QSS-DISSIPATIVITY AND FEEDBACK QS-PASSIVITY OF NONLINEAR DISCRETE-TIME SYSTEMS

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Abstract. Dissipativity and feedback passivity properties in nonlinear multiple-input multiple-output (MIMO) discrete-time systems are examined. Three main results are presented. First, necessary and sufficient conditions for the characterization of a class of dissipative nonlinear MIMO discrete-time systems in general form are proposed. The class of dissipativity treated is referred to as *Quadratic Storage Supply*-dissipativity. The conditions existing in the literature, addressed as Kalman-Yakubovich-Popov conditions, for the dissipative, passive or lossless cases, are derived from the proposed dissipativity characterization. Second, some relative degree-related properties of nonlinear MIMO *Quadratic Storage*-passive systems which are affine in the input are stated. Third, the problem of rendering a nonlinear affine-in-input MIMO discrete-time system passive using the properties of the relative degree and zero dynamics is analyzed. *Quadratic Storage*-passive systems are considered. The feedback passivity methodology is illustrated by means of a class of systems modelling different discrete dynamics with physical interpretation.

Keywords. Discrete-time systems, Nonlinear systems, Feedback passivity, Dissipativity, Feedback stabilization.

AMS (MOS) subject classification: 37N35, 93C10, 93C55, 93D05, 93B52.

1 Introduction

Two main problems will be considered in this paper. On the one hand, the proposal for a class of nonlinear discrete-time systems of what is referred to as Kalman-Yakubovich-Popov (KYP) conditions, that is, necessary and sufficient conditions for a system to meet in order to be dissipative or passive. On the other hand, the problem of rendering a system passive by means of a static state feedback control law. The dissipativity approach followed in this paper is the one based on the state-space dymamical representation and the use of storage and supply functions.

Dissipative (passive) systems present highly desirable properties which may simplify systems analysis and control design [7]. The concepts of dissipativity and passivity have been widely used for the stability analysis of continuous-time nonlinear systems and successfully applied in order to study

a great variety of electronic-type and electromechanical systems, see, for example [12,19]. These facts impel to translate well-known dissipativity properties in the continuous-time setting into the discrete-time framework.

Although the physicists almost always use differential equations for describing natural processes, many systems studied in different disciplines inherently operate in discrete time. For instance, systems involved in signal processing and data acquisition, some bank situations (i.e., a bank account behaviour), econometric models or electronic network systems, such as, threshold networks [21]. Furthermore, in the last few years, there has been an increasing interest in discrete representations and approximations of real continuous systems, for example, the control of continuous-time processes by means of discrete-time controllers or the use of new hybrid representations to describe more complex dynamical systems [20] in which discrete-time subsystems are present. Most of these examples may be described by the class of systems analyzed in this paper.

One of the most important formalizations of the characteristics of a dissipative or passive system is the KYP conditions or the KYP lemma. Although originally the KYP lemma established the connection between passivity and positive real transfer functions, the denomination of KYP conditions has been adopted in the nonlinear setting to address the set of necessary and sufficient conditions that a passive system satisfies. In the sequel, the KYP denomination is used in this sense.

The characterization of passive discrete-time systems is given for linear systems in [8] as the Positive Discrete Real Lemma, which is later extended by [6] to the (Q, S, R)-dissipativity case, i.e. dissipative systems with a supply function of the form $s(y, u) = y^T Q y + 2 y^T S u + u^T R u$. Necessary and sufficient conditions for affine-in-control discrete-time nonlinear systems to be lossless and passive are given in [2,3], respectively. Necessary conditions for a system to be passive are proposed in a different line in [11] which are generalized for the non-affine case in [10]. An interesting approach treating dissipativity-related properties in the nonlinear discrete-time setting is the one given in [21] where a generalized KYP lemma for (Q, S, R)-dissipativity and (Q, S, R)-losslessness is provided for nonlinear discrete-time affine-in-control systems. In this work, the definitions for lossless and dissipative systems are given in the framework of abstract dynamical energy systems. Another approach to passivity in the nonlinear discrete-time case is presented by Monaco and Normand-Cyrot's work [13,14]. They obtain KYP conditions for single-input multiple-output general non-affine-in-input and states discrete-time systems which can be expanded by exponential Lie series.

The present paper establishes the KYP conditions in the nonlinear setting for a class of dissipative discrete-time systems. These systems will be addressed as *Quadratic Storage Supply* (QSS)-dissipative systems. The dissipativity characterization given is an extension to the dissipativity case of that given in [16] for the losslessness case.

The KYP conditions here presented are different from previous results

given in [2,3,10]. On the one hand, in [3], the characterization of lossless systems which are affine in the control input is given, while the dissipativity case is treated for systems of general form in the present paper. On the other hand, [2,10] handle with the characterization of passive systems. The necessary conditions for the characterization of passive systems of general form presented in [10] are different from those given in this paper due to the fact that in [10], the basic passivity relation is used and no dissipation rate function is introduced. The same is for [2]. Furthermore, in [2], necessary and sufficient conditions for the characterization of QS-passive systems which are affine in the input are given.

