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"NONLINEAR PULLBACKS" OF FUNCTIONS AND L_∞-MORPHISMS FOR HOMOTOPY POISSON STRUCTURES

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ABSTRACT. We introduce mappings between spaces of functions on (super)manifolds that generalize pullbacks with respect to smooth maps but are, in general, nonlinear (actually, formal). The construction is based on canonical relations and generating functions. (The underlying structure is a formal category, which is a "thickening" of the usual category of supermanifolds; it is close to the category of symplectic micromanifolds and their micromorphisms considered recently by A. Weinstein and A. Cattaneo–B. Dherin–A. Weinstein.) There are two parallel settings, for even and odd functions. As an application, we show how such nonlinear pullbacks give L_∞-morphisms for algebras of functions on homotopy Schouten or homotopy Poisson manifolds.

INTRODUCTION

In this paper we introduce and study a certain notion of a "generalized pullback" for functions on smooth manifolds or supermanifolds. It has two characteristic features: it is, in general, a *nonlinear* mapping between the spaces of functions (actually, it is given by a formal nonlinear differential operator), and it contains the usual pullback of functions as a particular case (when it is of course linear). Being nonlinear, such a "generalized pullback" cannot be an algebra homomorphism; however, it has the property that its derivative at each point is an algebra homomorphism.

The underlying "generalized morphisms" of (super)manifolds are certain canonical relations between the corresponding cotangent bundles. Recall that a canonical relation (or a canonical correspondence) between symplectic manifolds is a Lagrangian submanifold in the direct product endowed with the difference of the symplectic forms. Such are the graphs of symplectomorphisms, hence canonical relations are usually seen as a generalization of symplectomorphisms. Our viewpoint is different (so thinking of symplectomorphisms is not useful for understanding) and can be explained as follows. Let $\varphi\colon M_1 \to M_2$ be a usual smooth map. It does not induce, in general, any map of the cotangent bundles. (The exception is the case of a diffeomorphism.) Nevertheless it gives a relation $R_\varphi \subset T^*M_1 \times (-T^*M_2)$, which, as one can see, is a Lagrangian submanifold (here the minus sign means the negative of the symplectic form). This is our point of departure. As a generalization of maps $\varphi\colon M_1 \to M_2$ we consider canonical relations $\Phi \subset T^*M_1 \times (-T^*M_2)$ of the type "closest to those of the form R_φ". The latter requirement, as we will see, forces us to consider *formal relations*, i.e., which live as submanifolds in the formal neighborhood

of the zero section. All our constructions are therefore *microformal* (based on power series in momentum variables).

Replacing ordinary maps by relations and considering categories of relations is a very classical idea; in particular, the unifying role of canonical relations has been stressed by Weinstein [29, 28].[1] The novelty is not in the use of relations as such, but in our choice of a particular class of relations and in their treatment for defining an analog of pullbacks. We see a relation $\Phi \subset T^*M_1 \times (-T^*M_2)$ as a 'morphism' not between T^*M_1 and T^*M_2, but between M_1 and M_2. This is crucial and leads to the idea of a **pullback**

$$\Phi^* \colon \ \mathbf{C}^\infty(M_2) \to \mathbf{C}^\infty(M_1) \,,$$

which we define as follows. Suppose $g \in \mathbf{C}^\infty(M_2)$ is a function on M_2. The pullback Φ^* should map it to a function $f \in \mathbf{C}^\infty(M_1)$. Consider the graph of the derivative $\Lambda_g \subset T^*M_2$. It is a Lagrangian submanifold. Then f can be defined by the formula:

$$\Lambda_f := \Phi \circ \Lambda_g \subset T^*M_1 \,,$$

where $\circ$ denotes the composition of relations; one needs to show that $\Phi \circ \Lambda_g$ has the form Λ_f. More precisely, this defines the function f up to a constant of integration, so our constructions involve a choice of constants. We assume that $\Phi \subset T^*M_1 \times (-T^*M_2)$ can be specified by a generating function $S(x, q)$ depending on position variables on M_1 and momentum variables on M_2, and we treat S as part of structure. Then the pullback $f = \Phi^*[g]$ introduced above in an abstract way is given explicitly by

$$\boxed{f(x) = g(y) + S(x, q) - y^i q_i \,,}$$

where to eliminate the variables q and y one should use the system of equations $q_i = \partial g/\partial y^i$ and $y^i = \partial S/\partial q_i$. It is solved by iterations, which gives y as a function of x depending in general on g and its derivatives. Therefore the resulting $f(x)$ is expressed in $g(y)$ with all its derivatives perturbatively and nonlinearly, as a formal power series. (This is one of the many instances where formality enters the picture. Actually, the very assumption that we can use $S(x, q)$ implies a formal framework.) Remarkably, when $\Phi = R_\varphi$, the equations decouple, the nonlinearity disappears and the formula gives the ordinary pullback $\varphi^* g$.

We came to this construction motivated by the following task. Suppose a supermanifold M is endowed with a homotopy analog of a Poisson structure. That means that — instead of a familiar bracket with two arguments — there is a whole sequence of brackets including binary, but also unary, ternary, etc., so that in particular the Jacobi identity for the binary bracket is satisfied up to an algebraic homotopy, where the ternary bracket is the homotopy and the unary bracket is the differential, and there are further identities involving the 'higher homotopies'. Speaking more formally, there is an L_∞-algebra structure on functions on M such that all the brackets are multiderivations with respect to the ordinary product of functions. (Actually, there are two different types of such structures on M, a 'homotopy Poisson' and a 'homotopy Schouten' structures, which differ by the parities of the brackets.) The problem is how to construct L_∞-morphisms of such homotopy structures. Let us concentrate on the homotopy Schouten case, for concreteness. From general theory it is

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[1]Other manifestations of this idea include additive relations in homological algebra [14] and the Berezin–Neretin spinor representations of classical categories [2], [17, 18].

known that the most efficient description for L_∞-algebras is that of Q-manifolds, i.e., super-manifolds endowed with homological vector fields (see, e.g., [11, 12]). L_∞-morphisms then correspond to maps of supermanifolds intertwining the corresponding homological vector fields. In the considered example, homological vector fields live on infinite-dimensional "functional" supermanifolds such as $\mathbf{C}^\infty(M)$ and one has to construct mappings between them. This is a problem formulated in terms of infinite-dimensional geometry. On the other hand, a homotopy Schouten structure has a convenient finite-dimensional description in terms of differential-geometric objects on M itself: it is specified by an odd function H on T^*M satisfying $(H, H) = 0$ for the canonical Poisson bracket. Hence the question: is it possible to construct L_∞-morphisms in these terms? Nonlinear pullbacks Φ^* that we introduce give the desired solution.

More precisely, we proved the following theorem. If two odd Hamiltonians H_1 and H_2 specifying homotopy Schouten structures on M_1 and M_2 satisfy $p_1^*H_1 = p_2^*H_2$ on $\Phi \subset T^*M_1 \times (-T^*M_2)$ as above (in other words, if they are 'Φ-related', for Φ regarded as a 'microformal morphism' from M_1 to M_2), then the pullback $\Phi^*\colon \mathbf{C}^\infty(M_2) \to \mathbf{C}^\infty(M_1)$ is an L_∞-morphism of the homotopy Schouten algebras. (Note that the possible nonlinearity of Φ^* is essential for obtaining nontrivial L_∞-morphisms.)

Nonlinear pullbacks Φ^* as above are applicable to **even** functions. There is a parallel construction that works for **odd** functions and gives a different 'formal thickening' of the category of smooth supermanifolds, with nonlinear pullbacks

$$\Psi^*\colon \ \mathbf{\Pi C}^\infty(M_2) \to \mathbf{\Pi C}^\infty(M_1)$$

(here Π is the parity reversion functor and 'points' of $\mathbf{\Pi C}^\infty(M)$ are odd functions on M). It is based on the anticotangent bundles ΠT^*M with the canonical odd symplectic structure. There is a similar application to homotopy Poisson manifolds.

The formal categorical structure outlined only briefly in this paper, and further applications to vector bundles and algebroids, are developed in [24]. A certain 'quantum' version is constructed in [25, 26].

1. Main construction

Consider smooth manifolds or supermanifolds M_1 and M_2. We shall define a mapping of smooth functions on M_2 to smooth functions on M_1 which generalizes the pullback w.r.t. a smooth map $\varphi\colon M_1 \to M_2$. This mapping is, in general, nonlinear — actually, it will be defined as a formal power series — and has an ordinary pullback as a particular case, in which is it is linear. (Moreover, as we shall see, ordinary pullbacks appear as linearizations or derivatives for our construction.)

Remark 1. Constructions in this section are not specifically super, so the reader may assume, initially, that we work with ordinary manifolds and ignore signs related with the supercase. Considering supermanifolds is necessary only for applications.
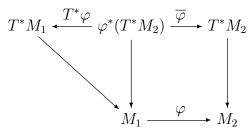
To emphasize that we consider spaces of smooth functions as infinite-dimensional manifolds rather than vector spaces, we use boldface letters for the notation, e.g., $\mathbf{C}^\infty(M)$ instead of $C^\infty(M)$.

Remark 2. This distinction is important for supermanifolds where $\mathbf{C}^\infty(M)$ is itself an infinite-dimensional supermanifold whose 'points' are, by definition, *even* functions on M possibly depending on auxiliary odd parameters, unlike elements of the $\mathbb{Z}_2$-graded vector space $C^\infty(M)$, which may be even or odd. Indeed, for consider unambiguously non-linear expressions involving $f(x)$, etc., the parity of f should be fixed, since even and odd functions satisfy different commutativity constraints. (Similarly, odd functions on M should be regarded as 'points' of the infinite-dimensional supermanifold $\mathbf{\Pi C}^\infty(M)$ corresponding to the $\mathbb{Z}_2$-graded vector space $\Pi C^\infty(M)$.)

We shall use the language of canonical relations. Consider the cotangent bundles T^*M_1 and T^*M_2. Denote local coordinates on M_1 and M_2 by x^a and y^i, and let p_a and q_i stand for the corresponding conjugate momenta; so the canonical symplectic forms are $\omega_1 = dp_a dx^a$ and $\omega_2 = dq_i dy^i$, respectively. We use local coordinates here because it is the most efficient language, though of course everything can be rephrased in a coordinate-free way. It is well known that canonical relations between symplectic manifolds arise as graphs of canonical transformations (symplectomorphisms) and may be seen as their generalizations. For cotangent bundles, they also arise when one considers the effect of smooth maps of the bases. This is our starting point.

