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IMPLICATIONS OF DISSIPATIVITY AND PASSIVITY
IN THE DISCRETE-TIME SETTING

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Abstract: This paper is endeavored to motivate the use and further research of dissipative and passive discrete-time systems exploiting their frequency-domain characteristics. Some important features and implications of the dissipativity property in the discrete-time setting are collected. These properties are mainly referred to the stability analysis (feedback stability systems and study of the zero dynamics), the relative degree, and the preservation of passivity under feedback and parallel interconnections. Dissipativity frequency-domain properties are related to some of the most important frequency-domain stability criteria.

Keywords: Discrete-time systems, Passive elements, Frequency responses, Interconnected systems, Absolute stability.

1. INTRODUCTION

Dissipativity and its particular case of passivity were born from the observation of physical systems behavior. They are the formalization of physical energy processes. Passivity ideas emerged in the circuit theory field, from the phenomenon of dissipation of energy across resistors. The abstraction of the connections between input-output behavior, internal system description and properties of energy functions is the basis for dissipative systems. Precisely, due to the fact that dissipativity merges all these concepts, it acts as a powerful tool for analyzing systems behavior.

Dissipativity and passivity implications in dynamical continuous-time systems have been broadly studied. Nevertheless, a lot of problems concerning dissipativity and passivity in the discrete-time setting remain unsolved, or they have not attracted as significant attention as in the continuous-time case. This is the case of the study of the interconnection of passive discrete-time systems or the study of the implications of dissipativity and passivity in the relative degree and the zero dynamics of discrete dynamics.

One of the most important passivity results is that a negative feedback loop consisting of two passive systems is passive. In addition, under an additional detectability condition, this feedback system is also stable. This result is well known for continuous-time systems (Sepulchre *et al.*, 1973), but it has not been broadly exploited for the discrete-time case. Passivity and dissipativity properties have been used in the framework of interconnected discrete-time systems for stability analysis purposes, see for example, (Wu and Desoer, 1970; Desoer and Vidyasagar, 1975). However, the study of passivity preservation under block interconnection has aroused less attention in the discrete-time setting. In the seminal work (Popov, 1973), among other things, the interconnection of passive systems is studied by means of the introduction of the concept of hyperstability, that is, a closed-loop

system consisting of a linear system with a nonlinear block in the feedback path is hyperstable when the nonlinear block satisfies a passivity-like characteristic and the linear block is positive real. This result is given either for the discrete-time or the continuous-time case.

The study of the properties of the relative degree and the zero dynamics of a passive system has played an important role in understanding problems such as feedback passivity or the stabilization of passive systems in the continuous-time setting, see (Byrnes *et al.*, 1991). For general discrete-time systems, the implications of dissipativity and passivity in the relative degree and the zero dynamics have not been established yet, these ones have only been studied for the losslessness case, see (Byrnes and Lin, 1994).

This paper exploits dissipativity and passivity concepts in discrete-time systems. Passivity preservation under block interconnection is studied. Furthermore, the relative degree and the zero dynamics of passive linear discrete-time systems are analyzed. In addition, the frequency-domain characteristics of dissipativity are related to some frequency-based nonlinear feedback stability criteria in the discrete-time domain.

The paper is organized as follows. Section (2) revisits the most commonly used definitions for dissipativity in the discrete-time setting in addition to its frequency-domain characteristics, wich will be used in the sequel. Section (3) is devoted to the study of the interconnection of passive discrete-time systems. Section (4) presents the special properties that the relative degree and the zero dynamics of passive discrete-time systems have. Section (5) proposes and conjectures dissipativity to be the key for analyzing the frequency properties of nonlinear discrete-time systems. Conclusions are given in the last section.

2. DEFINITIONS AND FREQUENCY-DOMAIN CHARACTERISTICS

Dissipativity can be formalized from two different points of view: considering the input-output description of the system via an operator on a function space or via the state-space or internal dynamical representation. The former endows the frequency-domain characterization of dissipativity; for the discrete-time case, see for example (Wu and Desoer, 1970; Popov, 1973; Goodwin and Sin, 1984). The latter interprets dissipativity by means of an energy balance equation; for the discrete-time case, see for example (Byrnes and Lin, 1994) and (Sengör, 1995).

