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## Robust consensus control of uncertain multi-agent systems with input delay: a model reduction method<sup>†</sup>

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### SUMMARY

This paper addresses the robust consensus control design for input-delayed multi-agent systems subject to parametric uncertainties. To deal with the input delay, the Artstein model reduction method is employed by a state transformation. The input-dependent integral term that remains in the transformed system, due to the model uncertainties, is judiciously analysed. By carefully exploring certain features of the Laplacian matrix, sufficient conditions for the global consensus under directed communication topology are identified using Lyapunov-Krasovskii functionals in the time domain. The proposed control only relies on relative state information of the subsystems via network connections. The effectiveness and robustness of the proposed control design is demonstrated through a numerical simulation example. Copyright © 2015 John Wiley & Sons, Ltd.

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**KEY WORDS:** Consensus Control; Input Delay; Multi-agent Systems; Parametric Uncertainty; Reduction Method

### 1. INTRODUCTION

Control of network-connected systems, including formation control [1], flocking [2] and synchronization [3], has attracted significant interests in control area and has been intensively studied in the literature, [4]–[9], to mention a few. In some applications involving multi-agent systems, groups of agents are required to reach an agreement upon certain quantities of interest, which is called consensus or agreement problem. One significant advance in consensus control is to use tools in graph theory, especially the use of Laplacian matrices [10], to characterize the network connection.

Time delays are inevitable in industrial processes due to the time taken for transmission of signals. The importance of addressing delay has been well recognized for a long time (see [11] and the references therein). Time delay is even more significant in network-connected systems as the control inputs depend on the signals transmitted through network communication. The presence of communication delays, if not considered in a controller design, may seriously deteriorate the performance of the multi-agent networks. This problem has attracted the considerate attention in control community, as evidenced by recent publications [12]–[16]. In [17], it is pointed out that the delay in the network communication can also be viewed as input delay in consensus control.

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The practical physical systems often suffer from uncertainties which may be caused by mutations in system parameters, modelling errors or some ignored factors [18]. The robust consensus problem of multi-agent systems has formed into a challenge topic in recent years. Han et al. investigate the robust consensus problem for multi-agent systems with continuous-time and discrete-time dynamics in [19] and [20], where the weighted adjacency matrix is a polynomial function of uncertain parameters. In particular, the  $H_\infty$  robust control problem is investigated in [21] for a group of autonomous agents governed by uncertain general linear node dynamics. However, most of the existing results on consensus control of uncertain multi-agent systems were often restricted to certain conditions, like single or double integrators [22], undirected network connections [23] or systems without delays [24]. The difficulty of solving the consensus problem with general uncertain model dynamics with time delay is mainly due to the fact that the systems can not be completely transformed to a delay-free one due to the existence of uncertainties. The nature of infinite-dimensionality of delay issues prevents the direct application of many well-know existing conventional control design tools. Further analysis is needed to tackle the influence of the extra integral terms under the transformations, including the ones for parametric uncertainty and input delay.

This paper systematically investigate the consensus control for general linear multi-agent systems with parametric uncertainties and communication delay. This kind of network communication delay can be formulated as the input delay when the inputs only depend on the relative state information transmitted via the network. A model reduction method, which was originally introduced by Artstein in [26], is used to deal with the input delay. The control design is only based on the neighbours' information obtained via the network connections, without local feedback of the subsystems. Further endeavors are made to ensure that the extra integral term, which remains in the system dynamics after transformation due to the parametric uncertainty and depends on the relative state due to the input, is properly considered. By transforming the Laplacian matrix into the real Jordan form, the consensus analysis is put in the framework of Lyapunov-Krasovskii functionals in real domain. A simulation example is included in the end of the paper.

The remainder of this paper is organised as follows. Some notations and the problem formulation are given in Section 2. Section 3 presents a couple of preliminary results for the stability analysis. Section 4 presents the main results on the consensus control design and stability analysis. Simulation results are included in Section 5. Section 6 concludes this paper.

## 2. PROBLEM STATEMENT

In this paper, we consider the control design for a set of  $N$  uncertain subsystems with input delay, of which the subsystems are described by

$$\dot{x}_i(t) = [A + \Delta A(t)]x_i(t) + [B + \Delta B(t)]u_i(t - h), \quad (1)$$

where, for subsystem  $i$ ,  $i = 1, 2, \dots, N$ ,  $x_i \in \mathbb{R}^n$  is the state vector,  $u_i \in \mathbb{R}^m$  is the control input vector,  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$  are constant matrices with  $(A, B)$  controllable,  $h > 0$  is the input delay,  $x_i(\theta)$ ,  $\theta \in [-h, 0]$ , are given and bounded,  $\Delta A(t)$  and  $\Delta B(t)$  are time-varying uncertain matrices which can be formulated in the form [27] as

$$\Delta A(t) = E\Sigma(t)F_1 \quad \text{and} \quad \Delta B(t) = E\Sigma(t)F_2, \quad (2)$$

where  $E$ ,  $F_1$  and  $F_2$  are real constant matrices with appropriate dimensions, and  $\Sigma(t)$  is an unknown real time-varying matrix that satisfies  $\Sigma^T(t)\Sigma(t) \leq I$ .

### Remark 1

It is worth noticing that the subsystems in the network are nominally identical and the model uncertainty matrices satisfy the same form as (2). Different from the existing works that focus on the identical agents in the network, the terms  $\Delta A$  and  $\Delta B$  in (1) allow the subsystems to have different dynamics and the uncertainty is characterised by the time-varying matrix  $\Sigma(t)$ , which implies that

each subsystem in the group can be non-identical. For the consensus design, only the bound of  $\Sigma(t)$  (i.e., the worst case) is needed.

*Assumption 1*

All subsystems in the network have known and identical input delays.

*Remark 2*

Assumption 1 is adopted for the convenience of illustration of the control design. The proposed method in this paper may be extended to the network-connected systems with non-identical constant delays and even unknown time-varying delays if the worst case is taken into account in the design.

The information flow among the subsystems is specified by a directed graph  $\mathcal{G}$ , which consists of a set of vertices denoted by  $\mathcal{V}$  and a set of edges denoted by  $\mathcal{E}$ . A vertex represents a subsystem, and each edge represents a connection. Associated with the graph is its adjacency matrix  $Q$ , where element  $q_{ij}$  denotes the connection among the subsystems. More specifically,  $q_{ij} = 1$  if there is a connection from subsystem  $j$  to subsystem  $i$ , and  $q_{ij} = 0$  otherwise. The Laplacian matrix  $L = \{l_{ij}\}$  is commonly defined by

$$l_{ij} = -q_{ij}, \text{ if } j \neq i,$$

$$l_{ii} = \sum_{j=1, j \neq i}^N q_{ij}.$$

From the definition of the Laplacian matrix, it is clear that

$$L\mathbf{1} = 0,$$

where  $\mathbf{1} = [1, \dots, 1]^T \in \mathbb{R}^N$ , which implies that the Laplacian matrix has 0 as an eigenvalue associated with the right eigenvector  $\mathbf{1}$ .

*Assumption 2*

0 is a simple eigenvalue of the Laplacian matrix.

In terms of the network collections, this condition implies that the network has a spanning tree to connect any two subsystems in the system. For consensus design, we only need that the eigenvalue at 0 is a single eigenvalue [28].

The consensus control problem considered in this paper is to design a control strategy, using the relative state information, to ensure that all input-delayed uncertain subsystems converge to an identical trajectory.

### 3. PRELIMINARY RESULTS

In this section, we present a couple of preliminary results which are useful for the stability analysis. We first present an overview of the Artstein model reduction method [25, 26]. Consider an input-delayed system

$$\dot{x}(t) = Ax(t) + Bu(t - h). \quad (3)$$

with  $x(\theta)$ ,  $\theta \in [-h, 0]$ , being given and bounded.

Introducing a new variable

$$z(t) = x(t) + \int_t^{t+h} e^{A(t-\tau)} Bu(\tau - h) d\tau \quad (4)$$



with  $I_2$  the identity matrix in  $\mathbb{R}^{2 \times 2}$  and

$$\nu(\alpha_k, \beta_k) = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}_{2 \times 2}.$$

We also need the following lemma from [29].

