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A Moderate Deviation Principle for 2-D Stochastic Navier-Stokes Equations Driven by Multiplicative Lévy Noises

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Abstract

In this paper, we establish a moderate deviation principle for two-dimensional stochastic Navier-Stokes equations driven by multiplicative *Lévy* noises. The weak convergence method introduced by Budhiraja, Dupuis and Ganguly in [3] plays a key role.

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1 Introduction

Consider the two-dimensional Navier-Stokes equation with Dirichlet boundary condition, which describes the time evolution of an incompressible fluid,

$$\frac{\partial u(t,x)}{\partial t} - \nu \Delta u(t,x) + (u(t,x)\cdot\nabla)u(t,x) + \nabla p(t,x) = f(t,x), \tag{1.1}$$

with the conditions

$$\begin{cases} (\nabla\cdot u)(t,x) = 0, & \text{for } \ x\in D, \quad t>0, \\ u(t,x)=0, & \text{for } \ x\in\partial D, t\geq 0, \\ u(0,x)=u_0(x), & \text{for } \ x\in D, \end{cases} \tag{1.2}$$

where D is a bounded open domain of $\mathbb{R}^2$ with regular boundary ∂D, $u(t,x)\in\mathbb{R}^2$ denotes the velocity field at time t and position x, $\nu>0$ is the viscosity coefficient, $p(t,x)$ denotes the pressure field, f is a deterministic external force.

To formulate the Navier-Stokes equation, we introduce the following standard spaces: let

$$V = \left\{ v\in H_0^1(D;\mathbb{R}^2) : \nabla\cdot v = 0, \text{a.e. in } D \right\},$$

with the norm

$$\|v\|_V := \left(\int_D |\nabla v|^2 dx \right)^{\frac{1}{2}} = \|v\|,$$

and let H be the closure of V in the L^2-norm

$$|v|_H := \left(\int_D |v|^2 dx \right)^{\frac{1}{2}} = |v|.$$

Define the operator A (Stokes operator) in H by the formula

$$Au := -\nu P_H \Delta u, \qquad \forall u\in H^2(D;\mathbb{R}^2)\cap V,$$

where the linear operator P_H (Helmhotz-Hodge projection) is the projection operator from $L^2(D;\mathbb{R}^2)$ to H, and define the nonlinear operator B by

$$B(u,v) := P_H((u\cdot\nabla)v),$$

with the notation $B(u):=B(u,u)$ for short.

By applying the operator P_H to each term of (1.1), we can rewrite it in the following abstract form:

$$du(t) + Au(t)dt + B(u(t))dt = f(t)dt \quad \text{ in } L^2([0,T],V'), \tag{1.3}$$

with the initial condition $u(0)=u_0$ for some fixed point u_0 in H.

Taking into account the random external forces, in this paper we consider stochastic Navier-Stokes equations (SNSE) driven by the multiplicative *Lévy* noise, that is, the following random perturbations of Navier-Stokes equation:

$$\begin{cases} du^\epsilon(t) = -Au^\epsilon(t)dt - B(u^\epsilon(t))dt + f(t)dt + \epsilon\int_{\mathbb{X}} G(u^\epsilon(t-),v)\widetilde{N}^{\epsilon^{-1}}(dtdv); \\ u^\epsilon(0) = u_0\in H. \end{cases} \tag{1.4}$$

Here $\mathbb{X}$ is a locally compact Polish space, G is a measurable mapping to be specified later, $N^{\epsilon^{-1}}$ is a Poisson random measure on $[0,T]\times\mathbb{X}$ with a σ-finite intensity measure $\epsilon^{-1}\lambda_T\otimes\vartheta$, λ_T is the Lebesgue measure on $[0,T]$ and ϑ is a σ-finite measure on $\mathbb{X}$, $\widetilde{N}^{\epsilon^{-1}}$ is the compensated Poisson random measure, i.e., for $O\in\mathcal{B}(\mathbb{X})$ with $\vartheta(O)<\infty$,

$$\widetilde{N}^{\epsilon^{-1}}([0,t]\times O)=N^{\epsilon^{-1}}([0,t]\times O)-\epsilon^{-1}t\vartheta(O).$$

As the parameter ε tends to zero, the solution u^ε of (1.4) will tend to the solution of the following deterministic Navier-Stokes equation at least in the mean sense

$$du^0(t)+Au^0(t)dt+B(u^0(t))dt=f(t)dt,\quad \text{with } u^0(0)=u_0\in H. \tag{1.5}$$

In this paper, we shall investigate deviations of u^ε from the deterministic solution u^0, as ε decreases to 0, that is, the asymptotic behavior of the trajectory,

$$Y^\varepsilon=\left(u^\varepsilon-u^0\right)/a(\varepsilon), \tag{1.6}$$

where $a(\varepsilon)$ is some deviation scale which strongly influences the asymptotic behavior of Y^ε. We will study the so-called moderate deviation principle (MDP for short, cf. [9]), that is when the deviation scale satisfies

$$a(\varepsilon)\to 0,\quad \varepsilon/a^2(\varepsilon)\to 0\quad \text{as } \varepsilon\to 0. \tag{1.7}$$

Throughout this paper, we assume that (1.7) is in place.

Large deviations for stochastic partial differential equations have been investigated in many papers, see [5], [6], [15], [19], etc.. Wentzell-Freidlin type large deviation results for the two-dimensional stochastic Navier-Stokes equations with Gaussian noise have been established in [1] and [20], and the case of Lévy noise has been established in [25] and [26].

Like the large deviations, the moderate deviation problems arise in the theory of statistical inference quite naturally. The estimates of moderate deviations can provide us with the rate of convergence and a useful method for constructing asymptotic confidence intervals, see [10], [11], [14], [16] and references therein. Results on the MDP for processes with independent increments were obtained in De Acosta [8], Chen [7] and Ledoux [17]. The study of the MDP estimates for other processes has been carried out as well, e.g., Wu [24] for Markov processes, Guillin and Liptser [12] for diffusion processes, Wang and Zhang [23] for stochastic reaction-diffusion equations. Wang *et al* [22] considered a MDP for 2-D stochastic Navier-Stokes equations driven by multiplicative Wiener processes.

The moderate deviation problems for stochastic evolution equations and stochastic partial differential equations driven by *Lévy* noise are drastically different because of the appearance of the jumps. There is not much study on this topic so far. Recently, Budhiraja, Dupuis and Ganguly [3] obtained the MDPs for stochastic differential equations driven by a Poisson random measure in finite dimensions and in some co-nuclear spaces, which can not cover SNSEs.

Our aim is to establish a moderate deviation principle for two-dimensional stochastic Navier-Stokes equations (SNSEs) driven by multiplicative *Lévy* noises. We will apply the abstract criteria (weak convergence approach) obtained in [3]. However, it is quite non-trivial to implement the weak convergence approach to SNSEs due to the highly non-linear term in the equation and the appearance of the jumps. The crucial step is to show the weak convergence of SNSEs driven by counting random measures with random intensity. To this end, we decompose the solutions into a sum of the solutions of several relatively simpler equations and prove the convergence/tightness of the solutions of each equations.

The organization of this paper is as follows. In Section 2, we recall the general criteria for a moderate deviation principle given in [3]. Section 3 is devoted to establishing the moderate deviation principle for two-dimensional stochastic Navier-Stokes equations driven by multiplicative *Lévy* noises.

Throughout this paper, $c_N, c_{f,T}, \cdots$ are positive constants depending on some parameters $N, f, T, \cdots$, independent of ε, whose value may be different from line to line.

2 Preliminaries

In this section, we will recall the general criteria for a moderate deviation principle given in [3], and to this end, we closely follow the framework and the notations in it.

2.1 Controlled Poisson random measure

Let $\mathbb{X}$ be a locally compact Polish space. Denote by $\mathcal{M}_{FC}(\mathbb{X})$ the space of all measures ϑ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ such that $\vartheta(K) < \infty$ for every compact K in $\mathbb{X}$, and let $C_c(\mathbb{X})$ be the space of continuous functions with compact supports. Endow $\mathcal{M}_{FC}(\mathbb{X})$ with the weakest topology such that for every $f \in C_c(\mathbb{X})$, the function $\vartheta \to \langle f, \vartheta \rangle = \int_{\mathbb{X}} f(u) d\vartheta(u), \vartheta \in \mathcal{M}_{FC}(\mathbb{X})$ is continuous. This topology can be metrized such that $\mathcal{M}_{FC}(\mathbb{X})$ is a Polish space (see e.g. [4]). Fix $T \in (0, \infty)$ and let $\mathbb{X}_T = [0, T] \times \mathbb{X}$. Fix a measure $\vartheta \in \mathcal{M}_{FC}(\mathbb{X})$, and let $\vartheta_T = \lambda_T \otimes \vartheta$, where λ_T is Lebesgue measure on $[0, T]$.

We recall that a Poisson random measure $\mathbf{n}$ on $\mathbb{X}_T$ with intensity measure ϑ_T is an $\mathcal{M}_{FC}(\mathbb{X}_T)$ valued random variable such that for each $B \in \mathcal{B}(\mathbb{X}_T)$ with $\vartheta_T(B) < \infty$, $\mathbf{n}(B)$ is Poisson distributed with mean $\vartheta_T(B)$ and for disjoint $B_1, \cdots, B_k \in \mathcal{B}(\mathbb{X}_T)$, $\mathbf{n}(B_1), \cdots, \mathbf{n}(B_k)$ are mutually independent random variables (cf. [13]). Denote by $\mathbb{P}$ the measure induced by $\mathbf{n}$ on $(\mathcal{M}_{FC}(\mathbb{X}_T), \mathcal{B}(\mathcal{M}_{FC}(\mathbb{X}_T)))$. Then letting $\mathbb{M} = \mathcal{M}_{FC}(\mathbb{X}_T)$, $\mathbb{P}$ is the unique probability measure on $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$ under which the canonical map, $N : \mathbb{M} \to \mathbb{M}$, $N(m) \doteq m$, is a Poisson random measure with intensity measure ϑ_T. We also consider, for $\theta > 0$, probability measures $\mathbb{P}_\theta$ on $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$ under which N is a Poissson random measure with intensity $\theta\vartheta_T$. The corresponding expectation operators will be denoted by $\mathbb{E}$ and $\mathbb{E}_\theta$, respectively.

