



On crack growth in functionally graded materials

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On Crack Growth in Functionally Graded Materials

by

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Malmö, December 1999

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Abstract

Stress intensity factors' behaviour is studied for long plane cracks interacting with a region of functionally graded elastic material. The region is assumed embedded into a large body treated as a homogeneous elastic continuum. The analysis is limited to small deviations of the graded region's elastic modulus from that of the surrounding body (Poisson's ratio is kept constant) and analytical solutions are sought using a perturbation technique. Emphasis is laid on the case of an infinite strip, which admits a closed form solution. A cosine change of the modulus of elasticity is treated, furnishing the solution for arbitrary variation in the form of a Fourier's expansion. Finite element analysis is subsequently performed for investigating the scope of validity of the analytical solution. The results for a set of finite changes of the elastic modulus are compared with the analytical predictions, and a remarkably wide range of validity is demonstrated. New functions, suitable for non-homogeneous material description, are introduced to approach the case of non-constant Poisson's ratio. The properties and possible applications of these functions are examined.

Keywords

Functionally graded materials, layered materials, composites, crack growth, stress intensity factors, perturbation technique, complex potentials, finite element method.

Preface

The prospects of designing new materials with desired performances by integration of components with different mechanical, thermal or chemical characteristics is a grate challenge for the contemporary engineering scientists. The new trends in creating these “smart” materials, along with the other studies, demand an exploration of advanced techniques for the analysis of their mechanical behaviour.

This thesis addresses two aspects of the wide area of mechanics of layered materials. It studies crack growth through a non-homogeneous layer incorporated into a homogeneous medium and it examines the advantages of a newly suggested functional description for non-homogeneous material’s characteristics.

At the bottom of it all, how strange, hides my supervisor, Professor Per Ståhle. I am much obliged to him for initiating the problems on which the research was carried out. Words fail me to express my gratitude to him for his patience during the endless discussions and for his many inspiring ideas for further progress in the studies.

I would like to extend my thanks to Professor Ali Massih, who introduced me into the Division and unveiled to me the remarkable power of the irreversible thermodynamics’ approach in solid mechanics. I promise to take advantage of it when the things go so complex that nothing else can help.

Many thanks to all my colleagues at the Division of Solid Mechanics for helping me with anything I possibly asked them, both at work and in everyday life. Without theirs support my stay in this impressive but still foreign country would not be as easy as it is. I owe a debt of gratitude also to the secretaries at the Division of Fluid Mechanics, who made my life even easier.

Not at the last place, I wish to thank my wife and my family back home for being so considerate with me and invariable in their support and encouragement.

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Andrey Jivkov

Malmö, December 1999

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Paper A

Jivkov, A.P. and Ståhle, P., *Analytical Perturbation Solutions for a Crack Interacting with a Functionally Graded Material*, Research report, Malmö University, 1999.

Paper B

Jivkov, A.P. and Ståhle, P., *Finite Element Analysis of a Crack In and Near a Strip of Elastically Graded Material*, to be submitted to 4-th EUROMECH Solid Mechanics Conference, Metz, France, 2000.

Paper C

Jivkov, A.P. and Ståhle, P., *Intrinsic Functions for Non-homogeneous Elastic Materials*, Research report, Malmö University, 1999.

There is a theory, which states that if anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another theory, which states that this has already happened.

Douglas Adams

Introduction

To meet the demands of new technologies the engineering community is constantly engaged in designing new materials with required performances. One direction of work has been the creation of materials, which at a macro scale can be considered homogeneous but at a micro scale have a specific composite structure, allowing certain control over their mechanical, thermal and chemical properties. For some new applications, however, it is becoming more and more difficult to employ such macro-scale-homogeneous materials. Examples are structural elements, which competitive industries require to be used at ever increasing temperatures or in more aggressive environments, where the intended characteristics of one or several of the material's components might be strongly reduced or eventually lost. Consequently, another direction of work consists in joining different materials, to take advantage of their respective favourable properties. The boundaries between the materials may be distinct or diffuse with gradient transitions between one material to another. Conventionally these are called functionally graded materials, but when a boundary is distinct the term layered materials is preferred. In all cases the improvement in the material's performance is paid by the initiation of non-homogeneity at a macro scale, which subsequently cause significant difficulties in the analytical description of the mechanical behaviour.

First attempt in studying the fracture mechanics aspects of layered material addressed the problem of a crack existing in one of the materials of a elastic bimaterial medium and perpendicularly approaching the interface between the two materials (Zak and Williams, 1963). The interface was assumed sharp, i.e. the elastic characteristics change abruptly across the interface. The continuum mechanics aspects of the same problem were investigated simultaneously and it was shown that the stress field in such a bimaterial medium depends only on two parameters, composed of four material characteristics for the two materials (Dundurs, 1969). Generally, the two parameters can be chosen in many ways but one particular choice, proposed by Dundurs, has been widely accepted and named after him.

Later, different treatments of the crack problem appeared in the literature, most of them based on integral equations for a dislocation density as the crack is presented by a distribution of dislocations. Closed form solutions were found for the stress around the crack tip when approaching an interface or free surface, and when passing the interface (Atkinson, 1975). The singular behaviour of the solution for a set of different cracks (edge, T-shaped and cross-shaped) terminating or crossing an interface, was

successfully investigated (Lu and Erdogan, 1983). The competition between crack deflection and penetration at an interface was examined (He and Hutchinson, 1989) and a range of interface toughness relative to the bulk material toughness is found giving a condition for the crack to deflect instead of to penetrate through the interface. The stress intensity factor and the crack opening displacement at the interface were calculated as functions of the distance from the crack tip to the interface and the Dundurs' parameters (Romeo and Ballarini, 1996).

The solution for the stresses around the crack tip shows that the strength of the singular stress field surrounding the crack tip is generally either weaker or stronger than the well-known square root singular field in a homogeneous material. Such strengths of the stress field create a phenomenon of either unboundedly growing or vanishing stress intensity factors K_I for a crack that approaches the interface (Schmauder and Müller, 1991), or differently the crack at the interface becomes either unconditionally unstable or impossible to drive through it. A possible way to overcome this anomaly is to adopt some non-linearity in the process description. A number of works consider plasticity and analyse the interaction of the plastic zone in front of the crack tip with the interface (Stähle and Shih, 1992, Romeo and Ballarini, 1993, Delfin et al., 1997, Chen and Atkinson, 1998, Kim et al., 1999). Another approach is to consider a model with a linear cohesive process region in front of the crack (Wäppling et al., 1998). For brittle materials, however, the predictions from this model are still unrealistic.

If the material is brittle, for example for layers of ceramics, one relief for the problem is to abandon the notion of sharp (or ideal) interface. Indeed, the manufacture process of the layered materials exposes their components to elevated temperatures and considerable diffusion across the material boundaries occurs. This naturally creates gradients in the new material properties around the interfaces. The diffusion, clearly, may be used also to develop functionally graded materials with tailored material property distribution.

A large class of solutions to the continuum mechanics problem of isotropic linear elastic medium with arbitrary changing (with respect to one co-ordinate only) elastic characteristics has been obtained (Abid Mian and Spencer, 1998). The approach is based on the concept of the "equivalent plate" – a hypothetical homogeneous plate with elastic properties appropriately averaging the properties of the original non-homogeneous plate. Basic fracture mechanics concepts in graded materials, like fracture toughness and R-curve, have also been studied (Jin and Batra, 1996).

The problem of a crack existing in a functionally graded material has been considered in several works, basically after a choice of a particular function describing the variation of the modulus of elasticity, while the change in Poisson's ratio has been neglected. Crack propagation has been studied in a material with exponential variation of elastic modulus with respect to one co-ordinate (Atkinson and List, 1978) and it has been shown that this is the only choice, which yields steady state solutions. Further, it has been shown (Delale and Erdogan, 1983) that an exponential variation with respect to one or both plane co-ordinates is the only choice which converts the differential equation for the plane problem into one with constant coefficients. In the same paper, the conventional square-root singularity is demonstrated for the accepted type of gradient function and the stress intensity factors are calculated. Other works (Erdogan et al., 1991, Erdogan, 1995) have studied the problem for a crack approaching and penetrating an interface between homogeneous and functionally graded materials. It has been proven there, that the phenomenon of either infinite or vanishing stress intensity factors is no longer present, as far as jumps in the material property distribution are not allowed.

The last two mentioned works investigate the stress field and the performance of the stress intensity factor in a sense which might be called "local", because the crack tip is interacting only with the interface between constant and functionally graded material.

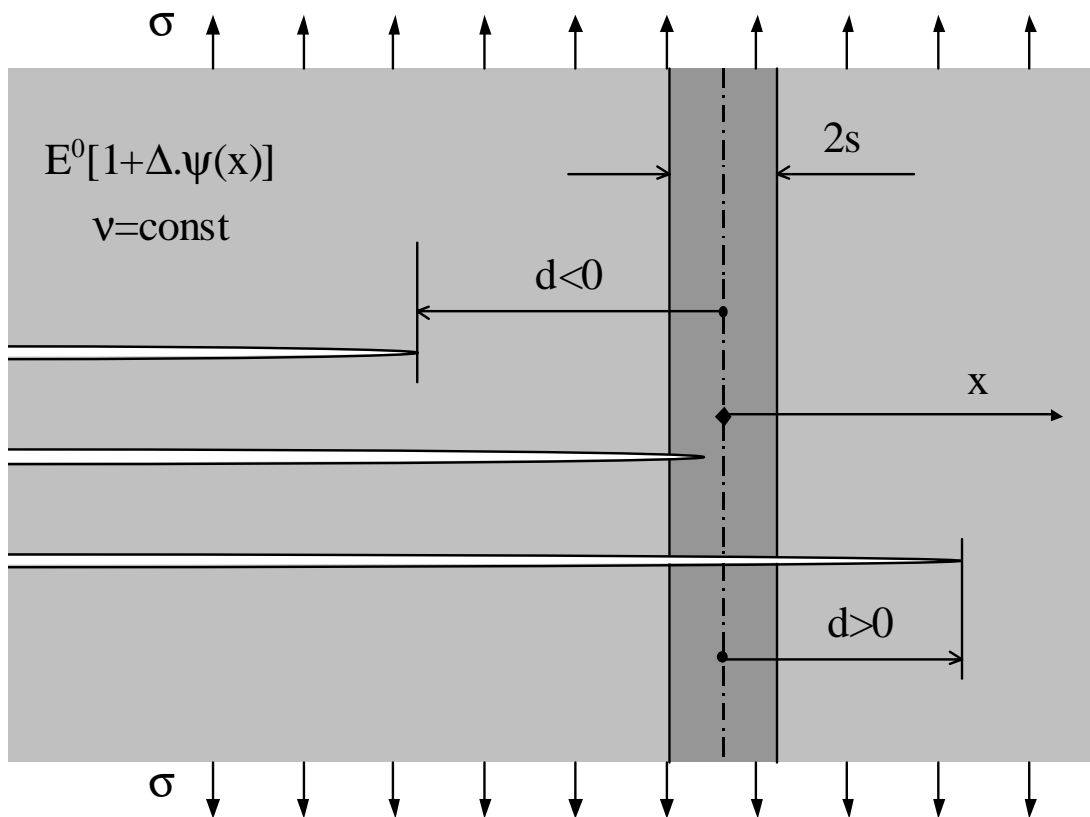
Distinctly, this thesis deals with a more "global" behaviour, where a whole layer of graded material is incorporated into a homogeneous media and the crack is considered passing through the entire composition. In the beginning of the study the layer is assumed to have a gradient in the elastic modulus alone, while the changes in the Poisson's ratio are neglected. Analytical solutions for the stress intensity factors behaviour are found for quite a general class of functions describing the elastic modulus gradient. They are consequently compared with results obtained by the finite element method simulations and the range of validity of the analytical solution is established.

For many materials a constant Poisson's ratio is quite an acceptable assumption, as commented in most of the quoted papers. There might be cases, however, when the changes in the Poisson's ratio are significant enough not to be discarded. If this have to be taken into account the gradient of the ratio enters into the problem formulation as a second function. There is one choice (among many) of two new functions, dependent on the elastic modulus and Poisson's ratio variations, which is introduced in the work as having certain advantages to all the other possible choices. These functions give the possibility for better formulations in a future work on layers with material gradients.

The thesis

Think about a two-dimensional homogeneous isotropic linear elastic body and a layer of a different material included into the body. The geometry and the characteristic lengths are shown schematically in Figure 1. The modulus of elasticity of the composed body is expressed with the formula $E = E^0 [1 + \Delta \psi(X_1)]$, where E^0 is the modulus of the homogeneous body, Δ is a coefficient and ψ is a function of one argument. Certain restrictions are imposed on the coefficient and the function to build an analytical solution. For example, the condition $|\Delta| \ll 1$ is required for the analytical treatment, while for the finite element solutions Δ might assume finite values. Similarly, the analytical derivation demands ψ to be a bounded continuous function with a first derivative being a function of bounded variation. In addition the function must vanish at the two ends of the layer, such that the transition from the one material to the other constitutes no jump in the elastic modulus. For a more thorough description of these prerequisites, consult Paper A. At these stage of the study the Poisson's ratio, ν , is assumed constant within the entire composite body. The possibility for considering a variation of ν will be discussed at a later stage in this thesis.

Figure 1. A crack interacting with a layer of functionally graded modulus of elasticity



Consider in addition a large crack, right tip of which is interfering with the strip. The crack surfaces are presumed traction free and the body is subjected remotely to a uniformly distributed load, which is perpendicular to the crack plane (see Figure 1).

The model, outlined shortly, constitutes a small perturbation of the elastic modulus of the homogeneous body. If the modulus of elasticity of the composed body is written in the form $E = E^0 + \Delta E^1$ ($E^1 = E^0 \cdot \psi(X_1)$), then the perturbation method (see for example Bush, 1992) allows one to assume that all the field variables can be expressed in the same way. Upper index zero will then denote the zero order solution (of the unperturbed problem) and upper index one will denote the first order solution for the respective variable. Evidently, the stress intensity factor for the composed problem can be written $K_I = K_I^0 + \Delta K_I^1$, where K_I^0 is the stress intensity factor for the homogeneous problem (caused by the stress field σ_{ij}^0), and K_I^1 - the stress intensity factor caused by the stress field σ_{ij}^1 .

The deviation of the stress intensity factor for the perturbed problem, K_I , from the stress intensity factor for the unperturbed one, K_I^0 , as a function of the crack tip position is of primary interest in this thesis. This function, expressed in a form not dependent on the magnitude of the perturbation coefficient, will be called stress intensity response:

$$(1) \quad \Pi(d) = \frac{1}{|\Delta|} \cdot \left(\frac{K_I}{K_I^0} - 1 \right) = \text{sgn}(\Delta) \frac{K_I^1}{K_I^0}$$

Essential part of the analysis is the use of the method of complex potentials (Muskhelishvili, 1963). Paper A has to be consulted for the full derivation. It is demonstrated there, that the solution for any function (under the imposed restrictions) may be written as:

$$(2) \quad \Pi(d) = \text{sgn}(\Delta) \cdot \frac{\kappa}{1 + \kappa} \cdot \left[\psi(d) - \frac{1}{2} \Psi(d) \right],$$

where $\kappa = 3 - 4\nu$ for plain strain and $\kappa = (3 - \nu)/(1 + \nu)$ for generalised plain stress conditions respectively, and $\Psi(d)$ is the Hilbert transform of the function $\psi(x)$.