*Lemma 2*

For a positive definite matrix  $P$ , and a function  $x : [a, b] \rightarrow \mathbb{R}^n$ , with  $a, b \in \mathbb{R}$  and  $b > a$ , the following inequality holds

$$\left( \int_a^b x^T(\tau) d\tau \right) P \left( \int_a^b x(\tau) d\tau \right) \leq (b-a) \int_a^b x^T(\tau) P x(\tau) d\tau. \quad (9)$$

#### 4. CONSENSUS CONTROL

For the multi-agent systems (1), we consider the linear transformation (4) by the reduction method. The original subsystems are transformed to

$$\dot{z}_i(t) = (A + \Delta A)z_i(t) + Du_i(t) + \Delta Bu_i(t-h) - \Delta A \int_t^{t+h} e^{A(t-\tau)} Bu_i(\tau-h) d\tau, \quad (10)$$

where  $D = e^{-Ah}B$ . As seen in (10), system (1) is not completely reduced to a free-of-delay system due to the model uncertainties.

We propose a control design using the relative state information. The control input takes the structure

$$u_i(t) = -K \sum_{j=1}^N q_{ij} [z_i(t) - z_j(t)] = -K \sum_{j=1}^N l_{ij} z_j(t), \quad (11)$$

where  $K \in \mathbb{R}^{m \times n}$  is a constant control gain matrix to be designed later.

*Remark 4*

It is worth noting from (4) that the proposed control in (11) only uses the relative state information of the subsystems via network connections.

*Remark 5*

Note that the information on each control input  $u_i(t)$  on the time interval  $[t-h, t]$  can be stored and used for control. In practical implementations, the discretization of an integral or some numerical quadrature method [31] can be used to approximate the integral term in the control input  $u_i(t)$ .

*Remark 6*

For unknown non-identical time-varying delays  $h_i(t)$  in (1) satisfying  $0 < h_i(t) < \bar{h}$ ,  $i = 1, 2, \dots, N$ , the upper bound of the delays that exist in all agents can be used in the predictor variable (4) as

$$z_i(t) = x_i(t) + \int_t^{t+\bar{h}} e^{A(t-\tau)} Bu_i(\tau-\bar{h}) d\tau. \quad (12)$$

Then, the original subsystems (1) can be transformed to

$$\begin{aligned} \dot{z}_i(t) &= (A + \Delta A)z_i(t) + e^{-A\bar{h}}Bu_i(t) - \Delta A \int_t^{t+\bar{h}} e^{A(t-\tau)} Bu_i(\tau-\bar{h}) d\tau + \Delta Bu_i(t-h_i(t)) \\ &\quad + B(u_i(t-h_i(t)) - u_i(t-\bar{h})) \\ &= (A + \Delta A)z_i(t) + \bar{D}u_i(t) - \Delta A \int_t^{t+\bar{h}} e^{A(t-\tau)} Bu_i(\tau-\bar{h}) d\tau + \Delta Bu_i(t-\bar{h}) \\ &\quad - BK \sum_{j=1}^N l_{ij} \int_{t-\bar{h}}^{t-h_i(t)} \dot{z}_j(\tau) d\tau - \Delta BK \sum_{j=1}^N l_{ij} \int_{t-\bar{h}}^{t-h_i(t)} \dot{z}_j(\tau) d\tau, \end{aligned} \quad (13)$$

where  $\bar{D} = e^{-A\bar{h}}B$ . Compared with (10), the use of the upper-bound of delays leads to additional integral terms in (13), which further complicates the design. Fortunately, with the same control structure (11), our proposed design method is still applicable if  $K$  is carefully identified by introducing additional Krasovskii functionals in consensus analysis. Similar technique can be found in [32] for a single system with unknown time-varying delay but without parametric uncertainties.

Let  $z(t) = [z_1, z_2, \dots, z_N]^T$ , and the closed-loop system is then written as

$$\dot{z}(t) = [I_N \otimes (A + \Delta A) - L \otimes DK]z(t) - (L \otimes \Delta BK)z(t-h) - (I_N \otimes \Delta A)\sigma(t), \quad (14)$$

where  $\sigma \triangleq [\sigma_1, \dots, \sigma_N]^T$  with the elements defined by

$$\sigma_i = - \int_t^{t+h} e^{A(t-\tau)} BK \sum_{j=1}^N l_{ij} z_j(\tau-h) d\tau, \quad (15)$$

and  $\otimes$  denotes the Kronecker product of matrices.

Let us define  $r \in \mathbb{R}^N$  as the left eigenvector of  $L$  corresponding to the eigenvalue at 0, that is,  $r^T L = 0$ . Furthermore, let  $r$  be scaled such that  $r^T \mathbf{1} = 1$ . It can be shown from Assumption 2 and Lemma 1 that there exists a non-singular matrix  $T$  with its first column being  $T_{(1)} = \mathbf{1}$  and the first row of  $T^{-1}$  being  $T_{(1)}^{-1} = r^T$ , such that

$$T^{-1}LT = J. \quad (16)$$

Based on the vector  $r$ , we introduce a state transformation

$$\xi_i(t) = z_i(t) - \sum_{j=1}^N r_j z_j(t) \quad (17)$$

for  $i = 1, \dots, N$ . With

$$\xi(t) = [\xi_1^T(t), \xi_2^T(t), \dots, \xi_N^T(t)]^T,$$

we have

$$\xi(t) = z(t) - [(\mathbf{1}r^T) \otimes I_n]z(t) = (M \otimes I_n)z(t),$$

where  $M \triangleq I_N - \mathbf{1}r^T$ . Since  $r^T \mathbf{1} = 1$ , it can be shown that  $M\mathbf{1} = 0$ . Therefore, the consensus of system (14) is achieved when  $\xi = 0$ , as  $\xi = 0$  implies  $z_1 = z_2 = \dots = z_N$ , due to the fact the null space of  $M$  is  $\text{span}\{\mathbf{1}\}$ . Then, the consensus problem is now converted to the stabilization problem.

The dynamics of  $\xi(t)$  can then be obtained as

$$\begin{aligned} \dot{\xi}(t) &= (M \otimes I_n)\dot{z}(t) \\ &= [I_N \otimes (A + \Delta A) - L \otimes DK]\xi(t) - (L \otimes \Delta BK)\xi(t-h) - (M \otimes I_n)(I_N \otimes \Delta A)\sigma. \end{aligned} \quad (18)$$

To explore the structure of  $L$  for stability analysis, let us introduce another state transformation

$$\eta(t) = (T^{-1} \otimes I_n)\xi(t). \quad (19)$$

Then, we have

$$\dot{\eta}(t) = [I_N \otimes (A + \Delta A) - J \otimes DK]\eta(t) - (J \otimes \Delta BK)\eta(t-h) - \Psi(z), \quad (20)$$

where  $\Psi(z) = (T^{-1} \otimes I_n)(M \otimes I_n)(I_N \otimes \Delta A)\sigma$ ,  $\eta(t) = [\eta_1(t), \eta_2(t), \dots, \eta_N(t)]^T$  and  $\Psi(z) = [\psi_1(z), \psi_2(z), \dots, \psi_N(z)]^T$  with  $\eta_i \in \mathbb{R}^n$  and  $\psi_i : \mathbb{R}^{nN} \rightarrow \mathbb{R}^n$ , for  $i = 1, 2, \dots, N$ .

With the state transformations (17) and (19), we have

$$\eta_1(t) = (r^T \otimes I_n)\xi(t) = [(r^T M) \otimes I_n]z(t) \equiv 0. \quad (21)$$

The consensus control can be guaranteed by showing that  $\eta$  converges to zero, which is sufficed by showing that  $\eta_i$  converge to zero for  $k = 2, 3, \dots, N$ , since we have shown that  $\eta_1(t) \equiv 0$ .

With the control law shown in (11), we design the control gain matrix  $K$  as

$$K = D^T P, \quad (22)$$

where  $P$  is a positive definite matrix to satisfy certain condition. In the remaining part of the paper, we will use Lyapunov-function-based analysis to identify conditions for  $P$  to ensure that the consensus control objective is achieved by using the control input (11) with the control gain (22).

The stability analysis will be carried out for  $\eta$ . Based on the structure of the Laplacian matrix shown in (8), we can see that

$$N_k = n_1 + \sum_{j=2}^k n_j, \quad (23)$$

with  $n_1 = 1$  and  $N_q = N$ , where  $k = 2, 3, \dots, q$ .

Due to the eigenstructure (8), the dynamics of the transformed state  $\eta$  in (20) will be discussed corresponding to the real and the complex eigenvalues, respectively.

