To appear in Transactions of the Institute of Measurement and Control

Transactions of the Institute of
Measurement and Control

Stochastic given-time H∞ consensus over Markov jump networks with disturbance constraint

Journal:	*Transactions of the Institute of Measurement and Control*
Manuscript ID	TIMC-15-0580
Manuscript Type:	Original Manuscript
Date Submitted by the Author:	05-Nov-2015
Complete List of Authors:	Luan, Xiaoli; Jiangnan University, Institute of Automation Min, Yang; Jiangnan University, Institute Ding, Zhengtao; University of Manchester, School of Electrical and Electronic Engineering Liu, Fei; Key Laboratory of Advanced Process Control for Light Industry (Ministry of Education), Jiangnan University
Keywords:	Directed communication graph, Markov jump topology, Give-time consensus, H∞ performance, Disturbance constraint
Abstract:	In this study, the given-time H∞ consensus problem over networks with directed information flow and Markov jump topologies is addressed. Our focus is on keeping the disagreement dynamics of networks remain confined within the prescribed bound in the fixed time interval. Compared with the asymptotical consensus in infinite settling time, the proposed algorithm is less conservative. In addition, the new model transformation approach is presented to make the design results more advantageous in commonality. Simulation results show the effectiveness of the proposed controller, and reveal that the prescribed boundary of the disagreement trajectory has the effect on disturbance rejection performance.

SCHOLARONE™
Manuscripts

Stochastic given-time H_∞ consensus over Markov jump networks with disturbance constraint

Xiaoli Luan, Yang Min, Zhengtao Ding and Fei Liu

For Peer Review

Abstract

In this study, the given-time H_∞ consensus problem over networks with directed information flow and Markov jump topologies is addressed. Our focus is on keeping the disagreement dynamics of networks remain confined within the prescribed bound in the fixed time interval. Compared with the asymptotical consensus in infinite settling time, the proposed algorithm is less conservative. In addition, the new model transformation approach is presented to make the design results more advantageous in commonality. Simulation results show the effectiveness of the proposed controller, and reveal that the prescribed boundary of the disagreement trajectory has the effect on disturbance rejection performance.

Index Terms

Directed communication graph, Markov jump topology, Give-time consensus, H_∞ performance

I. Introduction

In practical applications, dynamic systems are often connected with each other via communication and sensing networks to achieve some challenging control tasks, such as satellite formation flying, hazardous material handling, and industrial process controlling, etc (Fax and

Xiaoli Luan, Yang Min and Fei Liu are with the Key Laboratory of Advanced Control for Light Industry Processes, Ministry of Education, and the Institute of Automation, Jiangnan University, Wuxi 214122, China. E-mail: xlluan@jiangnan.edu.cn, yangmin1222@yeah.net, fliu@jiangnan.edu.cn

Zhengtao Ding is with the Control Systems Centre, School of Electrical and Electronic Engineering, University of Manchester, Sackville Street Building, Manchester M 13 9PL, UK. E-mail: Zhengtao.Ding@manchester.ac.uk

Murray, 2004; Huang et al., 2006; Dong and Farrell, 2009; Xie et al., 2015). For such network-connected systems, agreement and consensus protocol design are fundamental and interesting issues, which aim to enable the states of all subsystems converge to the same value using the local information obtained from the network. For example, an industrial heating furnace with multiple passes need to be controlled for all individual pass outlet temperature reaching the same value in order to achieve higher energy efficiency and obtain better profit. Since pioneering results on consensus control (Jadbabaie and Morse, 2003) and (Olifati-Saber and Murray, 2004), the consensus problem has been extensively investigated by numerous researchers from various perspectives (see (Scardovi and Sepulchre, 2009; Li et al., 2010; Ding, 2013; Liu et al., 2014) and the references therein).

Since systems are often operating under uncertain environments, stochastic consensus for networks with Markov jump topologies has been studied. For example, necessary and sufficient conditions of mean square consensus were presented in (Zhang and Tian, 2009; Miao et al., 2013) for networks in Markov switching topologies; Both continuous and discrete-time consensus problems were investigated in (You et al., 2013), which shows the effect of switching topologies on consensus is determined by the union of topologies associated with the positive recurrent states of the Markov process. For other uncertain environments with various disturbances and random communication noises, robust H_∞ control approach is usually brought to achieve consensus and attenuate the effect of external disturbance on the behavior of the system. For example, the H_∞ consensus performance problem for linear multi-agent systems with undirected topologies was analyzed in (Massioni and Verhaegen, 2009; Lin and Jia, 2010; Li et al., 2011). Considering physical systems in practice are inherently nonlinear, the consensus protocol with prescribed L_2 and H_∞ performance for multi-agent systems were investigated in (Wen et al., 2012; Li et al., 2012).

Carding the existing research results on consensus, including subsystems with linear and nonlinear dynamics, communications with fixed and stochastic topologies, protocols with state feedback and output feedback laws, most of which require that the disagreement dynamics of subsystems asymptotically converges to zero in infinite time interval. In practice, there are some cases where large value of state disagreement is not acceptable, for instance in the presence of saturation. Furthermore, some systems are required to operate satisfactorily only over fixed time interval, e.g., communication network systems, biochemistry reaction systems, and robot

For Peer Review

control systems. In this sense it appears more reasonable to investigate the network-connected systems whose state disagreement remains within prescribed bound in the fixed time interval. So far, to the best of authors knowledge, there is no result available yet on such problem, not to mention network-connected systems with Markov jump topologies and external disturbances. This situation actually motivates the present work.

The given-time consensus to be addressed in this paper deals with the network-connected systems whose operation is limited to a specified time interval, and the state disagreement does not exceed a given threshold. By means of the concept of short-time stability (finite-time stability) put forward by Dorato (Dorato, 1961), which has been developed well in the past few years (Amato and Ariola, 2005; Luan et al., 2013), a given-time H_∞ consensus protocol is designed for networks with stochastic Markov jump communication topologies and external disturbances. Compared with existing results in consensus, the proposed algorithm has three new features: 1) The proposed given-time H_∞ consensus reduces the conservativeness of the controller design from the perspective of engineering; 2) The new model transformation approach is presented by exploring certain features of Laplacian matrix in real Jordan form, which leads to more generality of the designed protocol; 3) Different from existing results in network topologies governed by Markov jump process, the main result obtained is determined not only by the jump mode, but also the jump rate.