The other problem treated is the action of rendering a system passive by means of a static state feedback, which is known as feedback passivity. Systems which can be rendered passive are referred to as feedback passive systems. The problem of feedback losslessness has been treated in [3] by means of the properties of the relative degree and zero dynamics of the system. Recently [18], the feedback passivity problem has been considered for nonlinear discrete-time systems of general form, however, only sufficient conditions to characterize MIMO feedback passive systems are proposed. In the present paper, the feedback passivity property is studied for MIMO nonlinear discrete-time systems which are affine in the control input and use is made of the relative degree and zero dynamics properties of the system, as in [3,18]. A class of passive systems addressed as *Quadratic Storage* (QS)-passive systems is analysed, see [15,16]. Necessary and sufficient conditions for aforementioned systems to be locally feedback QS-passive are proposed. The feedback passivity methodology presented here is an alternative to those proposed by the author in [15,16,17]. In these works, the local feedback dissipativity problem is treated for single-input single-output (SISO) nonlinear discrete-time systems which are non-affine in the states and the control input, and they are based on the basic dissipativity inequality. The results presented in this paper can be considered as an extension to the QS-passivity case of those given in [3] where the feedback QS-losslessness problem is treated and the properties of the relative degree and zero dynamics of QS-lossless systems are analyzed. The basic ideas are inherited from the continuous-time case [1].

The paper is organized as follows. The purpose of Section 2 is twofold. On the one hand, it revisits the basic definitions about dissipative systems for the discrete-time case. On the other hand, it presents necessary and sufficient conditions for the characterization of a class of dissipative nonlinear discrete-time systems addressed as QSS-dissipative. Section 3 is devoted to the properties of the relative degree and zero dynamics of MIMO nonlinear discrete-time systems which are QS-passive and affine in the control input. Section 4 deals with the feedback QS-passivity problem through the relative degree and zero dynamics properties for the systems presented in Section 3. In Section 5, a class of systems modelling different discrete dynamics with physical interpretation is used to illustrate the feedback passivity method.

Conclusions are given in the last section.

2 Dissipativity in nonlinear discrete-time systems

2.1 Preliminary definitions

Let a system of the form,

$$x(k+1) = f(x(k), u(k)), \tag{1a}$$

$$y(k) = h(x(k), u(k)), \tag{1b}$$

where $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ and $h : \mathbb{X} \times \mathbb{U} \to \mathbb{Y}$ are smooth maps, with $\mathbb{X} \subset \mathbb{R}^n$, $\mathbb{U}, \mathbb{Y} \subset \mathbb{R}^m$, $\mathbb{X} \times \mathbb{U}$ a neighbourhood of $x = 0$, $u = 0$, $k \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. Let $(\overline{x}, \overline{u})$ be an isolated fixed point of the system. There is no loss of generality in considering $(\overline{x}, \overline{u}) = (0, 0)$, $f(0, 0) = 0$ and $h(0, 0) = 0$.

A positive definite $\mathcal{C}^2$ function $V : \mathbb{X} \to \mathbb{R}$ such that $V(0) = 0 \Longleftrightarrow x = 0$ is addressed as storage function. A $\mathcal{C}^2$ function denoted by $s(y, u)$ with $s : \mathbb{Y} \times \mathbb{U} \to \mathbb{R}$ is addressed as supply function. A $\mathcal{C}^2$ function $\phi : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$, such that $\phi(\cdot, u)$ is positive for each $u \in \mathbb{U}$, with $\phi(0, 0) = 0$, is referred to as a dissipation rate function in the sense proposed in [7,16].

The dissipativity definition in the discrete-time nonlinear setting given in [3] will be rewritten in the following way.

Definition 2.1 [16] *System (1) with storage function $V(x)$ and supply function $s(y, u)$ is said to be locally (V, s)-dissipative if there exists a dissipation rate function ϕ such that*

$$V(f(x, u)) - V(x) = s(h(x, u), u) - \phi(x, u), \ \forall (x, u) \in \mathbb{X} \times \mathbb{U} \tag{2}$$

Definition 2.2 *System (1) is said to be locally V-passive if it is locally (V, s)-dissipative with a supply function of the form $s(y, u) = y^T u$.*

The dissipativity characterization proposed in this section is restricted to a class of dissipative systems, defined as follows.

Definition 2.3 [15] *System (1) is said to be locally QSS (Quadratic Storage Supply)-dissipative if it is locally (V, s)-dissipative with a storage function $V(x)$ and a supply function $s(y, u)$ such that $V(f(x, u))$ and $s(h(x, u), u)$ are quadratic in u.*

Now, locally QS-passive systems are introduced for nonlinear discrete-time systems which are affine in the control input.

