Multi-Level Monte Carlo Simulations with Importance Sampling

Multi-Level Monte Carlo Simulations with Importance Sampling

Przemyslaw Stan Stilger and Ser-Huang Poon*

March 21, 2014

Abstract

We present an application of importance sampling to multi-asset options under the Heston and the Bates models as well as to the Heston-Hull-White and the Heston-Cox-Ingersoll-Ross models. Moreover, we provide an efficient importance sampling scheme in a Multi-Level Monte Carlo simulation. In all cases, we explain how the Greeks can be computed in the different simulation schemes using the Likelihood Ratio Method, and how combining it with importance sampling leads to a significant variance reduction for the Greeks.

Keywords: Importance sampling; Simulation; Stochastic volatility

*Przemyslaw Stan Stilger (przemyslaw.stilger@postgrad.mbs.ac.uk) and Ser-Huang Poon (ser-huang.poon@mbs.ac.uk) are both at Manchester Business School, Crawford House, University of Manchester, Oxford Road, Manchester, M13 9PL, United Kingdom. Stilger gratefully acknowledges financial support from the ESRC.

1 Introduction

In practice, the valuation of multi-asset options typically involves the Monte Carlo simulation. The rate of convergence of this simulation is $\sqrt{n}$ where n is the number of sample paths. Hence, improving the accuracy of the simulation by a factor of 10 requires 100 times as many sample paths. For this reason, variance reduction techniques have become essential. Importance sampling reduces the variance by changing the drift of the simulated sample paths. The extent to which variance reduction is achieved through importance sampling very much depends on the change of drift. Much research effort focuses on how to change the drift to fully exploit the variance reduction potential of importance sampling.

The Multi-Level Monte Carlo was introduced in Giles (2008). It is a Monte Carlo simulation performed on different levels of uniform time discretizations. The main advantage of the Multi-Level Monte Carlo is that, for a given accuracy, it has lower computational cost due to reduced variance compared to the basic Monte Carlo. Here, we show that the variance of Multi-Level Monte Carlo can be further reduced by combining it with other variance reduction technique such as importance sampling.

In this paper, we focus on importance sampling for multi-asset options and incorporating importance sampling in the Multi-Level Monte Carlo simulation. Our contributions are as follows. First, we present an application of importance sampling with a stochastic change of drift to multi-asset options. Next, we provide an efficient importance sampling scheme in a Multi-Level Monte Carlo simulation. Then, we combine Multi-Level Monte Carlo with importance sampling to price multi-asset options. In all cases, we explain how the Greeks can be computed in the different simulation schemes using the Likelihood Ratio Method, and combine it with the importance sampling to reduce the variance of the Greeks.

There is relatively little work on variance reduction for multi-asset options in the literature. Barraquand (1995) introduces quadratic resampling and combines it with the importance sampling to price European multi-asset options. Avramidis (2002) proposes an algorithm that selects the importance sampling density as a mixture of multivariate Normal densities for best-of Asian and best-of barrier options. Neddermeyer (2011) develops non-parametric importance sampling in conjunction with quasi-random numbers to price basket and best-of options. The work of Barraquand (1995), Avramidis (2002), as well as Neddermeyer (2011) is done is done under the Black-Scholes model.

Su and Fu (1999), Arouna (2004), and Caprotti (2008) use importance sampling, with the optimal change of drift obtained by solving an optimization problem, to price basket options. In Su and Fu (1999) the change of drift based on a stochastic optimization. In Arouna (2004) the change of drift relies on the Robbins-Monro algorithms, whereas in Caprotti (2008), it depends on the least squares minimization. Finally, Pellizzari (1998) suggests the use of control variate based on unconditional and conditional expectations of asset prices as a variance reduction technique for multi-asset options in the Black-Scholes model.

The remainder of this paper is organized as follows. In Section 2, we present an application of importance sampling with a stochastic change of drift to multi-asset options in the Heston stochastic volatility model and the Bates stochastic volatility model with jumps. We consider basket, best-of, worst-of, spread, absolute, composite, and quotient options. In Section 3, we extend the Likelihood Ratio Method to multi-asset options and combine it with the importance sampling to reduce the variance of the Greeks. In Section 4, we derive the optimal change of drift for the Heston-Hull-White model and the Heston-Cox-Ingersoll-Ross model. In Section 5, apply importance sampling in a Multi-Level Monte Carlo using the Heston-Hull-White model and the Heston-Cox-Ingersoll-Ross model. We demonstrate that applying importance sampling only on the first level can significantly improve the effective performance of the Multi-Level Monte Carlo. In Section 6, we use the Likelihood Ratio Method to estimate the Greeks in a Multi-Level Monte Carlo and again combine it with the importance sampling to reduce the variance of the Greeks. In Section 7, we combine Multi-Level Monte Carlo with importance sampling for multi-asset options. Finally, Section 8 concludes the paper.

2 Importance Sampling for Multi-Asset Options

In this section, we apply importance sampling to price multi-asset options. The dynamics of the multi-asset Heston model under the risk neutral measure $\mathbb{Q}$ is given by

$$\begin{aligned} dS_{i,t} &= r_i S_{i,t} dt + \sqrt{v_{i,t}} S_{i,t} dW_{i,t}^{S_i} \\ dv_{i,t} &= \kappa_i \left(\theta_i - v_{i,t}\right) dt + \xi_i \sqrt{v_{i,t}} dW_{i,t}^{v_i} \end{aligned}$$

where $S_{i,t}$ is the i-th stock price, r_i is the i-th risk-free interest rate, $v_{i,t}$ is the i-th variance, κ_i is the i-th mean-reversion rate, θ_i is the i-th long-term variance, ξ_i is the i-th volatility of volatility, and $i = 1, \cdots, n$ denotes the number of underlying assets. The correlation matrix is

$$C = \begin{bmatrix} C_1 & C_2 \\ C_2^\top & C_3 \end{bmatrix} \tag{1}$$

where

$$C_1 = \begin{bmatrix} \rho_{1,1} & \cdots & \rho_{1,n} \\ \vdots & \ddots & \vdots \\ \rho_{n,1} & \cdots & \rho_{n,n} \end{bmatrix}$$

is the correlation between the stock price processes,

$$C_2 = \begin{bmatrix} \rho_{1,n+1} & \cdots & \rho_{1,2n} \\ \vdots & \ddots & \vdots \\ \rho_{n,n+1} & \cdots & \rho_{n,2n} \end{bmatrix}$$

is the correlation between the stock price processes and the variance processes, and

$$C_3 = \begin{bmatrix} \rho_{n+1,n+1} & \cdots & \rho_{n+1,2n} \\ \vdots & \ddots & \vdots \\ \rho_{2n,n+1} & \cdots & \rho_{2n,2n} \end{bmatrix}$$

is the correlation between the variance processes.

The difference between the multi-asset Heston model and the multi-asset Bates model is that in the multi-asset Bates model the stock price dynamics under the risk-neutral measure $\mathbb{Q}$ becomes

$$dS_{i,t} = S_{i,t} \left(r_i - \lambda_i \bar{k}_i \right) dt + S_{i,t} \sqrt{v_{i,t}} dW_{i,t}^{S_i} + S_{i,t} dZ_{i,t}$$

where $Z_{i,t}$ is a compound Poisson process with intensity λ_i and log-normal distribution of jump sizes such that if k_i is its jump size then $\ln(1 + k_i) \sim \mathcal{N}\left(\ln\left(1 + \bar{k}_i\right) - \frac{1}{2}\delta_i^2, \delta_i^2\right)$.

In matrix notation, the dynamics of the multi-asset Heston model is

$$dX_t = b(X_t) dt + a(X_t) d\eta_t \tag{2}$$

where $C = \Sigma\Sigma^\top$ is the correlation matrix in (1), η_t is a $2n$-dimensional correlated $\mathbb{Q}$-Brownian motion and

$$
\begin{aligned}
dX_t &= \begin{pmatrix} S_{1,t} \\ \vdots \\ S_{n,t} \\ v_{1,t} \\ \vdots \\ v_{n,t} \end{pmatrix} \\
b(x) &= \begin{pmatrix} r_1 s_1 \\ \vdots \\ r_n s_n \\ \kappa_1(\theta_1 - v_1) \\ \vdots \\ \kappa_n(\theta_n - v_n) \end{pmatrix} \\
a(x) &= \begin{pmatrix} \sqrt{v_1} s_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \sqrt{v_n} s_n & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \xi_1 \sqrt{v_1} & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \xi_n \sqrt{v_n} \end{pmatrix} \\
\eta_t &= \begin{pmatrix} W_{1,t}^{S_1} \\ \vdots \\ W_{n,t}^{S_n} \\ W_{1,t}^{v_1} \\ \vdots \\ W_{n,t}^{v_n} \end{pmatrix}
\end{aligned}
$$

Following Fouque and Tullie (2002), we derive the optimal change of measure for the multi-asset Heston model. First, we introduce the martingale

$$H_t = \exp\left(\int_0^T \Sigma^{-1} h(s, X_s) \cdot \Sigma^{-1} d\eta_t + \frac{1}{2} \int_0^T \Sigma^{-1} h(s, X_s) \cdot \Sigma^{-1} h(s, X_s) ds \right) \tag{3}$$