A trial function ψ in a form of truncated cosine, or one half-wave is adopted:

$$(3) \quad \psi(\xi) = \cos\left(\frac{\pi}{2} \cdot \xi\right) \cdot \frac{1 + \text{sgn}(1 - |\xi|)}{2}, \quad \text{where } \xi = \frac{x}{s}$$

The result for the stress intensity response is analytically derived:

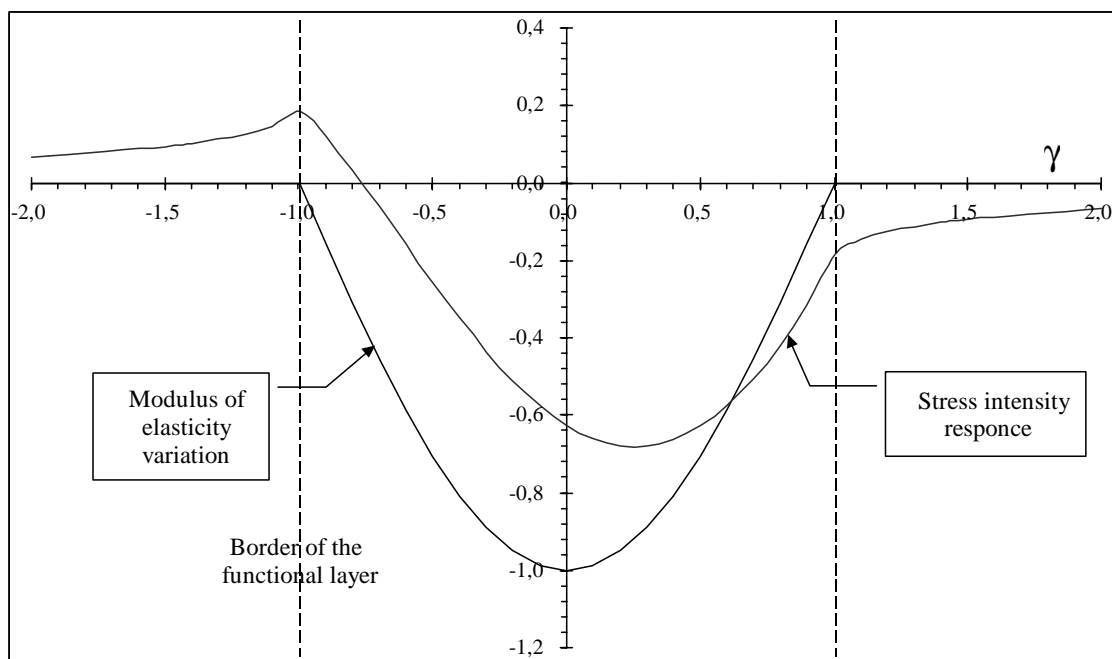
$$(4) \quad \Pi_1(\gamma) = \text{sgn}(\Delta) \frac{\kappa}{2(1+\kappa)} \left\{ \cos\left(\frac{\pi}{2} \gamma\right) [1 + \text{sgn}(1-|\gamma|)] - H(\gamma) \right\}, \text{ where } \gamma = \frac{d}{s}$$

The lower index one, attached to the function above, designates that the solution applies to one half-wave change of modulus of elasticity, and $H(\gamma)$ stays for the Hilbert transform of this half-wave, which utilises cosine and sine integrals:

$$(5) \quad H(\gamma) = \frac{1}{\pi} \cdot \cos\left(\frac{\pi}{2} \gamma\right) \cdot \left[Ci\left(\frac{\pi}{2}|1-\gamma|\right) - Ci\left(\frac{\pi}{2}|1+\gamma|\right) \right] - \frac{1}{\pi} \cdot \sin\left(\frac{\pi}{2} \gamma\right) \cdot \left[\text{sgn}(1-\gamma) \cdot Si\left(\frac{\pi}{2}|1-\gamma|\right) + \text{sgn}(1+\gamma) \cdot Si\left(\frac{\pi}{2}|1+\gamma|\right) \right]$$

Figure 2 displays the graphs of the trial function and the result $\Pi_1(\gamma)$, with a negative perturbation coefficient assumed for definiteness. Note that if a positive perturbation coefficient is to be used, both graphs will be symmetric to the plotted ones with respect to γ -axis.

Figure 2. Modulus of elasticity variation and produced stress intensity response



Paper B reports the support for the analytical solution supplied by finite element method (FEM) simulations of the posed problem. The FEM experiments have been performed using ABAQUS program. The layer has been created by the procedure for user's definition of a material (UMAT). Important feature of such a formulation is that it allows the permanent use of the initial geometry and crack configuration, while the strip is forced to change position for each simulation. As a matter of fact, the designed UMAT procedure permits to model an inclusion of arbitrary shape and arbitrary change of material characteristics. However, it has been used for modelling a layer with a change of elastic modulus as shown in Fig. 2, serving the purposes of Paper B. The complete description of the mesh and the calculation procedure should be found there.

The displacement field defined by the homogeneous problem stress intensity factor, K_I^0 , has been imposed as a boundary condition along the edge of the mesh. This approximation is claimed reasonable, because the layer's width is comparable to the node distances at the mesh boundary. Additional discussion on the matter is to be found in Paper B.

The stress intensity factor of the composed problem, for any given position of the layer relative to the crack tip, is calculated using the vertical displacements, u_2 , based on the first ten nodes behind the crack tip along the crack plane. The graphical procedure, explained thoroughly in the paper, can be summarised in the following form:

$$(6) \quad K_I^{FEM} = \lim_{r \rightarrow 0} c(r) \cdot \frac{u_2(r)}{\sqrt{r}} = c(r=0) \cdot \lim_{r \rightarrow 0} \frac{u_2(r)}{\sqrt{r}}$$

where r denotes the distance from the crack tip to the respective node and $c(r)$ is a parameter generally depending on the material characteristics at that node. In a limit, $c(r=0)$ is a constant depending on the modulus of elasticity at the crack tip, $E(r=0)$, and Poisson's ratio. The stress intensity factor found by this procedure can be conventionally written as $K_I^{FEM} = K_I^0 + \Delta K_I^1$, where K_I^0 is the factor obtained when $\Delta = 0$. In order to compare numerical and analytical results, an expression corresponding to (1) is constructed, namely $\Pi^{FEM}(\gamma) = \frac{1}{\Delta} \left(\frac{K_I^{FEM}}{K_I^0} - 1 \right)$.

Computational results confirm qualitatively the characteristic form of the response function (see Figures 5a and 5b in Paper B). The indefiniteness of the perturbation coefficient in the analytical solution makes the estimation of its validity range a substantially important task. Therefore, the numerical experiments aimed basically at evaluating the deviation (error) of the analytical prediction from the finite element results, obtained for finite values of the perturbation coefficient. The criteria for closeness between $\Pi_1(\gamma)$ and $\Pi^{FEM}(\gamma)$ curves is chosen to be the difference between

their minimal (for $\Delta < 0$) or maximal (for $\Delta > 0$) values, which corresponds to the well-known L^∞ -norm in functional spaces. The experiments are performed for a set of values of the perturbation coefficient, namely $\Delta = -0.9, -0.5, -0.1, -0.01, 0.01, 0.1, 0.5, 0.9$. The relative errors in the analytical solution minimum and maximum with respect to the FEM results are plotted in Fig.6 of Paper B. Because of the computational errors in the numerical results, additional reasoning is used to improve the error estimate for the analytical solution (see again Paper B).

The range of validity of the analytical solution is found to be surprisingly wide. For example, if an error of 5% is acceptable for certain applications, then the analytical solution is applicable for variations of the modulus of elasticity as large as about 40% for negative and about 60% for positive perturbation coefficient. Such variations would cause a maximal drop of around 28% or an increase of around 42% of the stress intensity factor respectively.

It is further demonstrated in Paper A, that the result (4) permits the solution for arbitrary number of half-waves with altering signs to be expressed in the form

$$(7) \quad \Pi_N(\gamma) = \sum_{n=1}^N (-1)^{n+1} \Pi_1(\gamma_n^N)$$

provided that the pattern of half-waves begins and ends at zero. This allows the response function for an arbitrary bounded variation of the elastic modulus to be expressed in the following trigonometric Fourier series:

$$(8) \quad \Pi(\gamma) = \sum_{N=0}^{\infty} a_N \cdot \Pi_{2N+1}(\gamma) \cdot \cos\left[(2N+1)\frac{\pi\gamma}{2}\right] + \sum_{N=1}^{\infty} b_N \cdot \Pi_{2N}(\gamma) \cdot \sin\left[2N\frac{\pi\gamma}{2}\right]$$

where the coefficients are calculated with the formulas:

$$(9) \quad \begin{aligned} a_N &= \int_{-1}^1 \psi(\gamma) \cdot \cos\left[(2N+1)\frac{\pi\gamma}{2}\right] d\gamma \\ b_N &= \int_{-1}^1 \psi(\gamma) \cdot \sin\left[2N\frac{\pi\gamma}{2}\right] d\gamma \end{aligned}$$

It is worth noting that $\psi(\gamma)$ is no longer required to be a continuous function. At the points of discontinuity of the elastic modulus, the result (8) will not converge, but it will be point-wise convergent for any other fixed point. Physically, it is hardly possible that a change of the elastic modulus is discontinuous, which will lead to a truncated series

expansion in the form (8), providing a nice connection between the model and the real situations.

Further in the studies, the possibilities of having non-constant Poisson's ratio are explored. Two functions, Γ_1 and Γ_2 , dependent on both elastic modulus and Poisson's ratio changes, are formally introduced with the help of differential relations:

$$(10a) \quad d\Gamma_1 = \frac{\kappa-1}{\kappa+1} \cdot d \ln \frac{\mu}{\kappa-1} = \frac{1}{2}(1-\hat{\nu}) \cdot d \ln \frac{\hat{E}}{1-\hat{\nu}}$$

$$(10b) \quad d\Gamma_2 = \frac{2}{\kappa+1} \cdot d \ln \mu = \frac{1}{2}(1+\hat{\nu}) \cdot d \ln \frac{\hat{E}}{1+\hat{\nu}}$$

Paper C explains the physical interpretation of these differentials as dimensionless quantities measuring the tendency of change of some material properties, compliances or stiffnesses, with respect to one chosen property.

The modified modulus of elasticity and Poisson's ratio, used in (10) are given by:

$$(11) \quad (\hat{E}, \hat{\nu}) = \begin{cases} (E, \nu) & \text{plane stress} \\ \left(\frac{E}{1-\nu^2}, \frac{\nu}{1-\nu} \right) & \text{plane strain} \end{cases}$$

In the case of discontinuous change of elastic characteristics, i.e. sharp boundary between two elastic materials, the introduced functions degenerate to constants, which are linear combinations of Dundurs' parameters α, β :

$$(12) \quad \Gamma_1 = 2\beta \quad \text{and} \quad \Gamma_2 = 2(\alpha - \beta)$$

This proposes the idea that the similarity between the two functions and Dundurs' parameters might be deeper. Indeed, a study of their properties shows many interesting features. A notion of separable functions is introduced in the paper to indicate the possibility each one of them to be written as a full differential of some expression not explicitly dependent on the other. It turns out that the differentials of the functions Γ_1 and Γ_2 are not separable and it is shown that this is an intrinsic feature of the functions adequate for gradient material description. This mutual dependence between two "permissible" functions arises from the consistency of the "measure" used to grade the

tendencies of change of the different compliances or stiffnesses. If different scaling is used for different parameters, the result will be an inapplicable couple of functions.

The inadequacy of separable functions arises from the fact, that they would describe, for example, a stress state, which could be identical to another state with generally different material characteristics. This situation may be avoided by using the introduced functions or any other non-separable couple, and it repeats the property of Dundurs' parameters. Hence the two functions are called intrinsic for a given material.

In regard to this property, it is important to describe the class of functional changes of the material parameters, say E and ν , corresponding to (or induced by) a given set of intrinsic functions. In the case of plane stress these classes of equivalence or of self-similar stress fields are derived:

$$(13) \quad \begin{aligned} E(x) &= \exp\left[\int d(\Gamma_1 + \Gamma_2) + k_1\right] = K_1 \cdot E^f(x) \\ \nu(x) &= E(x) \cdot \left[\int \frac{d(\Gamma_1 - \Gamma_2)}{E(x)} + k_2\right] = \nu^f(x) + K_2 \cdot E^f(x) \end{aligned}$$

The constants K_1, K_2 in (13) may be arbitrary, at least mathematically, although physics impose restrictions on their values. The functions $E^f(x), \nu^f(x)$ are some fundamental solutions to the integrals.

Quite an interesting conclusion can be drawn from (13). If the solution to some problem is found for one choice of elastic modulus variation and constant Poisson's ratio, then it must apply for identical shape of change, not necessarily same magnitude, of both characteristics. This should be true for the solutions found by different authors for exponential change of E and constant ν , and also for the cosine change of E and constant ν treated in Paper A. Equations (16) suggest also a natural division of the functionally graded materials into two classes. All materials with constant, or changing proportionally to elastic modulus Poisson's ratio, will belong to the first class and can be called uni-functional materials. Clearly, this class is achieved with constant fundamental function for ν . The second class covers all other materials with variations of ν not proportional to the change of E . This class is established with non-constant fundamental function for ν and the materials belonging to it can be called bi-functional.

Actually, most of the imaginable materials can be thought as uni-functional and thus the assumption of constant ν will prove correct. If there exists a bi-functional material, the analysis must employ the introduced intrinsic functions in more complex way. It is shown in Paper C how the functions will enter, for example, the equilibrium equations.

If the elastic properties vary in one direction, say x , than the only stress, which experiences change of its value (due to properties change) will be σ_y and it is given by:

$$(14) \quad d\sigma_y = (\sigma_x + \sigma_y) \cdot d\Gamma_1 - (\sigma_x - \sigma_y) \cdot d\Gamma_2$$

The increment of σ_y given by (14) will enter as an additional term in the equilibrium equations for the non-homogeneous continuum, turning them into a system of differential equations with variable coefficients. It is difficult to see a solution to this system in the general case, but an attempt combining the technique from Paper A and the discussion from Paper C deserves additional work.

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Paper A

**Analytical Perturbation Solutions for a Crack
Interacting with a Functionally Graded Material**

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Abstract — Stress intensity factors are calculated for long plane cracks with one tip interacting with a region of graded elastic properties. The material outside the region is assumed to be homogeneous and the material in the graded region to have continuous change of modulus of elasticity. The analysis is limited to small differences between stiffness of the surrounding body and of the graded region. Changes of Poisson's ratio are neglected. The body is assumed to be large as compared with the linear extent of the graded region. The crack tip, including the graded region, is assumed embedded in a square root singular stress field. Thus, a boundary layer solution is obtained with remote boundary conditions given by the stress intensity factor K_I . Solutions are given for rectangular regions but the limiting case of an infinite strip is devoted special interest. For a strip region with arbitrary variation of the modulus of elasticity an analytical solution is given in terms of the function describing the variation and its Hilbert's transform. Further, a particular primer is solved, allowing the solution for arbitrary variation to be found in the form of a Fourier's expansion. An interesting feature of the solution, supporting previously obtained results, is that the stress intensity factor remains finite and does neither vanish nor become unbounded as in the case where modulus of elasticity possesses jumps. The solution is compared with numerical results for finite changes of modulus of elasticity and is shown to have a surprisingly large range of validity. If an error of 5% is tolerated, modulus of elasticity may drop by around 40% or increase with around 60%.

Keywords — elastic material, inhomogeneous material, functionally graded material, layered material, crack growth, stress intensity factor

1. Introduction

Functional layers and protective coatings are increasingly used to improve mechanical, thermal and chemical performance of tools and devices. The bonding between dissimilar materials defines interfaces, which may have fixed or spatially varying

mechanical characteristics. As an example, fatigue cracks are attracted by a bimaterial interface to a weaker or softer material. Contrary to that fatigue cracks are deflected by a bimaterial interface to a stiffer or harder material. Thus a layer of soft or weak material will attract and trap fatigue cracks (Suresh et al., 1992). Another example is the improved wear resistance that successfully has been obtained by applying hard coating surfaces to tools. At wear the mechanisms for formation of fragments (debris) of the brittle coating is important in the understanding of the wear behaviour. Knowledge of initiation of flaws, formation of cracks and their subsequent growth will provide a full picture of the mechanisms. Here the effect of material changes on the crack tip driving force is of interest.