Let a system of the form,

$$x(k+1) = f(x(k)) + g(x(k))u(k) \tag{3a}$$

$$y(k) = h(x(k)) + J(x(k))u(k) \tag{3b}$$

where $f(x)$, $g(x)$, $h(x)$, $J(x)$ are smooth maps and $f(x) \in \mathbb{X} \subset \mathbb{R}^n$, $g(x) \in \mathbb{G} \subset \mathbb{R}^{n \times m}$, $h(x) \in \mathbb{Y} \subset \mathbb{R}^m$, $J(x) \in \mathbb{J} \subset \mathbb{R}^{m \times m}$, $x \in \mathbb{X} \subset \mathbb{R}^n$, $u \in \mathbb{U} \subset \mathbb{R}^m$. Consider $f(0) = 0$, $h(0) = 0$.

Definition 2.4 [15] *System (3) is said to be locally QS (Quadratic Storage)-passive if it is locally V-passive with a storage function $V(x)$ such that $V(f(x) + g(x)u)$ is quadratic in u $\forall f$, $\forall g$.*

Remark 2.5 *Although the denominations of QSS-dissipative, QSS-lossless and QS-passive systems are first introduced by the author in [15,16], these classes of systems are used in [2] for the passivity case and in [3] for the losslessness case.*

Remark 2.6 *Storage functions $V(x)$ such that $V(f(x) + g(x)u)$ is quadratic in u $\forall f$, $\forall g$ can be proposed as a quadratic form such that $V = x^T P x$, with P a constant positive definite matrix. In this case, the relation between function $V(x)$ and the dynamics of a locally QS-passive system of the form (3) can be obtained by means of the strict positive nature of the diagonal leading minors of the Hessian matrix of the function $V(x) + (h(x) + J(x)u)^T u - V(f(x) + g(x)u)$ evaluated at $x = 0$ and $u = 0$.*

2.2 Characterization of discrete-time dissipative systems in general form

The results presented in this section follow the same approach given in [16]. The difference is that, in [16], KYP-type conditions were established for QSS-lossless systems of the form (1), here, the QSS-dissipativity case is treated. Furthermore, the necessary and sufficient conditions proposed for the characterization of QSS-dissipative systems are an extension of those given in [2] for the QS-passivity case in affine-in-input systems.

Theorem 2.7 *Let $V(x)$ be a storage function and $s(y, u)$ be a supply function such that $V(f(x, u))$ and $s(h(x, u), u)$ are quadratic in u. Then, a system of the form (1) is locally QSS-dissipative with $V(x)$ and $s(y, u)$, if and only if, there exist real functions $l(x)$, $m(x)$ and $k(x)$, all of appropriate dimensions*

such that,

$$V(f(x,0)) - V(x) = s(h(x,0),0) - l^T(x)l(x) - m^T(x)m(x) \quad (4a)$$

$$\left.\frac{\partial V(z)}{\partial z}\right|_{z=f(x,0)} \left.\frac{\partial f(x,u)}{\partial u}\right|_{u=0} + 2l^T(x)k(x) = \left.\frac{\partial}{\partial u}s(h(x,u),u)\right|_{u=0} \quad (4b)$$

$$\left.\left(\frac{\partial f(x,u)}{\partial u}\right)^T\right|_{u=0} \left.\frac{\partial^2 V(z)}{\partial z^2}\right|_{z=f(x,0)} \left.\frac{\partial f(x,u)}{\partial u}\right|_{u=0} +$$

$$+ \left.\frac{\partial V(z)}{\partial z}\right|_{z=f(x,0)} \left.\frac{\partial^2 f(x,u)}{\partial u^2}\right|_{u=0} =$$

$$= \left.\frac{\partial^2}{\partial u^2}s(h(x,u),u)\right|_{u=0} - 2k^T(x)k(x) \quad (4c)$$

Proof. (Necessity): If system (1) is locally QSS-dissipative, there exists a dissipation rate function ϕ satisfying (2). Since $V(f(x,u))$ and $s(h(x,u),u)$ are quadratic in u, the function ϕ can be written as follows

$$\phi(x,u) = [l(x) + k(x)u]^T[l(x) + k(x)u] + m^T(x)m(x) \geq 0, \ \forall u \in \mathbb{U} \quad (5)$$

for some real functions $m(x)$, $l(x)$ and $k(x)$. Condition (4a) is obtained restricting (2) to $u = 0$, and taking $\phi(x,u)$ as defined in (5). Conditions (4b) and (4c) follow from the first-order derivative and the second-order derivative of (2) with respect to u, considering (5) and $u = 0$.