For the state variables associated with the Jordan blocks  $J_k$  of real eigenvalues, i.e., for  $2 \leq k \leq p$ , we have the dynamics

$$\begin{aligned} \dot{\eta}_i(t) = & (A + \Delta A - \lambda_i D D^T P) \eta_i(t) - D D^T P \eta_{i+1}(t) - \lambda_i \Delta B D^T P \eta_i(t-h) \\ & - \Delta B D^T P \eta_{i+1}(t-h) - \psi_i(z), \quad i = N_{k-1} + 1, N_{k-1} + 2, \dots, N_k - 1, \end{aligned} \quad (24)$$

$$\dot{\eta}_i(t) = (A + \Delta A - \lambda_i D D^T P) \eta_i(t) - \lambda_i \Delta B D^T P \eta_i(t-h) - \psi_i(z), \quad i = N_k. \quad (25)$$

For the state variables associated with the Jordan blocks  $J_k$  of conjugate complex eigenvalues, i.e., for  $k > p$ , we consider the dynamics of the state variables in pairs. For notational convenience, let

$$i_1(j) = N_{k-1} + 2j - 1 \quad \text{and} \quad i_2(j) = N_{k-1} + 2j, \quad (26)$$

where  $j = 1, 2, \dots, n_k/2$ . The dynamics of  $\eta_{i_1(j)}$  and  $\eta_{i_2(j)}$ , for  $j = 1, 2, \dots, n_k/2 - 1$ , are expressed by

$$\begin{aligned} \dot{\eta}_{i_1(j)}(t) = & (A + \Delta A - \alpha_k D D^T P) \eta_{i_1(j)}(t) - \beta_k D D^T P \eta_{i_2(j)}(t) - D D^T P \eta_{i_1(j)+2}(t) \\ & - \alpha_k \Delta B D^T P \eta_{i_1(j)}(t-h) - \beta_k \Delta B D^T P \eta_{i_2(j)}(t-h) - \Delta B D^T P \eta_{i_1(j)+2}(t-h) \\ & - \psi_{i_1(j)}(z), \end{aligned}$$

$$\begin{aligned} \dot{\eta}_{i_2(j)}(t) = & (A + \Delta A - \alpha_k D D^T P) \eta_{i_2(j)}(t) + \beta_k D D^T P \eta_{i_1(j)}(t) - D D^T P \eta_{i_2(j)+2}(t) \\ & - \alpha_k \Delta B D^T P \eta_{i_2(j)}(t-h) + \beta_k \Delta B D^T P \eta_{i_1(j)}(t-h) - \Delta B D^T P \eta_{i_2(j)+2}(t-h) \\ & - \psi_{i_2(j)}(z), \end{aligned}$$

and, for  $j = n_k/2$ ,

$$\begin{aligned} \dot{\eta}_{i_1(j)} = & (A + \Delta A - \alpha_k D D^T P) \eta_{i_1(j)}(t) - \beta_k D D^T P \eta_{i_2(j)}(t) \\ & - \alpha_k \Delta B D^T P \eta_{i_1(j)}(t-h) - \beta_k \Delta B D^T P \eta_{i_2(j)}(t-h) - \psi_{i_1(j)}(z), \\ \dot{\eta}_{i_2(j)} = & (A + \Delta A - \alpha_k D D^T P) \eta_{i_2(j)}(t) + \beta_k D D^T P \eta_{i_1(j)}(t) \\ & - \alpha_k \Delta B D^T P \eta_{i_2(j)}(t-h) + \beta_k \Delta B D^T P \eta_{i_1(j)}(t-h) - \psi_{i_2(j)}(z). \end{aligned}$$

Let

$$W_i = \eta_i^T(t) P \eta_i(t). \quad (27)$$



For  $i = N_{k-1} + 1, N_{k-1} + 2, \dots, N_k - 1$ , the time derivative of  $W_i$  along the trajectory (24) is

$$\begin{aligned}
\dot{W}_i &= \eta_i^T(t) P \dot{\eta}_i(t) \\
&= \eta_i^T(t) (A^T P + PA - 2\lambda_k P D D^T P) \eta_i(t) + 2\eta_i^T(t) P \Delta A \eta_i(t) - 2\eta_i^T(t) P D D^T P \eta_{i+1}(t) \\
&\quad - 2\lambda_k \eta_i^T(t) P \Delta B D^T P \eta_i(t-h) - 2\eta_i^T(t) P \Delta B D^T P \eta_{i+1}(t-h) - 2\eta_i^T(t) P \psi_i \\
&\leq \eta_i^T(t) (A^T P + PA - 2\lambda_k P D D^T P) \eta_i(t) + \frac{1}{\mu} \eta_i^T(t) P E E^T P \eta_i(t) + \mu \eta_i^T(t) F_1^T F_1 \eta_i(t) \\
&\quad + \eta_i^T(t) P D D^T P \eta_i(t) + \eta_{i+1}^T(t) P D D^T P \eta_{i+1}(t) + \frac{1}{\epsilon} \lambda_k \eta_i^T(t) P E E^T P \eta_i(t) \\
&\quad + \epsilon \lambda_k \eta_i^T(t-h) P D F_2^T F_2 D^T P \eta_i(t-h) + \frac{1}{\epsilon} \eta_i^T(t) P E E^T P \eta_i(t) \\
&\quad + \epsilon \eta_{i+1}^T(t-h) P D F_2^T F_2 D^T P \eta_{i+1}(t-h) - 2\eta_i^T(t) P \psi_i \\
&= \eta_i^T(t) \left[ A^T P + PA - (2\lambda_k - 1) P D D^T P + \left( \frac{\lambda_k + 1}{\epsilon} + \frac{1}{\mu} \right) P E E^T P + \mu F_1^T F_1 \right] \eta_i(t) \\
&\quad + \eta_{i+1}^T(t) P D D^T P \eta_{i+1}(t) + \epsilon(1 + \lambda_k) \eta_{i+1}^T(t-h) P D F_2^T F_2 D^T P \eta_{i+1}(t-h) \\
&\quad - 2\eta_i^T(t) P \psi_i, \tag{28}
\end{aligned}$$

and, for  $i = N_k$ , the time derivative of  $W_i$  along the trajectory (25) is

$$\begin{aligned}
\dot{W}_i &= \eta_i^T(t) P \dot{\eta}_i(t) \\
&= \eta_i^T(t) (A^T P + PA - 2\lambda_k P D D^T P) \eta_i(t) + 2\eta_i^T(t) P \Delta A \eta_i(t) - 2\lambda_k \eta_i^T(t) P \Delta B D^T P \eta_i(t-h) \\
&\quad - 2\eta_i^T(t) P \psi_i \\
&\leq \eta_i^T(t) (A^T P + PA - 2\lambda_k P D D^T P) \eta_i(t) + \frac{1}{\mu} \eta_i^T(t) P E E^T P \eta_i(t) + \mu \eta_i^T(t) F_1^T F_1 \eta_i(t) \\
&\quad + \frac{1}{\epsilon} \lambda_k \eta_i^T(t) P E E^T P \eta_i(t) + \epsilon \lambda_k \eta_i^T(t-h) P D F_2^T F_2 D^T P \eta_i(t-h) - 2\eta_i^T(t) P \psi_i \\
&= \eta_i^T(t) \left[ A^T P + PA - 2\lambda_k P D D^T P + \left( \frac{\lambda_k}{\epsilon} + \frac{1}{\mu} \right) P E E^T P + \mu F_1^T F_1 \right] \eta_i(t) \\
&\quad + \epsilon \lambda_k \eta_i^T(t-h) P D F_2^T F_2 D^T P \eta_i(t-h) - 2\eta_i^T(t) P \psi_i, \tag{29}
\end{aligned}$$

where the inequality  $\pm a^T b \leq a^T a + b^T b$ , for any vectors  $a$  and  $b$ , has been used.