II. Problem Description

The dynamics of N subsystems takes the following form:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) + B_w w_i(t) \tag{1}$$

for $i = 1, \cdots, N$, where $x_i(t) \in R^n$ and $u_i(t) \in R^m$ represent the state and control input of subsystem i, respectively. $w(t) \in L_2^p \Big[\begin{matrix} 0 & +\infty \end{matrix} \Big)$ is the external disturbance, and A, B, B_w are constant matrices with (A, B) controllable.

The connection among the subsystems is specified by a directed graph G which consists of a set of vertices denoted by Λ and a set of edges denoted by Υ. A vertex represents a subsystem, and each edge represents a connection. Associated with the graph, its adjacency matrix Q with elements q_{ij} denotes the connections such that $q_{ij} = 1$ if there is a connection from subsystem j to subsystem i, and $q_{ij} = 0$, otherwise. The Laplacian matrix $L = \{l_{ij}\}$ is commonly defined as

For Peer Review

$l_{ii} = \sum_{j=1,j\neq i}^{N} q_{ij}$, $l_{ij} = -q_{ij}$ for $i \neq j$. For a directed graph, some properties about the Laplacian matrix L will be given in the following lemma.

Lemma 2.1: (Yu et al., 2011) Zero is an eigenvalue of L with $\vec{1} = \begin{bmatrix} 1, & \cdots, & 1 \end{bmatrix}^{\mathrm{T}} \in R^N$ as the corresponding right eigenvector and all the non-zero eigenvalues have positive real parts. Furthermore, zero is a single eigenvalue of L if and only if the graph contains a directed spanning tree.

In this work, the considered network connection topology is switching, and the switching process $\{r_t, t \geq 0\}$ is governed by a homogeneous Markov chain. Let $(\Omega, E, \emptyset)$ be the probability space,where Ω is the sample space, E is the algebra of events and $\emptyset$ is the probability measure defined on E. Let the random form process $\{r_t, t \geq 0\}$ be the Markov stochastic process taking values in a finite set $S = \{1, 2, \cdots, s\}$ with transition rate matrix $\Pi = \{\pi_{rl}\}$, $r, l \in S$ and define the following transition probability from mode r at time t to mode l at time $t + \Delta t$ as

$$\emptyset_{rl} = \emptyset_r \{r_{t+\Delta t} = l | r_t = r\} = \begin{cases} \pi_{rl}\Delta t + o(\Delta t), & r \neq l \\ 1 + \pi_{rr}\Delta t + o(\Delta t), & r = l \end{cases}$$

with transition probability rates $\pi_{rl} \geq 0$ for $r, l \in S$, $r \neq l$ and $\sum_{l=1,l\neq r}^{s} \pi_{rl} = -\pi_{rr}$ where $\Delta t > 0$ and $\lim_{\Delta t \to 0} o(\Delta t)/\Delta t \to 0$.

To formulate the problem addressed in this paper, let us make the following assumptions:

Assumption 2.1: The external disturbance $w_i(t)$ for each subsystem is time varying and satisfies

$$\int_0^t w_i^{\mathrm{T}}(t) w_i(t) dt \leq d, d \geq 0 \tag{2}$$

where d is a given positive scalar.

Assumption 2.2: For the convenience of presentation of the proposed design, the eigenvalues of the Laplacian matrix are distinct (Cai et al., 2015).

In order to quantify the effect of disturbances to consensus, a controlled output function $z_i(t)$ is defined to measure the state disagreement variable as follows

$$z_i = x_i - \sum_{j=1}^{N} [\kappa(r_t)]_j x_j \tag{3}$$

where $\kappa(r_t) \in R^N$ is defined as the left eigenvector of jump Laplascian $L(r_t)$ associated with the eigenvalue 0 and satisfies $\kappa(r_t)^{\mathrm{T}}\vec{1} = 1$. It is desirable that the state disagreement $z_i(t)$ of the

For Peer Review

network-connected system (1) remains confined within the prescribed bound in the fixed time interval, which motivates the subsequent given-time consensus concept:

Definition 2.1: (Luan et al., 2013) For a given-time constant $T > 0$, the network-connected dynamic system (1) (setting $u_i(t) = 0, w_i(t) = 0$) is said to be given-time consentable with respect to $\begin{pmatrix} c_1 & c_2 & T & R \end{pmatrix}$, where $c_1 < c_2$ and $R > 0$, if

$$E\left\{z_i^{\mathrm{T}}(0)Rz_i(0)\right\} \leq c_1 \Rightarrow E\left\{z_i^{\mathrm{T}}(t)Rz_i(t)\right\} < c_2,\ \forall t \in \begin{bmatrix} 0 & T \end{bmatrix} \tag{4}$$

Remark 2.1: With the finite-time stability theory, the definition of given-time consensus for network-connected systems is presented to describe the transient behavior of state disagreement in finite-time interval. That is, for the given initial condition $E\left\{{z_i}^T(0)Rz_i(0)\right\} \leq c_1$, the disagreement trajectory is confined within the specified bound $E\left\{z_i^{\mathrm{T}}(t)Rz_i(t)\right\} < c_2$ over the fixed time interval $T > 0$. It should be noted that there is a significant difference between given-time consensus and asymptotical consensus studied in the existing literatures. In fact, a system is given-time consentable may not be asymptotically consentable if the disagreement trajectory does not converge to zero. Conversely, the asymptotical consensus of a system could not be given-time consensus if the disagreement trajectory exceeds the prescribed bound c_2.