The frequency-domain interpretation of passivity for linear systems is given by means of the positive realness property of a transfer function. Passivity is equivalent to positive realness, see for the discrete-time case (Hitz and Anderson, 1969). The concept of positive real tranfer functions is originated in the continuous-

time setting in network theory as the frequency-domain formulation of the fact that the time integral of the energy input to a passive network must be positive, in other words, a linear time-invariant passive circuit, having positive resistance, inductance, and capacitance values, has a positive real impedance function. This property can be easily identified via the Nyquist diagram of the associated transfer function of the system, which is confined in the right-hand side half of the Nyquist plane. In addition, positive real transfer functions do not have poles with modulus greater than one, and their poles lying on $|z| = 1$ are simple with positive real residues. These features will be used in the sequel.

The state-space formalization of dissipativity and passivity associates to the system a non-negative definite storage function V and a supply function s. Let the system,

$$x(k+1) = f(x(k), u(k)), \quad x \in \mathcal{X}, u \in \mathcal{U} \quad (1)$$
$$y(k) = h(x(k), u(k)), \quad y \in \mathcal{Y} \quad (2)$$

where $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ and $h : \mathcal{X} \times \mathcal{U} \to \mathcal{Y}$ are smooth maps, with $\mathcal{X} \subset \mathfrak{R}^n$, $\mathcal{U}, \mathcal{Y} \subset \mathfrak{R}^m$, $k \in \mathcal{Z}_+ := \{0, 1, 2, \ldots\}$.

Definition 1. (Byrnes and Lin, 1994) System (1)-(2) with supply rate $s : \mathcal{Y} \times \mathcal{U} \to \mathfrak{R}$ is said to be dissipative if there exists a positive definite function $V : \mathcal{X} \to \mathfrak{R}^+$, $V(0) = 0$, called the storage function, such that

$$V(x(k+1)) - V(x(k)) \leq s(y(k), u(k)),$$
$$\forall (x(k), u(k)) \in \mathcal{X} \times \mathcal{U}, \forall k \quad (3)$$

Definition 2. System (1)-(2) is said to be passive if it is dissipative with respect to the supply rate $s(y(k), u(k)) = y^T(k)u(k)$.

In the linear case, the relation between the input-output and the state-space representations of passivity properties is given by the Kalman-Yakubovich-Popov (KYP) lemma, which is proposed for the discrete-time setting in (Hitz and Anderson, 1969), and is obtained from the continuous-time result via a bilinear transformation. The generalized version of the KYP lemma, also called Discrete Positive Real lemma, for the dissipativity discrete-time case is given in (Goodwin and Sin, 1984) for supply functions of the form:

$$s(y, u) = y^T Q y + 2y^T S u + u^T R u, \quad (4)$$

where Q, S, R are appropriately dimensioned matrices, with Q and R symmetric.

Lemma 3. (Goodwin and Sin, 1984) Let $G(z)$ a transfer function description, and $M(z) = R + G^H(z)S + S^T G(z) + G^H(z)QG(z)$, with $G^H(z)$ denoting the hermitian transpose of $G(z)$. Let

$$x(k+1) = Ax(k) + Bu(k),$$
$$y(k) = Cx(k) + Du(k) \qquad (5)$$

a minimal realization of $G(z)$. Then $\forall z$ s.t. $|z| \geq 1$, $M(z) \geq 0$ if and only if there exist a real symmetric positive definite matrix P and real matrices L and W such that

$$A^T PA - P = C^T QC - L^T L \qquad (6)$$
$$A^T PB = C^T QD + C^T S - LW \qquad (7)$$
$$B^T PB = R + D^T S + S^T D + D^T QD - W^T W \qquad (8)$$