Set $\mathbb{Y} = \mathbb{X} \times [0, \infty)$ and $\mathbb{Y}_T = [0, T] \times \mathbb{Y}$. Similarly, let $\bar{\mathbb{M}} = \mathcal{M}_{FC}(\mathbb{Y}_T)$ and let $\bar{\mathbb{P}}$ be the unique probability measure on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}))$ under which the canonical map, $\bar{N} : \bar{\mathbb{M}} \to \bar{\mathbb{M}}, \bar{N}(\bar{m}) \doteq \bar{m}$, is a Poisson random measure with intensity measure $\bar{\vartheta}_T = \lambda_T \otimes \vartheta \otimes \lambda_\infty$, with λ_∞ being Lebesgue measure on $[0, \infty)$. The corresponding expectation operator will be denoted by $\bar{\mathbb{E}}$. Let $\mathcal{F}_t \doteq \sigma\{\bar{N}((0, s] \times O) : 0 \leq s \leq t, O \in \mathcal{B}(\mathbb{Y})\}$, and denote by $\bar{\mathcal{F}}_t$ the completion under $\bar{\mathbb{P}}$. Let $\bar{\mathcal{P}}$ be the predictable σ-field on $[0, T] \times \bar{\mathbb{M}}$ with the filtration $\{\bar{\mathcal{F}}_t : 0 \leq t \leq T\}$ on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}))$. Let $\bar{\mathcal{A}}_+$ (resp. $\bar{\mathcal{A}}$) be the class of all $(\bar{\mathcal{P}} \otimes \mathcal{B}(\mathbb{X}))/\mathcal{B}[0, \infty)$(resp. $(\bar{\mathcal{P}} \otimes \mathcal{B}(\mathbb{X}))/\mathcal{B}(\mathbb{R})$)-measurable maps $\varphi : \mathbb{X}_T \times \bar{\mathbb{M}} \to [0, \infty)$ (resp. $\varphi : \mathbb{X}_T \times \bar{\mathbb{M}} \to \mathbb{R}$). For

$\varphi \in \bar{\mathcal{A}}_+$, define a counting process N^φ on $\mathbb{X}_T$ by

$$N^\varphi((0,t]\times U) = \int_{(0,t]\times U}\int_{(0,\infty)} 1_{[0,\varphi(s,x)]}(r)\bar{N}(dsdxdr),\ t\in[0,T], U\in\mathcal{B}(\mathbb{X}). \tag{2.8}$$

N^φ is the controlled random measure, with φ selecting the intensity for the points at location x and time s, in a possibly random but non-anticipating way. When $\varphi(s,x,\bar{m})\equiv\theta\in(0,\infty)$, we write $N^\varphi = N^\theta$. Note that N^θ has the same distribution with respect to $\bar{\mathbb{P}}$ as N has with respect to $\mathbb{P}_\theta$.

We end this subsection with some notations. Define $l:[0,\infty)\to[0,\infty)$ by

$$l(r) = r\log r - r + 1,\quad r\in[0,\infty).$$

For any $\varphi\in\bar{\mathcal{A}}_+$ the quantity

$$L_T(\varphi) = \int_{\mathbb{X}_T} l(\varphi(t,x,\omega))\vartheta_T(dtdx) \tag{2.9}$$

is well defined as a $[0,\infty]$-valued random variable. Let $\{K_n\subset\mathbb{X},\ n=1,2,\cdots\}$ be an increasing sequence of compact sets such that $\cup_{n=1}^\infty K_n = \mathbb{X}$. For each n let

$$\begin{aligned}\bar{\mathcal{A}}_{b,n} \doteq \{\varphi\in\bar{\mathcal{A}}_+:\ & \textit{for all } (t,\omega)\in[0,T]\times\bar{\mathbb{M}},\ n\ge\varphi(t,x,\omega)\ge 1/n \textit{ if } x\in K_n\\ & \textit{and } \varphi(t,x,\omega)=1 \textit{ if } x\in K_n^c\},\end{aligned}$$

and let $\bar{\mathcal{A}}_b = \cup_{n=1}^\infty\bar{\mathcal{A}}_{b,n}$.

2.2 A General Moderate Deviation Result

In this subsection, we recall a general criteria for a moderate deviation principle introduced in [3].

Assume that $a(\varepsilon)$ satisfies (1.7). Let $\{\mathcal{G}^\epsilon\}_{\epsilon>0}$ be a family of measurable maps from $\mathbb{M}$ to $\mathbb{U}$, where $\mathbb{M}$ is introduced in Subsection 2.1 and $\mathbb{U}$ is a Polish space. We present below a sufficient condition for large deviation principle (LDP in abbreviation) to hold for the family $\mathcal{G}^\epsilon(\epsilon N^{\epsilon^{-1}})$ as $\epsilon\to 0$, with speed $\varepsilon/a^2(\varepsilon)$ and a rate function that is given though a suitable quadratic form, which is the so-called moderate deviation principle (MDP for short, cf. [9]).

For $\varepsilon>0$ and $M<\infty$, consider the spaces

$$\begin{aligned}\mathcal{S}^M_{+,\varepsilon} &= \{\varphi:\ \mathbb{X}\times[0,T]\to\mathbb{R}_+ \mid L_T(\varphi)\le Ma^2(\varepsilon)\}\\ \mathcal{S}^M_{\varepsilon} &= \{\psi:\ \mathbb{X}\times[0,T]\to\mathbb{R} \mid \psi = (\varphi-1)/a(\varepsilon),\ \varphi\in\mathcal{S}^M_{+,\varepsilon}\}.\end{aligned} \tag{2.10}$$

We also let

$$\begin{aligned}\mathcal{U}^M_{+,\varepsilon} &= \{\varphi\in\bar{\mathcal{A}}_b:\ \varphi(\cdot,\cdot,\omega)\in\mathcal{S}^M_{+,\varepsilon},\ \bar{\mathbb{P}}\text{-}a.s.\}\\ \mathcal{U}^M_{\varepsilon} &= \{\psi\in\bar{\mathcal{A}}:\ \psi(\cdot,\cdot,\omega)\in\mathcal{S}^M_{\varepsilon},\ \bar{\mathbb{P}}\text{-}a.s.\}\end{aligned} \tag{2.11}$$

The norm in the Hilbert space $L^2(\vartheta_T)$ will be denoted by $\|\cdot\|_2$ and $B_2(R)$ denotes the ball of radius R in $L^2(\vartheta_T)$. Throughout this paper $B_2(R)$ is equipped with the weak topology of $L^2(\vartheta_T)$ and it is therefore weakly compact. Given a map $\mathcal{G}_0:\ L^2(\vartheta_T)\to\mathbb{U}$ and $\eta\in\mathbb{U}$, let

$$\mathbb{S}^0_\eta = \{\psi\in L^2(\vartheta_T):\ \eta = \mathcal{G}_0(\psi)\}$$

and define I by

$$I(\eta)=\inf_{\psi\in\mathbb{S}^0_\eta}\Big[\frac{1}{2}\|\psi\|_2^2\Big]. \tag{2.12}$$

By convention, $I(\eta)=+\infty$ if $\mathbb{S}^0_\psi=\emptyset$.

Suppose $\varphi\in\mathbb{S}^M_{+,\varepsilon}$. By Lemma 3.2 in [3], there exists $\kappa_2(1)\in(0,\infty)$ that is independent of ε and such that $\psi 1_{\{|\psi|\le 1/a(\varepsilon)\}}\in B_2(\sqrt{M\kappa_2(1)})$, where $\psi=(\varphi-1)/a(\varepsilon)$. In this paper, we use the symbol " $\Rightarrow$ " to denote convergence in distribution.

Condition MDP: Let $\mathcal{G}_0: L^2(\vartheta_T)\to\mathbb{U}$ be measurable and satisfy:

(MDP-1) Given $M>0$, suppose that $g^\varepsilon,\ g\in B_2(M)$ and $g^\varepsilon\to g$. Then

$$\mathcal{G}_0(g^\varepsilon)\to\mathcal{G}_0(g)\quad \text{in } \mathbb{U}.$$

(MDP-2) Given $M>0$, let $\{\varphi^\varepsilon\}_{\varepsilon>0}$ be such that for every $\varepsilon>0$, $\varphi^\varepsilon\in\mathcal{U}^M_{+,\varepsilon}$ and for some $\beta\in(0,1]$, $\psi^\varepsilon 1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}}\Rightarrow\psi$ in $B_2(\sqrt{M\kappa_2(1)})$ where $\psi^\varepsilon=(\varphi^\varepsilon-1)/a(\varepsilon)$. Then

$$\mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi^\varepsilon})\Rightarrow\mathcal{G}_0(\psi)\quad \text{in } \mathbb{U}.$$

The following criteria was established in [3].

Theorem 2.1 *Suppose that the functionals $\mathcal{G}^\varepsilon$ and $\mathcal{G}_0$ satisfy* **Condition MDP**. *Then $\{Y^\varepsilon\equiv\mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}}),\varepsilon>0\}$ satisfies a large deviation principle with speed $\varepsilon/a^2(\varepsilon)$ and rate function I defined in (2.12).*

3 Moderate Deviation Principles

Let V,H be the Hilbert spaces introduced in Section 1. Denote by V' the dual of V. Identifying H with its dual H', we have the dense, continuous embedding

$$V\hookrightarrow H\cong H'\hookrightarrow V'.$$

In this way, we may consider A as a bounded operator from V to V'. The inner product in H is denoted by $\langle\cdot,\cdot\rangle$. Moreover, we denote by $(\cdot,\cdot)$, the duality between V and V'. Hence, for $u=(u_i)\in V$, $w=(w_i)\in V$, we have

$$(Au,w)=\nu\sum_{i,j=1}^2\int_D\partial_i u_j\partial_i w_j dx. \tag{3.13}$$

Define $b(\cdot,\cdot,\cdot): V\times V\times V\to\mathbb{R}$ by

$$b(u,v,w)=\sum_{i,j=1}^2\int_D u_i\partial_i v_j w_j dx. \tag{3.14}$$

In particular, if $u,v,w\in V$, then

$$(B(u,v),w)=((u\cdot\nabla)v,w)=\sum_{i,j=1}^2\int_D u_i\partial_i v_j w_j dx=b(u,v,w).$$

$B(u)$ will be used to denote $B(u,u)$. By integration by parts,

$$b(u,v,w) = -b(u,w,v), \tag{3.15}$$

therefore

$$b(u,v,v) = 0, \qquad \forall u,v \in V. \tag{3.16}$$

There are some well-known estimates for b (see [21] and [20] for example), which will be required in the rest of this paper.

$$|b(u,v,w)| \le 2\|u\|^{\frac{1}{2}} \cdot |u|^{\frac{1}{2}} \cdot \|v\|^{\frac{1}{2}} \cdot |v|^{\frac{1}{2}} \cdot \|w\|, \tag{3.17}$$

$$|b(u,u,v)| \le \frac{1}{2}\|u\|^2 + 32\|v\|_{L^4}^4 \cdot |u|^2, \tag{3.18}$$

$$|(B(u)-B(v), u-v)| \le \frac{1}{2}\|u-v\|^2 + c|u-v|^2 \cdot \|v\|_{L^4}^4, \tag{3.19}$$

where

$$\|v\|_{L^4}^4 \le \|v\|^2 |v|^2. \tag{3.20}$$

Now, we will state the precise assumptions on the coefficients and collect some preliminary results from [3], which will be used in the sequel.

