

# Adaptive Consensus Output Regulation of A Class of Nonlinear Systems with Unknown High-Frequency Gain <sup>★</sup>

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## Abstract

This paper deals with adaptive consensus output regulation of a class of network-connected nonlinear systems with completely unknown parameters, including the high frequency gains of the subsystems. The subsystems may have different dynamics, as long as the relative degrees are the same. A new type of Nussbaum gain is proposed to deal with adaptive consensus control of network-connected systems without any knowledge of the high frequency gains. Adaptive laws and internal models are designed for the subsystems to deal with unknown parameters for tracking trajectories and unknown system parameters. In the control design, only the relative information of subsystem outputs are used, provided that regulation error of one of the subsystems is available. The proposed control inputs and the adaptive laws are decentralized. If the relative degree is one, only the relative subsystem outputs are exchanged. For the case of higher relative degrees, the nonlinear model structure of the subsystems is exploited for backstepping control design, and some variables generated by the subsystem controllers are exchanged among the subsystems in the neighbourhood defined by the connection graph.

*Key words:* Adaptive control, Nonlinear systems, Consensus control, Output regulation, Disturbance rejection

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## 1 Introduction

Many dynamic systems are connected by networks to perform certain common or similar tasks that include formation control and cooperative control etc. Consensus control often refers to the situation where network-connected subsystems are controlled to achieve the same or very similar control objectives. A significant difference of consensus control to other control design is the use of the information collected from the subsystems in the neighbourhood, and the success of any proposed consensus control design depends on the structure of network connections which are described as connection graphs. A useful description of a connection graph is the Laplacian matrix that plays an important role in all the design methods on consensus control (Fax and Murray, 2004; R.Olfati-Saber and Murray, 2004; Qu, 2009).

In particular, some important properties of Laplacian matrices in relation to consensus control are well described in (Qu, 2009).

Consensus control started from the single-integrator dynamics of subsystems as shown in the review paper (Olfati-Saber, Fax and Murray, 2007) and then to subsystems with full linear system dynamics (Fax and Murray, 2004; R.Olfati-Saber and Murray, 2004; Xiang, Wei and Li, 2009; Seo, Shim and Back, 2009; Li, Duan, Chen and Huang, 2010; Yang, Roy, Wan and Saberi, 2009; Su and Huang, 2012; Grip, Yang, Saberi and Stoorvogel, 2012). Results on consensus control of systems with nonlinear subsystem dynamics have appeared in various publications (Hong, Hu and Gao, 2006; Chopra and Spong, 2008; Zhao, Hill and Liu, 2011; Münz, Papachristodoulou and Allgöwer, 2011; Li, Liu, Fu and Xie, 2012; Li, Ren, Liu and Fu, 2013; Ding, 2013a; Su and Huang, 2013). Most of the results on nonlinear dynamics are obtained for subsystems with relative degree one or the systems with Lipschitz nonlinearities. When there are uncertainties in the system, adaptive control

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strategies are naturally considered. One challenge is the implementation of adaptive laws in a decentralized manner (Yu and Xia, 2012). In the robust adaptive consensus control shown in (Das and Lewis, 2010; Zhang and Lewis, 2012), the adaptive laws are decentralized, with the influence of the uncertainties of the adjacent subsystems being treated as bounded disturbances, and the resultant consensus control errors are kept bounded instead of the convergence to zero due to the robust adaptive control treatment. Decentralized adaptive laws have been proposed for first-order nonlinear systems in (Yu and Xia, 2012).

We consider consensus output regulation of a class of network-connected nonlinear dynamic systems whose subsystems have all the system parameters completely unknown, including the high frequency gains. It is well known that Nussbaum gains can be used to tackle adaptive control with unknown high-frequency gains for single-input single-output systems including the case of nonlinear output regulation (Nussbaum, 1983; Ding, 2001). However, for a network connected system with multiple subsystems, the existing Nussbaum gain designed for individual systems would not be able to establish the boundedness of all the variables in the adaptive consensus control system, as Nussbaum gain parameters for different subsystems could move in different directions, and a usual contradiction could not be obtained. A very recent result (Chen, Li, Ren and Wen, 2014) proposes a Nussbaum gain for multi-agent systems with unknown control directions when the lower and upper bounds of the control coefficients are known. In this paper, we propose a new Nussbaum gain with a potentially faster rate such that the boundedness of the system parameters can be established by an argument of contradiction even if the Nussbaum gain parameter for only one of the subsystems goes unbounded. This new Nussbaum gain can be applied to the systems considered in (Chen et al., 2014) to remove the assumption of known lower and upper bounds of the control coefficients.

Tracking and disturbance rejection can be unified as an output regulation problem (Isidori, 1995). A recent result for consensus output regulation of linear systems is shown in (Grip et al., 2012) and for nonlinear systems in (Ding, 2013a; Su and Huang, 2013). The results shown in the later two are for the nonlinear subsystems with relative degree one. In particular, the result in (Su and Huang, 2013) extends the result for nonlinear output regulation with unknown exosystems (Ding, 2003) to consensus output regulation. The key step in designing an adaptive scheme for systems with consensus control is to ensure that only the information available in the local neighbourhood can be used for the adaptive control design. This is relatively easier when the subsystems are of relative degree one, for which no exchange of adaptive parameters are needed in the consensus control. With higher relative degrees,

filtered-transformation and backstepping, the tools to tackle high relative degrees, tend to propagate uncertainties in the network. In such a case, adaptive laws need to be designed with collaboration of the subsystems in the neighbourhood. We propose adaptive laws and control inputs with the information available from the subsystems in the neighbourhood, and therefore the adaptive laws and inputs are still viewed as decentralized, as no information from the subsystems outside the neighbourhood are needed. It is also noted that as for the subsystem outputs, the proposed design only use the relative information between the subsystems. Lyapunov function based analysis is used to establish the stability of the adaptive output regulation design for consensus control using the proposed adaptive laws and the new Nussbaum gain design. The proposed control can deal with the subsystems with different dynamics as long as the subsystems with the same relative degree. An example is included to demonstrate the proposed control design with the simulation results shown.

## 2 Problem Formulation

In this paper, we consider a set of  $N$  nonlinear subsystems, of which the subsystems are described by

$$\begin{aligned} \dot{x}_i &= A_{ci}x_i + \phi_i(y_i, w, \mu_i) + b_i u_i, \\ y_i &= C_i x_i, \end{aligned} \tag{1}$$

with  $b_i, C_i^T \in \mathbb{R}^{n_i}$  and

$$A_{ci} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, b_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_{i,\rho} \\ \vdots \\ b_{i,n} \end{bmatrix}, C_i^T = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

for  $i = 1, \dots, N$ , where  $x_i \in \mathbb{R}^{n_i}$  is the state vector, with  $n_i$  known positive constant integers denoting the order of the subsystems,  $y_i, u_i \in \mathbb{R}$  are the output and input respectively of the  $i$ th subsystem,  $\mu_i \in \mathbb{R}^{q_i}$  and  $b_i \in \mathbb{R}^{n_i}$  are vectors of unknown parameters, with  $b_i$  being a Hurwitz vector with  $b_{i,\rho} \neq 0$ , which implies the relative degree of the system is  $\rho$ ,  $\phi_i : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n_i}$  contains nonlinear functions with each element as polynomials of its variables and satisfies  $\phi_i(0, w, \mu_i) = 0$ , and  $w \in \mathbb{R}^m$  are disturbances, and they are generated from an unknown exosystem

$$\dot{w} = S(\sigma)w \tag{2}$$

with unknown  $\sigma \in \mathbb{R}^s$ , of which,  $S \in \mathbb{R}^{m \times m}$  is a constant matrix with distinct eigenvalues of zero real parts.