Example 1. Suppose $\varphi\colon M_1 \to M_2$ is a smooth map. It induces the following diagram for the cotangent bundles:

$$T^*M_1 \xleftarrow{\;T^*\varphi\;} \varphi^*(T^*M_2) \xrightarrow{\;\overline{\varphi}\;} T^*M_2$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$M_1 \xrightarrow{\;\varphi\;} M_2$$

where the map $\overline{\varphi}$ is fiberwise identical and at each point $x \in M_1$ the map $T^*\varphi(x) = (T\varphi)^*(x)$ is the adjoint of the tangent map. In coordinates, $\varphi^*(y^i) = \varphi^i(x)$ and

$$\overline{\varphi}\colon (x^a, q_i) \mapsto (y^i, q_i) \quad \text{where} \quad y^i = \varphi^i(x),$$

$$T^*\varphi\colon (x^a, q_i) \mapsto (x^a, p_a) \quad \text{where} \quad p_a = \frac{\partial \varphi^i}{\partial x^a}(x) q_i.$$

We define a relation $R_\varphi \subset T^*M_1 \times T^*M_2$ as follows:

$$R_\varphi = \left\{ (x^a, p_a, y^i, q_i) \;\middle|\; p_a = \frac{\partial \varphi^i}{\partial x^a}(x) q_i, \; y^i = \varphi^i(x) \right\}. \tag{1}$$

In other words, R_φ is the composition of the graphs of $T^*\varphi$ and $\overline{\varphi}$. Note that the dimension of R_φ is exactly half of the dimension of $T^*M_1 \times T^*M_2$. For the pullbacks of the symplectic forms ω_1 and ω_2 on R_φ we obtain

$$p_1^*\omega_1 = p_1^*(dp_a dx^a) = d\left(\frac{\partial \varphi^i}{\partial x^a} q_i \right) dx^a = d\left(\frac{\partial \varphi^i}{\partial x^a} q_i dx^a \right) = d\left(d\varphi^i q_i \right) = dq_i d\varphi^i,$$

and

$$p_2^*\omega_2 = p_2^*(dq_i dy^i) = dq_i d\varphi^i \,.$$

Therefore $p_1^*\omega_1 = p_2^*\omega_2$ on R_φ, and so R_φ is a Lagrangian submanifold in $T^*M_1 \times T^*M_2$ considered with the symplectic structure $p_1^*\omega_1 - p_2^*\omega_2$, i.e., a canonical relation.

This example serves as a model for our general construction. Denote by $T^*M_1 \times (-T^*M_2)$ the symplectic manifold $T^*M_1 \times T^*M_2$ considered with the symplectic form $\omega_1 - \omega_2$ (where we have suppressed p_1^* and p_2^*). Consider a canonical relation

$$\Phi \subset T^*M_1 \times (-T^*M_2)$$

of the form generalizing (1),

$$\Phi = \left\{ (x^a, p_a, y^i, q_i) \,\middle|\, p_a = \psi_a(x, q), \; y^i = \varphi^i(x, q) \right\} \,. \tag{2}$$

(We use the letter Φ is a reminiscent of a map of manifolds φ.) The condition that Φ is canonical implies that it can be described by a generating function $S = S(x, q)$ so that

$$p_a dx^a - (-1)^{\tilde{i}+1} y^i dq_i = dS \tag{3}$$

on Φ. (Note that $p_a dx^a - (-1)^{\tilde{i}+1} y^i dq_i = p_a dx^a - q_i dy^i + d(y^i q_i)$.) In other words, our relation has the form

$$\Phi = \left\{ (x^a, p_a, y^i, q_i) \,\middle|\, p_a = \frac{\partial S}{\partial x^a}(x, q), \; y^i = (-1)^{\tilde{i}} \frac{\partial S}{\partial q_i}(x, q) \right\}, \tag{4}$$

for a generating function $S = S(x, q)$. (In particular, (1) is recovered from (4) for $S = \varphi^i(x) q_i$.)

A good time to quote Arnold: "Before turning to the apparatus of generating functions, we remark that it is unfortunately noninvariant[2] and it uses, in an essential way, the coordinate structure in phase space" [1, p. 258].

Remark 3. Recall that if $\Lambda \subset N$ is a Lagrangian submanifold of a symplectic manifold, then in each local Darboux coordinate system on N one can choose half of the coordinates so that they are independent on Λ and the remaining canonically conjugated coordinates are expressed as the partial derivatives (up to signs) of a function of the coordinates from the first group. This function, by definition, is a *generating function* for Λ. By construction, it is a coordinate-dependent object. A particular type of such a function corresponds to a choice of Darboux coordinates on N taken as independent coordinates on the Lagrangian submanifold Λ. Not all choices may be possible. In our situation, we *postulate* the possibility of taking x^a, q_i as independent coordinates on $\Phi \subset T^*M_1 \times (-T^*M_2)$ and hence the possibility of expressing our canonical relation in the form (2), (4).

Now we shall define an analog of pullback of functions using a canonical relation Φ of the form (4) as a replacement of a smooth map $\varphi \colon M_1 \to M_2$. Consider an even function

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[2]In the Russian original it sounds even stronger: "depressingly noninvariant".

$g \in \mathbf{C}^{\infty}(M_2)$. The graph of its derivative is a Lagrangian submanifold $\Lambda_g \subset T^*M_2$. In coordinates,

$$\Lambda_g = \left\{ (y^i, q_i) \; \middle| \; q_i = \frac{\partial g}{\partial y^i}(y) \right\} . \tag{5}$$

Take the composition of the relation $\Phi \subset T^*M_1 \times (-T^*M_2)$ with the submanifold $\Lambda_g \subset T^*M_2$. It is a submanifold in T^*M_1,

$$\Lambda := \Phi \circ \Lambda_g = p_1 \left(\Phi \cap p_2^{-1}(\Lambda_g) \right) \subset T^*M_1 .$$

In coordinates,

$$\Lambda = \left\{ (x^a, p_a) \; \middle| \; p_a = \frac{\partial S}{\partial x^a}(x, q), \; q_i = \frac{\partial g}{\partial y^i}(y), \; y^i = (-1)^{\tilde{\imath}} \frac{\partial S}{\partial q_i}(x, q) \right\} . \tag{6}$$

Theorem-Definition 1 (preliminary version). *The submanifold $\Lambda \subset T^*M_1$ is Lagrangian and is the graph of the derivative of a function $f \in \mathbf{C}^{\infty}(M_1)$, $\Lambda = \Lambda_f$, which is given by*

$$f(x) = g(y) + S(x, q) - y^i q_i , \tag{7}$$

where q_i and y^i in (7) are determined from the equations

$$q_i = \frac{\partial g}{\partial y^i}(y) , \tag{8}$$

$$y^i = (-1)^{\tilde{\imath}} \frac{\partial S}{\partial q_i}\left(x, \frac{\partial g}{\partial y}(y)\right) . \tag{9}$$

*The function $f \in \mathbf{C}^{\infty}(M_1)$ defined by Eq. (7), together with Eqs. (8) and (9), is called the **generalized pullback** of a function $g \in \mathbf{C}^{\infty}(M_2)$ w.r.t. a canonical relation $\Phi \subset T^*M_1 \times (-T^*M_2)$. Notation: $f = \Phi^*[g]$.*

Proof. The fact that $\Lambda \subset T^*M_1$ is Lagrangian actually follows from general theory: the composition of Lagrangian relations, when it is well-defined, is Lagrangian. We shall check it directly together with establishing (7). On Φ, $p_a dx^a - q_i dy^i = dS - d(y^i q_i)$ (see (3)). By substituting $q_i dy^i = dg$, we obtain that on Λ, $p_a dx^a = dg + dS - d(y^i q_i)$, so $p_a dx^a = df$ with f given by (7). In particular, Λ has the correct dimension, so it is indeed Lagrangian and $\Lambda = \Lambda_f$, as claimed. Formulas (8) and (9) follow from the definitions of Φ and Λ_g. We shall see that the construction of f by formulas (7),(8),(9) does not depend on a choice of coordinates and f is globally defined. This requires some further clarifications that will be provided below after considering examples. $\qquad\square$

Let us analyze formulas (7), (8) and (9) defining together the map Φ^*. In order to find the function $f = \Phi^*[g] \in \mathbf{C}^{\infty}(M_1)$ from (7), one has to solve equation (9) so to express y as a function of x and then substitute $y = y(x)$ into (8) and (7). The function $g \in \mathbf{C}^{\infty}(M_2)$, the functional argument of the mapping Φ^*, enters the equation for y through the derivative $\partial g / \partial y$. Ultimately, the function g appears in (7) both explicitly as $g(y)$ and implicitly through the variables y and q. To see the whole procedure better, we consider examples (which will also lead us to promised clarifications).

Example 2 (image of zero). Let $g = 0 \in \mathbf{C}^{\infty}(M_2)$. We get $q_i = \partial g / \partial y^i = 0$, so

$$f(x) = S(x, 0) - 0 + 0 = S_0(x) ,$$

where we denoted $S_0(x) := S(x, 0)$. Therefore

$$\Phi^*[0] = S_0. \tag{10}$$

This is a fixed (even) function on M_1.

Let us write $S(x, q)$ in the form of a power expansion in q_i:

$$S(x, q) = S_0(x) + \varphi^i(x)q_i + \frac{1}{2}S^{ij}(x)q_jq_i + \frac{1}{3!}S^{ijk}(x)q_kq_jq_i + \ldots \tag{11}$$

(the reason for the choice of notation $\varphi^i(x)$ will become clear shortly).

Consider first a special case.

Example 3 (ordinary pullback with a shift). Suppose

$$S(x, q) = S_0(x) + \varphi^i(x)q_i \tag{12}$$

(no higher order terms). Then from (9),

$$y^i = \varphi^i(x),$$

so the coefficients φ^i define a smooth map $\varphi\colon M_1 \to M_2$. From (7) we obtain

$$f(x) = S_0(x) + \varphi^i(x)q_i - y^iq_i + g(y) = S_0(x) + g(\varphi(x)).$$

We see that in this case, the map Φ^* is the combination of an ordinary pullback w.r.t. a map $M_1 \to M_2$ and a 'constant shift' by a function on M_1,

$$\Phi^*[g] = S_0 + \varphi^*g. \tag{13}$$

In contrast with (12), it is not possible in a meaningful way to consider the generating function S with a finite number of higher order terms restricting their order by some $r > 1$ (as we shall see later). So we have to deal with the whole expansion (11). For that, we consider Φ^* perturbatively near $g \equiv 0$.

