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RESEARCH PAPER

Control of robust design in multiobjective optimization under uncertainties

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Abstract In design and optimization problems, a solution is called *robust* if it is stable enough with respect to perturbation of model input parameters. In engineering design optimization, the designer may prefer a use of robust solution to a more optimal one to set a stable system design. Although in literature there is a handful of methods for obtaining such solutions, they do not provide a designer with a direct and systematic control over a required robustness. In this paper, a new approach to robust design in multiobjective optimization is introduced, which is able to generate robust design with model uncertainties. In addition, it introduces an opportunity to control the extent of robustness by designer preferences. The presented method is different from its other counterparts. For keeping robust design feasible, it does not change any constraint. Conversely, only a special tunable objective function is constructed to incorporate the preferences of the designer related to the robustness. The effectiveness of the method is tested on well known engineering design problems.

Keywords Robust design optimization · Multiobjective optimization · Fuzzy uncertainty · Directed search domain

1 Introduction

The operational variations and uncertainties in the model parameters can affect the performance of the optimum design. Therefore, the designer may demand a stable (or *robust*) configuration, which is indifferent to these variations rather than an optimal solution.

Papers by Bryne (1987) and Taguchi et al. (1989) represent first efforts in finding robust design. In particular, they introduce a method to minimize the effects of uncontrollable parameters during design. The so called Taguchi loss function (Ross 1995) is used to make a design more tolerable to the model variations. To guarantee a less sensitive design, other researchers (e.g. Ramakrishnan and Rao 1991; Sundaresan et al. 1991; Mohandas 1989; Box and Fung 1986; Rao 1983) use optimization to minimize the variation of input parameters. They propose a robust design optimization with Taguchi loss function as an objective function subject to the model constraints. Implementing this, the constant and variable sensitivity from controllable and uncontrollable parameters are respectively minimized using nonlinear programming.

In single objective optimization, Parkinson et al. (1993) seek the robust solution by studying two main issues: feasibility of the design and the control of transmitted variation. In that work, the size of feasible space is reduced to find the robust solution unaffected by the model variation. In addition to keeping the design feasible, the sensitivity of the design is minimized during the optimization process. This work is done by a nonlinear optimal design formulation. The analysis of tolerance (Cox 1987), including both the worst case and statistical analysis, is implemented to calculate the transmitted variation of the parameters in the design functions. However, as mentioned by Gunawan and Azarm (2005), the information required for the probabilistic distributions of variations may not be easily obtained in practical problems.

Su and Renaud (1996), on the other hand, develop another kind of robust design approach to single objective

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optimization. They introduce sensitivity- and experimental-based robust design optimization to find a less sensitive optimum solution. The essence of these two approaches is to study the vicinity of current design and hence find the solution which is less sensitive to fluctuations of parameters and design variables. To study the sensitivity of the design, Ting and Long (1996) also use the sensitivity Jacobian based on the performance tolerances. As in paper by Parkinson et al. (1993), their approach exploits the analysis of tolerance to find a transmitted variation in the performance functions.

In multiobjective optimization, Messac and Ismail Yahaya (2002) develop a procedure to carry out a flexible robust optimization. Using the physical programming lexicon (Messac 1996), a system with a minimal variation in the input variability and uncertainties is found while the feasibility of the design solution is guaranteed. They consider robust design with both prescribed and variable tolerances, in which the variable tolerance level is determined optimally in the design phase.

In another study, Deb and Gupta (2005) extend an approach used in single objective optimization and develop a method for seeking robust solution in multiobjective optimization. Two types of robust solutions are considered in different prospectives. One type is related to optimization of the mean effective objective functions. In another type, the extent of robustness is taken into account by including an additional constraint in the problem with a free parameter as its right-hand side set by the designer. The essence of the work is to minimize the feasible space to keep the design solution feasible during the optimization. As argued by Deb and Gupta (2005), depending on the sample population in the vicinity of each design, this method is computationally expensive. In addition, as discussed by Shimoyama et al. (2009), the proposed approach faces difficulty in setting the right-hand side as the upper limit of the robust constraints for considering the degree of robustness.

Gunawan and Azarm (2005) proposed doubled loop optimization methodology for finding a robust solution in multiobjective optimization. An inner subproblem evaluates the sensitivity of the current design by maximizing the size of its worst case sensitivity region. Then, the outer loop optimization is performed for the nominal value of the parameters to find the robust solution. Although promising for certain implementations, it seems inappropriate to provide information of the objective function variation ranges in the case of general practical problems. Furthermore, since the method consists of two optimization problems for each design, the computational time increases for higher dimension problems. In addition, as argued by Gunawan and Azarm (2005), this method only provides a robust solution and it does not illustrate how much the solution is robust against another design. However, the robust measure is genuine to be considered for some certain optimization tasks with non-differentiable and discontinuous functions.

Response surface (RS) methodology is exploited by Chen et al. (1996) via the substitution of the original functions by simpler ones. To account for the robust solution, then, they simultaneously optimize the mean of the performance and minimize the variation of the response formulated in a bi-objective optimization problem. Clearly, this provides a trade-off between the optimality and robustness. However, the proposed approach is introduced in the context of a single objective optimization.