The problem for a cracked material attached to an elastic solid was first treated by Zak and Williams (1963) and consequently by Erdogan and Gupta (1975), Atkinson (1975), Lu and Erdogan (1983), He and Hutchinson (1989) and Romeo and Ballarini (1996). A series expansion for the stresses around the crack tip was deduced. One result of the analysis was that the strength of the singular stress field surrounding the crack tip is generally weaker or stronger than the well-known square root singular field. This causes analytical difficulties whereas it leads to either infinite or vanishing stress intensity factors K_I for a crack that approaches the interface (Schmauder and Müller, 1991). The prediction from that is that the crack either become unconditionally unstable or impossible to drive through the interface. One remedy to that is to consider the linear extent of the crack tip process region. However for brittle material predictions still become unrealistic (Wäppling et al., 1998).

Modern processes for diffusive or evaporative applications of thin layers to surfaces and thin internal layers in structures have provided possibilities to manufacture layers of size of few atomic distances. In conjunction with manufacturing of electronic devices the technique introduces new possibilities to increase the lifetime of components. During their production, as they are exposed to elevated temperatures, considerable diffusion across the material boundaries occurs, naturally creating material properties gradients. This might question the assumption regarding a distinct bimaterial interface. Further the diffusion may be taken advantage of, since graded materials with tailored material property distribution may be developed.

The fracture mechanics aspects of functionally graded materials have been also dealt with in a number of works. Exponential spatial variation of the elastic modulus has been assumed in the analysis by Atkinson and List (1978), and Delale and Erdogan (1983). Further it has been shown (Erdogan et al., 1991, Erdogan, 1995) that assuming a “smooth” material property distribution at the interface between a material with

constant properties and a functionally graded material, the anomalous behaviour of the stress state is eliminated. That is, the phenomenon of either infinite or vanishing stress intensity factors is no longer valid, as far as jumps in the material property distribution are not present.

In the present paper a graded material with a continuously varying stiffness is studied. A crack approaching, penetrating and passing a region of graded material is considered. The graded region is considered to have a rectangular shape with arbitrary sides, but special interest is devoted to a strip shaped layer with a material gradient only in the direction perpendicular to the extent of the strip. The analysis is divided into three main parts. In section 3 the solution for a crack interacting with a rectangular region with a material gradient in direction parallel to the crack plane is derived. The result for an infinite strip is shown to be a linear combination of the gradient function and its Hilbert's transform. In section 4 a particular choice of the gradient function is considered and the analytical solution for this case is compared to numerical results from FEM calculations (Jivkov and Ståhle, 1999). A very good agreement is shown for large range of gradient magnitudes. Finally, in section 5 it is demonstrated how this special choice of the gradient function allows the result for an arbitrary function to be written in the form of a trigonometric series expansion.

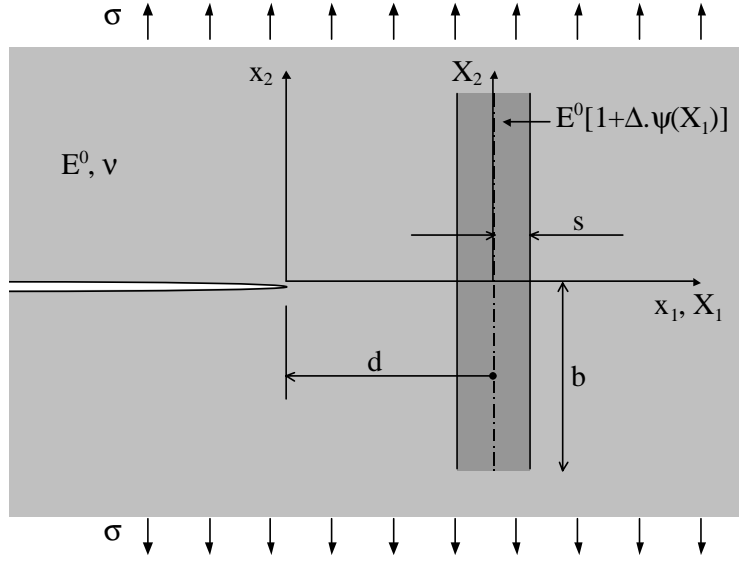
2. Model description

Consider an infinite two-dimensional homogeneous isotropic linear elastic body with a modulus of elasticity E^0 and a Poisson's ratio ν . A rectangular region of a different material is introduced into the body. A co-ordinate system (X_1, X_2) is initiated, such that the rectangle occupies a domain of the plane given by $\{|X_1| < s, |X_2| < b: s, b \in R^+\}$ (see Figure 1).

Suppose that the body is subjected remotely (at infinity) to a uniformly distributed load of intensity σ , which is perpendicular to the X_1 -axis, i.e. $\sigma_{22}^\infty = \sigma$, while $\sigma_{11}^\infty = \sigma_{12}^\infty = 0$.

Consider further a large crack existing into the body and occupying the domain $\{X_1 < d, X_2 = 0: d \in R\}$. In this description, obviously, d is the abscissa of the right tip of the crack with respect to the introduced co-ordinate system. It can also be thought of as the oriented distance from the central line, $X_1 = 0$, of the rectangular region to the crack tip: d is negative when the crack tip is approaching or penetrating the region up to its central line and positive after that. It must be remarked that the notion of largeness in the crack description expresses only the assumption that the crack length is much larger than the other characteristic length parameters, namely s, b, d .

Figure 1. A crack close to a region with spatially changing modulus of elasticity



In addition, the origin of a second co-ordinate system (x_1, x_2) is attached to the right tip of the crack. Note that the relation between the two co-ordinate systems is given by the transforms $x_1 = X_1 - d$ and $x_2 = X_2$.

The crack surfaces are presumed traction free, i.e. $\sigma_{22}(x_1, x_2) = \sigma_{12}(x_1, x_2) = 0$ whenever $x_1 < 0, x_2 = 0$.

Finally, suppose that the material of the rectangular region has the same Poisson's ratio, ν , as the surrounding body and a modulus of elasticity, E , changing only with respect to the co-ordinate X_1 according to the formula $E = E^0 [1 + \Delta \psi(X_1)]$. Here Δ is a coefficient, which may take positive or negative values under the requirement $|\Delta| \ll 1$.

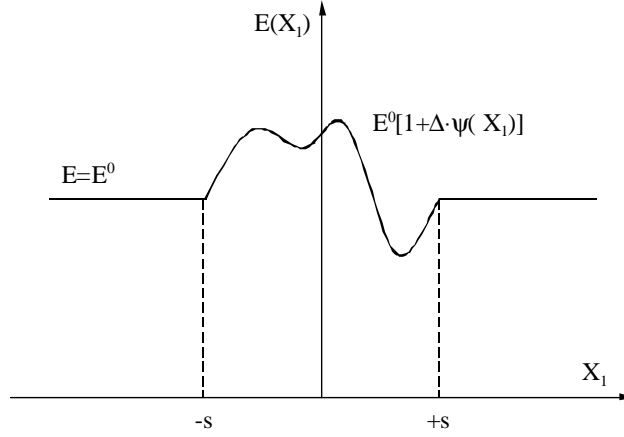
Further, ψ may be any bounded continuous function of one argument with the following properties (see Fig.2):

$$(1a) \quad \begin{aligned} \psi(X_1) &\neq 0, \text{ for } |X_1| < s \\ \psi(X_1) &= 0, \text{ for } |X_1| \geq s \end{aligned}$$

$$(1b) \quad \text{Max}_{X_1 \in R} \{ \psi(X_1) \} \leq 1$$

$$(1c) \quad \psi'(X_1) \text{ is a function of bounded variation, i.e. the derivative exists and is continuous everywhere except possibly at points of a set of measure zero, where it might have at most jumps. At these points the derivative is additionally defined to have the values } \psi'(X_1 -) = \lim_{\substack{t \rightarrow 0 \\ t < 0}} \frac{\psi(X_1 + t) - \psi(X_1)}{t} \text{ from the left and}$$

Figure 2. Change of modulus of elasticity of the medium within a strip of width $2s$



$\psi'(X_1 +) = \lim_{\substack{t \rightarrow 0 \\ t > 0}} \frac{\psi(X_1 + t) - \psi(X_1)}{t}$ from the right of the points respectively. This ensures the first derivative is integrable.

With these assumptions the formula describes a small perturbation of the elastic modulus of the initial body. The modulus of elasticity of the new body with the strip will be written in the more suitable form $E = E^0 + \Delta E^1$, where $E^1 = E^0 \cdot \psi(X_1)$. The problem with $\Delta = 0$ (i.e. $E = E^0$) will be referred to as unperturbed, while the one with $\Delta \neq 0$ as perturbed.

Because of the assumption for constant Poisson's ratio, the Lamé constants for the continuum of the perturbed problem, which are required in the analysis, are written in the same way as modulus of elasticity:

$$(2) \quad \begin{aligned} \mu &= \mu^0 [1 + \Delta \psi(X_1)] = \mu^0 + \Delta \mu^1 \\ \lambda &= \lambda^0 [1 + \Delta \psi(X_1)] = \lambda^0 + \Delta \lambda^1 \end{aligned} \quad , \text{ where } \quad \begin{aligned} \mu^1 &= \mu^0 \cdot \psi(X_1) \\ \lambda^1 &= \lambda^0 \cdot \psi(X_1) \end{aligned}$$

Since a small perturbation of these elastic characteristics is under consideration, the solutions to the perturbed problem are assumed to have the form:

$$(3a) \quad \begin{aligned} \sigma_{ij} &= \sigma_{ij}^0 + \Delta \sigma_{ij}^1 \\ \varepsilon_{ij} &= \varepsilon_{ij}^0 + \Delta \varepsilon_{ij}^1 \\ u_i &= u_i^0 + \Delta u_i^1 \end{aligned}$$

where $\sigma_{ij}^0, \varepsilon_{ij}^0, u_i^0$ (the zero order solutions) are the solutions of the unperturbed problem, and $\sigma_{ij}^1, \varepsilon_{ij}^1, u_i^1$ (the first order solutions) are unknown functions to be investigated in the following analysis. Consequently, the stress intensity factor for the perturbed problem can be written in the same manner:

$$(3b) \quad K_I = K_I^0 + \Delta K_I^1$$

where K_I^0 is the stress intensity factor for the unperturbed problem, caused by the stress field σ_{ij}^0 and K_I^1 is the stress intensity factor caused by the stress field σ_{ij}^1 .

The main objective of the work will be to estimate the deviation of the stress intensity factor for the perturbed problem, K_I , from the stress intensity factor for the unperturbed one, K_I^0 , as a function of the crack tip position. This function is normalised so that it does not depend on the magnitude of the perturbation coefficient, resulting in:

$$(4) \quad \Pi(d) = \frac{1}{|\Delta|} \cdot \left(\frac{K_I}{K_I^0} - 1 \right) = \text{sgn}(\Delta) \frac{K_I^1}{K_I^0}$$

However, the perturbation magnitude must still be small enough to ensure the assumptions made with equations (3) and all the subsequent derivations.

3. Mathematical analysis

The constitutive law for linear elasticity, used together with the equations (2) and (3a), affords the following representation of the stresses for the perturbed problem:

$$(5) \quad \begin{aligned} \sigma_{ij} &= \sigma_{ij}^0 + \Delta \sigma_{ij}^1 = 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij} = \\ &= 2(\mu^0 + \Delta \mu^1)(\varepsilon_{ij}^0 + \Delta \varepsilon_{ij}^1) + (\lambda^0 + \Delta \lambda^1)(\varepsilon_{kk}^0 + \Delta \varepsilon_{kk}^1) \delta_{ij} = \\ &= (2\mu^0 \varepsilon_{ij}^0 + \lambda^0 \varepsilon_{kk}^0 \delta_{ij}) + \\ &+ \Delta (2\mu^0 \varepsilon_{ij}^1 + \lambda^0 \varepsilon_{kk}^1 \delta_{ij}) + \Delta (2\mu^1 \varepsilon_{ij}^0 + \lambda^1 \varepsilon_{kk}^0 \delta_{ij}) + \\ &+ \Delta^2 (2\mu^1 \varepsilon_{ij}^1 + \lambda^1 \varepsilon_{kk}^1 \delta_{ij}) \end{aligned}$$

Equating coefficients of the zero powers of Δ produces the constitutive law for the unperturbed body, while equating coefficients of the first powers of Δ leads to an expression for the stresses of the first order solution which reads:

$$(6) \quad \sigma_{ij}^1 = (2\mu^0 \varepsilon_{ij}^1 + \lambda^0 \varepsilon_{kk}^1 \delta_{ij}) + (2\mu^1 \varepsilon_{ij}^0 + \lambda^1 \varepsilon_{kk}^0 \delta_{ij})$$

With the notations

$$(7a) \quad \Sigma_{ij}^K = (2\mu^1 \varepsilon_{ij}^0 + \lambda^1 \varepsilon_{kk}^0 \delta_{ij})$$

$$(7b) \quad \Sigma_{ij}^U = (2\mu^0 \varepsilon_{ij}^1 + \lambda^0 \varepsilon_{kk}^1 \delta_{ij})$$

the stresses of the first order solution can be rewritten as:

$$(8a) \quad \sigma_{ij}^1 = \Sigma_{ij}^K + \Sigma_{ij}^U$$

Hence these stresses consist of one known part, Σ_{ij}^K , and one unknown part, Σ_{ij}^U .

The purpose of this work, however, is not in finding the entire stress tensor of the first order solution but in analysing the behaviour of the stress intensity factor of this first order solution when the crack tip occupies different positions with respect to the perturbation region. It is clear from equation (8a) that the stress intensity factor of the first order solution will be, precisely like the stresses, a superposition of the stress intensity factors due to the two parts Σ_{ij}^K and Σ_{ij}^U , which will be hereafter denoted by K_I^K and K_I^U respectively. Consequently, one may write:

$$(8b) \quad K_I^1 = K_I^K + K_I^U$$

A substitution of the formulas for μ^1 and λ^1 from (2) into (7a) reduces the expression for Σ_{ij}^K to the form:

$$(9a) \quad \Sigma_{ij}^K = (2\mu^0 \psi(X_1) \varepsilon_{ij}^0 + \lambda^0 \psi(X_1) \varepsilon_{kk}^0 \delta_{ij}) = \psi(X_1) \cdot \sigma_{ij}^0$$

Since the zero order stresses induce the zero order stress intensity factor K_I^0 , i.e. the stress intensity factor for the unperturbed problem, it follows:

$$(9b) \quad K_I^K = \psi(X_1) \cdot K_I^0$$

In order to obtain the unknown terms in formulas (8a) and (8b) one observes that the first order stresses must obey the equilibrium equations $\sigma_{ij,j}^1 = 0$; hence the equality $\Sigma_{ij,j}^U + \Sigma_{ij,j}^K = 0$ ensues from (8a).

The derivative of the term Σ_{ij}^U reduces, after some well-known algebra, to:

$$(10a) \quad \Sigma_{ij,j}^U = \mu^0 u_{i,jj}^1 + (\mu^0 + \lambda^0) \mu_{j,jj}^1$$

while the derivative of the term Σ_{ij}^K is easily calculated from (9a) and from the equilibrium requirements $\sigma_{ij,j}^0 = 0$ for the stresses of the zero order solution:

$$(10b) \quad \Sigma_{ij,j}^K = (\psi(X_1) \cdot \sigma_{ij}^0)_j = \psi_{,j}(X_1) \cdot \sigma_{ij}^0 + \psi(X_1) \cdot \sigma_{ij,j}^0 = \psi_{,j}(X_1) \cdot \sigma_{ij}^0$$

Since the function $\psi(X_1)$ depends only on the first co-ordinate this result can be further reduced to $F_i = \Sigma_{ij,j}^K = \psi'(X_1) \cdot \sigma_{i1}^0$, where the prime denotes the ordinary derivative of the function $\psi(X_1)$. The new notation, F_i , in the last expression has been introduced for simplicity.

