Discounting the Subjective Present and Future

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Abstract

Liminal discounting, the model proposed here, generalises exponential discounting in a parsimonious way. It allows for well-known departures, whilst maintaining its elegance and tractability. A liminal discounter has a constant rate of time preference before and after some threshold time; the liminal point. If the liminal point is an absolute point in time, liminal discounting captures time consistent behaviour. If it is expressed in relative time, liminal discounting captures time invariant behaviour. We provide preference foundations for all cases, showing how the liminal point is derived endogenously from behaviour. We give applications to Rubinstein bargaining games, showcasing the model's use in microeconomic theory.

JEL Codes: D74, D90

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Intertemporal utility maximization is the dominant economic model used to analyse personal consumption and savings decisions, to propose policy implications and carry out welfare analyses. The neoclassical perspective on savings is based on Samuelson's (1937) additively separable utility model with exponential discounting. Under that model, the consumers will act such that consumption is smoothed out over their lifetime. Yet, empirical research has shown that consumption patterns are different. For example, at retirement a sudden decline in the consumption level is observed (Banks, Blundell and Tanner, 1998; Bernheim, Garrett and Maki, 2001). What is special about the retirement date is that, usually, it is a predictable point in future. If utility remains unchanged following retirement - a plausible assumption, we think - it appears as if individuals have deviated from consistent exponential discounting by having jumps in their discount factors: the utility for periods following the point of retirement has been discounted differently to periods preceding that point. The model that we propose captures precisely this sudden change in the discount factor.

Abrupt changes in behaviour do not fit well with exponential discounting. If this happens systematically when people retire, it is concerning. Even more so, we believe that there are many more examples, that arise far more frequently, all stemming from the same underlying cause. For example, consider an individual who plans, one month in advance, to start saving in the new year. When an individual decides to save, they express something about how they compare present and discounted future rewards. One who discounts the future less will save before one who discounts the future more. So, when our individual plans to start saving in the new year, something curious has happened. It is as if they know now that they will discount the future less after the new year. They know that their discount rate will change, they know when, but nothing has changed yet. This decision maker is in a state of

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liminality. That is, a state of being "in between" two distinct life phases; before new year and after new year. The new year is a *liminal point.* The type of reasoning we are referring to is surely common. Fixing on some date in advance, then forgiving one's self to overindulge in the meantime. Or, perhaps, punishing one's self to work harder until a certain date. It need not be the new year. Liminal points could depend on various personal and social factors. Examples include: one's 30th birthday, the due date for an expected child, the end of one's doctorate, the deadline to achieve tenure and, of course, the date one retires.

There is more to life than financial decisions. To study hard, to quit smoking, to start exercising, to not take drugs; these are important life decisions. Such decisions may well determine an individual's future wealth, health and happiness. The aggregation of many such decisions, made by many such individuals, will affect an entire society's economic prosperity, poverty levels, crime levels, even its natural environment. Each of these decisions compares smaller, sooner rewards with larger, later rewards. Thus, the extent to which a society is discounting delayed rewards has a profound effect on its future wellbeing. If decision makers' discount factors change dramatically at various, predictable points in their life, policy makers would do well to take notice.

This paper integrates the concept of liminality into intertemporal choice in the simplest possible way. We develop a model of *liminal discounting.* An individual with such preferences has a constant rate of time preference up to some threshold point in time; the liminal point. After this point the discount rate may change, but then remains constant afterwards. So, we maintain *all* the nice properties of exponential, constant discounting on very large subsets of timed-outcomes; all those that occur before the liminal point, and those that occur later. Violations of constant discounting occur only when comparing the near and distant future. Liminal dis-

counting can be seen, in a way we will make precise, as a generalisation of the popular quasi-hyperbolic discounting model (Phelps and Pollak, 1968; Laibson, 1997; Olea and Strzalecki, 2011). One may consider quasi-hyperbolic discounting as a model of *immediacy bias* and liminal discounting as a model of *present bias*, with "present" being subjectively defined.

Returning to our example, consider what happens when the new year actually arrives. It may be that the individual starts saving as planned. Such behaviour conforms with a principle known as *time consistency*. Suppose, instead, that the individual does not start saving when the new year arrives. Rather, they decide to start saving in the following month. Such behaviour conforms with a principle known as *time invariance*. The principle economic model of decision making over time, exponential discounting, satisfies both time consistency and time invariance (Halevy, 2012). It seems that the initial plan of "save next year" would not be made by an exponential discounter because, whatever they do next, they contradict one of these conditions. More directly, the plan itself admits that their preferences are not *stationary*.

The early empirical, typically experimental, literature on choice over time presented a strong case against exponential discounting, in favour of hyperbolic discounting (Thaler, 1981; Benzion, Rapoport and Yagil,1989; Kirby and Marakovic, 1995). Economic theory has benefited greatly from these insights (Laibson, 1997; O'Donoghue and Rabin, 2001), as has policy making (Thaler and Benartzi, 2004; Thaler and Sunstein, 2008). There is, undeniably, evidence against exponential discounting, in particular the stationarity axiom (Fishburn and Rubinstein, 1982; Frederick, Loewenstein and O'Donoghue, 2002). Recent work, however, has pointed out that the opinion, "real people are hyperbolic discounters" is unfounded. The typical data set includes a healthy amount of each class (Abdellaoui, Attema and Bleichrodt, 2010;

Andersen, Harrison, Lau and Rustrom, 2012).

In cases where violations of stationarity have been observed, static experiments do not say anything about time consistency or time invariance. Halevy (2012) investigated this experimentally. Remarkably, a significant proportion of subjects classified as time consistent did not exhibit stationary preferences. Put another way; there is no clear reason, theoretical or empirical, to presume that non-stationary choices lead to time inconsistent behaviour.

The second contribution of this paper is the extension of the liminal discounting model to dynamic choice. We develop time consistent and time invariant versions of the model. It turns out that whether the model captures time consistent or time invariant behaviour depends only on the interpretation of one parameter, the liminal point. If the point is expressed in "absolute time" then the model is time consistent. If the point is expressed in "relative time" the model is time invariant. These variations represent fundamentally different types of behaviour. In each case, however, we are able to give concrete, microeconomic applications of the model. For this purpose, we chose infinite horizon, sequential bargaining games. Rubinstein bargaining represents something of a benchmark for the applicability of any model of dynamic choice. This is where the liminal discounting model shines. There are few previous studies of time invariant, non-exponential discounting preferences in sequential bargaining (Akin, 2007; Ok and Masatlioglu, 2007; Noor, 2011; Kodritsch, 2012). We are not aware of any such applications with non-exponential discounting and time consistent preferences.

The outline of this paper is as follows: Section 1 contains the notation and definitions. Section 2 presents the exponential discounting model, 2.1 as applied to static choice and 2.2 as applied to dynamic decision making. In Section 3 we present the liminal

discounting model and in Section 3.1 we give a preference foundation for the model. Section 4 then considers applications of the liminal discounting model to dynamic choice. The time consistent version of the model is presented in Section 4.1. Two preference foundations are given, then the model is applied to sequential bargaining in Section 4.2. The time invariant version of the model is presented in Section 4.3. We give two preference foundations for this model, then it is applied to sequential bargaining in Section 4.4. All proofs are contained in the Appendices.

1 Definitions

Let $[0, X]$, with $X > 0$, denote the set of *outcomes* and $[0, T]$, with $T > 0$, be the set of times at which an outcome can occur. The set of *timed outcomes* is $[0, X] \times [0, T]$. A typical element of $[0, X] \times [0, T]$ is (x, t), which denotes the outcome x being received at time t. Such timed outcomes are the objects of choice.

We distinguish between: *instantaneous preferences*, *initial preferences*, and *dynamic preferences*. An *instantaneous preference* relation $\succcurlyeq_{\underline{t}}$ is a binary relation defined over $[0, X] \times [\underline{t}, T]$; the set of timed-outcomes occurring no sooner than time $\underline{t}$. An instantaneous preference relation characterises the preferences of our decision maker at time $\underline{t}$, as if they were making decisions at that time.[1] An *initial preference* relation $\succcurlyeq_0$ is an instantaneous preference relation for $t = 0$. For a set of *decision times* $\mathscr{D} \subseteq [0, T]$ with $0 \in \mathscr{D}$, a *dynamic preference structure* $\mathscr{R} := \{\succcurlyeq_{\underline{t}}\}_{\underline{t} \in \mathscr{D}}$ is a set of instantaneous preference relations indexed by $\mathscr{D}$.

Given an instantaneous preference $\succcurlyeq_{\underline{t}}$, the relations $\succ_{\underline{t}}$, $\preccurlyeq_{\underline{t}}$, $\prec_{\underline{t}}$ and $\sim_{\underline{t}}$ are defined in the usual way. An instantaneous preference $\succcurlyeq_{\underline{t}}$ is *complete* if, for any $(x, t), (x', t') \in$

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[1] We persist in underlining the decision time, $\underline{t}$, as it becomes useful in presenting what follows.

$[0, X] \times [\underline{t}, T]$, at least one of $(x, t) \succcurlyeq_{\underline{t}} (x', t')$ or $(x, t) \preccurlyeq_{\underline{t}} (x', t')$ hold. An instantaneous preference $\succcurlyeq_{\underline{t}}$ is *transitive* if, for any $(x, t), (x', t'), (x'', t'') \in [0, X] \times [\underline{t}, T]$, $(x, t) \succcurlyeq_{\underline{t}} (x', t')$ and $(x', t') \succcurlyeq_{\underline{t}} (x'', t'')$ jointly imply $(x, t) \succcurlyeq_{\underline{t}} (x'', t'')$. An instantaneous preference is a *weak order* if it is complete and transitive.

An instantaneous preference relation $\succcurlyeq_{\underline{t}}$ is *monotonic* if $x > x'$ implies $(x, t) \succ_{\underline{t}} (x', t)$; $\succcurlyeq_{\underline{t}}$ is *impatient* if $t > t'$ and $x > 0$ imply $(x, t) \prec_{\underline{t}} (x, t')$. We will always assume that $(0, t) \sim_{\underline{t}} (0, t')$, for any $t, t' \in [0, T]$, so include this condition in the definition of impatience. An instantaneous preference relation $\succcurlyeq_{\underline{t}}$ is *continuous* if, for any $(x, t) \in [0, X] \times [\underline{t}, T]$, the sets $\{(x', t') : (x, t) \succ_{\underline{t}} (x', t')\}$ and $\{(x', t') : (x, t) \prec_{\underline{t}} (x', t')\}$ are open subsets of $[0, X] \times [\underline{t}, T]$.

