Important Facts and Observations about Pairwise Comparisons

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Waldemar W. Koczkodaj [*] Ludmil Mikhailov [†]
Grzegorz Redlarski [‡] Michael Soltys [§] Jacek Szybowski [¶]
Eliza Wajch [‖] Kevin Kam Fung Yuen [**] Gaik Tamazian [††]

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Abstract

This study has been inspired by numerous requests from researchers who often confuse Saaty's AHP with the Pairwise Comparisons (PC) method, taking AHP as the only representation of PC. Most formal

[*]Computer Science, Laurentian University, Sudbury, Ontario P3E 2C6, Canada, wkoczkodaj@cs.laurentian.ca

[†]School of Computer Science, The University of Manchester, M13 9PL, United Kingdom, ludi.mikhailov@mbs.ac.uk

[‡]Electrical and Control Engeenering Department, Gdansk University of Technology, 80-297 Gdansk, Narutowicza st. 11/12, Poland, grzegorz.redlarski@pg.gda.pl

[§]California State University Channel Islands, Dept. of Computer Science, One University Drive, Camarillo, CA 93012, USA, michael.soltys@csuci.edu

[¶]AGH University of Science and Technology, Faculty of Applied Mathematics, al. Mickiewicza 30, 30-059 Krakow, Poland, szybowsk@agh.edu.pl; Research supported by the Polish Ministry of Science and Higher Education

[‖]Institute of Mathematics and Physics, Siedlce University of Natural Sciences and Humanities, 3 Maja 54, 08-110 Siedlice, Poland, eliza.wajch@wp.pl

[**]Department of Computer Science and Software Engineering. Xi'an Jiaotong-Liverpool University, Suzhou 215123, China, kevinkf.yuen@gmail.com

[††]Theodosius Dobzhansky Center for Genome Bioinformatics, St. Petersburg State University; This work was supported, in part, by Russian Ministry of Science Mega-grant no.11.G34.31.0068

results of this survey article are based on a recently published work by Koczkodaj and Szwarc. This article should be regarded as an interpretation and clarification of future theoretical investigations of PC. In addition, this article is a reflection on general PC research at a higher level of abstraction: the philosophy of science. It delves into the foundations and implications of pairwise comparisons. Finally, open problems have also been reported for future research.

Keywords: *pairwise comparisons, inconsistency, ratio scale*

1 Introduction

The first use of the method of Pairwise Comparisons (PC method) is often attributed to Ramon Llull, the 13th-century mystic and philosopher. His works created a basis for computation theory, considering his acknowledged influence on Gottfried Leibniz. Llull's contribution to PC, used for elections, was recently mentioned in [24]. Fechner (see [25]) is regarded as the precursor of using PC in Psychometrics, although it was Waber who really used PC in such way. Similar to Llull, Condorcet also used PC in [22], for voting (win or lose). In both cases, however, the PC method was not the main subject for the scientific investigation but rather its use. Thurstone (see [56]) proposed what is known as "The Law of Comparative Judgments" in 1927.

Saaty's seminal study [51] had a considerable impact on PC research. However, his AHP should not be equated with PC, despite using them. The fixed ratio scale (the scale in this presentation) assumed by Saaty has served its proponents well. However, the fixed scale makes AHP a subset of PC; PC is more general as it does not assume a particular scale, and allows for non-numerical rankings. For instance, the non-numerical rankings of [33] are relations, the "scales" in [45] are arbitrary groups, and abelian linearly ordered groups (alo-groups) are employed in [20, 21, 50].

Despite its long history, the PC method is still a very attractive subject for research. Ranking with a reference set of alternatives, as in [47, 48], is an example of such explorations. "To pair or not to pair" is not the question. "When and how to use PC" is the proper question. Due to the lack of popularity of the PC theory, basic concepts need to be presented in the next section; hence readers familiar with the theory may skip the next section.

2 Pairwise comparisons basics

Usually, we define an $n \times n$ *pairwise comparisons matrix* (abbreviated to *PC matrix*) simply as a square matrix $M = [m_{i,j}]$ such that $m_{i,j}$ are positive real numbers for all $i, j = 1, \ldots, n$. A pairwise comparison matrix $M = [m_{i,j}]$ is called:

(r) *reciprocal* if $m_{i,j} = \frac{1}{m_{j,i}}$ for all $i, j = 1, \ldots, n$ (then automatically $m_{i,i} = 1$ for each $i = 1, \ldots, n$);

(c) *consistent* (or *transitive*) if $m_{i,j} \cdot m_{j,k} = m_{i,k}$ for all $i, j, k = 1, 2, \ldots, n$.

Sometimes, it is very convenient to assign to a PC matrix $M = [m_{i,j}]$ the matrix $\ln(M) = [\ln(m_{i,j})]$ (cf. e.g. [41]). Then, for $a_{i,j} = \ln(m_{i,j}))$, we have $m_{i,j} = \frac{1}{m_{j,i}}$ if and only if $a_{i,j} = -a_{j,i}$. Moreover, $m_{i,j} \cdot m_{j,k} = m_{i,k}$ if and only if $a_{i,j} + a_{j,k} = a_{i,k}$. Therefore, in this case, we consider two kinds or reciprocity: multiplicative reciprocity for M and additive reciprocity for $\ln(M)$. We also consider multiplicative consistency for M and additive consistency for $\ln(M)$. In [20], a unified framework for both multiplicative and additive reciprocity and consistency was started by a general notion of a reciprocal PC matrix over an abelian linearly ordered group. This approach has been continued in [21] and [45] recently. In [45], among other results, notions of reciprocal and consistent PC matrices over a group were introduced. To avoid misunderstanding, let us recall that a group is an ordered pair $\langle X, \odot \rangle$, denoted briefly by X, where X is a set, while $\odot$ is a mapping from $X \times X$ to X such that the following conditions are satisfied:

(g1) if $a, b, c \in X$, then $(a \odot b) \odot c = a \odot (b \odot c)$ (associativity);

(g2) there exists exactly one element $1_X \in X$ (called the identity element of the group) such that, for each $a \in X$, the equality $a \odot 1_X = a$ holds;

(g3) for each $a \in X$, there exists $a^{-1} \in X$, called the inverse element of a, such that $a \odot a^{-1} = 1_X$.