Furthermore, considering the effect of external disturbances on the disagreement dynamics, the given-time H_∞ consensus protocol is designed such that the disagreement dynamics of system (1) not only satisfies (4), but also under zero initial condition, satisfies the following cost function inequality for a positive scalar $\gamma > 0$ and all admissible $w_i(t)$ with the constraint condition (2):

$$E\left\{\int_0^T z_i^{\mathrm{T}}(t)z_i(t)d\tau\right\} < \gamma^2 E\left\{\int_0^T w_i^{\mathrm{T}}(t)w_i(t)d\tau\right\} \tag{5}$$

In order to make the proposed given-time consensus controller suitable for network-connected system with more general topologies, the new model transformation method will be presented in the next section, which needs the following lemma:

Lemma 2.2: (Ding (2014)) For a Laplacian matrix that satisfies Lemma 2.1, there exists a transformation $F\left(r_t\right) = \begin{bmatrix} F_1\left(r_t\right) & \vec{1} \end{bmatrix}$ and $F(r_t)^{-1} = \begin{bmatrix} F_2\left(r_t\right) & \kappa\left(r_t\right) \end{bmatrix}^{\mathrm{T}}$ such that

$$F(r_t)^{-1}L\left(r_t\right)F\left(r_t\right) = J\left(r_t\right) \tag{6}$$

For Peer Review

where $J(r_t)$ is a block-diagonal matrix of blocks

$$J(r_t) = \begin{bmatrix} \lambda_1(r_t) & & & & & & \\ & \ddots & & & & & \\ & & \lambda_{n_\lambda}(r_t) & & & & \\ & & & \mu_1(r_t) & & & \\ & & & & \ddots & & \\ & & & & & \mu_{n_\mu}(r_t) & \\ & & & & & & 0 \end{bmatrix}$$

with $\lambda_i(r_t) \in R$ for $i = 1, \cdots, n_\lambda$ and $\mu_i(r_t) = \begin{bmatrix} \alpha_i(r_t) & \beta_i(r_t) \\ -\beta_i(r_t) & \alpha_i(r_t) \end{bmatrix} \in R^{2\times 2}$ for $i = 1, \cdots, n_\mu$. In the above expression, $\lambda_i(r_t)$, $\alpha_i(r_t)$ and $\beta_i(r_t)$ are positive real numbers with $\lambda_i(r_t)$ denoting real eigenvalues of $L(r_t)$ and $\alpha_i(r_t) \pm j\beta_i(r_t)$ denoting complex conjugate eigenvalues of $L(r_t)$ respectively and clearly we have $1 + n_\lambda + 2n_\mu = N$.

This paper relaxes the requirement that the disagreement dynamics asymptotically converges to zero, and only requires the disagreement trajectory stays in a prescribed bound with H_∞ performance specification over a given-time interval. To this end, the consensus protocol will be designed and sufficient conditions for the existence of given-time H_∞ consensus controller will be proposed in the next section.

III. Main Results

To achieve given-time consensus, the following protocol is adopted:

$$u_i = K(r_t) \sum_{j=1}^{N} q_{ij}(r_t)(x_j - x_i) \tag{7}$$

where $q_{ij}(r_t)$ are adjacency elements of the stochastic jump graph $G(r_t)$, and $K(r_t) \in R^{m\times n}$ is the mode-dependent control gain to be designed.

Then rewrite protocol (7) as follows:

$$u_i = -K(r_t) \sum_{j=1}^{N} l_{ij}(r_t) x_j \tag{8}$$

For Peer Review

With protocol (8) and denoting $x = \begin{bmatrix} x_1^{\mathrm{T}} & x_2^{\mathrm{T}} & \cdots & x_N^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, $w = \begin{bmatrix} w_1^{\mathrm{T}} & w_2^{\mathrm{T}} & \cdots & w_N^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, the closed-loop system is expressed as follows:

$$\dot{x} = (I_N \otimes A - L(r_t) \otimes BK(r_t))\, x + (I_N \otimes B_w)\, w \tag{9}$$

where $L(r_t)$ is the Laplacian matrix associated with jump graph $G(r_t)$, and $\otimes$ denotes the Kronecker product of matrices.

Denoting $z = \begin{bmatrix} z_1^{\mathrm{T}} & z_2^{\mathrm{T}} & \cdots & z_N^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, equality (3) is rewritten as follows:

$$\begin{aligned} z &= x - \left(\left(\vec{1} \cdot \kappa(r_t)^{\mathrm{T}} \right) \otimes I_n \right) x \\ &= M(r_t) \otimes I_n x \end{aligned} \tag{10}$$

where $M(r_t) = I_N - (\vec{1} \cdot \kappa(r_t)^T)$.

Making the following transformation $\xi = F(r_t)^{-1} \otimes I_n x$, $\tilde{z} = F(r_t)^{-1} \otimes I_n z$, $\tilde{w} = F(r_t)^{-1} \otimes I_p w$, the network dynamics (9) is transformed to the following system

$$\begin{cases} \dot{\xi} = (I_N \otimes A - J(r_t) \otimes BK(r_t))\, \xi + (I_N \otimes B_w)\, \tilde{w} \\ \tilde{z} = (F(r_t)^{-1} \otimes I_n)(M(r_t) \otimes I_n)(F(r_t) \otimes I_n)\xi \end{cases} \tag{11}$$

Performing the matrix calculation to equation (11), one has

$$\begin{aligned} \tilde{z} &= \left(F(r_t)^{-1} \otimes I_n \right) (M(r_t) \otimes I_n)(F(r_t) \otimes I_n)\, \xi \\ &= F(r_t)^{-1} M(r_t)\, F(r_t) \otimes I_n \xi \\ &= \left(F(r_t)^{-1} I_N F(r_t) - F(r_t)^{-1} (\vec{1} \cdot \kappa(r_t)^{\mathrm{T}}) F(r_t) \right) \otimes I_n \xi \\ &= \left(I_N - \left(F(r_t)^{-1} \vec{1} \right) \cdot \left(\kappa(r_t)^{\mathrm{T}} F(r_t) \right) \right) \otimes I_n \xi \end{aligned} \tag{12}$$

According to the definition of $F(r_t)$ in Lemma 2.2, we have

$$F(r_t)^{-1} F(r_t) = F(r_t)^{-1} \begin{bmatrix} F_1(r_t) & \vec{1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

For Peer Review

i.e.

$$F(r_t)^{-1}\cdot\vec{1}=\begin{bmatrix}0 & 0 & \cdots & 1\end{bmatrix}^{\mathrm{T}}$$

and

$$F(r_t)^{-1}F\left(r_t\right)=\begin{bmatrix}F_2\left(r_t\right) & \kappa(r_t)\end{bmatrix}^{\mathrm{T}}F\left(r_t\right)=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & & 0\\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 1\end{bmatrix}$$

i.e.