An instantaneous preference relation $\succcurlyeq_{\underline{t}}$ is *represented* by a real-valued function $V_{\underline{t}}$ if, for any $(x, t), (x', t') \in [0, X] \times [\underline{t}, T]$, the following holds:

$$(x, t) \succcurlyeq_{\underline{t}} (x', t') \iff V_{\underline{t}}(x, t) \geq V_{\underline{t}}(x', t').$$

A necessary condition for $\succcurlyeq_{\underline{t}}$ to admit such a representation is that $\succcurlyeq_{\underline{t}}$ is a weak order. It is reasonably straightforward to show that weak ordering, monotonicity, impatience and continuity of $\succcurlyeq_{\underline{t}}$ are sufficient for the existence of a *continuous* utility representation.

We call a set of functions $\mathscr{V} := \{V_{\underline{t}}\}_{\underline{t} \in \mathscr{D}}$, where $V_{\underline{t}} : [0, X] \times [\underline{t}, T] \to \mathbb{R}$ for each $\underline{t} \in \mathscr{D}$, a *dynamic model.* Finally, we say a dynamic preference structure $\mathscr{R}$ is *represented* by a dynamic model $\mathscr{V}$ if, for all $\underline{t} \in \mathscr{D}$, the preference relation $\succcurlyeq_{\underline{t}} \in \mathscr{R}$ is represented by $V_{\underline{t}} \in \mathscr{V}$.

2 Exponential Discounting

2.1 Instantaneous Choice & Exponential Discounting

In this section we briefly review the classical exponential discounting model, as applied to initial or instantaneous choice over timed-outcomes. A decision maker has exponential discounting initial preferences if they admit a representation of the following kind:

$$V_0(x,t) = \delta^t u(x)$$

for all $(x,t) \in [0,X] \times [0,T]$, with $\delta \in (0,1)$ and $u : [0,X] \to \mathbb{R}$ a continuous, strictly increasing function. The uniqueness properties pertaining to this representation are discussed later. Exponential discounting preferences exhibit impatience and monotonicity. These properties, more is better and sooner is better, are routinely assumed in economic models (Manzini and Mariotti, 2009). The key property of exponential discounting, that distinguishes it from other models, is *stationarity*:

Definition (Stationarity): An initial preference relation $\succcurlyeq_0$ satisfies *stationarity* if for all $(x,t), (y,t+\tau), (x,s), (y,s+\tau) \in [0,X] \times [0,T]$ the following holds:

$$(x,t) \succcurlyeq_0 (y,t+\tau) \iff (x,s) \succcurlyeq_0 (y,s+\tau).$$

The stationarity axiom asserts that a decision maker's preferences are unaffected by translations that preserve the time distance between two timed outcomes. The relative time distance between the timed outcomes and the decision time (0 in this case) does not affect preference. This formulation of stationarity is due to Fishburn and Rubinstein (1982). Koopmans (1960) gave the first foundation of exponential discounting. This was in a different framework, that of sequences of outcomes, hence

the condition was somewhat different. Koopmans (1960) (also see Bleichrodt, Rhode and Wakker, 2008) asked that preferences were independent of shifting every outcome later in time, inserting a new, common outcome at the start. That condition also implies an additive separability across coordinates, but the intuition regarding dependence only on delay is the same. The following result, characterising exponential discounting preferences for timed outcomes, is due to Fishburn and Rubinstein (1982):

Theorem 2.1.1 (Fishburn & Rubinstein, 1982)**.** *The following statements are equivalent:*

(i) *The initial preference relation $\succcurlyeq_0$ over $[0, X] \times [0, T]$ is a continuous, monotonic and impatient weak order that satisfies stationarity.*

(ii) *The initial preference relation $\succcurlyeq_0$ over $[0, X] \times [0, T]$ is represented by a real-valued function V_0 such that,*

$$V_0(x, t) = \delta^t u(x)$$

for some $\delta \in (0, 1)$ and $u : X \to \mathbb{R}$ a continuous, strictly increasing function.

The uniqueness results pertaining to Theorem 2.1.1 are important, in particular because they deviate from what one may expect. Much confusion can be avoided, in particular when it comes to interpreting the parameters, by recognising this. For a thorough discussion of this and further issues, see Benoit and Ok (2007).

Proposition 2.1.2 (Uniqueness Results)**.** *For the representation obtained in Theorem 2.1.1, the following hold:*

(i) *δ may be chosen arbitrarily from $(0,1)$.*

(ii) *Once δ is chosen, utility u is a ratio scale, that is, u may be replaced by v if and only if $v = \lambda u$ for some $\lambda > 0$.*

(iii) *The location of utility is fixed: $u(0) = 0$ for any representing u.*

Note two important points regarding the above proposition. First, since the discount factor δ is not unique, one cannot assign behavioural content to its magnitude. That is, one cannot say one decision maker is more or less impatient than another based on their discount factors. The only exception occurs when each decision maker has the same utility for outcomes in the representation. Such restricted interpersonal comparisons of discount factors are meaningful. Second, this lack of uniqueness is an artefact of the timed-outcomes framework we are using. For sequences of more than one non-zero outcome, one obtains uniqueness (see, for example, Theorem 2 of Bleichrodt, Rohde and Wakker, 2008). Olea and Strzalecki (2011) criticise the timed-outcome framework for this reason.

2.2 Dynamic Exponential Discounting

We now present the exponential discounting model as applied to dynamic choice. A dynamic preference structure $\mathscr{R}$ conforms to *dynamic exponential discounting* if it is represented by a dynamic model $\mathscr{V}$ where,

$$V_{\underline{t}}(x,t) = \delta^t u(x)$$

for all $V_{\underline{t}} \in \mathscr{V}$ and $(x,t) \in [0,X] \times [\underline{t}, T]$, with $\delta \in (0,1)$ and $u : [0,X] \to \mathbb{R}$ a continuous, strictly increasing function. That is, every instantaneous preference relation $\succcurlyeq_{\underline{t}} \in \mathscr{R}$

is represented by exponential discounting. Further, they are all represented by the same exponential discounting function.

Stationarity of a dynamic preference structure is defined in the obvious way: A dynamic preference structure $\mathscr{R} \coloneqq \{\succcurlyeq_t\}_{t \in \mathscr{D}}$ satisfies *stationarity* if every instantaneous preference relation $\succcurlyeq_t \in \mathscr{R}$ satisfies stationarity. Under dynamic exponential discounting, every preference relation in the dynamic preference structure admits an exponential discounting representation. As such, the dynamic preference structure is necessarily stationary.

The dynamic exponential discounting model assumes that the discount factor δ and utility function u are the same at every decision time. Recall the uniqueness properties of Theorem 2.1.1, expressed in Proposition 2.1.2. As the discount factors can be chosen arbitrarily at each point in time, they can obviously be chosen equal to each other. Each decision time's discount factor will determine, up to positive linear transformation, an associated utility function. That these utility functions must be linearly related does not follow from stationarity of the dynamic preference structure. Stationarity imposes no restrictions at all on comparing instantaneous preferences across the structure. Something further is required. We consider two properties, *time consistency* and *time invariance*. These are such that, taking either of one them will be sufficient for dynamic exponential discounting. Further, assuming both of them renders the separate assumption of stationarity redundant (Halevy, 2012).

Time consistency requires that the decision maker's instantaneous preference between two timed-outcomes does not depend on the decision time.

Axiom (Time Consistency): A dynamic preference structure $\mathscr{R}$ satisfies *time*

consistency if for all $\succcurlyeq_{\underline{t}}, \succcurlyeq_{\underline{t}'} \in \mathscr{R}$, $(x,t),(y,s) \in [0,X] \times [0,T]$ such that $\underline{t}, \underline{t}' \leqslant \min\{t,s\}$:

$$(x,t) \succcurlyeq_{\underline{t}} (y,s) \iff (x,t) \succcurlyeq_{\underline{t}'} (y,s).$$

Halevy (2012) used the term *time invariance* for the following condition:

Axiom (Time Invariance): A dynamic preference structure $\mathscr{R}$ satisfies *time invariance* if, for all $\succcurlyeq_{\underline{t}}, \succcurlyeq_{\underline{t}+\tau} \in \mathscr{R}$, $(x,t),(y,s),(x,t+\tau),(y,s+\tau) \in [0,X] \times [0,T]$ such that $\underline{t} \leqslant \min\{t,s\}$ and $\tau \geqslant 0$:

$$(x,t) \succcurlyeq_{\underline{t}} (y,s) \iff (x,t+\tau) \succcurlyeq_{\underline{t}+\tau} (y,s+\tau).$$

Time invariance captures the behaviour of a decision maker who evaluates timed outcomes in relative time. That is, only the delay between the decision time and the outcome time matters; not the "calendar time" of the outcome. Such a decision maker uses "stopwatch time". Time invariant sets of preference relations may fail to exhibit time consistency. The following theorem is due to Halevy (2012). Given that statements (i), (ii) and (iii) below are all equivalent to (iv), we can also express the theorem as "any two of stationarity, time consistency and time invariance of $\mathscr{R}$ imply the third". Its proof is elementary, but the theorem is of foundational importance.

Theorem 2.2.1 (Halevy, 2012). *Let* $\mathscr{R} := \{\succcurlyeq_t\}_{t \in \mathscr{D}}$ *be a dynamic preference structure. Let* $\mathscr{D} = [0,T]$. *Then, the following statements are equivalent:*

(i) $\mathscr{R}$ *satisfies stationarity and time consistency.*

(ii) $\mathscr{R}$ *satisfies stationarity and time invariance.*

(iii) $\mathscr{R}$ *satisfies time consistency and time invariance.*

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(iv) $\mathscr{R}$ *satisfies stationarity, time consistency and time invariance.*

Theorem 2.2.1 can be used to give various preference foundations for exponential discounting in the dynamic framework. This is achieved by combining the usual conditions with either of statements (i), (ii) or (iii). That is, any two of the three conditions for $\mathscr{R}$.

In our opinion, the characterisation of exponential discounting that assumes statement (iii) is the most convincing. It seems more natural to apply axioms that constrain the relationship between instantaneous preferences across time (time consistency and time invariance) than to any specific instantaneous preference. A case in point: compare the requirements of stationarity (one's instantaneous preferences are invariant under translations that preserve relative delays) and time consistency (one should not reverse one's preference for the same timed-outcomes). The latter condition is more suggestive of a prescriptive principle to guide economic agents. Avoiding preference reversals for the same objects provides a degree of immunity against certain economic ruin. Even Samuelson (1937), who assigned very little normative content to exponential discounting, mentions time consistency (of course, not using modern nomenclature) as a redeeming feature of the model.[2] Machina (1989; 1685) further argued that dynamic arguments in favour of expected utility, for choice under risk, were more "formidable" than those referring to static decisions. Those arguments also apply to choice over time with little translation.