Next, we define a new probability measure denoted by $\tilde{\mathbb{Q}}$ which is equivalent to $\mathbb{Q}$ by its Radon-

Nikodym derivative

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = (H_T)^{-1}$$

By the Girsanov theorem for correlated Brownian motions, the process

$$\tilde{\eta}_t = \eta_t + \int_0^t h(s, X_s)\, d\eta_s$$

is a $2n$-dimensional correlated $\tilde{\mathbb{Q}}$-Brownian motion. Using $\tilde{\eta}_t$, (2) and (3) can be written as

$$\begin{aligned} dX_t &= (b(X_t) - a(X_t)\, h(t, X_t))\, dt + a(X_t)\, d\tilde{\eta}_t \\ H_t &= \exp\left(\int_0^T \Sigma^{-1} h(s, X_s) \cdot \Sigma^{-1} d\tilde{\eta}_t - \frac{1}{2}\int_0^T \Sigma^{-1} h(s, X_s) \cdot \Sigma^{-1} h(s, X_s) ds\right) \end{aligned}$$

Using the analogous derivation to that presented in [13], the optimal choice of h for which the variance of the Monte Carlo estimator under $\tilde{\mathbb{Q}}$ is minimized is

$$h(t, X_t) = -\frac{1}{P(t, X_t)} a(t, X_t)^{\top} \nabla P(t, X_t) \tag{4}$$

This result is also valid for the Bates model with the difference that

$$a(x) = \begin{pmatrix} \left(\sqrt{v_1} + \frac{dZ_{1,t}}{dW_{1,t}^{S_1}}\right) s_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \left(\sqrt{v_n} + \frac{dZ_{n,t}}{dW_{n,t}^{S_n}}\right) s_n & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \xi_1 \sqrt{v_1} & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \xi_n \sqrt{v_n} \end{pmatrix}$$

Equation (4) requires the option price and its delta which are not known. Instead, we will use their Black-Scholes equivalents. Under the fast mean-reversion expansion, h for the multi-asset Heston model is given by

$$h_i = -\frac{1}{P_{FMR}} \begin{pmatrix} s_i \sqrt{v_i} \frac{\partial P_{FMR}}{\partial s} \\ 0 \end{pmatrix}$$

where P_{FMR} is the option price under the classic geometric Brownian motion dynamics with volatility $\sqrt{\sum_{i=1}^N \theta_i^2 - 2\sum_{1 \le i < j \le N}^N \rho_{i,j} \sqrt{\theta_i \theta_j}}$.

Similarly, under the fast mean-reversion expansion, h for the multi-asset Bates model is given by

$$h_i = -\frac{1}{P_{FMR}} \begin{pmatrix} s_i \left(\sqrt{v_i} + \frac{dZ}{dW^S} \right) \frac{\partial P_{FMR}}{\partial s} \\ 0 \end{pmatrix}$$

where P_{FMR} is the option price under the classic geometric Brownian motion dynamics with volatility $\sqrt{\sum_{i=1}^{N} \theta_i^2 - 2 \sum_{1 \le i < j \le N}^{N} \rho_{i,j} \sqrt{\theta_i \theta_j}}$.

2.1 Numerical Examples

In this section, we present the numerical results for spread, absolute, composite, quotient, basket, best-of, and worst-of options. We compared option prices simulated under the importance sampling using fast mean-reversion expansion (MC+IS) against the basic Monte Carlo (MC). All simulations are performed using the same sequence of pseudo-random numbers. We simulate 10,000 sample paths using a time increment of 0.001. For the numerical examples, we assume that the time to maturity is 1 year. For the Bates model, we assume in addition that the jump intensity is 1 jump per year, standard deviation of the jumps is 2%, and the mean jump size is -5%.

2.1.1 Spread, Absolute, Composite, and Quotient Options

Here, we consider options written on two underlying assets. Spread option depends on the difference between two underlying assets. Seller of such an option is long correlation which differentiates this option from the majority of multi-asset options that leave the seller short correlation. For example, the payoff of the spread call with maturity T is given by

$$\max\left(S_1(T) - S_2(T) - K, 0\right)$$

Absolute option is an option written on the absolute value of the difference between the two underlying assets at maturity. The holder of an absolute option benefits from the absolute change in price of the underlying assets. For example, the payoff of the spread call with maturity T is given by

$$\max\left(\max\left(S_1(T), S_2(T)\right) - \min\left(S_1(T), S_2(T)\right) - K, 0\right)$$

Composite option is an option on a foreign underlying asset with a strike denominated in the domestic currency. The holder of a composite option faces foreign exchange risk, but benefits from the strike being fixed in the domestic currency. For example, the payoff of the composite call with maturity T is given by

$$\max\left(S_1(T) S_2(T) - K, 0\right)$$

where $S_2(T)$ is the foreign exchange rate.

Quotient option, also known as ratio option, depends on the ratio of two underlying assets. The holder of a quotient option benefits from the relative change in price of the underlying assets. For example, the payoff of the quotient call with maturity T is given by

$$\max\left(\frac{S_1(T)}{S_2(T)} - K, 0\right)$$

The parameters used in the numerical examples are displayed in Table 1.

i	S	r	v_0	ξ	κ	θ
Spread						
1	30	0.05	0.04	0.4	3	0.09
2	5	0.05	0.09	0.3	0.5	0.25
Absolute						
1	30	0.05	0.04	0.4	3	0.09
2	35	0.05	0.09	0.3	0.5	0.25
Composite / Quotient						
1	30	0.05	0.04	0.4	3	0.09
2	2	0.05	0.09	0.3	0.5	0.25

Table 1: Model parameters for multi-asset options based on two underlying assets.

The correlation matrix is given by

$$\begin{bmatrix} 1 & 0.4 & -0.6 & -0.28 \\ 0.4 & 1 & -0.24 & -0.7 \\ -0.6 & -0.24 & 1 & 0.168 \\ -0.28 & -0.7 & 0.168 & 1 \end{bmatrix}$$

Tables 2 and 3 report the results for basic Monte Carlo (MC) and importance sampling (IS) for the Heston model and the Bates model. In all cases, importance sampling reduces the variance 5 to 13 times on average compared to the basic Monte Carlo.

Moneyness		0.7	0.8	0.9	1	1.1	1.2	1.3
Panel A: Spread								
Heston								
Price	MC	10.9116	8.7664	6.8078	5.0821	3.6278	2.4616	1.5897
	MC+IS	10.8962	8.7438	6.7701	5.0316	3.5795	2.4298	1.5730
Variance	MC	51.5956	47.0730	40.6259	32.9627	25.0319	17.7671	11.7748
	MC+IS	3.4245	3.7729	2.9216	1.7977	1.3673	1.0649	0.7246
Bates								
Price	MC	10.9352	8.8015	6.8525	5.1433	3.7056	2.5509	1.6764
	MC+IS	10.9055	8.7716	6.8252	5.1218	3.6918	2.5521	1.6944
Variance	MC	54.2643	49.5032	42.8706	34.9473	26.7347	19.1973	12.9460
	MC+IS	5.4703	4.7225	4.1768	3.4911	2.5868	1.9515	1.4214
Panel B: Absolute								
Heston								
Price	MC	0.8110	0.4772	0.2775	0.1561	0.0860	0.0470	0.0260
	MC+IS	0.8157	0.4738	0.2703	0.1517	0.0837	0.0457	0.0241
Variance	MC	8.4177	4.9509	2.8182	1.5635	0.8538	0.4603	0.2410
	MC+IS	4.3064	1.9271	0.9153	0.4107	0.1753	0.0765	0.0256
Bates								
Price	MC	0.8988	0.5351	0.3091	0.1733	0.0947	0.0509	0.0271
	MC+IS	0.8731	0.5180	0.3015	0.1726	0.0973	0.0530	0.0289
Variance	MC	9.2814	5.4544	3.0895	1.6995	0.9159	0.4850	0.2541
	MC+IS	3.9397	1.8787	0.8458	0.3651	0.1514	0.0554	0.0207

Table 2: Monte Carlo (MC) and Importance Sampling (MC+IS) price and variance of price for spread and absolute options based on two underlying assets under the Heston model and the Bates model.

Moneyness		0.7	0.8	0.9	1	1.1	1.2	1.3
Panel A: Composite								
Heston								
Price	MC	32.6826	28.1321	23.9924	20.2822	17.0000	14.1175	11.6163
	MC+IS	32.4569	27.8948	23.7450	20.0351	16.7593	13.9067	11.4605
Variance	MC	958.7243	888.7356	807.4117	719.3844	629.3578	541.8094	459.6539
	MC+IS	153.3319	155.1247	150.6481	140.7796	127.2177	111.4294	94.7747
Bates								
Price	MC	32.7633	28.2312	24.1100	20.4207	17.1358	14.2648	11.7747
	MC+IS	32.6670	28.1174	23.9634	20.2731	17.0256	14.2011	11.7800
Variance	MC	989.9160	918.5872	835.9709	746.4860	655.8963	567.2028	483.7828
	MC+IS	210.3686	203.9079	189.1749	177.2651	162.4286	145.2400	127.3492
Panel B: Quotient								
Heston								
Price	MC	7.2402	5.9534	4.8020	3.8233	3.0250	2.3880	1.8832
	MC+IS	7.2059	5.9192	4.7583	3.7677	2.9560	2.3126	1.8077
Variance	MC	43.3758	41.5555	38.6297	34.8073	30.5444	26.3034	22.3933
	MC+IS	3.9368	3.6635	3.5537	3.3123	2.9475	2.4886	2.0209
Bates								
Price	MC	7.2886	6.0117	4.8691	3.8986	3.0977	2.4557	1.9474
	MC+IS	7.2540	5.9816	4.8495	3.8777	3.0840	2.4433	1.9326
Variance	MC	44.9052	42.9381	39.8828	35.9234	31.5830	27.2593	23.2442
	MC+IS	6.9516	8.6092	10.8222	11.3142	12.2593	11.4338	8.0210

Table 3: Monte Carlo (MC) and Importance Sampling (MC+IS) price and variance of price for composite and quotient options based on two underlying assets under the Heston model and the Bates model.