Condition A: The coefficient $G : H \times \mathbb{X} \to H$ and the force term f satisfy the following hypothesis:

(A.1) for some $L_G \in L^2(\vartheta)$,

$$|G(x_1,y) - G(x_2,y)| \le L_G(y)|x_1 - x_2|, \quad x_1,\ x_2 \in H, \quad y \in \mathbb{X}; \tag{3.21}$$

(A.2) for some $M_G \in L^2(\vartheta)$,

$$|G(x,y)| \le M_G(y)(1+|x|), \quad x \in H, \quad y \in \mathbb{X}; \tag{3.22}$$

(A.3) $f \in L^2([0,T];V')$, i.e.,

$$\int_0^T \|f(s)\|_{V'}^2 ds < \infty. \tag{3.23}$$

The following result follows by standard arguments (see [2], [21]).

Theorem 3.1 *Fix $u_0 \in H$, and assume* **Condition A**. *Let u^ε be the unique solution of equation* (1.4) *in $L^2(\Omega; D([0,T];H)) \cap L^2(\Omega \times [0,T]; V)$, and u^0 the unique solution of equation* (1.5) *in $C([0,T],H) \cap L^2([0,T],V)$. Then, the following estimates hold: there exists $\varepsilon_0 > 0$ such that*

$$\sup_{\varepsilon \in (0,\varepsilon_0]} \Big[\mathbb{E}\Big(\sup_{t\in[0,T]} |u^\varepsilon(t)|^2\Big) + \mathbb{E}\Big(\int_0^T \|u^\varepsilon(t)\|^2 dt\Big)\Big] \le C_{f,T,u_0}; \tag{3.24}$$

and

$$\sup_{t\in[0,T]} |u^0(t)|^2 + \int_0^T \|u^0(t)\|^2 dt \le C_{f,T,u_0}. \tag{3.25}$$

We now state a LDP for $\{Y^\varepsilon\}$ (namely, the MDP for $u^\varepsilon, \varepsilon > 0$), where

$$Y^\varepsilon = (u^\varepsilon - u^0)/a(\varepsilon), \tag{3.26}$$

and $a(\varepsilon)$ is as in (1.7).

We define a class of functions by

$$\mathcal{H} = \Big\{ h : \mathbb{X} \to \mathbb{R} : \ \exists \delta > 0, s.t. \ \forall \Gamma \ with \ \vartheta(\Gamma) < \infty, \ \int_\Gamma \exp(\delta h^2(y))\vartheta(dy) < \infty \Big\}.$$

Condition B: The functions L_G and M_G are in the class $\mathcal{H}$.

The following theorem is our main result.

Theorem 3.2 *Suppose that Conditions A and B hold. Then $\{Y^\varepsilon\}$ satisfies a large deviation principle in $D([0,T],H) \cap L^2([0,T],V)$ with speed $\varepsilon/a^2(\varepsilon)$ and the rate function given by*

$$I(\eta) = \inf_\psi \Big\{ \frac{1}{2} \|\psi\|_2^2 \Big\},$$

where the infimum is taken over all $\psi \in L^2(\vartheta_T)$ such that (η, ψ) satisfies the following equation

$$\begin{aligned} d\eta(t) &= -A\eta(t)dt - B(\eta(t), u^0(t))dt - B(u^0(t), \eta(t))dt \\ &\quad + \int_{\mathbb{X}} \psi(y,t) G(u^0(t), y)\vartheta(dy)dt, \end{aligned} \tag{3.27}$$

with initial value $\eta(0) = 0$.

Proof: Proof of Theorem 3.2

According to Theorem 2.1, it suffices to prove that **Condition MDP** is fulfilled. The verification of Condition **MDP-1** will be given by Proposition 3.3. Condition **MDP-2** will be established in Proposition 3.6. ■

Let $\{T_t, t \geq 0\}$ denote the semigroup generated by $-A$. It is easy to see that $T_t, t \geq 0$ are compact operators. For $f \in L^1([0,T],H)$, define the mapping

$$Rf(t) = \int_0^t T_{t-s} f(s) ds, \ t \geq 0,$$

which is the mild solution of the equation:

$$Z(t) = -\int_0^t AZ(s)ds + \int_0^t f(s)ds.$$

We recall the following lemma proved in [18] (see Proposition 5.4 there).

Lemma 3.1 *If $\mathcal{D} \subset L^1([0,T],H)$ is uniformly integrable, then the image family $\mathcal{Y} = R(\mathcal{D})$ is relatively compact in $C([0,T],H)$.*

Denote $\mathcal{G}_0 : \ L^2(\vartheta_T) \to C([0,T],H) \cap L^2([0,T],V)$ by

$$\mathcal{G}_0(\psi) = \eta \text{ if for } \psi \in L^2(\vartheta_T), \text{ where } \eta \text{ solves (3.27).} \tag{3.28}$$

Proposition 3.3 *Suppose that Conditions A and B hold. Fix* $\Upsilon \in (0,\infty)$ *and* $g^\varepsilon, g \in B_2(\Upsilon)$ *such that* $g^\varepsilon \to g$*. Then* $\mathcal{G}_0(g^\varepsilon) \to \mathcal{G}_0(g)$ *in* $C([0,T],H) \cap L^2([0,T],V)$*.*

Proof: Set $f^\varepsilon(t) = \int_{\mathbb{X}} g^\varepsilon(y,t)G(u^0(t),y)\vartheta(dy), t \in [0,T]$. By (3.25),

$$\begin{aligned}\int_0^T \int_{\mathbb{X}} |G(u^0(t),y)|^2 \vartheta(dy)dt &\le \int_{\mathbb{X}} M_G^2(y)\vartheta(dy) \int_0^T (1+|u^0(t)|)^2 dt \\ &\le 2 \sup_{t\in[0,T]} (1+|u^0(t)|^2) \int_{\mathbb{X}} M_G^2(y)\vartheta(dy) \\ &< \infty,\end{aligned}$$

then, for every $v \in H$, $\langle G(u^0(t),y), v\rangle \in L^2(\vartheta_T)$. Combining $g^\varepsilon \to g$ in the weak topology on $L^2(\vartheta_T)$, we get

$$\lim_{\varepsilon\to 0}\langle \int_0^t f^\varepsilon(s)ds, v\rangle = \langle \int_0^t \int_{\mathbb{X}} g(y,s)G(u^0(s),y)\vartheta(dy)ds, v\rangle, \quad \forall v \in H,\ \forall t \in [0,T]. \quad (3.29)$$

Denote $\mathcal{D} = \{f^\varepsilon,\ \varepsilon > 0\}$. Since, for every measurable subset $O \subset [0,T]$

$$\begin{aligned}\int_O |f^\varepsilon(t)|dt &\le \int_O \int_{\mathbb{X}} |g^\varepsilon(y,t)||G(u^0(t),y)|\vartheta(dy)dt \\ &\le \Big(\int_0^T \int_{\mathbb{X}} |g^\varepsilon(y,t)|^2 \vartheta(dy)dt\Big)^{1/2} \Big(\int_O \int_{\mathbb{X}} |G(u^0(t),y)|^2\vartheta(dy)dt\Big)^{1/2} \\ &\le \Upsilon\Big(\int_{\mathbb{X}} M(y)^2\vartheta(dy)\int_O (1+|u^0(t)|)^2 dt\Big)^{1/2} \\ &\le \Upsilon \sup_{t\in[0,T]} (1+|u^0(t)|)\sqrt{\lambda_T(O)}, \end{aligned} \quad (3.30)$$

we see that the family $\mathcal{D} \subset L^1([0,T],H)$ is uniformly integrable in $L^1([0,T],H)$. Therefore, by lemma 3.1, $\{Z^\varepsilon,\ \varepsilon > 0\}$ is relatively compact in $C([0,T],H)$, here Z^ε satisfies

$$dZ^\varepsilon(t) = -AZ^\varepsilon(t)dt + f^\varepsilon(t)dt, \quad t \in [0,T],$$

with initial value $Z^\varepsilon(0) = 0$.

Let Z be any limit point of $\{Z^\varepsilon,\ \varepsilon > 0\}$ in $C([0,T],H)$, and combining (3.29), we have

$$\langle Z(t), v\rangle = -\int_0^t \langle Z(s), Av\rangle ds + \langle \int_0^t \int_{\mathbb{X}} g(y,s)G(u^0(s),y)\vartheta(dy)ds, v\rangle, \quad \forall v \in D(A).$$

This implies that Z is the unique solution of the following equation

$$\begin{cases} dZ(t) = -AZ(t)dt + \int_{\mathbb{X}} g(y,t)G(u^0(t),y)\vartheta(dy)dt, \quad t \in [0,T]; \\ Z(0) = 0. \end{cases}$$

Denote $\overline{Z^\varepsilon}(t) = Z^\varepsilon(t) - Z(t)$. Notice that (3.30) also holds for

$$f(t) = \int_{\mathbb{X}} g(y,t)G(u^0(t),y)\vartheta(dy)$$

and $\sup_{s\in[0,T]} |\overline{Z^\varepsilon}(s)| \to 0,\ as\ \varepsilon \to 0$, we obtain

$$|\overline{Z^\varepsilon}(t)|^2 + 2\nu \int_0^t \|\overline{Z^\varepsilon}(s)\|^2 ds$$

$$
\begin{aligned}
&= 2\int_0^t \Big\langle \overline{Z^\varepsilon}(s), \int_{\mathbb{X}} (g^\varepsilon(y,s) - g(y,s)) G(u^0(s),y)\vartheta(dy)\Big\rangle ds \\
&\le 2 \sup_{s\in[0,T]} |\overline{Z^\varepsilon}(s)| \\
&\quad \times \Big[\int_0^T \int_{\mathbb{X}} |g^\varepsilon(y,s)||G(u^0(s),y)|\vartheta(dy)ds + \int_0^T \int_{\mathbb{X}} |g(y,s)||G(u^0(s),y)|\vartheta(dy)ds \Big] \\
&\le 4\Upsilon \sup_{t\in[0,T]} (1+|u^0(t)|)\sqrt{T} \sup_{s\in[0,T]} |\overline{Z^\varepsilon}(s)| \to 0, \ \ as\ \varepsilon \to 0. \qquad (3.31)
\end{aligned}
$$