If, in addition, $a \odot b = b \odot a$ for all elements a, b of the group X, then the group is called abelian or commutative.

Now, let $X = \langle X, \odot \rangle$ be a group and let $M = [m_{i,j}]$ be an $n \times n$ matrix such that $m_{i,j} \in X$ for all $i, j \in \{1, \ldots, n\}$. Then we say that M is a *square matrix over the group* X. According to [45], the matrix M is called

a *reciprocal* PC *matrix over the group* X if $m_{i,i} = 1_X$ and $m_{j,i} = m_{i,j}^{-1}$ for all $i, j \in \{1, \ldots, n\}$. The matrix M over X is called *consistent* if it satisfies the following consistency (equivalently, transitivity) condition with respect to $\odot$:

$$m_{i,j} \odot m_{j,k} = m_{i,k},$$

for all $i, j, k = 1, 2, \ldots, n$.

We denote the group of all positive real numbers equipped with their standard multiplication by $\mathbb{R}_+$. Most applications of PC matrices have been found for PC matrices over the group $\mathbb{R}_+$. This is why, in the sequel, we assume that M is a reciprocal PC matrix over $\mathbb{R}_+$. Let M be of the form:

$$M = \begin{bmatrix} 1 & m_{1,2} & \cdots & m_{1,n} \\ \frac{1}{m_{1,2}} & 1 & \cdots & m_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{m_{1,n}} & \frac{1}{m_{2,n}} & \cdots & 1 \end{bmatrix}$$

where $m_{i,j}$ expresses a relative quantity, intensity, or preference of entity (or stimuli) E_i over E_j.

While every consistent matrix is reciprocal, the converse is false, in general. If the consistency condition does not hold, the matrix is *inconsistent* (or *intransitive*).

Consistent matrices correspond to the ideal situation, in which there are exact values for the quotients of entities: $E_1, \ldots, E_n$. Then the quotients $m_{ij} = E_i/E_j$ form a consistent PC matrix. For a consistent PC matrix, there is a vector $w = [w_1, \ldots w_n]$, unique up to a multiplicative constant, which generates this matrix, i.e., the condition $m_{ij} = w_i/w_j$ holds. We usually call w *the vector of weights*.

The main challenge for the PC method is the lack of consistency in the PC matrices, which in practice is the case very often — in terms of realistic inputs, most PC matrices are inconsistent. Only very simple or academic examples are fully consistent. "To err is human" and for such cases, a simplified PC matrix, proposed in [44], should be considered, as it requires only $n-1$ comparisons while the inconsistent PC matrix requires at least $n*(n-1)/2$. Given an $n \times n$ matrix M, which is not consistent, the theory attempts to provide a consistent $n \times n$ matrix M, which differs from matrix M "as little as possible". Such approximation is only acceptable when the inconsistency

is at an acceptable level. The acceptable level may be given by a rigorous application of a statistical p-value.

Given any positive real values $E_1, E_2, \ldots, E_n$, the matrix $M = [m_{ij}]$ where $m_{ij} = E_i/E_j$, is always consistent. It is an important observation since it implies that a problem of approximation is really a problem of a norm selection and distance minimization. For the Euclidean norm, the vector of geometric means (equal to the principal eigenvector for the transitive matrix) is the vector of weights which generates it. Oddly enough, it is also equal (when normalized) to an eigenvector corresponding to the principal eigenvalue. Needless to say, only optimization methods can approximate the given matrix for the assumed norm as addressed in Sec. 7. Such type of matrix is examined in [55] as an "error-free" PC matrix.

It is unfortunate that the singular form "comparison" is sometimes used, considering that a minimum of three comparisons are needed for the method to have a practical meaning. Comparing two entities (also stimuli or properties) in pairs is irreducible, since having one entity compared with itself trivially gives 1. Comparing only two entities (2×2 PC matrix) does not involve inconsistency. Entities and/or their properties are often called stimuli in the PC research but are rarely used in applications. On top of it, another form exists in the scientific literature: "paired comparison" (both hyphenated and not; singular or plural). Some researchers use also "pair comparison".

As was mentioned at the beginning of this section, the additive version of PC can be obtained by a logarithmic transformation; see [58] for a recent study. Non-numerical PC have been studied recently in [33] — the authors refer to the famous book [**?**] by the Nobelist K. Arrow, without commenting on Arrow's impossibility theorem.

3 The ratio scale problem

A scale may be considered as a mapping from a set of qualitative judgments to the set of real numbers. The qualitative judgments arise from comparing pairs of objects, they have a natural order, and are usually considered to form a constrained set. As a rule, one considers 7 ± 2 judgment gradations due to psychological limitations of human thinking [1]. Following [2], we formalize the notion of a scale in the following way. Let $\Lambda = \{ 0, \pm 1, \ldots, \pm 8 \}$ be the set of numbers representing qualitative judgments (see Table 1). We assume that for each pair of objects (C_i, C_j) being compared to each other, a number

$\lambda_{i,j} \in \Lambda$ is assigned in the following way: if the first object in the pair, C_i, is preferred to the second one, C_j, then a positive number from Λ is used, otherwise it is a negative one. Note that $\lambda_{i,j} = -\lambda_{j,i}$ since the judgments are reciprocal.