2.1.2 Basket, Best-of, and Worst-of Options

Here, we consider options written on three underlying assets. The payoff of a basket option depends on the performance of a basket of underlying assets, each with its own corresponding weight. The weights w_i must satisfy the constraints $0 \leq w_i \leq 1$ for all $i = 1, \cdots, n$ and $\sum_{i=1}^{n} w_i = 1$. For example, the payoff of the basket call with maturity T is given by

$$\max\left(w_1 S_1(T), \cdots, w_n S_n(T) - K, 0\right)$$

The main advantage of a basket option is that it offers a greater flexibility in the construction of the underlying basket and it is usually cheaper than buying vanilla options on each of the underlying assets. Basket option is mainly used for diversification purposes.

Best-of option depends on the performance of the best performing asset in a basket. For example, the payoff of the best-of call with maturity T is given by

$$\max\left(\max\left(S_1(T), \cdots, S_n(T)\right) - K, 0\right)$$

Best-of call has a higher upside potential compared to a call option on the same basket of underlying assets.

Worst-of option depends on the performance of the worst performing asset in a basket. For example, the payoff of the worst-of call with maturity T is given by

$$\max\left(\min\left(S_1(T), \cdots, S_n(T)\right) - K, 0\right)$$

Worst-of call has a lower upside potential compared to a call option on the same basket of underlying assets.

The parameters used in the numerical examples are displayed in Table 4.

i	w	S	r	v_0	ξ	κ	θ
Basket							
1	50%	70	0.05	0.04	0.4	3	0.09
2	30%	35	0.05	0.09	0.3	0.5	0.25
3	20%	40	0.05	0.25	0.2	5	0.04
Best-of and Worst-of							
1		30	0.05	0.04	0.4	3	0.09
2		35	0.05	0.09	0.3	0.5	0.25
3		40	0.05	0.25	0.2	5	0.04

Table 4: Model parameters for multi-asset options based on three underlying assets.

The correlation matrix is given by

$$\begin{bmatrix} 1 & 0.4 & 0.2 & -0.6 & -0.28 & -0.1 \\ 0.4 & 1 & 0.5 & -0.24 & -0.7 & -0.25 \\ 0.2 & 0.5 & 1 & 0.0282 & -0.35 & -0.5 \\ -0.6 & -0.24 & 0.0282 & 1 & 0.168 & 0.0294 \\ -0.28 & -0.7 & -0.35 & 0.168 & 1 & 0.175 \\ -0.1 & -0.25 & -0.5 & 0.0294 & 0.175 & 1 \end{bmatrix}$$

Table 5 reports the results for basic Monte Carlo (MC) and importance sampling (IS) for the Heston model and the Bates model. In all cases, importance sampling reduces the variance 3 to 5 times on average compared to the basic Monte Carlo.

Moneyness		0.7	0.8	0.9	1	1.1	1.2	1.3
Panel A: Basket								
Heston								
Price	MC	18.0939	13.4929	9.4321	6.1308	3.6583	2.0060	0.9916
	MC+IS	18.1301	13.5388	9.4302	6.0855	3.6288	1.9740	0.9759
Variance	MC	131.7882	116.7452	93.7079	66.5853	41.7007	22.8492	11.1104
	MC+IS	22.8241	27.9146	15.9959	11.1225	10.7502	5.6949	1.8306
Bates								
Price	MC	18.0972	13.5079	9.4722	6.1935	3.7289	2.0761	1.0521
	MC+IS	18.0389	13.4853	9.4724	6.2015	3.7453	2.0880	1.0719
Variance	MC	135.8888	120.4728	96.8471	69.1762	43.8538	24.5189	12.2881
	MC+IS	31.0928	26.9779	21.0307	14.3609	8.3187	4.1190	1.7322
Panel B: Best-of								
Heston								
Price	MC	20.0806	16.8117	13.6660	10.7524	8.1722	5.9940	4.2450
	MC+IS	20.0837	16.8275	13.6950	10.7276	8.1459	5.9882	4.2398
Variance	MC	114.2491	112.0612	106.5591	96.5682	82.5957	66.5238	50.5769
	MC+IS	49.6236	40.2004	31.9864	24.6266	19.9598	15.4075	11.2659
Bates								
Price	MC	20.2257	16.9608	13.8165	10.8969	8.3088	6.1219	4.3675
	MC+IS	20.1805	16.9499	13.8360	10.8988	8.3327	6.1667	4.4003
Variance	MC	117.5027	115.1506	109.5496	99.5909	85.5577	69.3061	53.0277
	MC+IS	49.5691	40.9534	33.3357	26.6619	21.6881	16.7958	12.4169
Panel C: Worst-of								
Heston								
Price	MC	5.0870	3.1251	1.7429	0.8760	0.3895	0.1544	0.0590
	MC+IS	5.1117	3.1181	1.7279	0.8628	0.3849	0.1524	0.0533
Variance	MC	32.2608	21.3585	12.1857	5.9854	2.5606	0.9838	0.3423
	MC+IS	13.8514	8.7862	4.7116	2.1009	0.7885	0.2509	0.0701
Bates								
Price	MC	5.0152	3.0794	1.7277	0.8753	0.3964	0.1649	0.0615
	MC+IS	5.0174	3.0583	1.6931	0.8489	0.3835	0.1561	0.0580
Variance	MC	32.4943	21.5437	12.3510	6.1418	2.6990	1.0523	0.3754
	MC+IS	14.3138	8.9745	4.7930	2.1609	0.8268	0.2723	0.0798

Table 5: Monte Carlo (MC) and Importance Sampling (MC+IS) price and variance of price for basket, best-of, and worst-of options based on three underlying assets under the Heston model and the Bates model.

3 Greeks for Multi-Asset Options

We begin with an option price under $\mathbb{Q}$ defined as

$$P(t,x) = \int_0^\infty \cdots \int_0^\infty e^{-r(T-t)} \phi\left(S_1(T), \cdots, S_n(T)\right) f(x_1, \cdots, x_n) dx_1 \cdots dx_n$$

where $\phi\left(S_1(T), \cdots, S_n(T)\right)$ is the payoff function and $f(x_1, \cdots, x_n)$ is the joint risk-neutral probability density function.

Next, consider, delta Δ, the first derivative of the option price with respect to $S_1(0)$

$$\begin{aligned}\Delta &= \frac{\partial}{\partial S_1(0)} \int_0^\infty \cdots \int_0^\infty B\phi(S_1(T), \cdots, S_n(T)) f(x_1, \cdots, x_n) dx_1 \cdots dx_n \\ &= \int_0^\infty \cdots \int_0^\infty B\phi(S_1(T), \cdots, S_n(T)) \frac{\frac{\partial}{\partial S_1(0)} f(x_1, \cdots, x_n)}{f(x_1, \cdots, x_n)} f(x_1, \cdots, x_n) dx_1 \cdots dx_n\end{aligned}$$

where $B = e^{-r(T-t)}$ and $\frac{\frac{\partial}{\partial S_1(0)} f(x_1, \cdots, x_n)}{f(x_1, \cdots, x_n)}$ is the likelihood ratio. By Sklar's Theorem there exists a copula C such that

$$F(x_1, \cdots, x_n) = C(F_1(x_1), \cdots, F_n(x_n)) = C(u_1, \cdots, u_n) \tag{5}$$

In [13], we showed that the cumulative distribution function (CDF) and the probability density function (PDF) for both the Heston model and the Bates model can be obtained as

$$\begin{aligned}F_1(x_1) &= Pr(S_1(T) \leq x_1) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty Re\left[\frac{\exp(-i\omega \ln(x_1)) \psi_T(\omega)}{i\omega}\right] d\omega \\ f_1(x_1) &\approx \frac{F_1(x_1 + \Delta x) - F_1(x_1)}{\Delta x}\end{aligned}$$

where ψ is the characteristic function. Taking n^{th} order differentiation of (5) gives an expression for the joint density.

$$f(x_1, \cdots, x_n) = \prod_{i=1}^{n} f_i(x_i) c(u_1, \cdots, u_n)$$

where $c(u_1, \cdots, u_n) = \frac{\partial^n C(u_1, \cdots, u_n)}{\partial u_1 \cdots \partial u_n}$. In order to estimate the Greeks we will use an analytical copula as an approximation of copula in (5). The same procedure as above can be followed to compute the other Greeks.