(Sufficiency): Assume there exist real functions $m(x)$, $l(x)$, $k(x)$ which satisfy conditions (4). Multiplying equality (4b) by u^T from the left and adding (4a), it is obtained

$$V(f(x,0)) - V(x) + u^T \left.\frac{\partial}{\partial u}V(f(x,u))\right|_{u=0} = s(h(x,0),0)+$$

$$+ u^T \left.\frac{\partial}{\partial u}s(h(x,u),u)\right|_{u=0} - 2l^T(x)k(x)u - l^T(x)l(x) - m^T(x)m(x) \quad (6)$$

Adding to the right-hand side term of (6) $u^T k^T k u - u^T k^T k u$, and using (4c), one yields

$$V(f(x,0)) + u^T \left.\frac{\partial}{\partial u}V(f(x,u))\right|_{u=0} + \frac{1}{2}u^T \left.\frac{\partial^2}{\partial u^2}V(f(x,u))\right|_{u=0} u - V(x) =$$

$$= s(h(x,0),0) + u^T \left.\frac{\partial}{\partial u}s(h(x,u),u)\right|_{u=0} + \frac{1}{2}u^T \left.\frac{\partial^2}{\partial u^2}s(h(x,u),u)\right|_{u=0} u-$$

$$\quad (7)$$

$$- \phi(x,u) \quad (8)$$

with $\phi(x,u)$ given in (5). By claiming that $V(f(x,u))$ and $s(h(x,u),u)$ are quadratic in u, the second-order Taylor expansion at $u = 0$ of $V(f(x,u))$ and $s(h(x,u),u)$ can be considered in (8), and (2) is obtained.

■

Remark 2.8 *As it is recommended in [21], a new function $m(x)$ has been considered, in comparison to the passivity conditions given in [2] for nonlinear discrete-time systems affine in the control input.*

Remark 2.9 *If $V(f(x,u))$ and $s(h(x,u),u)$ are not considered to be quadratic in u, conditions (4) yield only necessary conditions, as it is shown in [16].*

Necessary and sufficient KYP conditions appeared in the literature for dissipative MIMO discrete-time systems are strictly contained in the ones given in Theorem 2.7, either for the linear or the nonlinear case. For example, passivity conditions appearing in [2] for systems of the form (3) are obtained taking $s(y,u) = y^T u$ and $m(x) = 0$, and losslessness conditions given in [3] with $s(y,u) = y^T u$ and $\phi(x,u) = l(x) = k(x) = m(x) = 0$. In order to obtain the dissipativity conditions for nonlinear affine-in-input systems presented in [21], the left-hand side of equality (4a) would be $V(f(x) - x)$ and in (4b) and (4c), $z = f(x) - x$ should be considered with $V(x) = B^T(x) + x^T C(x)x$, with B and C matrices of appropriate dimensions, and $s(y,u) = y^T Qy + 2y^T Su + u^T Ru$, with Q, S, R constant matrices, Q and R symmetric. In order to obtain the losslessness conditions presented in this work, in addition, $\phi(x,u) = k(x) = l(x) = m(x) = 0$ must be considered. Conditions for the characterization of QSS-lossless systems of the form (1) defined in [16] can be also derived from conditions (4) considering $\phi(x,u) = l(x) = k(x) = m(x) = 0$. For the linear case, passivity conditions appearing in [8] are obtained taking $m(x) = 0$, $V(x) = \frac{1}{2}x^T Px$, $s(y,u) = y^T u$, and dissipativity conditions presented in [6] are obtained by taking $m(x) = 0$, $s(y,u) = y^T Qy + 2y^T Su + u^T Ru$, $V(x) = \frac{1}{2}x^T Px$, with P a real symmetric positive definite matrix.

3 Relative degree and zero dynamics of QS-passive nonlinear discrete-systems affine in the input

The local relative degree zero of QS-passive systems of the form (3) and the properties of the zero dynamics are studied in this section. These properties will be used to treat the feedback passivity problem in Section 4.

The basis of the analysis will be the QS-passivity characterization obtained as the restriction of QSS-dissipativity conditions (4) to $s(y,u) = y^T u$

and dynamics (3). Conditions (4) take the following form [2]:

$$V(f(x)) - V(x) = -l^T(x)l(x) - m^T(x)m(x) \tag{9a}$$

$$\left.\frac{\partial V(\alpha)}{\partial \alpha}\right|_{\alpha=f(x)} g(x) + 2l^T(x)k(x) = h^T(x) \tag{9b}$$

$$g^T(x) \left.\frac{\partial^2 V(\alpha)}{\partial \alpha^2}\right|_{\alpha=f(x)} g(x) = J^T + J(x) - 2k^T(x)k(x) \tag{9c}$$

Definition 3.1 [3] *System (3) is said to have local relative degree zero for all the outputs at $x = 0$ if $J(0)$ is nonsingular. The system has uniform relative degree zero for all the outputs if $J(x)$ is nonsingular $\forall\, x \in \mathbb{X}$.*

Definition 3.2 *A system of the form (3) has locally passive zero dynamics if there exists a storage function $V(x)$ locally defined in a neighbourhood $\mathbb{X}$ of $x = 0$ in $\mathbb{R}^n$ such that*

$$V(f^*(x)) \leq V(x), \quad \forall x \in \mathbb{X} \tag{10}$$

with $f^*(x)$ the zero dynamics of the system.

The properties of the relative degree and zero dynamics of locally QS-passive systems of the form (3) are established as follows.