Shimoyama et al. (2009) have recently presented a methodology to study the trade-off between optimality and robustness in multiobjective problems. They use RS methodology to represent a simpler formulation of the functions and avoid the computational efforts for the functions evaluations. Then, they introduce the robustness measure for each objective function derived through RS with its nominal value. The robustness measure is simply proposed as the standard deviation of a function with parameters dispersed around the current design. As presented in the paper, the method works quite reasonably but it does not provide any control over the level of robustness. In addition, for high dimension problems, the proposed method seems inefficient since the number of objective functions doubles comparing with the deterministic problem. Furthermore, the RS methodology is an exhaustive technique even for the small number of factors and levels (Gunawan and Azarm 2005). Moreover, to compute the mean and variance of the performance functions, sampling is required at each step. This makes the proposed method computationally expensive.

Another formulation of robust design is introduced by Lu and Li (2009) by solving two separate optimization tasks: minimizing the variation in design variables and minimizing the influence of model uncertainty. Obviously, this design methodology enables the designer to take into account both the model uncertainty and variations in the design variables. However, to elicit the model uncertainty, a sampling technique is required to find the bound of the perturbation sensitivity matrix. The accuracy of the proposed method depends on the estimation accuracy of this perturbation bound. Therefore, the technique seems computationally prohibitive for practical problems.

Independently from the Taguchi work and idea of reducing the feasible space, we propose a method, which not only ensures a robust design in uncertain environment but also controls the extent of robustness via the designer preferences. In particular, we consider the fuzzy and interval value uncertainty of the input parameters, which is quite common in engineering design. Furthermore, we extend a function as a measure for robustness for multiobjective optimization. It is shown that the proposed method is capable of obtaining different level of robust solutions only via the introduction of a new objective function. In addition, it enables the designer to set a bound for the robust solution set. It is important to note that we do not add any new constraint to the problem or even change the existing ones.

2 Handling uncertainty and robustness in optimization

In engineering design optimization problems, we often consider multiple goals which are required to be set at their optimum level subject to some inevitable constraints. A generic multiobjective optimization problem takes the following form:

$$\begin{aligned} Min\ \ \mathscr{F} &= \{F_1(x), F_2(x), \ldots, F_n(x)\}, \\ subject\ to\ \ & x \in \mathscr{D}^*, \end{aligned} \tag{1}$$

where $\mathscr{D}^*$ is the feasible space. We seek a set of solutions called the *Pareto*, which are based on the following definition:

Definition 1 (Pareto Optimality) Vector $\mathbf{x}^* \in \mathscr{D}^*$ is called a *Pareto* solution to problem (1) iff $\nexists\, \mathbf{x}^{**}$ such that, $F_i(\mathbf{x}^{**}) \leq F_i(\mathbf{x}^*)$ for any $i = 1, \ldots, n$ and exists j $(1 \leq j \leq n)$: $F_j(\mathbf{x}^{**}) < F_j(\mathbf{x}^*)$.

In the objective space, Pareto solutions form a *Pareto frontier*, which gives the best trade-off solutions to multiobjective optimization problem (1).

In real-life design, model uncertainty is caused due to some inevitable noise and uncertainties during the design process. In this study, we consider the uncertainty of the parameters using fuzzy variables though it is possible to implement any kind of uncertainty within the model. Hence, the problem (1) changes to:

$$\begin{aligned} Min\ \ \tilde{\mathscr{F}} &= \{\tilde{F}_1(x), \tilde{F}_2(x), \ldots, \tilde{F}_n(x)\}, \\ subject\ to\ \ \widetilde{g_i(x)} &\leq \tilde{b}_i,\ \ i = 1 \ldots m, \\ x &\in \mathscr{D}^*, \end{aligned} \tag{2}$$

where tilde implies that the parameters of the model are not precisely known but are modeled using fuzzy variables. The use of the fuzzy variables gives an opportunity to the designer to model a problem, in which there exists a doubt about the exactness of input parameters, the degree of credibility, and correctness of statements (Parkinson 1995). In return, the modeling can be done in a more flexible way, which can be used in practical applications.

To solve problem (2) for its optimum values, the model should be converted into a deterministic formulation. For this purpose, the nominal value of the problem parameters can be used. In this way, we substitute the fuzzy variables by their fuzzy possibilistic mean value (Erfani and Utyuzhnikov 2010b) using the definition in Appendix. Thus, problem (2) is reduced to

$$\begin{aligned} Min\ \ \mathscr{F}_{pm} &= \left\{F_1^{pm}(x), F_2^{pm}(x), \ldots, F_n^{pm}(x)\right\}, \\ subject\ to\ \ g_i^{pm}(x) &\leq b_i^{pm},\ \ i = 1 \ldots m, \\ x &\in \mathscr{D}^*, \end{aligned} \tag{3}$$

where the pm denotes the fuzzy possibilistic mean value of a parameter as its nominal value.

To solve the above problem for the nominal value of the parameters, the notion of a possibilistic Pareto solution is introduced (Erfani and Utyuzhnikov 2010b):

Definition 2 (Possibilistic Mean (PM) Pareto Optimality) x^* is called *Possibilistic Mean Pareto* solution to multiobjective optimization (3) iff $\nexists\, \mathbf{x}^{**}$ such that, $F_i^{pm}(\mathbf{x}^{**}) \leq F_i^{pm}(\mathbf{x}^*)$ for any $i = 1, \ldots, n$ and exists j $(1 \leq j \leq n)$: $F_j^{pm}(\mathbf{x}^{**}) < F_j^{pm}(\mathbf{x}^*)$.