Similarly, for  $j = n_k/2$  in (26)<sup>†</sup>, we have in pairs

$$\begin{aligned}
&\dot{W}_{i_1} + \dot{W}_{i_2} \\
&\leq \eta_{i_1}^T(t) \left[ A^T P + PA - 2\alpha_k P D D^T P + \left( \frac{\alpha_k + \beta_k}{\epsilon} + \frac{1}{\mu} \right) P E E^T P + \mu F_1^T F_1 \right] \eta_{i_1}(t) \\
&\quad + \eta_{i_2}^T(t) \left[ A^T P + PA - 2\alpha_k P D D^T P + \left( \frac{\alpha_k + \beta_k}{\epsilon} + \frac{1}{\mu} \right) P E E^T P + \mu F_1^T F_1 \right] \eta_{i_2}(t) \\
&\quad + \epsilon(\alpha_k + \beta_k) \eta_{i_1}^T(t-h) P D F_2^T F_2 D^T P \eta_{i_1}(t-h) - 2\eta_{i_1}^T(t) P \psi_{i_1} \\
&\quad + \epsilon(\alpha_k + \beta_k) \eta_{i_2}^T(t-h) P D F_2^T F_2 D^T P \eta_{i_2}(t-h) - 2\eta_{i_2}^T(t) P \psi_{i_2}, \tag{30}
\end{aligned}$$

<sup>†</sup>We omit without ambiguity the parameter  $j$  in  $i_1(j)$  and  $i_2(j)$  for simplicity.

and, for  $j = 1, 2, \dots, n_k/2 - 1$ ,

$$\begin{aligned}
 & \dot{W}_{i_1} + \dot{W}_{i_2} \\
 & \leq \eta_{i_1}^T(t) \left[ A^T P + PA - 2(\alpha_k - 1)PDD^T P + \left( \frac{\alpha_k + \beta_k + 1}{\epsilon} + \frac{1}{\mu} \right) PEE^T P + \mu F_1^T F_1 \right] \eta_{i_1}(t) \\
 & + \eta_{i_2}^T(t) \left[ A^T P + PA - 2(\alpha_k - 1)PDD^T P + \left( \frac{\alpha_k + \beta_k + 1}{\epsilon} + \frac{1}{\mu} \right) PEE^T P + \mu F_1^T F_1 \right] \eta_{i_2}(t) \\
 & + \epsilon(\alpha_k + \beta_k)\eta_{i_1}^T(t-h)PDF_2^T F_2 D^T P\eta_{i_1}(t-h) + \eta_{i_1+2}^T(t)PDD^T P\eta_{i_1+2}(t) - 2\eta_{i_1}(t)^T P\psi_{i_1} \\
 & + \epsilon(\alpha_k + \beta_k)\eta_{i_2}^T(t-h)PDF_2^T F_2 D^T P\eta_{i_2}(t-h) + \eta_{i_2+2}^T(t)PDD^T P\eta_{i_2+2}(t) - 2\eta_{i_2}(t)^T P\psi_{i_2} \\
 & + \epsilon\eta_{i_1+2}^T(t-h)PDF_2^T F_2 D^T P\eta_{i_1+2}(t-h) + \epsilon\eta_{i_2+2}^T(t-h)PDF_2^T F_2 D^T P\eta_{i_2+2}(t-h). \quad (31)
 \end{aligned}$$

where the inequality  $\pm a^T b \leq a^T a + b^T b$ , for any vectors  $a$  and  $b$ , has been used again.

The above inequalities will be used in the consensus analysis. However, we note that the extra integral term  $\psi_i(z)$  in the transformed system dynamic model (20) is expressed as a function of the state  $z$ . For the consensus analysis within the framework of Lyapunov-Krasovskii functionals, we need to establish a bound of the integral function  $-2\eta_i^T P\psi_i$  in terms of the transformed state  $\eta$ . The following lemma establishes a bound for the cross term  $-2\eta_i^T P\psi_i$  with respect to the transformed state  $\eta$ .

*Lemma 3*

For the integral term  $\Psi(z) = [\psi_1(z), \psi_2(z), \dots, \psi_N(z)]^T$  in (20), the summation of  $-2\eta_i^T P\psi_i$  is bounded by

$$\begin{aligned}
 -\sum_{i=1}^N 2\eta_i^T P\psi_i & \leq \frac{2}{\rho} \sum_{i=2}^N \eta_i^T(t) PEE^T P\eta_i(t) \\
 & + \rho\gamma_0^2 \sum_{i=2}^N h \int_0^h \eta_i^T(t-\tau) PDD^T e^{A^T \tau} F_1^T F_1 e^{A\tau} DD^T P\eta_i(t-\tau) d\tau, \quad (32)
 \end{aligned}$$

where  $\gamma_0^2 = 2\|T^{-1}\|_F^2(1 + N\|r\|_2^2)\|Q\|_F^2\|T\|_F^2$ .

*Proof*

See Appendix A. □

With the bound derived in Lemma 3, sufficient conditions can be identified respectively for the cases of the Laplacian matrix with distinct eigenvalues and multiple eigenvalues to guarantee the consensus. The following theorem summarises the results.

*Theorem 1*

Consider multi-agent systems (1) with Assumptions 1 and 2. The consensus control problem of system (1) can be solved by the control design (11) with the control gain  $K = D^T P$ , if there exist matrices  $X = P^{-1} > 0$ ,  $Y > 0$  and scalars  $\mu > 0$ ,  $\epsilon > 0$ ,  $\rho > 0$ , such that

$$\begin{bmatrix} Y & DF_2^T \\ F_2 D^T & \frac{1}{\epsilon} I \end{bmatrix} > 0, \quad (33)$$

$$\begin{bmatrix} U & XF_1^T & DD^T \\ F_1 X & -\frac{1}{\mu} I & 0 \\ DD^T & 0 & -\frac{1}{\rho\gamma_0^2} W \end{bmatrix} < 0, \quad (34)$$

where  $U$  is specified in one of the following two cases:

(i) If the eigenvalues of the Laplacian matrix  $L$  are distinct,

$$U = XA^T + AX - 2\underline{\alpha}DD^T + \left(\frac{1}{\mu} + \frac{\bar{\alpha} + \bar{\beta}}{\epsilon} + \frac{2}{\rho}\right) EE^T + (\bar{\alpha} + \bar{\beta})Y, \quad (35)$$

(ii) If the Laplacian matrix  $L$  has multiple eigenvalues,

$$U = XA^T + AX - 2(\underline{\alpha} - 1)DD^T + \left(\frac{1}{\mu} + \frac{\bar{\alpha} + \bar{\beta} + 1}{\epsilon} + \frac{2}{\rho}\right) EE^T + (\bar{\alpha} + \bar{\beta} + 1)Y, \quad (36)$$

and

$$\gamma_0^2 = 2\|T^{-1}\|_{\mathbb{F}}^2(1 + N\|r\|_2^2)\|Q\|_{\mathbb{F}}^2\|T\|_{\mathbb{F}}^2, \quad (37)$$

$$W^{-1} \geq h \int_0^h e^{A^T s} F_1^T F_1 e^{As} ds, \quad (38)$$

$$\bar{\alpha} = \max\{\lambda_2, \dots, \lambda_{n_\lambda}, \alpha_1, \dots, \alpha_{n_\nu}\}, \quad (39)$$

$$\bar{\beta} = \max\{\beta_1, \dots, \beta_{n_\nu}\}, \quad (40)$$

$$\underline{\alpha} = \min\{\lambda_2, \dots, \lambda_{n_\lambda}, \alpha_1, \dots, \alpha_{n_\nu}\}. \quad (41)$$

*Proof*

See Appendix B. □

*Remark 7*

The conditions shown in (33) to (34) can be checked by standard LMI routines for a set of fixed values  $R$  and  $W^{-1}$ . The iterative methods developed in [30] for single linear system may also be applied here.

*Remark 8*

Note from (35) and (36) that a more stringent condition is required for the case of the Laplacian matrix with multiple eigenvalues than the case with only distinct eigenvalues.

*Remark 9*

It can be seen from (38) that the matrix  $W^{-1}$  explicitly depends on the delay  $h$ , which implies that large input delays will lead to a difficulty in finding a feasible solution satisfying the conditions (33) and (34) simultaneously. Even if such a feasible solution  $P$  exists, a larger input delay results in a smaller  $P$  and therefore a smaller control gain  $K$ , which further implies a more sluggish consensus response.