In general, any positive function f such that $f(\lambda_1) < f(\lambda_2)$, provided that $\lambda_1 < \lambda_2$, is a scale. For a ratio scale, the following condition is also required: $f(-\lambda) = f^{-1}(\lambda)$. As an example, we consider Saaty's ratio scale [51] which is determined by the values $1/9, \ldots, 1/2, 1, 2, \ldots, 9$. In the terms given above, its representation is the following:

$$f_S(\lambda) = (1 + |\lambda|)^{\mathrm{sgn}\lambda}. \tag{1}$$

Note that for $\lambda \geq 0$, the function (1) is a linear function of λ.

In this section, we describe a number of alternative ratio scales proposed for pairwise comparisons and address the problem of choosing an appropriate ratio scale. The following scales are considered besides Saaty's scale:

- the Ma-Zheng scale;
- the Lootsma scale;
- the Donegan-Dodd scale;
- the Salo-Hämäläinen scale.

The Ma-Zheng scale [3] is defined in the following way:

$$f_{MZ}(\lambda) = \left(\frac{9}{9 - |\lambda|}\right)^{\mathrm{sgn}\lambda}. \tag{2}$$

The motivation for (2) was to propose a scale that would be linear for $\lambda \leq 0$ in the same way as Saaty's scale is linear for $\lambda \geq 0$.

The Lootsma scale [4] was proposed in the context of the multiplicative AHP and is based on psychological insights. It is defined as follows:

$$f_L(\lambda) = c^{\lambda}, \tag{3}$$

where $c > 1$ is a scale parameter.

$\lambda \in \Lambda$	Qualitative judgment
0	equivalence
2	weak superiority
4	strong superiority
6	very strong superiority
8	absolute superiority
1,3,5,7	intermediate judgments

Table 1: Representation of qualitative judgments as integer numbers.

The Donegan-Dodd scale [5] was suggested to handle the most extreme judgments ($\lambda = \pm 8$ in our notation). It is based on the inverse hyperbolic tangent function:

$$f_{DD}(\lambda) = \exp\left[\tanh^{-1}\left(\frac{\sqrt{3}|\lambda|}{14}\right)\right]^{\mathrm{sgn}\lambda}. \tag{4}$$

The Salo-Hämäläinen scale [6] is related to so-called *balanced scales* and was designed to give more uniform ('balanced') distribution of scale values compared to Saaty's scale. It is defined in the following way:

$$f_{SH} = \left(\frac{0.5 + |\lambda|s}{0.5 - |\lambda|s}\right)^{\mathrm{sgn}\lambda}, \tag{5}$$

where s is a scale parameter and is usually equal to 0.05 or 1/17.

A number of studies were carried out to compare the scales to each other and determine the most suitable one for practical use. According to [7], the Lootsma-like scale is the most appropriate based on the scale transitivity criterion; also the authors describe a way to derive the proper value of the scale parameter c. In [8], Saaty's, Ma-Zheng, Lootsma and Donegan-Dodd scales were compared to each other using a Monte-Carlo simulation study. The criterion for comparison was the symmetry of the priority value distribution; according to this criterion, the Ma-Zheng scale was optimal. The reader is also encouraged to reference [9], [10], and [2], for further investigation of the scale problem.

A mathematical proof that a small ratio scale (1 to 3) has the most desired mathematical properties (e.g., convexity) was provided in [28]. However, it is not the only reason postulated in this study for using the ratio scale in

PC. The strongest reason arises from human language, where comparisons are naturally given in triples: e.g., big → bigger → biggest.

Fulop's constant was introduced in [27]. Subsequently, it was employed in [28] for the derivation of the main result (the small scale). As proved in [27], there exists $a_0 > 0$ such that for any $a > 0$, the univariate function f_a defined as:

$$f_a(x) = (e^x - a)^2 + \left(e^{-x} - 1/a\right)^2 \tag{6}$$

is strictly convex if and only if $1/a_0 \leq a \leq a_0$.

When the condition $1/a_0 \leq a \leq a_0$ is fulfilled for all i,j, then f_a can be transformed into the convex programming problem:

$$\begin{array}{ll} \min & \sum_{i=1}^{n-1} f_{a_{in}}(x_i) + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} f_{a_{ij}}(x_{ij}) \\ \text{s.t.} & x_i - x_j - x_{ij} = 0,\ i = 1, \ldots, n-2,\ j = i+1, \ldots, n-1. \end{array} \tag{7}$$

with a strictly convex objective function to be minimized (see [27], Proposition 2). It implies that the programming problem (7) and the equivalent problem (6) have a unique solution which can be found using standard local search methods. The mentioned constant equals to $a_0 = ((123 + 55\sqrt{5})/2)^{1/4} = \sqrt{\frac{1}{2}\left(11 + 5\sqrt{5}\right)} \approx 3.330191$, what is a reasonable bound for real-life problems [27, 28].

However, an unbounded ratio scale is indispensable for measurable entities such as distances, areas, or temperatures if the precise measurements are available or could be obtained. There is an ongoing dispute about ratio scales for entities lacking the "unit" (e.g., emotions or safety). Usually, the ratio scale is assumed as arbitrary and the Likert scale is usually a ratio scale with small values "close enough" to 3, on each branch (negative and positive) as Fig. **??** demonstrates.

For the following PC matrix of the size 3×3:

$$M3 = \begin{bmatrix} 1 & 9 & x \\ 1/9 & 1 & 9 \\ 1/x & 1/9 & 1 \end{bmatrix}$$

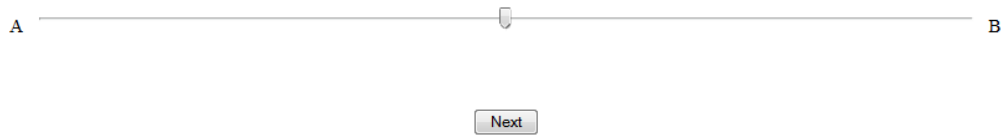

Figure 1: Comparing two scale items

there is no such x in AHP that the matrix M is fully consistent. The sum of sets $A = \{1, \ldots, 9\}$ and $B = \{\frac{1}{n} : \ n \in A\}$ of AHP matrices' elements is not closed under the multiplication, so it fails to create a group or even the weakest algebraic structure, which is a magma (or groupoid).