The connections between the subsystems are specified by an undirected graph  $\mathcal{G}$  that consists of a set of vertices denoted by  $\mathcal{V}$  and a set of edges denoted by  $\mathcal{E}$ . A vertex represents a subsystem, and each edge represents a connection. Associated with the graph, its adjacency matrix  $A$  with elements  $a_{ij}$  denotes the connections such that  $a_{ij} = 1$  if there is a path from subsystem  $j$  to subsystem  $i$ , and  $a_{ij} = 0$  otherwise. Since the connection is undirected, we have  $A = A^T$ . We define the Laplacian matrix  $L$  in a normal way as  $l_{ii} = \sum_{j=1}^N a_{ij}$  and  $l_{ij} = -a_{ij}$  when  $i \neq j$ .

We define the output regulation errors as

$$e_i = y_i - g(w) \quad (3)$$

with  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  being polynomials of its variables, for  $i = 1, \dots, N$ . In our set up, not every subsystem has access to  $g(w)$ , and the consensus output regulation will be achieved through the network connections among the subsystems.

The adaptive consensus output regulation problem considered in this paper is to design an adaptive control strategy using the relative output information  $y_i - y_j$ ,  $i \neq j$ , provided by the network connection to each subsystem to ensure the convergence to zero of output regulation errors  $e_i$  for  $i = 1, \dots, N$  under any initial condition of the system in the state space, i.e., the convergence of the subsystem outputs  $y_i$  to the common function  $g(w)$ .

Not all the subsystems have the access to the function value of  $g(w)$ . We use a diagonal matrix  $\Delta$  to denote the access to  $g(w)$  in the way that if  $\delta_{ii} = 1$ , the  $i$ th subsystem has access to the value of  $g(w)$  for the control design, and  $\delta_{ii} = 0$  otherwise. At least one subsystem has the access. The subsystems which do not have access to the tracking signal rely on the network connections to achieve the consensus tracking.

We make several assumptions about the dynamics of the subsystems, the exosystem and the connections between the subsystems.

**Assumption 1.** The invariant zeros of  $\{A_{ci}, b_i, C_i\}$  are stable, for  $i = 1, \dots, N$ , and all the subsystems have the same sign of the high-frequency gains.

**Assumption 2.** The eigenvalues of  $S$  are distinct and on the imaginary axis.

**Assumption 3.** The adjacency matrix  $A$  is irreducible.

*Remark 1.* When the disturbance term  $w$  disappears, each subsystem is in the standard output feedback form to which the geometric conditions for a nonlinear system to be transformed are specified in (Krstic, Kanelakopoulos and Kokotovic, 1995).  $A_{ci}$  and  $C_i$  are parts of

the standard form, and therefore as long as the systems are in the output feedback form even with different dynamics, we can always write the corresponding system matrices in these formats.  $\triangleleft$

*Remark 2.* In the formulation of an output regulation problem, the tracking trajectories and the disturbances are commonly assumed to be functions of the state that is generated by an exosystem. In such a formulation, disturbance rejection and output tracking can be treated in a unified way (Isidori, 1995; Ding, 2003). Assumption 2 on the eigenvalues of the exosystem dynamics is common in the formulation of output regulation, as the stable modes in the exosystem do not have an impact asymptotically. From a practical point of view, any periodic signals designed for the tracking trajectory can be approximated by sinusoidal functions with different frequencies, and those sinusoidal functions can be formulated as the state variables of the exosystem under Assumption 2.  $\triangleleft$

*Remark 3.* The adjacency matrix is irreducible if there exists a path between any two subsystems.  $\triangleleft$

### 3 Preliminaries

Several preliminary results on output regulation and consensus control are needed for proposing the adaptive control strategies later. We introduce a number of results for individual subsystems.

We introduce filtered transformation with the filter for the subsystem  $i$ , for  $i = 1, \dots, N$ ,

$$\begin{aligned} \dot{\xi}_{i,1} &= -\lambda_1 \xi_{i,1} + \xi_{i,2} \\ &\dots \\ \dot{\xi}_{i,\rho-1} &= -\lambda_{\rho-1} \xi_{i,\rho-1} + u_i, \end{aligned} \quad (4)$$

where  $\lambda_j > 0$  for  $j = 1, \dots, \rho - 1$  are the design parameters, and the filtered transformation

$$\bar{z}_i = x_i - [\bar{d}_{i,1} \dots \bar{d}_{i,\rho-1}] \xi_i, \quad (5)$$

where  $\xi_i = [\xi_{i,1} \dots \xi_{i,\rho-1}]^T$ ,  $\bar{d}_{i,j} \in \mathbb{R}^{n_i}$  for  $j = 1, \dots, \rho - 1$  and they are generated recursively by  $\bar{d}_{i,\rho-1} = b_i$  and  $\bar{d}_{i,j} = (A_{ci} + \lambda_{j+1}I)\bar{d}_{i,j+1}$  for  $j = \rho - 2, \dots, 1$ . The system (1) is then transformed to

$$\begin{aligned} \dot{\bar{z}}_i &= A_{ci} \bar{z}_i + \phi_i(y_i, w, \mu_i) + d_i \xi_{i,1} \\ y_i &= C \bar{z}_i, \end{aligned} \quad (6)$$

where  $d_i = (A_{ci} + \lambda_1 I)\bar{d}_{i,1}$ . It can be shown that  $d_{i,1} = b_{i,\rho}$  and

$$\sum_{j=1}^{n_i} d_{i,j} s^{n_i-j} = \prod_{j=1}^{\rho-1} (s + \lambda_j) \sum_{j=\rho}^{n_i} b_{i,j} s^{n_i-j}. \quad (7)$$