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[2]He writes, "The particular results we have reached are not subject to criticism on this score, having been carefully selected to take care of this provision. Contemplation of our particular equations will reveal that the results are unchanged even if the individual always discounts from the existing point of time rather than from the beginning of the period" (Samuelson, 1937; 160).

3 Liminal Discounting

In this section we present a model for instantaneous preferences over timed-outcomes. We will call this the *liminal discounting* model. The term 'liminal' is derived from the Latin *limen*, meaning a 'threshold'. The term is used in various fields (anthropology, sociology, medicine to name a few) to refer to being in an 'in-between' state before an anticipated change. We use the term to mean the following: the decision maker's discount rate will change at some known point. At the time of making a decision, they are in the 'in-between' stage. Their discount rate will change, they know when, but it remains the same for the present. We denote this time of change as h, for 'horizon'. The period from now to h is the *liminal period*; h is the *liminal point*. The decision maker has liminal discounting preferences if initial preferences $\succsim_0$ over $[0, X] \times [0, T]$ are represented by $V_0 : [0, X] \times [0, T] \to \mathbb{R}$ where,

$$V_0(x,t) = \begin{cases} \alpha^t u(x) & \text{if } t \leq h \\ (\alpha/\beta)^h \beta^t u(x) & \text{if } t > h \end{cases}$$

for all $(x,t) \in [0, X] \times [0, T]$, with $h \in [0, T]$, $\alpha, \beta \in (0,1)$ and $u : [0, X] \to \mathbb{R}$ a continuous, strictly increasing function.

When evaluating timed-outcomes that occur before h, the decision maker maximises an exponential discounting function with discount factor α. For timed-outcomes occurring later than h, the decision maker again uses an exponential discounting function but now with discount factor β. The weight $(\alpha/\beta)^h$ ensures that the evaluation function is continuous everywhere.[3] Before and after h, a liminal discounter uses the same utility function for outcomes.

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[3]This parametric discounting function was previously considered by Jamison and Jamison (2011), who called it *split-rate exponential discounting*. We thank Glenn Harrison for this reference.

Examples abound when one considers candidates for liminal points. Dates of cultural significance, such as New Year's Day have this property. It is not uncommon, for example, to plan to quit smoking, start saving, start exercising etc. after the New Year. These patterns of behaviour can all be modelled in the framework of instantaneous choice for timed outcomes or sequences of timed outcomes. In this sense, each example reflects a change in discount factors before and after some important date. Liminal points could be more personal: one's 30th birthday, the birth of a child etc. It will also be important, when we consider dynamic preference structures, whether the liminal point is described in *absolute* or *relative* time (see Section 4). The fact that we assume just one liminal point is, of course, a simplification of reality. The principle property of the liminal point, as far as we are concerned, is that it is anticipated by the decision maker.

Initial preferences are said to exhibit *present bias* if there are $x, y \in [0, X]$, $t \in [0, T]$ and $\tau > 0$ such that $(x, 0) \succ_0 (y, \tau)$ and $(x, t) \prec_0 (y, t + \tau)$. Present bias implies more than a "preference for the present", which is already captured by impatience. Present biased preferences reveal discount *factors* that are different between the present and the future. Consequently, present biased preferences are incompatible with the maximisation of an exponential discounting function.

The liminal discounting model can rationalise present biased choices. We say a liminal discounting function, with $h \in (0, T)$, exhibits *decreasing impatience* if $\alpha < \beta$ and *increasing impatience* if $\alpha > \beta$. Notice that, since the utility function does not change after h, we can make meaningful comparisons of the liminal discounter's discount factors. Consider a liminal discounting function that exhibits decreasing impatience. It can be shown that there exist some $x, y \in [0, X]$ and $0 < \tau < h$ such

that,

$$u(x) > \alpha^\tau u(y) \quad \& \quad u(x) < \beta^\tau u(y).$$

By choosing t such that $h - \tau < t < h$, it follows that these preferences must exhibit present bias. Whenever liminal discounting preferences exhibit decreasing impatience, they exhibit present bias. While it is possible for liminal discounting preferences to exhibit increasing impatience, it is impossible for present biased liminal discounting preferences to exhibit increasing impatience.

A closely related, similarly intentioned, model to liminal discounting is the popular *quasi-hyperbolic* discounting (Phelps and Pollak, 1968; Hayashi, 2003; Olea and Strzalecki, 2011). Under quasi-hyperbolic discounting preferences are represented by the following function,

$$V^{QH}(x,t) = \begin{cases} u(x) & \text{if } t = 0 \\ \lambda \delta^t u(x) & \text{if } t > 0 \end{cases}$$

where $\lambda, \delta \in (0,1)$ and $u : X \to \mathbb{R}$ is a continuous, strictly increasing function[4]. The quasi-hyperbolic discounting model is discontinuous in time, so is not a special case of the liminal discounting in our presentation. If we restrict attention to discrete time, for instance $T = 0, 1, \ldots$, then quasi-hyperbolic discounting can be seen as a special case of liminal discounting. Let $h = 1$, $\alpha = \lambda \delta$ and $\beta = \delta$ and liminal discounting coincides with the above quasi-hyperbolic discounting representation. Under quasi-hyperbolic discounting the "present" coincides with the immediate time. Liminal discounting adds some flexibility to the time period called "present". The liminal point h need not equal 1 and, moreover, it is a personal parameter. Different people

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[4]The quasi-hyperbolic discounting model is known affectionately in the literature as the (β, δ) model. Phelps and Pollak (1968) actually used α, not β. We have already used α and β, so use λ.

can have different liminal points. Indeed, one may consider a liminal discounter's liminal period $[0, h]$ to be what they consider the "present" and after h to be their "future"; a *subjective* present and future. This warrants the interpretation of liminal discounting as a model of "discounting the subjective present and future". In Section 3.1 we present a preference foundation for the liminal discounting model. We will not assume *a priori* that the liminal point exists. Our axiomatisation will guarantee the existence and uniqueness of the liminal point.

3.1   A Preference Foundation for Liminal Discounting

In this section we provide a preference foundation for the liminal discounting model. We present our result in the framework of initial choice over timed outcomes. Liminal discounting preferences coincide with exponential discounting preferences on large subsets of timed outcomes. In particular, those sets that contain only timed outcomes that occur before, or only occur after, the liminal point. In this sense, if we knew h, the preference conditions would be simple.

In practice, it may be difficult to know any individual's liminal point. Indeed, even if an individual's preferences admit a liminal discounting representation, they may well be unaware of the quantitative details. They behave only *as if* they know their liminal point, discount factors and utility function for outcomes. Once an individual's liminal point is known, eliciting discount factors and utility is routine. Knowing the liminal point is the problem. Our approach allows the liminal point to be detected from observed behaviour. We will need to first introduce two concepts: *stationarity-after-t* and *stationarity-before-t*.

Definition (Stationarity-after-t): A preference relation $\succsim_0$ satisfies *stationarity-*

after-t if for all $(x,t), (y, t + \tau), (x, s), (y, s + \tau) \in [0, X] \times [0, T]$ with $\tau > 0$ and $s > t$, the following holds:

$$(x, t) \succsim_0 (y, t + \tau) \implies (x, s) \succsim_0 (y, s + \tau).$$

Stationarity-after-t demands that, when comparing two timed outcomes with the *soonest* outcome occurring at time t, preferences are invariant under translations that put each outcome backward in time by the same amount. Note that it is only a one way implication; the preference regarding the earlier timed outcomes implying the preference regarding the later timed outcomes. Stationarity above was a two-way implication, or equivalence. One may readily verify that liminal discounting preferences satisfy stationarity-after-t when $t \geqslant h$.

Consider again our problem of uncovering an individual's liminal point from observed behaviour. The importance of stationarity-after-t is now apparent. Suppose we observe preferences that violate this condition. For liminal discounters, *this can only happen because their h is later than t*. Consequently, such an observation tells us *we do not need to look before t*. A violation of stationarity-after-t allows us to rule out an entire interval below t. A violation of the following condition, *stationarity-before-t*, will similarly allow us to rule out all times after that particular t:

Definition (Stationarity-before-t): A preference relation $\succsim_0$ satisfies *stationarity-before-t* if for all $(x, t), (y, t - \tau), (x, s), (y, s - \tau) \in [0, X] \times [0, T]$ with $0 < \tau < s < t$, the following holds:

$$(x, t) \succsim_0 (y, t - \tau) \implies (x, s) \succsim_0 (y, s - \tau).$$

In words: when comparing two timed outcomes, with the *latest* outcome occurring at time t, preferences are invariant under translations that bring each outcome forward in time by the same amount. Again, stationarity-before-t is a one-way implication. This time, the preference regarding the later timed outcomes implies a preference regarding earlier timed outcomes. One may also verify, by simple substitution of the preference functional, that Liminal Discounting preferences satisfy stationarity-before-t when $t \leqslant h$. We now present our key axiom, *weak stationarity*, that will be central in our characterisation of liminal discounting preferences:

Axiom (Weak Stationarity): At any time $t \in T$, preferences $\succcurlyeq_0$ are stationary-before-t, or stationary-after-t, or both.

Notice that the condition applies to *any* time $t \in T$.

To explain the necessity of the condition for liminal discounting preferences, suppose we consider an arbitrary time $t \in T$ and observe a violation of stationarity-before-t. Then, since preferences are assumed to be liminal discounting preferences, it must be that the liminal point is in the interval $[0, t)$. This is the only way in which a violation of stationarity-before-t could have happened. Then consider $[t, T]$; the interval *after* t. Since h cannot belong to this interval, stationarity-after-t must hold for this t under consideration. As we choose different $t \in T$, the joint contradiction of one condition (stationarity-before/after) with the assumption of liminal discounting always implies the other condition. Weak stationarity does not exclude violations of stationarity that occur when comparing the near and distant future. For example, take some t and suppose preferences are stationary-before-t. Then, it may well be the case that preference reversals occur when timed-outcomes before t are delayed to later than t. We are assured, however, that weak stationarity must necessarily hold

if preferences are liminal discounting preferences.[5]

We have discussed why weak stationarity must necessarily hold if preferences are liminal discounting preferences. When combined with the basic axioms, the weak stationarity axiom will imply the existence of a liminal point h and two exponential discounting functions. It will not, as we will discuss below, force these exponential discounting functions to have the same utility for outcomes. So, we will require one further axiom. At this point, however, we explain how the liminal point is derived.