Thus, the problem is deterministic and any suitable multiobjective algorithm can be used to generate the set of Pareto optimal solutions for it.

As stated earlier, variation in the model components makes the designer to look for a set of more stable solutions rather than the optimal ones, though the optimal solution yields a better design. A robust solution should be less sensitive to the uncertainty of the model. The concept is illustrated at Fig. 1. In this figure, fluctuations about design

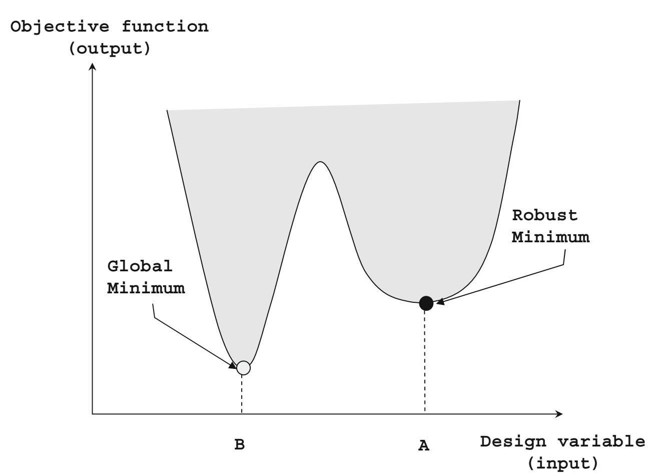

Fig. 1 Comparing robust and global optimum solution

A lead to less perturbation of the objective function in comparison to the variation of design **B**. Therefore, **A** is called to be more robust than **B**, while **B** yields a more favorite value of the objective function than **A** does. However, in multiobjective optimization, the problem is not as easy as in single objective optimization, while the idea can be extended and used. Deb and Gupta (2005) illustrate that in the multiobjective dimension context, the sensitivity near the solution should be checked for all of the objective functions. The robustness must also be defined not only for one solution, but for all the Pareto solutions. Bearing this in mind, next we define a robust measure in multiobjective optimization.

3 Construction of robust measure

In the literature, to realize a robust solution, the optimal solution is shifted into the feasible space by making the design space smaller and accepting the degradation of the optimal value (Parkinson et al. 1993). In this way, we should change the constraints and find the transmitted variation for each of them, which is time-consuming. In addition, unless the source of variability is known, changing the right hand side of the problem is an ad-hoc process for maintaining the feasibility percentage. In other words, in the case that the variability is an interval value or purely is specified by designer, there is no any control and even understanding of the extent of the robustness. Herein, we address a new approach for searching a robust solution without changing the constraint via the introduction of a function as a new objective function.

To take the robustness into account, following Fig. 1, we consider a robust measure in its general formulation as follows (Erfani and Utyuzhnikov 2010b):

$$\mathscr{R}_{\mathscr{F}} = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\sigma_{F_i}}{\sigma_{x_j}} \tag{4}$$

where m and n are the number of design components (parameters and variables, whichever is varied in accordance with design specification) and objective functions, respectively; σx_j denotes the variance of the (fuzzy or interval) parameter. To estimate the σF (i is omitted for the sake of simplicity), first order Taylor series is implemented as follows:

$$\sigma_F^2 = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \right)^2 \sigma_{p_i}^2. \tag{5}$$

where p is the uncertain parameter of the model. In (5), the designer states the variance of the parameters denoted by σ_p. For fuzzy or interval value parameters, the fuzzy variance may be used (Erfani and Utyuzhnikov 2010b). Having included this in formulation and solved the problem, one may expect a set of robust solutions. However, the problem, which is overlooked, is the control of the extent of robustness.

3.1 Control of robustness

In this Section, we propose an approach to control the extent of robust solution. The method provides a flexibility for the designer to generate and choose an appropriate robust solution. This work can be done via constructing a function and mapping robust measure (4) onto it. We construct a dimensionless positive decreasing/increasing (whichever is appropriate) convex function, which cannot be represented by high-order polynomials or cubic splines. Therefore, we introduce an approximation method, which was first suggested by Utyuzhnikov et al. (2005) for other purposes. We call this function the *tunable robust function (TRF)* shown by T onward. For our purpose, TRF is approximated piecewisely by mapping robust measure (4) scaled between 0 and 100 onto a dimensionless positive increasing function $T(R)$, where R is the scaled function of $\mathscr{R}$.

At the beginning, the designer proposes a level of desired robustness L_R. Then, TRF is constructed in four regions; represented by $\{0, L_R - d, L_R, L_R + d, 100\}$ as the boundaries. Here, d is a free parameter set by the designer, which can be changed appropriately. To guarantee the convexity of the TRF, suppose that the derivative of TRF has the following form:

$$\frac{dT}{dR} = Ae^{\alpha(R)}, \tag{6}$$

$$A > 0, \quad \alpha'(R) > 0. \tag{7}$$

Taking α as a smooth function of robust measure in (4) and integrating (7), we obtain

$$A \Delta_k R \int_0^1 e^{\alpha^{(k)}(\xi)} d\xi = \Delta_k T, \tag{8}$$

where $\Delta_k T = T(R_{k+1}) - T(R_k)$, $\alpha^{(k)} = \alpha^{(k)}(\xi^{(k)})$, $\xi^{(k)} = \frac{R - R_k}{R_{k+1} - R_k}$ ($R_k \le R \le R_{k+1}$) and $k = 1, 2, 3, 4$ show the region where the TRF is approximated. The sketch of the function is shown in Fig. 2.