Conditions (6)-(8) can be considered as the characterization of dissipativeness, Definition (1), for storage functions of the form $V = \frac{1}{2} x^T(k) Px(k)$, with P a positive definite symmetric matrix, and supply functions given in (4). Special cases of dissipativeness can be derived choosing different values for Q, S and R (Goodwin and Sin, 1984):

(1) Passivity: $Q = R = 0$, $S = \frac{1}{2}I$
(2) Input strict passivity (ISP): $Q = 0$, $S = \frac{1}{2}I$, $R = -\varepsilon I$
(3) Output strict passivity (OSP): $Q = -\delta I$, $S = \frac{1}{2}I$, $R = 0$
(4) Very strict passivity (VSP): $Q = -\delta I$, $S = \frac{1}{2}I$, $R = -\varepsilon I$
(5) Finite gain stable (FGS): $Q = -I$, $S = 0$, $R = k^2 I$,

with ε and δ small positive scalars, I the identity matrix and k an arbitrary constant.

3. IMPLICATIONS OF PASSIVITY IN INTERCONNECTED SYSTEMS

The purpose of this section is to show an alternative way in studying whether the feedback and the parallel interconnections (given in Figure (1)) of two discrete-time passive systems result in a passive system. It is inspired by the continuous results given in (Sepulchre *et al.*, 1973).
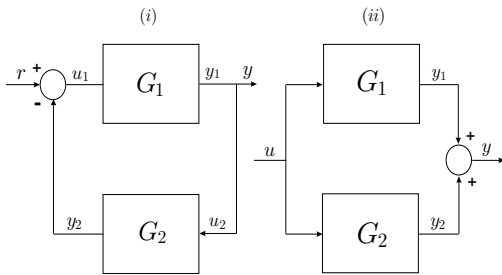

Fig. 1. (i) Feedback interconnection, (ii) Parallel interconnection.

Theorem 4. Consider the systems G_1 and G_2 (linear or nonlinear) to be passive. Then, the systems resulting

from the feedback and the parallel interconnections of systems G_1 and G_2 are passive.

Proof Let x_1 states of G_1, and x_2 states of G_2. Taking into account the dissipativity definition (1), and particularizing it for the passivity case, i.e., $s(y,u) = y^T u$, it is concluded that if G_1 and G_2 are passive, then there exist two storage functions $V_1(x_1)$ and $V_2(x_2)$, such that

$$V_1(x_1(k+1)) - V_1(x_1(k)) \leq y_1^T u_1 \qquad (9)$$
$$V_2(x_2(k+1)) - V_2(x_2(k)) \leq y_2^T u_2 \qquad (10)$$

A new state vector is defined as $x := (x_1, x_2)$, which will be the new state vector for the interconnected system, and a new positive definite storage function V is also considered

$$V(x) := V_1(x_1) + V_2(x_2) \qquad (11)$$

For the feedback interconnection (i), one has

$$V(x(k+1)) - V(x(k)) \leq y_1^T u_1 + y_2^T u_2$$

Taking into account that $u_2 = y_1$, $u_1 = r - y_2$, it follows that $y_1^T(r - y_2) + y_2^T y_1 = y_1^T r$. Consequently,

$$V(x(k+1)) - V(x(k)) \leq y_1^T r,$$

that is, the feedback interconnected system is passive.

For the parallel interconnection, the output of the system is $y_1 + y_2 = y$. If G_1 and G_2 are passive,

$$V_1(x_1(k+1)) - V_1(x_1(k)) \leq y_1^T u \qquad (12)$$
$$V_2(x_2(k+1)) - V_2(x_2(k)) \leq y_2^T u \qquad (13)$$

Adding (12) and (13), it is obtained

$$V(x(k+1)) - V(x(k)) \leq (y_1 + y_2)^T u = y^T u,$$

i.e., the system corresponding to the parallel interconnection is passive.

Remark 5. Following the same procedure, it can be easily checked that the property of OSP for supply functions of the form (4) is preserved under feedback block interconnection. Besides, ISP is preserved under parallel interconnection.