Proposition 3.3 *Let system (3) be locally QS-passive with a storage function $V(x)$. Assume that $x = 0$ is a nondegenerate critical point of $V(x)$. If $rank\{g(0)\} = m$ then,*

i) *The system has local relative degree zero at $x = 0$.*

ii) *If the system has local relative degree zero at $x = 0$ then the zero dynamics locally exists at $x = 0$ and is locally passive with $V(x)$ as storage function.*

Proof.

i) Evaluating condition (9c) at $x = 0$, and considering that the Hessian matrix of $V(x)$ at $x = 0$ is positive definite, it is concluded that if $rank\{g(0)\} = m$ then $J^T(0) + J(0)$ must be positive definite, consequently, $J(0)$ is nonsingular and the system has local relative degree zero at $x = 0$.

ii) If system (3) has local relative degree zero at $x = 0$ then there is an open neighbourhood $\mathbb{X}$ of $x = 0$ such that $J^{-1}(x)$ is well defined $\forall x \in \mathbb{X}$, therefore, the zero dynamics locally exists in $\mathbb{X}$. The zero dynamics of (3) is defined by $x(k + 1) = f^*(x(k)) = f(x(k)) + g(x(k))u^*(k)$, $\forall x(k) \in \mathbb{X}$, with $u^*(k) = -J^{-1}(x(k))h(x(k))$, $\forall x(k) \in \mathbb{X}$ the control which makes the output equal to zero. The result directly follows from relation (2).

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4 The feedback QS-passivity problem

This section is devoted to render a system of the form (3) locally QS-passive. Results presented in Section 3 will be used. The feedback passivity approach can be considered as an extension to the passivity case of that given in [3] where the feedback losslessness problem is treated.

Let $\alpha(x)$ and $\beta(x)$ be smooth functions, with $\alpha(0) = 0$. Consider a static state feedback control law of the form,

$$u = \alpha(x) + \beta(x)w \tag{11}$$

Definition 4.1 *A feedback control law of the form (11) is regular if for all $x \in \mathbb{X}$ it follows that $\beta(x)$ is invertible. System (3) with $u(k) = \alpha(x(k)) + \beta(x(k))w(k)$ is referred to as the feedback transformed system.*

Definition 4.2 *Consider system (3). Assume that there exists a storage function $V(x)$ and consider a supply function of the form $s = y^T w$. The system is said to be locally feedback QS-passive if there exists a regular static state feedback control law of the form (11) such that the feedback transformed system is locally QS-passive. w is the new input defined in a neighbourhood $\mathbb{X} \times \mathbb{U}$ of $x = 0$, $w = 0$.*

Theorem 4.3 *Let a system of the form (3). Suppose there exists a storage function $V(x)$ such that $V(f(x) + g(x)u)$ is quadratic in u, $\forall f$, $\forall g$ and $x = 0$ is a nondegenerate critical point of $V(x)$. Assume that $rank\{g(0)\} = m$. Then, system (3) is locally feedback QS-passive with $V(x)$ as storage function by means of a regular feedback control law of the form (11) if and only if the system has local relative degree zero at $x = 0$ and its zero dynamics is locally passive.*

Proof. (Necessity): Assume that there is a regular static state control law of the form (11) which renders system (3) locally QS-passive. Then the feedback transformed system,

$$x(k+1) = \overline{f}(x(k)) + \overline{g}(x(k))w(k) \tag{12a}$$
$$y(k) = \overline{h}(x(k)) + \overline{J}(x(k))w(k) \tag{12b}$$

is locally QS-passive, with $\overline{f}(x) = f(x) + g(x)\alpha(x)$, $\overline{g}(x) = g(x)\beta(x)$, $\overline{h}(x) = h(x) + J(x)\alpha(x)$, $\overline{J}(x) = J(x)\beta(x)$. On the one hand, since the function $\beta(0)$ is nonsingular, $rank\{\overline{g}(0)\} = m$. Considering Proposition 3.3, it is concluded that $\overline{J}(0)$ is nonsingular, therefore $J(0)$ is nonsingular and system (3) has local relative degree zero at $x = 0$. On the other hand, due to the fact that system (12) is locally QS-passive with $V(x)$ as a storage function, by Proposition 3.3, it has a locally passive zero dynamics with $V(x)$ as a storage function. It can be checked that the zero dynamics of (3) is identical to the zero dynamics of (12), and, in conclusion, is locally passive.

(Sufficiency): Since the system relative degree is zero at $x = 0$, $J(x)$ is invertible in a neighbourhood $\mathbb{X}$ of $x = 0$, then $J^{-1}(x)$ is well defined $\forall x \in \mathbb{X}$. It is chosen

$$u(k) = u^*(k) + J^{-1}(x)v(k) \qquad (13)$$

with $u^* = -J^{-1}(x)h(x)$. System (3) with (13) yields to $x(k+1) = f^*(x(k)) + g^*(x(k))v(k)$, $y(k) = v(x(k))$, where $f^*(x) = f(x) - g(x)J^{-1}(x)h(x)$ represents the zero dynamics of the original system and $g^*(x) = g(x)J^{-1}(x)$. Now, a new input control and a new output are defined as $y(k) = v(k) := \overline{h}(x(k)) + \overline{J}(x(k))w(k)$. Then, the new system dynamics is given by:

$$x(k + 1) = f^*(x(k)) + g^*(x(k))\overline{h}(x(k)) + g^*(x(k))\overline{J}(x)w(k)$$
$$y(k) = \overline{h}(x(k)) + \overline{J}(x)w(k) \qquad (14)$$