## 5. AN EXAMPLE

In this section, the scenario under consideration is a connection of six subsystems (i.e.,  $N = 6$ ) in the network as shown in Fig.1. The dynamics of each subsystem are described by (1) with

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Sigma(t) = \begin{bmatrix} \sin(t) & 0 \\ 0 & \sin(2t) \end{bmatrix},$$

$$E = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}.$$

The Laplacian matrix associated with the graph in Fig.1 is

$$L = \begin{bmatrix} 3 & 0 & 0 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

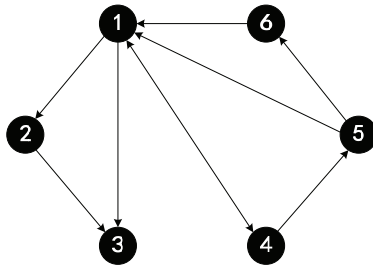


Figure 1. Network Connection Topology

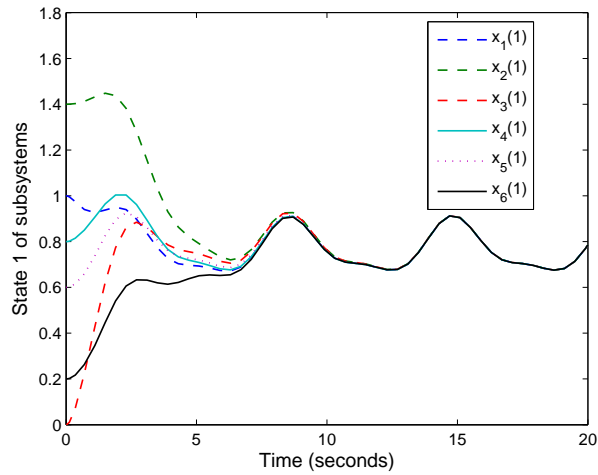


Figure 2. The state 1 of subsystems with  $h = 0.03$ .

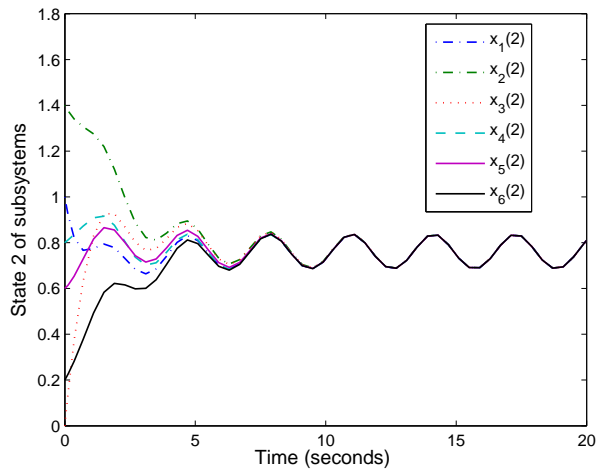


Figure 3. The state 2 of subsystems with  $h = 0.03$ .

where the eigenvalues of  $L$  are  $[0, 1, 2, 3.3247, 1.3376 \pm j0.5623]$ , which implies the case (i) in Theorem 1 is satisfied. Then, it can be straightforward to calculate the Jordan canonical form of

$L$  as

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.3247 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.3376 & 0.5623 \\ 0 & 0 & 0 & 0 & -0.5623 & 1.3376 \end{bmatrix}$$

with the matrices

$$T = \begin{bmatrix} 1 & 0 & 0 & -12.5635 & 0.2818 - j0.0145 & 0.2818 - j0.0145 \\ 1 & 0 & 1 & 5.4043 & -0.2022 + j0.3797 & -0.2022 - j0.3797 \\ 1 & 1 & 1 & 5.4043 & -0.2022 + j0.3797 & -0.2022 - j0.3797 \\ 1 & 0 & 0 & 5.4043 & -0.2022 + j0.3797 & -0.2022 - j0.3797 \\ 1 & 0 & 0 & -2.3247 & -0.3376 - j0.5623 & -0.3376 + j0.5623 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

$$T^{-1} = \begin{bmatrix} 0.1429 & 0 & 0 & 0.4286 & 0.2857 & 0.1429 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & -1.0000 & 0 & 0 \\ -0.0705 & 0 & 0 & 0.0229 & 0.0173 & 0.0303 \\ -0.0362 - j0.2945 & 0 & 0 & -0.2257 - j0.4693 & -0.1515 + j0.5799 & 0.4134 + j0.1839 \\ -0.0362 + j0.2945 & 0 & 0 & -0.2257 + j0.4693 & -0.1515 - j0.5799 & 0.4134 - j0.1839 \end{bmatrix}.$$

Thus, we have  $r = [0.1429, 0, 0, 0.4286, 0.2857, 0.1429]^T$ ,  $\underline{\alpha} = 1$ ,  $\bar{\alpha} = 3.3247$  and  $\bar{\beta} = 0.5623$ .

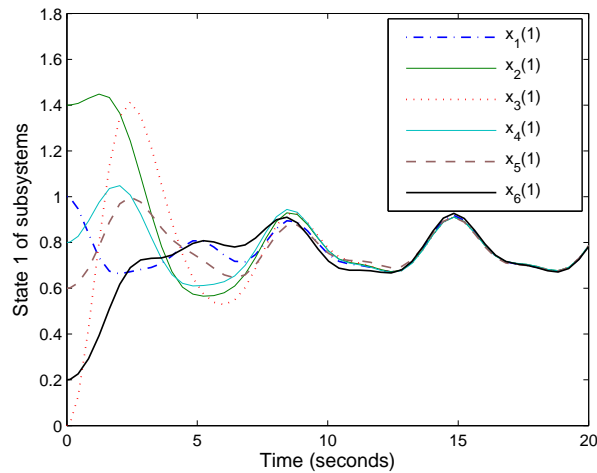


Figure 4. The state 1 of subsystems with  $h = 0.3\text{sec}$ .

The input delay of the system is  $h = 0.03\text{sec}$ . The positive definite matrix  $P$  can be computed with  $\mu = 1$ ,  $\epsilon = 1$  and  $\rho = 0.1$ , as

$$X^{-1} = P = \begin{bmatrix} 0.0002 & 0.0002 \\ 0.0002 & 0.5174 \end{bmatrix},$$

to satisfy the conditions of Theorem 1. Consequently, the control gain is obtained as

$$K = D^T P = [0.0002 \quad 0.5173].$$

Simulation study has been carried out with the results shown in Figures 2 and 3 for the states of each subsystem. Clearly the conditions specified in Theorem 1 are sufficient for the control gain

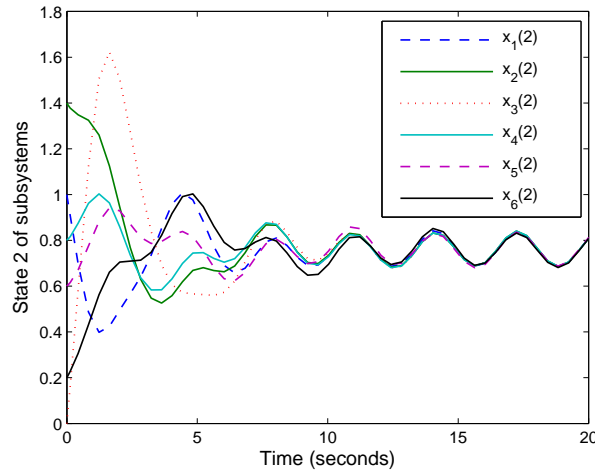


Figure 5. The state 2 of subsystems with  $h = 0.3\text{sec}$ .

to achieve consensus control. Without any re-tuning the control gain, the consensus control is still achieved for the multi-agent system with much a much larger delay  $h = 0.3\text{sec}$ , as shown in Figures 4 and 5, which imply the conditions could be conservative in the control gain design for a given input delay.

## 6. CONCLUSION

In this paper, we have solved the consensus problem of multi-agent systems in the presence of parametric uncertainty and input delay by exploiting the reduction method for delay together with consensus control design based on real Jordan form of the Laplacian matrix. Further analysis has been developed to tackle the influence of the extra integral term under transformations due to the model uncertainty. Sufficient conditions are derived for the closed-loop system to achieve global consensus using Lyapunov-Krasovskii method in the time domain. The significance of this research is to provide a feasible method to deal with the robust consensus control for input-delayed uncertain multi-agent systems.