Example 4 (linear approximation for Φ^*). Consider $g(y) = \varepsilon h(y)$ where $\varepsilon^2 = 0$. From (7) and (8), we see that we need to determine y from (9) only in the zero order approximation. Also, there is no input from the terms of order $\geqslant 2$ in (11). Hence $y^i = \varphi^i(x)$ and $f(x)$ is obtained similarly to Example 3:

$$f(x) = S_0(x) + \varphi^i(x)q_i - y^iq_i + \varepsilon h(y) = S_0(x) + \varepsilon h(\varphi(x)).$$

In other words,

$$\Phi^*[\varepsilon h] = S_0 + \varepsilon\,\varphi^*h \quad \mod \varepsilon^2. \tag{14}$$

Example 5 (quadratic approximation for Φ^*). To make one more step, let $g(y) = \varepsilon h(y)$ where $\varepsilon^3 = 0$. Now we see that we need y in the linear approximation. Writing $y_\varepsilon = y_0 + \varepsilon y_1$ mod ε^2, we obtain from (9) and (11)

$$y^i_\varepsilon = \varphi^i(x) + \varepsilon\, S^{ij}(x)\frac{\partial h}{\partial y^j}(y_0) = \varphi^i(x) + \varepsilon\, S^{ij}(x)\frac{\partial h}{\partial y^j}(\varphi(x)) \quad \mod \varepsilon^2.$$

To find $f(x)$, we substitute into (7) and simplify:

$$f(x) = S(x, q) - y^i q_i + \varepsilon h(y) =$$

$$S\left(x, \varepsilon \frac{\partial h}{\partial y}(y_0 + \varepsilon y_1)\right) - \varepsilon(y_0^i + \varepsilon y_1^i)\frac{\partial h}{\partial y^i}(y_0 + \varepsilon y_1) + \varepsilon h(y_0 + \varepsilon y_1) =$$

$$S_0(x) + \varepsilon\, \varphi^i(x)\frac{\partial h}{\partial y^i}(y_0 + \varepsilon y_1) + \varepsilon^2\, \frac{1}{2}\, S^{ij}(x)\frac{\partial h}{\partial y^j}(y_0)\frac{\partial h}{\partial y^i}(y_0) -$$

$$\varepsilon(y_0^i + \varepsilon y_1^i)\frac{\partial h}{\partial y^i}(y_0 + \varepsilon y_1) + \varepsilon h(y_0 + \varepsilon y_1) =$$

$$S_0(x) + \varepsilon^2\, \frac{1}{2}\, S^{ij}(x)\frac{\partial h}{\partial y^j}(y_0)\frac{\partial h}{\partial y^i}(y_0) - \varepsilon^2\, y_1^i\frac{\partial h}{\partial y^i}(y_0 + \varepsilon y_1) + \varepsilon h(y_0 + \varepsilon y_1) =$$

$$S_0(x) + \varepsilon^2\, \frac{1}{2}\, S^{ij}(x)\frac{\partial h}{\partial y^j}(y_0)\frac{\partial h}{\partial y^i}(y_0) - \varepsilon^2\, y_1^i\frac{\partial h}{\partial y^i}(y_0) + \varepsilon\, h(y_0) + \varepsilon^2\, y_1^i\frac{\partial h}{\partial y^i}(y_0) =$$

$$S_0(x) + \varepsilon\, h(\varphi(x)) + \varepsilon^2\, \frac{1}{2}\, S^{ij}(x)\frac{\partial h}{\partial y^j}(\varphi(x))\frac{\partial h}{\partial y^i}(\varphi(x)) \quad \mod \varepsilon^3\,.$$

Thus

$$\Phi^*[\varepsilon h] = S_0 + \varepsilon\, \varphi^* h + \varepsilon^2\, \frac{1}{2}\, S^{ij}\, \varphi^* \partial_j h\, \varphi^* \partial_i h \quad \mod \varepsilon^3\,. \tag{15}$$

Generalizing from these examples, we claim that in general the nonlinear transformation Φ^* *exists at least at the formal level* as a perturbation series around an ordinary pullback plus a shift (a transformation of the form (13)):

$$\Phi^*[g](x) = S_0(x) + \varphi^* g(x) + \sum_{r \geqslant 2} \Phi_r\big(x, \varphi^* \partial g(x), \varphi^* \partial^2 g(x), \ldots\big)\,, \tag{16}$$

where each term Φ_r is a homogeneous differential polynomial in g of order $\leqslant r$. (Also, Φ_r depends on derivatives of degrees $\leqslant r$ in g.) Here the 'shift' $S_0(x)$ is given by the zero order term of a generating function $S(x, q)$ and the ordinary pullback φ^* is with respect to a map $\varphi\colon M_1 \to M_2$, which is given the first order terms of the function $S(x, q)$.

That this is indeed so can be proved as follows. For a given g, equation (9) defines the variables y^i as functions of x^a, i.e., it defines a smooth map

$$\varphi_g\colon M_1 \to M_2 \tag{17}$$

depending on $g \in \mathbf{C}^\infty(M_2)$, which is a formal perturbation of a given map $\varphi = \varphi_0\colon M_1 \to M_2$. Solution of (9) may be obtained by an iterative procedure starting from $y^i = \varphi^i(x) = (-1)^{\tilde{\imath}} \partial S/\partial q_i(x, 0)$. In other words, we write

$$\varphi_g = \varphi_0 + \ldots\,,$$

where $\varphi_0 = \varphi$, so the map φ_g depending on g is expressed as a perturbation series around the map φ defined by the canonical relation Φ alone. Introduce parameter ε and consider εg instead of g. For the Nth iterative step we write

$$y_{(N)} = y_0 + \varepsilon\, y_1 + \varepsilon^2 y_2 + \ldots + \varepsilon^N y_N \qquad \mod \varepsilon^{N+1}\,,$$

where $y_0 = \varphi(x)$. The term y_N is defined via $y_0, y_1, \ldots, y_{N-1}$ from the equation

$$y_0^i + \varepsilon\, y_1^i + \varepsilon^2 y_2^i + \ldots + \varepsilon^N y_N^i =$$

$$\varphi^i(x) + \varepsilon\, S_1^i\left(x, \frac{\partial g}{\partial y}\left(y_0 + \varepsilon\, y_1 + \varepsilon^2 y_2 + \ldots + \varepsilon^{N-1} y_{N-1}\right)\right)$$

$$+ \varepsilon^2 S_2^i\left(x, \frac{\partial g}{\partial y}\left(y_0 + \varepsilon\, y_1 + \varepsilon^2 y_2 + \ldots + \varepsilon^{N-2} y_{N-2}\right)\right) + \ldots$$

$$+ \varepsilon^N S_N^i\left(x, \frac{\partial g}{\partial y}\left(y_0\right)\right) \qquad \mathrm{mod}\ \varepsilon^{N+1}. \quad (18)$$

Here

$$S_r^i(x, q) = (-1)^{\tilde{i}}\, \frac{\partial S_{r+1}}{\partial q_i}(x, q) = \frac{1}{r!}\, S^{i i_1 \ldots i_r}(x) q_{i_r} \ldots q_{i_1}\,,$$

where

$$S_r(x, q) = \frac{1}{r!}\, S^{i_1 \ldots i_r}(x) q_{i_r} \ldots q_{i_1}$$

is the rth homogeneous term in the expansion (11). To find y_N from (18), we expand the r.h.s. to order N in ε and notice that the terms of order $\leqslant N-1$ cancel automatically with the terms at the l.h.s. (since $y_1, \ldots, y_{N-1}$ were defined by exactly the same relation at the previous steps). Therefore, it is sufficient to collect the terms of order N at the r.h.s. and upon division by ε^N this gives y_N. By the way we see that y_N depends on the derivatives of g to order $\leqslant N$ (evaluated at $y_0 = \varphi(x)$). We may write symbolically

$$\varphi_g = \varphi + \varphi_1[g] + \varphi_2[g] + \ldots + \varphi_N[g] + \ldots \quad (19)$$

as a functional formal power series in g, where each term $\varphi_N[g]$ is of order N in g and $\varphi_N^i[g] = y_N^i$ defined by the above procedure. Hence

$$\Phi^*[g] = \varphi_g^* g + S\left(x, \varphi_g^* \frac{\partial g}{\partial y}\right) - \varphi_g^i \cdot \varphi_g^* \partial_i g\,, \quad (20)$$

which justifies the claim about the form of the expansion (16).

To summarize, we can now supplement Theorem-Definition 1 by saying that we should consider the generating function $S(x, q)$ specifying a canonical relation Φ as a formal power series in q_i; thus the relation Φ is itself formal. The operation Φ^* is defined by (20) via the above iterative procedure and is therefore a *formal* mapping between the spaces of functions. Also, the generating function $S(x, q)$ is regarded as a part of structure, which eliminates questions about a choice of 'constants of integration'. This is the first of the clarifications promised in the proof of Theorem-Definition 1. Another clarification concerns the (in)dependence of the procedure defining Φ^* of a choice of coordinates and will follow shortly.

In Example 4, we actually computed the derivative of the mapping Φ^* at $g \equiv 0$. It is possible to find the derivative of Φ^* at an arbitrary point.

Theorem 2. *The derivative of the formal mapping of functional manifolds*

$$\Phi^*\colon\ \mathbf{C}^\infty(M_2) \to \mathbf{C}^\infty(M_1)$$

at a point $g \in \mathbf{C}^\infty(M_2)$ *is given by the formula:*

$$(T\Phi^*)[g] = \varphi_g^*,$$

where

$$\varphi_g^*\colon \; C^\infty(M_2) \to C^\infty(M_1)$$

is the usual pullback with respect to the map $\varphi_g\colon M_1 \to M_2$ *defined by* g.

Proof. Consider a variation of a point $g \in \mathbf{C}^\infty(M_2)$, $g_\varepsilon(y) = g(y) + \varepsilon u(y)$. We need to find the corresponding $f_\varepsilon = \Phi^*[g_\varepsilon]$. We shall denote by q_i^ε and y_ε^i the solutions of (8) and (9) for the perturbed function g_ε, and by symbols without ε, the corresponding non-perturbed objects. We may write $y_\varepsilon^i = y^i + \varepsilon y_1^i$ and $q_i^\varepsilon = q_i + \varepsilon q_{1i}$. By substituting into (7), we obtain

$$f_\varepsilon(x) = S(x, q + \varepsilon q_1) - (y^i + \varepsilon y_1^i)(q_i + \varepsilon q_{1i}) + g_\varepsilon(y + \varepsilon y_1) =$$

$$S(x,q) + \varepsilon\, q_{1i}\frac{\partial S}{\partial q_i}(x,q) - y^i q_i - (-1)^{\tilde{\varepsilon}\tilde{\imath}}\varepsilon y^i q_{1i} - \varepsilon\, y_1^i q_i + g(y) + \varepsilon y_1^i \frac{\partial g}{\partial y^i}(y) + \varepsilon\, u(y) =$$

$$S(x,q) + \varepsilon\, q_{1i}(-1)^{\tilde{\imath}}y^i - y^i q_i - (-1)^{\tilde{\varepsilon}\tilde{\imath}}\varepsilon(-1)^{\tilde{\imath}(\tilde{\imath}+\tilde{\varepsilon})}q_{1i}y^i - \varepsilon\, y_1^i q_i + g(y) + \varepsilon y_1^i q_i + \varepsilon\, u(y) =$$

$$S(x,q) - y^i q_i + g(y) + \varepsilon\, u(y) = f(x) + \varepsilon\, u(y)\,.$$

Note that $y = \varphi_g(x)$. Therefore for a perturbation $\varepsilon\, u$, $u \in C^\infty(M_2)$, of $g \in \mathbf{C}^\infty(M_2)$, the corresponding perturbation of $\Phi^*[g] \in \mathbf{C}^\infty(M_1)$ is $\varepsilon\, \varphi_g^* u$, where $\varphi_g^* u \in C^\infty(M_1)$. $\qquad\square$

Finally, let us turn to the question of a transformation law of the generating function $S(x,q)$ under a change of coordinates. Geometrically, we have a Lagrangian submanifold $\Phi \subset T^*M_1 \times (-T^*M_2)$, which in given coordinates on M_1 and M_2 is described by (4) where $S = S(x,q)$ is a function of the variables x^a and q_i (the coordinates on the base of T^*M_1 and the standard fiber of T^*M_2, respectively). Suppose we change coordinates:

$$x^a = x^a(x')\,, \; p_a = \frac{\partial x^{a'}}{\partial x^a}\, p_{a'}\,, \; y^i = y^i(y')\,, \; q_i = \frac{\partial y^{i'}}{\partial y^i}\, q_{i'}\,. \tag{21}$$

We need to find a new function $S' = S'(x',q')$ of the variables $x^{a'}, q_{i'}$ such that in the new coordinates $x^{a'}, p_{a'}, y^{i'}, q_{i'}$ on $T^*M_1 \times (-T^*M_2)$ our Lagrangian submanifold Φ is specified by the equations of the same form:

$$p_{a'} = \frac{\partial S'}{\partial x^{a'}}(x',q')\,, \; y^{i'} = (-1)^{\tilde{\imath}'}\frac{\partial S'}{\partial q_{i'}}(x',q')\,. \tag{22}$$

Since local coordinates on M_1 and M_2 transform independently, the questions concerning the behavior of S w.r.t. transformations of x^a and y^i are separate. The behavior w.r.t. x^a is not problematic: it is easy to see that w.r.t. these variables S can be viewed as representing a genuine function on M_1, and so one has to simply perform a substitution in the arguments, $x^i = x^i(x')$. The real problem is with transformations of coordinates on M_2. The solution is given by the following statement. (Note that generating functions are generally defined up to constants, but we shall give a transformation law for S without such an ambiguity.)