For the sake of positivity and convexity of TRF, the following conditions on α should be held. In the given region, the lower value of the TRF in the current region coincides with the upper value of the afore region ($\alpha^{(k)}(0) = \alpha^{(k-1)}(1)$). This requirement guarantees the continuity and smoothness of the TRF, since the left- and right-hand side derivatives coincide. To satisfy this condition, a simple linear function

$$\alpha^{(k)}(\xi) = a_k \xi + b_k, \tag{9}$$

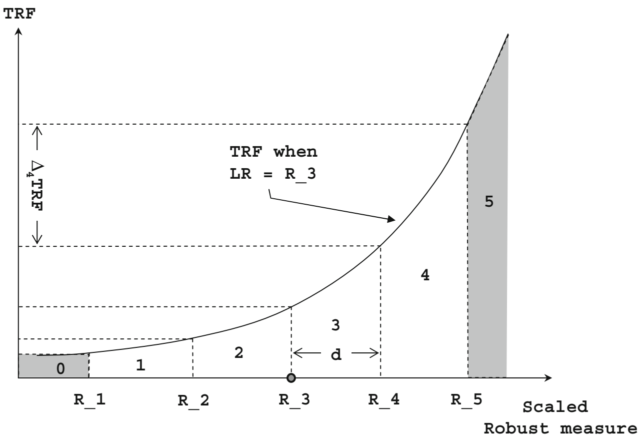

Fig. 2 Piecewise Robust Functions (TRF)

can be used with

$$a_k = \frac{A\Delta_k R e^{b_k}(e^{a_k} - 1)}{\Delta_k T}, \tag{10}$$

$$b_k = a_{k-1} + b_{k-1}. \tag{11}$$

This gives a recurrent relation for the calculation of a_k and b_k. As the values of TRF at the boundaries of each region do not depend on the value of scaled robust function (R), they are fixed if

$$\Delta_k T = \beta n \Delta_{k-1} T, \tag{12}$$

$$\beta > 1, \quad \Delta_0 T = 1, \tag{13}$$

where n denotes the number of objective functions. βn guarantees an exponentially increasing function, in which a k-th region is more preferable than the $k+1$-th one. Finally, using (8), the TRF can be approximated piecewise as

$$T^{(k)} = T_k + \Delta_k T \frac{e^{a_k \xi^{(k)}} - 1}{e^{a_k} - 1} \quad (R_k \le R \le R_{k+1}), \tag{14}$$

where $T_k = T(R_k)$ and R_k is the value of the scaled robust measure at the boundary of the region k as shown in Fig. 2.

To obtain a function defined in $\mathbb{R}$, it is possible to choose a simple exponential function

$$T^{(0)} = e^{A(R-R_1)}, \tag{15}$$

for the region of $R < R_1$. Then,

$$a_0 = 0, \quad b_0 = 0, \tag{16}$$

are the initial conditions, and hence, $T^{(0)} = 1$.

Parameter A can be chosen as $\frac{1}{R_5 - R_1}$ to have the same dimensionality with R^{-1}.

It is worth noting that, a positive solution to (10) can be obtained by the method of simple iterations if we choose

$$a_k^0 > -ln\left(A\Delta_k R e^{b_k}/\Delta_k T\right)$$

as the initial condition (Utyuzhnikov et al. 2005). Since the robust measure of (4) is introduced as an objective function to be minimized, the *increasing* TRF is implemented.

It should be noted that by this method the designer is able to cover all the robustness ranges as the robust measure is scaled between 0 to 100. At the extreme points of 0 and 100, the most robust solution (100 %) and the regular (optimum solution) are generated, respectively. Obviously, one may expect the overlap between the optimum solution and the robust one under some conditions. It goes without saying that the region with the overlap is of the most interest for the designer.

Figure 3 shows the TRF for different value of robustness.

It is worth noting that in addition to the flexibility the TRF function introduces, the final design is robust *at least* to a degree which the designer wishes to obtain. As one can investigate, the TRF is a soft constraint and the optimal value of robustness, therefore, is estimated during the optimization procedure.

In comparison with the existing methods, the proposed approach is efficient in finding robust design. The existing methods exploit the variance as a robust measure, for which a huge number of points is needed to calculate the variability. Obviously, for time consuming functions, this is exhaustive and not efficient. Meanwhile, in our approach to finding the robust solution, we only add one simple objective function to the original problem regardless of the problem dimension. Furthermore, it is to be noted that the proposed method does not require a presumed probability distribution of variation.

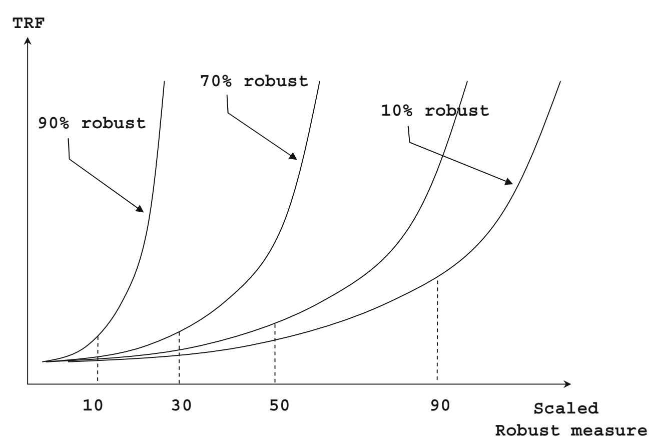

Fig. 3 Different Tunable Robust Functions (TRF)

The algorithm to summarize the methodology is shown in Algorithm 1.