Unfortunately, the same reasoning does apply to any other fixed value of the upper bound for a rating scale. Evidently, the scale $[1, \infty)$ (and its inverse $(0, 1]$) is a scale which does not suffer from the above lack of the mathematical closure problem. It appears that such a scale is consistent with the *zero, one, and infinity* rule often used in computer science having the interpretation of: the only reasonable numbers are zero, one, and infinity ([17]).

The 0-1-∞ rule implies that we should not impose arbitrary limits on the number of instances of a particular entity in the design of algorithms of heuristics. If more than a single instance of it is allowed, then the size should not have a fixed limit. Certainly, practicality may impose limits but this should be done by design and necessity rather than chance. Binary alternatives are the most commonly used in the decision making process (e.g., go left or right, buy or sell, pass or fail, etc.). There are well-defined cardinalities other than zero, one, and the arbitrary cardinality (represented by ∞ in the above rule). Most Indo-European languages have developed a concept of pairs or couples. It is also demonstrated by: "This" and "Not this" or "This" and "The other," or "One way" and "The other way". Without the use of binary choices there would be no computer technology since the binary system has two digits.

4 The eigenvalue problem

For the Euclidean norm, the vector of geometric means is the best approximation to a consistent matrix. However, the "best" approximation of an

inconsistent PC matrix is not generated by the vector of geometric means or by the eigenvector, although both are very accurate approximations for small inconsistencies. The problem is that no one actually knows "how small is small," although it should not be a surprise since a similar problem can be found in statistics with the p-value and α-value.

It is worth noticing that the normalized vector of the geometric means and the normalized eigenvector (corresponding to the principal eigenvalue) are identical for a consistent PC matrix. The Monte Carlo study was conducted to verify whether the eigenvector approximation is better than the geometric means as proposed in [30], and disproved such a claim on the basis of 1,000,000 matrices. However, its proponents still insists on the superiority of an eigenvector solution although the author's own Example 1, represented by Table 1 in [52], is better approximated by the geometric means than by an eigenvector. The same reasoning is applicable to most examples in Table 2.

The eigenvector approximation was criticized in the 1980s (when the geometric means method was proposed by [34]) and more recently in [18], who used better argumentation based on recent research. However, old habits die hard and the eigenvector method may survive for some time. In [19], it was established that the geometric mean is the only method for deriving weights from multiplicative PC that satisfy fundamental consistency requirements. According to [19], it is immune to the scale-inversion as well as to rank reversal.

The most considerable reasoning to the lack of the eigenvector "superiority" is provided by John Fichtner in his report in 1984 (see [26]). His eigenvector metric was misused by many researchers who have cited it. Fichtner never claimed that his eigenvector distance should be used instead of Euclidean distance.

Nevertheless here two methods for getting priority vectors from reciprocal matrices (geometric row mean and right eigenvector) and two possible scales (1;2;3;... and $1;\sqrt{2};2;2\cdot\sqrt{2};\ldots$) will be compared with the help of 4 qualitative comparison matrices given by Saaty (1980 ; relative visual brightness p. 38 f. and 58, wealth of nations, p. 40 f. and 59 and estimating distances p. 42 f.). The result can be seen in table 1. From these few examples it seems impossible to decide, which method or scale is better.

Figure 2: Excerpt from Fitchtner's 1984 report

In fact, his report findings were misused by twisting them around as Fig. 2 so well demonstrates. The Fichtner's distance for eigenvectors was defined as follows (keep in mind that this is taken from an old typewritten manuscript, where we transcribed the symbols as accurately as possible):

$$\delta(A,B) = \sqrt{(w_1(A) - w_1(B))^2 + \ldots + (w_n(A) - w_n(B))^2}$$

$$+\frac{1}{2(n-1)}|\lambda_{max}^{(A)} - \lambda_{max}^{(B)} + \frac{1}{2(n-1)}|\lambda_{max}^{(A)} - \lambda_{max}^{(B)} * \circledast$$

where:

$w_i(A)$ and $w_i(B)$ are, respectively, the i-th components of the normalized principal eigenvectors of PC matrix A and of PC matrix B;

$\lambda_{max}^{(A)}$ and $\lambda_{max}^{(B)}$ are the corresponding principal eigenvectors;

$\circledast$ is 0, if $A = B$, and 1, if $A \neq B$.

We leave it to the reader to assess whether it is "simpler and more accurate" than the Euclidean distance.

5 The inconsistency concept

Erroneously, [51] is often credited for providing a consistency definition of a PC matrix A. In fact, the condition "if and only if $a_{i,j} \cdot a_{j,k} = a_{i,k}$ for $i, j, k = 1, 2, ..., n$." was defined and examined before 1977 by at least these four studies published between 1939 and 1961: [36, 30, 29, 54].

Suppose that $X = \langle X, \odot \rangle$ is a group. Let $T = \langle x, y, z \rangle$ be a triad of elements x, y, z of X. We call T consistent with respect to $\odot$ if $x \odot z = y$. Otherwise, T is called inconsistent with respect to $\odot$ or an inconsistent triad of the group X. In particular, $\langle x, y, z \rangle$ is an inconsistent triad of the group $\mathbb{R}_+$ if and only if x, y, z are positive real numbers such that $x \cdot z \neq y$.