Theorem 3. *The 'new' generating function* $S'(x',q')$ *is given the formula*

$$S'(x',q') = S(x,q) - y^i q_i + y^{i'} q_{i'}\,, \tag{23}$$

where x^a, q_i, y^i and $y^{i'}$ are determined from the equations

$$y^{i'} = y^{i'}(y), \quad y^i = (-1)^{\tilde{i}}\frac{\partial S}{\partial q_i}(x,q), \quad q_i = \frac{\partial y^{i'}}{\partial y^i}(y)\,q_{i'}, \quad x^a = x^a(x').\qquad(24)$$

Proof. Differentiate both sides of (23):

$$dS' = dS - dy^i q_i - (-1)^{\tilde{i}} y^i dq_i + dy^{i'} q_{i'} + (-1)^{\tilde{i}'} y^{i'} dq_{i'}$$

$$= dS - (-1)^{\tilde{i}} y^i dq_i + (-1)^{\tilde{i}'} y^{i'} dq_{i'} = dx^a \frac{\partial S}{\partial x^a} + dq_i \frac{\partial S}{\partial q_i} - (-1)^{\tilde{i}} dq_i y^i + (-1)^{\tilde{i}'} dq_{i'} y^{i'},$$

and on the submanifold Φ,

$$dS' = dx^a\, p_a + (-1)^{\tilde{i}} dq_i\, y^i - (-1)^{\tilde{i}} dq_i y^i + (-1)^{\tilde{i}'} dq_{i'} y^{i'} = dx^{a'}\, p_{a'} + (-1)^{\tilde{i}'} dq_{i'} y^{i'},$$

which gives (22) as desired. To properly make use of formula (23), one has to express all the variables at the r.h.s. of it, i.e., x^a, q_i, y^i and $y^{i'}$ in terms of the variables at the l.h.s., i.e., $x^{a'}$ and $q_{i'}$. For x^a, we simply substitute $x^a = x^a(x')$. We also substitute $y^{i'} = y^{i'}(y)$ and use the standard transformation law for the momentum variables q_i, expressing them via $q_{i'}$ and y^i. The rest is subtler: for determining y^i we have a system of coupled equations

$$y^i = (-1)^{\tilde{i}}\frac{\partial S}{\partial q_i}(x,q), \quad q_i = \frac{\partial y^{i'}}{\partial y^i}(y)\,q_{i'},$$

which gives

$$y^i = (-1)^{\tilde{i}}\frac{\partial S}{\partial q_i}\Big(x, \frac{\partial y'}{\partial y}(y) q'\Big),$$

from where y is expressed as a function of x and q' by an iterative procedure similar to that defining the map φ_g above. The result is a formal power expansion in q'. $\qquad\square$

Formula (23) can be read as the composition of three transformations: the 'direct' Legendre transform from q_i to y^i, the substitution $y^i = y^i(y')$, and the 'inverse' Legendre transform from $y^{i'}$ to $q_{i'}$.[3] Namely, we pass from $S(q)$ (here the dependence on x is suppressed) to $S^*(y)$,

$$S^*(y) = y^i q_i - S(q),$$

where q_i is expressed from

$$y^i = (-1)^{\tilde{i}}\frac{\partial S}{\partial q_i}(q).$$

Then we substitute to obtain $S^{*\prime}(y') := S^*\big(y(y')\big)$. Finally, we pass from $S^{*\prime}$ to $S' = (S^{*\prime})^*$,

$$S'(q') = y^{i'} q_{i'} - S^*\big(y(y')\big),$$

where $y^{i'}$ is expressed from

$$q_{i'} = \frac{\partial}{\partial y^{i'}} S^*\big(y(y')\big).$$

---

[3]Note an analogy with pseudodifferential operators: the direct Fourier transform, then a multiplication operator, and then the inverse Fourier transform. It is not a random analogy because the Legendre transform can be seen as the 'classical limit' of the Fourier transform. We can treat formula (7) in a similar way. Compare with [25, 26].

Assembled together, these steps give equation (23). The possibility to make the Legendre transform from q to y puts a restriction on the generating function S (non-degeneracy in q). Such a restriction is not satisfied, for example, by S corresponding to a smooth map $M_1 \to M_2$. However, the restriction disappears for the composite transformation $S(q) \mapsto S'(q')$, because the two Legendre transforms compensate each other in a way. (It is illuminating to see how the inverse matrix S_{ij} eventually disappears from the final answer after one initially assumes the non-degeneracy of the quadratic form $S^{ij}q_jq_i$ in the expansion of S so to be able to apply the Legendre transform.) Theorem 3 does not require any non-degeneracy from S.

Remark 4. The transformation law for S given by (23) and (24) satisfies the *cocycle condition*, as one can immediately see: if $S'(x', q')$ is expressed from $S(x, q)$ by (23),(24), and $S''(x'', q'')$ is expressed from $S'(x', q')$ by the same formulas (with the necessary replacements), then the composite expression of $S''(x'', q'')$ via $S(x, q)$ coincides with the direct expression given by these formulas. This makes it possible to consider generating functions $S(x, q)$ (defined as power series in q) as geometric objects on $M_1 \times M_2$.

Example 6. Suppose the 'old' generating function $S(x, q)$ is given by the expansion (11). Under a change of coordinates (21), the 'new' generating function $S'(x', q')$ has the expansion

$$S'(x', q') = S_0\big(x(x')\big) + \varphi^{i'}(x')\, q_{i'} + \frac{1}{2}\, S^{i'j'}(x')\, q_{j'}q_{i'} + O(|q'|^3), \qquad (25)$$

where

$$\varphi^{i'}(x') = y^{i'}\big(\varphi(x(x'))\big), \qquad (26)$$

and

$$S^{i'j'}(x') = (-1)^{\tilde{i}(\tilde{i}'+1)}\frac{\partial y^{i'}}{\partial y^i}\big(\varphi'(x')\big)\, S^{ij}\big(x(x')\big)\, \frac{\partial y^{j'}}{\partial y^j}\big(\varphi'(x')\big). \qquad (27)$$

This can be obtained by a patient calculation along the lines above, which we leave to the pleasure of the reader. Note that (26) is just the expression of the map $\varphi\colon M_1 \to M_2$ in new coordinates on M_1 and M_2, and in equation (27) one recognizes the tensor law on M_2 at a point $\varphi(x)$. Non-tensor transformations depending, in particular, on higher derivatives of a coordinate transformation on M_2 appear in the higher order terms of S.

The following statement is a direct consequence of the definition of Φ^* and the transformation law given by Theorem 3. It deserves the name of a theorem because of its importance.

Theorem 4. *Suppose $g' = g'(y')$ is the expression of an even function $g = g(y)$ in new coordinates on M_2, i.e., $g'(y') = g(y(y'))$, and $S'(x', q')$ is the expression of a generating function $S(x, q)$ in new coordinates on M_1 and M_2 according to the transformation law (23),(24). Let the function $f' = f'(x')$ be obtained from g', S' and the function $f = f(x)$ be obtained from g, S, as the generalized pullbacks (in coordinates x', y' and x, y, respectively). Then the function $f' = f'(x')$ is the expression of the function $f = f(x)$ in the new coordinates on M_1, i.e., $f'(x') = f(x(x'))$.*

Proof. We are given that

$$f'(x') = g'(y') + S'(x', q') - y'q',$$

where
$$y^{i'} = (-1)^{i'} \frac{\partial S'}{\partial q_{i'}}(x', q'), \quad q_{i'} = \frac{\partial g'}{\partial y'}(y').$$

Also $g'(y') = g(y(y'))$, for an invertible change of variables $y' = y'(y)$, and
$$S'(x', q') = S(x, q) - yq + y'q',$$

where
$$y^{i'} = y^{i'}(y), \quad y^i = (-1)^{\tilde{i}} \frac{\partial S}{\partial q_i}(x, q), \quad q_i = \frac{\partial y^{i'}}{\partial y^i}(y) \, q_{i'}, \quad x^a = x^a(x').$$

Note that this transformation law for S implies $y^{i'} = (-1)^{i'} \partial S'/\partial q_{i'}(x', q')$ if $y^i = (-1)^i \partial S/\partial q_i(x, q)$. Hence we can 'compose' the formulas for f' and S' to obtain
$$f'(x') = g(y(y')) + S(x, q) - yq + y'q' - y'q' = g(y) + S(x, q) - yq,$$

where at the r.h.s.
$$y^i = (-1)^i \frac{\partial S}{\partial q_i}(x, q), \quad q_i = \frac{\partial g}{\partial y}(y),$$

and $x = x(x')$. This is exactly the equality $f'(x') = f(x(x'))$, as claimed. $\qquad\square$

To summarize, we may say that a generalized pullback Φ^*, a formal mapping of function spaces defined initially in local coordinates, *is independent of a choice of coordinates*. This finishes with all questions of substantiation.

We leave out discussion of compositions of formal canonical relations Φ given by formal generating functions $S(x, q)$. One should expect, in view of the analysis performed above, that they form what can be regarded as a *formal category*[4] and the usual formula
$$(\Phi_1 \circ \Phi_2)^* = \Phi_2^* \circ \Phi_1^* \tag{28}$$

holds, so our analog of pullbacks gives a nonlinear representation of (the dual of) this formal category. (Formula (28) should basically follow from the associativity of composition of relations.) These questions are considered fully in our forthcoming work [24].