3.1 Numerical Examples

In this section, we present delta, Δ, and gamma, Γ, of the multi-asset Heston model and the multi-asset Bates model calculated using Likelihood Ratio Method (MC) and Likelihood Ratio Method combined with importance sampling (MC+IS). We consider basket call option described in Section 2.1.2. As an approximation of the joint PDF for the Heston model we will use the t-copula with 62 degrees of freedom and correlation matrix

$$\begin{bmatrix} 1 & 0.4 & 0.2 \\ 0.4 & 1 & 0.5 \\ 0.2 & 0.5 & 1 \end{bmatrix} \tag{6}$$

As an approximation of the joint PDF for the Bates model we will use the t-copula with 30 degrees of freedom and the correlation matrix given by (6). For both models the correlation matrix and the number of degrees of freedom were estimated using maximum likelihood.

Table 6 presents results for delta and gamma, respectively, of a basket call option for the multi-asset Heston model. On average importance sampling reduces the variance of delta and gamma by factor of 2.

i	Moneyness		0.7	0.8	0.9	1	1.1	1.2	1.3
1	Delta	MC	0.4850	0.4572	0.4070	0.3346	0.2500	0.1678	0.1005
		MC+IS	0.4832	0.4528	0.4005	0.3290	0.2499	0.1702	0.1045
	Variance	MC	2.9080	2.0993	1.4687	0.9938	0.6445	0.3937	0.2223
		MC+IS	1.9176	1.1636	0.6470	0.3326	0.2544	0.1250	0.0439
	Gamma	MC	0.0009	0.0025	0.0048	0.0071	0.0084	0.0083	0.0067
		MC+IS	0.0026	0.0039	0.0060	0.0079	0.0089	0.0086	0.0071
	Variance	MC	0.0378	0.0291	0.0217	0.0157	0.0108	0.0070	0.0042
		MC+IS	0.0262	0.0174	0.0107	0.0061	0.0033	0.0018	0.0009
2	Delta	MC	0.2873	0.2671	0.2329	0.1870	0.1366	0.0906	0.0524
		MC+IS	0.2865	0.2673	0.2365	0.1910	0.1352	0.0874	0.0501
	Variance	MC	7.8665	5.3811	3.4908	2.1292	1.2105	0.6341	0.3048
		MC+IS	6.5965	4.1291	2.3213	1.1894	0.6826	0.2735	0.0868
	Gamma	MC	0.0115	0.0093	0.0077	0.0065	0.0054	0.0042	0.0027
		MC+IS	0.0169	0.0144	0.0120	0.0100	0.0079	0.0055	0.0036
	Variance	MC	0.3224	0.2279	0.1536	0.0977	0.0581	0.0322	0.0164
		MC+IS	0.2865	0.1852	0.1098	0.0587	0.0298	0.0127	0.0046
3	Delta	MC	0.1886	0.1728	0.1506	0.1219	0.0897	0.0583	0.0340
		MC+IS	0.1932	0.1789	0.1593	0.1315	0.0995	0.0689	0.0430
	Variance	MC	5.4444	3.6204	2.2803	1.3568	0.7634	0.4083	0.2102
		MC+IS	4.0029	2.3606	1.2534	0.6082	0.2816	0.1127	0.0388
	Gamma	MC	0.0082	0.0063	0.0047	0.0036	0.0029	0.0024	0.0020
		MC+IS	0.0103	0.0081	0.0062	0.0047	0.0034	0.0025	0.0017
	Variance	MC	0.1294	0.0883	0.0576	0.0360	0.0218	0.0130	0.0078
		MC+IS	0.0928	0.0554	0.0301	0.0150	0.0079	0.0032	0.0011

Table 6: Likelihood Ratio Method (MC) and Likelihood Ratio Method with Importance Sampling (MC+IS) Greeks (delta, variance of delta, gamma, and variance of gamma) for a basket call under the Heston model.

Table 7 presents results for delta and gamma, respectively, of a basket call option for the multi-asset Bates model. On average importance sampling reduces the variance of delta and gamma by factor of 2.

i	Moneyness		0.7	0.8	0.9	1	1.1	1.2	1.3
1	Delta	MC	0.4845	0.4567	0.4066	0.3347	0.2512	0.1706	0.1044
		MC+IS	0.4769	0.4489	0.3976	0.3292	0.2488	0.1722	0.1092
	Variance	MC	2.8944	2.1049	1.4874	1.0193	0.6713	0.4182	0.2421
		MC+IS	1.9074	1.1758	0.6710	0.3565	0.1826	0.0906	0.0463
	Gamma	MC	0.0028	0.0040	0.0060	0.0080	0.0091	0.0087	0.0071
		MC+IS	0.0040	0.0055	0.0071	0.0089	0.0100	0.0094	0.0078
	Variance	MC	0.0379	0.0295	0.0223	0.0163	0.0114	0.0076	0.0047
		MC+IS	0.0242	0.0168	0.0105	0.0063	0.0036	0.0020	0.0011
2	Delta	MC	0.2933	0.2708	0.2342	0.1862	0.1364	0.0906	0.0529
		MC+IS	0.2812	0.2598	0.2284	0.1803	0.1296	0.0833	0.0477
	Variance	MC	7.8325	5.3818	3.5098	2.1576	1.2400	0.6602	0.3245
		MC+IS	6.5213	4.1040	2.3413	1.2146	0.5691	0.2377	0.0889
	Gamma	MC	0.0174	0.0142	0.0115	0.0093	0.0072	0.0053	0.0033
		MC+IS	0.0227	0.0199	0.0163	0.0132	0.0098	0.0072	0.0048
	Variance	MC	0.3207	0.2294	0.1565	0.1009	0.0608	0.0340	0.0175
		MC+IS	0.2523	0.1679	0.1049	0.0561	0.0279	0.0123	0.0051
3	Delta	MC	0.1722	0.1588	0.1390	0.1136	0.0831	0.0543	0.0317
		MC+IS	0.1701	0.1611	0.1442	0.1206	0.0939	0.0655	0.0404
	Variance	MC	5.2603	3.4916	2.1943	1.3011	0.7289	0.3872	0.1981
		MC+IS	3.8632	2.2887	1.2276	0.6000	0.2648	0.1058	0.0389
	Gamma	MC	0.0096	0.0071	0.0050	0.0035	0.0025	0.0019	0.0014
		MC+IS	0.0094	0.0075	0.0057	0.0041	0.0032	0.0021	0.0013
	Variance	MC	0.1184	0.0801	0.0515	0.0315	0.0185	0.0105	0.0059
		MC+IS	0.0781	0.0472	0.0258	0.0129	0.0061	0.0026	0.0009

Table 7: Likelihood Ratio Method (MC) and Likelihood Ratio Method with Importance Sampling (MC+IS) Greeks (delta, variance of delta, gamma, and variance of gamma) for a basket call under the Bates model.

4 Importance Sampling for Heston with Stochastic Interest Rates

Here, we consider models with stochastic volatility and stochastic interest rates. The dynamics of the Heston-Hull-White model and the Heston-Cox-Ingersoll-Ross model under the risk neutral measure $\mathbb{Q}$ is given by

$$\begin{aligned} dS_t &= r_t S_t dt + \sqrt{v_t} S_t dW_t^S \\ dv_t &= \kappa\left(\bar{v} - v_t\right) dt + \gamma \sqrt{v_t} dW_t^v \\ dr_t &= \lambda\left(\theta_t - r_t\right) dt + \eta r_t^p dW_t^r \end{aligned}$$

where $\left\langle dW_t^S dW_t^v \right\rangle = \rho_{S,v} dt$, $\left\langle dW_t^S dW_t^r \right\rangle = \rho_{S,r} dt$, and $\langle dW_t^r dW_t^v \rangle = 0$. S_t is the stock price, r_t is the risk-free interest rate, v_t is the variance, κ is the variance mean-reversion rate, $\bar{v}$ is the long-term variance, γ is the volatility of volatility, λ is the interest rate mean-reversion rate, θ_t is the long-term interest rate, η is the volatility of interest rate, $\rho_{S,v}$ is the correlation between stock returns and changes in the variance, and $\rho_{S,r}$ is the correlation between stock returns and changes in the interest rate. If $p = 0$, we have the Heston-Hull-White model and if $p = 0.5$, we have the Heston-Cox-Ingersoll-Ross model.