With  $\xi_{i,1}$  as the input, the system (6) is with relative degree one and minimum phase. We introduce another state transformation to extract the internal dynamics of (6) with  $z_i \in \mathbb{R}^{n_i-1}$  given by

$$z_i = \bar{z}_{i,2:n_i} - \frac{d_{i,2:n_i}}{d_{i,1}} y_i, \quad (8)$$

where  $(\cdot)_{2:n_i}$  refers to the vector or matrix formed by the 2nd row to the  $n_i$ th row. With the coordinates  $(z_i, y_i)$ , (6) is rewritten as

$$\begin{aligned} \dot{z}_i &= D_i z_i + \psi_i(y_i, w, \theta_i) \\ \dot{y}_i &= z_{i,1} + \psi_{i,y}(y_i, w, \theta_i) + b_{i,\rho} \xi_{i,1}, \end{aligned} \quad (9)$$

where the unknown parameter vector  $\theta_i = [\mu_i^T, b_i^T]^T$ , and  $D_i$  is the left companion matrix of  $d_i$  given by

$$D_i = \begin{bmatrix} -d_{i,2}/d_{i,1} & 1 & \dots & 0 \\ -d_{i,3}/d_{i,1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -d_{i,n_i-1}/d_{i,1} & 0 & \dots & 1 \\ -d_{i,n_i}/d_{i,1} & 0 & \dots & 0 \end{bmatrix}, \quad (10)$$

and

$$\begin{aligned} \psi_i(y_i, w, \theta_i) &= D_i \frac{d_{i,2:n_i}}{d_{i,1}} y_i + \phi_{i,2:n_i}(y_i, w, \mu_i) \\ &\quad - \frac{d_{i,2:n_i}}{d_{i,1}} \phi_{i,1}(y_i, w, \mu_i), \\ \psi_{i,y}(y_i, w, \theta_i) &= \frac{d_{i,2}}{d_{i,1}} y_i + \frac{d_{i,2:n_i}}{d_{i,1}} \phi_{i,1}(y_i, w, \mu_i). \end{aligned}$$

Notice that  $D_i$  is Hurwitz, from (7), and that the dependence of  $d_i$  on  $b_i$  is reflected in the parameter  $\theta_i$  in  $\psi_i(y_i, w, \theta_i)$  and  $\psi_{i,y}(y_i, w, \theta_i)$ , and it is easy to check that  $\psi_i(0, w, \theta_i) = 0$  and  $\psi_{i,y}(0, w, \theta_i) = 0$ .

The solution of the output regulation problem depends on the existence of certain invariant manifold and feedforward input. From the structure of the exosystem, the disturbances are sinusoidal functions. Polynomials of sinusoidal functions are still sinusoidal functions, but with some high frequency terms. Since all the nonlinear functions involved in the system (1) are polynomials of their variables, the following results about the feedforward inputs and their immersions can be obtained.

**Lemma 3.1** For a subsystem  $i$  of (1), for  $i = 1, \dots, N$ , there exist an invariant manifold  $\pi_i(w) \in \mathbb{R}^{n_i-1}$  satisfies

$$\frac{\partial \pi_i(w)}{\partial w} S(\sigma) w = D_i \pi_i(w) + \psi_i(g(w), w, \theta_i). \quad (11)$$

Then there exists an immersion for the feedforward control input

$$\begin{aligned} \frac{\partial \tau_i(w, \theta_i, \sigma)}{\partial w} S(\sigma) w &= \Phi_i(\sigma) \tau_i(w, \theta_i, \sigma) \\ \alpha_i(w, \theta_i, \sigma) &= \Gamma_i \tau_i(w, \theta_i, \sigma), \end{aligned}$$

where

$$\begin{aligned} \alpha_i(w, \theta_i, \sigma) &= b_{i,\rho}^{-1} \left( \frac{\partial g(w)}{\partial w} S(\sigma) w - \pi_{i,1}(w) \right. \\ &\quad \left. - \psi_{i,y}(g(w), w, \theta_i) \right). \end{aligned}$$

Furthermore, this immersion can be re-parameterised as

$$\begin{aligned} \dot{\eta}_i &= (F_i + G_i b_{i,\rho}^{-1} l_i^T) \eta_i \\ \alpha_i &= b_{i,\rho}^{-1} l_i^T \eta_i, \end{aligned} \quad (12)$$

where  $(F_i, G_i)$  is a controllable pair with compatible dimensions,  $\eta_i = M_i \tau_i$  and  $l_i = \Gamma_i M_i^{-1}$  with  $M_i$  satisfying

$$M_i(\sigma) \Phi_i(\sigma) - F_i M_i(\sigma) = G_i \Gamma_i. \quad (13)$$

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We now introduce a state transformation based on the invariant manifold with

$$\tilde{z}_i = z_i - \pi_i \quad (14)$$

Finally we have the model for the control design

$$\begin{aligned} \dot{\tilde{z}}_i &= D_i \tilde{z}_i + \tilde{\psi}_i \\ \dot{e}_i &= \tilde{z}_{i,1} + \tilde{\psi}_{i,y} + b_{i,\rho} \xi_{i,1} - l_i^T \eta_i \\ \dot{\xi}_{i,1} &= -\lambda_1 \xi_{i,1} + \xi_{i,2} \\ &\dots \\ \dot{\xi}_{i,\rho-1} &= -\lambda_{\rho-1} \xi_{i,\rho-1} + u_i, \end{aligned} \quad (15)$$

where

$$\begin{aligned} \tilde{\psi}_i &= \psi_i(y_i, w, \theta_i) - \psi_i(q(w), w, \theta_i), \\ \tilde{\psi}_{i,y} &= \psi_{i,y}(y_i, w, \theta_i) - \psi_{i,y}(q(w), w, \theta_i). \end{aligned}$$

Since the nonlinear functions involved in  $\tilde{\psi}$  and  $\tilde{\psi}_y$  are polynomials with  $\tilde{\psi}(0, w, \theta, \sigma) = 0$  and  $\tilde{\psi}_y(0, w, \theta, \sigma) = 0$ ,  $w$  is bounded, and the unknown parameters are constants, it can be shown that

$$\|\tilde{\psi}_i\|^2 < \bar{r}_z (e_i^2 + e_i^{2p}), \quad (16)$$

$$|\tilde{\psi}_{i,y}|^2 < \bar{r}_y (e_i^2 + e_i^{2p}), \quad (17)$$

where  $p$  is a known positive integer, depending on the polynomials in  $\tilde{\psi}_i$  and  $\tilde{\psi}_{i,y}$ , and  $\bar{r}_z$  and  $\bar{r}_y$  are unknown positive real constants.

Since the state in the internal model  $\eta_i$  is unknown, we design the internal model

$$\dot{\hat{\eta}}_i = F_i \hat{\eta}_i + G_i \xi_{i,1}. \quad (18)$$

If we define the auxiliary error

$$\tilde{\eta}_i = \eta_i - \hat{\eta}_i + b_{i,\rho}^{-1} G_i e_i, \quad (19)$$

it can be shown that

$$\dot{\tilde{\eta}}_i = F_i \tilde{\eta}_i - b_{i,\rho}^{-1} F_i G_i e_i + b_{i,\rho}^{-1} G_i \tilde{z}_{i,1} + b_{i,\rho}^{-1} G_i \tilde{\psi}_{i,y}. \quad (20)$$

If the system (1) is of relative degree one, then  $\xi_{i,1}$  in (15) is the control input for the subsystem. For the systems with higher relative degrees, adaptive backstepping will be used to find the final control input  $u_i$  from the desirable value of  $\xi_{i,1}$ .

Before we introduce the control design, we need a result on the Laplacian matrix.