It is defined,

$$\overline{J}(x) = \left(\frac{1}{2}g^{*T} \left. \frac{\partial^2 V}{\partial z^2} \right|_{z=f^*(x)} g^*(x) \right)^{-1} \qquad (15)$$

$$\overline{h}(x) = -\overline{J}(x) \left(\left. \frac{\partial V}{\partial z} \right|_{z=f^*(x)} g^*(x) \right)^T \qquad (16)$$

System (14) with (15) and (16) will be shown to be locally QS-passive with a storage function $V(x)$. Since $V(f^*(x) + g^*(x)u)$ is quadratic in u, the Taylor expansion formula can be used on this function. Considering that the zero dynamics of (3) is locally passive, one yields to

$$V(f^*(x) + g^*(x)\overline{h}(x)) - V(x) = -r^T(x)r(x) - m^T(x)m(x) +$$
$$+ \left. \frac{\partial V}{\partial z} \right|_{z=f^*(x)} g^*(x)\overline{h}(x) + \frac{1}{2}\overline{h}^T(x)g^{*T}(x) \left. \frac{\partial^2 V}{\partial z^2} \right|_{z=f^*(x)} g^*(x)\overline{h}(x) \qquad (17)$$

with $r(x) = l(x) + k(x)\overline{h}(x)$. Differentiating both sides of (17) with respect to $\overline{h}(x)$, and multiplying the result by $\overline{J}(x)$, in addition to use (15) and (16), the passivity condition (9b) for system (14) follows.

Taking the second-order derivative with respect to $\overline{h}(x)$ in both sides of (17) and multiplying both sides of the result from the left by $\overline{J}^T(x)$ and from the right by $\overline{J}(x)$, using (15) and supposing $J(x)$ to be symmetric, one yields to the passivity condition (9c) for system (14). For the passivity conditions of system (14), the equivalent of functions $l(x)$ and $k(x)$ are $l(x) + k(x)\overline{h}(x)$ and $k(x)\overline{J}(x)$, respectively.

Besides, using (15) and (16) on the Taylor expansion of $V(f^*(x) + g^*(x)u)$, one yields to $V(f^*(x) + g^*(x)\overline{h}(x)) = V(f^*(x))$. Taking into account that the original system has locally passive zero dynamics, i.e., $V(f^*(x)) \leq V(x)$, the passivity condition (9a) for the system (14) is obtained. In conclusion, system (3) with the feedback passivity scheme (13)-(16) is locally QS-passive.

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Remark 4.4 *The control rendering the system QS-passive has the same structure as the control which renders a system QS-lossless in [3].*

5 Application of the feedback passivity method. Some remarks

Models of type (3) describe a great variety of systems, most of them with physical interpretation. In this section, the feedback passivity methodology proposed in the previous section will be applied to a class of bilinear systems. On the feedback transformed system, some of the properties satisfied by passive systems presented in Sections 2 and 3 will be checked.

5.1 A class of examples

Let a system of the form,

$$
x(k+1) = [A_0 x(k) + B_0] + \sum_{i=1}^{m} A_{1i} x(k) u_i(k) + B_1 u(k)
$$
$$
y(k) = h(x(k)) + J(x(k)) u(k)
\tag{18}
$$

where A_0, B_0, A_{1i} and B_1 is an $(n \times n)$, an $(n \times 1)$, an $(n \times n)$ and an $(n \times m)$-dimensional constant matrix, respectively, with u_i each component of the m-dimensional input vector u. For simplicity's sake, let us consider a SISO system ($m = 1$). Then, system (18) takes the form,

$$
x(k+1) = [A_0 x(k) + B_0] + [A_1 x(k) + B_1] u(k)
$$
$$
y(k) = h(x(k)) + J(x(k)) u(k)
\tag{19}
$$

Model (19) describes a great variety of discretized versions of systems, such as, several power converters (see [4]), DC (direct current) motors [23] and other kinds of nonlinear discrete-time models, see for example [22]. Let consider,

$$
V = \frac{1}{2} x^T P x
\tag{20}
$$

with P a constant symmetric positive definite matrix.

The fixed point $\overline{x}$ for system (19) with $u = 0$ is $\overline{x} = (I - A_0)^{-1} B_0$. Notice that, in general, $\overline{x}$ is not zero, then the coordinate change $x' = x - \overline{x}$ can be made in order to have $\overline{x} = 0$. Consequently, the system is rewritten as,

$$
x'(k+1) = [A_0 x'(k) + B_0'] + [A_1 x'(k) + B_1'] u(k)
$$
$$
y'(k) = h(x'(k) + \overline{x}) + J(x'(k) + \overline{x}) u(k)
\tag{21}
$$

with $B_0' = (A_0 - I)\overline{x} + B_0$, $B_1' = A_1 \overline{x} + B_1$.