Consequently the equilibrium equations for the stresses of the first order solution lead to a system of differential equations for the unknown displacements:

$$(11) \quad \mu^0 u_{i,jj}^1 + (\mu^0 + \lambda^0) u_{j,ji}^1 + F_i = 0$$

Equations (11) can be seen as an analogous system of equilibrium equations for a linearly elastic continuum, with characteristics as those for the unperturbed problem and subjected only to fictitious body forces F_i . Stated differently, the unknown term Σ_{ij}^1 in the first order solution of the perturbed problem can be found as a solution to a new elasticity problem for an infinite body with unperturbed elastic characteristics and subjected to body forces present only inside the initial perturbation region. The body forces, on their hand, are due to the stress field of the zero order solution, i.e. the solution for unperturbed homogeneous linear elastic body subjected to uniformly distributed load at infinity, perpendicular to the present crack.

The stress field around the crack tip for this body has a well-known solution, which in more elegant complex form yields the following representation of the fictitious body forces:

$$(12) \quad \begin{aligned} F_1 + iF_2 &= \psi'(X_1) \cdot (\sigma_{11}^0 + i\sigma_{21}^0) = \psi'(X_1) \frac{K_I^0}{2\sqrt{2\pi}} \left[\frac{1}{\sqrt{z}} + \frac{1}{\sqrt{\bar{z}}} + \frac{z - \bar{z}}{\bar{z}^{\frac{3}{2}}} \right] \\ F_1 - iF_2 &= \psi'(X_1) \cdot (\sigma_{11}^0 - i\sigma_{21}^0) = \psi'(X_1) \frac{K_I^0}{2\sqrt{2\pi}} \left[\frac{1}{\sqrt{z}} + \frac{1}{\sqrt{\bar{z}}} + \frac{\bar{z} - z}{z^{\frac{3}{2}}} \right] \end{aligned}$$

Note that in (12) $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$ are taken with respect to the co-ordinate system centred at the crack tip, while the argument of ψ' is taken with respect to the spatially fixed system. This is done for the sake of simplicity only and the transform to a single co-ordinate system will be performed when it becomes necessary.

The solution of (11) can be found using the results for the stress field due to concentrated horizontal and vertical forces and subsequent integration in a region S of the complex plane, defined by $S = \{(x_1, x_2) : |x_1 + d| \leq s, |x_2| \leq b\}$.

The stresses due to concentrated forces are easily obtained by Muskhelishvili's complex potentials method and the complex stress intensity factor is derived directly from them. Only the formulas of interest are included below:

$$(13a) \quad \Sigma_{22}^U - i\Sigma_{12}^U = \frac{1}{2\pi(1+\kappa)} \int_{z \in S} \left[(F_1 + iF_2) \left(\frac{\kappa}{\bar{w} - \bar{z}} - \frac{1}{w - z} \right) + (F_1 - iF_2) \left(\frac{w - z}{(\bar{w} - \bar{z})^2} - \frac{1}{\bar{w} - \bar{z}} \right) \right] dz$$

$$(13b) \quad K_I^U - iK_{II}^U = \frac{1}{\sqrt{2\pi}(1+\kappa)} \int_{z \in S} \left[(F_1 + iF_2) \left(\frac{1}{\sqrt{z}} - \frac{\kappa}{\sqrt{\bar{z}}} \right) + (F_1 - iF_2) \frac{\bar{z} - z}{2\bar{z}^{\frac{3}{2}}} \right] dz$$

The note for (12) is valid for (13) with x_1, x_2 being the co-ordinates of the application point of the forces F_1, F_2 with respect to the co-ordinate system centred at the crack tip.

A substitution of equations (12) into (13b) provides:

$$(14) \quad K_I^U - iK_{II}^U = \frac{K_I^0}{4\pi(1+\kappa)} \int_{z \in S} \psi'(X_1) \cdot \left(\frac{1-2\kappa}{2} \cdot \frac{1}{\sqrt{z\bar{z}}} + \frac{\bar{z}}{z\bar{z}} + \frac{1-\kappa}{2} \cdot \frac{z}{z\bar{z}} + \frac{1}{4} \cdot \frac{z^2 + \bar{z}^2}{(z\bar{z})^{\frac{3}{2}}} - \frac{1+\kappa}{2} \cdot \frac{z^3}{(z\bar{z})^2} \right) dz$$

A transition in (14) from the complex variable z to its algebraic representation $x_1 + ix_2$ and integration with respect to x_2 produces the intermediate result:

$$(15) \quad K_I^U - iK_{II}^U = \frac{K_I^0}{4\pi(1+\kappa)} \int_{-d-s}^{-d+s} \psi'(X_1) \cdot \left(\kappa \cdot \ln \frac{\sqrt{b^2 + x_1^2} - b}{\sqrt{b^2 + x_1^2} + b} + 4 \cdot \arctan\left(\frac{b}{x_1}\right) + \frac{2b}{\sqrt{b^2 + x_1^2}} - (1+\kappa) \cdot \frac{2bx_1}{b^2 + x_1^2} \right) dx_1$$

As seen in (15) the imaginary terms in the right hand side have cancelled out during the integration, thus leading to the expected zero stress intensity factor for the second mode.

Now the translation $x_1 = X_1 - d$, followed by the scaling substitution $\xi = X_1/s$ converts equation (15) into a more convenient form:

$$(16) \quad \mathbf{P} = \frac{K_I^U}{K_I^0} = \frac{1}{4\pi(1+\kappa)} \int_{-1}^{+1} \psi'(\xi \cdot s) \left(\kappa \cdot \ln \frac{\sqrt{\beta^2 + (\xi - \gamma)^2} - \beta}{\sqrt{\beta^2 + (\xi - \gamma)^2} + \beta} + 4 \cdot \arctan\left(\frac{\beta}{\xi - \gamma}\right) + \frac{2\beta}{\sqrt{\beta^2 + (\xi - \gamma)^2}} - (1 + \kappa) \cdot \frac{2\beta(\xi - \gamma)}{\beta^2 + (\xi - \gamma)^2} \right) s d\xi$$

where the notations $\beta = b/s$ and $\gamma = d/s$ have been introduced.

For any fixed β equation (16) gives \mathbf{P} as a function of the crack tip distance d , or normalised distance γ . Combining (16) with equation (9b) leads to an expression for the function defined in (4):

$$(17) \quad \Pi(d) = \text{sgn}(\Delta) \cdot \frac{K_I^1}{K_I^0} = \text{sgn}(\Delta) \cdot [\psi(d) + \mathbf{P}(d)]$$

The integral in (16) does not have a closed form solution for every particular choice of the function ψ . Therefore a numerical integration of $\mathbf{P}(d)$ will be used later for computing $\Pi(d)$ for different heights of the perturbation region (i.e. different β).

More interesting from practical point of view is the case when $\beta \gg \xi, \gamma$ or stated otherwise, when $b \gg s, d$. It can be seen as a limiting solution for $\beta \rightarrow \infty$, although the analysis is still confined within a range of $b \ll a$ according to the previous assumptions. The rectangular region, inducing the perturbation, will be hereafter called a strip, in order to underline this new condition. To obtain the solution for this case, consider the integral in (16) as two separate parts:

$$(18a) \quad I_1 = \kappa \int_{-1}^{+1} \psi'(\xi \cdot s) \cdot \ln \frac{\sqrt{\beta^2 + (\xi - \gamma)^2} - \beta}{\sqrt{\beta^2 + (\xi - \gamma)^2} + \beta} \cdot s d\xi$$

$$(18b) \quad I_2 = \int_{-1}^{+1} \psi'(\xi \cdot s) \cdot \left[4 \cdot \arctan\left(\frac{\beta}{\xi - \gamma}\right) + \frac{2\beta}{\sqrt{\beta^2 + (\xi - \gamma)^2}} - (1 + \kappa) \cdot \frac{2\beta(\xi - \gamma)}{\beta^2 + (\xi - \gamma)^2} \right] \cdot s d\xi$$

When passing β to approach infinity the integral in (18b) takes the form:

$$\begin{aligned}
I_2 &= \int_{-1}^{+1} \psi'(\xi \cdot s) \cdot [2\pi \cdot \text{sgn}(\xi - \gamma) + 2] \cdot s d\xi = 2\pi \int_{-1}^{+1} \psi'(\xi \cdot s) \cdot \text{sgn}(\xi - \gamma) \cdot s d\xi = \\
(19) \quad &= 2\pi [\psi(\xi \cdot s) \cdot \text{sgn}(\xi - \gamma)]_{-1}^{+1} - 4\pi \int_{-1}^{+1} \psi(\xi \cdot s) \cdot \delta(\xi - \gamma) d\xi = \\
&= -4\pi \int_{-\infty}^{+\infty} \psi(\xi \cdot s) \cdot \delta(\xi - \gamma) d\xi = -4\pi \cdot \psi(\gamma \cdot s) = -4\pi \cdot \psi(d)
\end{aligned}$$

where the properties $\psi(-s) = \psi(s) = 0$ and $\psi(x) \equiv 0$, for $|x| > s$ are used, together with some properties of the signum function and the Dirac's delta function.

To deal with the integral in (18a), integration by parts is initially performed to obtain:

$$\begin{aligned}
I_1 &= \kappa \left[\psi(\xi \cdot s) \cdot \ln \frac{\sqrt{\beta^2 + (\xi - \gamma)^2} - \beta}{\sqrt{\beta^2 + (\xi - \gamma)^2} + \beta} \right]_{-1}^{+1} - \\
(20) \quad &\kappa \int_{-1}^{+1} \psi(\xi \cdot s) \cdot \frac{2\beta}{(\xi - \gamma)\sqrt{\beta^2 + (\xi - \gamma)^2}} d\xi = \\
&= -\kappa \int_{-1}^{+1} \psi(\xi \cdot s) \cdot \frac{2\beta}{(\xi - \gamma)\sqrt{\beta^2 + (\xi - \gamma)^2}} d\xi
\end{aligned}$$

and by letting β to approach infinity

$$(21) \quad I_1 = -2\kappa \int_{-1}^{+1} \frac{\psi(\xi \cdot s)}{(\xi - \gamma)} d\xi$$

The last result, with the help of the property $\psi(x) \equiv 0$, for $|x| > s$, is written as:

$$(22) \quad I_1 = -2\kappa \int_{-\infty}^{+\infty} \frac{\psi(\xi \cdot s)}{(\xi - \gamma)} d\xi = -2\pi\kappa \cdot \Psi(\gamma \cdot s)$$

where $\Psi(\gamma \cdot s)$ is the Hilbert's transform of the function $\psi(\xi \cdot s)$, defined by:

$$(23) \quad \Psi(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\psi(t)}{(t - x)} dt \quad (\text{Cauchy principal value})$$

The general solution now follows from (16) and (17):

$$(24) \quad \Pi(d) = \text{sgn}(\Delta) \cdot \frac{\kappa}{1 + \kappa} \cdot \left[\psi(d) - \frac{1}{2} \Psi(d) \right]$$

An explanatory note

A question might arise about whether a function of the type:

$$(*) \quad \psi(x) = \begin{cases} 0, & |x| \geq s \\ 1, & |x| < s \end{cases}$$

can be used as a trial function for elastic modulus distribution in perturbation strip. The Hilbert's transform of (*) reads:

$$(**) \quad \Psi(d) = \frac{1}{\pi} \ln \left| \frac{s-d}{s+d} \right|$$

and causes infinite increase or decrease of the function $\Pi(d)$ when the crack tip is situated exactly at the borders of the strip. This outcome is in agreement, of course, with the previous results for cracks terminating at sharp interfaces. However, for the purposes of the present work such a function is not acceptable, since its derivative violates the assumption (1c) for being a function of bounded variation. A “nicer” function will be adopted in the next section, which solution could be used for resolving cases of more general elastic modulus distributions.

4. A primer

As a useful example consider a concrete choice of the perturbation function:

$$(25) \quad \psi(x) = \cos \frac{\pi x}{2s} \cdot \frac{1 + \operatorname{sgn}(s - |x|)}{2}$$

This choice fulfils all the prerequisites posed by equations (1). Inserting the chosen function into the general solution (24) yields to:

$$(26) \quad \Pi_1 = \operatorname{sgn}(\Delta) \frac{\kappa}{2(1 + \kappa)} \left\{ \cos\left(\frac{\pi}{2} \gamma\right) [1 + \operatorname{sgn}(1 - |\gamma|)] - \frac{1}{\pi} \int_{-1}^1 \frac{\cos\left(\frac{\pi}{2} \xi\right)}{\xi - \gamma} d\xi \right\}$$

where the lower index 1 has been introduced to underline the particularity of the solution for one half-wave shaped gradient function. Next section will clarify this point.

The integral in formula (26) can be solved using the translation $x = \xi - \gamma$, which gives:

$$\begin{aligned}
(27) \quad H &= \frac{1}{\pi} \cdot \int_{-1}^1 \frac{\cos\left(\frac{\pi}{2} \xi\right)}{\xi - \gamma} d\xi = \frac{1}{\pi} \cdot \int_{-1-\gamma}^{1-\gamma} \frac{\cos\left(\frac{\pi}{2} (x + \gamma)\right)}{x} dx = \\
&= \frac{1}{\pi} \cdot \cos\left(\frac{\pi}{2} \gamma\right) \cdot \int_{-1-\gamma}^{1-\gamma} \frac{\cos\left(\frac{\pi}{2} x\right)}{x} dx - \frac{1}{\pi} \cdot \sin\left(\frac{\pi}{2} \gamma\right) \cdot \int_{-1-\gamma}^{1-\gamma} \frac{\sin\left(\frac{\pi}{2} x\right)}{x} dx
\end{aligned}$$

and the substitution $t = \frac{n\pi}{2} x$, which leads to:

$$(28) \quad H = \frac{1}{\pi} \cdot \cos\left(\frac{\pi}{2} \gamma\right) \cdot \int_{\frac{\pi}{2}(-1-\gamma)}^{\frac{\pi}{2}(1-\gamma)} \frac{\cos(t)}{t} dt - \frac{1}{\pi} \cdot \sin\left(\frac{\pi}{2} \gamma\right) \cdot \int_{\frac{\pi}{2}(-1-\gamma)}^{\frac{\pi}{2}(1-\gamma)} \frac{\sin(t)}{t} dt$$

Several cases must be considered when solving the integrals in the last expression. All the results are obtained in terms of the sine and cosine integrals $Si(x)$ and $Ci(x)$.