Set $L^\varepsilon(t) = \mathcal{G}_0(g^\varepsilon)(t) - Z^\varepsilon(t)$ and $L(t) = \mathcal{G}_0(g)(t) - Z(t)$, and denote $\overline{L^\varepsilon}(t) = L^\varepsilon(t) - L(t)$. Then

$$
\begin{cases}
d\overline{L^\varepsilon}(t) = -A\overline{L^\varepsilon}(t)dt - B(\overline{L^\varepsilon}(t) + \overline{Z^\varepsilon}(t), u^0(t))dt - B(u^0(t), \overline{L^\varepsilon}(t) + \overline{Z^\varepsilon}(t))dt; \\
\overline{L^\varepsilon}(0) = 0.
\end{cases}
$$

We have

$$
\begin{aligned}
& |\overline{L^\varepsilon}(t)|^2 + 2\nu \int_0^t \|\overline{L^\varepsilon}(s)\|^2 ds \\
=\ & -2\int_0^t \big(B(\overline{L^\varepsilon}(s) + \overline{Z^\varepsilon}(s), u^0(s)), \overline{L^\varepsilon}(s)\big)\, ds \\
& -2\int_0^t \big(B(u^0(s), \overline{L^\varepsilon}(s) + \overline{Z^\varepsilon}(s)), \overline{L^\varepsilon}(s)\big)\, ds \\
=\ & 2\int_0^t \big(B(\overline{L^\varepsilon}(s), \overline{L^\varepsilon}(s)), u^0(s)\big)\, ds \\
& -2\int_0^t \big(B(\overline{Z^\varepsilon}(s), u^0(s)), \overline{L^\varepsilon}(s)\big)\, ds \\
& -2\int_0^t \big(B(u^0(s), \overline{Z^\varepsilon}(s)), \overline{L^\varepsilon}(s)\big)\, ds \\
=\ & I_1(t) + I_2(t) + I_3(t). \qquad (3.32)
\end{aligned}
$$

By (3.18) and (3.20),

$$
\begin{aligned}
|I_1(t)| &\le 2\int_0^t \Big| \big(B(\overline{L^\varepsilon}(s), \overline{L^\varepsilon}(s)), u^0(s)\big) \Big| ds \\
&\le \nu \int_0^t \|\overline{L^\varepsilon}(s)\|^2 ds + \frac{128}{\nu^3} \sup_{s\in[0,T]} |u^0(s)|^2 \int_0^t \|u^0(s)\|^2 |\overline{L^\varepsilon}(s)|^2 ds. \qquad (3.33)
\end{aligned}
$$

By (3.17) and (3.25),

$$
\begin{aligned}
|I_2(t)| &\le 2\int_0^t \Big| \big(B(\overline{Z^\varepsilon}(s), u^0(s)), \overline{L^\varepsilon}(s)\big) \Big| ds \\
&\le 4\int_0^t |\overline{Z^\varepsilon}(s)|^{1/2} \|\overline{Z^\varepsilon}(s)\|^{1/2} |u^0(s)|^{1/2} \|u^0(s)\|^{1/2} \|\overline{L^\varepsilon}(s)\| ds \\
&\le 4 \sup_{s\in[0,T]} |\overline{Z^\varepsilon}(s)|^{1/2} \sup_{s\in[0,T]} |u^0(s)|^{1/2} \int_0^t \|\overline{Z^\varepsilon}(s)\|^{1/2} \|u^0(s)\|^{1/2} \|\overline{L^\varepsilon}(s)\| ds \\
&\le C \sup_{s\in[0,T]} |\overline{Z^\varepsilon}(s)|^{1/2} \Big[\int_0^t \|\overline{L^\varepsilon}(s)\|^2 ds + \int_0^t \|\overline{Z^\varepsilon}(s)\| \|u^0(s)\| ds \Big]
\end{aligned}
$$

$$\leq \ C\sup_{s\in[0,T]}|\overline{Z^\varepsilon}(s)|^{1/2}\Big[\int_0^t\|\overline{L^\varepsilon}(s)\|^2ds+C\Big(\int_0^t\|\overline{Z^\varepsilon}(s)\|^2ds\Big)^{1/2}\Big]. \tag{3.34}$$

Similar to (3.34), we have

$$|I_3(t)|\leq C\sup_{s\in[0,T]}|\overline{Z^\varepsilon}(s)|^{1/2}\Big[\int_0^t\|\overline{L^\varepsilon}(s)\|^2ds+C\Big(\int_0^t\|\overline{Z^\varepsilon}(s)\|^2ds\Big)^{1/2}\Big]. \tag{3.35}$$

Combining (3.32)–(3.35), we get

$$\begin{aligned} & |\overline{L^\varepsilon}(t)|^2+(\nu-C\sup_{s\in[0,T]}|\overline{Z^\varepsilon}(s)|^{1/2})\int_0^t\|\overline{L^\varepsilon}(s)\|^2ds \qquad (3.36)\\ \leq \ & \frac{128}{\nu^3}\sup_{s\in[0,T]}|u^0(s)|^2\int_0^t\|u^0(s)\|^2|\overline{L^\varepsilon}(s)|^2ds+C\sup_{s\in[0,T]}|\overline{Z^\varepsilon}(s)|^{1/2}\Big(\int_0^T\|\overline{Z^\varepsilon}(s)\|^2ds\Big)^{1/2}. \end{aligned}$$

By (3.25)(3.31) and using Gronwall's lemma,

$$\lim_{\varepsilon\to 0}\left\{\sup_{t\in[0,T]}|\overline{L^\varepsilon}(t)|^2+\int_0^T\|\overline{L^\varepsilon}(t)\|^2dt\right\}=0. \tag{3.37}$$

Recall $L^\varepsilon(t)=\mathcal{G}_0(g^\varepsilon)(t)-Z^\varepsilon(t)$ and $L(t)=\mathcal{G}_0(g)(t)-Z(t)$. By (3.31) and (3.37) yield that

$$\lim_{\varepsilon\to 0}\left\{\sup_{t\in[0,T]}|\mathcal{G}_0(g^\varepsilon)(t)-\mathcal{G}_0(g)(t)|^2+\int_0^T\|\mathcal{G}_0(g^\varepsilon)(t)-\mathcal{G}_0(g)(t)\|^2dt\right\}=0.$$

■

Recall the definition of $\mathcal{U}^M_{+,\varepsilon}$ in (2.11). We note that for every $\varphi^\varepsilon\in\mathcal{U}^M_{+,\varepsilon}$, there exists unique process $X^\varepsilon\in D([0,T],H)\cap L^2([0,T],V)$ that solves the following equation

$$\begin{cases} dX^\varepsilon(t) &= -AX^\varepsilon(t)dt-B(X^\varepsilon(t),X^\varepsilon(t))dt+f(t)dt+\int_{\mathbb{X}}\varepsilon G(X^\varepsilon(t-),y)\widetilde{N}^{\varepsilon^{-1}\varphi^\varepsilon}(dydt)\\ &\quad +\int_{\mathbb{X}}G(X^\varepsilon(t),y)(\varphi^\varepsilon(y,t)-1)\vartheta(dy)dt;\\ X^\varepsilon(0) &= u_0. \end{cases}$$

We refer the reader to [3] for details. Moreover, the following lemmas 3.2-3.4 were also proved in [3].

Lemma 3.2 *Let $h\in L^2(\vartheta)\cap\mathcal{H}$ and fix $M>0$. Then there exists $\varsigma_h>0$ such that for any measurable subset I of $[0,T]$ and for all $\varepsilon>0$,*

$$\sup_{\varphi\in\mathcal{S}^M_{+,\varepsilon}}\int_{\mathbb{X}\times I}h^2(y)\varphi(y,s)\vartheta(dy)ds\leq\varsigma_h(a^2(\varepsilon)+\lambda_T(I)). \tag{3.38}$$

Lemma 3.3 *Let $h\in L^2(\vartheta)\cap\mathcal{H}$ and I be a measurable subset of $[0,T]$. Fix $M>0$. Then there exists $\Gamma_h,\rho_h:(0,\infty)\to(0,\infty)$ such that $\Gamma_h(u)\downarrow 0$ as $u\uparrow\infty$, and for all $\varepsilon,\beta\in(0,\infty)$,*

$$\sup_{\psi\in\mathcal{S}^M_\varepsilon}\int_{\mathbb{X}\times I}|h(y)\psi(y,s)|1_{\{|\psi|\geq\beta/a(\varepsilon)\}}\vartheta(dy)ds\leq\Gamma_h(\beta)(1+\sqrt{\lambda_T(I)}),$$

and

$$\sup_{\psi\in\mathcal{S}^M_\varepsilon}\int_{\mathbb{X}\times I}|h(y)\psi(y,s)|\vartheta(dy)ds\leq\rho_h(\beta)\sqrt{\lambda_T(I)}+\Gamma_h(\beta)a(\varepsilon).$$

Lemma 3.4 *Let $h \in L^2(\vartheta) \cap \mathcal{H}$ be positive. Then for any $\beta > 0$,*

$$\lim_{\varepsilon\to 0} \sup_{\psi\in\mathcal{S}_\varepsilon^M} \int_{\mathbb{X}\times[0,T]} |h(y)\psi(y,s)| 1_{\{|\psi|>\beta/a(\varepsilon)\}} \vartheta(dy)ds = 0. \tag{3.39}$$

Proposition 3.4 *There exists an $\varepsilon_0 > 0$ such that*

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \Big(\mathbb{E} \sup_{t\in[0,T]} |X^\varepsilon(t)|^2 + \mathbb{E} \int_0^T \|X^\varepsilon(t)\|^2 dt \Big) \le C_{\varepsilon_0} < \infty. \tag{3.40}$$

Proof: By Itô's formula,

$$\begin{aligned} & d|X^\varepsilon(t)|^2 + 2\nu \|X^\varepsilon(t)\|^2 dt \\ = \ & 2(f(t), X^\varepsilon(t))dt + 2\langle \int_{\mathbb{X}} \varepsilon G(X^\varepsilon(t-), y) \widetilde{N}^{\varepsilon^{-1}\varphi^\varepsilon}(dydt), X^\varepsilon(t-)\rangle \qquad (3.41) \\ & +2\langle \int_{\mathbb{X}} G(X^\varepsilon(t), y)(\varphi^\varepsilon(y,t) - 1)\vartheta(dy)dt, X^\varepsilon(t)\rangle + \int_{\mathbb{X}} \varepsilon^2 |G(X^\varepsilon(t-), y)|^2 N^{\varepsilon^{-1}\varphi^\varepsilon}(dydt). \end{aligned}$$

We have

$$\int_0^t |2(f(s), X^\varepsilon(s))| ds \le \nu \int_0^t \|X^\varepsilon(s)\|^2 ds + \frac{1}{\nu} \int_0^t \|f(s)\|_{V'}^2 ds. \tag{3.42}$$