Remark 5. We have worked so far with formal objects (power series). To extend consideration to non-formal objects may be possible but may require more work. Our central equation (9) defines a map $\varphi_g \colon M_1 \to M_2$ associated with a canonical relation Φ and a function g. We showed above how to solve it by iterations so to obtain a power series solution. At the same time, one can imagine that a Banach contraction mapping argument can be used for obtaining a non-formal solution of (9) in a neighborhood of the zero section. Since such a neighborhood is unspecified, a neat formulation would be to replace it by a *germ*. So the options are to work on a formal level with power series (infinite jets) or with germs. Considering germs of symplectic manifolds at Lagrangian submanifolds is Weinstein's idea dating back to [27]. If we follow this direction, our work will immediately meet the recent body of works on "symplectic microgeometry" such as [30], [3, 4, 5].

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[4]The underlying category, for which this formal category is a formal neighborhood, being the semi-direct product of the usual category of smooth supermanifolds and their smooth maps with algebras of smooth functions.

2. Hamilton–Jacobi vector fields

Consider a Hamiltonian function $H \in C^\infty(T^*M)$, which can be even or odd. We write $H = H(x,p)$, as usual. To such a function we assign a vector field $\boldsymbol{X}_H$ on the infinite-dimensional manifold $\mathbf{C}^\infty(M)$, as follows: for each $f \in \mathbf{C}^\infty(M)$, the variation of f is given by

$$f \mapsto f_\varepsilon = f + \varepsilon \boldsymbol{X}_H[f], \text{ where } f_\varepsilon(x) = f(x) + \varepsilon H\left(x, \frac{\partial f}{\partial x}(x)\right). \tag{29}$$

Here $\varepsilon^2 = 0$ and $\tilde{\varepsilon} = \tilde{H}$. The parity of the vector field $\boldsymbol{X}_H$ is the same as the parity of H. In standard terminology used in field theory or integrable systems, the vector field $\boldsymbol{X}_H$ is a 'first-order local vector field' on the space of functions. It can be written in terms of variational derivatives as

$$\boldsymbol{X}_H = (-1)^{\tilde{H}m} \int\limits_{M^{n|m}} Dx \, H\left(x, \frac{\partial f}{\partial x}(x)\right) \frac{\delta}{\delta f(x)}. \tag{30}$$

(The sign is required for linearity.)

The differential equation defining the flow of the vector field $\boldsymbol{X}_H$ on the manifold $\mathbf{C}^\infty(M)$ is a Hamilton–Jacobi equation.[5] It takes the familiar form

$$\frac{\partial f}{\partial t} = H\left(x, \frac{\partial f}{\partial x}\right) \tag{31}$$

when H is even. Here the time variable t in (31) is also even. (In (31), a function f depends on t in addition to x, so to give a curve in $\mathbf{C}^\infty(M)$.) For an odd H, the corresponding Hamilton–Jacobi equation takes the form

$$Df \equiv \left(\frac{\partial}{\partial \tau} + \tau \frac{\partial}{\partial t}\right) f = H\left(x, \frac{\partial f}{\partial x}\right), \tag{32}$$

with two time variables, even t and odd τ. (The operator D at the l.h.s. of (32) squares to $\partial/\partial t$.)

Theorem 5. *For arbitrary Hamiltonians H and F,*

$$[\boldsymbol{X}_H, \boldsymbol{X}_F] = -\boldsymbol{X}_{(H,F)}, \tag{33}$$

where the bracket at the l.h.s. is the commutator of vector fields on the infinite-dimensional manifold $\mathbf{C}^\infty(M)$ and the bracket at the r.h.s. is the canonical Poisson bracket on T^*M.

(The minus sign in (33) is of course completely inessential and depends on conventions.)

Proof. Direct calculation, but still worth giving here. For calculating the commutator, we start from a point $f_0 \in \mathbf{C}^\infty(M)$ and apply to it successively infinitesimal shifts along the vector fields $\boldsymbol{X}_H$ and $\boldsymbol{X}_F$. First we arrive at f_1, where

$$f_1(x) = f_0(x) + \varepsilon H\left(x, \frac{\partial f_0}{\partial x}\right)$$

---

[5]From arbitrary first order local vector fields, the vector fields $\boldsymbol{X}_H$ are distinguished by the dependence only on the values of the derivative but not the function itself. This is precisely what distinguishes the Hamilton–Jacobi equations among arbitrary first order partial differential equations.

and $\varepsilon^2 = 0$. Then we arrive at f_2, where

$$f_2(x) = f_1(x) + \eta F\left(x, \frac{\partial f_1}{\partial x}\right) =$$
$$f_0(x) + \varepsilon H\left(x, \frac{\partial f_0}{\partial x}\right) + \eta F\left(x, \frac{\partial f_0}{\partial x} + \frac{\partial}{\partial x}\varepsilon H\left(x, \frac{\partial f_0}{\partial x}\right)\right) =$$
$$f_0(x) + \varepsilon H\left(x, \frac{\partial f_0}{\partial x}\right) + \eta F\left(x, \frac{\partial f_0}{\partial x}\right) + \eta\frac{\partial}{\partial x^a}\varepsilon H\left(x, \frac{\partial f_0}{\partial x}\right) \cdot \frac{\partial F}{\partial p_a}\left(x, \frac{\partial f_0}{\partial x}\right)$$

and $\eta^2 = 0$. Next we arrive at f_3, where

$$f_3(x) = f_2(x) - \varepsilon H\left(x, \frac{\partial f_2}{\partial x}\right) =$$
$$f_0(x) + \varepsilon H\left(x, \frac{\partial f_0}{\partial x}\right) + \eta F\left(x, \frac{\partial f_0}{\partial x}\right) + \eta\frac{\partial}{\partial x^a}\varepsilon H\left(x, \frac{\partial f_0}{\partial x}\right) \cdot \frac{\partial F}{\partial p_a}\left(x, \frac{\partial f_0}{\partial x}\right) -$$
$$\varepsilon H\left(x, \frac{\partial f_0}{\partial x} + \frac{\partial}{\partial x}\eta F\left(x, \frac{\partial f_0}{\partial x}\right)\right) =$$
$$f_0(x) + \varepsilon H\left(x, \frac{\partial f_0}{\partial x}\right) + \eta F\left(x, \frac{\partial f_0}{\partial x}\right) + \eta\frac{\partial}{\partial x^a}\varepsilon H\left(x, \frac{\partial f_0}{\partial x}\right) \cdot \frac{\partial F}{\partial p_a}\left(x, \frac{\partial f_0}{\partial x}\right) -$$
$$\varepsilon H\left(x, \frac{\partial f_0}{\partial x}\right) - \varepsilon\frac{\partial}{\partial x^a}\eta F\left(x, \frac{\partial f_0}{\partial x}\right) \cdot \frac{\partial H}{\partial p_a}\left(x, \frac{\partial f_0}{\partial x}\right) =$$
$$f_0(x) + \eta F\left(x, \frac{\partial f_0}{\partial x}\right) + \eta\frac{\partial}{\partial x^a}\varepsilon H\left(x, \frac{\partial f_0}{\partial x}\right) \cdot \frac{\partial F}{\partial p_a}\left(x, \frac{\partial f_0}{\partial x}\right) -$$
$$\varepsilon\frac{\partial}{\partial x^a}\eta F\left(x, \frac{\partial f_0}{\partial x}\right) \cdot \frac{\partial H}{\partial p_a}\left(x, \frac{\partial f_0}{\partial x}\right).$$

Finally we arrive at f_4, where

$$f_4(x) = f_3(x) - \eta F\left(x, \frac{\partial f_3}{\partial x}\right) = f_0(x) + \eta\frac{\partial}{\partial x^a}\varepsilon H\left(x, \frac{\partial f_0}{\partial x}\right) \cdot \frac{\partial F}{\partial p_a}\left(x, \frac{\partial f_0}{\partial x}\right) -$$
$$\varepsilon\frac{\partial}{\partial x^a}\eta F\left(x, \frac{\partial f_0}{\partial x}\right) \cdot \frac{\partial H}{\partial p_a}\left(x, \frac{\partial f_0}{\partial x}\right) = f_0(x) + \eta\varepsilon\left((-1)^{\tilde{a}\tilde{H}}\frac{\partial}{\partial x^a}H\left(x, \frac{\partial f_0}{\partial x}\right)\cdot\right.$$
$$\left.\frac{\partial F}{\partial p_a}\left(x, \frac{\partial f_0}{\partial x}\right) - (-1)^{\tilde{F}(\tilde{a}+\tilde{H})}\frac{\partial}{\partial x^a}F\left(x, \frac{\partial f_0}{\partial x}\right) \cdot \frac{\partial H}{\partial p_a}\left(x, \frac{\partial f_0}{\partial x}\right)\right).$$

By examining the differential expression in the big bracket, we observe that the terms of the first order in f_0 assemble to

$$(-1)^{\tilde{a}\tilde{H}}\frac{\partial H}{\partial x^a}\frac{\partial F}{\partial p_a} - (-1)^{\tilde{F}(\tilde{a}+\tilde{H})}\frac{\partial F}{\partial x^a}\frac{\partial H}{\partial p_a} =$$
$$- (-1)^{\tilde{a}\tilde{H}}\left((-1)^{\tilde{a}}\frac{\partial H}{\partial p_a}\frac{\partial F}{\partial x^a} - \frac{\partial H}{\partial x^a}\frac{\partial F}{\partial p_a}\right) = -(H, F)\,,$$

the Poisson bracket of H and F, evaluated at $\left(x, \frac{\partial f_0}{\partial x}\right)$. At the same time, the terms of the second order in f_0 are

$$(-1)^{\tilde{a}\tilde{H}} \frac{\partial^2 f_0}{\partial x^a \partial x^b} \frac{\partial H}{\partial p_b} \frac{\partial F}{\partial p_a} - (-1)^{\tilde{F}(\tilde{a}+\tilde{H})} \frac{\partial^2 f_0}{\partial x^a \partial x^b} \frac{\partial F}{\partial p_b} \frac{\partial H}{\partial p_a}$$

and we can observe that they cancel by the symmetry of second partial derivatives. Hence

$$f_4(x) = f_0 - \eta\varepsilon\,(H, F)\left(x, \frac{\partial f_0}{\partial x}\right),$$

as claimed. $\square$

Corollary. *Let Q be an odd Hamiltonian and $(Q, Q) = -2H$, so H is even. Then the solution of the Hamilton–Jacobi equation for Q,*

$$Df = Q\left(x, \frac{\partial f}{\partial x}\right),$$

is given by $f(t, \tau) = f_0(t) + \tau f_1(t)$, where f_0 is the solution of the usual Hamilton–Jacobi equation for H,

$$\frac{\partial f_0}{\partial t} = H\left(x, \frac{\partial f_0}{\partial x}\right),$$

and $f_1 = Q\left(x, \frac{\partial f_0}{\partial x}\right)$. In particular, if $(Q, Q) = 0$, then the Hamilton–Jacobi equation for Q reduces to

$$\frac{\partial f}{\partial \tau} = Q\left(x, \frac{\partial f}{\partial x}\right),$$

and its solution is just an 'odd shift': $f = f_0 + \tau Q\left(x, \frac{\partial f_0}{\partial x}\right)$.