Case 1: $\gamma < -1$ (the crack tip is placed before the strip)

In this case both limits of the integrals are positive and the following argument is used:

$$\begin{aligned}
H &= \frac{1}{\pi} \cdot \cos\left(\frac{\pi}{2} \gamma\right) \cdot \left[\int_{\frac{\pi}{2}(-1-\gamma)}^{\infty} \frac{\cos(t)}{t} dt - \int_{\frac{\pi}{2}(1-\gamma)}^{\infty} \frac{\cos(t)}{t} dt \right] - \\
&\quad - \frac{1}{\pi} \cdot \sin\left(\frac{\pi}{2} \gamma\right) \cdot \left[\int_0^{\frac{\pi}{2}(1-\gamma)} \frac{\sin(t)}{t} dt - \int_0^{\frac{\pi}{2}(-1-\gamma)} \frac{\sin(t)}{t} dt \right] = \\
&= \frac{1}{\pi} \cdot \cos\left(\frac{\pi}{2} \gamma\right) \cdot \left[-Ci\left(\frac{\pi}{2}(-1-\gamma)\right) + Ci\left(\frac{\pi}{2}(1-\gamma)\right) \right] - \\
&\quad - \frac{1}{\pi} \cdot \sin\left(\frac{\pi}{2} \gamma\right) \cdot \left[Si\left(\frac{\pi}{2}(1-\gamma)\right) - Si\left(\frac{\pi}{2}(-1-\gamma)\right) \right]
\end{aligned}$$

Case 2: $\gamma > 1$ (the crack tip is placed after the strip)

In this case both limits are negative and first the substitution $t = -t$ is applied, succeeded by the same reasoning as in Case 1:

$$\begin{aligned}
H &= \frac{1}{\pi} \cdot \cos\left(\frac{\pi}{2} \gamma\right) \cdot \int_{\frac{\pi}{2}(1+\gamma)}^{\frac{\pi}{2}(-1+\gamma)} \frac{\cos(t)}{t} dt + \frac{1}{\pi} \cdot \sin\left(\frac{\pi}{2} \gamma\right) \cdot \int_{\frac{\pi}{2}(1+\gamma)}^{\frac{\pi}{2}(-1+\gamma)} \frac{\sin(t)}{t} dt = \\
&= \frac{1}{\pi} \cdot \cos\left(\frac{\pi}{2} \gamma\right) \cdot \left[-Ci\left(\frac{\pi}{2}(1+\gamma)\right) + Ci\left(\frac{\pi}{2}(-1+\gamma)\right) \right] + \\
&+ \frac{1}{\pi} \cdot \sin\left(\frac{\pi}{2} \gamma\right) \cdot \left[Si\left(\frac{\pi}{2}(-1+\gamma)\right) - Si\left(\frac{\pi}{2}(1+\gamma)\right) \right]
\end{aligned}$$

Case 3: $|\gamma| = 1$ (the crack tip is placed at one of the borders of the strip)

In this case the first term in (28) vanishes in limit, since the cosine function goes faster to zero than the cosine integral, which has a logarithmic singularity. The solution is hence given by:

$$H = -\operatorname{sgn}(\gamma) \cdot \frac{Si(\pi)}{\pi}$$

Note that this result implies a finite stress intensity factor when the crack tip is placed at the borders of the strip, differing from the phenomenon that occurs when the interface between two materials is sharp.

Case 4: $|\gamma| < 1$ (the crack tip is placed inside the strip)

The case is resolved by separating the integral of integration into two intervals and subsequently applying the logic from Case 1 and Case 2:

$$\begin{aligned}
H &= \frac{1}{\pi} \cdot \cos\left(\frac{\pi}{2} \gamma\right) \cdot \left[\int_{\frac{\pi}{2}(-1-\gamma)}^0 \frac{\cos(t)}{t} dt + \int_0^{\frac{\pi}{2}(1-\gamma)} \frac{\cos(t)}{t} dt \right] - \\
&- \frac{1}{\pi} \cdot \sin\left(\frac{\pi}{2} \gamma\right) \cdot \left[\int_{\frac{\pi}{2}(-1-\gamma)}^0 \frac{\sin(t)}{t} dt + \int_0^{\frac{\pi}{2}(1-\gamma)} \frac{\sin(t)}{t} dt \right] = \\
&= \frac{1}{\pi} \cdot \cos\left(\frac{\pi}{2} \gamma\right) \cdot \left[-\int_0^{\infty} \frac{\cos(t)}{t} dt + \int_{\frac{\pi}{2}(1+\gamma)}^{\infty} \frac{\cos(t)}{t} dt + \int_0^{\infty} \frac{\cos(t)}{t} dt - \int_{\frac{\pi}{2}(1-\gamma)}^{\infty} \frac{\cos(t)}{t} dt \right] - \\
&- \frac{1}{\pi} \cdot \sin\left(\frac{\pi}{2} \gamma\right) \cdot \left[\int_0^{\frac{\pi}{2}(1+\gamma)} \frac{\sin(t)}{t} dt + \int_0^{\frac{\pi}{2}(1-\gamma)} \frac{\sin(t)}{t} dt \right] = \\
&= \frac{1}{\pi} \cdot \cos\left(\frac{\pi}{2} \gamma\right) \cdot \left[-Ci\left(\frac{\pi}{2}(1+\gamma)\right) + Ci\left(\frac{\pi}{2}(1-\gamma)\right) \right] - \\
&- \frac{1}{\pi} \cdot \sin\left(\frac{\pi}{2} \gamma\right) \cdot \left[Si\left(\frac{\pi}{2}(1+\gamma)\right) + Si\left(\frac{\pi}{2}(1-\gamma)\right) \right]
\end{aligned}$$

Summarising the results for the examined cases the integral (27) can be written as:

$$(29) \quad H(\gamma) = \frac{1}{\pi} \cdot \cos\left(\frac{\pi}{2} \gamma\right) \cdot \left[Ci\left(\frac{\pi}{2}|1-\gamma|\right) - Ci\left(\frac{\pi}{2}|1+\gamma|\right) \right] - \\ - \frac{1}{\pi} \cdot \sin\left(\frac{\pi}{2} \gamma\right) \cdot \left[\operatorname{sgn}(1-\gamma) \cdot Si\left(\frac{\pi}{2}|1-\gamma|\right) + \operatorname{sgn}(1+\gamma) \cdot Si\left(\frac{\pi}{2}|1+\gamma|\right) \right]$$

which is apparently the Hilbert's transform of the chosen function.

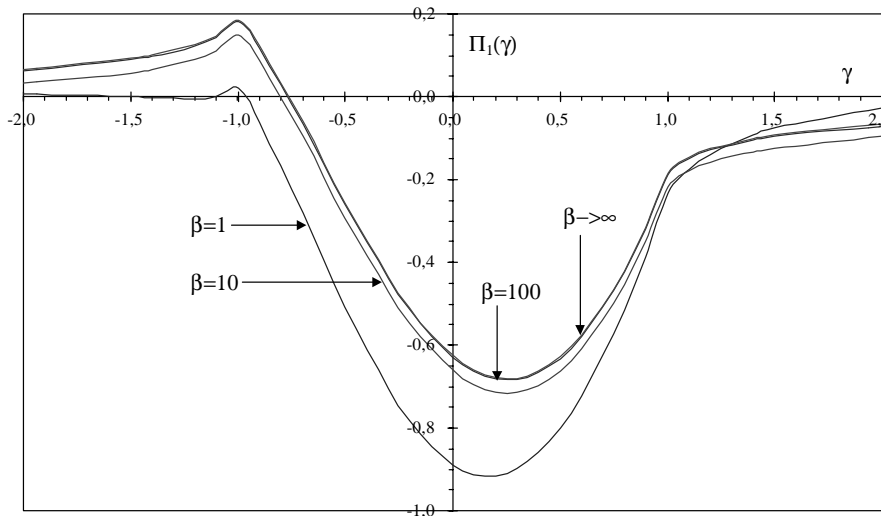
This completes the solution for the deviation of the stress intensity factor for the perturbed problem, with ψ given by (25), from the one for the unperturbed problem, as a function of the crack tip position:

$$(30) \quad \Pi_1 = \operatorname{sgn}(\Delta) \frac{\kappa}{2(1+\kappa)} \left\{ \cos\left(\frac{\pi}{2} \gamma\right) [1 + \operatorname{sgn}(1-|\gamma|)] - H(\gamma) \right\}$$

A graphical representation of the analytical solution (30) is shown in Fig. 3. The sign of the perturbation coefficient is taken negative (a drop in the modulus of elasticity), noting that a positive perturbation coefficient effects only in a change of the sign of the function $\Pi_1(\gamma)$ but not in its magnitudes.

Plane strain conditions are assumed with the comment that plane stress results differ from that for plain strain only by a factor $\frac{(3-\nu)(1-\nu)}{3-4\nu}$, easily derived from equation (30). Poisson's ratio, $\nu = 1/3$, is chosen for definiteness, assigning $\kappa = 5/3$ for plane strain and implying that the results for plain stress are 16/15 times greater.

Figure 3. Normalised stress intensity factors ratio for different crack tip positions



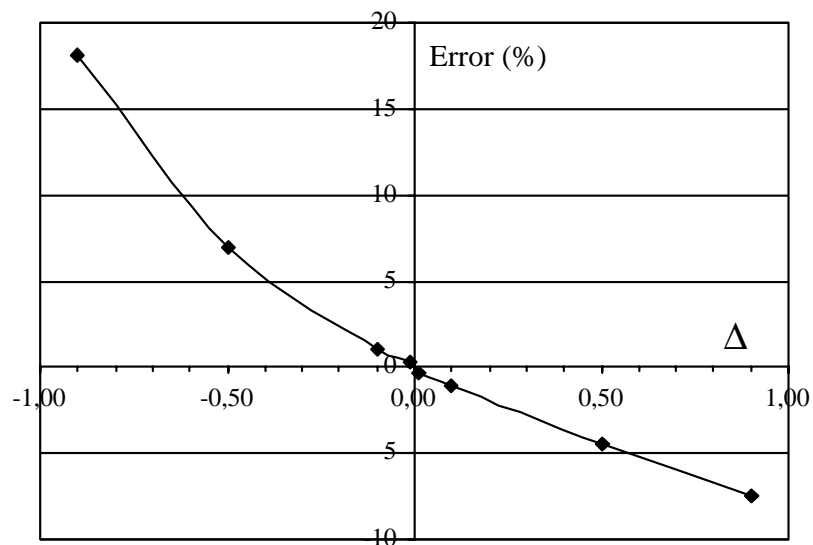
By using equation (17), with numerical integration of the part given by (16), solutions have been obtained for three different values of β : $\beta=1$; $\beta=10$; $\beta=100$. The resulting plots for the normalised stress intensity factors ratio Π as a function of γ are shown also in Fig.3.

As seen from the figure, the numerical solutions converge to the analytical one, and the curve for $\beta=100$ is practically indistinguishable from the analytical curve.

FEM experiments have been performed to check and support the analytical predictions (Jivkov and Ståhle, 1999). The experiments aimed also to show the deviation of the results from the analytical prediction for different finite values of the perturbation coefficient. To estimate this deviation (error) a comparison of the minimum, respectively maximum values of the functions $\Pi^{FEM}(\gamma)$ and $\Pi_1(\gamma)$ has been adopted. The experiments have been run for a set of values of the perturbation coefficient, namely $\Delta = -0.9, -0.5, -0.1, -0.01, 0.01, 0.1, 0.5, 0.9$. The relative errors in the analytical solution minimum and maximum with respect to the FEM results are presented in Fig.4.

From the graphics in Fig.4 the range of validity of the analytical solution is easily assessed. For example, if an error of 5% is acceptable for certain applications, then the analytical solution is applicable for variations of the modulus of elasticity as big as about 40% in negative and about 60% in positive direction.

Figure 4. Error in the analytical solution for finite values of Δ



5. Solution proposal for arbitrary elastic modulus gradient

The solution offered in the previous section, for a problem perturbed by one half-wave, can be extended to resolve the case of any trigonometric elastic modulus distribution by a simple reasoning. Consider, for this purpose, the scheme shown in Fig. 5. N half-waves with altering signs describe the variation of the modulus of elasticity inside the perturbation strip of width $2s$. The distances from the central lines of each of the half-waves to the crack tip are denoted by $d_i^N, i=1,2,\dots,N$, where the upper index stands for the number of half-waves and the lower index for the number of the half-wave being considered. The half-waves are counted from left to right so that $d_1^N < d_2^N < \dots < d_N^N$ holds. The distance from the central line of the strip to the crack tip is denoted as before by d .

It is easy to observe the following relation:

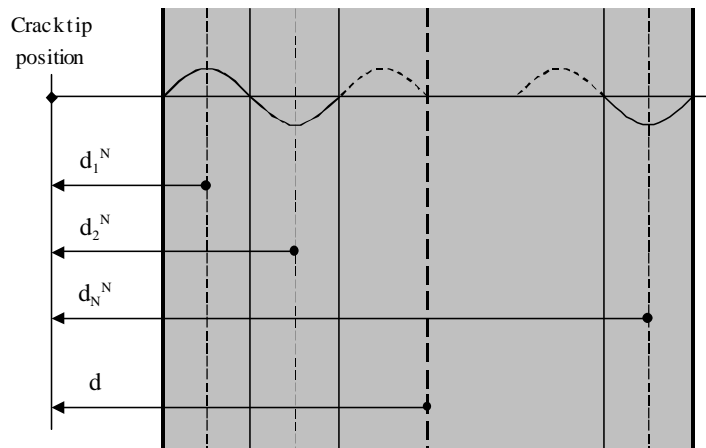
$$(31) \quad d_i^N = d + \frac{N+1-2i}{N} \cdot s, \quad i=1,2,\dots,N$$

By introducing the normalised distances $\gamma_i^N = \frac{d_i^N}{s/N}$; $\gamma = \frac{d}{s}$ equation (31) becomes:

$$(32) \quad \gamma_i^N = N(\gamma+1) + 1 - 2i, \quad i=1,2,\dots,N$$

The linearity of the problem allows the solution for the given N half-waves to be found by superimposing the solutions for each of them. Moreover the Hilbert's transform is also a linear operator. Consequently the solution is given by:

Figure 5. Distribution of modulus of elasticity in the form of N half-waves



$$(33) \quad \Pi_N(\gamma) = \sum_{i=1}^N (-1)^{i+1} \Pi_1(\gamma_i^N)$$

Now a Fourier's series may be defined as an expansion of a function $\psi(\gamma)$ in a series of sines and cosines such as:

$$(34) \quad \psi(\gamma) = \sum_{N=0}^{\infty} a_N \cos\left[(2N+1)\frac{\pi\gamma}{2}\right] + \sum_{N=1}^{\infty} b_N \sin\left[2N\frac{\pi\gamma}{2}\right]$$

There is no necessity any more for $\psi(\gamma)$ to be continuous, but it must remain bounded. All terms are orthogonal and the series converges everywhere in the interval $|\gamma| < 1$. The expansion's coefficients are computed as follows:

$$(35) \quad \begin{aligned} a_N &= \int_{-1}^1 \psi(\gamma) \cdot \cos\left[(2N+1)\frac{\pi\gamma}{2}\right] d\gamma \\ b_N &= \int_{-1}^1 \psi(\gamma) \cdot \sin\left[2N\frac{\pi\gamma}{2}\right] d\gamma \end{aligned}$$

The solutions (33) can now be combined with (35) to form a general solution for the case of an infinite strip with an arbitrary modulus of elasticity change. Thus, the deviation of the stress intensity factor may be written in a Fourier's series form as proceeds:

$$(36) \quad \Pi(\gamma) = \sum_{N=0}^{\infty} a_N \cdot \Pi_{2N+1}(\gamma) \cdot \cos\left[(2N+1)\frac{\pi\gamma}{2}\right] + \sum_{N=1}^{\infty} b_N \cdot \Pi_{2N}(\gamma) \cdot \sin\left[2N\frac{\pi\gamma}{2}\right]$$

A noteworthy generalisation obtained by that solution is that the requirement of a continuous $\psi(\gamma)$ is removed. Further one may reflect on the fact that a physical change of the elastic modulus is hardly atomically sharp, which calls for a representation via a truncated series expansion in the form (36). Hence, the required continuity of $\psi(\gamma)$ is granted. In the case of a mathematically sharp jump of the elastic modulus, the result (36) will be point-wise convergent for any fixed point except for that at the jump.

6. Discussion and conclusions

In the paper, the result for the stress intensity variation when a crack tip is interfering with an infinite strip of spatially changing modulus of elasticity is given in the form of a Fourier series expansion. The selected series converges to any continuous function defined in the region $|x| \leq s$, provided that the function vanishes at the points $|x| = s$. In

the case of discontinuous functions, the expansion converges at any fixed point excluding only the points of discontinuity. It is not known, at the present, whether the resulting stress intensity factor will have discontinuities in the form of finite jumps. However, at the jumps in elastic modulus the resulting stress intensity factor either become unbounded or vanish which of course in a limited region close to the jump in elastic modulus will violate the assumptions made for the present perturbation analysis.