3.1 Interconnection of passive linear discrete-time systems: An example.

For single-input single-output linear dynamics, a way of illustrating that the feedback and parallel interconnections of two passive systems result in a passive system is by means of the positive realness property of the transfer function of the interconnected resulting systems.

An example is considered. A discretized normalized model of the buck converter (Kassakian *et al.*, 1991), proved to be passive with respect to the current output in the continuous-time setting, it will be connected to itself by means of a negative feedback and a parallel interconnection. The discretization scheme used will be the trapezoidal or bilinear transformation, shown to preserve passivity under sampling in linear systems, see (Tsai, 1996). The continuous transfer function having the current through the inductor as the output, with a normalized load of $R_n = 0.3536$, takes the form:

$$G_c(s) = \frac{s + 0.3536}{s^2 + 0.3536s + 1} \qquad (14)$$

Applying the trapezoidal transformation on system (14) and choosing the sampling period time $T = 0.35355$, the following positive real transfer function in z is obtained:

$$G_c(z) = \frac{0.17173(z+1)(z-0.8823)}{z^2 - 1.771z + 0.8857} \qquad (15)$$

The Nyquist diagrams for the feedback and parallel interconnections of (15) are presented in Figure (2). They both correspond to positive real transfer functions or to passive systems.
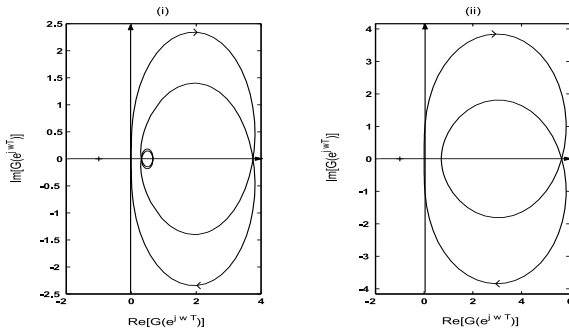

Fig. 2. (i) Nyquist plot for the feedback interconnection of (15), (ii) Nyquist plot for the parallel interconnection of (15).

4. IMPLICATIONS OF DISSIPATIVITY AND PASSIVITY IN THE RELATIVE DEGREE AND THE ZERO DYNAMICS OF A SYSTEM

The characteristics of the relative degree and the zero-dynamics of passive linear discrete-time systems will be analyzed. These properties give a valuable information of the relation between the input and the output of the system. As passivity property is an input-output property, the relative degree and zero dynamics of a passive system will present distinctive features.

The basis of our analysis will be the dissipativity conditions given in (6)-(8) particularized for the passivity case.

Proposition 6. (Hitz and Anderson, 1969) Suppose the storage function of the form $V = \frac{1}{2}x^T P x$, with P

a positive definite and symmetric matrix. A system of the form (5) is passive with respect to V, if and only if, there exists P such that

$$A^T P A - P \leq 0 \qquad (16)$$
$$B^T P A = C \qquad (17)$$
$$B^T P B - (D^T + D) \leq 0 \qquad (18)$$

Proposition 7. If system (5) is passive, then it has relative degree zero.

Proof Having relative degree zero is equivalent to $D \neq 0$, i.e., the output depends directly on the input. From condition (18), with P a positive definite matrix, one concludes that $B^T P B$ is a positive definite matrix, consequently $D^T + D$ must be a positive definite matrix, and therefore $D \neq 0$.

Remark 8. This result is not new, see (Byrnes and Lin, 1994).

Remark 9. In (Byrnes and Lin, 1994), it is stated that it does not make sense to study passivity and losslessness of discrete-time systems having outputs independent of u. This is the case for $s(y,u) = y^T u$. Indeed, dissipative systems can have relative degree greater than zero, that is, D can be zero. For example, considering dissipative systems with supply functions of the form (4), it can be concluded that ISP, VSP and FGS systems may have relative degree greater than zero.