$$\kappa(r_t)^{\mathrm{T}}F\left(r_t\right)=\begin{bmatrix}0 & 0 & \cdots & 1\end{bmatrix}$$

Hence, equation (12) can be restated as

$$\tilde{z}=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}\otimes I_n\xi \tag{13}$$

For the notational convenience, let us partition the state variable ξ into ξ_i for $i=1,\cdots,N$. Then for $i=1,\cdots,n_\lambda$, we have

$$\begin{cases}\dot{\xi}_i=(A-\lambda_i(r_t)BK(r_t))\xi_i+B_w\tilde{w}_i\\ \tilde{z}_i=\xi_i\end{cases} \tag{14}$$

and for $i=n_\lambda+1,\cdots,N-1$

$$\begin{cases}\dot{\xi}_{i_1}=\left(A-\alpha_i\left(r_t\right)BK\left(r_t\right)\right)\xi_{i_1}-\beta_i\left(r_t\right)BK\left(r_t\right)\xi_{i_2}+B_w\tilde{w}_{i_1}\\ \tilde{z}_{i_1}=\xi_{i_1}\end{cases} \tag{15}$$

$$\begin{cases}\dot{\xi}_{i_2}=\left(A-\alpha_i\left(r_t\right)BK\left(r_t\right)\right)\xi_{i_2}+\beta_i\left(r_t\right)BK\left(r_t\right)\xi_{i_1}+B_w\tilde{w}_{i_2}\\ \tilde{z}_{i_2}=\xi_{i_2}\end{cases} \tag{16}$$

For Peer Review

and for $i = N$

$$\begin{cases} \dot{\xi}_N = A\xi_N + B_w\tilde{w}_N \\ \tilde{z}_N = 0 \end{cases} \tag{17}$$

where $i_1 = 1 + n_\lambda + 2k - 1$, $i_2 = 1 + n_\lambda + 2k$, $k \in 1, \cdots, n_\mu$. For notational simplicity, when $r_t = r$, $r \in S$, $K(r_t)$, $\lambda_i(r_t)$, $\alpha_i(r_t)$ and $\beta_i(r_t)$ are respectively denoted as K_r, $\lambda_{i,r}$, $\alpha_{i,r}$ and $\beta_{i,r}$.

We shall design the mode-dependent control gain matrix K_r to ensure that the disagreement dynamics of system (1) remains confined within the prescribed bound in the fixed time interval with H_∞ performance specification.

Theorem 3.1: For the network-connected dynamic system (1) and a given-time constant $T > 0$, the given-time consensus problem can be solved under mode-dependent protocol (7) with respect to $\begin{pmatrix} c_1 & c_2 & T & R & d \end{pmatrix}$, if there exist positive constant $\eta > 0$ and $\gamma^2 > 0$, mode-dependent symmetric positive definite matrix X_r, mode-dependent matrix Y_r and $\lambda > 0$ satisfied the following LMIs for each $r \in S$:

$$\begin{bmatrix} X_rA^{\mathrm{T}} + AX_r - \sigma Y_r{}^{\mathrm{T}}B^{\mathrm{T}} - \sigma BY_r + \pi_{rr}X_r - \eta X_r & X_r & \Psi_{1r} & \Psi_{2r} \\ X_r & -1 & 0 & 0 \\ \Psi_{1r}{}^T & 0 & -\Psi_{3r} & 0 \\ \Psi_{2r}{}^T & 0 & 0 & -\Psi_{4r} \end{bmatrix} < 0 \tag{18}$$

$$\lambda\left(\tilde{R} \otimes R\right)^{-1} < (I_N \otimes X_r) < \left(\tilde{R} \otimes R\right)^{-1} \tag{19}$$

$$\begin{bmatrix} -e^{-\eta T}\eta c_2 + \lambda_{\max\left(F(r_t)^{-\mathrm{T}}F(r_t)^{-1}\right)}\gamma^2 d\left(1 - e^{-\eta T}\right) & \sqrt{\eta c_1} \\ \sqrt{\eta c_1} & -\lambda \end{bmatrix} < 0 \tag{20}$$

where

$$\Psi_{1r} = \begin{bmatrix} B_w & B_w & \cdots & B_w \end{bmatrix}$$

$$\Psi_{2r} = [\ \sqrt{\pi_{r1}}X_r \quad \cdots \quad \sqrt{\pi_{r(r-1)}}X_r \quad \sqrt{\pi_{r(r+1)}}X_r \quad \cdots \quad \sqrt{\pi_{rN}}X_r\ ]$$

$$\Psi_{3r} = diag\left\{ \gamma^2 I, \ \ \gamma^2 I, \ \ \cdots, \ \ \gamma^2 I \right\}$$

For Peer Review

$$\Psi_{4r} = diag\{\ X_1,\ \ \cdots,\ \ X_{r-1},\ \ X_{r+1},\ \ \cdots,\ \ X_N\ \}$$

$$\sigma = \min\left\{\lambda_{1,r}, \cdots, \lambda_{n_\lambda,r}, \alpha_{1,r}, \cdots, \alpha_{n_\mu,r}\right\}, \tilde{R} = F(r_t)^{\mathrm{T}} F(r_t)$$

If the above four LMIs are feasible, then the feedback matrix of the consensus protocol is given by $K_r = Y_r X_r^{-1}$.

Proof: Consider a stochastic Lyapunov function candidate as

$$V(\xi_i(t), r_t = r) = V(\xi_i, r) = \xi_i^{\mathrm{T}} P_r \xi_i \tag{21}$$

where P_r is mode-dependent positive definite symmetric matrix for each r. For $i = 1, \cdots, n_\lambda$, along the trajectories of system (14), the corresponding time derivative of $V(\xi_i, r)$ is given by

$$\begin{aligned} &\tfrac{d}{dt} E\left\{V(\xi_i, r)\right\} \\ &= \lim_{\Delta t \to 0} \tfrac{1}{\Delta t}\left[E\left\{V\left(\xi_i\left(t + \Delta t\right), r_{t+\Delta t}, t + \Delta t\right) \middle| \xi_i(t) = \xi_i, r_t = r\right\} - V\left(\xi_i(t), r, t\right)\right] \\ &\le \xi_i^{\mathrm{T}}\left((A - \lambda_{i,r} B K_r)^{\mathrm{T}} P_r + P_r\left(A - \lambda_{i,r} B K_r\right) + \sum_{l=1}^{s} \pi_{rl} P_l\right)\xi_i + 2\xi_i^{\mathrm{T}} P_r B_w \tilde{w}_i \end{aligned} \tag{22}$$