The feedback passivity methodology presented in Section 4 can be applied to systems of type (19). Two particular examples will be considered: (i) an approximated discrete model of a DC-to-DC boost converter, (ii) a discretized model of a DC motor.

As [4] shows, an approximated discretization of an averaged model of a DC-to-DC boost converter [5,9] can be obtained by the first-order Taylor series expansion of its exact discretization about a fixed duty cycle (for the example it is equal to 0.5 s). Then, $x^T = (x_1, x_2)$, with x_1 the current flowing through the inductor (L), x_2 the voltage across the capacitor (C), and u the duty cycle. For the following physical parameters,

$$L = 0.36\,mH,\ C = 28.2\,\mu F,\ v_{in} = 50\,V,\ R = 48\,\Omega,$$

with a sampling period $T = \frac{1}{20\,kHz}$, the following matrices are obtained [4],

$$
\begin{aligned}
A_0 &= \begin{pmatrix} a_{01} & a_{02} \\ a_{03} & a_{04} \end{pmatrix} = \begin{pmatrix} 0.934039 & 0.861448 \\ -0.0674801 & 0.969563 \end{pmatrix} \\
A_1 &= \begin{pmatrix} a_{11} & a_{12} \\ a_{13} & a_{14} \end{pmatrix} = \begin{pmatrix} 0 & -0.000327837 \\ -0.0000256806 & 0 \end{pmatrix} \\
B_0 &= v_{in} \begin{pmatrix} b_{01} \\ b_{02} \end{pmatrix} = \begin{pmatrix} 0.0889542 \\ 0.135042 \end{pmatrix} \\
B_1 &= v_{in} \begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix} = \begin{pmatrix} -0.00230366 \\ -0.00207904 \end{pmatrix}
\end{aligned}
\tag{22}
$$

The energy associated with the system is considered as the storage function,

$$V = \frac{1}{2}\left(Lx_1^2 + Cx_2^2\right) = \frac{1}{2}x^T \begin{pmatrix} L & 0 \\ 0 & C \end{pmatrix} x$$

The second model to be used is the first-order discretization of a simplified (singularly perturbed) model of a DC motor presented in [23], which takes the form (19) with,

$$
\begin{aligned}
A_0 &= \begin{pmatrix} a_{11} & 0 \\ 0 & a_{21} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & a_{12} \\ a_{22} & 0 \end{pmatrix} \\
B_0 &= \begin{pmatrix} a_{13} \\ 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
\tag{23}
$$

and

$$a_{11} = 1 + Tb_1,\ a_{12} = Tb_3$$
$$a_{13} = Tb_2,\ a_{21} = 1 + Tb_4,\ a_{22} = Tb_5$$

where $T = 0.01\,s$ is the sampling period and $b_1,\ b_2,\ b_3,\ b_4,\ b_5$ are constants related to system physical parameters ($L_r,\ J_m,\ R_s,\ R_r,\ L_s,\ V_r,\ F,\ K$). The system state vector is $x^T = (x_1, x_2)$, with x_1 and x_2 representing the rotor

current and the motor shaft angular velocity, respectively. The control variable u is the stator voltage. The storage function V is defined as the energy associated with the system, i.e.,

$$V = \frac{1}{2}\left(L_r x_1^2 + J_m x_2^2\right) = \frac{1}{2}x^T \begin{pmatrix} L_r & 0 \\ 0 & J_m \end{pmatrix} x$$

5.2   Application of the feedback passivity method

The feedback passivity methodology presented in Section 4 will be applied to system (19) in order to stabilize the system fixed point. For some cases, the fixed point $\overline{x}$ of the original system is not locally stable. Representation (21) must be used in order to apply the results presented in Sections 2, 3 and 4. Hereinafter, for the sake of simplicity in notation, x, y, B_0 and B_1 will denote x', y', B_0' and B_1', respectively.

Consider V as defined in (20) as storage function, $s = yu$ as supply function and $J(x) = 1$. If system (19) does not fulfill conditions (9), it is not QS-passive. However, the system can be rendered locally QS-passive by means of a static feedback control law (11), see Theorem 4.3, if and only if $J(x)$ is nonsingular and the zero dynamics of the system is locally passive. The first condition is satisfied. The control u^* rendering the output zero is $u^* = -h(x)$. Control u^* substituted in (19) gives the zero dynamics of the system. Condition (10) for the zero dynamics to be locally passive takes the following form for system (19):

$$(A_0 x + B_0)^T P(A_0 x + B_0) - 2(A_0 x + B_0)^T P(A_1 x + B_1)h(x)+$$
$$+ h^T(x)(A_1 x + B_1)^T P(A_1 x + B_1)h(x) - x^T P x \le 0 \tag{24}$$

The system output, more specifically $h(x)$, plays an important role in the fulfillment of relation (24). For example, for the boost converter model, if $h(x) = x_2$, the system has a non-passive zero dynamics and can not be rendered passive by the method. Then, a fictitious output without physical interpretation has to be proposed, for instance,

$$h(x) = \frac{a_{01} x_1 + a_{02} x_2 + v_{in} b_{01}}{a_{13} x_1 + a_{14} x_2 + v_{in} b_{12}}$$

For the discretized model of the DC motor, $h(x) = x_2$ is considered and the zero dynamics of the system is locally passive.