In decision making processes, given an inconsistent $n \times n$ PC matrix $A = [a_{i.j}]$ over the group $\mathbb{R}_+$, it might be desirable to find all inconsistent triads $\langle a_{i,k}, a_{i,j}, a_{k,j} \rangle$ to help the creators of the matrix to change their judgements to get an optimal consistent approximation of A. To do this, a common expectation is to define a mapping $ii : \mathbb{R}_+^3 \to \mathbb{R}$ which has the following properties:

(A.1) $ii(x, y, z) = 0$ if and only if $y = x \cdot z$,

(A.2) $ii(x, y, z) \in [0, 1)$ - by common sense, the "ideal inconsistency" (represented by 1) cannot be achieved,

(A.3) there exists a metric d on $\mathbb{R}_+$ such that if $\langle x, y, z\rangle$ and $\langle x', y', z'\rangle$ are triads of positive real numbers such that $d(x \cdot z, y) \leq d(x' \cdot z', y')$, then $ii(x, y, z) \leq ii(x', y', z')$.

If a mapping $ii : \mathbb{R}_+^3 \to \mathbb{R}$ satisfies conditions (A1)–(A3) and $x, y, z \in \mathbb{R}_+$, we say that ii is *an inconsistency indicator on the group* $\mathbb{R}_+$, while $ii(x, y, z)$ is the inconsistency indicator value of the triad $\langle x, y, z\rangle$. We can look at conditions (A.1)–(A.3) as defining axioms of inconsistency indicators over $\mathbb{R}_+$. The third axiom A.3 is crucial for any axiomatization of inconsistency indicators over $\mathbb{R}_+$. It seems that without this axiom, an inconsistency indicator would not make sense in practice. For any assumed definition of an inconsistency indicator, the inconsistency indicator value of a triad $T_1 = \langle x_1, y_1, z_1\rangle$ cannot be smaller than that of $T_2 = \langle x_2, y_2, z_2\rangle$ if, roughly speaking, the distance of T_2 from a consistent triad is smaller than the distance of T_1 from a consistent triad. The third axiom can be summarized by "the further we go away from $y = x \cdot z$ the bigger is the inconsistency value".

Based on the proposed axioms for inconsistency indicator and on [37], for positive real numbers x, y, z, let us define:

$$f(x, y, z) = 1 - \min\left\{\frac{y}{xz}, \frac{xz}{y}\right\}.$$

Then:

$$f(x, y, z) = 1 - e^{-\left|\ln\left(\frac{y}{xz}\right)\right|}.$$

The function f is called *Koczkodaj's inconsistency indicator map*. It was shown in [45] that the function $d_f : \mathbb{R}_+^2 \to \mathbb{R}$, defined by $d_f(x, y) = f(x, y, 1)$ whenever $x, y \in \mathbb{R}_+$, is a metric such that, for triads $\langle x_1, y_1, z_1\rangle$ and $\langle x_2, y_2, z_2\rangle$ of positive real numbers, we have $d_f(x_1 \cdot z_1, y_1) \geq d_f(x_2 \cdot z_2, y_2)$ if and only if $f(x_1, y_1, z_1) \geq f(x_2, y_2, z_2)$, since $f(x, y, z) = d_f(xz, y)$ for all $x, y, z \in \mathbb{R}_+$. Now, it is evident that f satisfies conditions (A.1)–(A.3).
For $ii = f$ and positive real numbers x, y, z, let us look at the following two examples:

- $ii(x, 2, z) \geq ii(1.5, 2, 2.5)$ if $x \geq 1.5$ and $z \geq 2.5$, since $d_f(x \cdot z, 2) \geq d_f(1.2{\cdot}2.5, 2)$ for $x \geq 1.5$ and $z \geq 2.5$. On the other hand, $ii(1.5, y, 2.5) \geq ii(1.5, 2, 2.5)$ if $y \leq 2$, since $d_f(1.5 \cdot 2.5, y) \geq d_f(1.5 \cdot 2.5, 2)$ for $y \leq 2$.

- $ii(1.5, y, 1.2) \geq ii(1.5, 2.5, 1.2)$ if $y \geq 2.5$, but if $x \leq 1.5$ and $z \leq 1.2$, then $ii(x, 2.5, z) \geq ii(1.5, 2.5, 1.2)$.

In [45], a more general notion of an inconsistency indicator map on a group $X = \langle X, \odot \rangle$ was introduced. Namely, a mapping $J : X^3 \to \mathbb{R}$ is called an *inconsistency indicator map on the group* X if there exists a metric $\rho : X^2 \to \mathbb{R}$ such that $J(x, y, z) = \rho(x \odot z, y)$ for all $x, y, z \in X$. If J is an inconsistency indicator map on the group X, then, for an arbitrary real number a such that $0 < a < 1$, we can define $J_a(x, y, z) = \min\{J(x, y, z), a\}$ to obtain an inconsistency indicator map J_a on the group X, such that $J_a(x, y, z) \in [0, 1)$ for all $x, y, z \in X$. Every inconsistency indicator map J on the group $\mathbb{R}_+$ satisfies conditions (A.1) and (A.3) (cf.[45]). Even if J does not satisfy (A.2), we can replace J by J_a, for an $a \in (0, 1)$, to get an inconsistency indicator map which satisfies (A.1)–(A.3). The notion of an incosistency indicator map on a group was generalized to a new concept of a $\mathcal{G}$-inconsostency indicator map over a group X where $\mathcal{G} = \langle G, \otimes, \leq \rangle$ is an abelian linearly ordered group such that $J(x, y, z) \in G$ for all $x, y, z \in X$. Namely, a mapping $J : X^3 \to G$ is a $\mathcal{G}$-inconsistency indicator map over X if and only if the function $d_J : X^2 \to G$, defined by $d_J(x, y) = J(x, y, 1_X)$ for all $x, y \in X$ is a $\mathcal{G}$-metric on X such that $J(x, y, z) = d_J(x \odot z, y)$ for all $x, y, z \in X$. For $\mathcal{G} = \langle \mathbb{R}, +, \leq \rangle$, where $+$ and $\leq$ are standard addition and linear order in $\mathbb{R}$, an example of a $\mathcal{G}$-inconsistency indicator map on the group $\mathbb{R}_+$ is Koczkodaj's inconsistency indicator map.