In matrix notation, the model dynamics is

$$dX_t = b(X_t)\,dt + a(X_t)\,d\eta_t \tag{7}$$

where η_t is a 3-dimensional $\mathbb{Q}$-Brownian motion and

$$\begin{aligned}
dX_t &= \begin{pmatrix} S_t \\ v_t \\ r_t \end{pmatrix} \\
b(x) &= \begin{pmatrix} rs \\ \kappa(\bar{v} - v) \\ \lambda(\theta_t - r) \end{pmatrix} \\
a(x) &= \begin{pmatrix} \sqrt{v}s & 0 & 0 \\ \gamma\sqrt{v}\rho_{S,v} & \gamma\sqrt{v\left(1-\rho_{S,v}^2\right)} & 0 \\ \eta r^p \rho_{S,r} & \eta r^p \frac{-\rho_{S,v}\rho_{S,r}}{\sqrt{\left(1-\rho_{S,v}^2\right)}} & \eta r^p \sqrt{1 - \left(\rho_{S,r}^2 + \frac{\rho_{S,v}^2 \rho_{S,r}^2}{1-\rho_{S,v}^2}\right)} \end{pmatrix} \\
\eta_t &= \begin{pmatrix} W_t^S \\ W_t^v \\ W_t^r \end{pmatrix}
\end{aligned}$$

Following Fouque and Tullie (2002), we derive the optimal change of measure for the Heston-Hull-White model and the Heston-Cox-Ingersoll-Ross model. First, we introduce the martingale

$$H_t = \exp\left(\int_0^T h^\top(s, X_s)\,d\eta_s + \frac{1}{2}\int_0^T \|h(s, X_s)\|^2\,ds\right) \tag{8}$$

where $h^\top$ denotes the transpose of h. Next, we define a new probability measure denoted by $\tilde{\mathbb{Q}}$ which is equivalent to $\mathbb{Q}$ by its Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = (H_T)^{-1}$$

By the Girsanov theorem, the process

$$\tilde{\eta}_t = \eta_t + \int_0^t h\left(s, X_s\right) d\eta_s$$

is a 3-dimensional $\tilde{\mathbb{Q}}$-Brownian motion. Using $\tilde{\eta}_t$, (7) and (8) can be written as

$$\begin{aligned} dX_t &= \left(b\left(X_t\right) - a\left(X_t\right) h\left(t, X_t\right)\right) dt + a\left(X_t\right) d\tilde{\eta}_t \\ H_t &= \exp\left(\int_0^T h^\top (s, X_s) d\tilde{\eta}_t - \frac{1}{2}\int_0^T ||h(s, X_s)||^2 ds\right) \end{aligned}$$

Using the analogous derivation to that presented in [13], the optimal choice of h for which the variance of the Monte Carlo estimator under $\tilde{\mathbb{Q}}$ is minimized is

$$h(t, X_t) = -\frac{1}{P(t, X_t)} a(t, X_t)^\top \nabla P(t, X_t) \tag{9}$$

From (9), under the fast mean-reversion expansion, h is given by

$$h = -\frac{1}{P_{FMR}} \begin{pmatrix} s\sqrt{\bar{v}}\frac{\partial P_{FMR}}{\partial s} \\ 0 \end{pmatrix}$$

where P_{FMR} is the option price under the classic geometric Brownian motion dynamics with volatility $\sqrt{\bar{v}}$.

5 Multi-Level Monte Carlo with Importance Sampling

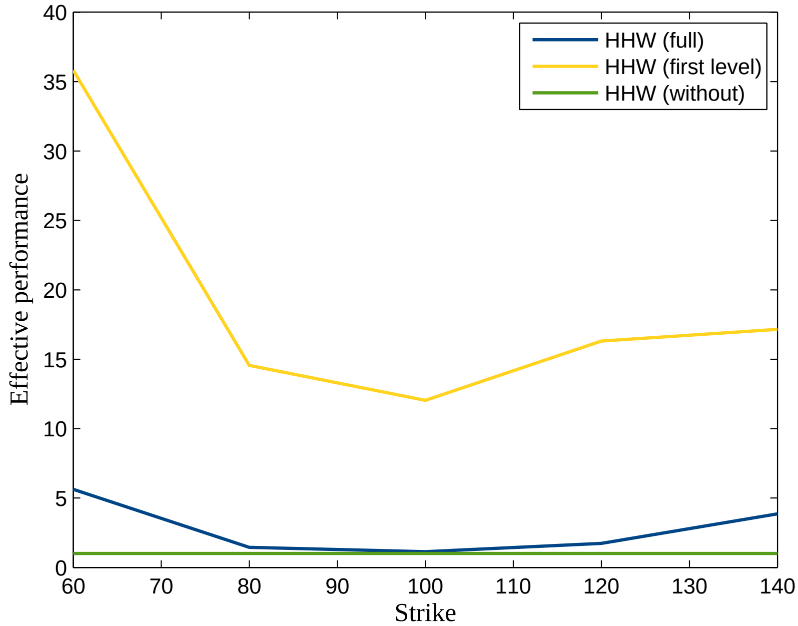
Multi-Level Monte Carlo is a Monte Carlo simulation with different number of time steps $h_l = 2^{-l}T$ on each level $l = 0, 1, \cdots, L$. For detailed discussion of Multi-Level Monte Carlo we refer to the Appendix. To illustrate the performance of Multi-Level Monte Carlo with the importance sampling, we will use the Heston-Hull-White model and the Heston-Cox-Ingersoll-Ross model described in the previous section. We compare the performance of the Multi-Level Monte Carlo with the importance sampling (MLMC+IS) and without importance sampling (MLMC) for the European call option against the semi-analytical solution (HHW/HCIR). The parameters are set as follows: $\kappa = 2$, $\gamma = 0.06$, $v_0 = \bar{v} = 0.04$, $\lambda = 0.05$, $r_0 = \theta = 0.07$, $\eta = 0.01$, $S_0 = 100$, $\rho_{S,v} = -0.3$, $\rho_{S,r} = 0.2$, $T = 1$. We set $\epsilon = 0.01$ and $L = 8$. We consider five strikes: 60, 80, 100, 120, and 140. Table 8 presents the price, variance, and relative error of the European call. The relative error is measured against the semi-analytical solution.

Strike		60	80	100	120	140
Heston-Hull-White						
Price	HHW	44.0682	26.0077	11.5943	3.7583	0.9221
	MLMC	44.0918	26.0258	11.6100	3.7636	0.9219
	MLMC+IS	44.0879	26.0256	11.6085	3.7648	0.9229
Variance	MLMC	413.4086	379.9792	238.5803	90.0735	23.6667
	MLMC+IS	44.9006	157.6493	124.6953	31.2749	3.4079
Relative error (%)	MLMC	0.05	0.07	0.13	0.14	0.02
	MLMC+IS	0.04	0.07	0.12	0.17	0.08
Heston-Cox-Ingersoll-Ross						
Price	HCIR	44.0686	25.9996	11.5668	3.7296	0.9071
	MLMC	44.0724	26.0067	11.5696	3.7320	0.9037
	MLMC+IS	44.0769	26.0045	11.5693	3.7326	0.9041
Variance	MLMC	413.1371	379.8149	238.3690	90.1313	23.6259
	MLMC+IS	43.8495	159.4084	121.6297	29.2775	3.5179
Relative error (%)	MLMC	0.01	0.03	0.02	0.06	0.37
	MLMC+IS	0.02	0.02	0.02%	0.08	0.33

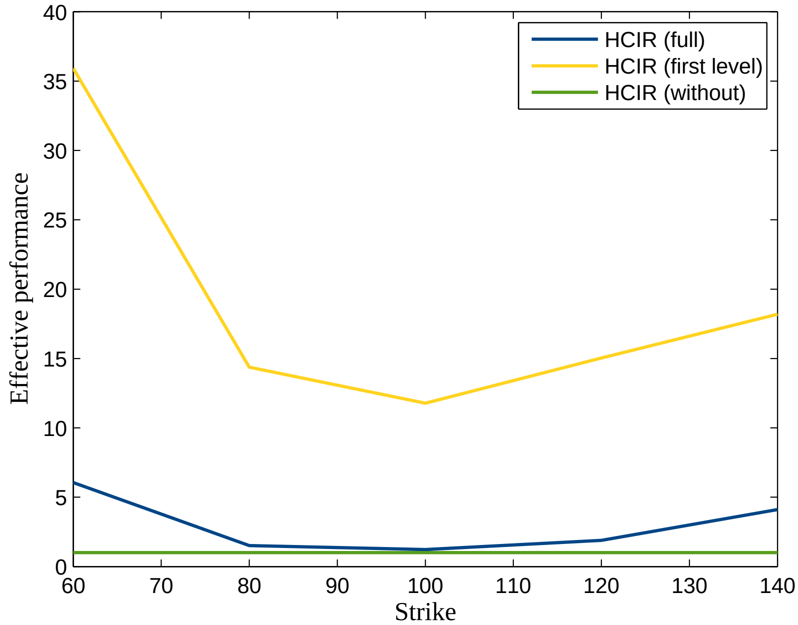
Table 8: Multi-Level Monte Carlo (MLMC) and Importance Sampling (MLMC+IS) price, variance of price, and relative error for a European call under the Heston-Hull-White model and the Heston-Cox-Ingersoll-Ross model. Relative error is measured against the semi-analytical solution (HHW/HCIR).

Given the nature of the Multi-Level Monte Carlo, we observe that it is possible to use importance sampling on all levels or some levels. We will refer to the former as full importance sampling. We note that the first level, where $l = 0$, is the coarsest, because there is only one step at this level with step size T. Variance at level l decreases as l increases because both P_{l-1} and P_l accurately approximate P as they are obtained using the same Brownian path. Therefore, as an alternative to the full importance sampling, we will consider importance sampling on the first level only.