Algorithm 1 Desired robust solution algorithm

1. Use fuzzy, interval or probabilistic mean to calculate the nominal value of parameters.
2. Calculate the robust measure ($\mathscr{R}_{\mathscr{F}}$).
3. Specify the level of desired robustness (L_R).
4. Scale the $\mathscr{R}_{\mathscr{F}}$ and map it onto the non-dimension exponential function based on L_R and call it TRF.
5. Include the TRF as a new objective function.
6. Solve the following problem using any optimizer i.e. DSD:

$$\begin{aligned} &Min \quad \mathscr{F}_{pm}(x),\\ &Min \quad TRF(R),\\ &s.t. \quad g_i^{pm}(x) \le b_i^{pm} \quad i = 1 \dots m,\\ &\qquad x \in \mathscr{D}^*. \end{aligned}$$

7. if another desired robustness is needed **then**
Go to Step 3

4 Engineering test problems

In this Section, we test the new approach to find the robust design for two bar trusses, welded beam and pressure vessel. For each case, the problem is discussed and the results are presented. All the Pareto and robust frontiers are generated using the Directed Search Domain (DSD) method (Erfani and Utyuzhnikov 2010a). The DSD method is able to generate an evenly distributed Pareto set in a general formulation, which is based on shrinking a search domain in a selected area on the Pareto frontier.

4.1 Case 1: two bar truss design

A symmetric two-bar truss structure is a popular structural design which we study for the best robust-configuration. The test case is taken from Messac and Ismail Yahaya (2002) and is shown in Fig. 4. The structure is supposed to support a load F. The truss consists of two steel tubes pinned together at one end and supported on the ground at the other. The design variables are the diameter of members (x_1) and the height (x_2) of structure. We consider two objectives of the design: (i) minimize total mass of truss members and (ii) minimize the vertical deflection due to the application of the load $F = (150, 20, 30)$ kN as a triangular fuzzy number (see the Appendix). As can be seen, these two objectives are in conflict. Thus, multiobjective optimization is needed for finding a trade-off set of solutions. The parameters of the problem are the two fuzzy numbers: member thickness $t = (2.5, 0.5, 1.5)$ mm and structure width $w = (750, 100, 50)$ mm, and the two crisp (real) numbers: mass density $\rho = 7.8 \times 10^{-3}$ gr/mm^3 and Elastic modulus $E = 210000$ Nmm^2. The normal stress has to be less than the buckling stress as constraint and $1 \le x_1 \le 100$

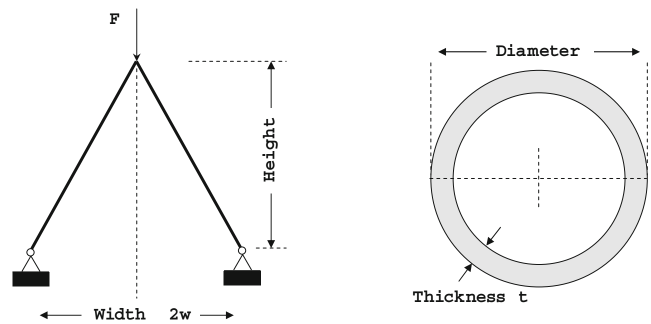

Fig. 4 Two-bar truss. Structure (*left*), section of member (*right*)

and $10 \le x_2 \le 1000$ must hold. Therefore, the formulation takes the following form:

$$Minimize \;\; F_1 = Mass = 2\pi\rho t x_1\sqrt{w^2 + x_2^2},$$

$$Minimize \;\; F_2 = Deflection = \frac{F(w^2 x_2^2)^{3/2}}{(2\pi t E x_1 x_2)^2},$$

$$\begin{aligned} s.t. \;\; & s \le \frac{1}{8}\pi^2 E \frac{t^2 + x_1^2}{w^2 + x_2^2},\\ & 1 \le x_1 \le 10,\\ & 100 \le x_2 \le 1000,\\ where & \\ & s = \frac{F}{2\pi t x_1 x_2}\sqrt{w^2 + x_2^2}. \end{aligned}$$

The triangular fuzzy parameters are substituted by their possibilistic mean value of $F = 151.6$ kN, $t = 2.66$ mm and $w = 741.6$ mm (see the Appendix). Thus, the constraints are modified. To formulate Step 6 of the algorithm for finding the robust frontier, the robust measure is constructed via considering the variance of 1020.6, 0.4 and 30.61 for F, t and w, respectively(see the Appendix).

Assume that F_1 and F_2 are the two objective functions representing mass and deflection. The robust measure given in (4), is then determined as follows, with F, t and w as the three uncertain parameters:

$$\mathscr{R}_{\mathscr{F}} = \frac{1}{2 \times 3}\left(\left(\frac{\sigma F_1}{\sigma F} + \frac{\sigma F_1}{\sigma t} + \frac{\sigma F_1}{\sigma w}\right) + \left(\frac{\sigma F_2}{\sigma t} + \frac{\sigma F_2}{\sigma w}\right)\right). \tag{17}$$

Here, σF_1 and σF_2 are calculated using (5) with respect to the uncertain parameters.