## ACKNOWLEDGEMENT

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## APPENDIX

### A. Proof of Lemma 3

From the state transformations (17) and (19), we have

$$\Psi(z) = \left(T^{-1} \otimes I_n\right) (M \otimes I_n) (I_N \otimes \Delta A) \sigma.$$

Let

$$\begin{aligned} \Phi &= [\phi_1, \dots, \phi_N]^T = (M \otimes I_n) \bar{\sigma}, \\ \bar{\sigma} &= [\bar{\sigma}_1, \dots, \bar{\sigma}_N]^T = (I_N \otimes \Delta A) \sigma. \end{aligned}$$

Recalling  $M = I_N - \mathbf{1}r^T$ , we have

$$\begin{aligned}\phi_k &= \bar{\sigma}_k - \sum_{j=1}^N r_j \bar{\sigma}_j = \Delta A \left( \sigma_k - \sum_{j=1}^N r_j \sigma_j \right), \\ \psi_i &= (\tau_i \otimes I_n) \Phi = \sum_{k=1}^N \tau_{ik} \phi_k = \sum_{k=1}^N \tau_{ik} \Delta A \left( \sigma_k - \sum_{j=1}^N r_j \sigma_j \right) \\ &= \Delta A \sum_{k=1}^N \tau_{ik} \sigma_k - \Delta A \sum_{k=1}^N \tau_{ik} \sum_{j=1}^N r_j \sigma_j,\end{aligned}$$

where  $\tau_i$  is the  $i$ th row of  $T^{-1}$ . It then follows that

$$\begin{aligned}-2\eta_i^T P \psi_i &= 2\eta_i^T P \Delta A \sum_{k=1}^N \tau_{ik} \sum_{j=1}^N r_j \sigma_j - 2\eta_i^T P \Delta A \sum_{k=1}^N \tau_{ik} \sigma_k \\ &\leq \frac{2}{\rho} \eta_i^T P E E^T P \eta_i + \rho \left( \sum_{k=1}^N \tau_{ik} \sigma_k \right)^T F_1^T F_1 \left( \sum_{k=1}^N \tau_{ik} \sigma_k \right) \\ &\quad + \rho \left( \sum_{k=1}^N \tau_{ik} \sum_{j=1}^N r_j \sigma_j \right)^T F_1^T F_1 \left( \sum_{k=1}^N \tau_{ik} \sum_{j=1}^N r_j \sigma_j \right) \\ &\leq \frac{2}{\rho} \eta_i^T P E E^T P \eta_i + \rho \left( \|\tau_i\|_2^2 + \|\tau_i\|_1 \|r\|_2^2 \right) \sum_{k=1}^N \|F_1 \sigma_k\|_2^2.\end{aligned}\tag{42}$$

From (11) and (15), we have

$$\begin{aligned}\sigma_k &= - \int_t^{t+h} e^{A(t-\tau)} B K \sum_{j=1}^N l_{kj} z_j(\tau-h) d\tau \\ &= \int_t^{t+h} e^{A(t-\tau)} B K \sum_{j=1}^N q_{kj} [z_j(\tau-h) - z_k(\tau-h)] d\tau \\ &= \int_t^{t+h} e^{A(t-\tau)} B K \sum_{j=1}^N q_{kj} [(t_j - t_k) \otimes I_n] \eta(\tau-h) d\tau \\ &= \int_t^{t+h} e^{A(t-\tau)} B K \sum_{j=1}^N q_{kj} \sum_{l=1}^N (t_{jl} - t_{kl}) \eta_l(\tau-h) d\tau \\ &= \sum_{j=1}^N q_{kj} \sum_{l=1}^N (t_{jl} - t_{kl}) \delta_l,\end{aligned}\tag{43}$$

where  $t_i$  is the  $i$ th row of  $T$  and

$$\delta_l = \int_t^{t+h} e^{A(t-\tau)} B K \eta_l(\tau-h) d\tau.\tag{44}$$

It then follows that

$$\begin{aligned}
 \sum_{k=1}^N \|F_1 \sigma_k\|_2^2 &= \sum_{k=1}^N \left\| \sum_{j=1}^N q_{kj} \sum_{l=1}^N (t_{jl} - t_{kl}) F_1 \delta_l \right\|_2^2 \\
 &\leq \sum_{k=1}^N \left\| \sum_{j=1}^N q_{kj} \sum_{l=1}^N t_{jl} F_1 \delta_l \right\|_2^2 + \sum_{k=1}^N \left\| \sum_{j=1}^N q_{kj} \sum_{l=1}^N t_{kl} F_1 \delta_l \right\|_2^2 \\
 &\leq 2 \|Q\|_F^2 \|T\|_F^2 \sum_{l=1}^N \|F_1 \delta_l\|_2^2.
 \end{aligned} \tag{45}$$

We next deal with  $\|F_1 \delta_l\|_2^2$  in (45). Then, using Lemma 2 we have

$$\begin{aligned}
 \|F_1 \delta_l\|_2^2 &= \delta_l^T F_1^T F_1 \delta_l \\
 &= \left( \int_t^{t+h} F_1 e^{A(t-\tau)} B K \eta_l(\tau-h) d\tau \right)^T \left( \int_t^{t+h} F_1 e^{A(t-\tau)} B K \eta_l(\tau-h) d\tau \right) \\
 &\leq h \int_t^{t+h} \eta_l^T(\tau-h) K^T D^T e^{A^T(t-\tau+h)} F_1^T F_1 e^{A(t-\tau+h)} D K \eta_l(\tau-h) d\tau \\
 &= h \int_0^h \eta_l^T(t-\tau) P D D^T e^{A^T \tau} F_1^T F_1 e^{A \tau} D D^T P \eta_l(t-\tau) d\tau.
 \end{aligned} \tag{46}$$

With (45)–(46) and  $\eta_1 \equiv 0$ , the summation of  $-2\eta_i^T P \psi_i$  can be obtained as

$$\begin{aligned}
 -\sum_{i=2}^N 2\eta_i^T P \psi_i &\leq \frac{2}{\rho} \sum_{i=2}^N \eta_i^T(t) P E E^T P \eta_i(t) + 2\rho \sum_{i=2}^N \left( \|\tau_i\|_2^2 + \|\tau_i\|_1^2 \|r\|_2^2 \right) \|Q\|_F^2 \|T\|_F^2 \sum_{l=2}^N \|F_1 \delta_l\|_2^2 \\
 &\leq \frac{2}{\rho} \sum_{i=2}^N \eta_i^T(t) P E E^T P \eta_i(t) \\
 &\quad + \rho \gamma_0^2 \sum_{i=2}^N h \int_0^h \eta_l^T(t-\tau) P D D^T e^{A^T \tau} F_1^T F_1 e^{A \tau} D D^T P \eta_l(t-\tau) d\tau,
 \end{aligned}$$

where  $2 \sum_{i=2}^N (\|\tau_i\|_2^2 + \|\tau_i\|_1^2 \|r\|_2^2) \|Q\|_F^2 \|T\|_F^2 \leq 2 \|T^{-1}\|_F^2 (1 + N \|r\|_2^2) \|Q\|_F^2 \|T\|_F^2$  has been inserted in the last inequality with  $\sum_{i=1}^N \|\tau_i\|_2^2 = \|T^{-1}\|_F^2$  being used.

## B. Proof of Theorem 1

For all the state variables associate with the Jordan blocks of real eigenvalues, we consider the following summation of (27):

$$V_k = \sum_{j=1}^{n_k} W_{j+N_{k-1}}, \tag{47}$$

and from (28)–(29) we then obtain

$$\begin{aligned}
 \dot{V}_k &\leq \sum_{j=1}^{n_k} \eta_{j+N_{k-1}}^T(t) \left[ A^T P + P A - 2(\lambda_k - 1) P D D^T P + \left( \frac{\lambda_k + 1}{\epsilon} + \frac{1}{\mu} \right) P E E^T P + \mu F_1^T F_1 \right] \\
 &\quad \times \eta_{j+N_{k-1}}(t) - \eta_{1+N_{k-1}}^T(t) P D D^T P \eta_{1+N_{k-1}}(t) - \eta_{N_k}^T(t) P D D^T P \eta_{N_k}(t) \\
 &\quad - \frac{1}{\epsilon} \eta_{N_k}^T(t) P E E^T P \eta_{N_k}(t) + \sum_{j=1}^{n_k} \epsilon (1 + \lambda_k) \eta_{j+N_{k-1}}^T(t-h) P D F_2^T F_2 D^T P \eta_{j+N_{k-1}}(t-h) \\
 &\quad - \epsilon \eta_{1+N_{k-1}}^T(t-h) P D F_2^T F_2 D^T P \eta_{1+N_{k-1}}(t-h) - 2 \sum_{j=1}^{n_k} \eta_{j+N_{k-1}}^T(t) P \psi_i.
 \end{aligned} \tag{48}$$



