Smooth densities of the laws of perturbed diffusion processes

DOI:
10.1016/j.spl.2016.07.016

Document Version
Accepted author manuscript

Citation for published version (APA):
Xu, L., Yue, W., & Zhang, T. (2016). Smooth densities of the laws of perturbed diffusion processes. *Statistics and Probability Letters*, *119*, 55-62. https://doi.org/10.1016/j.spl.2016.07.016

Published in:
Statistics and Probability Letters

SMOOTH DENSITIES OF THE LAWS OF PERTURBED DIFFUSION PROCESSES

LIHU XU, WEN YUE, AND TUSHENG ZHANG

ABSTRACT. Under some regularity conditions on b, σ and α, we prove that the solution of the following perturbed stochastic differential equation

$$X_t = x + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s + \alpha \sup_{0\le s\le t} X_s, \quad \alpha < 1 \tag{0.1}$$

admits smooth densities for all $0 < t \le t_0$, where $t_0 > 0$ is some finite number.
Keywords: Perturbed diffusion processes, Malliavin differentiability, Smooth density.
Mathematics Subject Classification (2000): 60H07.

1. INTRODUCTION

There have been a considerable body of literatures devoted to the study of perturbed stochastic differential equations(SDEs), see [1]-[7],[9], [11], [12]. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ be a filtered probability space with filtration$\{\mathcal{F}_t\}_{t\ge0}$, let $\{B_t\}_{t\ge0}$ be a one-dimensional standard $\{\mathcal{F}_t\}_{t\ge0}$-Brownian Motion. Suppose that $\sigma(x), b(x)$ are Lipschitz continuous functions on $\mathbb{R}$. It was proved in [5] that the following perturbed stochastic differential equation:

$$X_t = x + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s + \alpha \sup_{0\le s\le t} X_s, \ \ \forall\, \alpha < 1, \tag{1.1}$$

admits a unique solution. If $|\sigma(x)| > 0$, it was shown in [12] that the law of X_t is absolutely continuous with respect to Lebesgue measure, i.e. the law of X_t admits a density for $t > 0$.

There seem no results on the smoothness of the density of the law of a perturbed diffusion process. This paper aims to partly fill in this gap. The smoothness of densities is a popular topic in stochastic analysis and has been intensively studied for several decades. We refer readers to [8], [10] and references therein. Our approach to proving the smoothness of densities is by Malliavin calculus, so let us first recall some well known results on Malliavin calculus [8] to be used in this paper.

Let $\Omega = C_0(\mathbb{R}_+)$ be the space of continuous functions on $\mathbb{R}_+$ which are zero at zero. Denote by $\mathcal{F}$ the Borel σ-field on Ω and $\mathbb{P}$ the Wiener measure, then the canonical

Lihu Xu is supported by the following grants: Science and Technology Development Fund Macao S.A.R FDCT 049/2014/A1, MYRG2015-00021-FST.

coordinate process $\{\omega_t, t \in \mathbb{R}_+\}$ on Ω is a Brownian motion B_t. Define $\mathcal{F}_t^0 = \sigma(B_s, s \leq t)$ and denote by $\mathcal{F}_t$ the completion of $\mathcal{F}_t^0$ with respect to the $\mathbb{P}$-null sets of $\mathcal{F}$.

Let $H := L^2(\mathbb{R}_+, \mathcal{B}, \mu)$ where $(\mathbb{R}_+, \mathcal{B})$ is a measurable space with $\mathcal{B}$ being the Borel σ-field of $\mathbb{R}_+$ and μ being the Lebesgue measure on $\mathbb{R}_+$. We denote the norm of H by $\|.\|_H$. For any $h \in H$, $W(h)$ is defined by

$$W(h) = \int_0^\infty h(t)\mathrm{d}B_t. \tag{1.2}$$

Note that $\{W(h), h \in H\}$ is a Gaussian Process on H.