We shall refer to the vector fields on the infinite-dimensional manifold $\mathbf{C}^\infty(M)$ of the form $\boldsymbol{X}_H$ as to the *Hamilton–Jacobi vector fields*.

Consider an arbitrary relation $R \subset T^*M_1 \times T^*M_2$. We say that Hamiltonians $H_1 \in C^\infty(T^*M_1)$ and $H_2 \in C^\infty(T^*M_2)$ are *R-related* if $p_1^*H_1 = p_2^*H_2$, where p_i, $i = 1, 2$, are the restrictions of the canonical projections on T^*M_i. This terminology extends the classical notion of φ-related vector fields as shown by the following example.

Example 7. Suppose $R = R_\varphi$ corresponds to a smooth map $\varphi\colon M_1 \to M_2$ as in Example 1. Then the condition that $H_1 = H_1(x, p)$ and $H_2 = H_2(y, q)$ are R-related amounts to

$$H_1\left(x, \frac{\partial\varphi}{\partial x}(x)\,q\right) = H_2\big(\varphi(x), q\big).$$

In particular, if $H_1(x, p) = X^a(x)p_a$ and $H_2(y, q) = Y^i(y)q_i$ correspond to vector fields $X \in \operatorname{Vect}(M_1)$ and $Y \in \operatorname{Vect}(M_2)$, we recognize the familiar condition

$$X^a(x) \frac{\partial\varphi^i}{\partial x^a} = Y^i\big(\varphi(x)\big),$$

i.e., that the vector fields X and Y are φ-related.

Suppose there is a canonical relation $\Phi \subset T^*M_1 \times (-T^*M_2)$ of the form (4). Consider the pullback $\Phi^*\colon \mathbf{C}^\infty(M_2) \to \mathbf{C}^\infty(M_1)$.

Theorem 6. *If Hamiltonians $H_1 \in C^\infty(T^*M_1)$ and $H_2 \in C^\infty(T^*M_2)$ are Φ-related, then the Hamilton–Jacobi vector fields $\boldsymbol{X}_{H_2} \in \mathrm{Vect}(\mathbf{C}^\infty(M_2))$ and $\boldsymbol{X}_{H_1} \in \mathrm{Vect}(\mathbf{C}^\infty(M_1))$ are Φ^*-related.*

Proof. The condition that two vector fields are related by a smooth map means that the map intertwines the corresponding infinitesimal shifts. We shall check that for the vector fields $\boldsymbol{X}_{H_2}$ and $\boldsymbol{X}_{H_1}$. Note that the condition that H_1 and H_2 are Φ-related reads:

$$H_1(x,p) = H_2(y,q) \quad \text{for} \quad p_a = \frac{\partial S}{\partial x^a}(x,q) \text{ and } y^i = (-1)^{\tilde{\imath}} \frac{\partial S}{\partial q_i}(x,q). \tag{34}$$

Take an arbitrary $g \in \mathbf{C}^\infty(M_2)$ and apply to it the infinitesimal shift along $\boldsymbol{X}_{H_2}$. We obtain

$$g_\varepsilon(y) = g(y) + \varepsilon\, H_2\Big(y, \frac{\partial g}{\partial y}\Big).$$

Apply to the result the map Φ^*. By Theorem 2,

$$\Phi^*[g_\varepsilon] = \Phi^*[g] + \varepsilon\, \varphi_g^*\left(H_2\Big(y, \frac{\partial g}{\partial y}\Big)\right).$$

Recall that φ_g^* simply means that y^i should be found from the equation

$$y^i = (-1)^{\tilde{\imath}} \frac{\partial S}{\partial q_i}\Big(x, \frac{\partial g}{\partial y}\Big). \tag{35}$$

In the opposite direction, apply first Φ^* to g to obtain $\Phi^*[g]$ and then apply to it the infinitesimal shift along $\boldsymbol{X}_{H_1}$. We arrive at

$$\Phi^*[g] + \varepsilon H_1\Big(x, \frac{\partial \Phi^*[g]}{\partial x}\Big).$$

Denote $\Phi^*[g] =: f$. To calculate the derivative in the argument, write

$$f(x) = S(x,q) - y^i q_i + g(y),$$

where

$$q_i = \frac{\partial g}{\partial y} \quad \text{and} \quad y^i = (-1)^{\tilde{\imath}} \frac{\partial S}{\partial q_i}\Big(x, \frac{\partial g}{\partial y}\Big),$$

so

$$\frac{\partial f}{\partial x^a} = \frac{\partial S}{\partial x^a}\Big(x, \frac{\partial g}{\partial y}\Big) + \frac{\partial q_i}{\partial x^a}\frac{\partial S}{\partial q_i}\Big(x, \frac{\partial g}{\partial y}\Big) - \frac{\partial y^i}{\partial x^a} q_i - (-1)^{\tilde{\imath}\tilde{a}} y^i \frac{\partial q_i}{\partial x^a} + \frac{\partial y^i}{\partial x^a}\frac{\partial g}{\partial y^i} =$$

$$\frac{\partial S}{\partial x^a}\Big(x, \frac{\partial g}{\partial y}\Big) + (-1)^{\tilde{\imath}}\frac{\partial q_i}{\partial x^a} y^i - \frac{\partial y^i}{\partial x^a} q_i - (-1)^{\tilde{\imath}\tilde{a}+\tilde{\imath}(\tilde{a}+\tilde{\imath})}\frac{\partial q_i}{\partial x^a} y^i + \frac{\partial y^i}{\partial x^a} q_i = \frac{\partial S}{\partial x^a}\Big(x, \frac{\partial g}{\partial y}\Big).$$

Therefore, to prove our statement, we need to compare the infinitesimal increments of $f(x)$, $f = \Phi^*[g]$, given in one case by

$$H_2\Big(y, \frac{\partial g}{\partial y}\Big)$$

(after dropping ε) and in the other case by

$$H_1\Big(x, \frac{\partial S}{\partial x^a}\Big(x, \frac{\partial g}{\partial y}\Big)\Big),$$

where in both cases y^i is obtained from (35). We see that the equality in question

$$H_1\Big(x, \frac{\partial S}{\partial x^a}\Big(x, \frac{\partial g}{\partial y}\Big)\Big) = H_2\Big(y, \frac{\partial g}{\partial y}\Big)$$

follows from (34), which is valid for all q, in particular $q = \partial g/\partial y$. $\square$

This theorem may be seen as the main statement of our paper.

3. APPLICATION TO HOMOTOPY ALGEBRAS AND ALGEBROIDS

Let us recall some information concerning L_∞-algebras. We shall use the higher derived bracket construction [22]. A vector space L together with an infinite sequence of odd symmetric multilinear operations ('brackets')

$$\underbrace{L \times \ldots \times L}_{r \text{ times}} \to L\,,$$

where $r = 0, 1, 2, 3, \ldots$, is called an L_∞-*algebra* if the brackets satisfy the sequence of 'higher Jacobi identities'

$$J_n(v_1, \ldots, v_n) = 0\,, \tag{36}$$

for all $n = 0, 1, 2, 3, \ldots$,[6] where $J_n(v_1, \ldots, v_n)$ denotes the nth *Jacobiator* of the brackets defined by

$$J_n(v_1, \ldots, v_n) := \sum_{\substack{k, \ell \geqslant 0 \\ k+\ell = n}} \sum_{(k, \ell)\text{-shuffles}} (-1)^\alpha \big\{\{v_{\sigma(1)}, \ldots, v_{\sigma(k)}\}, v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\big\}\,. \tag{37}$$

(Here the sign $(-1)^\alpha$ is the usual Koszul sign depending on the parities of permuted arguments, e.g., $(-1)^0 = +1$, if all v_i are even.)

A sequence of symmetric multilinear operations of a given parity on a vector space L can be assembled into a formal vector field on the corresponding 'vector supermanifold' $\mathbf{L}$, where we use boldface for distinction. Conversely, given a vector field $X \in \mathrm{Vect}(\mathbf{L})$, a sequence of brackets on L is obtained as follows [22]:

$$i_{\{v_1, \ldots, v_r\}} = [\ldots [[X, i_{v_1}], i_{v_2}], \ldots, i_{v_2}](0)\,, \tag{38}$$

(evaluation at the origin), where i_v is the constant vector field corresponding to a vector $v \in L$. Suppose a sequence of odd brackets on L corresponds to an odd vector field $Q \in \mathrm{Vect}(\mathbf{L})$. Then the sequence of their Jacobiators J_n corresponds to the even vector field $Q^2 = \frac{1}{2}[Q, Q]$. (See [22] for a more general statement.) Therefore there is a one-to-one correspondence between L_∞-algebra structures on a vector space L and formal homological vector fields on $\mathbf{L}$. It is known that the language of homological vector fields is the most efficient way of working with L_∞-algebras (see, e.g. [11, 12]). In particular, an L_∞-*morphism* from an L_∞-algebra L_1 to an L_∞-algebra L_2 (in the above description) can be defined as a formal supermanifold map $\varphi \colon \mathbf{L}_1 \to \mathbf{L}_2$ (in general, nonlinear) such that the corresponding homological vector fields $Q_i \in \mathrm{Vect}(\mathbf{L}_i)$ are φ-related.

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[6]More precisely, this is an L_∞-algebra "in the symmetric version". In the original terminology [13], an L_∞-algebra or 'strongly homotopy Lie' algebra has antisymmetric brackets of alternating parities, namely, brackets with an even number of arguments being even and with an odd number of arguments, odd. These two notions transform to each other by the parity reversal of the underlying space, see [22].

We shall apply these general notions to the setup where brackets are introduced on the space of smooth functions on some (super)manifold.

A Hamiltonian $H \in C^\infty(T^*M)$ defines a sequence of symmetric brackets on the vector space $C^\infty(M)$ by the higher derived bracket construction [22]:

$$\{f_1, \ldots, f_r\}_H := (\ldots((H, f_1), f_2), \ldots, f_r)_{|M} . \tag{39}$$

The parity of these brackets is the same as the parity of H. All brackets (39) are multi-derivations w.r.t. the associative multiplications of functions. If we expand H as

$$H(x, p) = H_0(x) + H^a(x)p_a + \frac{1}{2} H^{ab}(x)p_b p_a + \frac{1}{3!} H^{abc}(x)p_c p_b p_a + \ldots , \tag{40}$$

with symmetric coefficients $H^{a_1 \ldots a_r}$, then

$$\{f_1, \ldots, f_r\}_H = \pm H^{a_1 \ldots a_r}(x) \partial_{a_r} f \ldots \partial_{a_1} f . \tag{41}$$

We shall refer to the Hamiltonian generating a given sequence of brackets as to the *master Hamiltonian*. It is natural to ask what is the corresponding vector field on the infinite-dimensional supermanifold $\mathbf{C}^\infty(M)$. The answer is given by the following statement.