For all the state variables corresponding to the conjugate eigenvalues in the Jordan blocks, we consider the following summation of (27) in pairs:

$$V_k = \sum_{j=1}^{n_k/2} [W_{i_1(j)} + W_{i_2(j)}]. \quad (49)$$

By (30)–(31) and after simple mechanical calculation, we have

$$\begin{aligned} \dot{V}_k &= \sum_{j=1}^{n_k/2} [\dot{W}_{i_1(j)} + \dot{W}_{i_2(j)}] \\ &\leq \sum_{j=1}^{n_k/2} \eta_{2j-1+N_{k-1}}^T(t) \left[ A^T P + PA - 2(\lambda_k - 1)PDD^T P + \left( \frac{\alpha_k + \beta_k + 1}{\epsilon} + \frac{1}{\mu} \right) PEE^T P + \mu F_1^T F_1 \right] \\ &\quad \times \eta_{2j-1+N_{k-1}}(t) - \sum_{j=1}^{n_k/2} 2\eta_{2j-1+N_{k-1}}^T(t) P \psi_{2j-1+N_{k-1}} \\ &\quad + \sum_{j=1}^{n_k/2} \eta_{2j+N_{k-1}}^T(t) \left[ A^T P + PA - 2(\lambda_k - 1)PDD^T P + \left( \frac{\alpha_k + \beta_k + 1}{\epsilon} + \frac{1}{\mu} \right) PEE^T P + \mu F_1^T F_1 \right] \\ &\quad \times \eta_{2j+N_{k-1}}(t) - \sum_{j=1}^{n_k/2} 2\eta_{2j+N_{k-1}}^T(t) P \psi_{2j+N_{k-1}} \\ &\quad - \eta_{1+N_{k-1}}^T(t) PDD^T P \eta_{1+N_{k-1}}(t) - \eta_{2+N_{k-1}}^T(t) PDD^T P \eta_{2+N_{k-1}}(t) - \eta_{N_{k-1}}^T(t) PDD^T P \eta_{N_{k-1}}(t) \\ &\quad - \eta_{N_k}^T(t) PDD^T P \eta_{N_k}(t) - \frac{1}{\epsilon} \eta_{N_{k-1}}^T(t) PEE^T P \eta_{N_{k-1}}(t) - \frac{1}{\epsilon} \eta_{N_k}^T(t) PEE^T P \eta_{N_k}(t) \\ &\quad + \sum_{j=1}^{n_k/2} \epsilon(\alpha_k + \beta_k + 1) \eta_{2j-1+N_{k-1}}^T(t-h) P D F_2^T F_2 D^T P \eta_{2j-1+N_{k-1}}(t-h) \\ &\quad + \sum_{j=1}^{n_k/2} \epsilon(\alpha_k + \beta_k + 1) \eta_{2j+N_{k-1}}^T(t-h) P D F_2^T F_2 D^T P \eta_{2j+N_{k-1}}(t-h) \\ &\quad - \epsilon \eta_{N_{k-1}}^T(t-h) P D F_2^T F_2 D^T P \eta_{N_{k-1}}(t-h) - \epsilon \eta_{N_k}^T(t-h) P D F_2^T F_2 D^T P \eta_{N_k}(t-h). \end{aligned} \quad (50)$$

With (47) and (49), we consider the following Lyapunov function

$$\begin{aligned} \mathcal{V}_1 &= \sum_{k=2}^q V_k = \sum_{k=2}^p V_k + \sum_{k=p+1}^q V_k \\ &= \sum_{k=2}^p \sum_{j=1}^{n_k} W_{j+N_{k-1}} + \sum_{k=p+1}^q \sum_{j=1}^{n_k/2} [W_{i_1(j)} + W_{i_2(j)}], \end{aligned} \quad (51)$$

where  $N_k$  is defined in (23) with  $N_1 = n_1 = 1$  and  $2 \leq k \leq q$ ,  $i_1(j)$  and  $i_2(j)$  are defined in (26) and (??), respectively. By (48) and (50), we can compute the time derivative of (51) as

$$\begin{aligned}
 \dot{V}_1 \leq & \sum_{k=2}^p \sum_{j=1}^{n_k} \eta_{j+N_{k-1}}^T(t) \left[ A^T P + PA - 2(\lambda_k - 1)PDD^T P + \left( \frac{\lambda_k + 1}{\epsilon} + \frac{1}{\mu} \right) PEE^T P + \mu F_1^T F_1 \right] \\
 & \times \eta_{j+N_{k-1}}(t) \\
 & + \sum_{k=p+1}^q \sum_{j=1}^{n_k/2} \eta_{2j-1+N_{k-1}}^T(t) \left[ A^T P + PA - 2(\lambda_k - 1)PDD^T P + \left( \frac{\alpha_k + \beta_k + 1}{\epsilon} \right. \right. \\
 & \quad \left. \left. + \frac{1}{\mu} \right) PEE^T P + \mu F_1^T F_1 \right] \eta_{2j-1+N_{k-1}}(t) \\
 & + \sum_{k=p+1}^q \sum_{j=1}^{n_k/2} \eta_{2j+N_{k-1}}^T(t) \left[ A^T P + PA - 2(\lambda_k - 1)PDD^T P + \left( \frac{\alpha_k + \beta_k + 1}{\epsilon} \right. \right. \\
 & \quad \left. \left. + \frac{1}{\mu} \right) PEE^T P + \mu F_1^T F_1 \right] \eta_{2j+N_{k-1}}(t) \\
 & - \sum_{k=2}^p \underbrace{\left[ \eta_{1+N_{k-1}}^T(t)PDD^T P \eta_{1+N_{k-1}}(t) + \eta_{N_k}^T(t)PDD^T P \eta_{N_k}(t) \right]}_{\geq 0} \\
 & - \sum_{k=p+1}^q \underbrace{\left[ \eta_{1+N_{k-1}}^T(t)PDD^T P \eta_{1+N_{k-1}}(t) + \eta_{2+N_{k-1}}^T(t)PDD^T P \eta_{2+N_{k-1}}(t) \right]}_{\geq 0} \\
 & - \sum_{k=p+1}^q \underbrace{\left[ \eta_{N_k-1}^T(t)PDD^T P \eta_{N_k-1}(t) + \eta_{N_k}^T(t)PDD^T P \eta_{N_k}(t) \right]}_{\geq 0} \\
 & - \sum_{k=2}^p \frac{1}{\epsilon} \underbrace{\eta_{N_k}^T(t)PEE^T P \eta_{N_k}(t)}_{\geq 0} - \sum_{k=p+1}^q \frac{1}{\epsilon} \underbrace{\left[ \eta_{N_k-1}^T(t)PEE^T P \eta_{N_k-1}(t) + \eta_{N_k}^T(t)PEE^T P \eta_{N_k}(t) \right]}_{\geq 0} \\
 & + \sum_{k=2}^p \sum_{j=1}^{n_k} \epsilon(1 + \lambda_k) \eta_{j+N_{k-1}}^T(t-h)PDF_2^T F_2 D^T P \eta_{j+N_{k-1}}(t-h) \\
 & + \sum_{k=p+1}^q \sum_{j=1}^{n_k/2} \epsilon(\alpha_k + \beta_k + 1) \eta_{2j-1+N_{k-1}}^T(t-h)PDF_2^T F_2 D^T P \eta_{2j-1+N_{k-1}}(t-h) \\
 & + \sum_{k=p+1}^q \sum_{j=1}^{n_k/2} \epsilon(\alpha_k + \beta_k + 1) \eta_{2j+N_{k-1}}^T(t-h)PDF_2^T F_2 D^T P \eta_{2j+N_{k-1}}(t-h) \\
 & - \sum_{k=2}^p \underbrace{\epsilon \eta_{1+N_{k-1}}^T(t-h)PDF_2^T F_2 D^T P \eta_{1+N_{k-1}}(t-h)}_{\geq 0} \\
 & - \sum_{k=p+1}^q \underbrace{\epsilon \left[ \eta_{N_k-1}^T(t-h)PDF_2^T F_2 D^T P \eta_{N_k-1}(t-h) + \eta_{N_k}^T(t-h)PDF_2^T F_2 D^T P \eta_{N_k}(t-h) \right]}_{\geq 0} \\
 & - 2 \sum_{i=2}^N \eta_i^T(t) P \psi_i(z). \tag{52}
 \end{aligned}$$