Figure 1 plots the effective performance against strikes for the different simulation schemes. Effective performance is defined as the ratio of variance reduction to speed. Speed itself is defined as the ratio of computational time of the Multi-Level Monte Carlo with the importance sampling to computational time of the Multi-Level Monte Carlo without importance sampling. Figure 1 compares Multi-Level Monte Carlo with full importance sampling, Multi-Level Monte Carlo with the importance sampling on the first level only, and Multi-Level Monte Carlo without importance sampling. The results indicate that the Multi-Level Monte Carlo with the importance sampling is more efficient than Multi-Level Monte Carlo without importance sampling. In addition, Multi-Level Monte Carlo with the importance sampling on the first level only is much more efficient than both Multi-Level Monte Carlo without importance sampling and Multi-Level Monte Carlo with full importance sampling. The performance improvement, compared to the Multi-Level Monte Carlo full importance sampling comes from two sources. The first one is variance reduction from importance sampling; the second is reduced computational time. This is due to the fact that the number of sample paths at level l which is given by (13) depends on the variance at level l. Since importance sampling reduces the variance at the first level, the required number of sample paths at this level is less, compared to the Multi-Level Monte Carlo without importance sampling.



(a) Heston-Hull-White



(b) Heston-Cox-Ingersoll-Ross

Figure 1: Effective performance for different strikes.

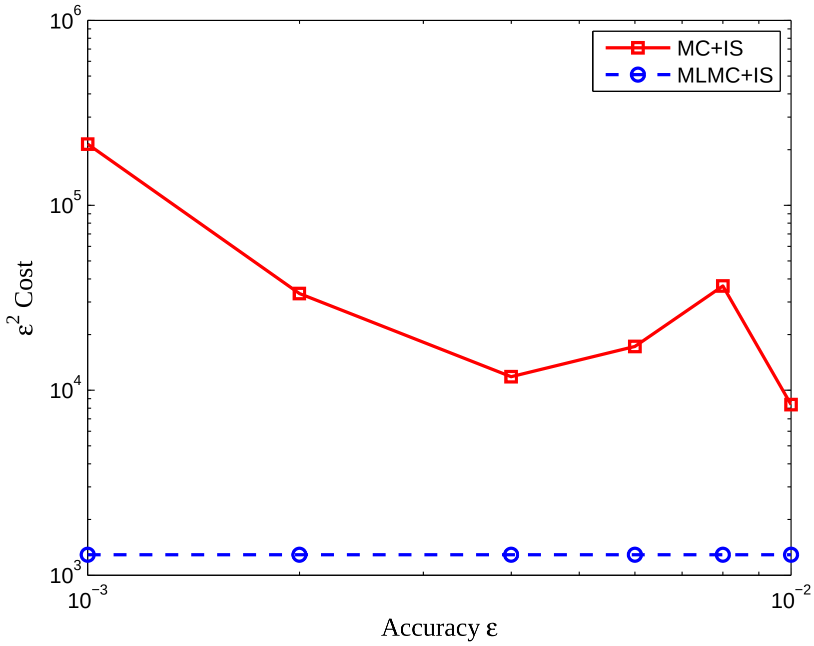
So far, we have demonstrated that importance sampling improves the efficiency of both basic Monte Carlo and Multi-Level Monte Carlo. It has been also shown by Giles (2008) that Multi-Level Monte Carlo is more efficient than basic Monte Carlo. To complete the picture, we compare basic Monte Carlo with Importance Sampling and Multi-Level Monte Carlo with Importance Sampling. Figure 2 compares the computational cost associated with the desired accuracy, ϵ, for basic Monte Carlo with Importance Sampling (MC+IS) and Multi-Level Monte Carlo with Importance Sampling on the first level (MLMC+IS). For Multi-Level Monte Carlo, the computational cost, C, is defined as the total number of time steps on all levels. For each sample path at level $l > 0$, there is one fine path with 2^l time steps and one coarse path with 2^{l-1} time steps. Hence,

$$C = N_0 + \sum_{l=1}^{L} N_l \left(2^l + 2^{l-1}\right)$$

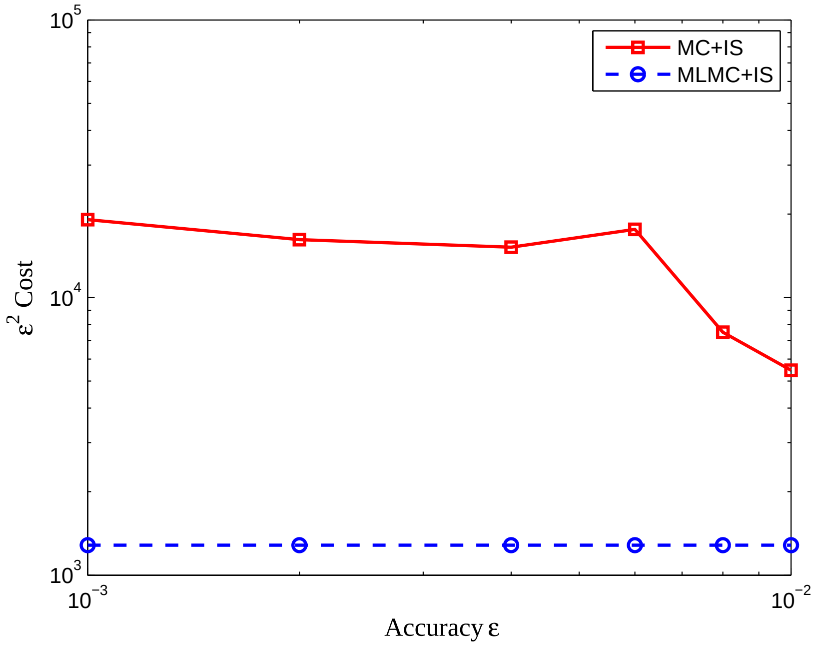
The computational of the basic Monte Carlo is calculated as

$$C^* = \sum_{l=0}^{L} N_l^* 2^l$$

where $N_l^* = 2\epsilon^{-2} Var\left[\phi_l\left(S\right)\right]$ so that the variance of the basic Monte Carlo estimator is also $\frac{1}{2}\epsilon^2$. Figure 2 shows that for a given accuracy, Multi-Level Monte Carlo with Importance Sampling on the first level has lower computational cost compared to the basic Monte Carlo with Importance Sampling. For Multi-Level Monte Carlo, $\epsilon^2 C$ is roughly constant which is consistent with theory that predicts computational cost of order ϵ^{-2}. For basic Monte Carlo, $\epsilon^2 C$ is approximately proportional to ϵ^{-1}, which is in line with the theoretical cost of order ϵ^{-3}.



(a) Heston-Hull-White



(b) Heston-Cox-Ingersoll-Ross

Figure 2: Computational cost associated with the desired accuracy (ϵ) for basic Monte Carlo with Importance Sampling (MC+IS) and Multi-Level Monte Carlo with Importance Sampling on the first level (MLMC+IS).

6 Greeks for Multi-Level Monte Carlo

It is also possible to use the Likelihood Ratio Method to estimate the Greeks in a Multi-Level Monte Carlo. Let us consider the first derivative of the Multi-Level Monte Carlo estimator (10) with respect to S_0.

$$\begin{aligned}
\Delta &= \frac{\partial}{\partial S_0}\mathbb{E}\left[\phi_L\left(S\right)\right] \\
&= \frac{\partial}{\partial S_0}\mathbb{E}\left[\phi_0\left(S\right)\right] + \sum_{l=1}^{L}\frac{\partial}{\partial S_0}\mathbb{E}\left[\phi_l\left(S\right) - \phi_{l-1}\left(S\right)\right] \\
&= \int_0^\infty e^{-r(T-t)}\phi_0\left(s\right)\frac{\frac{\partial}{\partial S_0}f(x)}{f(x)}f(x)dx + \\
&+ \sum_{l=1}^{L}\int_0^\infty e^{-r(T-t)}\left(\left[\phi_l\left(s\right) - \phi_{l-1}\left(s\right)\right]\right)\frac{\frac{\partial}{\partial S_0}f(x)}{f(x)}f(x)dx
\end{aligned}$$

where $\frac{\frac{\partial}{\partial S_0}f(x)}{f(x)}$ is the likelihood ratio which can be obtained from the characteristic function as in [13].

Table 9 reports delta and gamma computed by the Likelihood Ratio Method with the importance sampling (IS) and without importance sampling (L) for European option under the Heston-Hull-White model and the Heston-Cox-Ingersoll-Ross model. On average importance sampling reduces the variance of delta 8 times and the variance of gamma 7 times.

Strike		60	80	100	120	140
Heston-Hull-White						
Delta	MLMC	0.9595	0.9187	0.6691	0.3227	0.1068
	MLMC+IS	0.9621	0.9172	0.6690	0.3235	0.1058
Variance	MLMC	8.5492	4.7461	2.5166	1.1986	0.4324
	MLMC+IS	4.3355	1.1883	0.2358	0.0869	0.0379
Gamma	MLMC	0.0001	0.0042	0.0165	0.0175	0.0092
	MLMC+IS	0.0001	0.0042	0.0164	0.0174	0.0092
Variance	MLMC	0.0569	0.0370	0.0219	0.0119	0.0055
	MLMC+IS	0.0264	0.0108	0.0035	0.0011	0.0005
Heston-Cox-Ingersoll-Ross						
Delta	MLMC	0.9570	0.9179	0.6698	0.3231	0.1062
	MLMC+IS	0.9577	0.9195	0.6686	0.3224	0.1057
Variance	MLMC	8.6794	4.8099	2.5586	1.2152	0.4403
	MLMC+IS	4.4050	1.2070	0.2374	0.0861	0.0350
Gamma	MLMC	0.0001	0.0041	0.0165	0.0176	0.0092
	MLMC+IS	0.0002	0.0042	0.0164	0.0175	0.0091
Variance	MLMC	0.0591	0.0383	0.0227	0.0123	0.0057
	MLMC+IS	0.0271	0.0112	0.0036	0.0011	0.0005

Table 9: Likelihood Ratio Method (MLMC) and Likelihood Ratio Method with Importance Sampling (MLMC+IS) Greeks (delta, variance of delta, gamma, and variance of gamma) for a European call under the Heston-Hull-White model and the Heston-Cox-Ingersoll-Ross model.