Theorem 7. *The higher derived brackets* (39) *generated by* $H \in C^\infty(T^*M)$ *assemble to the Hamilton–Jacobi vector field* $\boldsymbol{X}_H \in \text{Vect}(\mathbf{C}^\infty(M))$,

$$\boldsymbol{X}_H = (-1)^{\tilde{H}m} \int\limits_{M^{n|m}} Dx\, H\Big(x, \frac{\partial f}{\partial x}(x)\Big) \frac{\delta}{\delta f(x)} . \tag{42}$$

Proof. Directly. One needs to apply (39) to $f_1 = \ldots = f_r = f$, for some even function $f \in \mathbf{C}^\infty(M)$. $\qquad\square$

When the master Hamiltonian H is odd, the derived brackets (39) are also odd and it is legitimate to ask whether the Jacobi identities (36) hold for them. As follows from a general theorem [22], if an odd Hamiltonian H obeys the classical *master equation*

$$(H, H) = 0 , \tag{43}$$

then all the Jacobi identities are satisfied for its derived brackets, so the space $C^\infty(M)$ with these brackets is an L_∞-algebra. Considered also with the ordinary multiplication of functions, it is a homotopy Schouten algebra. (By definition, a *homotopy Schouten algebra* or S_∞-*algebra* is a commutative associative algebra endowed with an infinite sequence of odd symmetric brackets that satisfy the higher Jacobi identities and also the Leibniz identity in each argument [22].) A supermanifold M whose algebra of functions is endowed with odd brackets making it a homotopy Schouten algebra will be called a *homotopy Schouten manifold* or an S_∞-*manifold*.

Remark 6. By Theorem 5, for an odd Hamiltonian H we have

$$[\boldsymbol{X}_H, \boldsymbol{X}_H] = -\boldsymbol{X}_{(H,H)} .$$

So if H satisfies $(H, H) = 0$, then $\boldsymbol{X}_H^2 = 0$. This gives a direct proof that such an H generates an L_∞-algebra.

Consider homotopy Schouten manifolds M_1 and M_2. Let H_1 and H_2 be the respective master Hamiltonians. Let $\Phi \subset T^*M_1 \times (-T^*M_2)$ be a canonical relation of the form (4).

Corollary (From Theorems 6 and 7). *If master Hamiltonians H_1 and H_2 are Φ-related, then the formal mapping of function supermanifolds*

$$\Phi^* \colon \ \mathbf{C}^\infty(M_2) \to \mathbf{C}^\infty(M_1)$$

(in general, nonlinear) is an L_∞-morphism of the corresponding L_∞-algebras.

Example 8. A very special case is that of Φ corresponding to an ordinary map $\varphi \colon M_1 \to M_2$, so that the master Hamiltonians are φ-related. Then $\varphi^* \colon C^\infty(M_2) \to C^\infty(M_1)$ is the usual pullback, hence linear. It gives a strict morphism of L_∞-algebras, for which all brackets are preserved separately.

This can also be applied to Lie algebroid theory, as follows.

For Lie bialgebroids, a Lie bialgebroid morphism $E_1 \to E_2$ is defined as a morphism of Lie algebroids $\varphi \colon E_1 \to E_2$ such that it is also a Poisson map for the Lie–Poisson brackets induced by the Lie algebroid structures on the dual bundles [15]. The latter condition is also equivalent to φ being a Poisson map for the Lie–Schouten brackets induced on ΠE_1 and ΠE_2. The conditions of a Lie algebroid morphism and a Poisson map naturally combine together into one condition that the odd Hamiltonians defining the QS-structures (see [21]) on ΠE_1 and ΠE_2 are φ-related. This is equivalent to $\varphi^* \colon C^\infty(\Pi E_2) \to C^\infty(\Pi E_1)$ being a morphism of differential Schouten algebras. The question arises, what should stand for all that in the homotopy case.

A structure of an L_∞-bialgebroid is defined on a vector bundle E by an odd master Hamiltonian H satisfying the master equation $(H, H) = 0$. It particular it makes the algebra of functions $C^\infty(\Pi E)$ is a homotopy Schouten algebra. How one should define morphisms of L_∞-bialgebroid? We should be looking for constructions leading to L_∞-morphisms of the algebras of functions. Ordinary morphisms of vector bundles can only lead to strict morphisms. This is clearly not sufficient. The correct notion should use nonlinear pullbacks (as can be showed).

Example 9. Let a supermanifold M have a homotopy Poisson structure (see, e.g., [10] and in the Appendix). (The difference with a homotopy Schouten structure is that the brackets are antisymmetric and have alternating parities, so that the binary bracket is even.) In [10] we showed that it induces the structure of an L_∞-algebroid on the cotangent bundle T^*M. (This is the analog of the Lie algebroid structure on T^*M for an ordinary Poisson manifold.) The corresponding sequence of odd brackets on functions on ΠTM are called the *higher Koszul brackets*. Recall that functions on ΠTM are (pseudo)differential forms on M. In the classical situation, there is only the binary Koszul bracket on forms induced by an ordinary Poisson structure and the pullback w.r.t. the Poisson anchor maps it to the canonical Schouten bracket of multivector fields. In [10], we posed the problem of extending this picture to the homotopy Poisson case, i.e., to find an L_∞-morphism between the higher Koszul brackets and the canonical Schouten bracket. The solution is given by a certain nonlinear pullback $\Phi^* \colon \mathbf{C}^\infty(\Pi TM) \to \mathbf{C}^\infty(\Pi T^*M)$ (see [24]). This question was the departure point of the present work.

Example 9 has an abstract form, which is an L_∞ version of 'triangular Lie bialgebroids' of Mackenzie–Xu [16] and in particular of the canonical Lie bialgebroid morphism $E^* \to E$ defined for them (which is an abstract analog of the Poisson anchor, see [15]). We elaborate these questions in [24] and a forthcoming paper with H. M. Khudaverdian.

APPENDIX: "NONLINEAR PULLBACKS" FOR ODD FUNCTIONS

In the main text we construct and study the mapping of *even* functions on supermanifolds

$$\Phi^*\colon \ \mathbf{C}^\infty(M_2) \to \mathbf{C}^\infty(M_1)$$

associated with a canonical relation $\Phi \subset T^*M_1 \times (-T^*M_2)$. There is a parallel construction of a similar mapping of *odd* functions

$$\Psi^*\colon \ \mathbf{\Pi C}^\infty(M_2) \to \mathbf{\Pi C}^\infty(M_1)\,.$$

Below we give a brief outline of the corresponding statements without repeating the proofs that generally go along the same lines. In the same way as the constructions in the main text are based on the symplectic geometry of the cotangent bundles of the (super)manifolds involved, the parallel constructions here make use of odd symplectic geometry. It is well known that there are fundamental differences between even and odd symplectic geometry (see, e.g., [7, 8], [9]), but up to a certain point everything remains similar and it suffices for our purpose.

For a supermanifold M consider the anticotangent bundle ΠT^*M. If x^a are local coordinates on M, then on ΠT^*M we obtain local coordinates x^a, x_a^*, where the variables x_a^* have the parities opposite to the parities of the corresponding x^a and they transform as

$$x_a^* = \frac{\partial x^{a'}}{\partial x^a}\, x_a^*\,.$$

The variables x^a, x_a^* form canonically conjugate pairs w.r.t. the odd bracket (the canonical Schouten bracket), where

$$[\![x_a^*, x^b]\!] = \delta_a^b\,,$$

and $[\![F, G]\!] = -(-1)^{(\tilde F + 1)(\tilde G + 1)}[\![G, F]\!]$ (see, e.g., [21]). It corresponds to the canonical odd symplectic form $\omega = d(dx^a\, x_a^*)$.

Let $\Psi \subset \Pi T^*M_1 \times (-\Pi T^*M_2)$ be a canonical relation such that it can be specified by an odd generating function $\Theta = \Theta(x, y^*)$,[7]

$$\Psi = \left\{ (x^a, x_a^*, y^i, y_i^*) \ \Big| \ x_a^* = \frac{\partial \Theta}{\partial x^a}\,(x, y^*)\,, \ \ y^i = \frac{\partial \Theta}{\partial y_i^*}\,(x, y^*) \right\}\,.$$

Here x^a, x_a^* are coordinates on ΠT^*M_1 and y^i, y_i^* are coordinates on ΠT^*M_2. Then a given odd function $g \in \mathbf{\Pi C}^\infty(M_2)$ is mapped to the odd function $f =: \Psi^*[g] \in \mathbf{\Pi C}^\infty(M_1)$ defined by the formula

$$f(x) = g(y) + \Theta(x, y^*) - y^i y_i^*\,,$$

where

$$y_i^* = \frac{\partial g}{\partial y^i}\,(y)\,,$$

---

[7]Unlike the even case, a Lagrangian submanifold Λ of an odd symplectic manifold N has a discrete invariant. Namely, if $\dim N = n|n$, then $\dim \Lambda$ can take any of the values $n - k|k$, where $k = 0, 1, \ldots, n$. The relations $\Psi \subset \Pi T^*M_1 \times (-\Pi T^*M_2)$ that we consider have this invariant equal to $m_1 + n_2$, where $\dim M_1 = n_1|m_1$ and $\dim M_2 = n_2|m_2$.

and y^i is determined from the equation

$$y^i = \frac{\partial \Theta}{\partial y_i^*}\left(x, \frac{\partial g}{\partial y}(y)\right)$$

(similarly to (7), (8), (9) above). This equation can be solved by iterations. If we expand

$$\Theta(x, y^*) = \Theta_0(x) + \varphi^i(x)y_i^* + \frac{1}{2}\Theta^{ij}(x)y_j^* y_i^* + \dots ,$$

then the zeroth order term Θ_0 is just a fixed odd function on M_1, the first order term corresponds to an ordinary smooth map $\varphi\colon M_1 \to M_2$, and the higher order terms give a 'perturbation'. As in the main text, we obtain $\varphi_g\colon M_1 \to M_2$ as a perturbative series

$$\varphi_g = \varphi + \varphi_1[g] + \varphi_2[g] + \dots$$

with terms of orders $1, 2, \dots$ in g.

Example 10. As an exercise, one can calculate the linear and quadratic terms in g to obtain

$$\varphi_g^i(x) = \varphi^i(x) \;+\; \underbrace{\Theta^{ij}(x)\frac{\partial g}{\partial y^j}(\varphi(x))}_{\varphi_1[g]} \;+$$

$$\underbrace{\Theta^{i(j}(x)\Theta^{k)l}(x)\frac{\partial g}{\partial y^l}(\varphi(x))\frac{\partial^2 g}{\partial y^k \partial y^j}(\varphi(x)) + \frac{1}{2}\Theta^{ijk}(x)\frac{\partial g}{\partial y^k}(\varphi(x))\frac{\partial g}{\partial y^j}(\varphi(x))}_{\varphi_2[g]} \;+\; \dots$$

(the round brackets in the indices denote symmetrization). The particular expression is not very important, but it gives a feeling of the general appearance of the terms in the expansion.