The result is interesting because the analysis is based on the assumption that the modulus of elasticity is continuous everywhere including across the boundaries of the strip. The solutions obtained for individual terms of a series expansion, provide us with a series point-wise convergent to the result for a discontinuous modulus of elasticity. A truncated series represents the result for a continuous modulus where the discontinuity is replaced with a fast changing wave. A truncated series expansion approximating a discontinuous modulus of elasticity thus supplies a possibility of examining a more realistic case where the jump in modulus occurs over a finite distance. This will give a physical meaning to truncating the series for terms with minimal period. Approximate finite distance over which the jump of modulus occurs should be around $1/4$ th of that minimal period.

The error due to the assumed small changes of the modulus of elasticity is examined only for the leading term of the Fourier series expansion. However direct comparison with numerical results for the leading term can be transformed into results for higher order terms following the method given in section 5. The numerical procedure for finite changes of the modulus of elasticity (Jivkov and Stähle, 1999), provides a possibility to estimate the error for any variation of modulus of elasticity. However the result must be treated with judgement whereas the numerical results cannot be superimposed.

In the present paper only the modulus of elasticity was depending on one spatial coordinate. A motivation for not introducing a spatial variation also of the Poisson's ratio is that the variation of ν between different materials is rather limited. In situations such as cracks approaching, running parallel with and on bimaterial interfaces the results has been shown to be fairly insensitive to the differences in ν across the interface. If this is the case also for gradient materials, may be a subject for a future study.

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Paper B

**Finite Element Analysis of a Crack In and Near
a Strip of Elastically Graded Material**

Finite Element Analysis of a Crack In and Near a Strip of Elastically Graded Material

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Abstract — The behaviour of the stress intensity factor is investigated for a long plane crack with one tip interacting with a strip of graded elastic properties. The material outside the strip is postulated to be homogeneous linear elastic and the material in the graded region is assumed to have continuous change of modulus of elasticity. Changes of the Poisson's ratio are ignored. The body is assumed to be large in comparison to the crack length, and the crack length itself to be large compared to the linear extent of the graded region. The study is performed using finite element method and the results are extrapolated to infinitesimal variations of the graded region's modulus of elasticity from the one of the surrounding body. The crack tip, including the graded region, is considered embedded in a square root singular stress field governed by the stress intensity factor for a body without a strip. Hence, the numerical solutions are obtained for remote boundary conditions in a form of prescribed displacements supplied by that stress intensity factor. The analytical solution to the problem for an infinitesimally small variation of the region's modulus of elasticity, obtained recently by the authors, is communicated in brief. A particular function describing the modulus of elasticity change is treated so that a comparison between the finite element solutions and the analytical results is facilitated. The analytical solution is shown to have a surprisingly large range of validity. If an error of 5% is tolerated, modulus of elasticity may decrease with around 40% or increase with around 60%.

Keywords — elastic material, functionally graded material, layered material, crack growth, stress intensity factor, finite element method

1. Introduction

Modern processes for diffusive or evaporative deposition of thin layers to surfaces and thin internal layers in structures have provided possibilities for manufacturing layers of size of few atomic distances. The technique introduces new prospects for increasing the

lifetime of components when the layers are properly designed for improvement of mechanical, thermal or chemical performance. During the production process, as the components are exposed to elevated temperatures, considerable diffusion across the material boundaries occurs, naturally creating material properties gradients. This might question the concept of a distinct bimaterial interface, which has been initially utilised for studying layered material. In fact, the diffusion may be taken advantage of since graded materials with tailored material property distribution may be developed.

In the present work an internal layer of a graded material with a continuously varying stiffness is considered. A crack perpendicularly approaching, penetrating and passing the layer is examined and the behaviour of the stress intensity factor is analysed. The case studied, is closely related to generally two directions of past works; those devoted to cracks at sharp interfaces and those connected with fracture of functionally graded materials.

The problem for a cracked material attached to an elastic solid was first treated by Zak and Williams (1963) and after that followed by Atkinson (1975), Lu and Erdogan (1983), He and Hutchinson (1989) and Romeo and Ballarini (1996), among others. A series expansion for the stresses around the crack tip was deduced, unveiling the phenomenon that the strength of the singular stress field surrounding the crack tip is generally weaker or stronger than the well-known square root singular field. This causes analytical difficulties because it leads to either infinite or vanishing stress intensity factors K_I for a crack that approaches the interface (Schmauder and Müller, 1991). The consequence from such an outcome is that the crack either become unconditionally unstable or impossible to drive through the interface. One possible approach to overcome this anomaly is to consider the linear extent of the crack tip process region. However for brittle material predictions still become unrealistic (Wäppling et al., 1998).

The fracture mechanics aspects of functionally graded materials have been also dealt with in a number of works. Exponential spatial variation of the elastic modulus has been adopted in the analysis by Atkinson and List (1978), and Delale and Erdogan (1983). Further it has been shown (Erdogan et al., 1991, Erdogan, 1995) that assuming a “smooth” material property distribution at the interface between a material with constant properties and a functionally graded material, the anomalous behaviour of the stress state is eliminated. That is, the phenomenon of either infinite or vanishing stress intensity factors is no longer valid, as far as jumps in the material property distribution are not present.

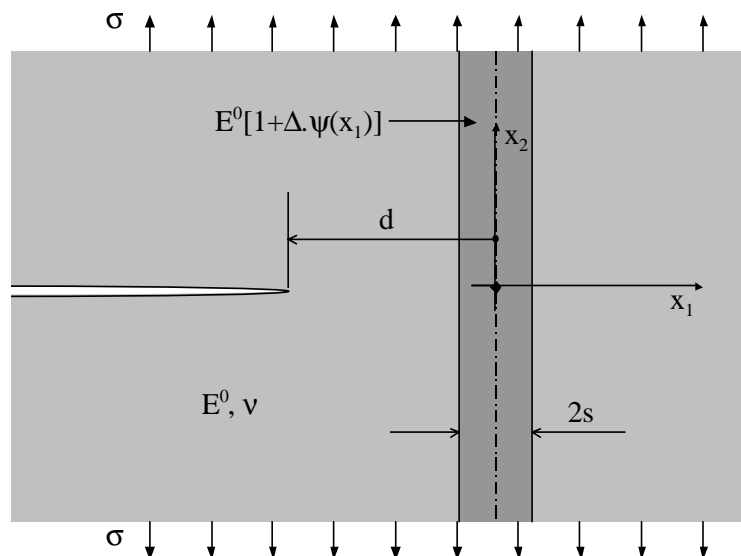
The emphasis in the mentioned works has been in analysing the “local” performance of the stress intensity factors, i.e. when the crack tip is interacting only with the interface between homogeneous and functionally graded material. Distinctly, the present work is devoted to studying a more “global” behaviour by including a whole layer of graded material as a media, through which the crack is passing.

The paper is divided into three main parts. In section 2, after posing the problem, a short description of the analytical solution is presented. A particular choice of the gradient function is considered, which allows the result for an arbitrary function to be written in the form of a trigonometric series expansion. In section 3 the finite element procedure is described, together with the necessary approximations adopted. Finally, in section 3 the numerical results for the stress intensity factor deviation are presented graphically and compared to the analytical predictions for the chosen material gradient function. A very good agreement is shown for large range of gradient magnitudes.

2. Model description and analytical solution

A large two-dimensional homogeneous isotropic linear elastic body is considered. The body is attributed a modulus of elasticity E^0 and a Poisson’s ratio ν . A strip of a different material and width $2s$ is introduced into the body. The material is assumed to have the same Poisson’s ratio, ν , as the surrounding body and a modulus of elasticity, E , changing with respect to the direction, perpendicular to the strip. A large crack is supposed to exist in the body with one tip interfering with the strip. The crack plane is

Figure 1. A crack close to a strip with spatially changing modulus of elasticity



perpendicular to the strip and the oriented distance from the central line of the strip to the crack tip is denoted by d . The crack surfaces are presumed traction free. Further, the body is subjected remotely (at infinity) to a uniformly distributed uniaxial stress field with intensity σ , which is perpendicular to the crack plane. Figure 1 displays the geometry and all the notations required for the subsequent discussion.

The modulus of elasticity of the composed body is expressed with the formula $E = E^0 [1 + \Delta \psi(X_1)]$. For the analytical treatment of the problem, Δ is a coefficient, obeying the requirement $|\Delta| \ll 1$. For the finite element solutions, however, Δ will take on finite values. Further, ψ is allowed to be a bounded continuous function of one argument with the following properties:

$$(1a) \quad \begin{aligned} \psi(X_1) &\neq 0, \text{ for } |X_1| < s \\ \psi(X_1) &= 0, \text{ for } |X_1| \geq s \end{aligned}$$

$$(1b) \quad \text{Max}_{X_1 \in R} \{\psi(X_1)\} \leq 1$$

$$(1c) \quad \psi'(X_1) \text{ is a function of bounded variation}$$

With these assumptions a model of a small perturbation of the elastic modulus of the initial body is formulated. The problem with $\Delta = 0$ (*i.e.* $E = E^0$) will be referred to as unperturbed, while the one with $\Delta \neq 0$ as perturbed. In a more suitable form the change of the modulus of elasticity reads:

$$(2) \quad E = E^0 + \Delta E^1 \quad \text{where } E^1 = E^0 \cdot \psi(X_1).$$

When a small enough perturbation is under consideration, all the field variables for the perturbed problem (displacements, strains and stresses) are presumed expressible in the same form as the elastic modulus in (2). Upper index zero denotes the zero order solution (of the unperturbed problem) and upper index one denotes the first order solution for the respective variable. Consequently, the stress intensity factor for the perturbed problem can be written as:

$$(3) \quad K_I = K_I^0 + \Delta K_I^1$$

where K_I^0 is the stress intensity factor for the unperturbed problem, caused by the stress field σ_{ij}^0 and K_I^1 is the stress intensity factor caused by the stress field σ_{ij}^1 .

The subject of this work is the deviation of the stress intensity factor for the perturbed problem, K_I , from the stress intensity factor for the unperturbed one, K_I^0 , as a function of the crack tip position. This function is normalised so that it does not depend on the magnitude of the perturbation coefficient, resulting in:

$$(4) \quad \Pi(d) = \frac{1}{|\Delta|} \cdot \left(\frac{K_I}{K_I^0} - 1 \right) = \text{sgn}(\Delta) \frac{K_I^1}{K_I^0}$$

The magnitude of the perturbation, however, must still be small enough to ensure the assumptions made for the field variables, resulting in equation (3).

The authors have resolved analytically the perturbation problem, posed here, in their recent work (Jivkov and Stähle, 1999). It has been demonstrated that the solution for any function, obeying the prerequisites (1), has the following form:

$$(5) \quad \Pi(d) = \text{sgn}(\Delta) \cdot \frac{\kappa}{1 + \kappa} \cdot \left[\psi(d) - \frac{1}{2} \Psi(d) \right],$$

where $\kappa = 3 - 4\nu$ for plain strain and $\kappa = (3 - \nu)/(1 + \nu)$ for generalised plain stress conditions respectively, and $\Psi(d)$ is the Hilbert transform of the function $\psi(x)$ given by:

$$(6) \quad \Psi(d) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\psi(x)}{(x-d)} dx \quad (\text{Cauchy principal value})$$

A trial function ψ of the kind (see Figure 2):

$$(7) \quad \psi(\xi) = \cos\left(\frac{\pi}{2} \cdot \xi\right) \cdot \frac{1 + \text{sgn}(1 - |\xi|)}{2}, \quad \xi = \frac{x}{s}$$

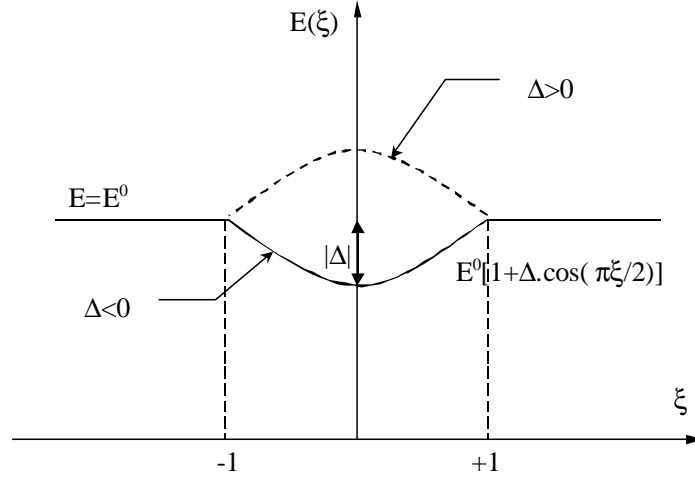
has been adopted.

The result for the normalised stress intensity factor deviation has been accomplished:

$$(8) \quad \Pi(\gamma) = \text{sgn}(\Delta) \frac{\kappa}{2(1 + \kappa)} \left\{ \cos\left(\frac{\pi}{2} \gamma\right) \left[1 + \text{sgn}(1 - |\gamma|) \right] - H(\gamma) \right\}, \quad \gamma = \frac{d}{s}$$

In the last equation $H(\gamma)$ stays for the Hilbert transform of the trial function given by:

Figure 2. Change of modulus of elasticity according to the trial function



$$(9) \quad H(\gamma) = \frac{1}{\pi} \cdot \cos\left(\frac{\pi}{2} \gamma\right) \cdot \left[Ci\left(\frac{\pi}{2} |1 - \gamma|\right) - Ci\left(\frac{\pi}{2} |1 + \gamma|\right) \right] - \frac{1}{\pi} \cdot \sin\left(\frac{\pi}{2} \gamma\right) \cdot \left[\operatorname{sgn}(1 - \gamma) \cdot Si\left(\frac{\pi}{2} |1 - \gamma|\right) + \operatorname{sgn}(1 + \gamma) \cdot Si\left(\frac{\pi}{2} |1 + \gamma|\right) \right]$$

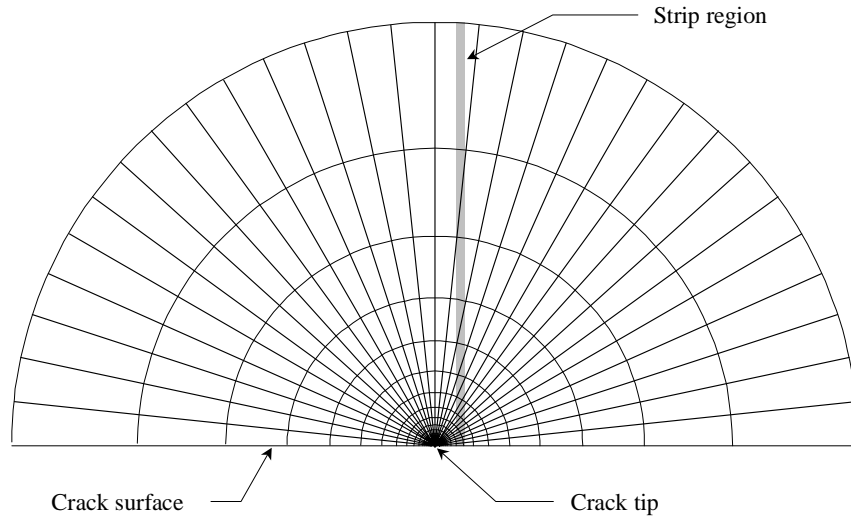
Further it has been explained how the result (8) can be used to obtain the solution for arbitrary function complying requirements (1).

3. Finite element formulation

A disk shaped body is initially examined for the numerical computations. The crack plane forms one diameter of the disk and the crack tip is fixed at the centre of the disc. Accounting for the symmetry of the problem only a half-disk, cut from the whole along the crack plane, has been considered for finite element method (FEM) modelling and calculations.

The finite element mesh for the half-disk is shown schematically in Fig. 3. The mesh is built up of 2250 eight-node biquadratic isoparametric plane strain finite elements. A half-disk region around the crack tip, called inner, is formed with 31 equally spaced nodes along each of 100 concentric circles centred at the tip. The first circle, closest to the crack tip, has a radius one fourth of the radius of the second circle, thus emulating a square root of the distance dependence of the displacements. All the other concentric circles in the inner region are equally spaced. The inner region has a radius one hundredth of the radius of the entire half-disk. The remaining part of the half-disk,

Figure 3. Schematic of the used finite element mesh



which has a half-ring shape, is called outer region. It is formed again with 31 equally spaced nodes along each of 50 concentric circles, but the distances between them are now gradually increasing with a factor of 1.5.