If system (5) has relative degree zero, its zero dynamics takes the following form

$$f^*(x(k)) = (A - BD^{-1}C)x(k) \qquad (19)$$

Definition 10. A system of the form (5) has locally passive zero dynamics, if there exists a positive definite function V, locally defined on the neighborhood $\mathscr{X}$ of $x = 0$ in $\Re^n$, $V(0) = 0$, such that

$$V(f^*(x)) \leq V(x), \ \forall x \in \mathscr{X}$$

Remark 11. A passive zero dynamics is a Lyapunov stable dynamics, also referred as *weakly minimum phase* dynamics, denomination proposed in (Byrnes *et al.*, 1991).

Proposition 12. Let a system of the form (5) be passive with a storage function V as defined above. Then, its zero dynamics is locally passive.

Proof Since system (5) is assumed to be passive, there exists P a positive definite and symmetric matrix satisfying equations (16)-(18). Consider $V = \frac{1}{2}x^T P x$. The zero dynamics of the system is given by (19), then $V(f^*(x)) - V(x) = \frac{1}{2}x^T M x$, where

$$M = (A - BD^{-1}C)^T P(A - BD^{-1}C) - P \quad (20)$$

Thus, it is needed to be proved that M is negative semi-definite. Considering condition (17), M can be written as follows

$$M = (A^T PA - P) - C^T \left[D^{-1} + (D^{-1})^T \right] C +$$
$$+ C^T (D^{-1})^T B^T PBD^{-1}C \quad (21)$$

Adding and substracting to (21) $C^T (D^{-1})^T (D^T + D)D^{-1}C = C^T[D^{-1} + (D^{-1})^T]C$, and using (16) and (18), it is concluded that M is negative semi-definite.

Remark 13. The properties of the relative degree zero and passive zero dynamics shown for linear discrete-time passive systems are accomplished by the passive or positive real transfer function (15) and its feedback and parallel interconnections proved to be passive in Section (3). It is interesting to notice that continuous-time passive systems present relative degree one (see (14)), while discrete-time passive systems have relative degree zero, see (15).

5. IMPLICATIONS OF DISSIPATIVITY AND PASSIVITY IN FEEDBACK SYSTEMS STABILITY

The study of stability of nonlinear systems using frequency criteria instead of Lyapunov's direct method has been proposed for linear systems with a nonlinearity in the feedback path. These methods, mainly, Popov's, Tsypkin's and the circle criteria establish stability criteria based upon the frequency response of the linear part. It is proposed (Popov, 1973) that if the transfer function corresponding to the linear block is positive real or passive and the non-linearity satisfies a Popov-like inequality, i.e., it is a sector bounded nonlinear function, then the resulting closed-loop system is said to be absolutely stable (the zero solution of the system is globally asymptotically stable).

This section tries to present the valuable importance that dissipativity and passivity concepts have in the stability analysis of nonlinear interconnected systems. The most interesting and remarkable property of passivity is that in linear systems (either discrete or continuous), the positive realness characteristic is equivalent to the passivity property, and in addition, it presents highly interesting stability properties in the frequency domain. The fact of having a Nyquist plot on the right-half plane, means that an infinite gain proportional control can be introduced without destabilizing the system.

Since the geometric interpretation of stability criteria such as Popov's, the circle and Tsypkin's ones are based on the positive realness of a transfer function, and a particular emplacement of the Nyquist plot, dissipativity formalism can be considered to have interesting relations with these stability criteria. Indeed, a passive nonlinear function has the property of falling in sector $[0, \infty)$ (Franklin *et al.*, 1990), consequently, the passivity property increases the validity of Popov's, the circle and Tsypkin's criteria. If a sector bounded non-linearity is passive, its sector boundaries are augmented in comparison to the boundaries proposed in the mentioned stability criteria.