7 Multi-Level Monte Carlo with Importance Sampling for Multi-Asset Options

Finally, we will use the Multi-Level Monte Carlo to price basket call on three underlying assets. We will use the same parameters as in Section 2 and $\lambda = 0.05$, $r_0 = \theta = 0.05$, $\eta = 0.01$, $\rho_{S,v} = -0.3$, $\rho_{S,r} = 0.2$, $\rho_{r,v} = 0$. We set $\epsilon = 0.05$ and $L = 8$. We note that a combination of Multi-Level Monte Carlo and hybrid stochastic volatility model such as Heston-Hull-White or Heston-Cox-Ingersoll-Ross is particularly suitable for pricing variable annuities which are in principle long-dated basket put options.

Table 10 reports the results for Multi-Level Monte Carlo with the importance sampling (MLMC+IS) and without importance sampling (MLMC) for a basket call on three underlying assets under the Heston-Hull-White model and the Heston-Cox-Ingersoll-Ross model. On average importance sampling reduces the variance 3 times compared to the Multi-Level Monte Carlo without importance sampling.

Moneyness		0.7	0.8	0.9	1	1.1	1.2	1.3
Heston-Hull-White								
Price	MLMC	17.9506	13.2379	9.1427	5.8453	3.3797	1.7681	0.8636
	MLMC+IS	17.9671	13.2424	9.1135	5.7541	3.3249	1.8456	0.9217
Variance	MLMC	123.9666	114.0966	94.7747	68.4085	43.3949	24.5796	12.3358
	MLMC+IS	41.6504	40.1116	33.2675	22.6223	12.5556	6.4589	2.4066
Heston-Cox-Ingersoll-Ross								
Price	MLMC	17.9789	13.2578	9.2080	5.8265	3.3508	1.7697	0.9293
	MLMC+IS	17.9381	13.1990	9.0608	5.7880	3.3921	1.8451	0.9390
Variance	MLMC	123.7446	114.3640	95.0163	68.7593	43.3255	24.1792	12.3348
	MLMC+IS	41.3532	40.3292	33.2685	22.9515	12.6570	6.1357	2.5758

Table 10: Multi-Level Monte Carlo (MLMC) and Importance Sampling (MLMC+IS) price and variance of price for a basket call under the Heston-Hull-White model and the Heston-Cox-Ingersoll-Ross model.

Tables 11 and 12 report delta and gamma of each underlying asset computed by the Likelihood Ratio Method with the importance sampling (IS) and without importance sampling (L) under the Heston-Hull-White model and the Heston-Cox-Ingersoll-Ross model. On average importance sampling reduces the variance of delta and gamma by factor of 3.

i	Moneyness		0.7	0.8	0.9	1	1.1	1.2	1.3
Heston-Hull-White									
1	Delta	MLMC	0.3564	0.3421	0.2985	0.2464	0.1807	0.1122	0.0639
		MLMC+IS	0.3562	0.3256	0.2956	0.2404	0.1776	0.1201	0.0680
	Variance	MLMC	2.1769	1.5899	1.1284	0.7723	0.4973	0.3056	0.1876
		MLMC+IS	1.3601	0.8669	0.5247	0.3050	0.1676	0.0910	0.0362
2	Delta	MLMC	0.5109	0.4553	0.3840	0.3004	0.2275	0.1624	0.1099
		MLMC+IS	0.5040	0.4634	0.3802	0.2979	0.2200	0.1698	0.1174
	Variance	MLMC	11.6235	8.5318	6.0153	4.2465	2.8229	1.7758	1.1957
		MLMC+IS	8.5810	5.7738	3.7107	2.0393	1.1347	0.6621	0.2517
3	Delta	MLMC	0.9084	0.8531	0.7083	0.6401	0.4732	0.3304	0.2159
		MLMC+IS	0.8841	0.8444	0.7431	0.6094	0.4847	0.3504	0.2311
	Variance	MLMC	38.4584	31.8220	25.7602	16.5950	15.8343	11.4764	8.1404
		MLMC+IS	22.4692	13.6315	12.6515	5.0691	3.3503	2.6284	0.6424
Heston-Cox-Ingersoll-Ross									
1	Delta	MLMC	0.3658	0.3407	0.3069	0.2444	0.1787	0.1158	0.0769
		MLMC+IS	0.3542	0.3327	0.3008	0.2462	0.1831	0.1182	0.0779
	Variance	MLMC	2.2403	1.6174	1.1399	0.7857	0.5084	0.3250	0.1949
		MLMC+IS	1.3789	0.8831	0.5337	0.3191	0.1751	0.0910	0.0530
2	Delta	MLMC	0.5209	0.4610	0.3938	0.3160	0.2363	0.1538	0.1066
		MLMC+IS	0.5021	0.4704	0.3833	0.3010	0.2218	0.1693	0.1113
	Variance	MLMC	11.9429	8.9477	6.1634	4.3160	2.8600	1.6796	1.0906
		MLMC+IS	8.6689	5.9153	3.7445	2.1748	1.1615	0.5881	0.3708
3	Delta	MLMC	0.9053	0.8298	0.7488	0.6466	0.4987	0.3463	0.2317
		MLMC+IS	0.9361	0.8567	0.7586	0.5951	0.5186	0.3498	0.2279
	Variance	MLMC	39.3159	29.1621	22.8414	16.5218	10.2947	5.8761	3.4462
		MLMC+IS	19.0239	14.6707	12.4880	3.7593	3.6790	0.7555	0.4583

Table 11: Likelihood Ratio Method (MLMC) and Likelihood Ratio Method with Importance Sampling (MLMC+IS) Greeks (delta and variance of delta) for a basket call under the Heston-Hull-White model and the Heston-Cox-Ingersoll-Ross model.

i	Moneyness		0.7	0.8	0.9	1	1.1	1.2	1.3
Heston-Hull-White									
1	Gamma	MLMC	-0.0208	-0.0147	-0.0096	-0.0033	0.0007	0.0022	0.0023
		MLMC+IS	-0.0219	-0.0158	-0.0089	-0.0034	0.0007	0.0026	0.0022
	Variance	MLMC	0.0190	0.0144	0.0110	0.0079	0.0053	0.0035	0.0025
		MLMC+IS	0.0111	0.0075	0.0047	0.0030	0.0018	0.0010	0.0005
2	Gamma	MLMC	0.0078	0.0145	0.0157	0.0195	0.0190	0.0157	0.0135
		MLMC+IS	0.0073	0.0157	0.0162	0.0173	0.0183	0.0174	0.0135
	Variance	MLMC	0.4305	0.3349	0.2444	0.1860	0.1268	0.0882	0.0640
		MLMC+IS	0.2910	0.2022	0.1398	0.0784	0.0477	0.0302	0.0117
3	Gamma	MLMC	0.2474	0.2027	0.1676	0.1338	0.0959	0.0687	0.0449
		MLMC+IS	0.2430	0.2081	0.1605	0.1269	0.0970	0.0718	0.0481
	Variance	MLMC	2.9352	1.9736	1.5644	0.7162	0.4576	0.3758	0.1866
		MLMC+IS	0.7470	1.2402	0.8445	0.1511	0.0870	0.0505	0.0206
Heston-Cox-Ingersoll-Ross									
1	Gamma	MLMC	-0.0205	-0.0156	-0.0091	-0.0038	0.0007	0.0029	0.0030
		MLMC+IS	-0.0212	-0.0154	-0.0086	-0.0032	0.0008	0.0023	0.0037
	Variance	MLMC	0.0204	0.0148	0.0111	0.0081	0.0057	0.0038	0.0025
		MLMC+IS	0.0112	0.0078	0.0049	0.0031	0.0019	0.0010	0.0010
2	Gamma	MLMC	0.0104	0.0150	0.0166	0.0197	0.0211	0.0163	0.0125
		MLMC+IS	0.0082	0.0165	0.0197	0.0171	0.0178	0.0170	0.0145
	Variance	MLMC	0.4728	0.3633	0.2544	0.1871	0.1365	0.0780	0.0546
		MLMC+IS	0.2982	0.2181	0.1403	0.0890	0.0502	0.0256	0.0223
3	Gamma	MLMC	0.2529	0.2013	0.1657	0.1364	0.0953	0.0777	0.0465
		MLMC+IS	0.2488	0.2099	0.1667	0.1265	0.1011	0.0683	0.0523
	Variance	MLMC	2.2489	1.8205	1.3654	1.2104	0.4645	0.3950	0.1975
		MLMC+IS	0.9539	0.4543	0.2702	0.2800	0.0928	0.1580	0.0188

Table 12: Likelihood Ratio Method (MLMC) and Likelihood Ratio Method with Importance Sampling (MLMC+IS) Greeks (gamma and variance of gamma) for a basket under the Heston-Hull-White model and the Heston-Cox-Ingersoll-Ross model.