Set $\psi^\varepsilon(y,s) = (\varphi^\varepsilon(y,s) - 1)/a(\varepsilon) \in \mathcal{U}_\varepsilon^M$. Then

$$\begin{aligned} & \Big| \int_0^t 2\langle \int_{\mathbb{X}} G(X^\varepsilon(s), y)(\varphi^\varepsilon(y,s) - 1)\vartheta(dy)ds, X^\varepsilon(s)\rangle \Big| \\ \le \ & 2a(\varepsilon) \int_0^t |X^\varepsilon(s)| \int_{\mathbb{X}} |G(X^\varepsilon(s), y)| |\psi^\varepsilon(y,s)| \vartheta(dy)ds \\ \le \ & 2a(\varepsilon) \int_0^t |X^\varepsilon(s)|(1 + |X^\varepsilon(s)|) \int_{\mathbb{X}} M_G(y) |\psi^\varepsilon(y,s)| \vartheta(dy)ds \\ \le \ & 4a(\varepsilon) \int_0^t (1 + |X^\varepsilon(s)|^2) \int_{\mathbb{X}} M_G(y) |\psi^\varepsilon(y,s)| \vartheta(dy)ds. \qquad (3.43) \end{aligned}$$

Combining (3.41)–(3.43), we have

$$\begin{aligned} & |X^\varepsilon(t)|^2 + \nu \int_0^t \|X^\varepsilon(s)\|^2 ds \\ \le \ & |u_0|^2 + \frac{1}{\nu} \int_0^T \|f(s)\|_{V'}^2 ds + \sup_{l\in[0,T]} \Big| 2 \int_0^l \langle \int_{\mathbb{X}} \varepsilon G(X^\varepsilon(s-), y) \widetilde{N}^{\varepsilon^{-1}\varphi^\varepsilon}(dyds), X^\varepsilon(s-)\rangle \Big| \\ & + \int_0^T \int_{\mathbb{X}} \varepsilon^2 |G(X^\varepsilon(s-), y)|^2 N^{\varepsilon^{-1}\varphi^\varepsilon}(dyds) + 4a(\varepsilon) \int_0^T \int_{\mathbb{X}} M_G(y) |\psi^\varepsilon(y,s)| \vartheta(dy)ds \\ & +4a(\varepsilon) \int_0^t |X^\varepsilon(s)|^2 \int_{\mathbb{X}} M_G(y) |\psi^\varepsilon(y,s)| \vartheta(dy)ds \\ = \ & I_1 + I_2 + I_3 + I_4 + I_5 + I_6(t). \qquad (3.44) \end{aligned}$$

Applying Gronwall' lemma and using Lemma 3.3, we get

$$|X^\varepsilon(t)|^2 + \int_0^t \|X^\varepsilon(s)\|^2 ds$$

$$\leq \Big(I_1+I_2+I_3+I_4+I_5\Big)\exp\Big[4a(\varepsilon)\Big(\rho_{M_G}(\beta)\sqrt{T}+\Gamma_{M_G}(\beta)a(\varepsilon)\Big)\Big]. \tag{3.45}$$

By (3.23) and Lemma 3.3,

$$I_1+I_2+I_5\leq C+4a(\varepsilon)\Big(\rho_{M_G}(\beta)\sqrt{T}+\Gamma_{M_G}(\beta)a(\varepsilon)\Big). \tag{3.46}$$

By Burkholder-Davis-Gundy inequality and Lemma 3.2,

$$\begin{aligned}
\mathbb{E}I_3 &\leq \mathbb{E}\Big(\int_0^T\int_{\mathbb{X}}4\varepsilon^2|X^\varepsilon(s-)|^2|G(X^\varepsilon(s-),y)|^2N^{\varepsilon^{-1}\varphi^\varepsilon}(dy,ds)\Big)^{1/2}\\
&\leq \mathbb{E}\Big[\sup_{s\in[0,T]}|X^\varepsilon(s)|\Big(\int_0^T\int_{\mathbb{X}}4\varepsilon^2|G(X^\varepsilon(s-),y)|^2N^{\varepsilon^{-1}\varphi^\varepsilon}(dy,ds)\Big)^{1/2}\Big]\\
&\leq \frac{1}{4}\mathbb{E}\sup_{s\in[0,T]}|X^\varepsilon(s)|^2+16\varepsilon\mathbb{E}\Big(\int_0^T\int_{\mathbb{X}}|G(X^\varepsilon(s),y)|^2\varphi^\varepsilon(y,s)\vartheta(dy)ds\Big)\\
&\leq \frac{1}{4}\mathbb{E}\sup_{s\in[0,T]}|X^\varepsilon(s)|^2+32\varepsilon\mathbb{E}\Big((\sup_{s\in[0,T]}|X^\varepsilon(s)|^2+1)\int_0^T\int_{\mathbb{X}}M_G^2(y)\varphi^\varepsilon(y,s)\vartheta(dy)ds\Big)\\
&\leq \frac{1}{4}\mathbb{E}\sup_{s\in[0,T]}|X^\varepsilon(s)|^2+32\varepsilon\varsigma_{M_G}(a^2(\varepsilon)+T)\mathbb{E}(\sup_{s\in[0,T]}|X^\varepsilon(s)|^2+1).
\end{aligned} \tag{3.47}$$

Similar to (3.47), we get

$$\begin{aligned}
\mathbb{E}I_4 &= \varepsilon\mathbb{E}\int_0^T\int_{\mathbb{X}}|G(X^\varepsilon(s),y)|^2\varphi^\varepsilon(y,s)\vartheta(dy)ds\\
&\leq 2\varepsilon\varsigma_{M_G}(a^2(\varepsilon)+T)\mathbb{E}(\sup_{s\in[0,T]}|X^\varepsilon(s)|^2+1).
\end{aligned} \tag{3.48}$$

Choosing $\varepsilon_0>0$ such that $34\varepsilon_0\varsigma_{M_G}(a^2(\varepsilon_0)+T)\leq 1/8$, and combining (3.45)–(3.48), we obtain (3.40). The proof is complete.

■

Recall (1.5). We have

Theorem 3.5

$$\lim_{\varepsilon\to 0}\Big(\mathbb{E}\sup_{t\in[0,T]}|X^\varepsilon(t)-u^0(t)|^2+\mathbb{E}\int_0^T\|X^\varepsilon(t)-u^0(t)\|^2dt\Big)=0. \tag{3.49}$$

Proof: Set $Z^\varepsilon(t)=X^\varepsilon(t)-u^0(t)$. Then

$$\begin{aligned}
dZ^\varepsilon(t) &= -AZ^\varepsilon(t)dt-B(X^\varepsilon(t),Z^\varepsilon(t))dt-B(Z^\varepsilon(t),u^0(t))dt \qquad (3.50)\\
&+ \varepsilon\int_{\mathbb{X}}G(X^\varepsilon(t-),y)\tilde{N}^{\varepsilon^{-1}\varphi^\varepsilon}(dydt)+\int_{\mathbb{X}}G(X^\varepsilon(t),y)(\varphi^\varepsilon(y,t)-1)\vartheta(dy)dt
\end{aligned}$$

with initial value $Z^\varepsilon(0)=0$.

Apply Ito's Formula,

$$\begin{aligned}
&d|Z^\varepsilon(t)|^2+2\nu\|Z^\varepsilon(t)\|^2dt \qquad (3.51)\\
=\ &2\Big\langle B(Z^\varepsilon(t),Z^\varepsilon(t)),u^0(t)\Big\rangle dt+2\varepsilon\int_{\mathbb{X}}\Big\langle G(X^\varepsilon(t-),y),Z^\varepsilon(t-)\Big\rangle\tilde{N}^{\varepsilon^{-1}\varphi^\varepsilon}(dydt)
\end{aligned}$$

$$+2\int_{\mathbb{X}}\Big\langle G(X^\varepsilon(t),y)(\varphi^\varepsilon(y,t)-1),Z^\varepsilon(t)\Big\rangle\vartheta(dy)dt+\varepsilon^2\int_{\mathbb{X}}|G(X^\varepsilon(t-),y)|^2N^{\varepsilon^{-1}\varphi^\varepsilon}(dydt).$$

By (3.18) and (3.20),

$$\begin{aligned}
&\int_0^t 2\Big|\Big\langle B(Z^\varepsilon(s),Z^\varepsilon(s)),u^0(s)\Big\rangle\Big|ds\\
\leq\ &\nu\int_0^t\|Z^\varepsilon(s)\|^2ds+\frac{64}{\nu^3}\int_0^t\|u^0(s)\|_{L^4}^4|Z^\varepsilon(s)|^2ds\\
\leq\ &\nu\int_0^t\|Z^\varepsilon(s)\|^2ds+\frac{64}{\nu^3}\sup_{l\in[0,T]}|u^0(l)|^2\int_0^t\|u^0(s)\|^2|Z^\varepsilon(s)|^2ds. \qquad (3.52)
\end{aligned}$$

Set $\psi^\varepsilon(y,t)=(\varphi^\varepsilon(y,t)-1)/a(\varepsilon)$. By (3.21) and (3.22),

$$\begin{aligned}
&2\int_0^t\Big|\int_{\mathbb{X}}\Big\langle G(X^\varepsilon(s),y)(\varphi^\varepsilon(y,s)-1),Z^\varepsilon(s)\Big\rangle\vartheta(dy)\Big|ds\\
\leq\ &2\int_0^t|Z^\varepsilon(s)|\int_{\mathbb{X}}|G(X^\varepsilon(s),y)-G(u^0(s),y)||\varphi^\varepsilon(y,s)-1|\vartheta(dy)ds\\
&+2\int_0^t|Z^\varepsilon(s)|\int_{\mathbb{X}}|G(u^0(s),y)||\varphi^\varepsilon(y,s)-1|\vartheta(dy)ds\\
\leq\ &2a(\varepsilon)\int_0^t|Z^\varepsilon(s)|^2\int_{\mathbb{X}}L_G(y)|\psi^\varepsilon(y,s)|\vartheta(dy)ds\\
&+a(\varepsilon)\int_0^t(1+|Z^\varepsilon(s)|^2)(1+|u^0(s)|)\int_{\mathbb{X}}M_G(y)|\psi^\varepsilon(y,s)|\vartheta(dy)ds\\
\leq\ &a(\varepsilon)\int_0^t|Z^\varepsilon(s)|^2\int_{\mathbb{X}}\Big(2L_G(y)+(1+\sup_{l\in[0,T]}|u^0(l)|)M_G(y)\Big)|\psi^\varepsilon(y,s)|\vartheta(dy)ds\\
&+a(\varepsilon)(1+\sup_{l\in[0,T]}|u^0(l)|)\int_0^t\int_{\mathbb{X}}M_G(y)|\psi^\varepsilon(y,s)|\vartheta(dy)ds. \qquad (3.53)
\end{aligned}$$