We denote by $C_p^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all of its partial derivatives have polynomial growth. Let $\mathcal{S}$ be the set of smooth random variables defined by

$$\mathcal{S} = \{F = f(W(h_1), ..., W(h_n));\ h_1, ..., h_n \in H, n \geq 1, f \in C_p^\infty(\mathbb{R}^n)\}.$$

Let $F \in \mathcal{S}$, define its Malliavin derivative D_tF by

$$D_tF = \sum_{i=1}^n \partial_i f(W(h_1), ..., W(h_n))h_i(t), \tag{1.3}$$

and its norm by

$$||F||_{1,2} = [\mathbb{E}(|F|^2) + \mathbb{E}(||DF||_H^2)]^{\frac{1}{2}},$$

where $||DF||_H^2 = \int_0^\infty |D_tF|^2\mu(\mathrm{d}t)$. Denote by $\mathbb{D}^{1,2}$ the completion of $\mathcal{S}$ under the norm $\|.\|_{1,2}$. We further define the norm

$$||F||_{m,2} = \left[\mathbb{E}(|F|^2) + \sum_{k=1}^m \mathbb{E}(\|D^kF\|_{H^{\otimes k}}^2)\right]^{\frac{1}{2}}.$$

Similarly, $\mathbb{D}^{m,2}$ denotes the completion of $\mathcal{S}$ under the norm $||.||_{m,2}$.

We shall use the following two propositions:

Proposition 1.1 (Proposition 1.2.3 of [8]). *Let $\phi : \mathbb{R}^d \to R$ be a continuously differentiable function with bounded partial derivatives. Suppose that $F = (F^1, \cdots, F^d)$ is a random vector whose components belong to the space $\mathbb{D}^{1,2}$. Then $\phi(F) \in \mathbb{D}^{1,2}$, and*

$$D(\phi(F)) = \sum_{i=1}^d \partial_i\phi(F)DF^i.$$

Proposition 1.2 (Proposition 2.1.5 of [8]). *If $F \in \mathbb{D}^{\infty,2}$ with $\mathbb{D}^{\infty,2} = \cap_{m\geq1}\mathbb{D}^{m,2}$ and $\|DF\|_H^{-1} \in \cap_{p\geq1}L^p(\Omega)$, then the density of F belongs to the space $C^\infty(\mathbb{R})$ of infinitely continuously differentiable functions.*

Throughout this paper, for a bounded measurable function f, we shall denote

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|.$$

2. Main Results

Throughout this paper, we need to assume $\alpha < 1$ to guarantee that Eq. (1.1) has a unique solution [5]. Furthermore, it is shown in [12] that

Theorem 2.1. *([12, Theorem 3.1]) Let $(X_t)_{t\geq 0}$ be the unique solution to Eq. (1.1). Then $X_t \in \mathbb{D}^{1,2}$ for all $t > 0$.*

Theorem 2.2. *([12, Theorem 3.2]) Assume that σ and b are both Lipschitz continuous, and $|\sigma(x)| > 0$ for all $x \in \mathbb{R}$. Then, for $t > 0$, the law of X_t is absolutely continuous with respect to Lebesgue measure.*

In this paper, we shall prove the following results about the smoothness of densities:

Theorem 2.3. *Assume that b is bounded smooth with $\|b'\|_\infty < \infty$ and that $\sigma(x) \equiv \sigma$. If $\alpha < 1$, $t_0 > 0$ and b satisfy*

$$\theta(t_0, \alpha, b) < 1/2,$$

with $\theta(t_0, \alpha, b) := \sqrt{2\|b'\|_\infty^2 t_0^2 + 8\alpha^2} + \|b'\|_\infty^2 t_0^2 + 4\alpha^2$, then the law of X_t in (1.1) admits a smooth density for all $t \in (0, t_0]$.

Theorem 2.4. *Assume that b is bounded smooth with $\|b'\|_\infty < \infty$ and that σ is bounded smooth with $\|\sigma'\|_\infty < \infty$, $\|\sigma''\|_\infty < \infty$ and $\inf_{x\in\mathbb{R}} |\sigma(x)| > 0$. Let*

$$F(y) = \int_x^y \frac{1}{\sigma(u)} \mathrm{d}u, \quad y \in (-\infty, \infty) \tag{2.1}$$

and $\tilde{b}(x) = \frac{b(F^{-1}(x))}{\sigma(F^{-1}(x))} - \frac{1}{2}\sigma'(F^{-1}(x))$, then $\tilde{b}$ is bounded smooth with $\|\tilde{b}'\|_\infty < \infty$. If $\alpha < 1$, $t_0 > 0$ and $\tilde{b}$ satisfy

$$\theta(t_0, \alpha, \tilde{b}) < 1/2$$

with $\theta(t_0, \alpha, \tilde{b}) := \sqrt{2\|\tilde{b}'\|_\infty^2 t_0^2 + 8\alpha^2} + \|\tilde{b}'\|_\infty^2 t_0^2 + 4\alpha^2$, then the law of X_t in (1.1) admits a smooth density for all $t \in (0, t_0]$.