To give the intuition for the main argument, suppose that stationarity-before-t holds for some $t \in T$. One can then show that stationarity-before-t' must hold for all $t' < t$. Similarly, stationarity-after-t implies stationarity-after-t' for all $t' > t$. It can then be shown that there is t^* such that *both* stationarity-before-t^* and stationarity-after-t^* hold simultaneously. Further, if $t^* \in (0, T)$, it is unique or else stationarity holds everywhere. This t^* is the point we are looking for; it must be h.

We have identified a weakening of the stationarity axiom that is necessary for liminal discounting. We claimed that weak stationarity does not force utility to be independent of time. This is assumed by the liminal discounting model. Consider an initial preference represented by,

$$
V_0(x, t) = \begin{cases} \alpha^t u(x) & \text{if } t \leqslant h \\ \phi(\beta^t v(x)) & \text{if } t > h \end{cases}
$$

with $\alpha, \beta \in (0, 1)$, $h \in [0, T]$, $u : X \to \mathbb{R}$ continuous and strictly increasing, and $\phi : \mathbb{R} \to \mathbb{R}$ a continuous, strictly increasing function. Such preferences necessarily satisfy weak stationarity, but need not be liminal discounting preferences. The stationarity

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[5]Note that stationarity is not implied by the simultaneous satisfaction of stationarity-before-t and stationarity-after-t for one $t \in [0, T]$. Full stationarity requires something more, namely the simultaneous satisfaction of stationarity-before-t and stationarity-after-t for all $t \in [0, T]$.

axiom would ensure that we could take ϕ as the identity function and that *both* $u = v$ and $\alpha = \beta$ hold. Weak stationarity implies *none* of this.[6] We use the following condition, *midpoint consistency*:

Axiom (Midpoint Consistency): An initial preference relation $\succsim_0$ satisfies *midpoint consistency* (or "$\succsim_0$ is *midpoint consistent*") if, for all $x, y, z \in [0, X]$ and $t, t', s, s' \in [0, T]$, any three of the following indifferences imply the fourth:

$$(x, t) \sim_0 (y, t') \qquad (x, s) \sim_0 (y, s')$$
$$(y, t) \sim_0 (z, t') \qquad (y, s) \sim_0 (z, s')$$

Midpoint consistency captures the idea that we may consistently measure utility ratios. Further, these utility ratios should not depend on which points in time are used. Köbberling and Wakker (2003) have presented such a condition. Similar techniques are discussed in Baillon, Driesen and Wakker (2012). Consider the first column of indifferences. The times t and t' act as gauges to aid in the measurement of appropriate x, y and z. The indifferences reveal that, replacing x with y at t (read down the column) has the same impact as replacing y with z at t'. The midpoint consistency axiom requires that the impact of this midpoint does not depend on the gauge times used. That is, if we use s and s', chosen appropriately to construct the third indifference, then the act of replacing x with y at s and y with z at s' should continue to maintain the indifference. The following theorem provides the preference foundation for the liminal discounting model:

Theorem 3.1.1. *The following statements are equivalent:*

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[6]Continuity of V_0 does ensure that $\alpha^h u(x) = \phi(\beta^h v(x))$ for all $x \in [0, X]$, so the utilities are certainly ordinally equivalent. Beyond this, no more can be said.

(i) *The initial preference relation $\succsim_0$ over $[0, X] \times [0, T]$ is a continuous, monotonic, impatient and midpoint consistent weak order that satisfies weak stationarity.*

(ii) *The initial preference relation $\succsim_0$ over $[0, X] \times [0, T]$ is represented by a real-valued function V_0 such that,*

$$V_0(x, t) = \begin{cases} \alpha^t u(x) & \text{if } t \leqslant h \\ (\alpha/\beta)^h \beta^t u(x) & \text{if } t > h \end{cases}$$

for some $\alpha, \beta \in (0, 1)$, $h \in [0, T]$ and a continuous, strictly increasing $u : X \to \mathbb{R}$.

The following Proposition outlines the uniqueness results pertaining to Theorem 3.1.1. As with Fishburn and Rubinstein's Theorem 2.1.1, the discount factors require careful interpretation. They are not uniquely determined. One cannot, however, choose *both* factors arbitrarily. One of the discount factors, α or β, may be chosen freely but, once chosen, the other factor will be unique; a "joint uniqueness" result. A liminal point is *meaningful* if it is in $(0, T)$ and $\alpha \neq \beta$. Suppose a liminal point is not meaningful. Then, $h \in \{0, T\}$ or $\alpha = \beta$ and liminal discounting collapses to exponential discounting. The liminal point h in the representation above, measured by observing preferences over timed outcomes, is uniquely determined when meaningful.

Proposition 3.1.2 (Uniqueness Results). *For the representation obtained in Theorem 3.1.1, when the liminal point is meaningful, the following hold:*

(i) *The liminal point h is uniquely determined.*

(ii) *One of α or β may be chosen arbitrarily from $(0, 1)$.*

(iii) *Once α or β is chosen, the remaining parameter is uniquely determined.*

(iv) *Once α or β is chosen, utility u is a ratio scale, that is, u may be replaced by v if and only if $v = \lambda u$ for some $\lambda > 0$.*

(v) *The location of utility is fixed: $u(0) = 0$ for any representing u.*

The proofs for Theorem 3.1.1 and Proposition 3.1.2 are in Appendices A.1 and A.2 respectively.

4   Dynamic Liminal Discounting Models

The liminal point is the key ingredient of the liminal discounting model. It is possible, however, to interpret the liminal point in more than one way. Firstly, one may consider it to be an *absolute* point in time (e.g. the birth of a child, retirement, 11th October 2012, etc.). Alternatively, it could be some point in time *relative* to the decision time (e.g. tomorrow, in six months, etc.). So far we have only presented the liminal discounting model as applied to initial preferences. That is, we have only considered a static decision maker assessing timed-outcomes. In this setting, it makes no difference if the liminal point is an absolute or relative time.[7]

When we consider *dynamic* preference structures, however, the distinction becomes relevant. If the liminal point is an absolute point in time h then, as time passes, the decision maker will get closer to it. Such a liminal point will eventually pass. But, the decision maker's preferences would then collapse to exponential discounting. Suppose, on the other hand, that the liminal point is some time relative to the decision maker's current position. Then it could, in absolute terms, be updated as

---

[7]Note that h is expressed in 'absolute' time units. What matters is if the liminal point is "decision time plus h" or just h. Of course, zero plus h is h, so the distinction between these interpretations is void in the initial choice framework.

time passes. It could, for example, remain h units ahead of the decision maker. Such a decision maker will never become an exponential discounter. These different interpretations will lead to dynamically consistent or dynamically inconsistent models of dynamic choice. We will consider these possibilities, absolute time and relative time, in Sections 4.1 and 4.3 respectively. In Sections 4.2 and 4.4 we study how each of these interpretations affects the application of liminal discounting to sequential bargaining.

4.1 Consistent Liminal Discounting

In this section, we present a model we call *consistent liminal discounting* (CLD). Suppose that our decision has liminal discounting initial preferences, as in Theorem 3.1.1. For a CLD preference structure, we ask that the liminal point be expressed in 'absolute' time. Formally, our decision maker is a CLD maximiser if their dynamic preference structure $\mathscr{R}$ is represented by a dynamic model $\mathscr{V}$ where,

$$V_{\underline{t}}(x,t) = \begin{cases} \alpha^t u(x) & \text{if } t \leqslant h \\ (\alpha/\beta)^h \beta^t u(x) & \text{if } t > h \end{cases}$$

for all $V_{\underline{t}} \in \mathscr{V}$ and $(x,t) \in [0,X] \times [\underline{t},T]$, with $h \in [0,T]$, $\alpha, \beta \in (0,1)$ and $u : [0,X] \to \mathbb{R}$ a continuous, strictly increasing function. Observe that, for CLD representations, α, β, h and u do not vary with the decision time. Indeed, each $V_{\underline{t}}$ is simply the restriction of the initial representing function V_0 to timed outcomes occurring later than $\underline{t}$. Since h is expressed in absolute time units, CLD preference structures exhibit time consistency.

Our Theorem 3.1.1 characterises those initial preference relations which admit a

liminal discounting representation. We may use time consistency to extend this result to the dynamic preference structure setting. The following theorem provides a characterisation of dynamic preference structures that exhibit consistent liminal discounting representations.

Theorem 4.1.1. *The following statements are equivalent:*

(i) *The initial preference relation $\succsim_0$ over $[0, X] \times [0, T]$ is a continuous, monotonic, impatient and midpoint consistent weak order that satisfies weak stationarity and the set of preference relations $\mathscr{R}$ satisfies time consistency.*

(ii) *Each preference relation $\succsim_{\underline{t}} \in \mathscr{R}$ can be represented by a real-valued function $V_{\underline{t}}$ where:*

$$V_{\underline{t}}(x,t) = \begin{cases} \alpha^t u(x) & \text{if } t \leqslant h \\ (\alpha/\beta)^h \beta^t u(x) & \text{if } t > h \end{cases}$$

for some $\alpha, \beta \in (0,1)$, $h \in T$ and a continuous, strictly increasing $u : X \to \mathbb{R}$.

Notice that, in Theorem 4.1.1, the set of decision times $\mathscr{D}$ was not endowed with any particular structure. Further, the uniqueness results are those of Proposition 3.1.2, applying across each decision time. The proof of Theorem 4.1.1 is in Appendix A.3.