To find $T^{(i)}$ for different regions using (14), $a = [0, 5.02, 0.24, 0]$ is approximated accordingly (Section 3.1), where $\beta = 1.5$, $n = 2$ and $A = \frac{1}{100}$ (10). Having scaled $\mathscr{R}_{\mathscr{F}}$, denoted by R, TRF is then piecewisely constructed for 50% robust frontier using (14). This is shown below with {0, 40, 50, 60, 100} representing the boundaries for each region.

$$TRF(R) = \begin{cases} 1 & \text{R} \le 40 \\ T^{(2)} & 40 < \text{R} \le 50 \\ T^{(3)} & 50 < \text{R} \le 60 \\ T^{(4)} & 60 < \text{R} \le 100 \\ T^{(5)} & \text{R} > 100 \end{cases}$$

Therefore, including the above $TRF(R)$ in the problem as the new objective function and using the possibilistic mean value of the uncertain parameters, we can generate the appropriate robust frontier. By changing the L_R value, one can find the new TRF and obtain the desired robust frontier accordingly. The generated Pareto and robust Pareto frontiers are shown in Fig. 5.

As can be seen, the robust frontiers displace inside the feasible space with respect to the true Pareto frontier. All these solutions are less sensitive to the parameters variations. However, as Fig. 5 shows, the design should be chosen with more value of $mass$ and less value of $deflection$. The reason is that the robust solution in this region (in bottom right of Fig. 5) deviates from the Pareto frontier less than the other design candidates. Thus, these solutions are not only near optimal but also robust. It is worth noting that, even before optimization, it is intuitive to design these trusses with large $mass$ for obtaining less $deflection$. It is equivalent to the choice of a solution from the indicated region in Fig. 5, as it exactly follows from the optimization formulation. The other thing to be noted is the extent of the deviation of the 100% robust frontier from the 50-and 0% frontiers. Although, using the new method, a robust solution with any desired level can be found, it is advised to choose a robustness degree by which the robust solutions are closer to the Pareto solutions. This selection simply gives the trade-off between optimality and robustness. Therefore, for the test case in question, it is better to choose the 50% robust frontier. In fact, these candidates are all *at least* 50% robust with respect to the constructed TRF objective function.

4.2 Case 2: welded beam design

In this optimization problem (Rao 1996), a beam is to be welded to a large case tolerating the force of P = (6000, 54.8, 54.8) lb (Fig. 6). The welding is to be done with the minimum cost (F_1) and the minimum deflection (F_2) of the beam. The shear stress in weld (τ), bending stress in the beam (σ) and the buckling load on the bar (Pc) are the design constraints adopted from Rao (1996). The design variables of the problem are the beam thickness (b), the beam width (t), the length of the welded joint (l) and the weld thickness (h); and L = (14,3.5,3.5) in is the uncertain fuzzy parameter. The design formulation is as follows:

$$Minimize F_1 = Cost = 1.105h^2l + 0.048tb(14 + l),$$
$$Minimize F_2 = Deflection = \delta = \frac{4PL^3}{Et^3b},$$
$$\begin{aligned} s.t.\ \ & \tau \le 13600, \\ & \sigma \le 30000, \\ & h \le b, \\ & Pc \ge P, \\ & 0.125 \le b \le 5, \quad 0.1 \le t \le 10, \\ & 0.1 \le l \le 10, \quad 0.125 \le h \le 5, \end{aligned}$$