Proofs of Theorems 2.3 and 2.4: The main idea is to use Proposition 1.2 to prove the two theorems. To verify the conditions in Proposition 1.2, it suffices to prove that $X_t \in D^{m,2}$ for all $m \geq 1$ and $\|DX_t\|_H \geq c > 0$ a.s. for some constant $c > 0$.

Theorem 2.3 immediately follows from Lemmas 3.1 and 3.4 below.

Now we prove Theorem 2.4. Recall $Y_t = \int_x^{X_t} \frac{1}{\sigma(u)} du$ in Lemma 3.5 below, by the condition of σ, F is a continuous and strictly increasing function with bounded derivative and thus

$$\|DY_t\|_H = \|DF(X_t)\|_H \leq \frac{1}{\inf_{x\in\mathbb{R}} |\sigma(x)|} \|DX_t\|_H. \tag{2.2}$$

Hence, by Lemmas 3.1 and 3.5 below, under the same condition as in Theorem 2.4 we have

$$\|DX_t\|_H \geq \inf_{x\in\mathbb{R}} |\sigma(x)| \cdot \|DY_t\|_H \geq \inf_{x\in\mathbb{R}} |\sigma(x)| \cdot \frac{[1 - 2\theta(t_0, \alpha, \tilde{b})]t}{2(1 + 2\|\tilde{b}'\|_\infty^2 t^2 + 2\alpha^2)} \quad t \in [0, t_0]. \tag{2.3}$$

Hence, X_t admits a smooth density for all $t \in (0, t_0]$. □

3. AUXILIARY LEMMAS

It is well known that $\|DX_t\|_H$ has the following representation [12] for all $t > 0$:

$$\|DX_t\|_H = \left(\int_0^t |D_r X_t|^2 \mathrm{d}r\right)^{\frac{1}{2}}$$

with $D_r X_t$ satisfying

$$D_r X_t = \sigma(X_r) + \int_r^t D_r b(X_s)\mathrm{d}s + \int_r^t D_r \sigma(X_s)\mathrm{d}B_s + \alpha D_r \left(\sup_{0\le s\le t} X_s\right). \tag{3.1}$$

We shall often use the following fact ([12], [8])

$$D_r X_t = 0 \quad \text{if} \;\; r > t, \tag{3.2}$$

$$\left\| D(\sup_{0\le s\le t} X_s)\right\|_H \le \sup_{0\le s\le t} \|DX_s\|_H\,, \tag{3.3}$$

where

$$\left\| D(\sup_{0\le s\le t} X_s)\right\|_H^2 = \int_0^t \left| D_r\left(\sup_{0\le s\le t} X_s\right)\right|^2 \mathrm{d}r, \quad \|DX_t\|_H^2 = \int_0^t |D_r X_t|^2 \mathrm{d}r.$$

3.1. X_t is an element in $\mathbb{D}^{m,2}$ for all $t > 0$ and $m \ge 1$.

Lemma 3.1. *Let X_t be the solution of the perturbed stochastic differential equation (1.1), and suppose that the coefficients b and σ are smooth with bounded derivatives of all orders. Then X_t belongs to $\mathbb{D}^{m,2}$ for all $t > 0$ and all $m \ge 1$.*

Proof. We shall use Picard iteration to prove the lemma. Letting $X_t^0 = x_0$ for all $t > 0$, define X_t^{n+1} be the unique, adapted solution to the following equation:

$$X_t^{n+1} = x_0 + \int_0^t \sigma(X_s^n)\mathrm{d}B_s + \int_0^t b(X_s^n)\mathrm{d}s + \alpha \max_{0\le s\le t}\left(X_s^{n+1}\right), \tag{3.4}$$

which obviously implies

$$\max_{0\le s\le t}\left(X_s^{n+1}\right) = x_0 + \max_{0\le s\le t}\left(\int_0^t \sigma(X_s^n)\mathrm{d}B_s + \int_0^t b(X_s^n)\mathrm{d}s\right) + \alpha \max_{0\le s\le t}\left(X_s^{n+1}\right).$$