It is important to notice here that the definition of an inconsistency indicator map given above allows us to localize the inconsistency in a PC matrix in the following sense: if X is a group, $\mathcal{G}$ is an alo-group, while J is a $\mathcal{G}$-inconsistency indicator map on X, then, for a matrix $A = [a_{i,j}]$ over the group X, we can localize all inconsistent triads $\langle a_{i,k}, a_{i,j}, a_{k,j} \rangle$ by looking whether $J(a_{i,k}, a_{i,j}, a_{k,j})$ is the identity element of G. For a $\mathcal{G}$-inconsistency indicator map J on a group X and for a matrix A over X, we define the J-inconsistency indicator of A by the following formula:

$$J[A] = \max\{J(a_{i,k}, a_{i,j}, a_{k,j}) : i, j, k \in \{1, \ldots, n\}\}.$$

Another possible definition of the inconsistency of a reciprocal PC matrix has a global character and needs explanations. Let $A = [a_{i,j}]_{i,j=1}^n$ be an $n \times n$ reciprocal PC matrix over a group $X = \langle X, \odot \rangle$. Suppose that $\mathcal{G} = \langle G, \otimes, \leq \rangle$ is an alo-group and that J is a $\mathcal{G}$-inconsistency indicator map on X. The

matrix A is consistent if and only if for any $1 \leq i < j \leq n$ the following equation holds:

$$a_{ij} = a_{i,i+1} \odot a_{i+1,i+2} \odot \ldots \odot a_{j-1,j}.$$

Therefore, a global J-inconsistency indicator $J_g[A]$ of A can be defined by the formula:

$$J_g[A] = \max_{1 \leq i < j \leq n} J(a_{i,j}, a_{i,i+1} \odot a_{i+1,i+2} \odot \ldots \odot a_{j-1,j}, 1).$$

In particular, in much the same way, as in [43], if X is the group $\mathbb{R}_+$ while f is Koczkodaj's inconsistency indicator map on $\mathbb{R}_+$, then, for every reciprocal matrix $A = [a_{i,j}]_{i,j=1}^n$ over $\mathbb{R}_+$, we can define the global inconsistency indicator $f_g[A]$ as follows:

$$f_g[A] = 1 - \min_{1 \leq i < j \leq n} \min \left(\frac{a_{ij}}{a_{i,i+1} a_{i+1,i+2} \ldots a_{j-1,j}}, \frac{a_{i,i+1} a_{i+1,i+2} \ldots a_{j-1,j}}{a_{ij}} \right).$$

It is obvious that:

$$f_g[A] = \max_{1 \leq i < j \leq n} \left(1 - e^{-\left| \ln\left(\frac{a_{ij}}{a_{i,i+1} a_{i+1,i+2} \cdots a_{j-1,j}} \right) \right|} \right)$$

For every reciprocal PC matrix A over the group $\mathbb{R}_+$, both definitions of $f[A]$ and of $f_g[A]$ have some advantages and disadvantages. The first definition allows us to find the localization of the most inconsistent triad elements of A. It follows what is adequately described by the idiom: "one bad apple spoils the barrel". The second definition may be useful when the global inconsistency is more important.

The CPC counter-example for the incorrectness of the eigenvalue-based inconsistency was supported by the mathematical reasoning in [43]. In all likeliness, all *panoptic*-type inconsistency indicators and all inconsistency indicators based on *central tendency* suffer from the similar problem because the consistency condition has (by the definition) the requirement "for all triads" and with the growing PC matrix size ($n \longrightarrow \infty$), the value of inconsistency indicator vanishes but the local estimation error does not.

If we accept the inconsistency "vanishing problem" as normal, by the same logic, we should not worry about one "nuke" left behind by the collapsed Great Empire since it is only one lost weapon for approximately 7,000,000,000 inhabitants of this planet. The nuke must be located and possibly destroyed. The same goes for inconsistency. Once located and found

as unacceptable, the inconsistency must be reduced before any further computations take place. Looking the other way around at the unacceptable inconsistency and computing the approximation is as wrong idea as the following "proof". Let us assume x=0 giving us $x = 2 * x$ since $0 = 2 * 0$. If so, we ca simplify it by dividing both side of the equation $x = 2 * x$ by x, getting 0=1, which we know is not exactly so. Our error comes from $k - n - o - w - i - n - g$ that $x = 0$ and dividing over it, despite knowing that we should not do it.

Evidently, we should not use unacceptably large inconsistent assessment to estimate our solution but AHP (and other inconsistency based on 'central tendency") ignores such principle. As a consequence, an error of the arbitrarily large value (e.g., 1,000,000% or more) is tolerated. The opponents of the above point may say: "...but dividing by 0 is an exceptional phenomenon". Apparently, $\frac{1}{10^(-n)}$ is not just one case. It generates arbitrary large values for growing n although the divisor is not exactly 0 and there are ∞ of them.

Drawing mechanical conclusions, based on the "central tendency," may be very dangerous, as the following example demonstrates. Let us look at the average weight of 80kg (200lbs) in a given population of 100 men. It does not imply that there is even a single man with the weight of 80kg nor even near that average. For example, it may happen that 50 men may each weigh 70kg and 50 weigh 90kg, which results in an average of 80kg. In the case of inconsistency indicators, the unacceptably high inconsistency in one triad may not contribute much to "the average" but it is unacceptable. Indeed, most of us the local approximation error aberration occurs and the argument that "we only lost one nuclear weapon" (which may be "accidentally found" by terrorists) is unacceptable to most of us. However, it does take place for the eigenvalue-based inconsistency as demonstrated by [43] for the CPC counter-example.