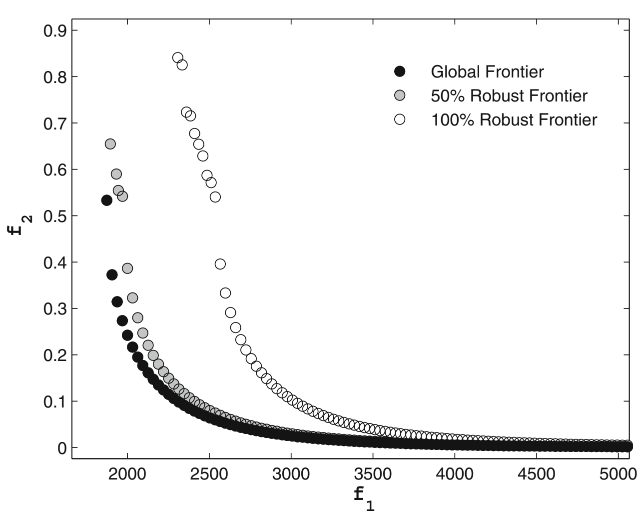

Fig. 5 Pareto solution for two bar truss design

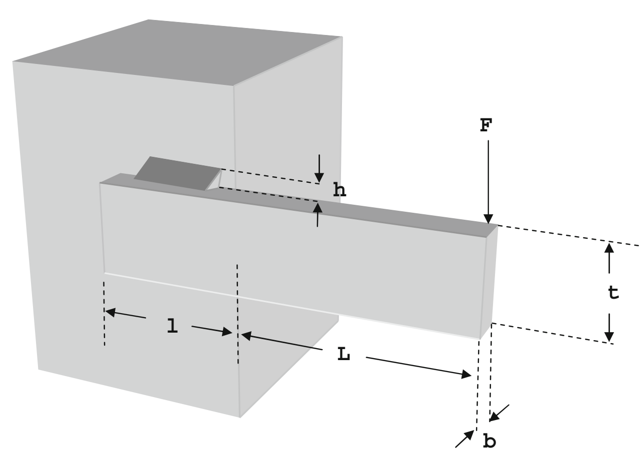

Fig. 6 Welded beam design

where

$$\sigma = \frac{6PL}{t^2 b},$$
$$Pc = 64746.02(1-0.3t)tb^3,$$
$$\tau = \sqrt{\tau'^2 + \tau''^2 + l\tau' \frac{\tau''}{\sqrt{0.25(l^2+(h+t)^2)}}},$$
$$\tau'' = P(14+0.5l)\frac{\sqrt{0.25(l^2+(h+t)^2)}}{2(0.71hl(l^2/12+0.25(h+t)^2))},$$
$$\tau' = \frac{P}{\sqrt{2}hl},$$
$$P = (6000, 54.8, 54.8)lb,\ L = (14, 3.5, 3.5)in,$$
$$E = 30e6psi.$$

To find the robust solutions, using the procedure from the first test case, the TRF is constructed and Step 6 of the Algorithm is formulated. In Fig. 7, the generated 50% and 100% robust frontiers are shown. As discussed before, all the robust solutions are situated between the non-dominated frontier and 100% robust one. These generated solutions provide useful information to the designer. As can be seen, the least sensitive part of Pareto frontier corresponds to the best solution. The solutions in this region are optimum and meanwhile show less variation in the presence of uncertainty. A designer may choose these solutions provided that there are enough resources. It can be seen that, using the TRF as well as the fuzzy variance of 500 and 2, respectively for P and L, an appropriate design can be made. Furthermore, it is obvious that the acceptance of the cost between 12 to 16 pounds does not bring any significant change to both optimal and 100% robust deflection (F_2). Thus, incurring the cost in the given range, the designer can obtain an optimal structure, which is utmost in robustness.

In addition, as can be seen in Fig. 7, some of the solutions degrade more as the robustness increases from 50% to 100% than it does from 0% to 50%. Hence, the designer can accept the robustness range of 50% as the designs in this range are closer to optimal. For the same trade-off analysis in welded beam design, one may refer to the paper by Amarchinta and Grandhi (2008), where the designer's preferences are also taken into account.

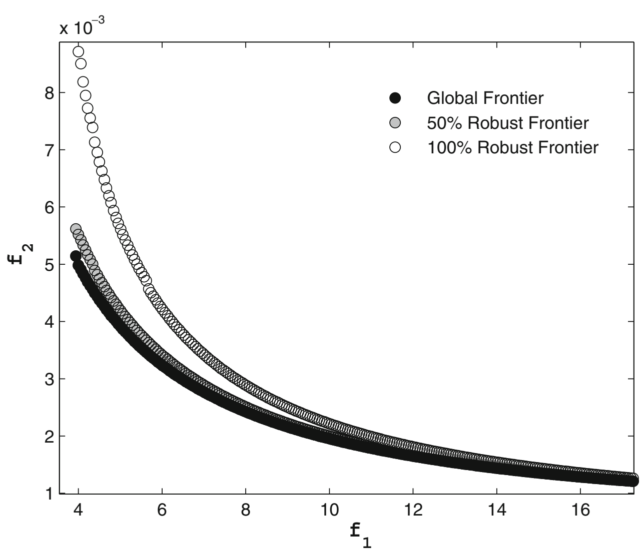

Fig. 7 Design of welded beam

4.3 Case 3: pressure vessel design

In this test case, the optimization consists of finding the best design for a cylindrical vessel with two hemispherical heads at both ends (Kannan and Kramer 1994). As shown in Fig. 8, $x_1 = T_s$ (shell thickness), $x_2 = T_h$ (head thickness), $x_3 = R$ (inner radius) and $x_4 = L$ (vessel cylindrical section length without the head) are to be optimized in order to be applied for the working pressure of 3,000 psi and the minimum volume of 750 $feet^3$. The problem is formulated as follows:

$$Minimize\ \ F_1 = 2\pi DCsx_1x_3x_4 + 2\pi DChx_2x_3^2 + 3.17x_1^2x_4 + 19.84x_1^2x_3,$$
$$Minimize\ \ F_2 = -x_1 - x_2,$$
$$s.t.\ \ 0.02x_3 \leq x_1,$$
$$0.00954x_3 \leq x_2,$$
$$\pi x_3^2 x_4 + \frac{4}{3}\pi x_3^3 \geq 750 \times 1728,$$
$$x_4 \leq 240,$$
$$1.1 \leq x_1 \leq 2, \quad 0.6 \leq x_2 \leq 2,$$
$$0 \leq x_3 \leq 100, \quad 0 \leq x_4 \leq 240,$$