Therefore,

$$\max_{0\le s\le t}\left(X_s^{n+1}\right) = \frac{x_0}{1-\alpha} + \frac{1}{1-\alpha}\max_{0\le s\le t}\left(\int_0^t \sigma(X_s^n)\mathrm{d}B_s + \int_0^t b(X_s^n)\mathrm{d}s\right),$$

this and (3.4) further gives

$$X_t^{n+1} = \frac{x_0}{1-\alpha} + \int_0^t \sigma(X_s^n)\mathrm{d}B_s + \int_0^t b(X_s^n)\mathrm{d}s + \frac{\alpha}{1-\alpha}\max_{0\le s\le t}\left(\int_0^s \sigma(X_u^n)\mathrm{d}B_u + \int_0^s b(X_u^n)\mathrm{d}u\right).$$

By the above representation of X_t^{n+1} and a standard method [5], for every $t > 0$ we have

$$\lim_{n\to\infty} X_t^n = X_t \quad \text{in } L^2(\Omega). \tag{3.5}$$

Let $m \ge 1$, it is standard to check that $X_t^n \in \mathbb{D}^{m,2}$ for every $t > 0$ and $n \ge 1$ [12, Theorem 3.1]. By a similar argument as in [12, Theorem 3.1], we have

$$\sup_{n\ge 1}\mathbb{E}\left[\|D^k X_t^n\|_{H^{\otimes k}}^2\right] < \infty, \qquad k = 1, ..., m. \tag{3.6}$$

Next we prove $X_t \in \mathbb{D}^{m,2}$ by the argument of [8, Lemma 1.2.3]. Indeed, by (3.6), there exists some subsequence $D^k X_t^{n_j}$ weakly converges to some α_k in $L^2(\Omega, H^{\otimes k})$ for $k = 1, ..., m$. By (3.5) and the remark immediately below [8, Proposition 1.2.2], the projections of $D^k X_t^{n_j}$ on any Wiener chaos converge in the weak topology of $L^2(\Omega)$, as n_j tends to infinity, to those of α_k for $k = 1, ..., m$. Hence, $X_t \in \mathbb{D}^{m,2}$ and $D^k X_t = \alpha_k$ for $k = 1, ..., m$. Moreover, for any weakly convergent subsequence the limit must be equal to $\alpha_1, ..., \alpha_m$ by the same argument as above, and this implies the weak convergence of the whole sequence. □

3.2. **Additive noise case.** If $\sigma(x) \equiv \sigma$, then Eq. (3.1) reads as

$$D_r X_t = \sigma + \int_r^t D_r b(X_s)\mathrm{d}s + \alpha D_r\left(\sup_{0\le s\le t} X_s\right). \tag{3.7}$$

Lemma 3.2. *Let $t > 0$ be arbitrary and b be bounded smooth with $\|b'\|_\infty < \infty$. For all $0 < t_1 < t_2 \le t$, we have*

$$\left|\|DX_{t_2}\|_H^2 - \|DX_{t_1}\|_H^2\right| \le 2\left[\sqrt{2\|b'\|_\infty^2(t_2-t_1)^2 + 8\alpha^2} + \|b'\|_\infty^2(t_2-t_1)^2 + 4\alpha^2\right]\sup_{0\le s\le t}\|DX_s\|_H^2.$$

Proof. It is easy to see that

$$\left|\|DX_{t_2}\|_H^2 - \|DX_{t_1}\|_H^2\right| = \left|\int_0^{t_2}(D_r X_{t_2})^2\mathrm{d}r - \int_0^{t_1}(D_r X_{t_1})^2\mathrm{d}r\right| \le I_1 + I_2,$$

where

$$I_1 := \int_{t_1}^{t_2}(D_r X_{t_2})^2\mathrm{d}r, \qquad I_2 := \int_0^{t_1}\left|(D_r X_{t_2})^2 - (D_r X_{t_1})^2\right|\mathrm{d}r.$$

We claim that

$$\int_0^{t_2}(D_rX_{t_2}-D_rX_{t_1})^2\mathrm{d}r \le 2\left[\|b'\|_\infty^2(t_2-t_1)^2+4\alpha^2\right]\sup_{0\le s\le t}\|DX_s\|_H^2 . \tag{3.8}$$

and we will prove it in the last part of this proof.