Provided that (24) is satisfied, the feedback passivity scheme (13)-(16) will be applied to (19) and feedback transformed system (12) will be obtained

with:

$$\overline{f}(x) = (A_0 x + B_0) + (A_1 x + B_1)\left[\overline{h}(x) - h(x)\right]$$

$$\overline{g}(x) = (A_1 x + B_1)\overline{J}(x)$$

$$\overline{J}(x) = 2\left[(A_1 x + B_1)^T P(A_1 x + B_1)\right]^{-1}$$

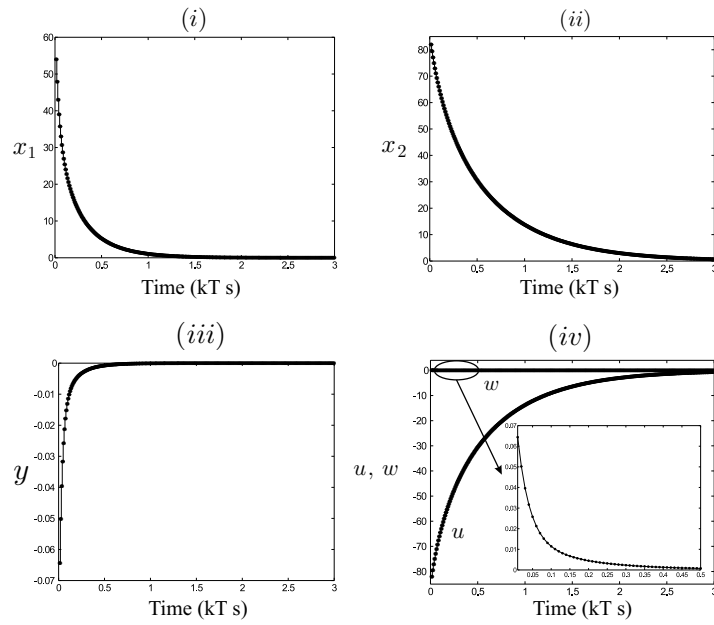$$\overline{h}(x) = -\overline{J}(x)\left[(A_1 x + B_1)^T P(A_0 x + B_0) - (A_1 x + B_1)^T P(A_1 x + B_1)h(x)\right]$$

$$(25)$$

Figure 1: Application of the feedback passivity scheme. System response for a discretized model of a DC motor with the form (21) with (23). k is the number of iterations, $T = 0.01\,s$ is the sampling period and $x_0 = (54, 82)^T$: (i) x_1, (ii) x_2, (iii) system output, (iv) control u rendering the system QS-passive and stabilizing control w.

As it is proposed in [11], the control $w(k) = -y(k)$ can be substituted into the feedback transformed system (12) with (25). This control locally asymptotically stabilizes a V-passive discrete-time system. In this case, $w(k) = -\left[\overline{J}(x(k)) + 1\right]^{-1}\overline{h}(x(k))$. For the DC motor, with $h(x) = x_2$, the feedback transformed system response is depicted in Figure (1). The coordinate change $x' = x - \overline{x}$ has been made. The following parameters obtained

from a real system have been used: $b_1 = -30$, $b_2 = 1500$, $b_3 = -0.15$, $b_4 = -0.002$, $b_5 = 0.03$, $L_r = 0.1$, $J_m = 0.5$. As it can be appreciated from the figure, the states and the output are stabilized to the origin $x = (0, 0)^T$, $y = 0$. It can be also checked that the feedback transformed system fulfills the characteristics of a locally QS-passive shown in Proposition 3.3.

For particular cases of system (19) corresponding to systems with physical interpretation, applications specialists could improve the performance of the feedback transformed system by means of introducing a different control scheme w.

6   Conclusions

Some properties of MIMO nonlinear discrete-time dissipative systems have been studied. On the one hand, necessary and sufficient conditions fulfilled by a class of dissipative systems referred to as QSS-dissipative systems have been derived. On the other hand, the properties of the relative degree and zero dynamics of a class of passive systems have been related to its feedback passivity property, and a feedback passivity methodology has been proposed for a class of MIMO nonlinear discrete-time systems affine in the control input. The class of systems for which the feedback passivity problem has been considered are referred to as QS-passive systems, that is, passive systems with $V(x(k+1))$ quadratic in u. The feedback passivity methodology has been illustrated by means of a class of systems describing different discrete dynamics with physical interpretation. The results here presented are an extension to the passivity case of the ones given in [3] where the losslessness feedback problem is reported.

7   Acknowledgements

8   References

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