We have obtained a simple characterisation of consistent liminal discounting structures. CLD preferences satisfy weak stationarity and time consistency. By Halevy's result (Theorem 2.2.1), this model cannot satisfy time invariance without collapsing to exponential discounting. We will now dispense with weak stationarity and consider an axiom that applies to the dynamic preference structure, rather than any instantaneous preference relation. Our new condition here is *weak time invariance*. We must first introduce the notions of *time-invariance-before-t* and *time-invariance-after-t*. The former is as follows:

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Definition (Time-Invariance-before-t): A dynamic preference structure $\mathscr{R}$ satisfies *time-invariance-before-t* if, for all $(x, t'), (y, t''), (x, t'-\tau), (y, t''-\tau) \in [0, X] \times [0, T]$ with $0 \leqslant \tau \leqslant \underline{t} < t', t'' \leqslant t$, the following holds:

$$(x, t') \succcurlyeq_{\underline{t}} (y, t'') \implies (x, t' - \tau) \succcurlyeq_{\underline{t}-\tau} (y, t'' - \tau).$$

Time-invariance-before-t of a dynamic preference structure asks that, given $t \in [0, T]$, the dynamic preference structure is time invariant whenever the timed-outcomes and decision times are no later than t. That is, if we observe a preference between two outcomes with the latest occurring before time t, then the preference does not change if we bring both outcomes and the decision time forward in time by the same amount. Notice that this is a one-way implication; the later preference implying the earlier. The analogous condition, *time-Invariance-after-t*, is defined as follows:

Definition (Time-Invariance-after-t): A dynamic preference structure $\mathscr{R}$ satisfies *time-invariance-after-t* if, for all $(x, t'), (y, t''), (x, t'+\tau), (y, t''+\tau) \in [0, X] \times [0, T]$ with $0 \leqslant \tau$ and $\underline{t} \leqslant t \leqslant t', t''$, the following holds:

$$(x, t') \succcurlyeq_{\underline{t}} (y, t'') \implies (x, t' + \tau) \succcurlyeq_{\underline{t}+\tau} (y, t'' + \tau).$$

Time-invariance-after-t of a dynamic preference structure asks that, for a given t, the structure is time invariant whenever the timed-outcomes and decision time are no earlier than t. That is, if we observe a preference between two outcomes with the earliest occurring after time t, then the preference does not change if we delay both outcomes and the decision time by the same amount. Notice again that this is a one-way implication; the preference regarding earlier outcomes implying that for the later ones.

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Consider the consistent liminal discounting model again and a simple property emerges. Recall that, in this model, the liminal point h is an absolute point in time. Consistent liminal discounting preferences must, therefore, satisfy time-invariance-before-h and time-invariance-after-h. Of course, we cannot state this as an axiom. We do not know h *a priori*. Our axiom must be formulated so that the existence of h and the appropriate representation are implied. We offer the following axiom, *Weak Time Invariance*:

Axiom (Weak Time Invariance): At any time $t \in T$, the set of preference relations $\mathscr{R}$ satisfies time-invariance-before-t, or time-invariance-after-t, or both.

Notice that weak time invariance provides a testable condition that must hold at *any* t in $[0, T]$. The following theorem proves the equivalence of our previous axiom set, using weak stationarity, and an axiom set using weak time invariance.

Theorem 4.1.2. *Let the set of decision times $\mathscr{D} = [0, T]$. Then, the following statements are equivalent:*

(i) *The initial preference relation $\succcurlyeq_0$ over $[0, X] \times [0, T]$ is a continuous, monotonic, impatient and midpoint consistent weak order and the set of preference relations $\mathscr{R}$ satisfies time consistency and weak time invariance.*

(ii) *Each preference relation $\succcurlyeq_{\underline{t}} \in \mathscr{R}$ can be represented by a real-valued function $V_{\underline{t}}$ where:*

$$V_{\underline{t}}(x, t) = \begin{cases} \alpha^t u(x) & \text{if } t \leqslant h \\ (\alpha/\beta)^h \beta^t u(x) & \text{if } t > h \end{cases}$$

for some $\alpha, \beta \in (0, 1)$, $h \in T$ and a continuous, strictly increasing $u : X \to \mathbb{R}$.

The uniqueness results pertaining to Theorem 4.1.2 are the same as those in Proposition 3.1.2, applied at each decision time. The proof of Theorem 4.1.2 is in Appendix

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A.4.

4.2   Consistent Liminal Discounting in Bargaining

In this section we present an application of consistent liminal discounting (CLD). We study CLD preferences in the infinite-horizon, alternating offers bargaining model of Rubinstein (1982). This application highlights how new results may be generated in many areas by exploiting the special structure of CLD preferences. The main insight can be easily explained before we proceed. Suppose we have an infinite-horizon game where the players have CLD preferences. Then *eventually* (that is, after the last of the players' liminal points) the game will become an infinite-horizon game where each player has exponential discounting preferences. One may then use existing results to find the subgame perfect Nash equilibrium (SPE) of this subgame. Then, one may find the SPE of the overall game by: truncating the game at the latest liminal point, assigning each player their (sub-)SPE payoff instead, and then solving the truncated game by simple backward induction. We now apply this 'trick' to the Rubinstein model to gain new insights into bargaining theory.

The original game $\mathcal{G}$ is as follows. There are two players, 1 and 2, and a surplus of (normalised) size 1. The players have exponential discounting (ED) preferences with linear utility, that is, an outcome of x at time t gives utility at time zero of $\delta_i^t x$, where $\delta_i \in (0,1)$ is player i's discount factor. The players alternate in proposing and considering offers regarding how the surplus should be divided. Player 1 proposes first at $t = 0$ and player 2 may accept or reject the proposal. If player 2 accepts the proposal, the game ends at that point and the payoffs are those specified in player 1's offer. If player 2 rejects the proposal, the game continues to $t = 1$ at which time the players' previous roles are exchanged. The game continues, perhaps indefinitely,

with players 1 and 2 making offers at every even and odd t, respectively.

As a special case of what Rubinstein (1982) studied, the game $\mathcal{G}$ has a unique SPE. The SPE prescribes an immediate agreement; player 1 offers the division $(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2})$, which player 2 accepts and the game ends.

We now consider a game $\mathcal{G}'$ that is the same as $\mathcal{G}$, except that player 2 has CLD preferences with $h = 1$. If the game were to continue to $t = 2$, then the remaining game is a standard Rubinstein bargaining model with player 1 as first mover. So, the game starting at $t = 2$ has a unique SPE with payoffs $(\frac{1-\beta_2}{1-\delta_1\beta_2}, \frac{\beta_2(1-\delta_1)}{1-\delta_1\beta_2})$. To solve the game $\mathcal{G}'$ we construct a truncated game $\mathcal{G}'(2)$ with the following structure. $\mathcal{G}'(2)$ is the same as $\mathcal{G}'$ at times 0 and 1. If the game reaches $t = 1$ and player 1 rejects player 2's offer, instead of continuing to $t = 2$ the game ends and players 1 and 2 are assigned payoffs $\frac{1-\beta_2}{1-\delta_1\beta_2}$ and $\frac{\beta_2(1-\delta_1)}{1-\delta_1\beta_2}$, respectively. These are SPE payoffs of the usual Rubinstein game with players 1 and 2 having constant discount factors δ_1 and β_2.

A payoff profile $(x_1, 1 - x_1)$, paid immediately, occurs as a SPE of $\mathcal{G}'$ if and only if it occurs as a SPE of $\mathcal{G}'(2)$. The truncated game $\mathcal{G}'(2)$ is solved by backward induction. At $t = 1$, by standard arguments, player 2 should offer player 1 a share of $x^1 = \delta_1 \frac{1-\beta_2}{1-\delta_1\beta_2}$. At $t = 0$ player 1 should offer player 2 a share of $1 - x^0 = \alpha(1 - x^1)$ implying x^0 for himself. This is immediately accepted by player 2, so the unique SPE involves immediate agreement and payoffs $(x^0, 1 - x^0)$ where:

$$x^0 = \frac{1 - [\delta_1\beta_2 + (1 - \delta_1)\alpha_2]}{1 - \delta_1\beta_2}.$$

Notice that $x^0 = \frac{1-\beta_2}{1-\delta_1\beta_2}$ if and only if $\alpha_2 = \beta_2$. The agreement is still reached immediately, however the SPE is different as the incentives to delay agreement are different. The change in discount rate that occurs after player 2's liminal point affects the SPE

payoffs in a predictable way.

We can construct examples where the player who is more impatient *at the time of agreement* does better. In such cases the player in question will, at some point, become less impatient than his opponent. This fact must, therefore, be integrated into the determination of the equilibrium payoffs *even though the equilibrium prescribes agreement without any delay.* We have considered the simplest Rubinstein game here (linear utility, pie of size 1, only one player having non-ED preferences) but it is straightforward to extend the analysis.

4.3   Invariant Liminal Discounting

In the previous sections we have presented the foundations and an application of the consistent liminal discounting model. In this section we present our second dynamic model. We call this model *invariant liminal discounting.* In this case, we think of the liminal point as being expressed in relative time. Rather than a fixed point in time that the decision maker approaches, the liminal point will be some fixed distance in front of them *at all decision times.* It will be "decision time plus h", rather than just h. This type of decision maker lives in a state of constant limbo. They consistently fix on their (subjective) present and never get past it. The liminal period, from current decision time to h units ahead, can be thought of as representing a level of *myopia.*

Formally, our decision maker is a invariant liminal discounter if their dynamic preference structure $\mathscr{R}$ is represented by a dynamic model $\mathscr{V}$ where,

$$V_{\underline{t}}(x,t) = \begin{cases} \alpha^t u(x) & \text{if } t \leqslant \underline{t} + h \\ (\alpha/\beta)^{\underline{t}+h}\beta^t u(x) & \text{if } t > \underline{t} + h \end{cases}$$

for all $V_{\underline{t}} \in \mathscr{V}$ and $(x,t) \in [0,X] \times [\underline{t},T]$, with $h \in [0,T]$, $\alpha, \beta \in (0,1)$ and $u : [0,X] \to \mathbb{R}$ a continuous, strictly increasing function.

The main difference between invariant and consistent liminal discounting is that the decision time $\underline{t}$ now plays an important role. At each decision time, an invariant liminal discounter will evaluate timed outcome (x,t) using discount factor α if t is less than h units from the current decision time $(t \leqslant \underline{t} + h)$. They will use discount factor β if t is more than h units in front of the current decision time $(t > \underline{t} + h)$. Such preferences are not time consistent (unless $\alpha = \beta$) because the evaluation of a timed outcome (x,t) changes with the decision time. They are, however, time invariant.

We use time invariance, combined with our initial conditions for liminal discounting, to characterise those sets of preference relations of the invariant liminal discounting class:

Theorem 4.3.1. *The following statements are equivalent:*

(i) *The initial preference relation $\succsim_0$ over $[0,X] \times [0,T]$ is a continuous, monotonic, impatient and midpoint consistent weak order that satisfies weak stationarity and the set of preference relations $\mathscr{R}$ satisfies time invariance.*

(ii) *Each preference relation $\succsim_{\underline{t}} \in \mathscr{R}$ can be represented by a real-valued function $V_{\underline{t}}$ where:*

$$V_{\underline{t}}(x,t) = \begin{cases} \alpha^t u(x) & \text{if } t \leqslant \underline{t} + h \\ (\alpha/\beta)^{\underline{t}+h} \beta^t u(x) & \text{if } t > \underline{t} + h \end{cases}$$

for some $\alpha, \beta \in (0,1)$, $h \in T$ and a continuous, strictly increasing $u : X \to \mathbb{R}$.