Combining (3.51)(3.52)(3.53),

$$|Z^\varepsilon(t)|^2+\nu\int_0^t\|Z^\varepsilon(s)\|^2ds\leq M_1(T)+M_2(T)+M_3(T)+\int_0^t J(s)|Z^\varepsilon(s)|^2ds,$$

here

$$M_1(T)=2\varepsilon\sup_{s\in[0,T]}\Big|\int_0^s\int_{\mathbb{X}}\Big\langle G(X^\varepsilon(l-),y),Z^\varepsilon(l-)\Big\rangle\tilde{N}^{\varepsilon^{-1}\varphi^\varepsilon}(dydl)\Big|,$$

$$M_2(T)=\varepsilon^2\int_0^T\int_{\mathbb{X}}|G(X^\varepsilon(t-),y)|^2N^{\varepsilon^{-1}\varphi^\varepsilon}(dydt),$$

$$M_3(T)=a(\varepsilon)(1+\sup_{l\in[0,T]}|u^0(l)|)\int_0^T\int_{\mathbb{X}}M_G(y)|\psi^\varepsilon(y,s)|\vartheta(dy)ds,$$

$$J(s)\ =\ \frac{64}{\nu^3}\sup_{l\in[0,T]}|u^0(l)|^2\|u^0(s)\|^2+2a(\varepsilon)\int_{\mathbb{X}}L_G(y)|\psi^\varepsilon(y,s)|\vartheta(dy)$$

$$+a(\varepsilon)(1+\sup_{l\in[0,T]}|u^0(l)|)\int_{\mathbb{X}}M_G(y)|\psi^\varepsilon(y,s)|\vartheta(dy).$$

By Gronwall' lemma, Lemma 3.3 and (3.25),

$$\begin{aligned}
&|Z^\varepsilon(t)|^2+\nu\int_0^t\|Z^\varepsilon(s)\|^2ds\\
\le\ &\Big(M_1(T)+M_2(T)+M_3(T)\Big)\exp\Big(\int_0^T J(s)ds\Big)\\
\le\ &C\Big(M_1(T)+M_2(T)+M_3(T)\Big). \qquad (3.54)
\end{aligned}$$

By Lemma 3.2 and (3.40)

$$\begin{aligned}
&\mathbb{E}M_1(T)\\
\le\ &\mathbb{E}\Big(\int_0^T\int_{\mathbb{X}}4\varepsilon^2|G(X^\varepsilon(l-),y)|^2|Z^\varepsilon(l-)|^2N^{\varepsilon^{-1}\varphi^\varepsilon}(dydl)\Big)^{1/2}\\
\le\ &1/2\mathbb{E}\Big(\sup_{t\in[0,T]}|Z^\varepsilon(t)|^2\Big)+8\varepsilon\mathbb{E}\Big(\int_0^T\int_{\mathbb{X}}M_G^2(y)(1+|X^\varepsilon(l)|)^2\varphi^\varepsilon(y,l)\vartheta(dy)dl\Big)\\
\le\ &1/2\mathbb{E}\Big(\sup_{t\in[0,T]}|Z^\varepsilon(t)|^2\Big)+8\varepsilon\mathbb{E}\Big(\sup_{t\in[0,T]}(1+|X^\varepsilon(t)|)^2\int_0^T\int_{\mathbb{X}}M_G^2(y)\varphi^\varepsilon(y,l)\vartheta(dy)dl\Big)\\
\le\ &1/2\mathbb{E}\Big(\sup_{t\in[0,T]}|Z^\varepsilon(t)|^2\Big)+16\varepsilon\varsigma_{M_G}(a^2(\varepsilon)+T)C. \qquad (3.55)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathbb{E}M_2(T)\ &=\ \varepsilon\mathbb{E}\int_0^T\int_{\mathbb{X}}|G(X^\varepsilon(t),y)|^2\varphi^\varepsilon(y,t)\vartheta(dy)dt\\
&\le\ 2\varepsilon\mathbb{E}\Big(\sup_{t\in[0,T]}(1+|X^\varepsilon(t)|^2)\int_0^T\int_{\mathbb{X}}M_G^2(y)\varphi^\varepsilon(y,t)\vartheta(dy)dt\Big)\\
&\le\ \varepsilon\varsigma_{M_G}(a^2(\varepsilon)+T)C. \qquad (3.56)
\end{aligned}$$

By (3.25) and Lemma 3.3,

$$M_3(T)\le Ca(\varepsilon)\Big(\rho_{M_G}(\beta)\sqrt{T}+\Gamma_{M_G}(\beta)a(\varepsilon)\Big). \qquad (3.57)$$

Combining (3.54)–(3.57), we have

$$\lim_{\varepsilon\to 0}\Big(\mathbb{E}\sup_{t\in[0,T]}|Z^\varepsilon(t)|^2+\mathbb{E}\int_0^T\|Z^\varepsilon(t)\|^2dt\Big)=0. \qquad (3.58)$$

The proof is complete. ■

Define

$$\mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi^\varepsilon}):=Y^\varepsilon=\frac{1}{a(\varepsilon)}(X^\varepsilon-u^0). \qquad (3.59)$$

Then Y^ε satisfies

$$\begin{cases} dY^\varepsilon(t) = -AY^\varepsilon(t)dt - B(Y^\varepsilon(t), u^0(t))dt - B(X^\varepsilon(t), Y^\varepsilon(t))dt \\ \qquad\qquad +\frac{\varepsilon}{a(\varepsilon)}\int_{\mathbb{X}} G(X^\varepsilon(t-), y)\widetilde{N}^{\varepsilon^{-1}\varphi^\varepsilon}(dydt) \\ \qquad\qquad +\frac{1}{a(\varepsilon)}\int_{\mathbb{X}} G(X^\varepsilon(t), y)(\varphi^\varepsilon(y,t)-1)\vartheta(dy)dt, \\ Y^\varepsilon(0) = 0. \end{cases} \tag{3.60}$$

Proposition 3.6 *Given $M < \infty$. Let $\{\varphi^\varepsilon\}_{\varepsilon>0}$ be such that $\varphi^\varepsilon \in \mathcal{U}^M_{+,\varepsilon}$ for every $\varepsilon > 0$. Let $\psi^\varepsilon = (\varphi^\varepsilon - 1)/a(\varepsilon)$ and $\beta \in (0,1]$. Then the family $\{Y^\varepsilon, \psi^\varepsilon 1_{\{|\psi^\varepsilon| \le \beta/a(\varepsilon)\}}\}_{\varepsilon>0}$ is tight in $D([0,T],H) \times B_2\Big(\sqrt{M\kappa_2(1)}\Big)$, and any limit point (Y,ψ) solves the equation (3.27).*

Proof: The proof is divided into four steps.

Step 1. Let Z^ε be the solution of the following equation

$$dZ^\varepsilon(t) = -AZ^\varepsilon(t)dt + \frac{\varepsilon}{a(\varepsilon)}\int_{\mathbb{X}} G(X^\varepsilon(t-), y)\widetilde{N}^{\varepsilon^{-1}\varphi^\varepsilon}(dydt)$$

with initial value $Z^\varepsilon(0) = 0$.

Applying Ito's formula to $|Z^\varepsilon(t)|^2$,

$$\begin{aligned} & d|Z^\varepsilon(t)|^2 + 2\nu\|Z^\varepsilon(t)\|^2 dt \\ = & \frac{2\varepsilon}{a(\varepsilon)}\int_{\mathbb{X}} \langle G(X^\varepsilon(t-), y), Z^\varepsilon(t-)\rangle \widetilde{N}^{\varepsilon^{-1}\varphi^\varepsilon}(dydt) + \frac{\varepsilon^2}{a^2(\varepsilon)}\int_{\mathbb{X}} |G(X^\varepsilon(t-), y)|^2 N^{\varepsilon^{-1}\varphi^\varepsilon}(dydt). \end{aligned} \tag{3.61}$$

By Burkholder-Davis-Gundy inequality, Lemma 3.2 and (3.40), we have

$$\begin{aligned} & \mathbb{E}\Big(\sup_{t\in[0,T]}\Big|\int_0^t \int_{\mathbb{X}} \frac{2\varepsilon}{a(\varepsilon)}\langle G(X^\varepsilon(s-), y), Z^\varepsilon(s-)\rangle \widetilde{N}^{\varepsilon^{-1}\varphi^\varepsilon}(dyds)\Big|\Big) \\ \le\ & C\mathbb{E}\Big(\int_0^T \int_{\mathbb{X}} \frac{\varepsilon^2}{a^2(\varepsilon)}|G(X^\varepsilon(s-), y)|^2 |Z^\varepsilon(s-)|^2 N^{\varepsilon^{-1}\varphi^\varepsilon}(dyds)\Big)^{1/2} \\ \le\ & 1/2\mathbb{E}\Big(\sup_{t\in[0,T]}|Z^\varepsilon(t)|^2\Big) + C\mathbb{E}\Big(\int_0^T \int_{\mathbb{X}} \frac{\varepsilon^2}{a^2(\varepsilon)}|G(X^\varepsilon(s-), y)|^2 N^{\varepsilon^{-1}\varphi^\varepsilon}(dyds)\Big) \\ \le\ & 1/2\mathbb{E}\Big(\sup_{t\in[0,T]}|Z^\varepsilon(t)|^2\Big) + \frac{C\varepsilon}{a^2(\varepsilon)}\mathbb{E}\Big(\int_0^T \int_{\mathbb{X}} M_G^2(y)(1+|X^\varepsilon(s)|^2)\varphi^\varepsilon(y,s)\vartheta(dy)ds\Big) \\ \le\ & 1/2\mathbb{E}\Big(\sup_{t\in[0,T]}|Z^\varepsilon(t)|^2\Big) + \frac{C\varepsilon}{a^2(\varepsilon)}\varsigma_{M_G}(a^2(\varepsilon)+T), \end{aligned} \tag{3.62}$$

and similarly

$$\begin{aligned} & \mathbb{E}\Big(\int_0^T \int_{\mathbb{X}} \frac{\varepsilon^2}{a^2(\varepsilon)}|G(X^\varepsilon(t-), y)|^2 N^{\varepsilon^{-1}\varphi^\varepsilon}(dydt)\Big) \\ =\ & \frac{\varepsilon}{a^2(\varepsilon)}\mathbb{E}\Big(\int_0^T \int_{\mathbb{X}} |G(X^\varepsilon(t), y)|^2 \varphi^\varepsilon(y,t)\vartheta(dy)dt\Big) \\ \le\ & \frac{\varepsilon}{a^2(\varepsilon)}\varsigma_{M_G}(a^2(\varepsilon)+T). \end{aligned} \tag{3.63}$$

