doi:10.1016/j.cnsns.2013.01.008.

[13] S. Asghar, A. Ahmad, Unsteady Couette flow of viscous fluid under a non-uniform magnetic field, Appl. Math. Lett. 25 (2012) 1953–1958. doi:10.1016/j.aml.2012.03.008.

[14] T. Ting, Certain non-steady flows of second-order fluids, Arch. Ration. Mech. Anal. 14 (1963) 1–26. doi:10.1007/BF00250690.

[15] D. Bernardin, C. Nouar, Transient Couette flows of Oldroyd's fluids under imposed torques, J. Nonnewton. Fluid Mech. 77 (1998) 201–231. doi:10.1016/S0377-0257(97)00130-4.

[16] Y.S. Muzychka, M.M. Yovanovich, Unsteady viscous flows and Stokes's first problem, Int. J. Therm. Sci. 49 (2010) 820–828. doi:10.1016/j.ijthermalsci.2009.11.013.

[17] C. Lei, L. Gang, Z. Guozhong, T. Yuannan, J. Chai, Transient stage comparison of Couette flow under step shear stress and step velocity boundary conditions, Int. Commun. Heat Mass Transf. 75 (2016) 232–239.

[18] M. Emin Erdoğan, On the unsteady unidirectional flows generated by impulsive motion of a boundary or sudden application of a pressure gradient, Int. J. Non. Linear. Mech. 37 (2002) 1091–1106. doi:10.1016/S0020-7462(01)00035-X.