**Lemma 3.2** If the adjacency matrix  $A$  is irreducible, and the non-negative diagonal matrix  $\Delta$  has at least one positive diagonal element, the matrix  $(L + \Delta)$  is positive definite.  $\triangleleft$

The proof of this lemma can be found in (Qu, 2009).

Let us denote

$$e = [e_1, e_2, \dots, e_N]^T \quad (21)$$

and the consensus regulation error

$$\zeta = Qe \quad (22)$$

where  $Q = L + \Delta$ . Since  $Q$  is invertible, the control objective is equivalent to  $\lim_{t \rightarrow \infty} \zeta = 0$ . It is worth noting that (22) implies that

$$\zeta_i = \sum_{j=1}^N a_{ij} (y_i - y_j) + \delta_{ii} (y_i - g(w)) \quad (23)$$

for  $i = 1, \dots, N$ . Clearly,  $\zeta_i$  is available to the control design for the  $i$ th subsystem.

We have another result relating  $\zeta$  and  $e$  that is needed later for the stability analysis.

**Lemma 3.3** With  $\zeta = Qe$ , the following inequality holds for any positive integer  $p$ ,

$$\sum_{i=1}^N e_i^{2p} \leq N^{p-1} \lambda_{max}^{2p}(Q) \sum_{i=1}^N \zeta_i^{2p} \quad (24)$$

where  $\lambda_{max}(Q)$  denotes the singular value of  $Q$ .

**Proof.** A direct evaluation gives

$$\begin{aligned} \sum_{i=1}^N \zeta_i^{2p} &= N \left( \left[ \frac{1}{N} \sum_{i=1}^N (\zeta_i^2)^p \right]^{1/p} \right)^p \\ &\geq N \left( \left[ \frac{1}{N} \sum_{i=1}^N (\zeta_i^2) \right] \right)^p \\ &= N^{1-p} (\|\zeta\|^2)^p \\ &= N^{1-p} \|Qe\|^{2p} \\ &\geq N^{1-p} \lambda_{max}^{-2p}(Q) \|e\|^{2p} \\ &\geq N^{1-p} \lambda_{max}^{-2p}(Q) \left( \sum_{i=1}^N e_i^2 \right)^p \\ &\geq N^{1-p} \lambda_{max}^{-2p}(Q) \sum_{i=1}^N e_i^{2p} \end{aligned}$$

from which (24) is obtained.  $\triangleleft$

## 4 Nussbaum Gain for Consensus Control

When high-frequency gains are completely unknown, Nussbaum gains are used in adaptive control. The basic idea of Nussbaum gain design is to construct a function  $\mathcal{N}(k)$  where  $k$  is a variable, and the control input takes the form  $u = \mathcal{N}(k)\bar{u}$ . Then the control design is continued with  $\bar{u}$  such that a condition in the following form is obtained, for a single-input system,

$$V(t) \leq V(0) + \int_0^k (b_\rho \mathcal{N}(s) - 1) ds + r(t) \quad (25)$$

where  $V$  is a positive definite function,  $k(t)$  is a continuous function with  $k(0) = 0$ , and  $r(t)$  is a bounded function and  $b_\rho$  is the unknown high-frequency gain. The boundedness of  $k$  and subsequently the boundedness of  $V$  can be established by seeking a contradiction using (25) if the Nussbaum function satisfies the two-sided Nussbaum properties

$$\lim_{k \rightarrow \pm\infty} \sup \frac{1}{k} \int_0^k \mathcal{N}(s) ds = +\infty, \quad (26)$$

$$\lim_{k \rightarrow \pm\infty} \inf \frac{1}{k} \int_0^k \mathcal{N}(s) ds = -\infty, \quad (27)$$

Typical choices of Nussbaum functions  $\mathcal{N}(k)$  are  $k^2 \cos(k)$  and  $k^2 \sin(k)$  etc.

For consensus control, there are  $N$  unknown high-frequency gains, and we can aim at a condition

$$V(t) \leq V(0) + \sum_{i=1}^N \int_0^{k_i} (b_{i,\rho} \mathcal{N}(s_i) - 1) ds_i + r(t) \quad (28)$$

similar to (25), but with multiple continuous functions  $k_i$ 's. The problem is that the Nussbaum function that satisfies the conditions (26) and (27) is no longer able to establish the boundedness of  $k_i$ 's in (28) as  $k_i$ 's are independent. Intuitively, we expect a function which grows faster such that one of the  $k_i$ 's is dominant for the positive definite condition for consensus control in (28).

We consider

$$\mathcal{N}(k) = e^{k^2/2} (k^2 + 2) \sin(k) \quad (29)$$

and indeed, it can be used as a Nussbaum gain for consensus control.

**Lemma 4.1** With the Nussbaum gain shown in (29), The boundedness of  $k_i$ 's and  $V$  can be established from (28).

**Proof.** Let

$$f(k) = \int_0^k \mathcal{N}(s) ds$$

It can be obtained that

$$f(k) = e^{k^2/2} (k \sin(k) - \cos(k)) + 1$$

Furthermore, it can be shown that  $f(k)$  takes local minimums at  $k = 2j\pi$  and local maximums at  $k = (2j+1)\pi$ , with  $j \in \mathbb{N}^0$ , a natural number. Hence for  $2j\pi < k \leq 2(j+1)\pi$ , we have

$$-e^{(2(j+1)\pi)^2/2} + 1 \leq f(k) \leq e^{((2j+1)\pi)^2/2} + 1$$

Now let us establish the boundedness of  $k_i$  in (28) by seeking a contradiction. Consider the case  $b_{i,\rho} > 0$  first. Assume that at least one of  $k_i(t)$ s becomes unbounded at  $t^*$ . Since  $k_i$ 's are continuous, we can define a sequence of time in the interval  $[0, t^*)$  as

$$t_j = \arg \min_t \{ \max_{i=1}^N \{ k_i(t) \} = 2(j+1)\pi \} \quad (30)$$

for  $j \in \mathbb{N}^0$ . Based on (30), we have  $\lim_{j \rightarrow \infty} t_j = t^*$ . Let

$$U(t) = V(0) + \sum_{i=1}^N \int_0^{k_i} (b_{i,\rho} \mathcal{N}(s_i) - 1) ds_i + r(t) \quad (31)$$