The uniqueness results pertaining to Theorem 4.3.1 are the same as those in Proposition 3.1.2, applied at each decision time. The proof of Theorem 4.3.1 is in Appendix A.5.

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Invariant liminal discounting preference structures may violate time consistency. Time consistency is the most important normative property of the classical exponential discounting model. It is important, therefore, to obtain a complete understanding of precisely how and when invariant liminal discounting preferences violate this property. We will now provide a foundation for invariant liminal discounting, this time with the key axioms applying to the dynamic preference structure. That is, we will dispense with the weak stationarity axiom assumed in Theorem 4.3.1. We then introduce a relaxation of the time consistency axiom we call *weak time consistency*. Weak time consistency will permit violations of time consistency only when comparing timed outcomes that are subjectively far enough apart in time.

Before presenting our weak time consistency axiom, it is necessary to develop two conditions: *Time-consistency-within-t-from-now* and *time-Consistency-beyond-t-from-now*. The former is as follows:

Definition (Time-consistency-within-t-from-now): A set of preference relations $\mathscr{R}$ satisfies *time-consistency-within-t-from-now* if for all $\succcurlyeq_{\underline{t}} \in \mathscr{R}$ and all $(x,t),(y,t') \in [0,X] \times [0,T]$ with $\underline{t} \leqslant t' \leqslant t$ the following holds:

$$(x,t) \succcurlyeq_0 (y,t') \;\Rightarrow\; (x,t) \succcurlyeq_{\underline{t}} (y,t').$$

Time-consistency-within-t-from-now of a dynamic preference structure demands that, for timed-outcomes occurring before t, initial preferences are not later reversed. The initial preference relation $\succcurlyeq_0$ plays a key role in this axiom. Preference relations at later points in time, $\succcurlyeq_{\underline{t}}$ with $\underline{t} \leqslant t$, are forced to agree with the initial preference relation. This is a one-way implication, with initially expressed preferences implying the later ones. Consider the invariant liminal discounting representation again and

suppose $t \leqslant h$. Then, the initial preference is formed comparing $\alpha^t u(x)$ and $\alpha^{t'} u(y)$ (since $t' \leqslant t$). For any later decision time $\underline{t}$, of course we have $t \leqslant h \leqslant \underline{t} + h$. Essentially, the timed-outcome that was t-from-now when "now" is time zero, must be within-t-from-now when "now" is later. So, an invariant liminal discounter would keep using discount factor α and the preference cannot be reversed. That is, invariant liminal discounting preference structures must satisfy time-consistency-within-t-from now whenever $t \leqslant h$. This condition need not hold for invariant liminal discounters when $t > h$, but in this case there is an analogous condition:

Definition (Time-consistency-beyond-t-from-now): A set of preference relations $\mathscr{R}$ satisfies *time-consistency-beyond-t-from-now* if for all $\succcurlyeq_{\underline{t}} \in \mathscr{R}$ and all $(x, t + \underline{t}), (y, t' + \underline{t}) \in [0, X] \times [0, T]$ with $t \leqslant t'$ the following holds:

$$(x, t + \underline{t}) \succcurlyeq_{\underline{t}} (y, t' + \underline{t}) \implies (x, t + \underline{t}) \succcurlyeq_0 (y, t' + \underline{t}).$$

Time-consistency-beyond-t-from-now of a dynamic preference structure demands that, for timed-outcomes occurring no sooner than t after decision time $\underline{t}$, preferences expressed are respected by initial preferences. Again, it is a one-way implication with the later preference implying the earlier, initial preference. Time-consistency-beyond-t-from-now, when combined with time invariance, will be equivalent to stationarity-*after-t*. The distance between decision time and the timed outcomes increases when comparing the first and second preference expressions. That is, at time decision $\underline{t}$ a timed-outcome occurring at time $t + \underline{t}$ is "t-from-now". As "now" becomes earlier, back to time zero, the same timed-outcome is beyond-t-from-now. Under time invariance, the former preference implies $(x, t) \succcurlyeq_0 (y, t')$. We see that the implied preference above completes the requirement for stationarity-after-t. We are now set

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to define the characteristic axiom for invariant liminal discounting.

Axiom (Weak Time Consistency): At any time $t \in T$, the set of preference relations $\mathscr{R}$ satisfies time-consistency-within-t-from-now, or time-consistency-beyond-t-from-now, or both.

We have already explained how time-consistency-beyond-t-from-now, under the assumption of time invariance of $\mathscr{R}$, is equivalent to stationarity-after-t. Under the same assumptions, time-consistency-within-t-from-now of $\mathscr{R}$ can be shown to be equivalent stationarity-before-t of $\mathscr{R}$. Weak time consistency, although by itself completely distinct from weak stationarity, must be equivalent to weak stationarity for time invariant structures. This is summarised in the following equivalence theorem:

Theorem 4.3.2. *Let the set of decision times $\mathscr{D} = [0, T]$. Then, the following statements are equivalent:*

(i) *The initial preference relation $\succsim_0$ over $[0, X] \times [0, T]$ is a continuous, monotonic, impatient and midpoint consistent weak order and the set of preference relations $\mathscr{R}$ satisfies time invariance and weak time consistency.*

(ii) *Each preference relation $\succsim_{\underline{t}} \in \mathscr{R}$ can be represented by a real-valued function $V_{\underline{t}}$ where:*

$$V_{\underline{t}}(x, t) = \begin{cases} \alpha^t u(x) & \text{if } t \leqslant \underline{t} + h \\ (\alpha/\beta)^{\underline{t}+h} \beta^t u(x) & \text{if } t > \underline{t} + h \end{cases}$$

for some $\alpha, \beta \in (0, 1)$, $h \in T$ and a continuous, strictly increasing $u : X \to \mathbb{R}$.

The uniqueness results pertaining to Theorem 4.3.2 are the same as those in Proposition 3.1.2, applied at each decision time. The proof of Theorem 4.3.2 is in Appendix A.6.

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4.4 Invariant Liminal Discounting in Bargaining

In this section we present an application of the invariant liminal discounting model to the Rubinstein bargaining game. This application will highlight, firstly, how simple it is to apply the model and, secondly, the differences from the consistent liminal discounting case.

A problem arises once time consistency is dropped. A time inconsistent decision maker has different preferences at different decision times. Effectively, they are a different decision maker at each decision time. Although the decision maker is a collection of selves, the "decision time self" gets to decide how to treat the other selves. So, one must consider how this decision time self handles the time inconsistency. Three strategies advanced in the literature are *naive*, *resolute* and *sophisticated* choice (Blackorby, Nissen, Primont and Russell, 1973; Hammond, 1976; Machina, 1989; O'Donogue and Rabin, 2001; Hey and Lotito, 2009; Hey and Panaccione, 2011). We address each of these as applied to invariant liminal discounting and sequential bargaining.

The naive approach to address time inconsistency is, essentially, to ignore it. That is, the decision time self acts as if they are ignorant of the fact that their preferences will change. In general, this leads to different outcomes to time consistent choice. In the case of the Rubinstein bargaining game, however, the outcome will be as the time consistent case. The players act now *as if* they will never violate time consistency. Then, all the hypothesising about what they will do at subsequent decision times will lead to the consistent liminal discounting outcome. Since the naive subgame perfect equilibrium prescribes immediate agreement, the players will never learn of their preferences changing.

A *resolute* decision maker (Machina, 1989) achieves time consistent behaviour by sticking to their initial plan, regardless of their evolving preferences. As with the naive case, the imposition of the initial preferences leads to immediate agreement. So, the consistent liminal discounting outcome is implemented. Agreement occurs immediately, so we need not appeal to any form of commitment device. In this application, the difference between naive and resolute choice is only the underlying reasoning. A naive decision maker does not know his preferences will change; a resolute decision maker does not care.

The final case we consider is *sophisticated* choice. In this case, the time inconsistent decision makers are fully aware of their future preferences. They fully anticipate the optimal choices of their future selves and integrate this into their current strategy. To address this case, we will consider a minor variation of the game studied in Section 4.2. Instead of the alternating offers occurring at times 0,1,2, and so on we will suppose there is a delay of Δ. So, offers occur at times $0,\Delta,2\Delta$, and so on. We will assume, for ease of exposition, that player 1 has exponential discounting preferences and player 2 is a sophisticated, invariant liminal discounter.

A thorough analysis of time invariant and multiplicatively separable preferences has been conducted by Kodritsch (2012). We begin by studying a simple type of solution, Kodritsch (2012) calls a *Rubinstein Equilibrium*. Some notation is required. Let $f_i : [0,1] \times [0,T] \rightarrow [0,1]$ be such that, for all $(x,t) \in [0,1] \times [0,T]$, $f_i(x_i,t) = u^{-1}(V_0^i(x_i,t))$. So, $f_i(x_i,t)$ is the smallest share that player i will accept immediately over share x_i at time t. A *Rubinstein pair* is any pair of profiles $(x^*, 1-x^*)$ and $(y^*, 1-y^*)$ such that:

$$y^* = f_1(x^*,\Delta) \quad \& \quad (1-x^*) = f_2((1-y^*),\Delta)$$

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A Rubinstein Equilibrium occurs when: $(x^*, 1-x^*)$ and $(y^*, 1-y^*)$ form a Rubinstein pair, player 1 uses the strategy "always offer x^* and always accept offers of at least y^*", and player 2 uses the strategy "always offer y^* and accept offers of at least $1 - x^*$". Rubinstein equilibrium is determined entirely by the players attitudes to single period delays (of length Δ). Under time invariance, attitudes to single period delays never change.

Suppose that the bargaining delay Δ is longer than player 2's horizon h. Then, every time player 2 has to make a decision of the accept or reject now, they must be using discount factor β. In each round, by time invariance, they effectively reset their stopwatch and use the same factor again. Because of this, it must be that the Rubinstein equilibrium is the same if we construct a new game, replacing player 2 with an exponential discounter with discount factor β. Suppose, instead, that the delay Δ is smaller than h. Then, the Rubinstein equilibrium of the game will be the same if we replace player 2 with an exponential discounter with discount factor α. Summing up, the Rubinstein equilibrium of the bargaining game with player 2 a sophisticated, invariant liminal discounter, involves immediate agreement and payoffs $(x^0, 1 - x^0)$ where:

$$
x^0 = \begin{cases} \frac{1-\alpha^\Delta}{1-\delta^\Delta \alpha^\Delta} & \text{if } \Delta \leqslant h \\ \frac{1-\beta^\Delta}{1-\delta^\Delta \beta^\Delta} & \text{if } \Delta > h \end{cases}
$$

It seems that the departure from exponential discounting is more pronounced when the players are consistent rather than invariant liminal discounters. This seems unusual, especially because the consistent model is the one with which we bestow normative connotations. Of course, this is not quite the full story. While the invariant case does seem closer to the standard case, the outcome is much more sensitive to

the specification of the game. In particular, the payoffs do not vary continuously with the bargaining delay. There is a pronounced jump in the equilibrium partition as Δ becomes longer or shorter than the liminal period.