For $i = n_\lambda + 1, \cdots, N-1$, we consider $V(\xi_i, r)$ in pairs for the complex conjugate eigenvalues. It can be obtained in a similar way to the real eigenvalue case that

$$\begin{aligned} &\tfrac{d}{dt} E\left\{V(\xi_{i_1}, r)\right\} + \tfrac{d}{dt} E\left\{V(\xi_{i_2}, r)\right\} \\ &\le \xi_{i_1}^{\mathrm{T}}\left((A - \alpha_{i,r} B K_r)^{\mathrm{T}} P_r + P_r\left(A - \alpha_{i,r} B K_r\right) + \sum_{l=1}^{s} \pi_{rl} P_l\right)\xi_{i_1} \\ &+\xi_{i_2}^{\mathrm{T}}\left((A - \alpha_{i,r} B K_r)^{\mathrm{T}} P_r + P_r\left(A - \alpha_{i,r} B K_r\right) + \sum_{l=1}^{s} \pi_{rl} P_l\right)\xi_{i_2} \\ &+2\xi_{i_1}^{\mathrm{T}} P_r B_w \tilde{w}_{i_1} + 2\xi_{i_2}^{\mathrm{T}} P_r B_w \tilde{w}_{i_2} \end{aligned} \tag{23}$$

Let $V\left(\xi(t), r\right) = \sum\limits_{i=1}^{N-1} V(\xi_i, r)$, if the following condition holds

$$\begin{bmatrix} (A - \sigma B K_r)^T P_r + P_r\left(A - \sigma B K_r\right) + \sum\limits_{l=1}^{s} \pi_{rl} P_l - \eta P_r + I & P_r B_w \\ * & -\gamma^2 I \end{bmatrix} < 0 \tag{24}$$

it follows

$$\frac{d}{dt} E\left\{V\left(\xi(t), r\right)\right\} < \eta E\left\{V\left(\xi(t), r\right)\right\} + \gamma^2 \tilde{w}(t)^{\mathrm{T}} \tilde{w}(t) - \tilde{z}(t)^{\mathrm{T}} \tilde{z}(t) \tag{25}$$

For Peer Review

Multiplying both sides of (25) by $e^{-\eta t}$ and integrating from 0 to t, it follows that

$$\begin{aligned} E\left\{V\left(\xi(t),\ r\right)\right\} &< e^{\eta t}\left(V\left(\xi(0),\ r_0\right)+\gamma^2\int_0^t e^{-\eta\tau}\tilde{w}(\tau)^{\mathrm{T}}\tilde{w}(\tau)d\tau\right) \\ &< e^{\eta t}\left(V\left(\xi(0),\ r_0\right)+\left(N-1\right)\gamma^2\lambda_{\max\left(F(r_t)^{-\mathrm{T}}F(r_t)^{-1}\right)}d\frac{\left(1-e^{-\eta t}\right)}{\eta}\right) \end{aligned} \tag{26}$$

where $\lambda_{\max\left(F(r_t)^{-\mathrm{T}}F(r_t)^{-1}\right)}$ is the maximum eigenvalue of $F(r_t)^{-\mathrm{T}}F(r_t)^{-1}$.

Defining

$$I_N\otimes R=\left(F(r_t)^{-1}\otimes I_n\right)^{\mathrm{T}}\left(\tilde{R}\otimes R\right)\left(F(r_t)^{-1}\otimes I_n\right)$$

$$I_N\otimes\tilde{P}_r=\left(\tilde{R}\otimes R\right)^{-1/2}\left(I_N\otimes P_r\right)\left(\tilde{R}\otimes R\right)^{-1/2}$$

we have

$$E\left\{V\left(\xi(t),\ r\right)\right\}<e^{\eta t}\left(N-1\right)\left(\lambda_{\max}\left(I_N\otimes\tilde{P}_r\right)c_1+\gamma^2\lambda_{\max\left(F(r_t)^{-\mathrm{T}}F(r_t)^{-1}\right)}d\frac{\left(1-e^{-\eta t}\right)}{\eta}\right) \tag{27}$$

On the other hand, the following condition holds

$$\begin{aligned} E\left\{V\left(\xi(t),\ r\right)\right\} &= E\left\{\tilde{z}(t)^{\mathrm{T}}\left(I_N\otimes P_r\right)\tilde{z}(t)\right\} \\ &= E\left\{\tilde{z}(t)^{\mathrm{T}}\left(\tilde{R}\otimes R\right)^{1/2}\left(I_N\otimes\tilde{P}_r\right)\left(\tilde{R}\otimes R\right)^{1/2}\tilde{z}(t)\right\} \\ &> \lambda_{\min}\left(I_N\otimes\tilde{P}_r\right)E\left\{\tilde{z}^{\mathrm{T}}\left(\tilde{R}\otimes R\right)\tilde{z}\right\} \end{aligned} \tag{28}$$

Putting together (27) and (28), we have

$$E\left\{\tilde{z}^{\mathrm{T}}\left(\tilde{R}\otimes R\right)\tilde{z}\right\}<\frac{e^{\eta t}\left(N-1\right)\left[c_1\eta\lambda_{\max}\left(I_N\otimes\tilde{P}_r\right)+\lambda_{\max\left(F(r_t)^{-\mathrm{T}}F(r_t)^{-1}\right)}\gamma^2 d\left(1-e^{-\eta t}\right)\right]}{\lambda_{\min}\left(I_N\otimes\tilde{P}_r\right)\eta}$$

If the following condition is satisfied

$$\frac{e^{\eta t}\left[c_1\eta\lambda_{\max}\left(I_N\otimes\tilde{P}_r\right)+\lambda_{\max\left(F(r_t)^{-\mathrm{T}}F(r_t)^{-1}\right)}\gamma^2 d\left(1-e^{-\eta t}\right)\right]}{\lambda_{\min}\left(I_N\otimes\tilde{P}_r\right)\eta}<c_2 \tag{29}$$

which implies that for $\forall\, t\in\begin{bmatrix}0 & T\end{bmatrix}$, $E\left\{z_i^{\mathrm{T}}(t)Rz_i(t)\right\}<c_2$, then the network-connected dynamical system (1) is said to be given-time consentable with respect to $\begin{pmatrix}c_1 & c_2 & T & R & d\end{pmatrix}$. Further more, according to condition (25) and in zero initial condition, we can get

$$E\left\{\int_0^T\tilde{z}(t)^{\mathrm{T}}\tilde{z}(t)d\tau\right\}<\gamma^2E\left\{\int_0^T\tilde{w}(t)^{\mathrm{T}}\tilde{w}(t)d\tau\right\} \tag{30}$$

For Peer Review

which is equivalent to the cost function (5). To obtain the gains of the designed protocol, next we will reduce conditions (24) and (29) to feasibility problems involving LMIs.