Evaluating  $U$  at  $t_j$  gives

$$\begin{aligned} U(t_j) &= V(0) + \sum_{i=1}^N (b_{i,\rho} f(k_i(t_j))) - \sum_{i=1}^N k_{i,\rho} + r(t_j) \\ &\leq V(0) + \underline{b} (-e^{(2(j+1)\pi)^2/2} + 1) \\ &\quad + (N-1) \bar{b} (e^{((2j+1)\pi)^2/2} + 1) + r(t_j) \end{aligned} \quad (32)$$

where  $\underline{b} = \min_{i=1}^N \{ b_{i,\rho} \}$  and  $\bar{b} = \max_{i=1}^N \{ b_{i,\rho} \}$ . With

$$\begin{aligned} &- \underline{b} e^{(2(j+1)\pi)^2/2} + (N-1) \bar{b} e^{(2j+1)\pi)^2/2} \\ &= - \underline{b} e^{(2j+1)\pi)^2/2} (e^{(4j+3)\pi^2/2} - \frac{(N-1)\bar{b}}{\underline{b}}) \end{aligned} \quad (33)$$

we have

$$\begin{aligned} U(t_j) &\leq - \underline{b} e^{(2j+1)\pi)^2/2} (e^{(4j+3)\pi^2/2} - \frac{(N-1)\bar{b}}{\underline{b}}) \\ &\quad + \bar{r}(t) \end{aligned} \quad (34)$$

where  $\bar{r}(t_j)$  is bounded. As  $e^{(4j+3)\pi^2/2}$  will dominate any bounded functions with sufficient large  $j$ , we can conclude from (34),  $U(t_j) < 0$  for sufficient large  $j$ 's. This is a contradiction, as  $V(t)$  is a positive definite function. Hence, none of the  $k_i$ 's becomes unbounded, and therefore boundedness of  $k_i$ 's and  $V$  are established.

For the case  $b_{i,\rho} < 0$ , we define

$$t_j = \arg \min_t \{ \max_{i=1}^N \{ k_i(t) \} = (2j+1)\pi \} \quad (35)$$

The rest of the proof can be carried out in the same way as for the case  $b_{i,\rho} > 0$ .  $\triangleleft$

## 5 Adaptive Consensus Regulation, $\rho = 1$

For the subsystems with relative degree  $\rho = 1$ , the filtered transformation is not needed and we have  $\xi_i = u_i$  in (15). The dynamics of the subsystems are now expressed as

$$\begin{aligned} \dot{z}_i &= D_i \tilde{z}_i + \tilde{\psi}_i \\ \dot{e}_i &= \tilde{z}_{i,1} + \tilde{\psi}_{i,y} + b_{i,\rho} u_i - l_i^T \eta_i \end{aligned} \quad (36)$$

Note that  $e_i$  is not available for control design. The control design is based on  $\zeta = Qe$ . Since the high-frequency gain  $b_{i,\rho}$  is completely unknown, we use the Nussbaum gain proposed in the previous section for adaptive control

$$u_i = \gamma \mathcal{N}(k_i) \bar{u}_i \quad (37)$$

$$\dot{k}_i = \zeta_i \bar{u}_i, \quad k_i(0) = 0 \quad (38)$$

where the Nussbaum gain  $\mathcal{N}$  is defined in (29), and  $\gamma$  is a positive real design parameter.

From (36) and the definition of the Nussbaum gain in (38), we have

$$\dot{e}_i = \tilde{z}_{i,1} + (\gamma b_{i,\rho} \mathcal{N}(k_i) - 1) \bar{u}_i + \bar{u}_i - l_i^T \eta + \tilde{\psi}_{i,y}.$$

We now design  $\bar{u}_i$  as,

$$\bar{u}_i = -c_1 \zeta_i - \hat{\kappa}_0 (\zeta_i + \zeta_i^{2p-1}) + \hat{l}_i^T \hat{\eta}_i.$$

where  $c_1 \geq 2$  is a constant design parameter,  $\hat{l}_i$  is an estimate of  $l_i$ ,  $\hat{\kappa}_0$  is an estimate of an unknown positive constant  $\kappa_0$  and  $\hat{\eta}_i$  is generated from (18) with  $\xi_{i,1} = u_i$ , because of the relative degree  $\rho = 1$ .

Using (19), we have the resultant error dynamics

$$\begin{aligned} \dot{e}_i = & -c_1 \zeta_i - \hat{\kappa}_0 (\zeta_i + \zeta_i^{2p-1}) + \tilde{z}_1 + (\gamma b_{i,\rho} \mathcal{N}(k_i) - 1) \bar{u}_i \\ & - l_i^T \tilde{\eta}_i - \tilde{l}_i^T \hat{\eta}_i + b_{i,\rho}^{-1} l_i^T G e + \tilde{\psi}_{i,y}. \end{aligned} \quad (39)$$

where  $\tilde{l}_i = l_i - \hat{l}_i$ . The adaptive laws are given by

$$\dot{\hat{\kappa}}_0 = \zeta_i^2 + \zeta_i^{2p}, \quad (40)$$

$$\dot{\hat{l}}_i = -\hat{\eta}_i \zeta_i, \quad (41)$$

Note that the control design and adaptive laws for subsystem  $i$  only use  $\zeta_i$  that is available via network connection to the subsystem, and therefore the control design and adaptive laws are decentralized.

**Theorem 5.1** The decentralized control inputs (38) and adaptive laws (40) and (41) solve the adaptive consensus regulation problem when the relative degree  $\rho = 1$  in the sense that the regulation error  $e$  converges to zero asymptotically.

**Proof.** Let

$$V_e = e^T Q e + \frac{1}{2} \sum_{i=1}^N (\tilde{l}_i^T \tilde{l}_i + \tilde{\kappa}_i^2) \quad (42)$$

where  $\tilde{\kappa}_i = \kappa_i - \hat{\kappa}_i$ . It can be obtained that

$$\dot{V}_e = \sum_{i=1}^N (\zeta_i \dot{e}_i - \tilde{l}_i^T \dot{\tilde{l}}_i - \tilde{\kappa}_i \dot{\tilde{\kappa}}_i) \quad (43)$$

From (39), (40) and (41), we have

$$\begin{aligned} \dot{V}_e = & \sum_{i=1}^N (-c_1 \zeta_i^2 - \kappa_0 (\zeta_i^2 + \zeta_i^{2p}) + (\gamma b_{i,\rho} \mathcal{N}(k_i) - 1) \zeta_i \bar{u}_i \\ & + \zeta_i \tilde{z}_{i,1} - \zeta_i l_i^T \tilde{\eta}_i + \zeta_i b_{i,\rho}^{-1} l_i^T G_i e_i + \zeta_i \tilde{\psi}_{i,y}) \end{aligned}$$

$$\begin{aligned} \leq & \sum_{i=1}^N (-\kappa_0 (\zeta_i^2 + \zeta_i^{2p}) + (\gamma b_{i,\rho} \mathcal{N}(k_i) - 1) \zeta_i \bar{u}_i + \frac{1}{2} \|\tilde{z}_i\|^2 \\ & + \frac{1}{2} \|l_i\|^2 \|\tilde{\eta}_i\|^2 + \frac{1}{2} |b_{i,\rho}^{-1} l_i^T G_i|^2 |e_i|^2 + \frac{1}{2} \|\tilde{\psi}_{i,y}\|^2) \end{aligned} \quad (44)$$