We have fully characterised the Rubinstein equilibrium. It remains to justify the use of this solution concept for the bargaining game under consideration. One can consider the bargaining game with sophisticated, time inconsistent players as a game with an infinite number of players. *Strotz-Pollak equilibrium* is the subgame perfect Nash equilibrium of that game. As such, Strotz-Pollak equilibrium is the appropriate solution concept for sophisticated choice. For more on Strotz-Pollak equilibrium, see Shefrin (1998). The Rubinstein equilibrium we have discussed is clearly a Strotz-Pollak equilibrium. Although the Rubinstein equilibrium is unique, there may be other Strotz-Pollak equilibria. Kodritsch (2012) provides, however, in his Theorem 4 a sufficient condition for the unique Rubinstein equilibrium we derived to be the unique Strotz-Pollak equilibrium of the bargaining game. This sufficiency condition is indeed elegant, asking only that preferences satisfy a weak present-bias condition. Call preferences *separable* if they are represented by a function that maps (x, t) to $D(t)u(x)$, with D strictly decreasing and positive with $D(0)=1$, and u strictly increasing. Liminal discounting preferences are separable. Kodritsch's present bias condition requires that $D(\Delta) \leqslant D(t + \Delta)/D(t)$ holds for all $t \in [0, T]$. This is equivalent, for Liminal Discounters, to $\alpha \leqslant \beta$. A natural, but in any case testable, requirement.

5   Further Comments

We presented the liminal discounting model in the timed-outcomes framework. This is ideally suited for bargaining applications, although many economic applications will require the model to be extended to sequences of outcomes. A *sequence*, denoted $(x_0, t_0; ...; x_n, t_n)$ gives a payment x_i at time t_i with $i = 0, \ldots, n$. When considering initial choice over sequences of outcomes, the obvious formula is,

$$V_0(x_0, t_0; \ldots; x_n, t_n) = \sum_{t_i \leqslant h} \alpha^{t_i} u(x_i) + \sum_{t_i > h} \left(\frac{\alpha}{\beta}\right)^h \beta^{t_i} u(x_i)$$

for $h \in [0, T]$, $\alpha, \beta \in (0, 1)$ and $u : [0, X] \to \mathbb{R}$ a continuous, strictly increasing function. Extending this model to the dynamic choice setting is the same as for timed-outcomes. Time consistency and time invariance have natural definitions for sequences of outcomes. Under the assumption that h is an absolute point in time, the function represents time consistent preferences for sequences. Alternatively, as before, the assumption that h is expressed relative to decision time represents time invariant preferences. A simple example, separating the consistent and invariant case, is that of *procrastination*. Following Prelec (2004), we say a decision maker procrastinates if, for $x, y > 0$, $\underline{t}, \underline{t} + s \in [0, T]$ and $\underline{t} > s > 0$, we have $(-x, \underline{t}; y, \underline{t} + s) >_0$ $(-x, 0; y, \underline{t} + s) >_0 0$ and $(-x, \underline{t}; y, \underline{t} + s) <_{\underline{t}} 0$. That is, at time 0 the decision maker prefers to face an immediate cost of x to get y later over doing nothing, but delaying the cost is better. At decision time $\underline{t}$, however, he prefers to do nothing. We see immediately that these preferences are not time consistent, so the absolute h version of this model cannot explain procrastination. The time invariant version, with h

between s and $\underline{t}$ from current decision time, allows for procrastination whenever:

$$\alpha^s < \frac{-u(-x)}{u(y)} \leqslant \beta^s.$$

A moving liminal point, combined with decreasing impatience, can explain procrastination. We leave the full investigation of the axiomatic characterisation and application of these extensions for further research.

6   Conclusion

In this paper we have presented the liminal discounting model, given it an axiomatic foundation, extended it to the dynamic framework in two different ways, and in each dynamic interpretation provided two different axiomatic foundations. These characterisations provide simple, testable conditions (weak stationarity, weak time consistency and weak time invariance) that merit empirical study. We have also shown how the model may be applied, in each of its dynamic incarnations, to sequential bargaining. The Rubinstein bargaining game represents something of a benchmark for demonstrating a non-exponential discounting model's applicability. That the applications to bargaining turned out to be straightforward is encouraging; further applications should be achievable.

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A Appendices

A.1 Proof of Theorem 3.1.1

First suppose that the initial preference relation is represented as in statement (ii) of the theorem. That this implies statement (i) is straightforward. weak ordering and continuity are immediate. Monotonicity follows as u is strictly increasing and $\alpha, \beta > 0$, and impatience follows as α and β are both in $(0,1)$. Define a function $D : [0,T] \to \mathbb{R}$ such that $D(t) = \alpha^t$ if $t \leqslant h$ and $D(t) = (\alpha/\beta)^h \beta^t$ if $t > h$. So, preferences are represented by a function that maps (x,t) to $D(t)u(x)$. For midpoint consistency, suppose (for instance) that the first three indifferences of the condition hold. Under statement (ii) this is equivalent to:

$$D(t)u(x) = D(t')u(y) \ \& \ D(t)u(y) = D(t')u(z) \ \& \ D(s)u(x) = D(s')u(y)$$

First notice that, if any of x, y or z are zero, then they are all zero. In that case, the condition holds, given our extended definition of impatience. Suppose now they are all non-zero, so $u(x), u(y), u(z) > 0$. The first two equalities jointly imply, $u(x)/u(y) = u(y)/u(z) = D(t')/D(t) := \lambda$. Given this, the third implies $D(s')/D(s) = \lambda$. The equality $D(s)u(y) = D(s')u(z)$ follows immediately, as does the equivalent, required, fourth indifference. The necessity of weak stationarity, given statement (ii), follows from the fact that we know h. If h is zero or T then all of the conditions of weak stationarity hold at all times. Suppose $h \in (0,T)$. Taking any time $t \in [0,T]$, Stationarity-before-t holds whenever $t \leqslant h$ and stationarity-after-t holds when $t > h$. Both conditions hold at h. This covers all cases, so establishing weak stationarity.

For the remaining part of the proof we assume statement (i) of the theorem and derive statement (ii). We first outline some of the implications of weak stationarity.

Under weak ordering, and using the definitions behind weak stationarity, if $\succeq_0$ satisfies stationarity-after-t then it satisfies stationarity-after-t' for any $t' > t$. Similarly, if $\succeq_0$ satisfies stationarity-before-t then it satisfies stationarity-before-t' for any $t' < t$. We will now show that if stationarity-before-t and stationarity-after-t' hold with $t > t'$, then stationarity holds everywhere. To see this, suppose that the conditions of the claim are true. The restriction of preferences to $[0, X] \times [0, t]$ satisfies all the conditions of Theorem 2.1.1, so admits an exponential discounting representation. The same holds for preferences restricted to $[0, X] \times [t', T]$. By the uniqueness results attached to Fishburn and Rubinstein's theorem, we can choose the same δ for each case. Then, there will be a u such that (x, t) mapped to $\delta^t u(x)$ represents preferences on $[0, X] \times [0, t]$, and a $\tilde{u}$ such that (x, t) mapped to $\delta^t \tilde{u}(x)$ represents on $[0, X] \times [t', T]$. By assumption, there is a set $[0, X] \times [t', t]$ where both functions must represent preferences, so they can be chosen to be equal. Then, preferences over the whole set of timed-outcomes admit one exponential discounting representation, so stationarity must necessarily hold everywhere.

We now show the existence of h. Firstly, if the initial preference relation is stationary, then we may choose either $h = 0$ or $h = T$. Suppose now that the conditions of (ii) hold, but stationarity does not hold. We use the following definitions:

$$t_* = \sup\{t \in [0, T] : \succeq_0 \text{ satisfies Stationarity-before-}t\}$$

$$t^* = \inf\{t \in [0, T] : \succeq_0 \text{ satisfies Stationarity-after-}t\}$$

Weak stationarity demands that $[0, T] = [0, t_*] \cup [t^*. T]$. By connectedness, if the

union of $[0, t_*]$ and $[t^*, T]$ cover $[0, T]$, they must have a non-empty intersection. We cannot have $t_* > t^*$, or else stationarity would hold everywhere as argued above. Then, there is a unique point in this intersection, $t_* = t^* := h$ as required.

To complete the theorem when $h = 0$ or $h = T$, simply notice that the conditions coincide with Theorem 2.1.1. For the remainder of the proof, we consider the $h \in (0, T)$ case. One may proceed in various ways from this point. We apply Observation 4.1 of Bleichrodt, Kothiyal, Prelec and Wakker (2012) to derive a separable representation: preferences are represented by a function that maps (x, t) to $D(t)u(x)$, with D continuous, strictly decreasing and positive with $D(0) = 1$, and u continuous and strictly increasing.[8] To determine the structure of $D : [0, T] \to \mathbb{R}_{++}$ we consider its behaviour on $[0, h]$ and $[h, T]$ separately. As shown, preferences satisfy stationarity-before-h. Then, for $t, s, t+s \leqslant h$ and $x, x' \in [0, X]$ the following equivalence holds: $(0, x) \sim (t, x')$ if and only if $(s, x) \sim (t+s, x')$. The existence of suitable x and x' is straightforward. Substituting the separable representation we obtain: $u(x) = D(t)u(x')$ if and only if $D(s)u(x) = D(t+s)u(x')$. Equivalently, D satisfies the following local functional equation:

$$D(t+s) = D(t)D(s) \quad t, s, t+s \in [0, h].$$

This is the second of Cauchy's functional equations, restricted to a connected subset of the reals. The classic approach to solving this applies to the case where the equation holds on all of $\mathbb{R}$. So, one must show that there is an extension of D that preserves the functional equation. This has been addressed by Aczel and Skof (2007), whose results apply here as D is strictly positive. The general, continuous solution gives $D(t) = \lambda\alpha^t$ for all $t \in [0, h]$ for non-zero α and λ. The initial condition,

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[8]What we call midpoint consistency is referred to as the hexagon condition, and as 1-unit invariance in Bleichrodt, Kothiyal, Prelec and Wakker (2012).