Combining (3.61) (3.62) and (3.63), we obtain

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big(\sup_{t\in[0,T]}|Z^\varepsilon(t)|^2 + \int_0^T \|Z^\varepsilon(t)\|^2 dt\Big) = 0. \tag{3.64}$$

Step 2. Recall $\psi^\varepsilon = (\varphi^\varepsilon - 1)/a(\varepsilon)$. Let $L^\varepsilon(t)$ be the unique solution of

$$\begin{cases} dL^\varepsilon(t) = -AL^\varepsilon(t)dt + \int_{\mathbb{X}} G(X^\varepsilon(t), y)\psi^\varepsilon(y,t)1_{\{|\psi^\varepsilon|>\beta/a(\varepsilon)\}}\vartheta(dy)dt, \\ L^\varepsilon(0) = 0. \end{cases}$$

We have

$$\begin{aligned} & |L^\varepsilon(t)|^2 + 2\nu \int_0^t \|L^\varepsilon(s)\|^2 ds \\ = \ & 2\int_0^t \int_{\mathbb{X}} \langle G(X^\varepsilon(s), y)\psi^\varepsilon(y,s)1_{\{|\psi^\varepsilon|>\beta/a(\varepsilon)\}}, L^\varepsilon(s)\rangle \vartheta(dy)ds \\ \leq \ & 2\int_0^T \int_{\mathbb{X}} |G(X^\varepsilon(s), y)||\psi^\varepsilon(y,s)|1_{\{|\psi^\varepsilon|>\beta/a(\varepsilon)\}}|L^\varepsilon(s)|\vartheta(dy)ds \\ \leq \ & 2\sup_{t\in[0,T]}|L^\varepsilon(t)| \sup_{t\in[0,T]}(1+|X^\varepsilon(t)|)\int_0^T\int_{\mathbb{X}} M_G(y)|\psi^\varepsilon(y,s)|1_{\{|\psi^\varepsilon|>\beta/a(\varepsilon)\}}\vartheta(dy)ds \\ \leq \ & 1/2 \sup_{t\in[0,T]}|L^\varepsilon(t)|^2 \\ & + C\sup_{t\in[0,T]}(1+|X^\varepsilon(t)|^2)\Big[\int_0^T\int_{\mathbb{X}} M_G(y)|\psi^\varepsilon(y,s)|1_{\{|\psi^\varepsilon|>\beta/a(\varepsilon)\}}\vartheta(dy)ds\Big]^2. \end{aligned}$$

By (3.40) and Lemma 3.4,

$$\begin{aligned} & \mathbb{E}\Big(\sup_{t\in[0,T]}|L^\varepsilon(t)|^2 + \nu\int_0^T \|L^\varepsilon(t)\|^2 dt\Big) \\ \leq \ & C\mathbb{E}\Big(\sup_{t\in[0,T]}(1+|X^\varepsilon(t)|^2)\Big)\Big[\sup_{\psi\in\mathcal{S}_\varepsilon^M}\int_0^T\int_{\mathbb{X}} M_G(y)|\psi(y,s)|1_{\{|\psi|>\beta/a(\varepsilon)\}}\vartheta(dy)ds\Big]^2 \\ \to \ & 0, \quad as\ \varepsilon \to 0. \end{aligned} \tag{3.65}$$

Step 3. Denote by U^ε the unique solution of the following equation

$$dU^\varepsilon(t) = -AU^\varepsilon(t)dt + \int_{\mathbb{X}}\Big(G(X^\varepsilon(t), y) - G(u^0(t), y)\Big)\psi^\varepsilon(y,t)1_{\{|\psi^\varepsilon|\leq\beta/a(\varepsilon)\}}\vartheta(dy)dt,$$

with initial value $U^\varepsilon(0) = 0$. Then

$$\begin{aligned} & |U^\varepsilon(t)|^2 + 2\nu\int_0^t \|U^\varepsilon(s)\|^2 ds \\ = \ & 2\int_0^t\int_{\mathbb{X}}\Big\langle\Big(G(X^\varepsilon(s), y) - G(u^0(s), y)\Big), U^\varepsilon(s)\Big\rangle\psi^\varepsilon(y,s)1_{\{|\psi^\varepsilon|\leq\beta/a(\varepsilon)\}}\vartheta(dy)ds \\ \leq \ & 2\int_0^T\int_{\mathbb{X}}\Big|G(X^\varepsilon(s), y) - G(u^0(s), y)\Big||U^\varepsilon(s)||\psi^\varepsilon(y,s)|\vartheta(dy)ds \\ \leq \ & 2\sup_{s\in[0,T]}|U^\varepsilon(s)|\sup_{s\in[0,T]}|X^\varepsilon(s) - u^0(s)|\int_0^T\int_{\mathbb{X}} L_G(y)|\psi^\varepsilon(y,s)|\vartheta(dy)ds \\ \leq \ & 1/2\sup_{s\in[0,T]}|U^\varepsilon(s)|^2 + C\sup_{s\in[0,T]}\Big|X^\varepsilon(s) - u^0(s)\Big|^2\Big(\sup_{\psi\in\mathcal{S}_\varepsilon^M}\int_0^T\int_{\mathbb{X}} L_G(y)|\psi(y,s)|\vartheta(dy)ds\Big)^2. \end{aligned}$$

By Lemma 3.3 and (3.49), we have

$$\lim_{\varepsilon\to 0}\Big[\mathbb{E}\Big(\sup_{s\in[0,T]}|U^\varepsilon(s)|^2\Big)+\mathbb{E}\Big(\int_0^T\|U^\varepsilon(s)\|^2ds\Big)\Big]=0. \tag{3.66}$$

Step 4. Set $K^\varepsilon=Z^\varepsilon+L^\varepsilon+U^\varepsilon$ and denote $\Upsilon^\varepsilon=Y^\varepsilon-K^\varepsilon$. By (3.60), we have

$$\begin{cases} d\Upsilon^\varepsilon(t)=-A\Upsilon^\varepsilon(t)dt-a(\varepsilon)B\Big(\Upsilon^\varepsilon(t)+K^\varepsilon(t),\Upsilon^\varepsilon(t)+K^\varepsilon(t)\Big)dt,\\ \qquad -B\Big(u^0(t),\Upsilon^\varepsilon(t)+K^\varepsilon(t)\Big)dt-B\Big(\Upsilon^\varepsilon(t)+K^\varepsilon(t),u^0(t)\Big)dt,\\ \qquad +\int_{\mathbb{X}}G(u^0(t),y)\psi^\varepsilon(y,t)1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}}\vartheta(dy)dt,\\ \Upsilon^\varepsilon(0)=0. \end{cases} \tag{3.67}$$

Set

$$\Pi=\Big(D([0,T],H)\cap L^2([0,T],V);\ C([0,T],H)\cap L^2([0,T],V);\ B_2\Big(\sqrt{M\kappa_2(1)}\Big)\Big).$$

By (3.64), (3.65) and (3.66), and notice that $(\psi^\varepsilon 1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}})_{\varepsilon>0}$ is tight in $B_2\Big(\sqrt{M\kappa_2(1)}\Big)$ (see Lemma 3.2 in [3]), $(Z^\varepsilon,\ L^\varepsilon+U^\varepsilon,\ \psi^\varepsilon 1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}})_{\varepsilon>0}$ is tight in Π, and let $(0,0,\psi)$ be any limit point of the tight family, and denote by $Y=\mathcal{G}_0(\psi)$ the solution of equation (3.27).

It follows from the Skorokhod representation theorem that there exist a stochastic basis $(\Omega^1,\mathcal{F}^1,\{\mathcal{F}^1_t\}_{t\in[0,T]},\mathbb{P}^1)$ and, on this basis, Π-valued random variables $(\widetilde{Z}^\varepsilon,\ \widetilde{LU}^\varepsilon,\ \widetilde{\psi}^\varepsilon)$, $(0,0,\widetilde{\psi})$, $\epsilon\in(0,\epsilon_0)$, such that $(\widetilde{Z}^\varepsilon,\ \widetilde{LU}^\varepsilon,\ \widetilde{\psi}^\varepsilon)$ (respectively $(0,0,\widetilde{\psi})$) has the same law as $(Z^\varepsilon,\ L^\varepsilon+U^\varepsilon,\ \psi^\varepsilon 1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}})$ (respectively $(0,0,\psi)$), and $(\widetilde{Z}^\varepsilon,\ \widetilde{LU}^\varepsilon,\ \widetilde{\psi}^\varepsilon)\to(0,0,\widetilde{\psi})$ in Π, $\mathbb{P}^1$-a.s..

Set $\widetilde{K}^\varepsilon=\widetilde{Z}^\varepsilon+\widetilde{LU}^\varepsilon$. Denote by $\widetilde{\Upsilon}^\varepsilon$ the unique solution of (3.67) with $(K^\varepsilon,\psi^\varepsilon)$ replaced by $(\widetilde{K}^\varepsilon,\widetilde{\psi}^\varepsilon)$. Then $(\widetilde{K}^\varepsilon,\widetilde{\Upsilon}^\varepsilon)$ has the law as $(K^\varepsilon,\Upsilon^\varepsilon)$. Hence, $\widetilde{Y}^\varepsilon=\widetilde{K}^\varepsilon+\widetilde{\Upsilon}^\varepsilon$ has the same law as $Y^\varepsilon=K^\varepsilon+\Upsilon^\varepsilon$ in $D([0,T],H)\cap L^2([0,T],V)$. Denote by $\widetilde{Y}$ the solution of equation (3.27) with $\psi(y,t)$ replaced by $\widetilde{\psi}(y,t)$. $\widetilde{Y}$ must have the same law as Y.

Thus, the proof of the Proposition will be complete if we can show that

$$\sup_{t\in[0,T]}|\widetilde{Y}^\varepsilon(t)-\widetilde{Y}(t)|^2+\int_0^T\|\widetilde{Y}^\varepsilon(t)-\widetilde{Y}(t)\|^2dt\to 0,\ \mathbb{P}^1-a.s.,\ as\ \varepsilon\to 0. \tag{3.68}$$

This is the task of the remaining proof.

Consider the following equation

$$\begin{cases} d\widetilde{\Gamma}^\varepsilon(t)=-A\widetilde{\Gamma}^\varepsilon(t)dt+\int_{\mathbb{X}}G(u^0(t),y)\widetilde{\psi}^\varepsilon(y,t)\vartheta(dy)dt,\\ \widetilde{\Gamma}^\varepsilon(0)=0. \end{cases} \tag{3.69}$$

Using similar arguments as in the proof of (3.31), we have

$$\lim_{\varepsilon\to 0}\Big(\sup_{t\in[0,T]}|\widetilde{\Gamma}^\varepsilon(t)-\widetilde{\Gamma}(t)|^2+\int_0^T\|\widetilde{\Gamma}^\varepsilon(t)-\widetilde{\Gamma}(t)\|^2dt\Big)=0, \tag{3.70}$$

here $\widetilde{\Gamma}$ satisfies (3.69) with $\widetilde{\psi}^\varepsilon(y,t)$ replaced by $\widetilde{\psi}(y,t)$.