To analyse the dynamics of  $\tilde{z}_i$ , let

$$V_z = \sum_{i=1}^N \tilde{z}_i^T P_{i,z} \tilde{z}_i \quad (45)$$

where  $P_{i,z}$  is a positive definite matrix that satisfies

$$D_i^T P_{i,z} + P_{i,z} D_i = -3I.$$

From (36), we obtain that

$$\begin{aligned} \dot{V}_z = & \sum_{i=1}^N (-3 \|\tilde{z}_i\|^2 + 2 \tilde{z}_i P_{i,z} \tilde{\psi}_i) \\ \leq & \sum_{i=1}^N (-2 \|\tilde{z}_i\|^2 + \|P_{i,z}\|^2 \|\tilde{\psi}_i\|^2) \end{aligned} \quad (46)$$

Next, we consider the stability of  $\tilde{\eta}_i$ . Let

$$V_\eta = \sum_{i=1}^N \tilde{\eta}_i^T P_{i,\eta} \tilde{\eta}_i \quad (47)$$

where  $P_{i,\eta}$  is a positive definite matrix that satisfies

$$F_i^T P_{i,\eta} + P_{i,\eta} F_i = -5I.$$

From (20), it can be obtained that

$$\begin{aligned} \dot{V}_\eta = & \sum_{i=1}^N (-2 \|\tilde{\eta}_i\|^2 + \|b_{i,\rho}^{-1} P_{i,\eta} F_i G_i\|^2 |e_i|^2 \\ & + \|b_{i,\rho}^{-1} P_{i,\eta} G_i\|^2 \|\tilde{z}_i\|^2 + \|b_{i,\rho}^{-1} P_{i,\eta} G_i\|^2 \|\tilde{\psi}_{i,y}\|^2) \end{aligned} \quad (48)$$

Finally, Let

$$V = V_e + \beta_1 V_z + \beta_2 V_\eta \quad (49)$$

where  $\beta_1$  and  $\beta_2$  are positive constants satisfying

$$\begin{aligned} \beta_2 & \geq \frac{1}{2} \|l_i\|^2 \\ \beta_1 & \geq \frac{1}{2} + \beta_2 \|b_{i,\rho}^{-1} P_{i,\eta} G_i\|^2. \end{aligned}$$

From (44), (46) and (48), we have

$$\begin{aligned} \dot{V} \leq & \sum_{i=1}^N [-\kappa_0(\zeta_i^2 + \zeta_i^{2p}) + (\gamma b_{i,\rho} \mathcal{N}(k_i) - 1)\zeta_i \bar{u}_i \\ & + (\frac{1}{2} |b_{i,\rho}^{-1} l_i^T G_i|^2 + \beta_2 \|b_{i,\rho}^{-1} P_{i,\eta} F_i G_i\|^2) |e_i|^2 \\ & + (\frac{1}{2} + \beta_2 \|b_{i,\rho}^{-1} P_{i,\eta} G_i\|^2) |\tilde{\psi}_{i,y}|^2 \\ & + \beta_1 \|P_{i,z}\|^2 \|\tilde{\psi}_i\|^2] \end{aligned} \quad (50)$$

From (16) and (17), there exist positive constant  $r_0$  such that

$$\begin{aligned} \dot{V} \leq & \sum_{i=1}^N [-\kappa_0(\zeta_i^2 + \zeta_i^{2p}) + (\gamma b_{i,\rho} \mathcal{N}(k_i) - 1)\zeta_i \bar{u}_i \\ & + r_0(e_i^2 + e_i^{2p})]. \end{aligned} \quad (51)$$

From Lemma 3.3, we can set

$$\kappa_0 \geq r_0 \max\{\lambda_{max}^2(Q), N^{p-1} \lambda_{max}^{2p}(Q)\} \quad (52)$$

which results in

$$\dot{V} \leq \sum_{i=1}^N (\gamma b_{i,\rho} \mathcal{N}(k_i) - 1) \dot{k}_i. \quad (53)$$

Integrating both sides of (53) gives

$$V(t) \leq V(0) + \sum_{i=1}^N \int_0^{k_i} (\gamma b_{i,\rho} \mathcal{N}(s_i) - 1) ds_i. \quad (54)$$

The boundedness of  $V$  and  $k_i$ 's can be concluded from Lemma 4.1. Therefore, we can further conclude the boundedness of all variables in the adaptive control system and  $\lim_{t \rightarrow \infty} e(t) = 0$ , by following the standard procedures of stability analysis of adaptive control systems (Ding, 2013b).  $\triangleleft$

*Remark 4.* Note that the control design is based on  $e_i$  and  $\zeta_i$ , and does not use  $w$ . For the subsystems with  $\delta_{ii} = 1$ ,  $e_i$  denotes the regulation error, or the tracking error, of the subsystem, and the solution of an output regulation problem does require the measurement of the tracking error for the control design as in the standard formulation (Isidori, 1995).  $\triangleleft$

## 6 Adaptive Consensus Regulation, $\rho > 1$

When the relative degree  $\rho > 1$ , we use adaptive backstepping for the control design. The first step is almost the same as the control design for  $\rho = 1$ . In this case, we cannot directly assign values to  $\xi_{i,1}$ . Instead, we can

start the control design from the desired values for  $\xi_{i,1}$  denoted by  $\hat{\xi}_{i,1}$  as

$$\hat{\xi}_{i,1} = \gamma \mathcal{N}(k_i) \bar{\xi}_{i,1} \quad (55)$$

$$\dot{k}_i = \zeta_i \bar{\xi}_{i,1}, \quad k_i(0) = 0 \quad (56)$$

where the Nussbaum gain  $\mathcal{N}$  is defined in (29). From (15) and the definition of the Nussbaum gain in (55), we have

$$\begin{aligned} \dot{e}_i = & \tilde{z}_{i,1} + (\gamma b_{i,\rho} \mathcal{N}(k_i) - 1) \bar{\xi}_{i,1} + \bar{\xi}_{i,1} + \tilde{b}_{i,\rho} \tilde{\xi}_{i,1} \\ & + \hat{b}_{i,\rho} \tilde{\xi}_{i,1} - l_i^T \eta_i + \tilde{\psi}_{i,y} \end{aligned}$$

where  $\hat{b}_{i,\rho}$  is an estimate of  $b_{i,\rho}$ ,  $\tilde{b}_{i,\rho} = b_{i,\rho} - \hat{b}_{i,\rho}$  and  $\tilde{\xi}_{i,1} = \xi_{i,1} - \hat{\xi}_{i,1}$ .

We design

$$\bar{\xi}_{i,1} = -c_1 \zeta_i - \hat{\kappa}_i (\zeta_i + \zeta_i^{2p-1}) + \tilde{l}_i^T \hat{\eta}_i.$$

where  $\hat{\kappa}_i$  is an estimate of an unknown constant  $\kappa_0$ .

The resultant dynamics of  $e_i$  are obtained as

$$\begin{aligned} \dot{e}_i = & -c_1 \zeta_i - \hat{\kappa}_i (\zeta_i + \zeta_i^{2p-1}) + \tilde{z}_{i,1} + (\gamma b_{i,\rho} \mathcal{N}(k_i) - 1) \bar{\xi}_{i,1} \\ & + \tilde{b}_{i,\rho} \tilde{\xi}_{i,1} + \hat{b}_{i,\rho} \tilde{\xi}_{i,1} - l_i^T \tilde{\eta}_i - \tilde{l}_i^T \hat{\eta}_i \\ & + \tilde{b}_{i,\rho}^{-1} l_i^T G e + \tilde{\psi}_{i,y}. \end{aligned} \quad (57)$$

From  $\xi_i$ , we will design the final control design  $u_i$  in  $\rho - 1$  steps using adaptive backstepping.