Let us now estimate I_1 and I_2 by (3.8). Observe

$$I_1=\int_{t_1}^{t_2}(D_rX_{t_2}-D_rX_{t_1})^2\mathrm{d}r \le \int_0^{t_2}(D_rX_{t_2}-D_rX_{t_1})^2\mathrm{d}r,$$

by (3.8) we have

$$I_1 \le 2\left[\|b'\|_\infty^2(t_2-t_1)^2+4\alpha^2\right]\sup_{0\le s\le t}\|DX_s\|_H^2 . \tag{3.9}$$

Further observe

$$\begin{aligned}
I_2 &\le \left[\int_0^{t_1}(D_rX_{t_2}-D_rX_{t_1})^2\mathrm{d}r\right]^{\frac12}\left[\int_0^{t_1}|D_rX_{t_2}+D_rX_{t_1}|^2\mathrm{d}r\right]^{\frac12}\\
&\le \sqrt{2}\left[\int_0^{t_1}(D_rX_{t_2}-D_rX_{t_1})^2\mathrm{d}r\right]^{\frac12}\left[\int_0^{t_1}|D_rX_{t_2}|^2+|D_rX_{t_1}|^2\mathrm{d}r\right]^{\frac12}\\
&\le \sqrt{2}\left[\int_0^{t_1}(D_rX_{t_2}-D_rX_{t_1})^2\mathrm{d}r\right]^{\frac12}\left[\int_0^{t_2}|D_rX_{t_2}|^2\mathrm{d}r+\int_0^{t_1}|D_rX_{t_1}|^2\mathrm{d}r\right]^{\frac12}\\
&\le 2\left[\int_0^{t_1}(D_rX_{t_2}-D_rX_{t_1})^2\mathrm{d}r\right]^{\frac12}\sup_{0\le s\le t}\|DX_s\|_H\\
&\le 2\left[\int_0^{t_2}(D_rX_{t_2}-D_rX_{t_1})^2\mathrm{d}r\right]^{\frac12}\sup_{0\le s\le t}\|DX_s\|_H ,
\end{aligned}$$

this inequality and (3.8) gives

$$I_2 \le 2\sqrt{2[\|b'\|_\infty^2(t_2-t_1)^2+4\alpha^2]}\sup_{0\le s\le t}\|DX_s\|_H^2 .$$

Combining the estimates of I_1 and I_2, we immediately get the desired inequality in the lemma.

It remains to prove (3.8). By (3.7), we have

$$\begin{aligned}
(D_rX_{t_2}-D_rX_{t_1})^2 &\le 2\left|\int_{t_1}^{t_2}D_rb(X_s)\mathrm{d}s\right|^2+2\alpha^2\left|D_r\left(\sup_{0\le s\le t_1}X_s\right)-D_r\left(\sup_{0\le s\le t_2}X_s\right)\right|^2\\
&\le 2\left|\int_{t_1}^{t_2}D_rb(X_s)\mathrm{d}s\right|^2+4\alpha^2\left|D_r\left(\sup_{0\le s\le t_1}X_s\right)\right|^2+4\alpha^2\left|D_r\left(\sup_{0\le s\le t_2}X_s\right)\right|^2 .
\end{aligned}$$

By Hölder inequality, (3.2) and Proposition 1.1, we have

$$\begin{aligned}\int_0^{t_2}\left|\int_{t_1}^{t_2} D_r b(X_s)\mathrm{d}s\right|^2 \mathrm{d}r &\le \|b'\|_\infty^2 \int_0^{t_2}(t_2-t_1)\int_{t_1}^{t_2}|D_rX_s|^2\mathrm{d}s\mathrm{d}r\\ &= \|b'\|_\infty^2(t_2-t_1)\int_{t_1}^{t_2}\int_0^s |D_rX_s|^2\mathrm{d}r\mathrm{d}s\\ &\le \|b'\|_\infty^2(t_2-t_1)^2\sup_{0\le s\le t}\|DX_s\|_H^2\,.\end{aligned}$$