Substituting (32) into (52) and using the definitions (39)–(41), we have the following two cases:

(i) when the Laplacian matrix  $L$  has distinct eigenvalues, i.e.,  $n_k = 1$  for all  $k \in \{2, 3, \dots, q\}$ ,

$$\begin{aligned} \dot{V}_1 \leq & \sum_{i=2}^N \eta_i^T(t) \left[ A^T P + PA - 2\alpha PDD^T P + \left( \frac{1}{\mu} + \frac{\bar{\alpha} + \bar{\beta}}{\epsilon} + \frac{2}{\rho} \right) PEE^T P + \mu F_1^T F_1 \right] \eta_i(t) \\ & + \sum_{i=2}^N \epsilon (\bar{\alpha} + \bar{\beta}) \eta_i^T(t-h) P D F_2^T F_2 D^T P \eta_i(t-h) \\ & + \rho \gamma_0^2 \sum_{i=2}^N h \int_0^h \eta_i^T(t-\tau) P D D^T e^{A^T \tau} F_1^T F_1 e^{A \tau} D D^T P \eta_i(t-\tau) d\tau; \end{aligned} \quad (53)$$

(ii) when the Laplacian matrix  $L$  has multiple eigenvalues, i.e.,  $n_k > 1$  for any  $k \in \{2, 3, \dots, q\}$ ,

$$\begin{aligned} \dot{V}_1 \leq & \sum_{i=2}^N \eta_i^T(t) \left[ A^T P + PA - 2\alpha PDD^T P + \left( \frac{1}{\mu} + \frac{\bar{\alpha} + \bar{\beta} + 1}{\epsilon} + \frac{2}{\rho} \right) PEE^T P + \mu F_1^T F_1 \right] \eta_i(t) \\ & + \sum_{i=2}^N \epsilon (\bar{\alpha} + \bar{\beta} + 1) \eta_i^T(t-h) P D F_2^T F_2 D^T P \eta_i(t-h) \\ & + \rho \gamma_0^2 \sum_{i=2}^N h \int_0^h \eta_i^T(t-\tau) P D D^T e^{A^T \tau} F_1^T F_1 e^{A \tau} D D^T P \eta_i(t-\tau) d\tau. \end{aligned} \quad (54)$$

For the delayed term shown in (53) and (54), we consider the following Krasovskii functionals for both cases, respectively,

$$\begin{aligned} \text{(i)} \quad \mathcal{V}_2 &= \sum_{i=2}^N (\bar{\alpha} + \bar{\beta}) \int_{t-h}^t \eta_i^T(\tau) R \eta_i(\tau) d\tau, \\ \text{(ii)} \quad \mathcal{V}_2 &= \sum_{i=2}^N (\bar{\alpha} + \bar{\beta} + 1) \int_{t-h}^t \eta_i^T(\tau) R \eta_i(\tau) d\tau. \end{aligned}$$

where

$$R - \epsilon P D F_2^T F_2 D^T P > 0. \quad (55)$$

A direct calculation gives, respectively, that

$$\text{(i)} \quad \dot{\mathcal{V}}_2 = \sum_{i=2}^N (\bar{\alpha} + \bar{\beta}) [\eta_i^T(t) R \eta_i(t) - \eta_i^T(t-h) R \eta_i(t-h)], \quad (56)$$

$$\text{(ii)} \quad \dot{\mathcal{V}}_2 = \sum_{i=2}^N (\bar{\alpha} + \bar{\beta} + 1) [\eta_i^T(t) R \eta_i(t) - \eta_i^T(t-h) R \eta_i(t-h)]. \quad (57)$$

For the integral term shown in (53) or (54), we consider the following Krasovskii functional

$$\mathcal{V}_3 = \rho h \gamma_0^2 \sum_{i=2}^N \int_0^h \int_{t-s}^t \eta_i^T(\tau) P D D^T e^{A^T s} F_1^T F_1 e^{A s} D D^T P \eta_i(\tau) d\tau ds.$$

A direct calculation gives that

$$\begin{aligned}
 \dot{\mathcal{V}}_3 &= \rho h \gamma_0^2 \sum_{i=2}^N \int_0^h \eta_i^T(t) P D D^T e^{A^T s} F_1^T F_1 e^{As} D D^T P \eta_i(t) ds \\
 &\quad - \rho h \gamma_0^2 \sum_{i=2}^N \int_0^h \eta_i^T(t-s) P D D^T e^{A^T s} F_1^T F_1 e^{As} D D^T P \eta_i(t-s) ds \\
 &\leq \rho \gamma_0^2 \sum_{i=2}^N \eta_i^T(t) P D D^T W^{-1} D D^T P \eta_i(t) \\
 &\quad - \rho h \gamma_0^2 \sum_{i=2}^N \int_0^h \eta_i^T(t-s) P D D^T e^{A^T s} F_1^T F_1 e^{As} D D^T P \eta_i(t-s) ds, \tag{58}
 \end{aligned}$$

where

$$W^{-1} \geq h \int_0^h e^{A^T s} F_1^T F_1 e^{As} ds. \tag{59}$$

Let

$$\mathcal{V}_0 = \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3.$$

From (53), (57) and (58), we obtain

$$\dot{\mathcal{V}}_0 = \dot{\mathcal{V}}_1 + \dot{\mathcal{V}}_2 + \dot{\mathcal{V}}_3 \leq \sum_{i=2}^N \eta_i^T(t) H \eta_i(t), \tag{60}$$

where

$$\begin{aligned}
 \text{(i)} \quad H &\triangleq A^T P + P A - 2\alpha P D D^T P + \mu F_1^T F_1 + (\bar{\alpha} + \bar{\beta}) R \\
 &\quad + \left( \frac{1}{\mu} + \frac{\bar{\alpha} + \bar{\beta}}{\epsilon} + \frac{2}{\rho} \right) P E E^T P + \rho \gamma_0^2 P D D^T W^{-1} D D^T P, \tag{61}
 \end{aligned}$$

$$\begin{aligned}
 \text{or (ii)} \quad H &\triangleq A^T P + P A - 2(\alpha - 1) P D D^T P + \mu F_1^T F_1 + (\bar{\alpha} + \bar{\beta} + 1) R \\
 &\quad + \left( \frac{1}{\mu} + \frac{\bar{\alpha} + \bar{\beta} + 1}{\epsilon} + \frac{2}{\rho} \right) P E E^T P + \rho \gamma_0^2 P D D^T W^{-1} D D^T P. \tag{62}
 \end{aligned}$$

From the analysis above, the control (11) stabilises  $\eta(t)$  if the conditions (55), (59) and  $H < 0$  in (60) are satisfied. Indeed, it is easy to see that the conditions (55) and (59) are equivalent, respectively, to the conditions specified in (33) and (38) with  $Y = P^{-1} R P^{-1}$ . From (60), it can be shown that  $H < 0$  is equivalent to

$$\begin{aligned}
 \text{(i)} \quad &P^{-1} A^T + A P^{-1} - 2\alpha D D^T + \left( \frac{1}{\mu} + \frac{\bar{\alpha} + \bar{\beta}}{\epsilon} + \frac{2}{\rho} \right) E E^T \\
 &+ \mu P^{-1} F_1^T F_1 P^{-1} + (\bar{\alpha} + \bar{\beta}) P^{-1} R P^{-1} + \rho \gamma_0^2 D D^T W^{-1} D D^T < 0 \tag{63}
 \end{aligned}$$

$$\begin{aligned}
 \text{or (ii)} \quad &P^{-1} A^T + A P^{-1} - 2(\alpha - 1) D D^T + \left( \frac{1}{\mu} + \frac{\bar{\alpha} + \bar{\beta} + 1}{\epsilon} + \frac{2}{\rho} \right) E E^T \\
 &+ \mu P^{-1} F_1^T F_1 P^{-1} + (\bar{\alpha} + \bar{\beta} + 1) P^{-1} R P^{-1} + \rho \gamma_0^2 D D^T W^{-1} D D^T < 0 \tag{64}
 \end{aligned}$$

which is further equivalent to (34) with  $X = P^{-1}$ . Hence, we conclude that  $\eta(t)$  converges to zero asymptotically. This completes the proof.

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