In the second step, we have

$$\dot{\tilde{\xi}}_{i,1} = \lambda_1 \xi_{i,1} + \xi_{i,2} - \dot{\hat{\xi}}_{i,1} \quad (58)$$

Note that  $\hat{\xi}_{i,1}$  is a function of  $\zeta_i$ ,  $k_i$ ,  $\hat{b}_{i,\rho}$ ,  $\hat{\kappa}_i$  and  $\hat{\eta}_i$ . Hence, we have

$$\begin{aligned} \dot{\tilde{\xi}}_{i,1} = & -\lambda_1 \xi_{i,1} + \xi_{i,2} \\ & - \frac{\partial \hat{\xi}_{i,1}}{\partial k_i} \dot{k}_i - \frac{\partial \hat{\xi}_{i,1}}{\partial \hat{b}_{i,\rho}} \dot{\hat{b}}_{i,\rho} - \frac{\partial \hat{\xi}_{i,1}}{\partial \hat{\kappa}_i} \dot{\hat{\kappa}}_i - \frac{\partial \hat{\xi}_{i,1}}{\partial \hat{\eta}_i} \dot{\hat{\eta}}_i \\ & - \frac{\partial \hat{\xi}_{i,1}}{\partial \zeta_i} \sum_{j=1}^{n_i} q_{ij} (\tilde{z}_{j,1} + \tilde{\psi}_{j,y} + b_{j,\rho} \xi_{j,1} - l_j^T \eta_j) \end{aligned} \quad (59)$$

Based on (57) and (59), we design  $\hat{\xi}_{i,2}$  as

$$\begin{aligned} \hat{\xi}_{i,2} = & \lambda_1 \xi_{i,1} - \hat{b}_{i,\rho} \tilde{\xi}_{i,1} - c_{2,1} \tilde{\xi}_{i,1} - c_{2,2} \left( \frac{\partial \hat{\xi}_{i,1}}{\partial \zeta_i} \right)^2 \tilde{\xi}_{i,1} \\ & + \frac{\partial \hat{\xi}_{i,1}}{\partial k_i} \dot{k}_i + \frac{\partial \hat{\xi}_{i,1}}{\partial \hat{b}_{i,\rho}} \dot{\hat{b}}_{i,\rho} + \frac{\partial \hat{\xi}_{i,1}}{\partial \hat{\kappa}_i} \dot{\hat{\kappa}}_i + \frac{\partial \hat{\xi}_{i,1}}{\partial \hat{\eta}_i} \dot{\hat{\eta}}_i \end{aligned}$$



$$+ \frac{\partial \hat{\xi}_{i,1}}{\partial \zeta_i} \sum_{j=1}^N q_{ij} (\hat{b}_{j,\rho} \xi_{j,1} - \tilde{l}_j^T \hat{\eta}_j) \quad (60)$$

where  $c_{2,1}$  and  $c_{2,2} \geq \frac{3}{2}$  are positive constant design parameters. The resultant dynamics of  $\tilde{\xi}_{i,2}$  are obtained as

$$\begin{aligned} \dot{\tilde{\xi}}_{i,1} = & -\hat{b}_{i,\rho} \tilde{\zeta}_i - c_{2,1} \tilde{\xi}_{i,1} - c_{2,2} \left( \frac{\partial \hat{\xi}_{i,1}}{\partial \zeta_i} \right)^2 \tilde{\xi}_{i,1} \\ & - \frac{\partial \hat{\xi}_{i,1}}{\partial \zeta_i} \sum_{j=1}^N q_{ij} (\tilde{z}_{j,1} + \tilde{\psi}_{j,y} - l_j^T \tilde{\eta}_j \\ & - \tilde{l}_j^T \hat{\eta}_j + b_{j,\rho}^{-1} l_j^T G_j e_j) \end{aligned} \quad (61)$$

If the relative degree  $\rho = 2$ , we have  $\tilde{\xi}_{i,2} = 0$  and we can set

$$u_i = \hat{\xi}_{i,2}. \quad (62)$$

In such a case, we design the adaptive laws as

$$\dot{\hat{b}}_{i,\rho} = \zeta_i \tilde{\xi}_{i,1} - \left( \sum_{j=1}^N q_{ji} \frac{\partial \hat{\xi}_{j,1}}{\partial \zeta_j} \tilde{\xi}_{j,1} \right) \xi_{i,1}, \quad (63)$$

$$\dot{\hat{l}}_i = -\zeta_i \hat{\eta}_i + \left( \sum_{j=1}^N q_{ji} \frac{\partial \hat{\xi}_{j,1}}{\partial \zeta_j} \tilde{\xi}_{j,1} \right) \hat{\eta}_i \quad (64)$$

Note that the control inputs shown in (60) and adaptive laws in (63) and (64) are still decentralized, because the summation of the variables of other subsystems are through  $q_{ji}$ . This means the control and adaptive laws of subsystem  $i$  use the information of the subsystems that are connected the subsystem  $i$ . Of course, for high relative degrees, we need more information from other subsystems. For the case of relative degree  $\rho = 2$ , we need the information of  $\hat{b}_{j,\rho}$ ,  $\hat{\eta}_j$ ,  $\xi_{j,1}$ , and  $\tilde{\xi}_{j,1}$  etc. It should also be noted that those variables are generated by the local controllers, and there is no need to pass the information of the system output, except the relative system outputs.

Clearly, control design can be carried out for  $\rho > 2$  by following the same design procedures as shown for  $\rho = 2$ , but more tedious. We will not show the details of the design here, for the sake of the page limits. Instead, we show the result of the stability analysis for the case of  $\rho = 2$ .

**Theorem 6.1** The decentralized control inputs (62) and adaptive laws (56), (63) (64) solve the adaptive consensus regulation problem when the relative degree  $\rho = 2$  in the sense that the regulation error  $e$  converges to zero asymptotically.