In (Goodwin and Sin, 1984), the generalized KYP or Discrete Positive Real Lemma is proposed for dissipative discrete-time linear systems with supply function (4), see Lemma (3). In addition, the characteristics of the Nyquist plot of $G(e^{j\omega})$ for single-input single-output systems are presented depending on the form of the supply function. Two cases are analyzed: Q being negative definite and $Q = 0$. On the one hand, if $Q < 0$, the Nyquist plot of $G(e^{j\omega})$ lies inside the circle with center $S/ \mid Q \mid$ and radius $(1/ \mid Q \mid) \sqrt{S^2 + R \mid Q \mid}$. On the other hand, if $Q = 0$, the Nyquist plot of $G(e^{j\omega})$ lies to the right (if $S > 0$) or to the left (if $S < 0$) of the vertical line $\text{Re}\, z = -R/2S$.

From the characteristics of the Nyquist plot of $G(e^{j\omega})$, dissipativity frequency-domain properties could be considered as the generalization of the stability conditions of the mentioned criteria for the discrete-time setting.

Tsypkin's criterion for nonlinear sampled-data systems establishes that the closed-loop system consisting of a linear transfer function with a nonlinear function in the feedback path is absolutely stable if the nonlinear function falls in a sector bounded by two straight lines with slopes 0 and b, and the Nyquist plot of the discrete transfer function lies to the right of the vertical line $\text{Re}\, z = -1/b$ (Tsai, 1996). Considering dissipative systems with supply function (4), it is easy to check that the geometric interpretation of the Tsypkin's criterion in the framework of the frequency domain is a special case of dissipativity with $Q = 0$, $S = I/2, R = I/b$.

The **circle criterion** gives a sufficient condition for the absolutely stability of a linear system with a nonlinear function gain in the feedback path wich falls in a sector bounded by two straight lines with slopes a and b. This class of system will be absolutely stable if the Nyquist plot of the transfer function associated to the linear block does not intersect a region C defined by the points $(-1/a + 0j)$ and $(-1/b + j0)$. In case $a, b \neq 0$ the region C will be a circle. On the other hand, if $a = 0, b \neq 0$ or $b = 0, a \neq 0$, the critical disk is converted into a critical line which the Nyquist plot must not cross.

The discrete-time version of the circle criterion is obtained from the continuous-time result and via the bilinear transformation, and using $z = e^{j\omega T}$, with T the sampling period, see (Franklin *et al.*, 1990). Considering the frequency-domain characteristics of dissipativity, the conditions that the linear block of the nonlinear

system under consideration must accomplish can be seen as different classes of dissipativity. For example, the case of having $a = 0, b \neq 0$ corresponds to the dissipativity case considering the supply function (4) with $Q = 0, S = I/2, R = I/b$ where the Nyquist plot of the transfer function corresponding to the linear part lies to the right of the vertical line $\mathrm{Re}\, z = -1/b$. The case of having $b = 0, a \neq 0$ corresponds to the dissipativity case considering the supply function (4) with $Q = 0, S = -I/2, R = -I/a$ where the Nyquist plot of the transfer function corresponding to the linear part lies to the left of the vertical line $\mathrm{Re}\, z = -1/a$. When the critical region corresponds to the interior or the outside of the circle determined by the points $(-1/a + 0j)$ and $(-1/b + j0)$, the stability conditions proposed by the circle criterion may also be obtained from the dissipativity frequency-domain properties, considering supply functions of the form (4) with Q negative definite.

Dissipativity characterization in the frequency domain can also be used in order to extend **Popov's stability criterion** to the discrete-time setting, however, a more complicated analysis than the one made for the Tsypkin's and the circle criteria is required; probably, another kind of supply functions different to (4) are suspected to be proposed.

6. CONCLUSIONS

Some implications of dissipativity and passivity properties for the discrete-time case have been presented, mainly: the preservation of passivity under feedback and parallel interconnections, the study of the the relative degree and the zero dynamics of linear discrete-time passive systems. Dissipativity characterization in the frequency domain has been used to illustrate the preservation of passivity under feedback and parallel interconnections by means of an example. The frequency-domain characteristics of dissipative systems have also been used to present dissipativity as an interesting tool for the study of systems stability in the discrete-time setting, and it can be considered as the key for obtaining frequency-based stability criteria types, such as: Tsypkin's, the circle and Popov's criteria, for nonlinear discrete-time systems.

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