Then the image of Ψ^* in a greater detail is

$$\Psi^*[g](x) = g(\varphi_g(x)) + \Theta\left(x, \frac{\partial g}{\partial y}(\varphi_g(x))\right) - \varphi_g^i(x)\frac{\partial g}{\partial y^i}(\varphi_g(x)),$$

and, as in the main text, we can obtain that to the second order

$$\Psi^*[g](x) = \Theta_0(x) \;+\; g(\varphi(x)) \;+\; \frac{1}{2}\Theta^{ij}(x)\frac{\partial g}{\partial y^j}(\varphi(x))\frac{\partial g}{\partial y^i}(\varphi(x)) \;+\; \dots$$

Composition of canonical relations of the considered form leads to another 'formal category' extending the category of smooth supermanifolds and their smooth maps, different from the one considered in the main text.[8] One should expect

$$(\Psi_1 \circ \Psi_2)^* = \Psi_2^* \circ \Psi_1^*,$$

so "nonlinear pullbacks" give a nonlinear representation of this formal category on the spaces of odd functions.

Similarly to Theorem 2 of the main text, we have

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[8]Working in a formal framework allows to go around the standard difficulties with composition. Compare remark at the end of section 1. Moreover, for ordinary (purely even) manifolds, the fibers of $\Pi T^* M$ are odd, hence there is no difference between formal and non-formal treatments. It was Ševera [20] who first noted that in that case Weinstein's symplectic "category" is a genuine category without quotes.

Theorem. *The derivative of the formal nonlinear mapping*

$$\Psi^*\colon \ \mathbf{\Pi C}^\infty(M_2) \to \mathbf{\Pi C}^\infty(M_1)$$

at a point $g \in \mathbf{\Pi C}^\infty(M_2)$ *is given by the formula:*

$$(T\Psi^*)[g] = \varphi_g^*,$$

where

$$\varphi_g^*\colon \ C^\infty(M_2) \to C^\infty(M_1)$$

is the ordinary pullback w.r.t. the map $\varphi_g\colon M_1 \to M_2$ *depending on* g. $\qquad\square$

Analogs of the Hamilton–Jacobi vector fields introduced in the main text, in the 'odd' setup take the form

$$\boldsymbol{X}_H = (-1)^{\tilde{H}(m+1)} \int\limits_{M^{n|m}} Dx \, H\left(x, \frac{\partial f}{\partial x}\right) \frac{\delta}{\delta f(x)}\,,$$

where $H \in C^\infty(\Pi T^*M)$ is a multivector (or 'pseudomultivector') field on M. Here we need to emphasize that the function f is *odd*, so in particular the substitution of its derivatives $\partial f / \partial x^a$ for the antimomenta x_a^* makes good sense. In other words, we have infinitesimal shifts of odd functions on M of the form

$$f \mapsto f_\varepsilon = f + \varepsilon \boldsymbol{X}_H[f]\,, \text{ where } \boldsymbol{X}_H[f](x) = H\left(x, \frac{\partial f}{\partial x}(x)\right).$$

Here $\varepsilon^2 = 0$ and $\tilde{\varepsilon} = \tilde{H} + 1$. The parity of the vector field $\boldsymbol{X}_H$ on $\mathbf{\Pi C}^\infty(M)$ is the opposite to the parity of H.

Theorem. *For arbitrary multivector fields* H *and* F,

$$[\boldsymbol{X}_H, \boldsymbol{X}_F] = (-1)^{\tilde{H}} \boldsymbol{X}_{[\![H,F]\!]}\,,$$

where the bracket at the l.h.s. is the commutator of vector fields on the infinite-dimensional supermanifold $\mathbf{\Pi C}^\infty(M)$ *and the bracket at the r.h.s. is the canonical Schouten bracket on* ΠT^*M. $\qquad\square$

Multivector fields $H_1 \in C^\infty(\Pi T^*M_1)$ and $H_2 \in C^\infty(\Pi T^*M_2)$ are said to be R-related for a relation $R \subset \Pi T^*M_1 \times \Pi T^*M_2$ if $p_1^*H_1 = p_2^*H_2$. For a canonical relation $\Psi \subset \Pi T^*M_1 \times (-\Pi T^*M_2)$ as above the analog of Theorem 6 holds:

Theorem. *If multivector fields* $H_1 \in C^\infty(\Pi T^*M_1)$ *and* $H_2 \in C^\infty(\Pi T^*M_2)$ *are* Ψ-*related, then the vector fields* $\boldsymbol{X}_{H_2} \in \mathrm{Vect}(\mathbf{\Pi C}^\infty(M_2))$ *and* $\boldsymbol{X}_{H_1} \in \mathrm{Vect}(\mathbf{\Pi C}^\infty(M_1))$ *are* Ψ^*-*related, for* $\Psi^*\colon \mathbf{\Pi C}^\infty(M_2) \to \mathbf{\Pi C}^\infty(M_1)$. $\qquad\square$

An even multivector field $P \in C^\infty(\Pi T^*M)$ satisfying $[\![P,P]\!] = 0$ defines a *homotopy Poisson structure* (or a P_∞-*structure*) on M via the higher derived bracket construction [22]. That means antisymmetric brackets of alternating parities on $C^\infty(M)$ that make it into an L_∞-algebra in the "antisymmetric version" and which are multiderivations w.r.t. ordinary multiplication. On the vector space $\Pi C^\infty(M)$ this induces an L_∞-algebra structure in the "symmetric version". With an abuse of language we still refer to P as to a 'Poisson tensor'

on M. The homological vector field $\boldsymbol{Q}$ on the supermanifold $\boldsymbol{\Pi C}^\infty(M)$ corresponding to this L_∞-structure has the Hamilton–Jacobi form

$$\boldsymbol{Q} = \int\limits_{M^{n|m}} Dx\, P\Big(x, \frac{\partial f}{\partial x}\Big)\frac{\delta}{\delta f(x)}\,.$$

Let $M_1 = (M_1, P_1)$ and $M_2 = (M_2, P_2)$ be two homotopy Poisson manifolds and let $\Psi \subset \Pi T^* M_1 \times (-\Pi T^* M_2)$ be a canonical relation as above.

Corollary. *If the Poisson tensors P_1 and P_2 are Ψ-related, then the mapping*

$$\Psi^*\colon\ \boldsymbol{\Pi C}^\infty(M_2) \to \boldsymbol{\Pi C}^\infty(M_1)$$

is an L_∞-morphism of the corresponding L_∞-algebras. $\qquad\square$

References

[1] V. I. Arnol′d. *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1989. Translated from the Russian by K. Vogtmann and A. Weinstein.

[2] F. A. Berezin. *The method of second quantization.* Translated from the Russian by Nobumichi Mugibayashi and Alan Jeffrey. Pure and Applied Physics, Vol. 24. Academic Press, New York, 1966.

[3] A. S. Cattaneo, B. Dherin, and A. Weinstein. Symplectic microgeometry I: Micromorphisms. *J. Symplectic Geom.*, 8(2):205–223, 2010.

[4] A. S. Cattaneo, B. Dherin, and A. Weinstein. Symplectic microgeometry II: Generating functions. *Bull. Braz. Math. Soc. (N.S.)*, 42(4):507–536, 2011.

[5] A. S. Cattaneo, B. Dherin, and A. Weinstein. Symplectic microgeometry III: Monoids. *J. Symplectic Geom.*, 11(3):319–341, 2013.

[6] A. S. Cattaneo, B. Dherin, and A. Weinstein. Integration of Lie algebroid comorphisms. *Port. Math.*, 70(2):113–144, 2013. `arXiv:1210.4443 [math.DG]`.

[7] H[O]. M. Khudaverdian. Geometry of superspace with even and odd brackets. Preprint of the Geneva University, UGVA-DPT 1989/05-613, 1989. Published in: *J. Math. Phys.* 32 (1991), 1934–1937.

[8] H[O]. M. Khudaverdian and A. P. Nersessian. On geometry of Batalin-Vilkovisky formalism. *Mod. Phys. Lett*, A8(25):2377–2385, 1993.

[9] H. M. Khudaverdian and Th. Th. Voronov. On odd Laplace operators. *Lett. Math. Phys.*, 62:127–142, 2002.

[10] H. M. Khudaverdian and Th. Th. Voronov. Higher Poisson brackets and differential forms. In *XXVII Workshop on Geometrical Methods in Physics*, volume 1079 of *AIP Conf. Proc.*, pages 203–215. Amer. Inst. Phys., Melville, NY, 2008.

[11] M. Kontsevich. Deformation quantization of Poisson manifolds, I. `math.QA/9709180`.

[12] M. Kontsevich. Deformation quantization of Poisson manifolds. *Lett. Math. Phys.*, 66(3):157–216, 2003.

[13] T. Lada and J. Stasheff. Introduction to SH Lie algebras for physicists. *Internat. J. Theoret. Phys.*, 32(7):1087–1103, 1993.

[14] S. Mac Lane. *Homology.* Die Grundlehren der mathematischen Wissenschaften, Bd. 114. Academic Press Inc., Publishers, New York, 1963.

[15] K. C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, volume 213 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.

[16] K. C. H. Mackenzie and P. Xu. Lie bialgebroids and Poisson groupoids. *Duke Math. J.*, 73(2):415–452, 1994.

[17] Yu. A. Neretin. Extension of representations of classical groups to representations of categories. *Algebra i Analiz*, 3(1):176–202, 1991.

[18] Yu. A. Neretin. *Categories of symmetries and infinite-dimensional groups*, volume 16 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 1996. Translated from the Russian by G. G. Gould, Oxford Science Publications.

[19] A. Schwarz. Geometry of Batalin-Vilkovisky quantization. *Comm. Math. Phys.*, 155(2):249–260, 1993.

[20] P. Ševera. Noncommutative differential forms and quantization of the odd symplectic category. *Lett. Math. Phys.*, 68(1):31–39, 2004.

[21] Th. Th. Voronov. Graded manifolds and Drinfeld doubles for Lie bialgebroids. In *Quantization, Poisson Brackets and Beyond*, volume 315 of *Contemp. Math.*, pages 131–168. Amer. Math. Soc., Providence, RI, 2002.

[22] Th. Th. Voronov. Higher derived brackets and homotopy algebras. *J. of Pure and Appl. Algebra*, 202(1–3):133–153, 2005.

[23] Th. Th. Voronov. Higher derived brackets for arbitrary derivations. In *Travaux mathématiques. Fasc. XVI*, Trav. Math., XVI, pages 163–186. Univ. Luxemb., Luxembourg, 2005.

[24] Th. Th. Voronov. Microformal geometry. `arXiv:1411.6720 [math.DG]`.

[25] Th. Th. Voronov. Thick morphisms of supermanifolds and oscillatory integral operators. *Russian Math. Surveys*, 71 (6), 2016 *(in print)*. `arXiv:1506.02417 [math.DG]`.

[26] Th. Th. Voronov. Quantum microformal morphisms of supermanifolds: an explicit formula and further properties. `arXiv:1512.04163 [math-ph]`.

[27] A. Weinstein. Symplectic manifolds and their Lagrangian submanifolds. *Advances in Math.*, 6:329–346 (1971), 1971.

[28] A. Weinstein. The symplectic "category". In *Differential geometric methods in mathematical physics (Clausthal, 1980)*, volume 905 of *Lecture Notes in Math.*, pages 45–51. Springer, Berlin-New York, 1982.

[29] A. Weinstein. Symplectic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 5(1):1–13, 1981.

[30] A. Weinstein. Symplectic categories. *Port. Math.*, 67(2):261–278, 2010.

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