**Proof.** Let

$$V_e = e^T Q e + \frac{1}{2} \sum_{i=1}^N (\tilde{b}_{i,\rho}^2 + \tilde{\kappa}_i^2 + \tilde{l}_i^T \tilde{l}_i + \tilde{\xi}_{i,1}^2) \quad (65)$$

From (57), (61), (56), (63) and (64), we have

$$\begin{aligned} \dot{V}_e = & \sum_{i=1}^N (-c_1 \zeta_i^2 - \kappa_0 (\zeta_i^2 + \zeta_i^{2p}) + (\gamma b_{i,\rho} \mathcal{N}(k_i) - 1) \zeta_i \bar{\xi}_{i,1} \\ & + \zeta_i \tilde{z}_{i,1} - \zeta_i l_i^T \tilde{\eta}_i + \zeta_i l_i^T G_i e_i + \zeta_i \tilde{\psi}_{i,y} \\ & - c_{2,1} \tilde{\xi}_{i,1}^2 - c_{2,2} \left( \frac{\partial \hat{\xi}_{i,1}}{\partial \zeta_i} \right)^2 \tilde{\xi}_{i,1}^2 \\ & - \frac{\partial \hat{\xi}_{i,1}}{\partial \zeta_i} \tilde{\xi}_{i,1} \sum_{j=1}^N q_{ij} (\tilde{z}_{j,1} + \tilde{\psi}_{j,y} + b_{j,\rho}^{-1} l_j^T G_j e_j)) \\ \leq & \sum_{i=1}^N (-\kappa_0 (\zeta_i^2 + \zeta_i^{2p}) + (\gamma b_{i,\rho} \mathcal{N}(k_i) - 1) \zeta_i \bar{\xi}_{i,1} \\ & + \frac{1}{2} (\|\tilde{z}_i\|^2 + \|l_i\|^2 \|\tilde{\eta}_i\|^2 + |b_{i,\rho}^{-1} l_i^T G_i|^2 |e_i|^2 + \|\tilde{\psi}_{i,y}\|^2) \\ & - c_{2,1} \tilde{\xi}_{i,1}^2 - (c_{2,2} - \frac{3}{2}) \left( \frac{\partial \hat{\xi}_{i,1}}{\partial \zeta_i} \right)^2 \tilde{\xi}_{i,1}^2 \\ & + \frac{1}{2} \sum_{j=1}^N q_{ij}^2 (\|\tilde{z}_j\|^2 + \|l_j\|^2 \|\tilde{\eta}_j\|^2 + |b_{j,\rho}^{-1} l_j^T G_j|^2 |e_j|^2 \\ & + \|\tilde{\psi}_{j,y}\|^2)) \\ \leq & \sum_{i=1}^N (-\kappa_0 (\zeta_i^2 + \zeta_i^{2p}) + (\gamma b_{i,\rho} \mathcal{N}(k_i) - 1) \zeta_i \bar{\xi}_{i,1} \\ & - c_{2,1} \tilde{\xi}_{i,1}^2 + \frac{1}{2} (1 + \|Q\|^2) (\|\tilde{z}_i\|^2 + \|l_i\|^2 \|\tilde{\eta}_i\|^2 \\ & + |b_{i,\rho}^{-1} l_i^T G_i|^2 |e_i|^2 + \|\tilde{\psi}_{i,y}\|^2)) \end{aligned} \quad (66)$$

The dynamics of  $\tilde{z}_i$  and  $\tilde{\eta}_i$  can be analysed in the same way as in the proof of Theorem 5.1 by using the same  $V_z$  and  $V_\eta$ . The rest of the proof follows similarly to the proof of Theorem 5.1.  $\square$

## 7 Example

The proposed design works for the subsystems of different orders as long as the relative degrees are the same. We use an example to demonstrate the application of the proposed control design to a network connected systems with 5 subsystems of different orders. For  $i = 1, 3, 5$ , each subsystem is described by a third-order nonlinear model

$$\begin{aligned} \dot{x}_{i,1} &= x_{i,2} \\ \dot{x}_{i,2} &= x_{i,3} + \mu_{i,1} y_i^2 w_1 + b_{i,2} u_i \\ \dot{x}_{i,3} &= \mu_{i,2} y_i^3 + b_{i,3} u_i \\ y_i &= x_{i,1} \end{aligned}$$

and for  $i = 2, 4$ , each subsystem is described by a second-order nonlinear model

$$\begin{aligned}\dot{x}_{i,1} &= x_{i,2} + \mu_{i,1}y_i^2w_1 \\ \dot{x}_{i,2} &= \mu_{i,1}x_{i,2}^3 + \mu_{i,3}w_2 + b_{i,2}u_i \\ y_i &= x_{i,1}\end{aligned}$$

with the exosystem

$$\dot{w} = \begin{bmatrix} 0 & \sigma \\ -\sigma & 0 \end{bmatrix} w, \quad g(w) = [1 \ 0]w.$$

Note that the exosystem generates sinusoidal functions with frequency  $\sigma$ . The desired trajectory  $q(w) = w_1$  which is only available to the second subsystem. The adjacency matrix and the result  $Q$  are given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 4 & -1 & -1 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

The system considered here is in the format of (1), and it can be checked that the system satisfies Assumptions 2 and 3. We can choose the unknown parameters in the simulation so that Assumption 1 is also satisfied. The desired feedforward controls  $\alpha_i$  can be shown to contain the sinusoidal functions of frequencies  $\sigma$  and  $3\sigma$ . The frequency components of  $3\sigma$  are due to the nonlinear functions in the system. For the internal model design, we need to choose  $F_i \in \mathbb{R}^{4 \times 4}$ . It can also be shown that  $p = 3$  is enough. The design of control inputs then follows the exact steps shown in section 6.

Simulation study has been carried out with the unknown system parameters set as  $\mu_{i,j} = 0.1$ , and other unknown parameters as 1, including  $b_{i,2}$  and  $\sigma$ . In the simulation study, we used

$$F_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16 & -32 & -24 & -8 \end{bmatrix}, \quad G_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix},$$

for the internal models, and the control parameters were set as 1 except  $\gamma = 0.001$ , and  $c_{2,2} = 2$ . Simulation results for the output of the subsystems are shown in Figure 1, which shows that the outputs for subsystems 1, 3, 5, and subsystems 2, 4 converge to the same value.

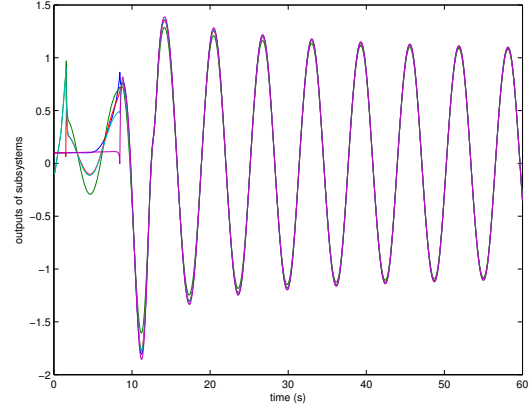


Fig. 1. The subsystem outputs.

## 8 Conclusions

In this paper, we have proposed a new type of Nussbaum gains that can be used for adaptive consensus of network connected systems. This newly proposed Nussbaum gain has been used together with carefully designed adaptive laws and internal models for adaptive consensus output regulation of a class of network-connected systems with possibly different nonlinear dynamics. The success of the proposed Nussbaum gain design depends on a new fast-growing Nussbaum function. The adaptive laws depend on the information available from the subsystems in the neighbourhood. The proposed adaptive consensus control method can deal with tracking and disturbance rejection under the formation of adaptive output regulation. The example demonstrates that proposed design does ensure the convergence of the subsystem outputs to a common trajectory.

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