The described geometry is fixed for all calculations and does not include a description of the strip with spatially changing modulus of elasticity. A modulus of elasticity E^0 and a Poisson's ratio $\nu = 1/3$ have been initially ascribed to all elements. The FEM experiments have been performed using ABAQUS program. The procedure for user's definition of a material (UMAT) has been utilised for simulation of the strip. The design of the code allows a geometrical description of an inclusion with arbitrary shape and having arbitrary change of material characteristics, and it has been used in this work for simulating a strip region with the modulus of elasticity change shown in Fig. 2. More important, indeed, is the possibility, which the code gives for using only one configuration of the mesh and the crack. Instead of modelling the body with the strip as a half-disk with spatially fixed strip region, and then considering different locations of the crack tip with respect to the strip, the procedure allows the use of the initial configuration of the crack, with a strip changing its distance from the crack tip. The half-width of the strip, s , has been taken as one hundredth of the half-disk radius, or equal to the inner region radius. The strip has been displaced at discrete set of distances in the crack tip vicinity, ranging from $2s$ before the tip to $2s$ after the tip with a step of $s/10$.

The idea of convective solver, i.e. "moving strip", has been used to carry out all the calculations in the following way. For every position of the strip, the composed body is loaded by imposing one and the same displacements along the edge of the outer region

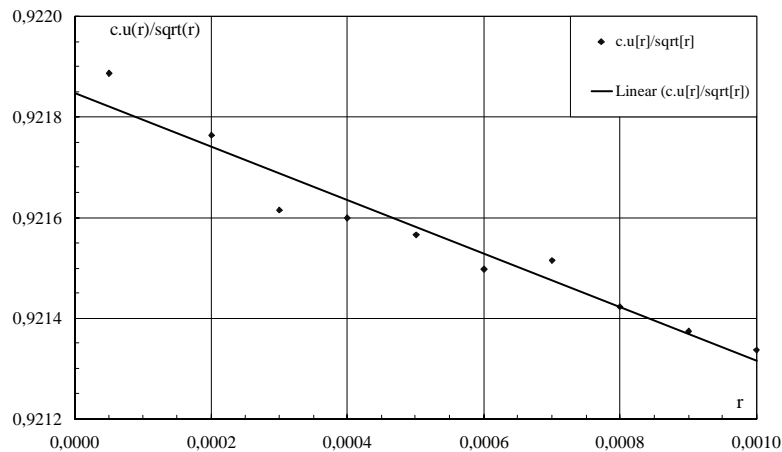
of the mesh. These displacements are calculated from the stress intensity factor for the unperturbed problem, K_I^0 . Hence, it is assumed that the presence of the strip does not affect the prescribed boundary conditions. Such an assumption is reasonable, since the strip width is comparable to the node distances at the outer boundary, and at most one of the nodes could possibly fall into the strip zone. More precise discussion on this matter is to be found in section 5 of the work.

The displacements found along the crack plane are used for calculating the stress intensity factor of the perturbed problem for the given position of the strip relative to the crack tip. The accepted operation is employing the vertical displacements, u_2 , of the first ten nodes behind the crack tip along the crack plane. A graph of the ratio u_2/\sqrt{r} as a function of the distance r from the crack tip to the respective node is drawn and an extrapolation to the limit $r = 0$ is used to establish:

$$(10) \quad K_I^{FEM} = \lim_{r \rightarrow 0} c(r) \cdot \frac{u_2(r)}{\sqrt{r}} = c(r=0) \cdot \lim_{r \rightarrow 0} \frac{u_2(r)}{\sqrt{r}}$$

where $c(r)$ is a parameter depending on the material characteristics at the corresponding node. In the cases when the crack tip position is outside the strip $c(r)$ is a constant depending on the unperturbed modulus of elasticity and Poisson's ratio. When crack tip position is inside the strip, clearly $c(r)$ is not a constant, since the modulus of elasticity change. In a limit, however, $c(r=0)$ is a constant depending on the modulus of elasticity at the crack tip, $E(r=0)$, and Poisson's ratio. A typical example of the procedure for estimating $E(r=0)$ is shown in Fig. 4, where the crack tip is situated in the middle of the strip and $E(r=0) = 0.9E^0$.

Figure 4. An example of estimating the stress intensity factor by linear regression



4. Results and error estimation of the analytical solution

The FEM experiments, reported in this work, have been performed primarily to check and support the analytical predictions given by equations (8) and (9). The experiments aimed also to show the deviation of the results from the analytical prediction for different values of the perturbation coefficient. Since the analytical solution is obtained with the assumption for small Δ , a good agreement should be expected at least for such coefficients. When performing FEM experiments, the stress intensity factor found for each position of the crack tip is actually the one of the perturbed problem, which can be written as $K_I^{FEM} = K_I^0 + \Delta K_I^1$.

The stress intensity factor for the unperturbed problem, K_I^0 , has been found with letting $\Delta = 0$ in the calculation procedure. In order to compare these results with the analytical ones, the expression $\frac{1}{\Delta} \left(\frac{K_I^{FEM}}{K_I^0} - 1 \right) = \frac{K_I^1}{K_I^0} = \Pi^{FEM}(\gamma)$ has been plotted for different values of Δ . Here $\gamma = d/s$ is a dimensionless position of the crack tip used in the analytical solution.

Figure 5a shows $\Pi^{FEM}(\gamma)$ for $\Delta = -0.01; \Delta = -0.1; \Delta = -0.5; \Delta = -0.9$, and Figure 5b for $\Delta = 0.01; \Delta = 0.1; \Delta = 0.5; \Delta = 0.9$. On the same figures the graphs of the analytical solution (8) are given for negative and positive perturbation coefficient respectively.

To estimate the error in analytical prediction for finite magnitudes of Δ a comparison of the minimal, respectively maximal values of the functions $\Pi^{FEM}(\gamma)$ and $\Pi(\gamma)$ has been adopted. The calculated minimum for negative Δ (maximum for positive Δ) of the analytical solution is -0.6854 (0.6854) achieved at point $\gamma \approx 0.26$. The FEM solution

Figure 5a. FEM results for the function $\Pi^{FEM}(\gamma)$ for different negative perturbation coefficients and analytic curve $\Pi(\gamma)$ for $\text{sgn}(\Delta) = -1$.

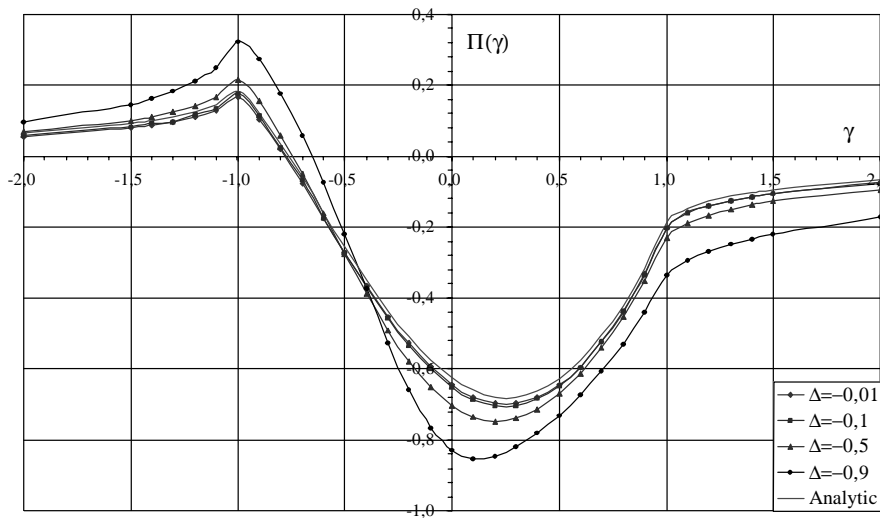
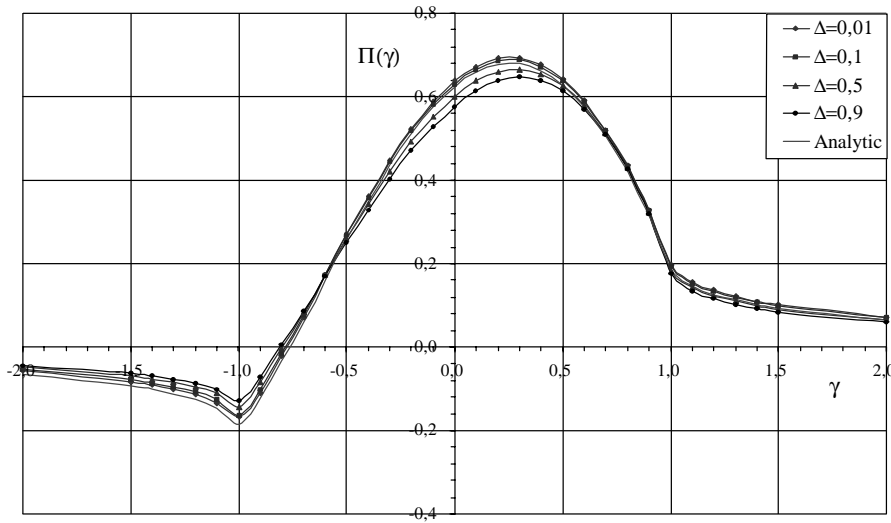


Figure 5b. FEM results for the function $\Pi^{FEM}(\gamma)$ for different positive perturbation coefficients and analytic curve $\Pi(\gamma)$ for $\text{sgn}(\Delta) = +1$.

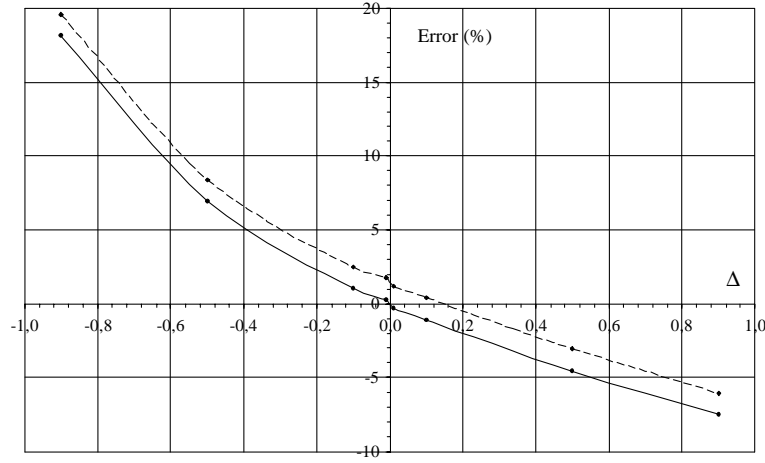


curves cannot offer the exact position of the minimum or maximum because of the discreteness of the set of crack locations. As seen from the figures there is a tendency for shifting the extremum with changing Δ . Nevertheless, the minimal and maximal values of $\Pi^{FEM}(\gamma)$ has been taken as follows: $-0.6976, -0.7031, -0.7483, -0.8525$ for $\Delta = -0.01, -0.1, -0.5, -0.9$ respectively and $0.6936, 0.6881, 0.6651, 0.6464$ for $\Delta = 0.01, 0.1, 0.5, 0.9$ respectively. The relative errors in the analytical solution minimum and maximum with respect to the FEM results is presented in Fig.6 by the dashed line. Since the FEM results inevitably carry computational errors, additional reasoning is used to improve the error estimate for the analytical solution.

First of all it is expected that the error in the analytical solution must vanish when the perturbation coefficient is approaching zero. This must result in an error estimate curve passing trough the origin. Further it might be expected a linear behaviour of the error for small non-zero values of the perturbation coefficient. This proposes a translation of the error estimate curve by the mean value of the errors for the smallest positive and negative perturbation coefficients. The effect of these considerations is the bold line in Fig. 6, presenting an improved error estimate curve.

From the graphics in Fig.6 the range of validity of the analytical solution is easily assessed. For example, if an error of 5% is acceptable for certain applications, then the analytical solution is applicable for variations of the modulus of elasticity as big as about 40% in negative and about 60% in positive direction.

Figure 6. Error in analytical solution for finite values of the perturbation coefficient



5. Discussion

In the present paper a numerical solution is found applying the dominating term of the Williams expansion for mode I crack in a homogeneous material. Boundary conditions are applied to a disk shaped region surrounding the crack tip. For a case with an inclusion of different elastic material, the under laying assumption is that the distance between the crack tip and the remote boundary is large compared to the distance between the crack tip and the inclusion or the linear extent of the inclusion. This assumption is of course violated when an infinite strip is examined. Accept for a moment that the strip is more compliant than the surrounding material (the case with negative perturbation coefficient). Then, imposed displacements from the Williams solution would underestimate energy release rate in the disk selected for the calculations. This is due to underestimated boundary tractions. As opposed to that, imposed boundary tractions also from the Williams solution will overestimate the energy release rate due to overestimated displacements at the boundary. The difference between the stress intensity factor, resulting from prescribed boundary tractions, and that resulting from prescribed displacements can be found by computing the stress intensity factor arising from boundary tractions acting along the intersection between the strip and the boundary. These tractions give a total force that is of the magnitude:

$$\delta F = C \cdot \frac{K_I}{\sqrt{R}} \cdot s$$

where s is the strip width, R is the radius of the considered disk, and C is a constant dependent on the particular choice of the elastic modulus variation in the strip.

The stress intensity factor caused by that force is of the order:

$$\delta K_I = \frac{\delta F}{\sqrt{R}}$$

which leads to the expression:

$$\delta K_I = C \cdot K_I \cdot \frac{s}{R}.$$

The computed deviation of the stress intensity factor due to the presence of the strip is of the order $C \cdot K_I$, yielding an error due to approximated boundary conditions of the order s/R . This fact is taken into account in the finite element modelling, where $s/R = 0.01$, which is considered fulfilling reasonable requirements on accuracy.

The error due to approximated boundary conditions can also be directly related to the difference in results between a finite rectangular region with a length equal to the linear extent of the treated disk and an infinite strip. The fact that the result for a strip of a length to width ratio of 100 almost coincides with the result for an infinite strip (Jivkov and Stähle, 1999) lends confidence to the presently assumed boundary conditions.

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Paper C

**Intrinsic Functions for Non-homogeneous
Elastic Materials**

Intrinsic Functions for Non-homogeneous Elastic Materials

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Abstract — A special choice of two functions is proposed for analysing non-homogeneous materials, when both modulus of elasticity and Poisson’s ratio experience spatial variations. It is shown that in the case of abrupt change of the two basic material’s characteristics, these two functions degenerate to constants, which are linear combinations of Dundurs’ parameters. The properties of the two functions are studied and their potential applications are discussed.

Keywords — non-homogeneous elastic materials, functionally graded materials, layered materials, composites

1. Introduction

The advantages of materials deliberately fabricated with spatially varying mechanical properties have been recognised in the recent years for a bulk of applications. These advantages consist in exploiting valuable characteristics of a set of different materials to produce an integrated material, having a desirable performance under a particular regime of exploitation. The materials have been called functionally graded materials (FGM) to underline the idea behind their design and manufacturing. The “disadvantage” from the mechanical point of view is that such materials can not be analysed in the frame of classical homogeneous isotropic linear elastic theory, since at least the homogeneity is lost by convention.

Possibly the simplest examples of FGM are layered (or laminated) materials, for which by definition the mechanical characteristics experience discontinuous changes at discrete surfaces. Consequently, the case of two different elastic materials, ideally joined along a plane has been initially investigated (Dundurs, 1969). The major result of this work is the observation that the stress field in such a bimaterial medium depends

only on two parameters, but not on four characteristics for given stress boundary conditions. Generally, the two parameters can be chosen in many ways as proper combinations of the four constants of the two materials but one particular choice, proposed by Dundurs, has been widely accepted and named after him.