Moreover, by (3.3) and (3.2) we have

$$\int_0^{t_2}\left|D_r\left(\sup_{0\le s\le t_2}X_s\right)\right|^2\mathrm{d}r \le \sup_{0\le s\le t_2}\|DX_s\|_H^2 \le \sup_{0\le s\le t}\|DX_s\|_H^2,$$

$$\int_0^{t_2}\left|D_r\left(\sup_{0\le s\le t_1}X_s\right)\right|^2\mathrm{d}r = \int_0^{t_1}\left|D_r\left(\sup_{0\le s\le t_1}X_s\right)\right|^2\mathrm{d}r \le \sup_{0\le s\le t}\|DX_s\|_H^2.$$

Collecting the above four inequalities, we immediately get the desired (3.8). □

Lemma 3.3. *Let b be bounded smooth with $\|b'\|_\infty<\infty$, we have*

$$\sup_{0\le s\le t}\|DX_s\|_H^2 \ge \frac{\sigma^2 t}{2(1+2\|b'\|_\infty^2 t^2+2\alpha^2)},\qquad t>0. \tag{3.10}$$

Proof. By (3.7) and using $(a+b)^2\ge\frac{1}{2}a^2-b^2$, we have

$$\begin{aligned}(D_rX_t)^2 &\ge \frac{1}{2}\sigma^2-\left[\int_r^t D_rb(X_s)\mathrm{d}s+\alpha D_r\left(\sup_{0\le s\le t}X_s\right)\right]^2\\ &\ge \frac{1}{2}\sigma^2-2\left(\int_r^t D_rb(X_s)\mathrm{d}s\right)^2-2\alpha^2\left[D_r\left(\sup_{0\le s\le t}X_s\right)\right]^2.\end{aligned}$$

Further observe

$$\begin{aligned}\int_0^t\left(\int_r^t D_rb(X_s)\mathrm{d}s\right)^2\mathrm{d}r &\le \int_0^t(t-r)\int_r^t|D_rb(X_s)|^2\mathrm{d}s\mathrm{d}r\\ &\le \int_0^t(t-r)\|b'\|_\infty^2\int_r^t|D_rX_s|^2\mathrm{d}s\mathrm{d}r\\ &\le t\|b'\|_\infty^2\int_0^t\int_r^t|D_rX_s|^2\mathrm{d}s\mathrm{d}r\\ &= t\|b'\|_\infty^2\int_0^t\|DX_s\|_H^2\mathrm{d}s\\ &\le t^2\|b'\|_\infty^2\sup_{0\le s\le t}\|DX_s\|_H^2,\end{aligned}\tag{3.11}$$

where the second inequality is by Proposition 1.1. Hence,

$$\begin{aligned}\|DX_t\|_H^2 &\geq \frac{\sigma^2 t}{2} - 2\|b'\|_\infty^2 t^2 \sup_{0\leq s\leq t} \|DX_s\|_H^2 - 2\alpha^2 \|D(\sup_{0\leq s\leq t} X_s)\|_H^2 \\ &\geq \frac{\sigma^2 t}{2} - 2\|b'\|_\infty^2 t^2 \sup_{0\leq s\leq t} \|DX_s\|_H^2 - 2\alpha^2 \sup_{0\leq s\leq t} \|DX_s\|_H^2,\end{aligned}$$

where the last inequality is by (3.3).

This clearly implies

$$\sup_{0\leq s\leq t} \|DX_s\|_H^2 \geq \frac{\sigma^2 t}{2} - 2\|b'\|_\infty^2 t^2 \sup_{0\leq s\leq t} \|DX_s\|_H^2 - 2\alpha^2 \sup_{0\leq s\leq t} \|DX_s\|_H^2,$$

which immediately yields the desired bound. □

Lemma 3.4. *Let b is bounded smooth with $\|b'\|_\infty < \infty$ and $\sigma(x) \equiv \sigma$ with $\sigma \neq 0$. If $\alpha < 1$, $t_0 > 0$ and b satisfy*

$$\theta(t_0, \alpha, b) < 1/2$$

with $\theta(r, \alpha, b) := \sqrt{2\|b'\|_\infty^2 r^2 + 8\alpha^2} + \|b'\|_\infty^2 r^2 + 4\alpha^2$ for $r > 0$, then

$$\|DX_t\|_H^2 \geq \frac{[1 - 2\theta(t_0, \alpha, b)]\sigma^2 t}{2(1 + 2\|b'\|_\infty^2 t^2 + 2\alpha^2)}, \qquad t \in [0, t_0]. \tag{3.12}$$