Common sense dictates a simple rule: "locate the worse inconsistency and check it; if unacceptable, reduce it". Certainly, if we are unable to reduce the inconsistency or there is n important reason (e.g., time) to continue with the inconsistency, she may do it. It is also important to notice that there are no inconsistencies in the simplified version of PC recently introduced in [44]. However, the zero inconsistency indicator does not guarantee that assessments are accurate. The consistent (or doctored data) ignorance can give result in 0 inconsistency. In fact, playing safe and giving all 1s for all entries of the entire PC matrix gives 0 inconsistency, but may be a sign

of the total ignorance, where everything is of equal importance. Certainly, everything may be of equal (or unknown) importance and in such case, all PC matrix entries should be equal to one.

Much has been done in this direction but more research is definitely still needed. Firstly, the input values should be extended to indicate the certainty of the assessment. It seems that fuzzy logic is the best approach to express uncertainty. Although the fuzzy logic is widely accepted for modelling uncertainly and approximate reasoning, Saaty vigorously opposed its use in two of his publications by using "On the invalidity of fuzzifying numerical judgments" in [16] and "There Is No Mathematical Validity for Using Fuzzy Number Crunching" in [16, 15], despite hundreds of well-documented successful applications (e.g., [14] with 695 citations as of 2015-03-27). The hierarchical classifier has not been given enough attention. In 1960s and 1970s, the hierarchical approach was popularized by many researchers in systems analysis. The top-down design (based on a hierarchical structure) was promoted by IBM researchers Harlan Mills and Niklaus Wirth in the late 1960s and early 1970s. The top-down strategy results in a hierarchical structure. The HIPO model (abbreviation for Hierarchical Input Process Output model) was introduced by IBM in early 1970s in [32] with a follow-up book [35]. However, the hierarchical classifier has been assumed rather than computed in most applications. The hierarchical clustering seems to offer the best solution for a non-supervised learning.

Finally, there has not been a single publication on PCs in any of the flagship AMS journals. The solid mathematical basis is recently proposed in [45].

6 The inconsistency reduction process

Erroneously, [51] is often given credited for providing a consistency definition of a PC matrix A. In fact, the condition "if and only if $a_{i,j} * a_{j,k} = a_{i,k}$ for $i, j, k = 1, 2, ..., n$." was defined and examined before 1977 by at least these four studies published between 1939 and 1961: [36, 30, 29, 54].

Localizing the inconsistency was proposed in [37]. The consistency-driven process was postulated since it comes from the GIGO principle and common sense. However, it is still not certain whether we should begin the reduction process, starting with the worse triad (as the common sense may dictate) or to improve the triad with the smallest inconsistency. In the second case,

smaller errors may propagate through the PC matrix. A Monte Carlo study has been launched and the preliminary finding support the foreknowledge in 1993.

It is safe to assume that processing (e.g., approximating) random numbers gives random results. In the case of PC, the "randomness" of the input is associated with the inconsistency. It is evidenced in Fig. 3.

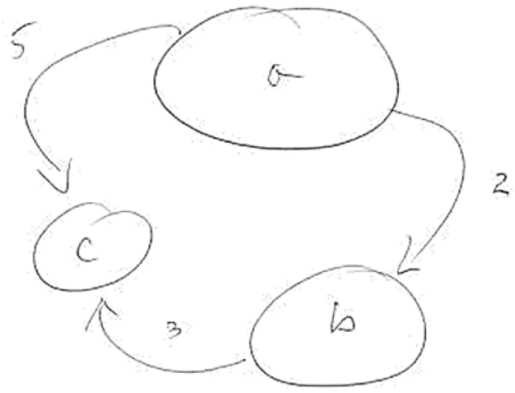

Figure 3: [Triad (2,5,3

If we assume that $A/B = 2$ and $B/C = 3$, evidently A/C cannot be 5 but rather 6. The problem is that these three comparisons might have been conducted, without problems due to the Internet, on three different continents, based on the "undisputed taste" (as earlier noted). No one can assert which of these three comparisons is (or is not) accurate.

The illustration of the inconsistency is even more drastic by the PC matrix $M3$, which is generated by the triad $(2, 1/2, 2)$. It gives unnormalized weights $v = [1, 1, 1]$ ($[1/3, 1/3, 1/3]$ in its normalized form). Vector v generates a trivial matrix with all entries equal to 1 representing ignorance (lack of knowledge). For this reason, we have to improve all individual triads. Which strategy should be employed, is the key issue for future research.

Many PC method studies fail to see the real problem with the goal of approximation. When we try to test the PC method on real examples related to subjective data (say, public satisfaction), how can we be sure what is and what is not "the right solution"? After all, it is said that the taste is indisputable.

The reduction process in [37] was based on the common sense approach. It called for finding the most inconsistent triad and reducing it. In this

study, the inconsistency reduction process was examined. It was based on the reduction of the triad with the smallest inconsistency. Why? It is suspected that the error propagation may be smaller than the triad with the worst inconsistency. Some research partners of the author have even questioned why bother with the inconsistency reduction when an approximation could be achieved in one step by geometric means. It may not be easy to provide a good response to such question, since it is probably impossible to provide an analytical answer why an inconsistency is undesirable (other than the GIGO adage but is it not a proof or even an unspoken rule since it is not true, in general). It is not easy to conduct a Monte Carlo experimentation although such preliminary study was conducted (see [38]) with the randomly generated images for its area estimation.

According to [42], finding consistent approximations of PC matrices with a high level of inconsistency makes little practical sense. From the standard mathematical logic, we know that only falsehood can generate both truth or falsehood. The research in [23] demonstrated that a "little falsehood" should still largely lead to the truth. However, the old adage that "one bad apple spoils the barrel" seems to be more applicable here: even a little falsehood may contribute to significant errors and misjudgments. An approximation of a PC matrix is meaningful, only if the initial inconsistency is acceptable (that is, located, brought under control and/or reduced to a certain predefined minimum; in our analogy, always remove an overripe fruit promptly if it is possible to find it).