Pre- and post-multiplying the inequality (24) by block-diagonal matrix $diag\left\{\begin{array}{cc} P_r^{-1} & I \end{array}\right\}$, letting $X_r = P_r^{-1}$, $Y_r = K_r X_r$ and applying Schur complement, it leads to inequality (18) .

According to $I_N \otimes \tilde{P}_r = \left(\tilde{R} \otimes R\right)^{-1/2} (I_N \otimes P_r) \left(\tilde{R} \otimes R\right)^{-1/2}$, one has

$$\left(I_N \otimes \tilde{P}_r\right)^{-1} = \left(\tilde{R} \otimes R\right)^{1/2} (I_N \otimes P_r)^{-1} \left(\tilde{R} \otimes R\right)^{1/2}$$

$$\lambda_{\max}\left(I_N \otimes \tilde{P}_r\right)^{-1} = \frac{1}{\lambda_{\min}\left(I_N \otimes \tilde{P}_r\right)}$$

Condition (29) follows that

$$\frac{c_1 \eta}{\lambda_{\min}(\tilde{X}_r)} + \lambda_{\max\left(F(r_t)^{-\mathrm{T}} F(r_t)^{-1}\right)} \gamma^2 d \left(1 - e^{-\eta t}\right) < \frac{e^{-\eta t} \eta c_2}{\lambda_{\max}(\tilde{X}_r)} \tag{31}$$

Define

$$\lambda_{\max}(\tilde{X}_r) < 1, \lambda < \lambda_{\min}(\tilde{X}_r) \tag{32}$$

From definition(32), we can have condition (19). Putting (31) and (32) together, the desired condition (20) can be obtained, which completes the proof.

Remark 3.1: Note that inequality (25) with $\eta > 0$ in the proof relaxes the requirement on Lyapunov energy function by allowing it to increase, which leads to less conservativeness of the results. If we set $\eta = 0$ and make the assumption that the communication topology is fixed, then the condition derived in (19) will reduce to asymptotical consensus results in literatures (Li et al., 2010; Ding, 2014). In this case, the state of network-connected dynamic system (1) can also asymptotically reach to an identical value via the designed control protocol (7).

Remark 3.2: Upon exploring certain features of Laplacian matrix in real Jordan form, and defining the state disagreement in the form of (3), the closed-loop system (9) is transformed into system (11), which makes the designed protocol suitable for more general networks with directed information flow. Considering that the Laplacian matrix of a directed graph is not symmetric positive semi-definite, the transformation method employed in (Lin and Jia, 2010; Liu and Jia, 2011) is not applicable here. Therefore, with the new transformation proposed in this paper, sufficient conditions for achieving given-time H_∞ consensus are presented in Theorem 3.1.

Remark 3.3: From Theorem 3.1 it can be seen that the given-time H_∞ consensus of system (1) is influenced by not only the eigenvalues of Laplacian matrix, but also the transition rate of

For Peer Review

communication topology. Furthermore, to obtain an optimized given-time consensus controller, the bound of disagreement trajectory can be reduced to the minimum possible value such that LMIs (18-20) are satisfied. The optimization problem can be described as follows:

$$\min_{X_r, Y_r, \lambda, c_2} c_2 \\ s.\,t.\;\; LMIs\ (18-20) \tag{33}$$

IV. Illustrative Example

In this section, we will provide an example to show the advantage of the proposed method. The system under consideration is a connection of four subsystems, and each of them is described by a state-space model as

$$\dot{x}_i(t) = \begin{bmatrix} -1.48 & 0.96 \\ 1.57 & 1.95 \end{bmatrix} x_i(t) + \begin{bmatrix} 1 \\ 5 \end{bmatrix} u_i(t) + \begin{bmatrix} 0.1 \\ -0.3 \end{bmatrix} w_i(t)$$

The interconnection topology jumps between G_1 and G_2 with adjacency matrices described as:

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, Q_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and the transition rate between G_1 and G_2 is

$$\Pi = \begin{bmatrix} -0.7 & 0.7 \\ 0.4 & -0.4 \end{bmatrix}$$

The external disturbance is supposed to be

$$\begin{aligned} w(t) &= \begin{bmatrix} w_1(t) & w_2(t) & w_3(t) & w_4(t) \end{bmatrix}^{\mathrm{T}} \\ &= \begin{bmatrix} 2\sin(t) & -\sin(t) & 3\sin(t) & -1.5\sin(t) \end{bmatrix}^{\mathrm{T}} \end{aligned}$$

According to the relationship between Q and L, the resultant Laplasian matrices are obtained as

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}, L_2 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

According to the eigenvalues of L_r for $r = 1, 2$, it is easy to get that

$$J_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3/2 & \sqrt{3}/2 & 0 \\ 0 & -\sqrt{3}/2 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, J_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with the transformation matrices