Proof. Let $t \in [0, t_0]$. For all $0 \leq t_1 \leq t_2 \leq t$, by Lemma 3.2, we have

$$\left| \|DX_{t_2}\|_H^2 - \|DX_{t_1}\|_H^2 \right| \leq 2\theta(t_2 - t_1, \alpha, b) \sup_{0\leq s\leq t} \|DX_s\|_H^2.$$

Hence, for all $s \in [0, t]$,

$$\begin{aligned}\|DX_s\|_H^2 &\leq \left| \|DX_s\|_H^2 - \|DX_t\|_H^2 \right| + \|DX_t\|_H^2 \\ &\leq 2\theta(t - s, \alpha, b) \sup_{0\leq s\leq t} \|DX_s\|_H^2 + \|DX_t\|_H^2,\end{aligned}$$

and consequently

$$\sup_{0\leq s\leq t} \|DX_s\|_H^2 \leq 2\theta(t, \alpha, b) \sup_{0\leq s\leq t} \|DX_s\|_H^2 + \|DX_t\|_H^2.$$

The above inequality and (3.10) further give

$$\begin{aligned}\|DX_t\|_H^2 &\geq [1 - 2\theta(t, \alpha, b)] \sup_{0\leq s\leq t} \|DX_s\|_H^2 \\ &\geq [1 - 2\theta(t_0, \alpha, b)] \sup_{0\leq s\leq t} \|DX_s\|_H^2.\end{aligned}$$

Combining the above inequality and Lemma 3.3 immediately gives the desired inequality. □

3.3. **Multiplicative noise case.** By the condition of σ, we have $\sup_{x\in\mathbb{R}}\sigma(x) < 0$ or $\inf_{x\in\mathbb{R}}\sigma(x) > 0$. Without loss of generality, we assume that

$$\inf_{x\in\mathbb{R}}\sigma(x) > 0.$$

Let us consider the following well known transform

$$F(X_t) = \int_x^{X_t} \frac{1}{\sigma(u)} \mathrm{d}u, \tag{3.13}$$

it is easy to see that F is a strictly increasing function with bounded derivative. Hence,

$$\sup_{0\le s\le t} F(X_s) = F\left(\sup_{0\le s\le t} X_s\right). \tag{3.14}$$

By Itô formula, we have

$$F(X_t) = \int_0^t \left(\frac{b(X_s)}{\sigma(X_s)} - \frac{1}{2}\sigma'(X_s)\right) \mathrm{d}s + B_t + \alpha \int_0^t \frac{1}{\sigma(X_s)} \mathrm{d}M_s \tag{3.15}$$

where $M_t = \sup_{0\le s\le t} X_s$. It is easy to see that M_t is an increasing function of t and that $\frac{1}{\sigma(X_s)}$ has a contribution to the related integral only when $X_s = M_s$. Hence,

$$F(X_t) = \int_0^t \left(\frac{b(X_s)}{\sigma(X_s)} - \frac{1}{2}\sigma'(X_s)\right) \mathrm{d}s + B_t + \alpha \int_0^t \frac{1}{\sigma(M_s)} \mathrm{d}M_s. \tag{3.16}$$

Since M_t is a continuous increasing function with respect to t, we have

$$F(X_t) = \int_0^t \left(\frac{b(X_s)}{\sigma(X_s)} - \frac{1}{2}\sigma'(X_s)\right) \mathrm{d}s + B_t + \alpha \int_0^{M_t} \frac{1}{\sigma(u)} \mathrm{d}u. \tag{3.17}$$

By (3.14),

$$F(X_t) = \int_0^t \left(\frac{b(X_s)}{\sigma(X_s)} - \frac{1}{2}\sigma'(X_s)\right) \mathrm{d}s + B_t + \alpha \sup_{0\le s\le t} F(X_s). \tag{3.18}$$

Denote $Y_t = F(X_t)$, it solves the following perturbed SDE:

$$Y_t = \int_0^t \tilde{b}(Y_s)\mathrm{d}s + B_t + \alpha \sup_{0\le s\le t} Y_s \tag{3.19}$$

where $\tilde{b}(x) = \frac{b(F^{-1}(x))}{\sigma(F^{-1}(x))} - \frac{1}{2}\sigma'(F^{-1}(x))$. Applying Lemma 3.4, we get the following lemma about the dynamics Y_t:

Lemma 3.5. *Assume that b is bounded smooth and that σ is bounded smooth with $\|\sigma'\|_\infty < \infty$, $\|\sigma''\|_\infty < \infty$ and $\inf_{x\ge 0}|\sigma(x)| > 0$. Then $\tilde{b}$ is bounded smooth. If $\alpha < 1$, $t_0 > 0$ and b satisfy*

$$\theta(t_0, \alpha, \tilde{b}) < 1/2$$

with $\theta(r,\alpha,\tilde{b}) := \sqrt{2\|\tilde{b}'\|_\infty^2 r^2 + 8\alpha^2} + \|\tilde{b}'\|_\infty^2 r^2 + 4\alpha^2$ *for* $r > 0$*, then*

$$\|DY_t\|_H^2 \geq \frac{[1 - 2\theta(t_0,\alpha,\tilde{b})]t}{2(1 + 2\|\tilde{b}'\|_\infty^2 t^2 + 2\alpha^2)}, \qquad t \in (0, t_0]. \tag{3.20}$$

Proof. It is easy to check that under the conditions in the lemma $\tilde{b}$ is bounded smooth with $\|\tilde{b}'\|_\infty < \infty$. Hence, the lemma immediately follows from applying Lemma 3.4 to Y_t. □

REFERENCES

1. Ph. Carmona, F. Petit, M. Yor, Beta variables as times spent in $[0,\infty)$ by certain perturbed Brownian motions, J. London Math. Soc. 58 (1998), 239-256.
2. L. Chaumont, R.A. Doney, Pathwise uniqueness for perturbed versions of Brownian motion and reflected Brownian motion, Probab. Theory Related Fields 113 (1999), 519-534.
3. B. Davis, Brownian motion and random walk perturbed at extrema, Probab. Theory Related Fields 113(1999)501-518.
4. R.A. Doney, Some calculations for perturbed Brownian motion. Séminaire de Probabilités, XXXII, 231-236, Lecture Notes in Math., 1686, Springer, Berlin, 1998.
5. R.A. Doney, T.S. Zhang, Perturbed Skorohod equations and perturbed reflected diffusion processes, Ann. Inst. H. Poincaré Probab. Statist. 41 (2005), no. 1, 107-121.
6. J.F. Le Gall, M. Yor, Excursions browniennes et carrés de processus de Bessel, C.R. Acad. Sci. Paris Sér. I 303(1986), 73-76.
7. J.F. Le Gall, M. Yor, Enlacements du mouvement brownien autour des courbes de l'espace, Trans. Amer. Math. Soc. 317 (1990), 687-722.
8. D. Nualart, The Malliavin calculus and related topics. Second edition. Probability and its Applications (New York). Springer-Verlag, Berlin, 2006. xiv+382 pp.
9. M. Perman, W. Werner, Perturbed Brownian motions, Probab. Theory Related Fields 108 (1997), no. 3, 357-383.
10. M. Sanz-Sole, Malliavin calculus. With applications to stochastic partial differential equations. Fundamental Sciences. EPFL Press, Lausanne; distributed by CRC Press, Boca Raton, FL, 2005. viii+162 pp.
11. W. Werner, Some remarks on perturbed reflecting Brownian motion. Séminaire de Probabilités, XXIX, 37-43, Lecture Notes in Math., 1613, Springer, Berlin, 1995.
12. W. Yue, T.S. Zhang, Absolute continuity of the laws of perturbed diffusion processes and perburbed reflected diffusion processes, J. Theoret. Probab. 28 (2015), no. 2, 587-618.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, UNIVERSITY OF MACAU, E11 AVENIDA DA UNIVERSIDADE, TAIPA, MACAU, CHINA
E-mail address: lihuxu@umac.mo

INSTITUTE OF ANALYSIS AND SCIENTIFIC COMPUTING, VIENNA UNIVERSITY OF TECHNOLOGY, WIEDNER HAUPTSTRASS 8-10, 1040 WIEN, AUSTRIA
E-mail address: wenyue@hotmail.co.uk

SCHOOL OF MATHEMATICS,UNIVERSITY OF MANCHESTER OXFORD ROAD, MANCHESTER M13 9PL, UNITED KINGDOM
E-mail address: tusheng.zhang@manchester.ac.uk