In practical applications, a high value of the inconsistency indicator is a "red flag," or a sign of potential problems. A distance-based inconsistency reduction algorithm focuses, at each step, on an inconsistent triad and "corrects" it by replacing it with a consistent (or, more generally, a less inconsistent) triad. It resembles "Whac-a-Mole," a popular arcade game. One difference is that instead of one mole, we have three array elements, as explained above. After hitting the mole (which generally results in some other "moles" appearing), the next triad is selected according to some rule (which may be for example the greedy algorithm) and the process is repeated. Numerous practical implementations (e.g., a hazard ratio system for abandoned mines in Northern Ontario) have shown that the inconsistency converges relatively quickly. However, the need for rigorous prove of the convergence (that is, showing that whacked moles always have the tendency of coming out less and less eagerly) was evident. An approximation of a PC matrix is meaningful only if the inconsistency in it is acceptable. This means that

we need to localize the inconsistency and reduce it to a certain predefined minimum, if it is too high.

6.1 Not-so-inconsistent matrices

Using completely random PC matrices for testing has very little scientific merits, since they are simply random numbers and they defy all principles of learning (machine or natural). Common sense dictates to use matrices somehow inconsistent but not just a proverbial "bunch of random numbers". We will call such a PC matrix "not-so-inconsistent" (NSI) PC matrix. The NSI matrix was defined in [31] as follows. We obtain NSI PC matrix M from a random vector v with positive coordinates by: $M = [v_i/v_j]$ where $i, j = 1, 2, \ldots, n$. We deviate M by random multipliers $m_{ij} := m_{ij} * rand()$.

NSI PC matrix have been recently used in [39] to check how quick is the convergence to consistency by the reduction process. The results were very positive. The number of iterations to reduce the inconsistency below the accepted level (assumed as 1/3 for the distance-based inconsistency) turned to be at most 10.

7 Deriving priorities from PC matrices

In PC we compare two elements, and assign a value, which represents an assessment of the relative preference of E_i over E_j . If E_i is preferred to E_j then $a_{ij} > 1$, otherwise $0 < a_{ij} < 1$. A full set of assessments for n elements requires $n(n-1)/2$ comparisons. In order to derive a priority vector from a given set of assessments, [51] constructs a positive reciprocal matrix, whose elements satisfy the reciprocal property $a_{ji} = 1/a_{ij}$, $a_{ij} > 0$, $i, j \in 1, 2, \ldots, n$.

The eigenvector method (EV), proposed in [51] is based on the Perron-Frobenius theorem. Saaty proves that the principal right eigenvector of A can be used as a priority vector, so the EV solves the equation: $Aw = \lambda_{max} w$, where λ_{max} is the principal eigenvalue of A. However, EV method suffers from numerous drawbacks, discussed in the previous sections. Numerous alternative prioritization methods, have been proposed. Many of them are based on the optimization approach. These methods need an objective function, which measures the accuracy of the approximation of the initial comparison assessments by the candidate solution. Thus, the problem of priority derivation is formulated as an optimization task of minimizing the objective

function, subject to normalization and possible additional constraints.

The Direct Least Squares (DLS) method (probably for the first time introduced in [34]) is based on the assumption that the errors between the initial assessments and solution ratios should be minimized, so it uses the Euclidean distance metric (or its square) as an objective function. The prioritization task is formulated as a constrained optimization problem:

$$MinED = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - w_i/w_j)^2 \right)^{1/2}$$

subject to: $\sum_{i=1}^{n} w_i = 1$, $w_i > 0$.

On page 312 in [53], Saaty and Vargas state:

> Remark: The most frequently used metric to measure closeness and accuracy in n-dimensional spaces is the Euclidean metric.

Common sense dictates that there is a reason why the Euclidean metric is the most frequently used. It is used for hundreds of years since it is simple, elegant and accurate. The above quotation was not only true in 1984 but it is still true at the time of drafting this study (after 31 yeas). No wonder that Saaty and Vargas followed it by an "explanation":

> However, the Euclidean metric does not address the question of inconsistency.

The above quotation leads us to the logical conclusion that the inconsistency needs to be handled separately from approximation. In fact, no approximation can really give a reliable solution for heavily polluted data hence the inconsistency analysis should precede any approximation as recommended in former sections. With the proper inconsistency analysis, MinED is the most practical solution since it is simple and accurate. After all, the statistical results (based on 1,000,000 randomly generated case in [31] and numerous smaller Monte Carlo experimentation) support it.

8 Notable applications

The number of applications of PC is ever growing. It is currently one of a few valid methods for processing subjective data. Listing any applications here

would be risky since it may imply more importance over other applications. To illustrate, we know that a society equipped with only medical doctors would not survive for long since they may starve without farmers. Medicine and farming can be, and should be, supported by PC. In the past, the PC method was used for decision making on the national level, related to nuclear weapons or energy [46]. Certainly, making such decisions must be guided by many common sense "rules" and GIGO (garbage-in, garbage-out) is one of them.

It is also important to point out that one of the sizable countries in the European Union has already passed a law to use PCs for evaluations at the national level as documented in [40], where scientific entities have been evaluated. Credits for introducing PC to the ministerial regulations in Poland should be given to Prof. R. Slowinskii, a Member of the Polish Academy of Science.

9 Conclusions

There is still much to be done in PC. The biggest challenge is not so much the technology: hardware and software, but rather the theory itself. However, complicating the existing theory is not the solution. "Best practices," as they are often used in the Software Engineering approach, are needed ([11, Section 4.5]).

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