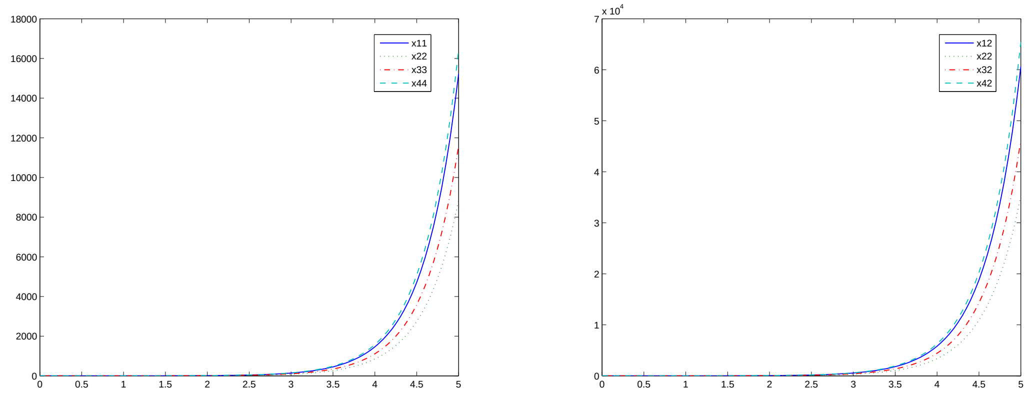
$$T_1 = \begin{bmatrix} 0 & 0.5 & \sqrt{3}/2 & 1 \\ 0 & -1 & 0 & 1 \\ -3/2 & 1/2 & \sqrt{3}/2 & 1 \\ 0 & 1/2 & -\sqrt{3}/2 & 1 \end{bmatrix}, T_2 = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

and

$$T_1{}^{-1} = \begin{bmatrix} 2/3 & 0 & -2/3 & 0 \\ 1/3 & -2/3 & 0 & 1/3 \\ \sqrt{3}/3 & 0 & 0 & -\sqrt{3}/3 \\ 1/3 & 1/3 & 0 & 1/3 \end{bmatrix}, T_2{}^{-1} = \begin{bmatrix} 1/2 & 0 & 0 & -1/2 \\ 0 & -1/2 & 0 & 1/2 \\ -1/2 & 0 & 1 & -1/2 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$

Our aim here is to design a given-time H_∞ consensus controller in the form of (7) such that the disagreement trajectory stays within the given bound in the presence of external disturbance and Markov jump topology. Introducing the initial value for $c_1 = 0.1$, $c_2 = 0.3$, $T = 5$, $R = 0.1I_2$,

For Peer Review



(a) State x_{i1} of free system (b) State x_{i2} of free system

Fig. 1: State response of free system

For Peer Review

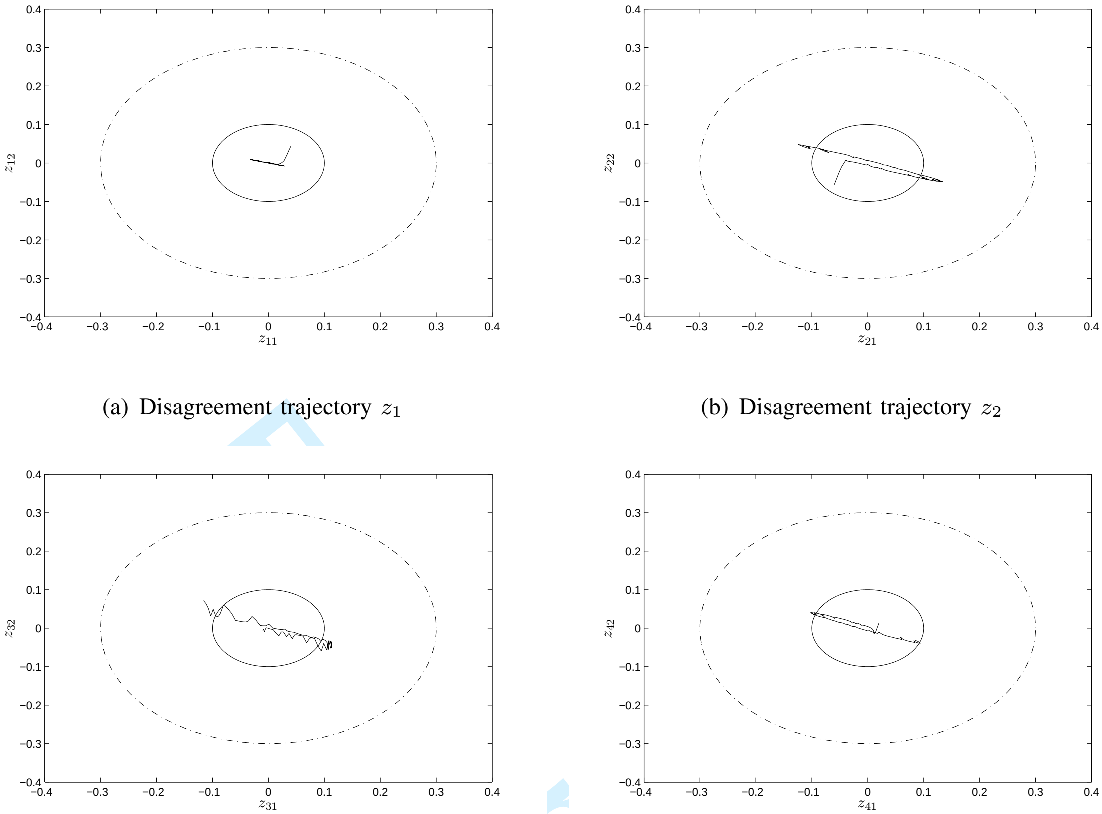
$d = 2$, $\gamma = 1$ and $\eta = 1$, and by solving (18)-(20) in Theorem 3.1, the corresponding mode-dependent controller gains are computed as follows:

$$K_1 = [\ 0.7177 \quad 02.7305\ ]$$

$$K_2 = [\ 0.9636 \quad 03.5157\ ]$$

To demonstrate the effectiveness of the design method, the state response of free system without control input is shown in Fig.1 and the disagreement trajectory under control protocol (7) is shown in Fig. 2 for given initial states $x_{10} = [\ 0.3 \quad 0.5\ ]^{\mathrm{T}}$, $x_{20} = [\ 0.2 \quad 0.4\ ]^{\mathrm{T}}$, $x_{30} = [\ 0.25 \quad 0.45\ ]^{\mathrm{T}}$ and $x_{40} = [\ 0.38 \quad 0.47\ ]^{\mathrm{T}}$. It can be seen that even for the unstable system, the state disagreement stays within the specified bound $c_2 = 0.3$ over the given-time horizon $T = 5$ with the designed controller. In order to investigate the effect of γ and η in the search of minimum upper bound of c_2 and maximum lower bound of c_1 , the results of the optimization problem are summarized in Table I. In addition, from Theorem 3.1 it can be seen that the values of c_1 and c_2 have effect on the disturbance rejection level γ, which are given in Table II.