Set $\widetilde{M}=\widetilde{Y}-\widetilde{\Gamma}$ and $\widetilde{M}^\varepsilon=\widetilde{Y}^\varepsilon-\widetilde{K}^\varepsilon-\widetilde{\Gamma}^\varepsilon$. Then

$$\begin{cases} d\widetilde{M}(t)=-A\widetilde{M}(t)dt-B\Big(u^0(t),\widetilde{M}(t)+\widetilde{\Gamma}(t)\Big)dt-B\Big(\widetilde{M}(t)+\widetilde{\Gamma}(t),u^0(t)\Big)dt,\\ \widetilde{M}(0)=0.\end{cases} \tag{3.71}$$

and

$$\begin{cases} d\widetilde{M}^\varepsilon(t) &= -A\widetilde{M}^\varepsilon(t)dt-a(\varepsilon)B\Big(\widetilde{M}^\varepsilon(t)+\widetilde{\Gamma}^\varepsilon(t)+\widetilde{K}^\varepsilon(t),\widetilde{M}^\varepsilon(t)+\widetilde{\Gamma}^\varepsilon(t)+\widetilde{K}^\varepsilon(t)\Big)dt,\\ &\quad -B\Big(u^0(t),\widetilde{M}^\varepsilon(t)+\widetilde{\Gamma}^\varepsilon(t)+\widetilde{K}^\varepsilon(t)\Big)dt\\ &\quad -B\Big(\widetilde{M}^\varepsilon(t)+\widetilde{\Gamma}^\varepsilon(t)+\widetilde{K}^\varepsilon(t),u^0(t)\Big)dt,\\ \widetilde{M}^\varepsilon(0) &= 0.\end{cases} \tag{3.72}$$

Since

$$\lim_{\varepsilon\to0}\Big[\sup_{t\in[0,T]}|\widetilde{K}^\varepsilon(t)|^2+\int_0^T\|\widetilde{K}^\varepsilon(t)\|^2dt\Big]\to0,\ \mathbb{P}^1-a.s., \tag{3.73}$$

taking into account (3.70), by standard arguments (see [21]), we have

$$\begin{aligned}&\sup_{\varepsilon\in(0,\varepsilon_0]}\Big[\sup_{t\in[0,T]}|\widetilde{M}^\varepsilon(t)|^2+\int_0^T\|\widetilde{M}^\varepsilon(t)\|^2dt\Big]+\Big[\sup_{t\in[0,T]}|\widetilde{M}(t)|^2+\int_0^T\|\widetilde{M}(t)\|^2dt\Big]\\ \leq\ & C(\omega^1)<\infty,\ \mathbb{P}^1-a.s..\end{aligned} \tag{3.74}$$

Set $\overline{\widetilde{M}^\varepsilon}=\widetilde{M}^\varepsilon-\widetilde{M}$ and $\overline{\widetilde{\Gamma}^\varepsilon}=\widetilde{\Gamma}^\varepsilon-\widetilde{\Gamma}$. Now the proof of (3.68) reduces to the proof of

$$\lim_{\varepsilon\to0}\Big[\sup_{t\in[0,T]}|\overline{\widetilde{M}^\varepsilon}(t)|^2+\int_0^T\|\overline{\widetilde{M}^\varepsilon}(s)\|^2ds\Big]=0,\ \ \mathbb{P}^1-a.s., \tag{3.75}$$

We have

$$\begin{aligned}&|\overline{\widetilde{M}^\varepsilon}(t)|^2+2\nu\int_0^t\|\overline{\widetilde{M}^\varepsilon}(s)\|^2ds\\ =\ & -2a(\varepsilon)\int_0^t\Big(B\Big(\widetilde{M}^\varepsilon(s)+\widetilde{\Gamma}^\varepsilon(s)+\widetilde{K}^\varepsilon(s),\widetilde{M}^\varepsilon(s)+\widetilde{\Gamma}^\varepsilon(s)+\widetilde{K}^\varepsilon(s)\Big),\overline{\widetilde{M}^\varepsilon}(s)\Big)\,ds\\ & -2\int_0^t\Big(B\Big(u^0(s),\overline{\widetilde{M}^\varepsilon}(s)+\overline{\widetilde{\Gamma}^\varepsilon}(s)+\widetilde{K}^\varepsilon(s)\Big),\overline{\widetilde{M}^\varepsilon}(s)\Big)\,ds\\ & -2\int_0^t\Big(B\Big(\overline{\widetilde{M}^\varepsilon}(s),u^0(s)\Big),\overline{\widetilde{M}^\varepsilon}(s)\Big)\,ds\\ & -2\int_0^t\Big(B\Big(\overline{\widetilde{\Gamma}^\varepsilon}(s)+\widetilde{K}^\varepsilon(s),u^0(s)\Big),\overline{\widetilde{M}^\varepsilon}(s)\Big)\,ds\\ =\ & I_1(t)+I_2(t)+I_3(t)+I_4(t).\end{aligned} \tag{3.76}$$

Fix $\omega^1\in\Omega^1$. By (3.17) and (3.74), we have

$$\begin{aligned}&|I_1(t)|\\ \leq\ & 4a(\varepsilon)\int_0^t|\widetilde{M}^\varepsilon(s)+\widetilde{\Gamma}^\varepsilon(s)+\widetilde{K}^\varepsilon(s)|\|\widetilde{M}^\varepsilon(s)+\widetilde{\Gamma}^\varepsilon(s)+\widetilde{K}^\varepsilon(s)\|\|\overline{\widetilde{M}^\varepsilon}(s)\|ds\end{aligned}$$

$$
\begin{aligned}
&\leq a(\varepsilon)\int_0^t \|\widetilde{\overline{M^\varepsilon}}(s)\|^2 ds \\
&\quad +2a(\varepsilon)\int_0^t |\widetilde{M}^\varepsilon(s)+\widetilde{\Gamma}^\varepsilon(s)+\widetilde{K}^\varepsilon(s)|^2\|\widetilde{M}^\varepsilon(s)+\widetilde{\Gamma}^\varepsilon(s)+\widetilde{K}^\varepsilon(s)\|^2 ds \\
&\leq a(\varepsilon)\int_0^t \|\widetilde{\overline{M^\varepsilon}}(s)\|^2 ds + a(\varepsilon)C(\omega^1),
\end{aligned}
\tag{3.77}
$$

and

$$
\begin{aligned}
&|I_2(t)| \\
&= 2\Big|\int_0^t \Big(B\Big(u^0(s),\widetilde{\overline{\Gamma^\varepsilon}}(s)+\widetilde{K}^\varepsilon(s)\Big),\widetilde{\overline{M^\varepsilon}}(s)\Big)\,ds\Big| \\
&\leq 4\int_0^t |u^0(s)|^{1/2}\|u^0(s)\|^{1/2}|\widetilde{\overline{\Gamma^\varepsilon}}(s)+\widetilde{K}^\varepsilon(s)|^{1/2}\|\widetilde{\overline{\Gamma^\varepsilon}}(s)+\widetilde{K}^\varepsilon(s)\|^{1/2}\|\widetilde{\overline{M^\varepsilon}}(s)\| ds \\
&\leq \frac{1}{2}\nu\int_0^t \|\widetilde{\overline{M^\varepsilon}}(s)\|^2 ds + C\int_0^t |u^0(s)|\|u^0(s)\||\widetilde{\overline{\Gamma^\varepsilon}}(s)+\widetilde{K}^\varepsilon(s)|\|\widetilde{\overline{\Gamma^\varepsilon}}(s)+\widetilde{K}^\varepsilon(s)\| ds \\
&\leq \frac{1}{4}\nu\int_0^t \|\widetilde{\overline{M^\varepsilon}}(s)\|^2 ds + C(\omega^1)\Big[\int_0^T \|\widetilde{\overline{\Gamma^\varepsilon}}(s)+\widetilde{K}^\varepsilon(s)\|^2 ds\Big]^{1/2},
\end{aligned}
\tag{3.78}
$$

similar to (3.78),

$$
|I_4(t)| \leq \frac{1}{4}\nu\int_0^t \|\widetilde{\overline{M^\varepsilon}}(s)\|^2 ds + C(\omega^1)\Big[\int_0^T \|\widetilde{\overline{\Gamma^\varepsilon}}(s)+\widetilde{K}^\varepsilon(s)\|^2 ds\Big]^{1/2}. \tag{3.79}
$$

By (3.18)(3.20) and (3.74),

$$
\begin{aligned}
|I_3(t)| &= 2\Big|\int_0^t \Big(B\Big(\widetilde{\overline{M^\varepsilon}}(s),\widetilde{\overline{M^\varepsilon}}(s)\Big),u^0(s)\Big)\,ds\Big| \\
&\leq \nu\int_0^t \|\widetilde{\overline{M^\varepsilon}}(s)\|^2 ds + C\int_0^t \|u^0(s)\|^2|\widetilde{\overline{M^\varepsilon}}(s)|^2 ds.
\end{aligned}
\tag{3.80}
$$

Combining (3.76)–(3.80), we have

$$
\begin{aligned}
&|\widetilde{\overline{M^\varepsilon}}(t)|^2 + \Big(1/2-a(\varepsilon)\Big)\nu\int_0^t \|\widetilde{\overline{M^\varepsilon}}(s)\|^2 ds \\
&\leq a(\varepsilon)C(\omega^1) + C(\omega^1)\Big[\int_0^T \|\widetilde{\overline{\Gamma^\varepsilon}}(s)+\widetilde{K}^\varepsilon(s)\|^2 ds\Big]^{1/2} \\
&\quad +C\int_0^t \|u^0(s)\|^2|\widetilde{\overline{M^\varepsilon}}(s)|^2 ds.
\end{aligned}
$$

Since $\lim_{\varepsilon\to 0} a(\varepsilon)=0$ and

$$
\lim_{\varepsilon\to 0}\Big[\int_0^T \|\widetilde{\overline{\Gamma^\varepsilon}}(s)+\widetilde{K}^\varepsilon(s)\|^2 ds\Big]=0,\quad \mathbb{P}^1-a.s.,
$$

by Gronwall's lemma we obtain

$$
\lim_{\varepsilon\to 0}\Big[\sup_{t\in[0,T]}|\widetilde{\overline{M^\varepsilon}}(t)|^2 + \int_0^T \|\widetilde{\overline{M^\varepsilon}}(s)\|^2 ds\Big]=0,\quad \mathbb{P}^1-a.s.
$$

The proof is complete. ■

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