8 Conclusion

We have presented an application of importance sampling with stochastic change of drift to multi-asset options. We have illustrated the use of importance sampling with spread, absolute, composite, quotient, basket, best-of, and worst-of options as examples. Based on our results, importance sampling reduces variance of multi-asset options by a factor of 3-13 on average.

The paper has also provided an extension of the Likelihood Ratio Method to multi-asset options, and combined it with the importance sampling to reduce the variance of the Greeks. Based on our results, importance sampling reduces variance of the Greeks of multi-asset options by a factor of 2 on average.

We applied importance sampling in a Multi-Level Monte Carlo and have demonstrated that applying importance sampling on the first level significantly improves its effective performance. For the European option in the Multi-Level Monte Carlo with full importance sampling the effective performance is on average almost 3 times better than that of the Multi-Level Monte Carlo without importance sampling. For the same option in the Multi-Level Monte Carlo with the importance sampling on the first level only the effective performance is on average almost 19 times better than that of the Multi-Level Monte Carlo without importance sampling. We have also used the Likelihood Ratio Method to estimate the Greeks in a Multi-Level Monte Carlo, and combined it with the importance sampling to reduce the variance of the Greeks. Based on our results, importance sampling reduces variance of the Greeks by a factor of 7-8 on average for the European option and by a factor of 3 for a basket option.

Appendix A. Multi-Level Monte Carlo

Multi-Level Monte Carlo is a Monte Carlo simulation with different number of time steps of size $h_l = 2^{-l}T$ on each level $l = 0, 1, \cdots, L$. For example, when $l = 0$, there is only one time step of size $h_0 = T$. When $l = 1$, there are two times steps each of size $h_1 = \frac{T}{2}$. Finally, when $l = L$, there are 2^L times steps each of size $h_L = \frac{T}{2^L}$.

Let P denote the true derivative price

$$P = \mathbb{E}\left[\phi_L\left(S\right)\right] = \mathbb{E}\left[\phi_0\left(S\right)\right] + \sum_{l=1}^{L} \mathbb{E}\left[\phi_l\left(S\right) - \phi_{l-1}\left(S\right)\right] \tag{10}$$

where $\phi_l\left(s\right)$ denotes the payoff on level l. The Multi-Level Monte Carlo estimator is given by

$$\hat{P} = \sum_{l=0}^{L} \hat{P}_l$$

$\hat{P}_0$ is an estimator for $\mathbb{E}\left[\phi_0\left(S\right)\right]$ calculated as a mean of N_l independent sample paths

$$\hat{P}_0 = \frac{1}{N_0} \sum_{i=1}^{N_0} \phi_0\left(s_i\right)$$

and $\hat{P}_l$ is an estimator for $\mathbb{E}\left[P_l - P_{l-1}\right]$ calculated as a mean of N_l independent sample paths

$$\hat{P}_l = \frac{1}{N_l} \sum_{i=1}^{N_l} \left(\phi_l\left(s_i\right) - \phi_{l-1}\left(s_i\right)\right)$$

The variance of the combined Multi-Level Monte Carlo estimator on level l is given by

$$\begin{aligned} Var\left[\hat{P}_l\right] &= Var\left[\frac{1}{N_l} \sum_{i=1}^{N_l} \left(\phi_l\left(s_i\right) - \phi_{l-1}\left(s_i\right)\right)\right] \\ &= \frac{1}{N_l^2} \sum_{i=1}^{N_l} Var\left[\phi_l\left(s_i\right) - \phi_{l-1}\left(s_i\right)\right] \\ &= \frac{V_l}{N_l} \end{aligned}$$

Thus, the variance of the combined Multi-Level Monte Carlo estimator is

$$Var\left[\hat{P}\right] = \sum_{l=0}^{L} Var\left[\hat{P}_l\right] = \sum_{l=0}^{L} \frac{1}{N_l} V_l$$

where $V_l = Var\left[\phi_l(S) - \phi_{l-1}(S)\right]$. Furthermore,

$$\begin{aligned} V_l &= Var\left[\phi_l(S) - \phi_{l-1}(S)\right] \\ &= Var\left[\phi_l(S)\right] + Var\left[\phi_{l-1}(S)\right] - 2Cov\left[\phi_l(S), \phi_{l-1}(S)\right] \end{aligned}$$

so the higher the correlation between $\phi_l(S)$ and $\phi_{l-1}(S)$, the lower the variance of the Multi-Level Monte Carlo estimator.

In order to minimize the variance of the Multi-Level Monte Carlo estimator for a given computational cost $C = \sum_{l=0}^{L} N_l \frac{T}{h_l}$, it is possible to use the Lagrange multiplier method. The Lagrangian is given by

$$\mathcal{L} = \sum_{l=0}^{L} \frac{1}{N_l} V_l + \lambda \left(\sum_{l=0}^{L} N_l \frac{T}{h_l} - C \right) \tag{11}$$

Differentiating (11) with respect to N_l and applying the first order condition shows that the variance is minimized at

$$N_l^* = \sqrt{\frac{V_l h_l}{\lambda T}} \tag{12}$$

With such choice of N_l the variance of the combined Multi-Level Monte Carlo estimator becomes

$$Var\left[\hat{P}\right] = \sum_{l=0}^{L} \frac{1}{N_l^*} V_l = \sum_{l=0}^{L} \sqrt{\frac{V_l \lambda T}{h_l}}$$

By Theorem 3.1 in Giles (2008), $Var\left[\hat{P}\right] \leq \frac{\epsilon^2}{2}$ where ϵ is a user-specified accuracy. It follows that,

$$\sum_{l=0}^{L} \sqrt{\frac{V_l \lambda T}{h_l}} \leq \frac{\epsilon^2}{2}$$

Using the definition of λ in (12) this becomes

$$N_l \geq 2\epsilon^{-2} \sqrt{V_l h_l} \left(\sum_{l=0}^{L} \sqrt{\frac{V_l}{h_l}} \right)$$

Therefore, the optimal number of sample paths for level l, in order to minimize the variance of the Multi-Level Monte Carlo estimator for a given computational cost C, is

$$N_l = \left\lceil 2\epsilon^{-2} \sqrt{V_l h_l} \left(\sum_{l=0}^{L} \sqrt{\frac{V_l}{h_l}} \right) \right\rceil \tag{13}$$

Overall, Monte Carlo has computational cost proportional to ϵ^{-3}, whereas that of the Multi-Level Monte Carlo is proportional to $\epsilon^{-2} (\log \epsilon)^2$ due to reduced variance.

References

[1] Arouna, B. (2004) Robbins-Monro Algorithms and Variance Reduction in Finance. Journal of Computational Finance, 7(2):35-62.

[2] Avramidis, A. (2002) Importance Sampling for Multimodal Functions and Applications to Pricing Exotic Options. Proceedings of the 2002 Winter Simulation Conference, pp. 1493-1501.

[3] Barraquand, J. (1995) Numerical Valuation of High Dimensional Multivariate European Securities. Management Science, 41(12):1882-1891.

[4] Bates D. (1996) Jumps and Stochastic Volatility: The Exchange Rate Processes Implicit in Deutschemark Options. Review of Financial Studies, 9(1):69-107.

[5] Bouzoubaa M. and A. Osseiran (2010) Exotic Options and Hybrids: A Guide to Structuring, Pricing and Trading. John Wiley & Sons.

[6] Capriotti, L. (2008) Least Squares Importance Sampling for Monte Carlo Security Pricing. Quantitative Finance, 8(5):485-497.

[7] Fouque, J.P. and T. Tullie (2002) Variance Reduction for Monte Carlo Simulation in a Stochastic Volatility Environment. Quantitative Finance, 2(1):24-30.

[8] Giles, M.B. (2008) Multi-Level Monte Carlo Path Simulation. Operations Research, 56(3):607-617.

[9] Grzelak, L. and C. Oosterlee (2011) On the Heston Model with Stochastic Interest Rates. SIAM Journal on Financial Mathematics, 2(1):255-286.

[10] Heston H. L. (1993) A Closed-form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. Review of Financial Studies, 6(2):327-343.

[11] Neddermeyer, J.C. (2011) Non-Parametric Partial Importance Sampling for Financial Derivative Pricing. Quantitative Finance, 11(8):1193-1206.

[12] Pellizzari, P. (1998) Efficient Monte Carlo Pricing of Basket Options. Working Paper.

[13] Stilger, P.S., S. Acomb and S.-H. Poon (2012) Pricing and Risk Management with Stochastic Volatility Using Importance Sampling. Working Paper.

[14] Su, Y. and M.C. Fu (2002) Optimal Importance Sampling in Securities Pricing. Journal of Computational Finance, 5(4):27-50.

[15] Zhang, P. G. (1998) Exotic Options: A Guide to Second Generation Options. World Scientific.