As shown in Table I , the increasing of γ and decreasing of η lead to the smaller bound of c_1 and bigger bound of c_2. From Table II we can conclude that the disturbance rejection level is improved for bigger value of c_1 and smaller value of c_2 , which is consistent with the practical situation.



(a) Disagreement trajectory z_1 (b) Disagreement trajectory z_2

(c) Disagreement trajectory z_3 (d) Disagreement trajectory z_4

Fig. 2: Disagreement trajectory of the controlled system

TABLE I: Minimum upper bound of c_2 and maximum lower bound of c_1 for different values of γ and η

η	γ	c_1	c_2
5	1.2	0.0061	3.4332
5	1.3	0.0061	3.4240
5	1.4	0.0062	3.4077
5	1.5	0.0062	3.3906
3	1.5	0.0097	3.2883
3	1.6	0.0097	3.2749
1	1.6	0.0256	3.1284

TABLE II: Minimum value of γ for different values of c_1 and c_2

c_1	c_2	γ
0.6	0.9	1.2681
0.6	1.8	1.2678
0.6	2.7	1.2661
0.3	2.7	1.1526
0.1	2.7	0.6284

For Peer Review

V. Conclusion

The given-time H_∞ consensus problem has been proposed and solved in this paper for network-connected dynamical systems with directed communication graph and Markov jump topologies. To keep the disagreement dynamics of networks within the prescribed bound in the fixed time interval with a guaranteed H_∞ disturbance rejection performance, sufficient conditions for the existence of the controller have been derived. Future work will focus on the design of output feedback control protocol for achieving given-time H_∞ consensus for networks under communication delays with fixed or switching directed topologies.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China (Grant No. 61473137) and the 111 project (Grant No. B12018).

References

Amato F and Ariola M (2005) Finite-time control of discrete-time linear systems. *IEEE Transactions on Automatic Control* 50(5): 724–729.

Cai N, Cao JW and Khan MJ (2015) Almost decouplability of any directed weighted network topology. *Physica A: Statistical Mechanics and its Applications* 436: 637–645.

Ding Z (2013) Consensus output regulation of a class of heterogeneous nonlinear systems. *IEEE Transactions on Automatic Control* 58: 2648–2653.

Ding Z (2014) Consensus control of a class of Lipschitz nonlinear systems. *International Journal of Control* 87(11): 2372–2382.

Dong WJ and Farrell JA (2009) Cooperative control of multiple nonholonomic mobile agents. *IEEE Transactions on Automatic Control* 53(6): 262–268.

Dorato P (1961) Short time stability in linear time-varying systems. In: *Proceedings of the IRE International Convention Record Part 4*. New York, pp. 83–87.

Fax JA and Murray RM (2004) Information flow and cooperative control of vehicle formation. *IEEE Transactions on Automatic Control* 49: 1465–1476.

Huang J, Farritor SM, Qadi A and Goddard S (2006) Localization and follow-the-leader control of a heterogeneous group of mobile robots. *IEEE/ASME Transactions on Mechatronics* 11(2): 205–215.

Jadbabaie A and Morse A (2003) Coordination of groups of autonomous mobile agents using nearest rules. *IEEE Transactions on Automatic Control* 48(6): 988–1001.

Li Z, Duan Z and Chen G (2011) On H_∞ and H_2 performance regions of multi-agent systems. *Automatica* 47(4): 797–803.

Li Z, Duan Z, Chen G and Huang L (2010) Consensus of multi-agent systems and synchronization of complex networks, a unified viewpoint. *IEEE Transactions on Circuits and Systems I* 57(1): 213–224.

Li Z, Liu X, Fu M and Xie L (2012) Global H_∞ consensus of multiagent systems with Lipschitz nonlinear dynamics. *IET Control Theory and applications* 6(13): 2041–2048.

Lin P and Jia Y (2010) Robust H_∞ consensus analysis of a class of second-order multi-agent systems with uncertainty. *IET Control Theory and applications* 4(3): 487–498.

Liu Y and Jia Y (2011) Robust H_∞ consensus control of uncertain multi-agent systems with time delays. *International Journal of Control, Automation, and Systems* 9(6): 1086–1094.

Liu Y, Min HB, Wang SC, Liu ZG and Liao SY (2014) Distributed consensus of a class of networked heterogeneous multi-agent systems. *Journal of the Franklin Institute* 351(3): 1700–1716.

Luan XL, Shi P and Liu F (2013) Finite-time stabilization for Markov jump systems with Gaussian transition probabilities. *IET Control Theory and applications* 7(2): 298–304.

Massioni P and Verhaegen M (2009) Distributed control for identical dynamically coupled systems: a decomposition approach. *IEEE Transactions on Automatic Control* 55(1): 124–135.

Miao GY, Xu SY and Zou Y (2013) Necessary and sufficient conditions for mean square

For Peer Review

consensus under Markov switching topologies. *International Journal of System Science* 44(1): 178–186.

Olifati-Saber R and Murray RM (2004) Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control* 49(9): 1520–1533.

Scardovi L and Sepulchre R (2009) Synchronization in networks of identical linear systems. *Automatica* 45(11): 2557–2562.

Wen G, Duan Z, Li Z and Chen G (2012) Consensus and its L_2-gain performance of multi-agent systems with intermittent information transmissions. *International Journal of Control* 85(4): 384–396.

Xie DS, Xu SY, Chu YM and Zou Y (2015) Event-triggered average consensus for multi-agent systems with nonlinear dynamics and switching topology. *Journal of the Franklin Institute* 352(3): 1080–1098.

You KY, Li ZK and Xie LH (2013) Consensus condition for linear multi-agent systems over randomly switching topologies. *Automatica* 49(10): 3125–3132.

Yu WW, Chen GR and Gao M (2011) Consensus in directed networks of agents with nonlinear dynamics. *IEEE Transactions on Automatic Control* 56(6): 1436–1441.

Zhang Y and Tian YP (2009) Consentability and protocol design of multi-agent systems with stochastic switching topology. *Automatica* 45: 1195–1201.

For Peer Review