$D(0) = 1$, gives $\lambda = 1$.

The existence of the separable representation of preferences, when combined with stationarity-after-h, will lead to a local functional equation on $[h, T]$. A minor subtlety is needed as there is no $t \in [h, T]$ with $D(t) = 1$. So, define a function $\tilde{D}$ such that $\tilde{D}(t) = D(t)/D(h)$ for all $t \in [h, T]$. Notice that $\tilde{D}()u()$ still represents preferences and that $\tilde{D}(h) = 1$. Stationarity-after-h guarantees that, for $t, s, t + s \geqslant h$ and $x, x' \in [0, X]$, $(h, x) \sim (t, x)$ if and only if $(h + s, x) \sim (t + s, x')$. Substituting the rescaled representation gives:

$$\tilde{D}(t + s) = \tilde{D}(t)\tilde{D}(s) \quad t, s, t + s \in [h, T]$$

The general, continuous solution is of the form $\tilde{D}(t) = \tilde{\lambda}\beta^t$ for all $t \in [h, T]$, for non-zero $\tilde{\lambda}$ and β. That $\tilde{\lambda} = \beta^{-h}$ follows immediately from the initial condition. Recall that $D = D(h)\tilde{D}$ on $[h, T]$. Summing up, we have shown that:

$$V_{\underline{t}}(x, t) = D(t)u(x) = \begin{cases} \alpha^t u(x) & \text{if } t \leqslant h \\ (\alpha/\beta)^h \beta^t u(x) & \text{if } t > h \end{cases}$$

as required. ∎

A.2 Proof of Proposition 3.1.2

Under the conditions of Theorem 3.1.1, preferences admit a liminal discounting representation $V_0 : [0, X] \times [0, T] \to \mathbb{R}$ for some parameters $\alpha, \beta \in (0, 1)$, $h \in [0, T]$ and utility function $u : [0, X] \to \mathbb{R}$. The uniqueness of h, when $h \notin \{0, T\}$, has been explained in the proof of Theorem 3.1.1 in Appendix A.1. Either h is unique, or else stationarity must hold everywhere.

Since V_0 represents preferences, it can be replaced by $f \circ V_0$ whenever f is strictly increasing. In general, such transformations need not retain the separable form. To remain within the class, one may replace α with any $\alpha' \in (0,1)$ by applying the transformation $f \circ V_0 = \exp(ln V_0^{ln(\alpha')/ln(\alpha)})$ and defining a new utility $u' = u^{ln(\alpha')/ln(\alpha)}$. Alternatively, one may choose any $\beta' \in (0,1)$, again defining a different utility function based on a similar transformation.

Once α or β are chosen, utility must be a ratio scale. This follows from well-known results on separable representations. Typically one may replace utility with an affine transformation, although in this case the location of utility is fixed. To see this, recall that we included the condition $(0,t) \sim_{\underline{t}} (0,t')$, for any $t,t' \in [0,T]$, in the definition of impatience. Then $u(0) = 0$ holds, or else the representation would not exhibit impatience.

Having chosen either of α or β, however, the other is uniquely determined. To see this, one may take any $x < y$ and find a unique t such that $(x,0) \sim (y,t)$. Substituting the representation and rearranging gives:

$$\beta = \left[\frac{u(x)}{\alpha^h u(y)} \right]^{\frac{1}{t-h}}$$

Given that u is a ratio scale, the right hand side of the above equation is dimensionless. So β for given α (or vice versa) is uniquely determined. $\blacksquare$

A.3   Proof of Theorem 4.1.1

First assume statement (ii) of the theorem holds. At each decision time, the instantaneous preference relation admits a liminal discounting representation. By the Theorem 3.1.1, the initial preference is a continuous, monotonic, impatient and mid-

point consistent weak order that satisfies weak stationarity. The assumed dynamic model is such that, each decision time's representation is the restriction of the initial representation to timed-outcomes occurring no sooner than that decision time. At no time, therefore, can initial preferences be reversed. As such, the dynamic preference structure is time consistent and statement (i) is proved.

We now assume the conditions of statement (i) and derive statement (ii). For $\underline{t} \in \mathscr{D}$, time consistency allows us to identify $\succcurlyeq_{\underline{t}}$ with $\succcurlyeq_0 |_Z$ where $Z = X \times [\underline{t}, T]$; the restriction of initial preferences $\succcurlyeq_0$ to the set of timed-outcomes occurring no sooner than time $\underline{t}$. Hence, by Theorem 3.1.1, each $\succcurlyeq_{\underline{t}}$ may be represented by $V_{\underline{t}} := V_0 |_Z$ as required. ∎

A.4   Proof of Theorem 4.1.2

For the proof of Theorem 4.1.2 we fix a set of weakly ordered, monotonic, impatient, midpoint consistent and continuous preference relations $\mathscr{R}$. We prove that if $\mathscr{R}$ is time consistent then, stationarity-before-t of $\succcurlyeq_0$ is equivalent to time-invariance-before-t of $\mathscr{R}$. Let $(x, t), (y, t'), (x, t - \tau), (y, t' - \tau) \in [0, X] \times [0, T]$ with $\tau \geqslant 0$ and $t \geqslant t'$. The following diagram aids the proof of the theorem:

$$(x, t) \succcurlyeq_0 (y, t') \quad \Rightarrow^1 \quad (x, t - \tau) \succcurlyeq_0 (y, t' - \tau)$$

$$\Updownarrow^2 \qquad\qquad\qquad\qquad \Updownarrow^3$$

$$(x, t) \succcurlyeq_{\underline{t}} (y, t') \quad \Rightarrow^4 \quad (x, t - \tau) \succcurlyeq_{\underline{t} - \tau} (y, t' - \tau)$$

Note that implication 1 is stationarity-before-t, equivalences 2 and 3 are time consistency, and implication 4 is time-invariance-before-t. Notice that implication 1 may

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be deduced by starting at the top left preference, then 2, then 4, and then 3. Implication 4 may be deduced by starting with the bottom left preference, then 2, then 1, and then 3. The equivalence of time consistency with stationarity-after-t, and time consistency with time-invariance-after-t may be similarly shown. Then, statement (i) of Theorem 4.1.1 and statement (i) of Theorem 4.1.2 are equivalent and the CLD representation follows from the proof in Appendix A.3. ∎

A.5   Proof of Theorem 4.3.1

First assume statement (ii) of the theorem holds. At each decision time, the instantaneous preference relation admits a liminal discounting representation. By the Theorem 3.1.1, the initial preference is a continuous, monotonic, impatient and midpoint consistent weak order that satisfies weak stationarity. The assumed dynamic model is such that, each decision time $\underline{t}$ has a representation obtained by translating the initial representation, and appropriately restricting it, as follows:

$$V_{\underline{t}}(x,t) = V_0(x, t - \underline{t}) \quad \text{on } [0, X] \times [\underline{t}, T].$$

As such, initial preferences cannot be reversed when the timed-outcomes and decision time are all translated by a fixed amount. That is, the dynamic preference structure is time invariant and statement (i) is proved.

We now assume the conditions of statement (i) and derive statement (ii). For $\underline{t} \in \mathscr{D}$, define $\succcurlyeq_{0,\underline{t}}$ according to:

$$(x,t) \succcurlyeq_{0,\underline{t}} (x', t') \iff (x, t - \underline{t}) \succcurlyeq_0 (x', t' - \underline{t})$$

for all $(x,t), (x',t'), (x, t - \underline{t}), (x', t' - \underline{t}) \in X \times [0,T]$. By Theorem 3.1.1, $\succcurlyeq_0$ is represented by a liminal discounting function V_0. Construct $\tilde{V}_{\underline{t}} : X \times [\underline{t}, T] \to \mathbb{R}$ such that $\tilde{V}_{\underline{t}}(\cdot, t) \equiv V_0(\cdot, t - \underline{t})$. Clearly, $\tilde{V}_{\underline{t}}$ represents $\succcurlyeq_{0, \underline{t}}$. Time invariance allows us to identify $\succcurlyeq_{\underline{t}}$ with the restriction of $\succcurlyeq_{0, \underline{t}}$ to the set of timed-outcomes occuring no sooner than time $\underline{t}$. Hence, $\succcurlyeq_{\underline{t}}$ may be represented by a function $V_t := \tilde{V}_{\underline{t}}\big|_Z$ where $Z = X \times [\underline{t}, T]$ as required. ∎

A.6   Proof of Theorem 4.3.2

For the proof of Theorem 4.3.2 we fix a set of weakly ordered, monotonic, impatient, midpoint consistent and continuous preference relations $\mathcal{R}$. We prove that if $\mathcal{R}$ is time invariant then, stationarity-before-t of $\succcurlyeq_0$ is equivalent to time-consistency-within-t-from-now of $\mathcal{R}$. Let $(x,t), (y,t'), (x, t - \underline{t}), (y, t' - \underline{t}) \in [0, X] \times [0, T]$. The following diagram contains the proof of this claim:

$$(x,t) \succcurlyeq_{\underline{t}} (y, t')$$

$$\nearrow{}^1$$

$$(x,t) \succcurlyeq_0 (y,t') \qquad\qquad \Updownarrow^3$$

$$\searrow{}_2$$

$$(x, t - \underline{t}) \succcurlyeq_0 (y, t' - \underline{t})$$

Note that implication 1 is time-consistency-within-t-from-now, implication 2 is stationarity-before-t and equivalence 3 is time invariance. We next show that, if $\mathcal{R}$ satisfies time invariance, then stationarity-after-t of $\succcurlyeq_0$ and time-consistency-beyond-t-from-now of

$\mathscr{R}$ are equivalent. The following diagram contains the proof of this claim:

$$(x,t) \succsim_0 (y,t')$$

$$\searrow {}^2$$

$$\updownarrow {}^1 \qquad\qquad (x,t+\underline{t}) \succsim_0 (y,t'+\underline{t})$$

$$\nearrow {}^3$$

$$(x,t+\underline{t}) \succsim_{\underline{t}} (y,t'+\underline{t})$$

Note that equivalence 1 is time invariance, implication 2 is stationarity-after-t and implication 3 is time-consistency-beyond-t-from-now. We have, therefore, established that time invariance and weak stationarity are jointly equivalent, given the other conditions, to time invariance and weak time consistency. Then, statement (i) of Theorem 4.3.1 and statement (i) of Theorem 4.3.2 are equivalent. So, the proof of Theorem 4.3.1 in Appendix A.5 applies here. ∎.