where

$$D = 0.284,\ Cs = \text{ï£¡}0.35/lb,\ Ch = \text{ï£¡}1/lb.$$

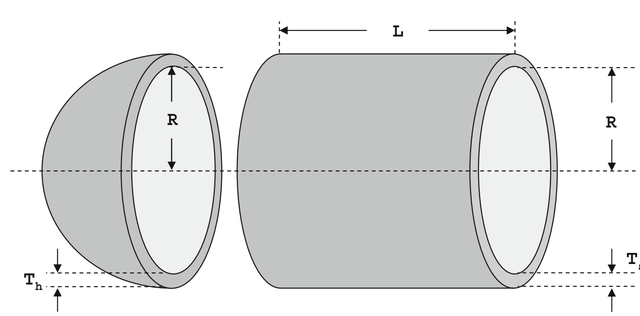

Fig. 8 Design of a pressure vessel

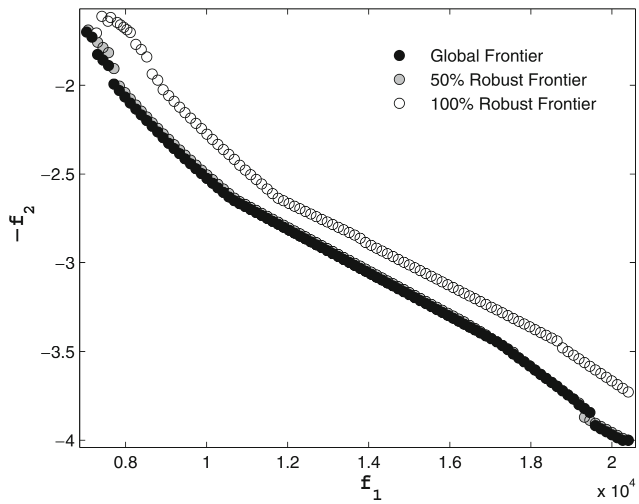

Fig. 9 Design of pressure vessel

where D is the density of carbon steel from the ASME codes, Cs = (0.35,1.55,1.55) is the cost of rolled plate and Ch = (1,1.9,1.9) donates the cost of forged plates as the fuzzy parameters. Using the same side constraints as proposed by Kannan and Kramer (1994), the thickness of the shell is not to be less than 1.1 inches and the thickness of the head is not to be less than 0.6 inches. The variance of C_s and C_h are calculated as 0.4 and 0.6, respectively (see the Appendix).

One objective maximizes the thickness of the vessel shell and the other keeps the cost of vessel construction as low as possible. The maximization of the shell guarantees the minimization of the chance of fracture. In Fig. 9, the global frontier is shown for this problem. In addition, using the same detailed approach reported for the first test case, it is illustrated that the 50%-generated robust frontier is close enough to the global frontier. Thus, the solutions are not sensitive even when the robustness changes from 0% to 50%. However, as the robustness changes to its extreme of 100%, the solutions shift into the feasible space and distant from the global frontier without any overlap. For the example in question, the bounds for the robustness is explicitly shown in the Fig. 9.

5 Conclusion

The current paper has introduced a flexible robust design methodology in uncertain environment. The method allows the designer to obtain a desired robust design via implementation of one additional objective function to the problem. Although any kind of uncertainty can be handled, we have implemented the fuzzy value uncertainty on the parameters. Using the mean and variance of fuzzy parameters, an uncertain formulation is converted to a deterministic formulation. To realize the robust design, a robust measure is developed, and a tunable robust function (TRF) is constructed. The TRF maps the proposed robust measure into a dimensionless positive convex function. The shape of the TRF depends on the level of robustness the designer demands. The proposed approach is tested on engineering problems and the results are promising. It is shown that we are able to set an upper bound for robust frontier using the proposed approach. All the other robust solutions, with any level of robustness, are situated between the introduced frontiers.

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Appendix: Fuzzy numbers

The α-cut of a fuzzy number is defined as follows.

Definition 3 (α-cut of fuzzy set) If $\tilde{A}$ is a fuzzy set, the crisp set of the elements:

$$\tilde{A_\alpha} = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\},$$

is called α-cut of $\tilde{A}$. Here, $\mu_{\tilde{A}}(x)$ is the membership function of the fuzzy number A.

As an example, suppose A is a *triangular* fuzzy number shown by membership function $A = (a, b, c)$, where b and c are the left-width and right-width of the fuzzy number centered at a. Therefore, the α-cut of A is defined by

$$A_\alpha = [a - (1-\alpha)b\ ,\ a + (1-\alpha)c], \quad \forall \alpha \in [0, 1].$$

Definition 4 (Possibilistic Mean value of fuzzy number) Following the paper by Carlsson and Fuller (2001), if A is a fuzzy number, its possibilistic mean value is the arithmetic mean of its lower and upper possibilistic mean value, i.e.

$$Mean(A) = \frac{\underline{M}(A) + \overline{M}(A)}{2},$$

where

$$\underline{M}(A) = 2\int_0^1 \alpha \underline{A} d\alpha, \qquad \overline{M}(A) = 2\int_0^1 \alpha \overline{A} d\alpha,$$

and $\overline{A}$ and $\underline{A}$ are the lower and upper bounds of α-cut of fuzzy number A, respectively.

Definition 5 (Variance of fuzzy number) If A is a fuzzy number, the variance of it is defined as (Carlsson and Fuller 2001)

$$Var(A) = \frac{1}{2}\int_0^1 \alpha(\overline{A} - \underline{A})^2 d\alpha.$$

If A is a *triangular* fuzzy number shown by $A = (a, b, c)$, it is easy to prove that the possibilistic mean value of the fuzzy number is given by

$$M(A) = a + \frac{c-b}{6},$$

and also its variance is calculated as

$$Var(A) = \frac{(b+c)^2}{24}.$$

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