The case of discontinuous change of elastic properties can hardly be supported with physical reasons. During the manufacture process considerable diffusion across the interfacial planes might occur, questioning the idea of abrupt properties' change. The diffusion is also used to create a prescribed distribution of the material properties, thus producing more general illustrations of FGM. In both cases the notion of sharp (or ideal) interfaces can be deserted and the variation of material characteristics can be accepted continuous.

Some studies of FGM address the continuum mechanics problems. For example a concept of "equivalent homogeneous plate" has been successfully applied (Abid Mian and Spencer, 1998) to obtain a class of solutions for stress field in functionally graded and layered materials with arbitrarily changing (with respect to one co-ordinate only) elastic characteristics.

For the present, the crack problem and some other fracture mechanics concepts in FGM have been studied basically after a choice of a particular function describing the variation of the modulus of elasticity, while the changes in Poisson's ratio have been neglected. For example, steady state solutions for crack propagating in a material with exponential change of elastic modulus have been found (Atkinson and List, 1978) and it has been demonstrated that this is the only choice of variation leading to them. It has been shown later (Delale and Erdogan, 1983, Erdogan et al., 1991, Erdogan, 1995) that an exponential variation with respect to one or both plane co-ordinates is the only choice, which converts the coefficients of the differential equation for the plane problem to constants.

Based on the results, it has been discussed in most of the mentioned papers that the assumption of constant Poisson's ratio, ν , is not very restrictive. This is because of the insensitivity of the solutions to reasonable changes of ν . For the sake of completeness and also for dealing with the potential cases when the changes in Poisson's ratio are significant enough not to be discarded, the influence of these changes deserves attention. In such cases the spatial change of ν enters the problem formulation as a second function in addition to the change of modulus of elasticity, E . In this paper two functions, dependent on the variations of elastic modulus and Poisson's ratio, are introduced. It is initially demonstrated that these functions degenerate to constants in the

case of discontinuous change of elastic characteristics, and these constants are linear combinations of Dundurs' parameters. It is clarified that this is not the only possible choice of functions, but it is shown to have certain advantages to other possible choices. Any particular mathematical form of these functions defines two classes of functions, for E and ν variations respectively, for which the stress fields will be self-similar. Thus a solution for one couple of the introduced functions will generate solutions for infinitely many choices (belonging to the initiated class) of the modulus of elasticity and Poisson's ratio spatial changes. The goal of the present paper is to explore the prospects for improved formulations in a future work on cracks in functionally graded materials.

2. Opening motives in functional description

Consider a plane state of an elastic homogeneous continuum and note that the constitutive law can be written for both plane strain and generalised plane stress as follows:

$$(1) \quad \varepsilon_{ij} = \frac{1}{2\mu} \left[\sigma_{ij} + \frac{\kappa-3}{4} \sigma_{kk} \delta_{ij} \right], \quad \langle i, j = 1, 2 \rangle$$

where μ is the shear modulus, and $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress respectively.

Introduce also for the sake of clarity the modified modulus of elasticity and Poisson's ratio:

$$(2) \quad (\hat{E}, \hat{\nu}) = \begin{cases} (E, \nu) & \text{plane stress} \\ \left(\frac{E}{1-\nu^2}, \frac{\nu}{1-\nu} \right) & \text{plane strain} \end{cases}$$

When $\sigma_{22} = 0$ equation (1) can be written in one of the following forms:

$$(3) \quad \varepsilon_{11} = \frac{\kappa+1}{8\mu} \sigma_{11} = \frac{1}{\hat{E}} \sigma_{11} = C_1 \cdot \sigma_{11}$$

It is natural to call C_1 the uniaxial compliance of the continuum at plane state.

When $\sigma_{22} \neq 0$, a contraction of equation (1) leads to any of the presentations:

$$(4) \quad \varepsilon_{ii} = \frac{\kappa-1}{4\mu} \sigma_{ii} = \frac{1-\hat{\nu}}{\hat{E}} \sigma_{ii} = C_2 \cdot \sigma_{ii}$$

Again C_2 could be called biaxial (bulk or real) compliance of the continuum at plane state. Initiate for convenience a new constant C_3 using the relations:

$$(5) \quad C_3 = 2C_1 - C_2 = \frac{1}{2\mu} = \frac{1+\hat{\nu}}{\hat{E}}$$

which can be called a shift of the real from the uniaxial compliance.

Now examine the case of a linear but inhomogeneous continuum with elastic properties changing as arbitrary functions of one of the plane co-ordinates, say x . Then the constants C_1 , C_2 and C_3 , defined by equations (3)-(5), can be regarded as instantaneous uniaxial, biaxial and shift compliance respectively at the spatial position under consideration, for which the constitutive law (1) is valid. They could also be regarded as functions of the spatial co-ordinate along which the elastic properties vary as their definitions show.

Introduce formally two functions, Γ_1 and Γ_2 , through the differential relations:

$$(6a) \quad d\Gamma_1 = -\frac{dC_2}{2C_1} = -\frac{\kappa-1}{\kappa+1} \cdot \frac{d\frac{\kappa-1}{\mu}}{\frac{\kappa-1}{\mu}} = \frac{\kappa-1}{\kappa+1} \cdot d \ln \frac{\mu}{\kappa-1} = \frac{1}{2}(1-\hat{\nu}) \cdot d \ln \frac{\hat{E}}{1-\hat{\nu}}$$

$$(6b) \quad d\Gamma_2 = -\frac{dC_3}{2C_1} = -\frac{2}{\kappa+1} \cdot \frac{d\frac{1}{\mu}}{\frac{1}{\mu}} = \frac{2}{\kappa+1} \cdot d \ln \mu = \frac{1}{2}(1+\hat{\nu}) \cdot d \ln \frac{\hat{E}}{1+\hat{\nu}}$$

The differentials of Γ_1 and Γ_2 can be thought of as dimensionless quantities measuring the tendency of change of the biaxial and the shift compliances with respect to the instantaneous uniaxial compliance at a given spatial position.

For the case of a continuum, consisting of two bonded half-planes of different homogeneous linear elastic materials, it can be easily shown that the two introduced functions are simply related to the known Dundurs' parameters. Indeed these parameters can be written in terms of the biaxial and the shift compliances modifying Dundurs (1969):

$$(7a) \quad \alpha = \frac{C_1^1 - C_1^2}{C_1^1 + C_1^2} = \beta + \frac{1}{2} \cdot \frac{C_3^1 - C_3^2}{C_1^1 + C_1^2}$$

$$(7b) \quad \beta = \frac{1}{2} \cdot \frac{C_2^1 - C_2^2}{C_1^1 + C_1^2}$$

where all terms are constants with the superscripts denoting the number of the material.

On the other side the three compliances can be written as functions of the above constants using Heaviside's step function, $\theta(x)$:

$$(8) \quad C_i(x) = C_i^1 + (C_i^2 - C_i^1) \cdot \theta(x)$$

which introduces the jump in the compliances at $x = 0$.

If postulated that the instantaneous uniaxial compliance at the point of jump is the mean of the uniaxial compliances of the two materials, i.e. $C_1(x=0) = (C_1^1 + C_1^2)/2$, then it can be concluded from equation (6) that the following relations hold:

$$(9a) \quad \frac{d\Gamma_1}{dx} = \frac{C_2^1 - C_2^2}{C_1^1 + C_1^2} \cdot \delta(x) = 2\beta\delta(x)$$

$$(9b) \quad \frac{d\Gamma_2}{dx} = \frac{C_3^1 - C_3^2}{C_1^1 + C_1^2} \cdot \delta(x) = 2(\alpha - \beta)\delta(x)$$

where $\delta(x)$ is Dirac's delta function. If equations (9) are integrated over the entire real axis, the result shows that in the case of sharp interface between two homogeneous materials, the chosen functions degenerate to constants, which are linear combinations of the Dundurs' parameters:

$$(10) \quad \Gamma_1 = 2\beta \quad \text{and} \quad \Gamma_2 = 2(\alpha - \beta)$$

Note that the symmetry property typical for Dundurs' parameters is preserved in the introduced functions, namely an alternation in tendency of change of the shift and the biaxial compliances from negative to positive (or vice versa) alters the sign of the respective function.

3. An applicability study and discussion

A first observation is that the functions from the previous section are mutually dependent and not separable in the sense that each of one of the functions is not expressible as a full differential of some expression not explicitly dependent on the other. It is easily shown for example that the following implications hold:

$$(11) \quad \begin{aligned} \Gamma_1 = const &\Rightarrow \Gamma_2 = \ln \frac{\kappa-1}{\kappa+1} = \Gamma_2^1(\kappa) \\ \Gamma_2 = const &\Rightarrow \Gamma_1 = \ln \frac{2}{\kappa+1} = \Gamma_1^2(\kappa) \end{aligned}$$

which may be used to write the differential form of the functions differently

$$(12) \quad \begin{aligned} d\Gamma_1 &= \frac{\kappa-1}{\kappa+1} \cdot d \ln \frac{\mu}{\kappa-1} = \exp(\Gamma_2^1) \cdot d \ln \frac{\mu}{\kappa-1} \\ d\Gamma_2 &= \frac{2}{\kappa+1} \cdot d \ln \mu = \exp(\Gamma_1^2) \cdot d \ln \mu \end{aligned}$$

Equations (12) demonstrate the intrinsic property of the proposed functions, namely each one of them depends on one unique choice of the other (the choice for which the first is constant). The claim here is that two mutually independent characteristics have not to be used in a mechanical description of a non-homogeneous medium (for example $E(x), \nu(x)$) since another choice of these characteristics leading to the same state of the continuum can always be found. Thus the solution to a particular problem, solved for two separable parameters will be a solution to a class of problems but with unknown originating property of the class. In such a case another solution for separable parameters, by chance belonging to the same class, would well cause an unnecessary effort. Two characteristics, which avoid this possible situation, like the proposed functions, will be called intrinsic.

Additional debate is suggested to clarify the notion of non-separable couples of functions. The mutual dependence arises from the consistency of the “measure” used to grade the tendencies of change of the different compliances. In the previous section this measure was chosen to be the uniaxial compliance. The tendencies of change of all the three compliances, measured with this scale will give three reasonable choices and any two of them, a matter of taste, can be chosen for intrinsic functions, the third one remaining a linear combination of the two. If another measure is used, say the biaxial compliance, another set of intrinsic functions will be initiated provided the measure is used consistently for every possible tendency of change. This is not the case, however,

if different measures to scale different tendencies of change of the compliances are applied. For example, if C_2 -measure and C_3 -measure are used to scale the tendencies of change of the biaxial and the shift compliances respectively, the result will be a couple of quasi-intrinsic functions given by:

$$(13) \quad d\Gamma_1^q = d \ln \frac{\mu}{\kappa - 1}; \quad d\Gamma_2^q = d \ln \mu$$

These functions are obviously inadequate, since they are separable (already separated), and the state of the continuum described by them might be described by many others.

Therefore, if two candidate functions are to be used the first property to be inspected is that they are inseparable, or mutually dependent, which can be achieved for example by applying a consistent measure for all types of changes to be scaled.

It is important to identify the “classes of equivalence”, which the intrinsic functions induce. The case of plane stress will be considered for definiteness, but obviously plane strain conditions can be treated similarly. The two functions may be written as:

$$(14) \quad \begin{aligned} d\Gamma_1 &= \frac{1}{2}(1 - \nu) \cdot d \ln E + \frac{1}{2} d\nu \\ d\Gamma_2 &= \frac{1}{2}(1 + \nu) \cdot d \ln E - \frac{1}{2} d\nu \end{aligned} \quad \Rightarrow$$

$$(15) \quad \begin{aligned} d(\Gamma_1 + \Gamma_2) &= d \ln E \\ d(\Gamma_1 - \Gamma_2) &= d\nu - \nu \cdot d \ln E \end{aligned}$$

Equations (15) lead to the following solutions for $E(x), \nu(x)$ if the intrinsic functions are given:

$$(16) \quad \begin{aligned} E(x) &= \exp \left[\int d(\Gamma_1 + \Gamma_2) + k_1 \right] = K_1 \cdot E^f(x) \\ \nu(x) &= E(x) \cdot \left[\int \frac{d(\Gamma_1 - \Gamma_2)}{E(x)} + k_2 \right] = \nu^f(x) + K_2 \cdot E^f(x) \end{aligned}$$

where K_1, K_2 are arbitrary constants and $E^f(x), \nu^f(x)$ are some fundamental solutions to the integrals. If the integration constants are chosen as $K_1 = 1; K_2 = 0$, i.e. a definite choice of modulus of elasticity and Poisson’s ratio variations is made, then the solution for the corresponding definite intrinsic functions found by (14), will cover the solutions for the family of elastic modulus and Poisson’s ratio variations given by (16). In other

words, the class of equivalent material parameters' gradients induced by the intrinsic functions is described by equations (16).

To see the application of the intrinsic functions, imagine an infinitesimal element of a non-homogeneous continuum, with change of the elastic properties in one direction, say x . For this element one can write the following conditions:

$$(17) \quad \begin{aligned} \delta\sigma_x = \delta\sigma_{xy} = 0; \delta\sigma_y \neq 0 \\ \delta\varepsilon_y = \delta\varepsilon_{xy} = 0; \delta\varepsilon_x \neq 0 \end{aligned}$$

where δ denotes the variation of the respective quantity due to the variation of the elastic properties in x -direction. It is interesting to find the change in the stresses for future applications. Using the constitutive law for elasticity one may express:

$$(18) \quad \delta\varepsilon_y = 0 = \frac{\kappa+1}{8\mu} \cdot \delta\sigma_y + (\sigma_x + \sigma_y) \cdot \delta\left(\frac{\kappa-1}{8\mu}\right) - (\sigma_x - \sigma_y) \cdot \delta\left(\frac{1}{4\mu}\right)$$

Now it is easy to see that the change of σ_y due to the elastic properties' variation (the only stress experiencing such a change) may be expressed with the help of the intrinsic functions and the initial stress state:

$$(19) \quad d\sigma_y = (\sigma_x + \sigma_y) \cdot d\Gamma_1 - (\sigma_x - \sigma_y) \cdot d\Gamma_2$$

where the variations are replaced with derivatives.

The increment of σ_y given by (19) will be an additional term in, for example, the equilibrium equations for the non-homogeneous continuum, turning them into a system of differential equations with variable coefficients.

As can be deduced from (16) the only choice making these coefficients constant is an exponential variation of the elastic modulus and a constant Poisson's ratio, provided that $K_2 = 0$. In the case of nonzero constant, $K_2 \neq 0$, it is clear that a solution for exponential variation of the elastic modulus will be also a solution to all the problems with identical exponential variation of Poisson's ratio, scaled by that nonzero constant. This should apply for the solutions found by different authors for exponential change of E and constant ν , or for cosine change of E and constant ν (Jivkov and Ståhle, 1999), which is a rather interesting observation.

As a matter of fact, equations (16) suggest one natural separation of the graded materials into two classes. The first class covers all materials with constant, or changing

similarly to elastic modulus Poisson's ratio. By similarly changing is meant having the same shape, but not necessarily the same scale. Obviously, this class is established from constant fundamental function for ν . The materials belonging to this class can be called uni-functional. The second class covers all other materials, for which the change of ν has essentially dissimilar shape from the change of E . This class is created from non-constant fundamental function for ν and the materials belonging to it can be called bi-functional.

Thus the solutions found for exponential, or cosine variation of modulus of elasticity hold for uni-functional materials with the corresponding functional change. In fact, bi-functional materials are difficult to imagine. But if there are such materials, it is hard to see how a closed form solution to the differential equations with variable coefficients can be found. One attempt with the help of perturbation analysis, however, might be worth to explore.

Finally, it is important to note that, although mathematically the two constants K_1, K_2 may be arbitrary, physically only a bounded collection of choices is of practical interest. This is especially valid for the choices of K_2 since Poisson's ratio potential variations are much more restricted than those of the modulus of elasticity are.

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